PRIMITIVE LIE PI-ALGEBRAS

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Dedicated to the memory of Ottmar Loos

Lie algebra, Jordan element, polynomial identity, primitive Lie algebra

Abstract. A Lie algebra L is called primitive if it is prime, nondegenerate, and contains a nonzero Jordan element a such that the attached Jordan algebra L_a is primitive. In this paper we prove that every primitive Lie PI-algebra over a field of zero characteristic is simple and finite-dimensional over its centroid.

Introduction

Whilst primitive associative PI-algebras and primitive Jordan PI-algebras are well described by Kaplansky's theorem [10, Theorem 8.3.6] and Zelmanov's theorem [16, Theorems 7 and 8] respectively, so far there is no a structure theorem for simple Lie PI-algebras, and it is not seem to be possible to get such a classification because, unlike the associative and Jordan cases, simple Lie PI-algebras (even over an algebraically closed field of zero characteristic) may lack minimal (abelian, in the Lie case) inner ideals (extremal elements when the base field is algebraically closed). An example that confirms this phenomenon is the Lie algebra of derivations of the algebra of polynomials $\mathbb{F}[x]$ in one variable over a field $\mathbb{F}$ (see [4, Example 2.1.3(v)] and [11, Example 2.10]). Therefore some additional condition is required for a simple Lie PI-algebra to be finite-dimensional. An example of the type of conditions is *local finiteness* [2, Theorem 2], which turns out to be central in all our study. Another example is speciality (recall that a Lie algebra L is said to be *special* if there exists an associative PI-algebra A such that L is embedded in the Lie algebra A^-). As proved in Theorem 1.9 of [7], a prime Lie algebra L over a field of zero characteristic is special if and only if its central closure $\tilde{L}$ is simple and of finite dimension over its centroid (equals the extended centroid of L). Note that this result is the special Lie version of Posner's theorem for prime associative PI-algebras.

In this paper we study conditions under which a simple Lie PI-algebra over a field of zero characteristic is finite-dimensional over its centroid, and prove that any primitive Lie PI-algebra (according to the notion of primitivity introduced in [11]) over a field of zero characteristic is simple and finite-dimensional over its centroid.

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1. Preliminaries

Algebras considered here are over a field $\mathbb{F}$ of zero characteristic, although some of the results do not require this scalar restriction. Most of the definitions and notations are taken from the book [11].

• Let L be a Lie algebra and let $a \in L$. We denote by ad_a the linear map $\mathrm{ad}_a : L \to L$ given by $\mathrm{ad}_a x = [a, x]$ for all $x \in L$, and by $\mathrm{Ad}(L)$ the subalgebra of the associative algebra $\mathrm{End}_{\mathbb{F}}(L)$ generated by the set $\{\mathrm{ad}_a : a \in L\}$.

• Any associative algebra A becomes a Lie algebra A^- under the Lie product $[x, y] = xy - yx$, $x, y \in A$.

• A Lie algebra L over a field $\mathbb{F}$ is called *finitary* (over $\mathbb{F}$) if it is a subalgebra of Lie algebra $\mathcal{F}(X)^-$ of finite rank linear operators on an $\mathbb{F}$-vector space X. It is clear that every simple finite-dimensional Lie algebra is finitary.

• A Lie algebra L over $\mathbb{F}$ is called a *special* Lie algebra or a Lie *SPI-algebra* if there exists an associative PI-algebra A over $\mathbb{F}$ such that L can be embedded in the Lie algebra A^-. It is proved in [14, Theorem 2.1.1] that if L is a Lie SPI-algebra, then $\mathrm{Ad}(L)$ is an associative PI-algebra.

• A Lie algebra L is said to be *nondegenerate* if $\mathrm{ad}_x^2 L = 0 \Rightarrow x = 0$, for $x \in L$. It is well known [11, Corollary 3.24] that any simple Lie algebra over a field of zero characteristic is nondegenerate.

• A Lie algebra L is said to be *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for I, J ideals of L. And L is called *strongly prime* if it is prime and nondegenerate.

• Let A be an associative algebra with center $Z(A)$. If A is semiprime (resp. prime) then the Lie algebra $A^-/Z(A)$ is nondegenerate (resp. strongly prime) [11, Proposition 3.35].

• Let A be a *prime* nonassociative algebra over an arbitrary field $\mathbb{F}$. Then A is said to be *centrally closed* if its *centroid* $\Gamma(A)$ coincides with its *extended centroid* $\mathcal{C}(A)$ (see [11, Sections 1.2 and 1.3] for definitions). A nonassociative algebra A is said to be *multiplicatively prime* (in short *m.p.*) whenever both A and its *multiplication algebra* $\mathcal{M}(A)$ are prime. It follows from Pikhtilkov's theorem quoted above that prime special Lie algebras are multiplicatively prime.

The following lemma will be used in the proof of our main result for the particular case of a Lie algebra.

Lemma 1.1. *Let A be a multiplicatively prime nonassociative algebra and let M be an ideal of A which is simple as an algebra. Then $\mathcal{C}(A) = \mathcal{C}(M) = \Gamma(M)$.*

Proof. By [8, Proposition 1], the condition $F(M) = 0$, for $F \in \mathcal{M}(A)$, implies $F = 0$. Then, by [6, Theorem 2.2], we have that $\mathcal{C}(A) = \mathcal{C}(M)$. Moreover, since M is a simple algebra, it is clear that $\mathcal{C}(M) = \Gamma(M)$. □

Remark 1.2. If A is not m.p., then the equality of its extended centroid with the centroid of any ideal M of A which is simple as an algebra cannot be guaranteed, as shown by the following example inspired in the celebrated Albert example of a unital prime 3-dimensional algebra which is not m.p. (see [9, Proofs of Lemma 4.4.83(iii) and Proposition 4.4.84(ii)]).

Let A be the three-dimensional real algebra with basis $\{u, v, w\}$ and multiplication table given by

	u	v	w
u	u	v	0
v	v	-u	v
w	0	0	u

It is not difficult to verify that A has only a nonzero proper ideal $M = \mathbb{R}u + \mathbb{R}v$, with $M^2 \neq 0$, so A is *prime*. It is also easy to see that M is isomorphic, as real algebra, to the complex field, with $u = 1$ and $v^2 = -1$, so $\Gamma(M) = \mathbb{C}$, i.e. any $\gamma \in \Gamma(M)$ is determined by a fixed pair (λ, μ) of real numbers: $\gamma_{(\lambda,\mu)}(\alpha u + \beta v) := (\lambda u + \mu v)(\alpha u + \beta v)$, $\alpha, \beta \in \mathbb{R}$. Finally, the reader can check that $\gamma_{(\lambda,\mu)} \in \mathcal{C}(A)$ if and only if $\mu = 0$, so $\mathcal{C}(A) = \mathbb{R}$, and that $\mathcal{M}(A)$ is not even semiprime: $L_w \neq 0$ and $L_w \mathcal{M}(A) L_w = 0$.

• An element a in a Lie algebra L is said to be *Jordan* if $\mathrm{ad}_a^3 L = 0$. It is easy to see that any zero square element of an associative algebra A is a Jordan element of its associated Lie algebra A^-.

• An *abelian inner ideal* of a Lie algebra L is a subspace B of L such that $[B, [B, L]] \subset B$ and $[B, B] = 0$. Any element in an abelian inner ideal of a Lie algebra L is a Jordan element of L, and conversely, for any Jordan element $a \in L$, $\mathrm{ad}_a^2 L$ is an abelian inner ideal of L [11, Lemma 4.5 and Proposition 4.6].

• An element $e \in L$ is called *von Neumann regular* if it is Jordan and $e \in \mathrm{ad}_e^2 L$. Again it is easy to check that for a zero square element a in an associative algebra A, a is a von Neumann element in A if and only if it is a von Neumann regular element in A^-.

• Let L be a Lie algebra over a field $\mathbb{F}$. A nonzero element $e \in L$ is called *extremal* if $\mathrm{ad}_e^2 L = \mathbb{F}e$. It is clear that any extremal element is von Neumann regular and generates a one-dimensional inner ideal.

• To any Jordan element $a \in L$ we attach a Jordan algebra L_a called the Jordan algebra of L at a [11, Theorem 8.43]. Although most of the properties of a Lie algebra can be transferred to its Jordan algebras, it is an open question whether the Jordan algebras of a simple Lie algebra are simple. However, the question is answered in the affirmative in some particular cases.

Proposition 1.3. *Let L be a simple Lie algebra over a field $\mathbb{F}$ of zero characteristic.*

(1) *If L has a nontrivial finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$, then any nonzero element $a \in L_n$ is Jordan and the Jordan algebra L_a is simple.*

(2) *For any nonzero von Neumann regular element e in L, L_e is a simple unital Jordan algebra.*

Proof. (1) As a consequence of the grading properties, a is a Jordan element in L. Now we have (see [11, Theorem 11.32]) that the associated Jordan pair $V = (L_n, L_{-n})$ is simple, and by [11, Proposition 11.42], for any nonzero element $a \in L_n$, the Jordan algebra L_a is isomorphic to the Jordan algebra V_a, which is simple by [1, Theorem 2.5].

(2) Using [11, Lemma 5.8 and Theorem 5.11], extend e to an $\mathfrak{sl}_2$-triple (e, f, h) with associated 5-grading $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ and Jordan pair $V = (\mathrm{ad}_e^2 L, \mathrm{ad}_f^2 L)$. Since $e \in L_2$, it follows from (1) that L_e is a simple Jordan algebra; that L_e is unital follows from [11, Proposition 8.61(i)]. □

2. Simple Lie PI-algebras

In this section we study conditions under which a simple Lie PI-algebra over a field of zero characteristic is finite-dimensional over its centroid. We begin with a technical lemma which will be frequently used in what follows. (As will be seen next, it proves something stronger than what it states.)

Lemma 2.1. *Let L be a simple Lie PI-algebra over a field of zero characteristic. If L contains a Jordan element a such that the Jordan algebra L_a is simple, then L is locally finite over its centroid.*

Proof. Let $a \in L$ be a Jordan element such that its attached Jordan algebra L_a is simple. Since L is PI, we have by [11, Proposition 8.57] that L_a is a Jordan PI-algebra. By Zelmanov's PI-theorem [16, Theorem 7], a simple Jordan PI-algebra is either finite-dimensional over its (associative) center Z, or isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form. In both cases, L_a contains minimal inner ideals and hence, by [11, Lemma 8.50], L itself contains a minimal abelian inner ideal. Since for any division element b in L, the division Jordan algebra L_b is PI ([11, Proposition 8.57] again), it follows from [11, Theorem 14.7] that L is *finitary* over its centroid and therefore locally finite by [11, Proposition 8.74]. □

The following theorem, due to Yu. A. Bahturin, is the key tool to prove all the others results of this paper. Condition (i) does not appear in Bahturin's theorem, but we think it is relevant for better understanding the important role played by Jordan elements.

Theorem 2.2. [2, Theorem 2] *Let L be a simple Lie algebra over a field of zero characteristic. Then the following conditions are equivalent:*

(i) *L is* PI *and contains an ad-algebraic element which is not ad-nilpotent,*
(ii) *L is* PI *and locally finite over its centroid,*
(iii) *L is finite-dimensional over its centroid.*

Proof. (i) ⇒ (ii). Let K be the centroid of L and $\overline{K}$ its algebraic closure. Then $\overline{L} := \overline{K} \otimes_K L$ is a central simple Lie PI-algebra over $\overline{K}$. Moreover, any ad-algebraic element in L remains ad-algebraic in $\overline{L}$, so $\overline{L}$ contains an algebraic element which is not ad-nilpotent. Then, by [11, Example 1.4], $\overline{L}$ has a nontrivial finite $\overline{K}$-grading, which without loss of generality we may assume it to be a $\mathbb{Z}$-grading $\overline{L} = \overline{L}_{-n} \oplus \cdots \oplus \overline{L}_n$. Hence, by Proposition 1.3(1), $\overline{L}$ contains an element a such that the Jordan algebra $\overline{L}_a$ is simple, which implies by Lemma 2.1 that $\overline{L}$ is locally finite over $\overline{K}$, equivalently, L is locally finite over K.

(ii) ⇒ (iii). Let K, $\overline{K}$, and $\overline{L}$ be as above. Then $\overline{L}$ is a simple locally finite Lie PI-algebra over $\overline{K}$. By [11, Lemma 10.1], $\overline{L}$ is finite-dimensional over $\overline{K}$ and hence $\dim_K L = \dim_{\overline{K}} \overline{L}$ is also finite.

(iii) ⇒ (i). Since L has finite dimension, it is PI and any of its elements is ad-algebraic. But not all them can be ad-nilpotent, for in that case L would be nilpotent by the Engel–Jacobson Theorem [11, Theorem 2.54], which leads to a contradiction. □

Remarks 2.3. (1) It is proved in [3, Proposition 1.8] that any simple SPI-algebra is finite-dimensional over its centroid.

(2) Let L be an n-dimensional central simple Lie algebra over a field $\mathbb{K}$. Then L satisfies the nontrivial polynomial identity

$$\sum_{\sigma\in S_{n+1}} (-1)^{\sigma}[x_{\sigma(1)},\cdots,x_{\sigma(n)},x_{\sigma(n+1)},x]=0,$$

where for any positive integer m, S_m denotes the group of permutations of m elements, and the left commutator $[x_1,\cdots,x_m]$ is defined recursively:

$$[x_1]=x_1,\ [x_1,x_2,\cdots,x_{m+1}]=[x_1,[x_2,\cdots,x_{m+1}]].$$

Thus L is a PI-algebra over any subfield of $\mathbb{K}$.

(3) Take the field of fractions $\mathbb{F}(x)$, where $\mathbb{F}$ is a field of zero characteristic. For $n>1$, the Lie algebra $\mathfrak{s}l(n,\mathbb{F}(x))$ is finite-dimensional and central simple over $\mathbb{F}(x)$, but it is not locally finite over $\mathbb{F}$: the subalgebra generated by the entry matrices $x[12]$ and $x[21]$ is infinite-dimensional over $\mathbb{F}$.

Corollary 2.4. *Let L be a simple Lie PI-algebra over a field of zero characteristic. If L contains a Jordan element a such that the Jordan algebra L_a is simple, then L is finite-dimensional over its centroid.*

Proof. By Lemma 2.1, L is locally finite over its centroid. Now Theorem 2.2 applies. □

Corollary 2.5. *Let L be a simple Lie* PI-*algebra over a field of zero characteristic. If L contains a nonzero von Neumann regular element, then L is finite-dimensional over its centroid.*

Proof. Let e be a nonzero von Neumann regular element in L. By Proposition 1.3(2), the Jordan algebra L_e is simple. Now Corollary 2.4 applies. □

Corollary 2.6. *Let L be a simple Lie* PI-*algebra over a field of zero characteristic. If L contains a minimal abelian inner ideal, then L is finite-dimensional over its centroid.*

Proof. Let B be a minimal abelian inner ideal of L. Since L is nondegenerate, we have by [11, Lemma 4.19] that any element in B is von Neumann regular. Now Corollary 2.5 applies. □

Remark 2.7. Corollary 2.6 can be directly derived from Corollary 2.4 (avoiding thus the use of Jordan pair theory). Indeed, it follows from [11, Proposition 8.63] that for any nonzero element b in a minimal abelian inner ideal of L, L_b is a division, and therefore simple, Jordan algebra.

Corollary 2.8. *Let L be a simple Lie* PI-*algebra over a field $\mathbb{F}$ of zero characteristic. If L contains an extremal element, then L is finite-dimensional over $\mathbb{F}$.*

Proof. Let $e\in L$ be an extremal element. Then $\mathrm{ad}_e^2 L$ is a minimal abelian inner ideal, so, by Corollary 2.6, L is finite-dimensional over its centroid, which coincides with $\mathbb{F}1_L$ by [11, Lemma 6.5]. □

Remark 2.9. Corollary 2.8 can be directly derived from Theorem 2.2 and the fact that L is central over $\mathbb{F}$. Indeed, by [11, Proposition 5.21], L is generated by any extremal element and hence it is locally finite by [11, Corollary 6.16].

3. Primitive Lie PI-algebras

A Lie algebra L is said to be *primitive* if it is strongly prime and contains a nonzero Jordan element a such that its attached Jordan algebra L_a is primitive (see [11, 8.34] for the definition of primitive Jordan algebra). Primitive Lie algebras were introduced in [11], proving among other results that this definition is consistent with the usual one for associative algebras [11, Propositions 8.70 and 8.72].

Remark 3.1. There is another notion of primitivity for Lie algebras, in analogy with the case of associative algebras: a Lie algebra L is primitive if it has a faithful irreducible representation. As shown in [13], the class of these primitive Lie algebras is quite ample. For instance, any finite-dimensional semisimple Lie algebra over an algebraically closed field of zero characteristic is primitive. Therefore, according to this definition, a primitive Lie algebra is not necessarily prime.

Theorem 3.2. *Every primitive Lie* PI-*algebra L over a field $\mathbb{F}$ of zero characteristic is simple and finite-dimensional over its centroid.*

Proof. By [11, Proposition 8.69], L has nonzero socle, say $\mathrm{Soc}(L) = S$. And by [11, Theorem 5.22], S is a simple Lie PI-algebra containing minimal abelian inner ideals, what in virtue of Corollary 2.6 implies that S is finite-dimensional over its centroid, say $\Gamma(S) = K$. The proof will be complete by proving that L and S have equal centroids, since in this case, via adjoint representation, L is embedded into $\mathrm{Der}(S)$, with every derivation of S being inner.

Suppose for the moment that L is centrally closed over $\mathbb{F}$, i.e. the centroid extended $\mathcal{C}(L)$ of L coincides with $\mathbb{F}$. It is clear that K is a field extension of $\mathbb{F} = \mathcal{C}(L)$. We claim that $K = \mathbb{F}$. Let $n = \dim_K(S)$. We have $L \leq \mathrm{Der}(S) \leq \mathrm{M}_n(K)^-$ as $\mathbb{F}$-algebras. Thus L is a special Lie algebra over $\mathbb{F}$. But any prime special Lie PI-algebra over a field of zero characteristic is multiplicatively prime. Hence K coincides with $\mathbb{F}$ by Lemma 1.1.

Coming back to the general case, let $\tilde{L}$ denote the central closure of L. It is known (see [5]) that $\tilde{L}$ is a prime Lie algebra which is generated as a $\mathcal{C}(L)$-vector space by L, and $\tilde{L}$ is centrally closed, i.e. $\Gamma(\tilde{L}) = \mathcal{C}(\tilde{L}) = \mathcal{C}(L)$. Hence $\tilde{L}$ is PI, and since K is a field extension of $\mathcal{C}(L)$, S remains as an ideal of $\tilde{L}$. Now it follows from [11, Propositions 4.12 and 8.67] that $\tilde{L}$ is primitive. Then, by the previous case, $S \leq L \leq \tilde{L} = S$ forces $L = S$, so L is simple and finite-dimensional over its centroid. □

Remark 3.3. The converse of Theorem 3.2 does not hold in general (the orthogonal real algebra $\mathfrak{o}(n)$ does not contain nonzero Jordan elements). Nevertheless, if $\mathbb{F}$ is algebraically closed, then L is primitive.

Corollary 3.4. *Let L be a strongly prime Lie PI-algebra over a field $\mathbb{F}$ of zero characteristic. If L contains a minimal abelian inner ideal, then L is simple and finite-dimensional over its centroid.*

Proof. By [11, Proposition 8.68], L is primitive. Now Theorem 3.2 applies. □

Remark 3.5. If in Corollary 3.4 "minimal abelian inner ideal" is replaced by "extremal element", then both S and L (in the proof of Theorem 3.2) are automatically centrally closed over $\mathbb{F}$ [11, Lemma 6.5 and Corollary 6.6]. Thus it is not necessary to involve neither special nor multiplicatively prime algebras in the proof of this particular case.

A Lie algebra L over a field $\mathbb{F}$ is called *algebraic* if for any $a \in L$, the adjoint map ad_a is algebraic over $\mathbb{F}$.

Corollary 3.6. [11, Corollary 10.3] *Let L be a strongly prime Lie PI-algebra over an algebraically closed field $\mathbb{F}$ of zero characteristic. If L is algebraic over $\mathbb{F}$, then L is simple and finite-dimensional.*

Proof. By [11, Corollary 4.32], L contains a nonzero Jordan element, say a. Then the Jordan algebra L_a is strongly prime and PI. But such a Jordan algebra (see [11, Remark 8.42]), is simple and contains minimal inner ideals. As we have just seen, this implies that L itself contains minimal abelian inner ideals and hence extremal elements since $\mathbb{F}$ is algebraically closed. Now the argument of Remark 3.5 applies. □

Remark 3.7. Actually, a strongly prime algebraic Lie PI-algebra over a field $\mathbb{F}$ of zero characteristic (not necessarily algebraically closed) is simple and finite-dimensional over its centroid, which is an algebraic field extension of $\mathbb{F}$. This result was proved in [12, Theorem 1.1] combining Corollary 3.6 with Zelmanov's theorem for algebraic Lie PI-algebras [15, Theorem 1], and in [11, Theorem 10.11] assuming that L is algebraic of bounded degree (a condition stronger than being PI [11, Lemma 10.4]).

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