Imbedding Jordan Systems in Primitive Systems [1]

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Abstract: We show that special Jordan systems J which are free of torsion over a scalar ring imbed in primitive (hence strongly prime) Jordan systems $\widetilde{J}=J\oplus Heart(\widetilde{J})$ with simple primitive heart. We show examples of exceptional Jordan systems which cannot be imbedded in a nondegenerate system.

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Introduction

In [14], every nondegenerate Jordan algebra was shown to imbed in a semiprimitive Jordan algebra with the same polynomial identities.

In [3], it is shown how to "paste" a simple heart to an arbitrary associative system R over a field, so that the original system imbeds in a primitive system $\widetilde{R}$. Unlike [14], the polynomial identities of R are not preserved in this procedure and, in fact, looking at the precise way of building $\widetilde{R}$, one can check that $\widetilde{R}$ and its heart are not PI. Recently, in [4], the results obtained in [3] were improved in two ways: allowing more general rings of scalars, and considering systems with involution. Analogues for Lie algebras have been obtained in [5]. Both the results of [5] and those obtained here play a fundamental role in the forthcoming paper [8], devoted to the study of the inheritance of nondegeneracy by semiprime quotients in Lie and Jordan systems.

We will investigate whether the process for pasting a simple heart also applies to Jordan systems. Unlike the Lie algebra case [5], we will not need to start with the

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study of systems over fields, and will proceed directly working over general rings of scalars. This is due to the simpler form of Herstein's Theorems for Jordan systems built out of associative systems [6, 15], and also to the deeper knowledge in the context of Jordan systems of our regularity conditions. Our results will revealed linked to the question of the existence of absolute zero divisors in free objects.

The paper is divided into two sections, plus a preliminary one devoted to list some basic notions and properties. In the first section we obtain versions of the main results of [4] for special Jordan systems. We will use Herstein-type theorems on the transfer of regularity conditions from an associative system to the Jordan system built out of it, and also some results on the inheritance of regularity conditions by ideals. The second section is devoted to showing that speciality cannot be removed as a condition in our main theorem of the previous section. We will give examples of exceptional Jordan algebras over a field that cannot be imbedded in a nondegenerate Jordan algebra. In a philosophical sense, this shows that, unlike what happens in associative systems or Lie algebras, even nondegenerate systems are not wide enough to contain full information about the entire class of Jordan systems.

0. Preliminaries

0.1 We will deal with associative and quadratic Jordan Φ-systems (algebras, triple systems and pairs over an arbitrary ring of scalars Φ). The reader is referred to [1, 9, 13, 17, 18] for basic facts and notions not explicitly mentioned in this section.

— Associative products will be denoted by juxtaposition.

— Given a Jordan algebra J, its products will be denoted by x^2, U_xy, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z}y = \{x, y, z\} = V_{x,y}z$, respectively.

— For a Jordan pair $V = (V^+, V^-)$, we have products $Q_xy \in V^\varepsilon$, for any $x \in V^\varepsilon$, $y \in V^{-\varepsilon}$, $\varepsilon = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$.

— A Jordan triple system J is given by its products P_xy, for any $x, y \in J$, with linearizations denoted by $P_{x,z}y = \{x, y, z\} = L_{x,y}z$.

In the case of pairs $R = (R^+, R^-)$ or $J = (J^+, J^-)$, a module always means a pair of modules $M = (M^+, M^-)$, and module endomorphisms T are always pairs of endomorphisms $T = (T^+, T^-)$ for $T^\sigma \in \mathrm{End}_\Phi(M^\sigma)$.

0.2 Given an associative or Jordan system M, the *heart* $\mathrm{Heart}(M)$ of M is the intersection of all nonzero ideals of M.

If J is a nondegenerate Jordan system, Heart(J) is the unique simple ideal of J when it is nonzero (cf. [7, 2.6, 3.6, 3.8]).

Conversely, if a strongly prime Jordan system J has an ideal I which is a simple system, then Heart$(J) = I$ [7, 0.9].

0.3 (i) A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [13, 1.13, 1.14].

(ii) An associative system R gives rise to a Jordan system $R^{(+)}$ by *symmetrization*: over the same Φ-module, we define $x^2 = xx$, $U_x y = xyx$, for any $x, y \in R$ in the case of algebras, $P_x y = xyx$ in the case of triple systems, and $Q_{x^\sigma} y^{-\sigma} = x^\sigma y^{-\sigma} x^\sigma$, $\sigma = \pm$ in the pair case.

0.4 A Jordan system J is said to be *special* if there exists an associative system R such that J is a subsystem of $R^{(+)}$. If, in addition, R is generated as an associative system by J, then R is called an *associative envelope* of J.

An associative envelope R of J is said to be *tight* if every nonzero ideal of R hits J ($I \cap J \neq 0$, for any nonzero ideal I of R).

A Jordan system which is not special is called *exceptional.*

0.5 If J is a special Jordan system, there exists an associative envelope of J which is tight. Indeed, if J is a subsystem of $R^{(+)}$, for some associative system R, the associative subsystem S of R generated by J is an envelope of J. Moreover, S can be *tighten* by factoring out a maximal ideal of S not hitting J, whose existence follows from a straightforward application of Zorn's lemma.

0.6 Let M be a nonzero Φ-module (by our conventions a pair of modules in the case of pairs).

(i) The *annihilator of M in* Φ, $\mathrm{Ann}_\Phi(M) := \{\lambda \in \Phi \mid \lambda M = 0\}$ is the kernel of the natural ring homomorphism of Φ into $\mathrm{End}_{\mathbf{Z}}(M)$. Let $\overline{\Phi}$ denote the quotient $\Phi / \mathrm{Ann}_\Phi(M)$. Notice that $\overline{\Phi}$ is isomorphic to the image of Φ in $\mathrm{End}_{\mathbf{Z}}(M)$ and M becomes a $\overline{\Phi}$-module. The $\overline{\Phi}$-submodules, subalgebras, ideals, etc. are exactly the same as the Φ-modules, subalgebras, etc. But now M becomes a *faithful* $\overline{\Phi}$-module (cf. [10, Lemmas 1.1.1, 1.1.2]). Moreover, if M is an associative or Jordan system over Φ, then it is also an associative or Jordan system over $\overline{\Phi}$.

(ii) Conversely, if we take any proper ideal I of Φ, any Φ/I-module, or associative or Jordan system over Φ/I can be viewed as a Φ-module, or associative or Jordan system over Φ, respectively, in the obvious manner.

(iii) A stronger condition is that M is free of Φ-torsion or that Φ acts without torsion on M: $\alpha x = 0 \Longrightarrow \alpha = 0$ or $x = 0$. Note that if $M \neq 0$ this implies that Φ is an integral domain: $\alpha \neq 0, \alpha\beta = 0, x \neq 0 \Longrightarrow \alpha(\beta x) = 0 \Longrightarrow \beta x = 0 \Longrightarrow \beta = 0$.

This paper is aimed at studying Jordan analogues of the following results for associative systems.

0.7 If a nonzero associative Φ-system R imbeds in a prime associative Φ-system $\widetilde{R}$ then $\overline{\Phi} = \Phi/\operatorname{Ann}_\Phi(R)$ acts without torsion on R (cf. [4, 2.1]).

0.8 Let R be a nonzero associative Φ-system such that $\overline{\Phi} = \Phi/\operatorname{Ann}_\Phi(R)$ acts without torsion on R. Then there exists a *hearty primitive envelope of* R, an associative Φ-system $\widetilde{R}$ such that:

(i) R is a subsystem of $\widetilde{R}$,

(ii) $\widetilde{R}$ is a left primitive system, hence it is prime,

(iii) $\operatorname{Heart}(\widetilde{R})$ is simple and left primitive,

(iv) $\widetilde{R} = R \oplus \operatorname{Heart}(\widetilde{R})$, hence $\widetilde{R}/\operatorname{Heart}(\widetilde{R}) \cong R$ (cf. [4, 2.2]).

1. Growing Hearts in Special Jordan Systems

An analogue of (0.7) holds for Jordan systems.

1.1 Proposition. *If J is a nonzero Jordan Φ-system which imbeds in a prime Jordan Φ-system $\widetilde{J}$, then $\overline{\Phi} = \Phi/\operatorname{Ann}_\Phi(J)$ acts without torsion on J.*

Proof: Without loss of generality, we may assume that $J \subseteq \widetilde{J}$. If $0 \neq \alpha + \operatorname{Ann}_\Phi(J) \in \overline{\Phi}$ and there exists $0 \neq r \in J$ satisfying $\alpha r = 0$, then $Id_{\widetilde{J}}(r)$ and $\alpha\widetilde{J}$ would be nonzero orthogonal ideals of $\widetilde{J}$, contradicting primeness. ■

1.2 We will need some known facts about regularity conditions in Jordan systems:

(i) A primitive Jordan system (algebra, triple system, or pair) is strongly prime [11, 5.5; 9, 0.8; 1, 3.7, 3.9].

(ii) If J is a primitive Jordan system and I is a nonzero ideal of J, then I is primitive [9, 3.1; 1, 5.4, 4.2; 2, 2.4, 2.2].

(iii) If J is a prime Jordan system and I is a nonzero ideal of J such that I is primitive, then J is primitive [9, 3.2; 1, 5.6, 4.4; 2, 2.4, 2.2].

1.3 Lemma. *Let I be a nonzero ideal of a strongly prime Jordan system J, and J_0 be a subsystem of J such that $I \subseteq J_0$. Then J_0 is also strongly prime.*

Proof: Let us assume that we are dealing with triple systems. The annihilator of I in J is $\mathrm{Ann}_J(I) = \{x \in J|\ P_x I = 0\}$ (by [16, 1.7(i)]), which is 0 since J is strongly prime and $I \neq 0$ [16, 1.6(iii)]. Thus J_0 is nondegenerate: clearly any absolute zero divisor of J_0 belongs to $\mathrm{Ann}_J(I)$, hence is zero. Moreover, J_0 is prime: any nonzero ideal K of J_0 hits I (since $K \cap I = 0 \Longrightarrow P_K I \subseteq K \cap I = 0 \Longrightarrow K \subseteq \mathrm{Ann}_J I = 0$); if K_1 and K_2 were nonzero orthogonal ideals of J_0, then $K_1 \cap I$ and $K_2 \cap I$ would be nonzero orthogonal ideals of I, which is impossible since I is strongly prime by [16, 2.5]. Thus J_0 is strongly prime.

For pairs and algebras, one just need to apply the result for Jordan triple systems using the functors T and V, and [6, 0.5]. ∎

1.4 Theorem. *If J is a nonzero special Jordan Φ-system free of Φ-torsion, then there exists a special Jordan Φ-system $\widetilde{J}$ such that*

(i) *J is a subsystem of $\widetilde{J}$,*

(ii) *$\widetilde{J}$ is primitive (hence strongly prime),*

(iii) *$\mathrm{Heart}(\widetilde{J})$ is simple and primitive as Jordan system,*

(iv) *$\widetilde{J} = J \oplus \mathrm{Heart}(\widetilde{J})$ (hence $\widetilde{J}/\mathrm{Heart}(\widetilde{J}) \cong J$),*

that will be called hearty primitive Φ-envelope of J.

Proof: Let R be an associative tight envelope of J (see (0.4) and (0.5)). Then R is free of Φ-torsion: if $\alpha r = 0$ for $0 \neq \alpha \in \Phi$ and $0 \neq r \in R$ then $K := \{x \in R \mid \alpha x = 0\}$ is a nonzero ideal of the associative system R, hence $K \cap J \neq 0$ by tightness, and there exists a nonzero element $x \in J$ such that $\alpha x = 0$, contradicting that J is free of Φ-torsion.

Now, $\overline{\Phi} = \Phi$, and we can apply (0.8) to find a hearty primitive Φ-envelope $\widetilde{R} = R \oplus H \supseteq R$ such that

(1) $H := \mathrm{Heart}(\widetilde{R})$ is simple and left primitive;

(2) $\widetilde{R}$ is left primitive, hence prime.

We are not claiming that $\widetilde{J} := J \oplus H$ is all of $\widetilde{R}$. But $J \subseteq R \subseteq \widetilde{R}$ so J is a subsystem of $\widetilde{R}^{(+)}$, and the associative ideal H is also a Jordan ideal of $\widetilde{R}^{(+)}$, so $\widetilde{J}$ is at least a Jordan subsystem of $\widetilde{R}^{(+)}$.

Moreover,

(3) $\widetilde{R}^{(+)}$ is primitive

by [9, 4.2; 1, 5.8, 4.5] since $\widetilde{R}$ is left primitive, hence

(4) $\widetilde{R}^{(+)}$ is strongly prime

by (1.2)(i). Then

(5) $H^{(+)}$ is primitive, hence strongly prime

by (1.2)(ii). Moreover, using [15, Th. 4; 6, 1.7(ii)], we have

(6) $H^{(+)}$ is simple.

We have that $\widetilde{J}$ is a subsystem of $\widetilde{R}^{(+)}$, which is strongly prime (4), and $\widetilde{J}$ contains the nonzero ideal $H^{(+)}$ of $\widetilde{R}^{(+)}$, hence $\widetilde{J}$ is strongly prime by (1.3). Since the ideal $H^{(+)}$ of $\widetilde{J}$ is simple (6), $H^{(+)} = \mathrm{Heart}(\widetilde{J})$ by (0.2), and we have (iii) by (5) and (6). Moreover, $H^{(+)}$ being primitive (5) implies that $\widetilde{J}$ is primitive by (1.2)(iii), and we have (ii). ■

Finally we obtain the desired version of (0.8), which is the converse of (1.1) for special Jordan systems.

1.5 COROLLARY. *If J is a nonzero special Jordan Φ-system such that $\overline{\Phi} = \Phi/\operatorname{Ann}_\Phi(J)$ acts without torsion on J, then there exists a hearty primitive Φ-envelope $\widetilde{J}$ of J.*

PROOF: We just need to apply (1.4) to J as a $\overline{\Phi}$-system, and read the result in terms of Φ-systems (0.6)(ii). ■

2. Exceptional Examples

In this section we will give examples of Jordan systems (necessarily exceptional in view of (1.4)) which cannot be imbedded in strongly prime (or even merely nondegenerate) systems.

2.1 Recall the Jacobson Counterexample [12, ex. 3, p. 12]: If Φ is a ring of scalars such that $2\Phi = 0$, and $A = \Phi[x]$ is the usual unital commutative associative algebra of polynomials in a variable x, then $J := A^{(+)}/I = \Phi\bar{1} \oplus \Phi\bar{x} \oplus \Phi\bar{x}^3$ is an i-special Jordan algebra for $I = \Phi x^2 \oplus x^4 A$ (the span of $x^2, x^4, x^5, \ldots$), which is an ideal of the Jordan algebra $A^{(+)}$. The element $a = \bar{x}$ has $a^2 = 0, a^3 \neq 0$ and is an absolute zero divisor since $U_{a^3} = U_a U_{a^2} = U_a 0 = 0$. As a consequence, J is a degenerate exceptional Jordan algebra. Moreover, it cannot be imbedded in any nondegenerate Jordan Φ-algebra $\widetilde{J}$ since a^3 will still be nonzero (by the imbedding) and still $\widetilde{U}_{a^3} = \widetilde{U}_a \widetilde{U}_{a^2} = 0$.

Call a nonzero element z of a Jordan Φ algebra J a *Jacobson obstacle* if $z^2 = 0, z^3 \neq 0$. Such an element instantly renders all envelopes $\widetilde{J} \supseteq J$ exceptional, and degenerate since z^3 is an absolute zero divisor of $\widetilde{J}$.

2.2 Notice that (2.1) shows that (1.4) cannot be extended to arbitrary Jordan systems, not even to i-special Jordan systems (remember that our 3-dimensional example J of (2.1) is the quotient of a special algebra, hence is i-special).

2.3 The existence of Jacobson obstacles in the example above depends strongly on the fact that Φ is a ring such that $2\Phi = 0$. To find examples in the linear setting where $\frac{1}{2} \in \Phi$, we can make use of absolute zero divisors in free systems which behave as a^3 in (2.1) in the sense that they are absolute zero divisors in an intrinsic way, no matter the system where they are imbedded.

A *generic Medvedev-Zelmanov* Φ*-obstacle* is a nonzero element $p(x_1, \ldots, x_n)$ in a free Jordan system (algebra, triple, pair) $FJ_\Phi[X]$ which is an absolute zero divisor,

$$U_{p(x_1,\ldots x_n)} = 0,$$

and it does not involve all the variables in X. Therefore any nonzero value $z = p(a_1, \ldots, a_n)$ for a_i in a Φ-system J will be an absolute zero divisor in all envelopes $\widetilde{J} \supseteq J$ of J: $\widetilde{U}_z \widetilde{a} = 0$ for all $\widetilde{a} \in \widetilde{J}$ since there exists a homomorphism $\varphi : FJ_\Phi[X] \to \widetilde{J}$ with $x_i \to a_i$ for $1 \leq i \leq n$ and $x \to \widetilde{a}$ for any $x \in X \setminus \{x_1, \ldots, x_n\}$, so that $0 = \varphi\big(U_{p(x_1,\ldots,x_n)} x\big) = \widetilde{U}_{p(a_1,\ldots,a_n)} \widetilde{a} = \widetilde{U}_z \widetilde{a}$.

Such an element $z \in J$ will be called a *Medvedev-Zelmanov* Φ*-obstacle*: it renders all envelopes of J degenerate. It renders all envelopes exceptional too if J is free of Φ-torsion since any J having Medvedev-Zelmanov Φ-obstacles is itself exceptional by (1.4).

Notice that, as soon as a generic Medvedev-Zelmanov Φ-obstacle p exists, we can always find a Jordan Φ-system J which is finitely spanned as a Φ-module and contains Medvedev-Zelmanov Φ-obstacles. Indeed, we just need to take the span I of all Jordan monomials m for which there exists a variable $x \in X$ such that the degree of m in x is bigger than the degree of p in x. Such I is an ideal of $FJ_\Phi[X]$ and the quotient $J := FJ_\Phi[X]/I$ is finitely spanned as a Φ-module. Moreover, the natural epimorphism $\varphi : FJ_\Phi[X] \longrightarrow J$ satisfies $\varphi(p) \neq 0$, so that $\varphi(p)$ is the desired Medvedev-Zelmanov Φ-obstacle.

2.4 Zelmanov shows in [20, 21] that the existence of nonzero absolute zero divisors in a free Jordan algebra $FJ_{alg}[X]$ on an infinite countable set of variables X is equivalent to the existence of nilpotent elements if the ring of scalars Φ is a field of characteristic not two. Medvedev [19, Th. 2] shows that there exist nilpotent elements in $FJ_{alg}[X]$ if $|X| \geq 32$, the ring of scalars Φ contains $\frac{1}{2}$, and $(14!)^2 \neq 0$ in Φ. Putting together the results of Zelmanov and Medvedev, the existence of generic Medvedev-Zelmanov obstacles is proved if Φ is a field of characteristic not dividing 14! (so characteristic 0 or bigger than 13).

Concerning Jordan pairs, Zelmanov proves the existence of absolute zero divisors in the free Jordan pair $FJ_{pair}[X]$ for an infinite countable set of variables X if Φ is a domain without 28!-torsion [22, Th. 9], so that again generic Medvedev-Zelmanov obstacles exist if Φ is a field of characteristic not dividing 28! (so characteristic 0 or bigger than 23).

The existence of absolute zero divisors or nilpotent elements in free Jordan systems in a general quadratic setting, or over less restrictive rings of scalars (for example, over fields of small characteristic) is not known.

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