# Homotope polynomial identities in prime Jordan systems

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**Abstract**: We prove an analogue of Posner-Rowen's theorem for strongly prime Jordan pairs and triple systems: the central closure of a strongly prime Jordan system satisfying a homotope polynomial identity is simple with finite capacity. We also prove that if a Jordan system satisfies a homotope polynomial identity it also satisfies a strict homotope polynomial identity.

Keywords: Jordan triple system, Jordan pair, polynomial identity

## Introduction

Polynomial identities play a fundamental role Zelmanov's structure theory of Jordan systems and even more explicitely in D'Amour and McCrimmon's cuadratic extension. Those results lead, in its applications to problems concerning strongly prime systems to the case-splitting of PI-systems and non-PI (hence hermitian) systems. For algebras, the efectiveness of this approach lies in the possibility of reducing the the study in PI-case to the study of algebras with finite capacity thanks to the analogue of Posner-Rowen's theorem for Jordan algebras [4]. For Jordan pairs and triple systems, in view of D'Amour and McCrimmon's results [2], the distinction is based on the notion of homotope polynomial identity (see [1]). Here there is also a weak analogue of Posner-Rowen's theorem proved in [17] that allows the use of socle-related techniques and that makes use of the extended centroid, and its attached central extension, the extended central closure, rather than of the usual central closure. That the stronger version with the cnetral closure holds was cojectured in [17] and its proof is the aim of the present paper.

Before addressing that problem, we turn to the original weak version of the theorem that makes use of the extended central closure [17]. Although the result immediately follws from the results of [17] if the system strictly satisfies a homotope polynomial identity, the proof of the more general case, where the identity is not supposed to hold strictly, requires some extra work. We complete that proof in section 1. In section 2 we prove that a nondegenerate Jordan pair satisfying a homotope

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polynomial identity, strictly satisfies some homotope polynomial identity, but, unless what one obtains with the usual linearization process, which does not preserve homotope polynomials, the homotope polynomial identity strictly satisfied by the system has bigger degree that the original identity. This does not make superfluous the general case considered before, since its proof is based on that result.

Finally, in section 3, we prove that in a strongly prime homotope-PI Jordan system $J$, for every nonzero ideal $I$ of $J$ there is an element $\gamma$ belonging to the centroid (and to the multiplication algebra of $J$) such that $\gamma J \subseteq I$, which yields the analogue of Posner-Rowen's theorem for Jordan systems.

## 0. Preliminaries

Throughout $\Phi$ will be a fixed unital commutative ring.

**0.1** We will work with Jordan pairs, triple systems, and algebras over $\Phi$. We refer to [7, 8, 15] for notation, terminology, and basic results. We record in this section some of those notations and results.

A Jordan algebra $J$ has products $x^2$ and $U_x y$, quadratic in $x$ and linear in $y$, whose linearizations are $x \circ y = V_x y = (x+y)^2 - x^2 - y^2$, and $U_{x,y}z = V_{x,z}y = \{x,z,y\} = U_{x+y}z - U_x z - U_y z$, respectively

A Jordan Pair $V = (V^+, V^-)$ has products $Q_x y$ for $x \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = D_{x,y}z = \{x,y,z\} = Q_{x+z}y - Q_x y - Q_z y$.

A Jordan triple system $T$ has products $P_x y$ whose linearizations are $P_{x,y}z = L_{x,z}y = \{x,z,y\} = P_{x+y}z - P_x z - P_y z$.

A Jordan algebra gives rise to a Jordan triple systems with $P = U$. If a Jordan triple systems has an element 1 with $P_1 = Id$, the identity, then it is a unital Jordan algebra with square $x^2 = P_x 1$.

We will make use of the identities of Jordan pairs (and their corresponding triple versions) proved in [8], and of the identities of Jordan algebras proved in [7]. We refer to those identities by the labellings JPx of [8] and QJx of [7].

**0.2** Doubling a Jordan triple system $T$ produces a Jordan pair $V(T) = (T,T)$ with $Q_x y = P_x y$. Reciprocally, each Jordan pair $V = (V^+, V^-)$ gives rise to a polarized triple system $T(V) = V^+ \oplus V^-$ with product $P_{x^+ \oplus x^-} y^+ \oplus y^- = Q_{x^+} y^- \oplus Q_{x^-} y^+$. Niceness conditions such as nondegeneracy, primeness, strong primeness and others are inherited by the polarized triple system of a Jordan pair. However this does no longer hold in the reverse direction, from Jordan triple systems to their double Jordan pairs. To remedy that situation, D'Amour and McCrimmon [1, p. 229], and Anquela and Cortés [3, p. 667] defined tight doubles:

Given a Jordan triple system $T$, a tight double of $T$ is a quotien pair $V(T)/I = (T/I^+, T/I^-)$ where $I$ is an ideal $(I^+, I^-)$ of $V(T)$ which is maximal with respect to $I^+ \cap I^- = 0$ (so that the $I^\sigma$ are semi-ideals of $T$, but they may not be ideals). These always exist and share niceness properties with $T$ (see 5.2 and 5.3 of [3]). Moreover, for a strongly prime $J$, the ideal $I$ is unique up to the exchange involution: if $V(J)/L$ is another tight double, then either $L = I$ or $L^{op} = (L^-, L^+) = I$ [18, 0.2].

**0.3** Let $V = (V^+, V^-)$ be a Jordan pair over $\Phi$. Recall that the centroid $\Gamma(V)$ of $V$ is the set of all pairs $\gamma = (\gamma^+, \gamma^-) \in \mathrm{End}_\Phi(V^+) \times \mathrm{End}_\Phi(V^-)$ which satisfy:

$$\gamma^\sigma Q_{x^\sigma} = Q_{x^\sigma}\gamma^{-\sigma}, \quad \gamma^\sigma L_{x^\sigma,y^{-\sigma}} = L_{x^\sigma,y^{-\sigma}}\gamma^\sigma \quad \text{and} \quad Q_{\gamma^\sigma(x^\sigma)} = (\gamma^\sigma)^2 Q_{x^\sigma},$$

for all $x^\sigma \in V^\sigma, y^{-\sigma} \in V^{-\sigma}$, and $\sigma = \pm$.

For a Jordan triple $J$, the centroid and the outer centroid consist of the sets of $\Phi$-endomorphisms $\gamma : J \to J$ which satisfy the versions of the above equalities without superscripts.

If $J$ is a Jordan system (pair or triple) $\Gamma(J)$ is a reduced commutative ring if $J$ is nondegenerate, a domain actig faithfully on $J$ if $J$ is strongly prime, and a field if $J$ is simple [14]. In case $J$ is strongly prime we can always form the central closure $\Gamma(J)^{-1}J$, which is a system of the same type as $J$ over the field $\Gamma(J)^{-1}\Gamma(J)$.

The centroid of a Jordan algebra $J$ is defined similarly as for Jordan triple systems, but taking into account the squaring of $J$: a $\Phi$-linear $\gamma : J \to J$ will belong to the centroid if, in addition to the above equalities, it satisfies

$$(\gamma(x))^2 = \gamma^2 x^2 \quad \text{and} \quad \gamma(x) \circ y = \gamma(x \circ y),$$

for all $x, y \in J$.

For linear Jordan algebras ($\frac{1}{2} \in \Phi$), there is also a classical notion of center, which has been recently extended to quadratic Jordan algebras in two different directions: the scalar center, and the weak center (see [6]). Here we will only make use of the latter, since it is linked to polynomial identities (although in the situations that we will consider, both notions coincide [6, Theorem 6]). An element $z \in J$ belongs to the weak center $C_w(J)$ of $J$ if the operators $U_z$ and $V_z$ belong to the centroid of $J$.

**0.4** Let $J$ be a Jordan triple system. The multiplication algebra of $J$, denoted $\mathcal{M}(J)$, is the subalgebra of $\mathrm{End}_\Phi(J)$ generated by all $P_x$ and $L_{x,y}$ for all $x, y \in J$. We denote the unital version, generated by $\mathcal{M}(J)$ and the identity mapping, by $\mathcal{M}^1(J)$.

For a Jordan pair $V$, the multiplication algebra $\mathcal{M}(V)$ is defined as the subalgebra of

$$\operatorname{End}(V^+ \oplus V^-, V^+ \oplus V^-) = \begin{pmatrix} \operatorname{End}(V^+) & \operatorname{Hom}(V^-, V^+) \\ \operatorname{Hom}(V^+, V^-) & \operatorname{End}(V^-) \end{pmatrix}$$

generated by all

$$\begin{pmatrix} 0 & Q_{x^+} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Q_{x^-} & 0 \end{pmatrix}, \begin{pmatrix} D_{x^+,x^-} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D_{x^-,x^+} \end{pmatrix},$$

with $x^+ \in V^+$ and $x^- \in V^-$. So, in fact, we have that the multiplication algebra of the Jordan pair $V$ is just the multiplication algebra of its attached polarized Jordan triple system $T(V)$: $\mathcal{M}(V) = \mathcal{M}(T(V))$.

The subalgebra of $\operatorname{End}(V^- \oplus V^-)$ generated by $\mathcal{M}(V)$ and the identity mapping will be denoted by $\mathcal{M}^1(V)$, and called the unital multiplication algebra.

If $V$ is a Jordan pair, we can make the identifications:

$$\Gamma(V) \subseteq \Gamma_{\text{out}}(V) \subseteq \begin{pmatrix} \operatorname{End}(V^+) & 0 \\ 0 & \operatorname{End}(V^-) \end{pmatrix} \subseteq \operatorname{End}(V^- \oplus V^-),$$

and view $\Gamma(V)$ and $\Gamma_{\text{out}}(V)$ as subsets of $\Gamma(T(V))$ and $\Gamma_{\text{out}}(T(V))$ respectively. More precisely, one has:

**0.5** Lemma. *Let $V$ be a Jordan pair. If $V$ is nondegenerate, $\Gamma(V)$ can be identified to $\Gamma(T(V))$ via*

$$\gamma = (\gamma^+, \gamma^-) \mapsto \gamma^+ \oplus \gamma^-.$$

Proof: If $0 \neq (\gamma^+, \gamma^-) \in \Gamma(V)$, it is clear that $0 \neq \gamma^+ \oplus \gamma^- \in \Gamma(T(V)$, and the above mapping defines a monomorphism. To see that it is surjective, take $\gamma \in \Gamma(T(V))$. For $x^+ \in V^+$ write $\gamma(x^+) = y^+ \oplus y^-$ with $y^\sigma \in V^\sigma$, $\sigma = \pm$. Then, for all $a^+ \in V^+$ we have $0 = \gamma(P_{a^+} x^+) = P_{a^+} \gamma(x^+) = Q_{a^+} y^-$. therefore $Q_{V^+} y^- = 0$, and $y^- = 0$ follows from the nondegeneracy of $V$. Thus $\gamma(V^+) \subseteq V^+$, and similarly $\gamma(V^-) \subseteq V^-$. Thus $\gamma = \gamma^+ \oplus \gamma^-$ for the restrictions $\gamma^+$, $\gamma^-$ of $\gamma$. ■

**0.6** Let $(V^+, V^-)$ be a Jordan pair and $a \in V^\sigma$, where $\sigma = \pm$. The *$a$-homotope* of $V$, denoted by $(V^{-\sigma})^{(a)}$, is the Jordan algebra over the $\Phi$-module $V^{-\sigma}$ with operations $U^{(a)}_{x^{-\sigma}} y^{-\sigma} = Q_{x^{-\sigma}} Q_a y^{-\sigma}$ (linearized to $\{x^{-\sigma}, y^{-\sigma}, z^{-\sigma}\}^{(a)} = \{x^{-\sigma}, Q_a y^{-\sigma}, z^{-\sigma}\}$), and $(x^{-\sigma})^2 = Q_{x^{-\sigma}} a$ (linearized to $x^{-\sigma} \circ_{(a)} y^{-\sigma} = \{x^{-\sigma}, a, y^{-\sigma}\}$.

The set $\operatorname{Ker} a$ of all $x^{-\sigma} \in V^{-\sigma}$ such that $Q_a x^{-\sigma} = Q_a Q_{x^{-\sigma}} a = 0$ (or simply $Q_a x^{-\sigma} = 0$ if $V$ is nondegenerate) is an ideal of $(V^{-\sigma})^{(a)}$, so that the quotient

$V_a^{-\sigma} = (V^{-\sigma})^{(a)}/\mathrm{Ker}\, a$ is again a Jordan algebra. This is called the *local algebra of $V$ at $a$*.

For triple systems and Jordan algebras, homotopes and local algebras are defined in the same way: just delete the superscripts $\sigma$ from the previous definitions. We refer to [1] for a thorough study of local algebras.

Local algebras of Jordan pairs can be viewed through the theory of subquotients as developed by Loos and Neher [12]: If $V = (V^+, V^-)$ is a Jordan pair and $M \subseteq V^\sigma$ is an inner ideal of $V$, the subquotient of $V$ determined by $M$ is the pair $S$ given by $S^\sigma = M$ and $S^{-\sigma} = V^{-\sigma}/\mathrm{Ker}_V M$, where $\mathrm{Ker}_V M$ (or simply $\mathrm{Ker}\, M$ if there is not ambiguity) is the set of $x \in V^{-\sigma}$ for wich $Q_M x = Q_M Q_x M = 0$. (Again, the second condition is superfluous if $V$ is nondegenerate.)

When $M = \Phi a + Q_a V^{-\sigma}$ is the principal ideal determined by $a \in V^\sigma$, the subquotient $S$ determined by $M$ in $V$ has $S^{-\sigma} = V_a^{-\sigma}$, and $S$ is isomorphic to the double $(V_a^{-\sigma}, V_a^{-\sigma})$ (see [16, 0.4]). Moreover, if $a \in V^\sigma$ is regular, we can complete it to an idempotent $e = (e^+, e^-)$ with $e^\sigma = a$, and the subquotient determined by $M$ is (isomorphic to) the Peirce space $V_2(e)$ [12, 1.12].

**0.7** We denote by $\kappa(J)$ the capacity [11] of a Jordan system $J$ (defined as the capacity of the Jordan pair $V(J)$ for algebras and triple systems). Recall [16, 0.7] that if $V = (V^+, V^-)$ is a Jordan pair, $\sigma = \pm$, and $a \in V^\sigma$, $\kappa(V_a^{-\sigma})$ equals the rank $\mathrm{rk}(a)$ of $a$ [10], and therefore, the socle $\mathrm{Soc}(V^\sigma)$ of $V$ can be characterized as the set of all $a \in V^\sigma$ whose local algebra has finite capacity: $\kappa(V_a^{-\sigma}) < \infty$.

recall that an idempotent $e = (e^+, e^-)$ of a Jordan pair $V$ is principal if its Peirce 0-component is zero $V_0(e) = 0$. If $V$ has capacity, this is equivalent to the fact that $\mathrm{rk}(e) = \kappa(V)$.

**0.8** We finally mention some facts from Jordan PI-theory. Recall that a polynomial $f(x_1, \ldots, x_n) \in \mathrm{FJ}[X]$, the free Jordan algebra on the set $X$, is called essential if its image in the free special Jordan algebra $\mathrm{SJ}[X]$ under the natural homomorphism has a monic leading term (as an associative polynomial). A Jordan PI-algebra is a Jordan algebra which satisfies some essential $f(x_1, \ldots, x_n)$. From [4, 1.1 and 5.2] together with Corollary to Theorem 3 of [11], analoges of Kaplansky's Theorem and Posner's Theorem follow:

**0.9** THEOREM. *Let $J$ be a Jordan PI-algebra. If $J$ is primitive then it is simple with finite capacity. If $J$ is strongly prime, then the central closure $\Gamma^{-1}J$ is simple with finite capacity.*

Moreover, this has been extended to nondegenerate Jordan algebras in the following analogue of Posner-Formanek-Rowen's theorem [5, 3.6]:

**0.10** Theorem. *Let $J$ be a nondegenarate Jordan PI-algebra. Then any nonzero ideal $I$ of $J$ hits $C_w(J)$: $C_w(J) \cap I \neq 0$.*

**0.11** The operant notion of Jordan PI-triple system or pair, is that of homotope-PI triple system or pair. We will use the notations of [1] and [3]. In particular, if $f(x_1, \ldots, x_n)$ is a polynomial in the free Jordan algebra $\mathrm{FJ}[X]$ on a countable set of generators $X$, and $z$ is an element of the free Jordan triple system $\mathrm{FJT}[X]$, the polynomial

$$f(z; x_1, \ldots, x_n) = f^{(z)}(x_1, \ldots, x_n)$$

is the image of $f$ under the only homomorphism $\mathrm{FJ}[X] \to \mathrm{FJT}[X]^{(z)}$ extending the identity on $X$. If $T(X) \subseteq \mathrm{FJ}(X)$, and $Y \subseteq \mathrm{FJ}[X]$, we denote by $T(Y; X)$ the subset of $\mathrm{FJT}[X]$ formed by the polynomials $f(y; x_1, \ldots, x_n)$ for $f(x_1, \ldots, x_n) \in T(X)$ and $y \in Y$.

A Jordan triple system $T$ satisfies a homotope polynomial identity (homotope-PI, for short) if there is a polynomial $f(x_1, \ldots, x_n)$ in $\mathrm{FJ}[X]$ whose image in the free special Jordan algebra $\mathrm{SJ}[X]$ has a monic term of highest degree (as an associative polynomial) and such that the polynomial $f(y; x_1, \ldots, x_n)$ with $y \in X$ different from the $x_i$, vanishes under all substitutions of elements $y, x_i \in T$.

That definition extends to Jordan pairs $V$ by considering their associated triple system $T(V)$. Notice that, since for all $a^+ \oplus a^- \in T(V)$ the homotope $T(V)^{(a^+ \oplus a^-)}$ is isomorphic to the product $V^{+(a^-)} \times V^{-(a^+)}$, a polynomial $f(x_1, \ldots, x_n) \in \mathrm{FJ}[X]$ is an identity of all homotopes of $T(V)$ if and only if it is an identity of all homotopes of $V$. We can rephrase it in the following way. Choose disjoint sets $X^+$ and $X^-$, and bijections $X \to X^\sigma$, $x \mapsto x^\sigma$, $\sigma = \pm$, and consider the free Jordan Pair $\mathrm{FJP}[X^+, X^-]$ (see [19]). For any $y^{-\sigma} \in X^{-\sigma}$, there is a homomorphism $\psi_{y^{-\sigma}}$ : $\mathrm{FJ}[X] \to \mathrm{FJP}[X^+, X^+]^{\sigma(y^{-\sigma})}$ induced by the bijection $X \to X^\sigma$. We denote the image of a polynomial $h = h(x_1, \cdots, x_n) \in \mathrm{FJ}[X]$ by $\psi_{y^{-\sigma}}(h) = h(y^{-\sigma}; x_1^\sigma, \ldots, x_n^\sigma)$. Now if $V$ and $f$ are as before, setting $f^\sigma = f(y^{-\sigma}; x_1^\sigma, \ldots, x_n^\sigma) \in \mathrm{FJP}[X^+, X^-]^\sigma$ for $\sigma = \pm$, where $y^\sigma \in X^\sigma$ and $y^\sigma \neq x_i^\sigma$, $f(y; x_1, \ldots, x_n)$ is an identity of $T(V)$ if and only if $(f^+, f^-)$ is an identity of $V$.

**0.12** The fact that a Jordan system $J$ satisfies a homotope-PI means that all homotopes, hence all local algebras, satisfy a given identity. Often, we are interested in a weaker assertion, the existence of some $a \in J$ for wich the local algebra $J_a$ is PI. We call such an element a PI-element, and write $\mathrm{PI}(J)$ for the set of PI-elements of $J$ ($\mathrm{PI}(V) = (\mathrm{PI}(V^+), \mathrm{PI}(V^-))$ if $J = V = (V^+, V^-)$ is a Jordan pair). Thus, the fact that $J$ has a nonzero PI-element can be abbreviated to $\mathrm{PI}(J) \neq 0$. We recall here the main results of [16].

**0.13** THEOREM. *Let $J$ be a nondegenerate Jordan system. Then* $\mathrm{PI}(J)$ *is an ideal of* $J$.

A Jordan system $J$ is said to be rationally primitive if it is primitive and has a nonzero PI-element. This is the Jordan analogue of strongly primitive associative algebras. Rational primitivity is characterized in the following analogue of Amitsur's theorem on generalized identities [16, Theorems 4.5 and 4.6]

**0.14** THEOREM. *Let $J$ be a Jordan system. The following are equivalent*

(a) *$J$ is rationally primitive,*

(b) *$J$ is strongly prime and* $\mathrm{Soc}(T) = \mathrm{PI}(T) \neq 0$

(c) *$J$ is strongly prime and the local algebra at some nonzero element is a simple unital PI-algebra.*

As a consequence one has an analogue for Jordan systems of Kaplansky's theorem, with homotope polynomial identities on Jordan systems playing the role of polynomial identities on algebras.

**0.15** THEOREM. *Let $J$ be a primitive Jordan pair or triple system.*

(i) *If the local algebra at each element of $J$ is PI, then $J$ is simple equal to its socle.*

(ii) *If $J$ satisfies a homotope-PI, then $J$ is simple with finite capacity.*

## 1. Local PI-theory and prime Jordan systems

**1.1** Strongly prime Jordan systems having nonzero PI-elements have been studided in [17]. Their description makes use of the notions of extended centroid $\mathcal{C}(J)$ and of extended central closure $\mathcal{C}(J)J$ of a (quadratic) Jordan system $J$, for which we refer to [17]. With these notions, the main result on strongly prime Jordan systems with nonzero PI-elements is [17, theorem 5.1 ]

**1.2** THEOREM. *Let $J$ be a strongly prime Jordan system. If* $\mathrm{PI}(J) \neq 0$, *then the extended central closure $\mathcal{C}(J)J$ of $J$ is rationally primitive; hence it has nonzero socle equal to* $\mathrm{PI}(\mathcal{C}(J)J)$, *and* $\mathrm{PI}(J) = J \cap \mathrm{Soc}(\mathcal{C}(J)J)$.

**1.3** Let $J$ be a nondegenerate Jordan system. The centroid $\Gamma(J)$ of $J$ can be seen in a natural way as a subring of the extended centroid $\mathcal{C}(J)$. If, in addition, $J$ is prime, then $\mathcal{C}(J)$ is a field [17, 1.15], and the field of fractions $\Gamma(J)^{-1}\Gamma(J)$ is contained in $\mathcal{C}(J)$. Therefore, one can define a monomorphism $\Gamma(J)^{-1}J = \Gamma(J)^{-1}\Gamma(J) \otimes_{\Gamma(J)} J \to \mathcal{C}(J)J$ in the obvious way, and view the central closure $\Gamma(J)^{-1}J$ as a subsystem of the extended central closure $\mathcal{C}(J)J$. Moreover, both systems coincide if and only if $\Gamma(J)^{-1}\Gamma(J) = \mathcal{C}(J)$.

**1.4** Extended central closures are very close to the original systems since, for instance, nonzero inner ideals of the extended central closure always hit the original algebra. A more precise picture of that proximity can be obtained by using the notion of denominator inner ideal of an element $\tilde{a} \in \tilde{J}$ of an extension $\tilde{J} \subseteq J$ into an ideal $I$ of $J$, $\mathcal{D}_J(\tilde{a}, I)$ [17], consisting of the set of $x \in J$ such that the elements $P_x\tilde{a}, P_{\tilde{a}}x$ and the sets $P_xP_{\tilde{a}}J$, $P_{\tilde{a}}P_xJ$, $L_{x,\tilde{a}}J$ and $L_{\tilde{a},x}J$ (together with $x \circ \tilde{a}$, $U_x\tilde{a}^2$ and $U_{\tilde{a}}x^2$ if $J$ is an algebra) are all contained in $I$. When $I = J$ this is simply denoted $\mathcal{D}_J(\tilde{a})$.

One can then introduce the following notion, defined originally for algebras in [5]:

Let $J \subseteq \tilde{J}$ be Jordan systems. Then $\tilde{J}$ will be called an *innerly tight* extension of $J$ if it satisfies:

(T1) $P_{\tilde{a}}J \cap J \neq 0$ for all $\tilde{a} \in \tilde{J}$

(T2) $\mathcal{D}_J(\tilde{a})$ is an essential inner ideal of $J$ for all $\tilde{a} \in \tilde{J}$.

**1.5** LEMMA. *If $J$ is a nondegenerate Jordan system, its extended central closure is in an innerly tight extension of $J$.*

PROOF: By [17, 4.3(2)], for all $a \in \mathcal{C}(J)J$, $\mathcal{D}_J(\tilde{a})$ contains an essential ideal of $J$. From this, T2 follows. To have T1 it suffices to show that for any essential ideal $I$ of $J$, $P_{\tilde{a}}I = 0$ implies $\tilde{a} = 0$. Now, $P_{\tilde{a}}I = 0$ implies $P_{\tilde{a}}\mathcal{C}(J)J = 0$, hence $\tilde{a} \in \mathrm{ann}_{\mathcal{C}(J)J}(\mathcal{C}(J)I)$. Thus $\mathrm{ann}_{\mathcal{C}(J)J}(\mathcal{C}(J)I)$ is a nonzero ideal of $\mathcal{C}(J)J$, and by tightness [17, 3.8] $0 \neq J \cap \mathrm{ann}_{\mathcal{C}(J)J}(\mathcal{C}(J)I) \subseteq \mathrm{ann}_J(I)$ which contradicts essentiality of $I$, since an essential ideal in a nondegenerate system has zero annihilator. ■

**1.6** REMARK: A stronger version of T2 is, in fact, satisfied by the extended central clodure: If $I$ is an essential ideal of a Jordan triple system $J$, then, for any $\tilde{a} \in \mathcal{C}(J)J$, $\mathcal{D}_J(\tilde{a}, I)$ contains an essential ideal of $J$. For a Jordan pair $V$, any essential ideal $I$ of $V$, and any $\tilde{a} \in \mathcal{C}(V)V^\sigma$, considering the attached polarized system $T(V)$, and taking into account the obvious identification $T(\mathcal{C}(V)V) = \mathcal{C}(T(V))T(V)$ (see 3.9 of [5]), and that if, say, $\sigma = +$, $\mathcal{D}_{T(V)}(\tilde{a} \oplus 0, I^+ \oplus I^-) = V^+ \oplus \mathcal{D}_V(\tilde{a}, I)$, we also have that $\mathcal{D}_V(\tilde{a}, I)$ contains the $(-)$-part of an essential ideal of $V$.

**1.7** LEMMA. *Let $J \subseteq \tilde{J}$ be an innerly tight extension of Jordan systems, and assume that $\tilde{a} \in \mathrm{Soc}(A)$. Then there is an element $a \in P_{\tilde{a}}J \cap J$ with $\mathrm{rk}(a) = \mathrm{rk}(\tilde{a})$.*

PROOF: First note that if $J \subseteq \tilde{J}$ is an innerly tight extension of Jordan triple systems, then $V(J) \subseteq V(\tilde{J})$ is an innerly tight extension of Jordan pairs, therefore it suffices to prove the result for Jordan pairs.

Thus set $J = (V^+, V^-) \subseteq \tilde{J} = (\tilde{V}^+, \tilde{V}^-)$, and suppose that $\tilde{a} \in \tilde{V}^+$. Since $\tilde{a} \in \mathrm{Soc}(\tilde{J})$ we can complete $\tilde{a}$ to an idempotent $e = (e^+, e^-)$ with $e^+ = \tilde{a}$. Take a strong frame $F = \{e_1, \ldots, e_n\}$ of $\tilde{V}_2(e)$. Since the extension is innerly tight, for $i = 1 \ldots, n$ we can find elements $0 \neq c_i = Q_{e_i^-} x_i \in Q_{e_i^-} V^+ \cap V^+$. Now, $Q_{c_i} V^+$ is a nonzero inner ideal, so again by inner tightness, there is a nonzero $b_i = Q_{c_i} y_i \in Q_{c_i} V^+ \cap \mathcal{D}_J(\tilde{a})$. Set $a_i = Q_{e^+} b_i = Q_{\tilde{a}} b_i \in V^+ \cap Q_{e^+} V^-$, and $a = a_1 + \cdots + a_n = Q_{\tilde{a}}(b_1 + \cdots + b_n) \in V^+ \cap Q_{\tilde{a}} V^-$.

Since $b_i \in Q_{e_i^-} V^+ = V_2^-(e_i) \subseteq V_2^-(e)$, we have $Q_{e^-} a_i = Q_{e^-} Q_{e^+} b_i = b_i \neq 0$, hence $a_i \neq 0$, and $1 \leq \mathrm{rk}(Q_{e^-} a_i) = \mathrm{rk}(Q_{e^-} Q_{e^+} b_i) = \mathrm{rk}(b_i) = \mathrm{rk}(Q_{c_i} y_i) \leq \mathrm{rk}(c_i)$ (by [10, Proposition 3(5)]) $= \mathrm{rk}(Q_{e_i^-} x_i)$ (by [10, Proposition 3(5)]) $\leq \mathrm{rk}(e_i) = 1$. Thus $1 \leq \mathrm{rk}(a_i) = \mathrm{rk}(Q_{e^+} Q_{e^-} a_i) \leq \mathrm{rk}(Q_{e^-} a_i)$ (by [10, Corollary 1(a)]) $= 1$. Therefore $\mathrm{rk}(a_i) = 1$ for all $i$, and $\mathrm{rk}(a) = \mathrm{rk}(a_1) + \cdots + \mathrm{rk}(a_n) = n$ (by [10, Proposition 3(7)], since $a_i \in V_2^+(e_i)$, and $e_i \perp e_j$ for $i \neq j$ implies $a_i \perp a_j$). ■

Now we can complete the proof of [17, 6.1]:

**1.8** THEOREM. *Let $J$ be a strongly prime Jordan system (triple, pair or algebra).*

(i) *If the local algebra at each element of $J$ is PI, then the extended central closure of $J$ is simple, equal to its socle.*

(ii) *If $J$ satisfies a homotope PI $f^{(y)} = 0$ for an essential $f$ of degree less than or equal to $n$, then the extended central closure of $J$ is simple with finite capacity at most $n$.*

PROOF: (i) By [17, 5.1], $\mathcal{C}(J)J$ is rationally primitive with $\mathrm{Soc}(\mathcal{C}(J)J) = \mathrm{PI}(\mathcal{C}(J)J)$. Also $J = \mathrm{PI}(J) \subseteq \mathrm{PI}(\mathcal{C}(J)J)$, and since $\mathrm{PI}(\mathcal{C}(J)J$ is an ideal over $\mathcal{C}(J)$ (in fact, over the centroid of $\mathcal{C}(J)J$), we have $\mathcal{C}(J)J = \mathcal{C}(J)\mathrm{PI}(J) \subseteq \mathcal{C}(J)\mathrm{PI}(\mathcal{C}(J)J) = \mathrm{PI}(\mathcal{C}(J)J)$, hence $\mathcal{C}(J)J = \mathrm{PI}(\mathcal{C}(J)J) = \mathrm{Soc}(\mathcal{C}(J)J)$, is simple, equal to its socle.

(ii) By (i), $\tilde{J} = \mathcal{C}(J)J$ is simple, equal to its socle. We can always assume that the essential $f$ is multilinear. We claim that the rank of each element of $\tilde{J}$ is at most $n$.

Indeed, take an arbitrary $\tilde{a} \in \tilde{J}$. By 1.7, there exists some $b \in P_{\tilde{a}} J \cap J$ with $\mathrm{rk}(b) = \mathrm{rk}(\tilde{a})$. Now, $J_b \subseteq \tilde{J}_b$ is a scalar extension, hence $\tilde{J}_b$ satisfies $f = 0$. By [17, 2.1] we have $\kappa(\tilde{J}_b) \leq n$, and since $\kappa(\tilde{J}_b) = \mathrm{rk}(b)$, we get $\mathrm{rk}(\tilde{a}) = \mathrm{rk}(b) \leq n$.

Having a common bound for the rank of all its elements implies by [16, 4.9] that $J$ has finite capacity. Now, if $e \in V(\tilde{J})$ is a complete idempotent (or $e \in \tilde{J}$, if $\tilde{J}$ is a pair), then the principal length $\pi(V(\tilde{J})_2(e))$ (see [11]) coincides with $\mathrm{rk}(e)$, and $\pi(V(\tilde{J})_2(e)) = \kappa(V(\tilde{J})_2(e))$ (by [11, Theorem 3]) $= \kappa(V(\tilde{J}))$ (by [11, Lemma 5], since $e$ is principal, hence $V(\tilde{J})_0(e) = 0) = \kappa(\tilde{J})$. Therefore $\kappa(\tilde{J}) = \mathrm{rk}(e)$. ■

## 2. Strict homotope polynomial identities

As mentioned before, the linealization process of identities does not produce a homotope polynomial when applied to a homotope polynomial. As a consequence, it is not straighforward that a scalar extension of a Homotope-PI Jordan system is again homotope PI. A homotope-PI $f^{(y)} = 0$ which is again satisfied by any scalar extension is said to be a strict homotope polynomial identity and we say that the system strictly satisfies $f^{(y)} = 0$ . We show in this section that if a nondegenerate system satisfied some homotope-PI $f^{(y)} = 0$, for an essential algebra polynomial $f$, then it strictly satisfies some homotope-PI. It must be pointed out, however, that, in contrast with what happens with the usual linearization process, the strict homotope-PI that we find has bigger degree than the original identity.

**2.1** We consider the following family of polynomials parametrized by the positive integer $m$:

$$F_m(x,y,z) = \sum_{\sigma\in S_{m+1}} (-1)^\sigma V_{x^{\sigma(1)},y} \cdots V_{x^{\sigma(m+1)},y} z.$$

of degree $\frac{(m+2)(m+3)}{2}$. This is an essential polynomial in the free Jordan algebra $FJ[x,y,z]$ (see [16, 2.2]). We will also consider $G_m(x,y,z) = F_m(x,y,z)^3$.

**2.2** Lemma. *Let $V$ be a simple Jordan pair of finite capacity $n$ over a large algebraically closed field $\Omega$: $|\Omega| > \dim_\Omega V + 2$. Then $V$ satisfies the homotope polynomial identity $G_m^{(t)}(x,y,z) = 0$ for all $m \geq n$.*

Proof: This means that every homotope of $V$ satisfies $G_m = 0$ for all $m \geq n$, and since for any element $a$ of $V$ the local algebra at $a$ is the quotient of the $a$-homotope by an ideal with cube zero, it suffices to show that every local algebra satisfies $F_m = 0$ for $m \geq n$.

Now take $a$ in $V$, $a \in V^+$ say. Then, the local algebra $V_a^-$ has finite capacity $\kappa(V_a^-) = \mathrm{rk}(a) \leq n$ (0.7), and $V_a^-$ is a Jordan algebra over $\Omega$ which is large for $V_a^-$: $|\Omega| > \dim_\Omega V + 2 \geq \dim_\Omega V_a^- + 2$. Thus by [16, 2.4], $V_a^-$ satisfies $F_m = 0$ for all $m \geq n$. ■

**2.3** Lemma. *let $V$ be a simple Jordan pair with finite capacity, and let $e \in V$ be a complete idempotent such that $V_{e^-}^+$ satisfies a polynomial identity of degree $n$. Then $V$ strictly satisfies the homotope polynomial identity $G_m^{(t)}(x,y,z) = 0$ for $m \geq n$.*

Proof: Take an algebraically closed field extension $\Omega$ of $\Gamma(V)$ (which is a field since $V$ is simple) with cardinaltity $|\Omega| > \dim_{\Gamma(V)} V + 2$, and form a tight scalar extension $\tilde{V} = \Omega V$. We have $\dim_\Omega \tilde{V} \leq \dim_{\Gamma(V)} V$, hence $|\Omega| > \dim_\Omega \tilde{V} + 2$, and $\Omega$ is large for $\tilde{V}$.

Now the local algebra $\tilde{V}^+_{e^-}$ is a scalar extension of $V^+_{e^-}$, and therefore it satisfies a PI of degree $n$. Then $\mathrm{PI}(\tilde{V}) \neq 0$, hence $\mathrm{PI}(\tilde{V}) = \tilde{V}$ (since $\tilde{V}$ is simple), and $\tilde{V}$ is primitive: it has a nonzero idempotent $e$, and $V_0^+(e)$ is easily seen to be a primitizer with modulus $(e^+, e^-)$ (the argument is the same as in [3, 3.5]). Thus, $\tilde{V}$ is rationally primitive, hence $\tilde{V} = \mathrm{PI}(\tilde{V}) = \mathrm{Soc}(\tilde{V})$ by 0.14. On the other hand $e$ is also a complete idempotent of $\tilde{V}$ (since the projection onto the 0-Pierce component, the Bergmann operator $B_{e^+,e^-} = Id_{V^+} - L_{e^+,e^-} + Q_{e^+}Q_{e^-}$, vanishes on $V$, hence on $\tilde{V}$ by linearity). Thus $\kappa(\tilde{V}) = \kappa(\tilde{V}_2(e))$ (by [11, Theorem 3]) $= \kappa(\tilde{V}^+_{e^-})$ (by 0.6) $\leq n$ (by [16, 2.1], since $\tilde{V}^+_{e^-}$ satisfies an identity of degree $n$). By 2.2, $\tilde{V}$ satisfies $G_m^{(t)}(x, y, z) = 0$ for all $m \geq n$, and since $\Omega$ is infinite, $\tilde{V}$ strictly satisfies that identity. ∎

**2.4** LEMMA. *Let $V$ be a strongly prime Jordan pair. If $V$ satisfies a homotope PI $f^{(y)} = 0$ for some essential $f$ of degree $n$, then $V$ strictly satisfies $G_m^{(t)}(x, y, z) = 0$ for $m \geq n$.*

PROOF: By 1.8(ii), the extended central closure $\mathcal{C}(V)V$ of $J$ is simple with capacity at most $n$. Now, if $e \in \mathcal{C}(V)V$ is a complete idempotent, we can choose $a \in Q_{e^-}V^+ \cap V^-$ with $\mathrm{rk}(a) = \mathrm{rk}(e^-)$ by 1.7, and complete it to an idempotent $(b, a)$ which is necesarily complete since $\mathrm{rk}(a) = \mathrm{rk}(e^-)$. Therefore we can assume that $e^- \in V$. Now $V^+_{e^-} \subseteq (\mathcal{C}(V)V)^+_{e^-}$ is a scalar extension, hence $(\mathcal{C}(V)V)^+_{e^-}$ also satisfies a polynomial identity of degree $n$. From 2.3 it follows that $\mathcal{C}(V)V$, hence $V$, strictly satisfies $G_m^{(t)}(x, y, z) = 0$ for $m \geq n$. ∎

**2.5** THEOREM. *If a nondegenerate Jordan system $J$ satisfies a homotope PI $f^{(y)} = 0$ for an essential $f$ of degree $n$, then $J$ strictly satisfies $G_m^{(t)}(x, y, z) = 0$ for $m \geq n$.*

PROOF: If $J$ is a triple system or an algebra, the pair $V(J)$ satisfies the same strict homotope identities as $J$, and is again nondegenerate, therefore it suffices to prove the assertion for Jordan pairs.

Since $J$ is nondegenerate, it is a subdirect product of a family of strongly prime Jordan pairs $J_i$, $i \in I$, and each $J_i$ satisfies the homotope PI $f^{(y)} = 0$. By 2.4, each $J_i$ strictly satisfies $G_m^{(t)}(x, y, z) = 0$ for $m \geq n$, hence $J$ itself strictly satisfies $G_m^{(t)}(x, y, z) = 0$ for $m \geq n$. ∎

## 3. Posner-Rowen Theorem for Jordan systems

If $R$ is a prime associative algebra , and $\tilde{R}$ is its central closure, it is easy to see that for any $\tilde{a} \in \tilde{R}$, and any $z \in \tilde{R}$ with $\bar{z} \in Z(\tilde{R}_{\tilde{a}})$, there is $\gamma \in \mathcal{C}(R)$, the extended centroid of $R$, such that $\bar{z}\bar{r} = \gamma\bar{r}$ for all $\bar{r} \in \tilde{R}_{\tilde{a}}$. It was proved in [17; 5.12] that, for Jordan systems, the centroid of a local algebra at a PI-element of the extended central closure can also be related to the extended centroid of the system. We quote that result in the following Lemma.

**3.1** Lemma. *Let $V$ be a strongly prime Jordan pair, and let $\tilde{V} = \mathcal{C}(V)V$ be its extended central closure. Then, for any $\tilde{a} \in \mathrm{PI}(\tilde{V})$, the local centroid $\Gamma_{\tilde{a}} = \Gamma(\tilde{V}_{\tilde{a}})$ has $\Gamma_{\tilde{a}}^4 \subseteq \mathcal{C}(V)$.*

**3.2** Theorem. *Let $V$ be strongly prime Jordan pair satisfying a homotope polynomial identity. If $I$ is a nonzero ideal of $V$, there is a nonzero $\gamma \in \Gamma(V) \cap \mathcal{M}(V)$ such that $\gamma V \subseteq I$.*

Proof: By 1.8(ii), the extended central closure $\tilde{V} = \mathcal{C}(V)V$ is simple of finite capacity. We can take an element $\tilde{a} \in \tilde{V}^+$ of rank $\mathrm{rk}(\tilde{a}) = \kappa(\tilde{V})$. Since $V \subseteq \tilde{V}$ is an innerly tight extension by 1.5, we can find an $a \in V$ such that $\mathrm{rk}(a) = \mathrm{rk}(\tilde{a})$ by 1.7. Now, since $\tilde{V}$ has finite capacity, we can comlete $a$ to an idempotent $e = (e^+, e^-) \in \tilde{V}$ with $e^+ = a$. Since $\mathrm{rk}(a) = \kappa(\tilde{V})$, $e$ is a complete idempotent and therefore it induces a Peirce decomposition $\tilde{V} = \tilde{V}_2(e) + \tilde{V}_1(e)$. For $\tilde{x} \in \tilde{V}$ we denote by $\tilde{x}_i \in \tilde{V}_i(e)$, $i = 1, 2$ its Peirce components.

Now, by 1.6 there is a nonzero ideal $L$ of $V$ such that $L^+ \subseteq \mathcal{D}_V(e^-, I)$. Set $N = I \cap L$, which is a nonzero ideal of $V$. Since $V$ satisfies a homotope PI, the local algebra $V_a^-$ is PI. We denote with bars the images of elements of $V^-$ in $V_a^-$. Now, with the usual notational convention, $N_a^-$ is a nonzero ideal of $V_a^-$, hence by 0.10 there is $z \in N^-$ such that $0 \neq \bar{z} = z + \mathrm{Ker} a \in C_w(V_a^-)$. Note now that $z_2 = Q_{e^-} Q_a z \in Q_{e^-} Q_a L^- \subseteq Q_{e^-} L^+ \subseteq Q_{e^-} \mathcal{D}(e^-, I) \subseteq I^-$ and $\bar{z} = \bar{z}_2$ so that we can assume that $0 \neq z = z_2 \in \tilde{V}_2(e)^- \cap I^-$.

By 3.1 , we have $\Gamma(\tilde{V}_a^-)^4 \subseteq \mathcal{C}(V)$, that is, for any $\mu \in \Gamma(\tilde{V}_a^-)$ there exists $\delta \in \mathcal{C}(V)$ such that for any $x^- \in \tilde{V}^-$, $\mu^4(\bar{x}^-) = \delta \bar{x}^-$. In particular, since $U_{\bar{z}} \in \Gamma(V_a^-)$ and $V_a^- \subseteq \tilde{V}_a^-$ is a scalar extension, $U_{\bar{z}} \in \Gamma(V_a^-)$ implies $U_{\bar{z}} \in \Gamma(\tilde{V}_a^-)$, and there is $\delta \in \mathcal{C}(V)$ such that $U_{\bar{z}}^4 \bar{e}^- = \delta e^-$. Now $0 \neq U_{\bar{z}}^4 \bar{e}^- = \bar{z}^8$, hence, setting $c = z^{(8,a)}$, we get $0 \neq \bar{c} = \delta \bar{e}^-$. Moreover, since $z \in \tilde{V}_2(e)^- \cap I^-$, we also have $c \in \tilde{V}_2(e)^- \cap I^-$. Therefore $\bar{c} = \delta \bar{e}^-$ implies $Q_a c = Q_a \delta e^- = \delta a$, hence $0 \neq c = c_2 = Q_{e^-} Q_a c = \delta Q_{e^-} a = \delta e^-$.

We define a pair of mappings $\gamma^\sigma : V^\sigma \to V^\sigma$, $\sigma = \pm$, by

$$\gamma^+(x^+) = \{Q_a c, c, x^+\} - Q_a Q_c x^+$$

fro $x^+ \in V^+$, and

$$\gamma^-(x^-) = \{c, Q_a c, x^-\} - Q_c Q_a x^-$$

for $x^- \in V^-$.

Then, computing in $\tilde{V}$, we get $\gamma^+(x^+) = \{Q_{e^+}\delta e^-, \delta e^-, x^+\} - Q_{e^+}Q_{\delta e^-}x^+ = \delta^2(\{e^+, e^-, x^+\} - Q_{e^+}Q_{e^-}x^+) = \delta^2(2x_2^+ + x_1^+ - x_2^+) = \delta^2 x^+$, and similarly, $\gamma^-(x^-) = \delta^2 x^-$. It follows then that $0 \neq \gamma = (\gamma^+, \gamma^-) \in \Gamma(V)$, and since $c \in I$ we have $\gamma V \subseteq I$, and clearly $\gamma \in \mathcal{M}(V)$. ■

We include the previous results in:

**3.3** THEOREM. *Let $J$ be a strongly prime Jordan pair or triple system satisfying a homotope PI, then for any nonzero ideal $I$ of $J$ there is a nonzero $\gamma \in \Gamma(J) \cap \mathcal{M}(J)$ such that $\gamma(J) \subseteq I$. In particular, the extended centroid $\mathcal{C}(J)$ of $J$ coincides with the field of fractions $\Gamma(J)^{-1}\Gamma(J)$, and the central closure $\Gamma(J)^{-1}J$ is simple with finite capacity. Moreover, if the homotope polynomial identity is $f^{(y)} = 0$ for some essential algebra polynomial $f$ of degree $d$, then the capacity of $\Gamma(J)^{-1}J$ is at most $d$.*

PROOF: The first assertion for Jordan pairs in 3.2. Let us then assume that $J$ is a Jordan triple system. Take the double $V(J)$ of $J$, and a tight double $W = V(J)/L$ for some ideal $L = (L^+, L^-)$ of $V(J)$. We distinguish two cases, according to whether $L = 0$ or $\neq 0$.

*Case* $L = 0$. Here $W = V(J)$ is already tight, hence $V(J)$ is strongly prime, and it satisfies the same homotope PI as $J$. Now, if $I$ is a nonzero ideal of $J$, then $V(I) = (I, I)$ is a nonzero ideal of $V(J)$, and by the pair case, there is a nonzero $\gamma \in \Gamma(V(J)) \cap \mathcal{M}(V(J))$ such that $\gamma V(J) \subseteq V(I)$, i.e. if $\gamma = (\gamma^+, \gamma^-)$, then $\gamma^\sigma \in \mathcal{M}(J)$ and $\gamma^\sigma J \subseteq I$, for $\sigma = \pm$. Set $\delta = \gamma^+\gamma^-$. If we write $\gamma^* = (\gamma^-, \gamma^+)$, then $\gamma^* \in \Gamma(V(J))$, which is a commutative ring, hence $\gamma\gamma^* = \gamma^*\gamma$, and we have $\gamma^+\gamma^- = \gamma^-\gamma^+$, and $\gamma^*\gamma = (\delta, \delta)$. From this it readily follows that $\delta \in \Gamma(J)$, and we also have $\delta \in \mathcal{M}(J)$, and $\delta J \subseteq I$.

*Case* $L \neq 0$. Take $\tilde{L} = L^+ \oplus L^-$, which is an ideal of $J$, then $V(\tilde{L})/L = ((L^+ \oplus L^-)/L^+, (L^+ \oplus L^-)/L^-)$ is an ideal of $W$ which is isomorphic to $(L^-, L^+) = L^{op}$ as a Jordan pair since $L^+ \cap L^- = 0$. We denote by $\eta = (\eta^+, \eta^-) : V(\tilde{L})/L \to L^{op}$ that isomorphism. Now, by the pair case considered above, since $W$ is strongly prime with a homotope PI, there is $\gamma_0 = (\gamma_0^+, \gamma_0^-) \in \Gamma(W) \cap \mathcal{M}(W)$ with $\gamma_0 W \subseteq V(\tilde{L})/L$. Define $\gamma_1 : J \to J$ by $\gamma_1(x) = \eta^+(\gamma_0^+(x + L^+)) + \eta^-(\gamma_0^-(x + L^-)) \in L^- \oplus L^+ = \tilde{L} \subseteq J$. It is easy to see that $\gamma_1$ is then a nonzero element of $\Gamma(J)$ and it satisfies $\gamma_1 J \subseteq \tilde{L}$.

Now let $I$ be a nonzero ideal of $J$. By primeness, $I \cap \tilde{L}$ is a nonzero ideal of $J$, and $K = \tilde{L} * (I \cap \tilde{L}) = P_{\tilde{L}}(I \cap \tilde{L}) + P_J P_{\tilde{L}}(I \cap \tilde{L})$ is a nonzero ideal of $J$ [13, p. 221], and a polarized ideal of $\tilde{L}$: $K = K^+ \oplus K^-$ with $K^\sigma = P_{L^\sigma}(I \cap \tilde{L}) + P_J P_{L^{-\sigma}}(I \cap \tilde{L}) \subseteq L^\sigma$, $\sigma = \pm$. Now, $L^{op} = (L^-, L^+)$ is a strongly prime Jordan pair with a homotope PI since it is isomorphic to the ideal $V(\tilde{L})/L$ of $W$. Then, by the pair case, there is a nonzero $\delta = (\delta^+, \delta^-) \in \Gamma(L^{op}) \cap \mathcal{M}(L^{op})$ such that $\delta L^{op} \subseteq (K^+, K^-)$. Set $\mu = \delta^+ \oplus \delta^- : \tilde{L} \to \tilde{L}$. then $\mu \in \Gamma(\tilde{L})$, and $\mu\tilde{L} \subseteq K^+ \oplus K^- = K \subseteq I$. Now, consider $\gamma = \mu\gamma_2$ with $\gamma_2 = \gamma_1^2$, and $\gamma_1$ as before. Then $\gamma \in \Gamma(J)$: Indeed, if $x, y \in J$, $\gamma_2 P_{\gamma(x)} y = \gamma_2 P_{\mu\gamma_2(x)} y = P_{\mu\gamma_2(x)}\gamma_2(y) = \mu^2 P_{\gamma_2(x)}\gamma_2(y)$ (since $\mu \in \Gamma(\tilde{L})$ and $\gamma_2(J) \subseteq \tilde{L}) = \mu^2\gamma_2^2(\gamma_2 P_x y) = (\gamma_2\mu)^2\gamma_2 P_x y$ (since $\Gamma(\tilde{L})$ is commutative, and $\gamma_2 J \subseteq \tilde{L}$) $= \gamma_2(\mu\gamma_2)^2 P_x y$. Therefore $\gamma_2(P_{\gamma(x)} y - \gamma^2 P_x y) = 0$, hence $P_{\gamma(x)} y = \gamma^2 P_x y$ since $J$ does not have $\Gamma(J)$-torsion. Similarly it is easy to see that $\gamma_2^2 P_x \gamma(y) = \gamma_2^2\gamma P_x y$, hence $P_x\gamma(y) = \gamma P_x y$ for all $x, y \in J$, and that $\gamma_2^2\{x, y, \gamma(z)\} = \gamma_2^2\gamma\{x, y, z\}$, hence $\{x, y, \gamma(z)\} = \gamma\{x, y, z\}$ for all $x, y \in J$.

Also, the element $\gamma \in \Gamma(J)$ has $\gamma J = \mu\gamma_2 J \subseteq \mu\tilde{L} \subseteq I$. Thus, it only remains to show that $\gamma \in \mathcal{M}(J)$. To see that, consider the subalgebra $\mathcal{M}_{\tilde{L}}(J)$ of $\mathcal{M}(J)$ generated by all operators $P_x$ and $L_{x,y}$ defined on $J$, for $x, y \in \tilde{L}$. Restriction to $\tilde{L}$ defines a homomorphism $\phi : \mathcal{M}_{\tilde{L}}(J) \to \mathcal{M}(\tilde{L})$ which is obviously surjective. Moreover, if $F \in \mathcal{M}_{\tilde{L}}(J)$ has $\phi(F) = 0$, then $\gamma_1 F = F\gamma_1 = \phi(F)\gamma_1 = 0$, hence $F = 0$, and therefore $\phi$ is an isomorphism. Now $\mu \in \mathcal{M}(\tilde{L})$ has $\phi^{-1}(\mu) = \sum M_1 \cdots M_n$, a sum of compositions of elements $M_i = P_{x_i}$ or $L_{x_i,y_i}$ with $x_i, y_i \in \tilde{L}$. Therefore $\gamma = \mu\gamma_2 = \phi^{-1}(\mu)\gamma_2 = \sum M_1 \dots (M_n\gamma_2) \in \mathcal{M}(J)$, since $M_n\gamma_2 \in \mathcal{M}(J)$: indeed, $P_x\gamma_2 = P_x\gamma_1^2 = P_{\gamma_1(x)} \in \mathcal{M}(J)$, and $L_{x,y}\gamma_2 = L_{x,y}\gamma_1^2 = L_{\gamma_1(x),\gamma_1(y)} \in \mathcal{M}(J)$.

Finally, to see that $\mathcal{C}(J)$ is the field of fractions of $J$ for a Jordan pair or triple system, take $\lambda \in \mathcal{C}(J)$ and a representative $(f, I) \in \lambda$. By what we have proved, there is a nonzero $\gamma \in \Gamma(J)$ with $\gamma J \subseteq I$. Then it is straighforward that $\delta = \lambda\gamma \in \mathcal{C}(J)$ has a representative whose domain is $J$, hence $\delta \in \Gamma(J)$, and $\lambda = \delta\gamma^{-1} \in \Gamma(J)^{-1}\Gamma(J)$.

As mentioned in 1.3 this implies that the natural inclusion of $\Gamma(J)^{-1}J$ into $\mathcal{C}(J)J$ is an isomorphism, and therefore that $\Gamma(J)^{-1}J$ is simple with finite capacity (at most the degree $d$ of $f$) by 1.8. ■

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