The alternative Dunford-Pettis Property in the predual of a von Neumann algebra *

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Abstract

Let A be a be a type II von Neumann algebra with predual A_*. We prove that A_* does not satisfy the alternative Dunford-Pettis property introduced by W. Freedman [7], i.e., there is a sequence (φ_n) converging weakly to φ in A_* with $\|\varphi_n\| = \|\varphi\| = 1$ for all $n \in \mathbb{N}$ and a weakly null sequence (x_n) in A such that $\varphi_n(x_n) \nrightarrow 0$. This answers a question posed in [7].

1 Introduction

A Banach space X is said to have the *Dunford-Pettis property (DP)* if for any weakly null sequences (x_n) in X and (f_n) in X^*, it holds $f_n(x_n) \to 0$. It is a classical result [6, 8] that the spaces $C(K)$ and $L_1(\mu)$ have DP. We refer to [5] as a good survey on DP.

In the 90's, the Dunford-Pettis property has been studied in the setting of some algebraic structures. The von Neumann algebras having DP were characterized by C-H. Chu and B. Iochum [3] as the finite direct sums of type I_n von Neumann algebras. In [2], L. Bunce shows that the predual A_* of a von Neumann algebra A has DP if, and only if, A is of type I finite.

Another property which has been studied in the setting of preduals of von Neumann algebras, is the so-called *Kadec-Klee property* (*KKP* in the sequel). Recall that a Banach space has KKP if any sequence in the unit

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sphere whose weak limit is also in the unit sphere, is norm convergent. It is known [7, Theorem 3.4] that a C*-algebra has KKP if, and only if, it is finite dimensional. In the case of the predual of a semifinite von Neumann algebra A, G. Dell'Antonio showed that A_* satisfies KKP if, and only if, A is of type I with atomic center [4] (see also [7, Remark 2.3]).

Recently, W. Freedman [7] has introduced a new property weaker than DP and KKP, the DP1. A Banach space X has the *DP1* if for any weakly convergent sequences $x_n \to x$ in X, and $f_n \to 0$ in X^*, such that $\|x_n\| = \|x\| = 1$, it holds $f_n(x_n) \to 0$. Of course, the condition $\|x_n\| = \|x\| = 1$ can be replaced by $\|x_n\| \to \|x\|$. This property has been also studied under some algebraic assumption. For instance, W. Freedman also proves that the DP1 is equivalent to the DP for von Neumann algebras [7, Theorem 3.5], but is is strictly weaker than DP and KKP for preduals of von Neumann algebras [7, Example 2.4]. In a more general setting, it is shown in [1, Corollary 2] that a JBW*-triple satisfies DP1 if, and only if, it satisfies KKP or DP.

In the already quoted paper [7], it is asked if every predual of a von Neumann algebra satisfies DP1. The aim of this paper is to show that this is not the case. Indeed, we prove that the predual of every type II von Neumann algebra fails to have DP1 (Theorem 3).

2 The results

Let A be a von Neumann algebra. In the sequel A_{sa} will denote the set of all hermitian elements in A. We recall that a *spin system* in A is a set S of at least two symmetries not equal to ± 1 verifying $st + ts = 0$ whenever $s \neq t$ in S. By a *normal state* on A we mean a weak*-continuous positive linear functional on A. The next proposition gives a sufficient condition for the predual of a von Neumann algebra to fail DP1.

Proposition 1. *Let A be a von Neumann algebra. Suppose that there exist a countable spin system $\{s_n\}_{n\in\mathbb{N}}$ in A_{sa} and a normal state ρ on A, such that $\rho(s_n x) = \rho(x s_n)$ for every $x \in A$ and $n \in \mathbb{N}$. Then A_* does not satisfy DP1.*

Proof. For $n \in \mathbb{N}$, let ρ_n be the element of A_* defined by

$$\rho_n(x) = \rho\big((1 + s_n)x\big) \qquad (x \in A).$$

Since for every positive element $x \in A$, the inequality

$$0 \leq \rho\big((1 + s_n)x(1 + s_n)\big) = \rho\big((1 + s_n)^2 x\big) = 2\rho\big((1 + s_n)x\big)$$

holds, we deduce that each ρ_n is a positive linear functional in A_*. Therefore,

$$\|\rho_n\| = \rho_n(1) = 1 + \rho(s_n) \tag{1}$$

for every $n \in \mathbb{N}$. It is well known (see [9]) that the real Banach subspace V of A_{sa} generated by $\{s_n\}$ (the so-called spin factor) is isomorphic to a real Hilbert space containing $\{s_n\}$ as an orthonormal system. Then, the sequence (s_n) is weakly null in V and hence in A. In particular, it follows from (1) that

$$\|\rho_n\| \to 1 = \|\rho\|.$$

On the other hand, since ρ is positive, the mapping

$$(a, b) \mapsto (a|b)_\rho := \rho(ab^*) \qquad (a, b \in A)$$

is a positive sesquilinear form on A. If we write $N_\rho = \{a \in A \ : \ \rho(aa^*) = 0\}$, then the quotient A/N_ρ can be completed to a Hilbert space denoted by H_ρ. The natural quotient map from A to H_ρ will be denoted by J_ρ. It is easy to check that $(s_n|s_m)_\rho = \delta_{nm}$ for $n, m \in \mathbb{N}$, and therefore $\{J_\rho(s_n)\}$ is an orthonormal sequence in H_ρ. Now, for every $x \in A$, we have

$$\|x\|^2 \geq \|J_\rho x\|_\rho^2 = (x|x)_\rho \geq \sum_{n\in\mathbb{N}} |(x|s_n)_\rho|^2 = \sum_{n\in\mathbb{N}} |\rho(xs_n)|^2.$$

Then $\rho(s_n x) \to 0$, and hence (ρ_n) converges weakly to ρ in A_*. Finally, since $(\|\rho_n\|)$ goes to $\|\rho\|$, (s_n) is weakly null, and

$$\rho_n(s_n) = 1 + \rho(s_n) \nrightarrow 0,$$

we conclude that A_* does not satisfy DP1. $\square$

In [4, Lemma 4], G. Dell'Antonio established that the predual of a von Neumann algebra A does not satisfy KKP if there exists a projection p in A such that the predual of pAp does not satisfy KKP. The next lemma shows that the result still true for DP1.

Lemma 2. *Let A be a von Neumann algebra on a Hilbert space H and p a projection in A. If $\big(pAp\big)_*$ does not satisfy DP1, then neither does A_*.*

Proof. By hypothesis and [7, Proposition 2.1], there exist $(\rho_n) \to \rho$ weakly in $(pAp)_*$ and a weakly null sequence (x_n) in pAp, such that ρ_n, ρ are normal states, and $\rho_n(x_n) \nrightarrow 0$. Let $H_1 = p(H)$. Then pAp is a von Neumann algebra on H_1. By [10, Proposition II.3.20], there exist sequences $(\xi_m^n)_m$, $(\xi_m)_m$ in H_1 such that

$$\rho_n = \sum_{m=1}^{\infty} \omega_{\xi_m^n} \qquad\qquad \rho = \sum_{m=1}^{\infty} \omega_{\xi_m},$$

where for arbitrary $\xi \in H_1$, ω_ξ denotes the map given by $\omega_\xi(a) = (a(\xi)|\xi)$. From this, considering ρ_n and ρ as normal states on A, it is easy to check that

$$\rho_n(x) = \rho_n(pxp), \quad \rho(x) = \rho(pxp) \qquad\quad (x \in A).$$

Therefore, $\rho_n \to \rho$ weakly in A_*. Since (x_n) is also weakly null in A, and $\rho_n(x_n) \nrightarrow 0$, we conclude that A_* does not satisfy DP1. $\square$

Now, we can assure the existence of von Neumann algebras whose preduals do not satisfy DP1.

Theorem 3. *Let A be a type II von Neumann algebra. Then A_* does not satisfy DP1.*

Proof. It is well known (see [10]) that there exists a projection p on A such that pAp is of type II_1. Therefore, using the above lemma, we can assume that A is a type II_1 von Neumann algebra.

Now, by [9, 11], A contains a countable infinite spin system $\{s_n\}$, consisting of nontrivial symmetries s_n in A_{sa} satisfying $s_n s_m + s_m s_n = 0$ whenever $n \neq m$. On the other hand, it is easy to see that the (unique) normal trace ρ on A is a normal state on A which satisfy $\rho(s_n x) = \rho(x s_n)$ for every $x \in A$ and every $n \in \mathbb{N}$ (see [10, §V.2]). To finish the proof, just apply Proposition 1. $\square$

Remark 4. We do not know if the predual of every type I von Neumann algebra has DP1.

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