Polynomial functions on Jordan pairs

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In a recent preprint [**2**], F. Vandebrouck proved by a case-by-case verification that a certain rational function on hermitian Jordan triple systems, defined in terms of the generic norm, is in fact a polynomial. We give an elementary proof of this fact which is valid for all finite-dimensional Jordan pairs over arbitrary fields.

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We will need some basic facts on generic points and polynomial and rational functions on a finite-dimensional vector space, say V, over a field k. To keep the exposition simple, we fix a basis $b_1, \ldots, b_n$ of V and thus identify V with k^n. A basis-free approach may be found in [**1**, §18]. Let $X = (\xi_1, \ldots, \xi_n)$ be an n-tupel of indeterminates. The polynomial ring $k[X] = k[\xi_1, \ldots, \xi_n]$ in n variables may be identified with the ring of polynomial functions on V, and the n-tuple X with the generic point of V. Its quotient field $k(X)$ is then the field of rational functions on V. By a rational map on V with values in a finite-dimensional k-vector space W we mean an element $f = f(X) \in W \otimes_k k(X)$. Since $k[X]$ is a factorial ring, f admits a reduced expression $f(X) = g(X)/h(X)$ where the numerator $g(X) \in W \otimes k[X]$ and the denominator $h(X) \in k[X]$, also called an exact denominator of f, have greatest common divisor 1 and are uniqe up to a nonzero scalar in k. All this has obvious extensions to the case where V is replaced by a finite direct product $V_1 \times V_2 \times \cdots$ with generic point $(X_1, X_2, \ldots)$.

Let now $V = (V^+, V^-)$ be a finite-dimensional Jordan pair over k, with $X = (\xi_1, \ldots, \xi_m)$ the generic point of V^+ and $Y = (\eta_1, \ldots, \eta_n)$ the generic point of V^- (the dimensions of V^+ and V^- need not be the same). Then $k(X, Y)$ is the field of rational functions on $V^+ \times V^-$. The pair (X, Y) is quasi-invertible in $V \otimes_k k(X, Y)$ because $\det B(X, Y) \neq 0$ in $k(X, Y)$. Indeed, $\det B(X, Y)$ is a polynomial and $\det B(0, 0) = 1$. Hence (X, Y) is quasi-invertible, and the quasi-inverse X^Y is a rational map from $V^+ \times V^-$ to V^+. The exact denominator $N(X, Y) \in k[X, Y]$ of X^Y, normalized by $N(0, 0) = 1$, is the generic norm of V [**1**, 16.9].

Consider $V^+ \times V^- \times V^+ \times V^-$, with generic point (X, Y, Z, T). We claim that

$$N_4(X, Y, Z, T) = N(X, Y)N(X^Y + Z, T) \tag{1}$$
$$= N(Z, T)N(X, Y + T^Z) \tag{2}$$
$$= N(X, Y)N(Z, T)N(X^Y, T^Z) \tag{3}$$

is a polynomial function on $V^+ \times V^- \times V^+ \times V^-$.

Indeed, by the cocycle relations for the generic norm [**1**, 16.11.1, 16.11.2] and quasi-invertibility of (X, Y) and (Z, T) in $V \otimes k(X, Y, Z, T)$ we have

$$N(X, Y + T^Z) = N(X, Y)N(X^Y, T^Z),$$
$$N(X^Y + Z, T) = N(Z, T)N(X^Y, T^Z).$$

By multiplying these equations with $N(Z,T)$ and $N(X,Y)$, respectively, we see that the right hand sides of (1)–(3) are equal. Now (1) shows that $N_4(X,Y,Z,T) \in k(X,Y)[Z,T]$ is a polynomial in (Z,T) with coefficients in $k(X,Y)$, while (2) shows, similarly, that $N_4(X,Y,Z,T) \in k(Z,T)[X,Y]$. Thus

$$N_4(X,Y,Z,T) \in k(X,Y)[Z,T] \cap k(Z,T)[X,Y] = k[X,Y,Z,T]$$

is indeed a polynomial in all variables.

Next, consider the rational function

$$f(X,Y,Z,W,T) = N(X,Y)N(Z,W)N(X^Y + Z^W, T) \tag{4}$$

on $V^+ \times V^- \times V^+ \times V^- \times V^-$ (with generic point (X,Y,Z,W,T)). This, too, is a polynomial: By symmetry in (X,Y) and (Z,W) we have

$$f(X,Y,Z,W,T) = N(Z,W)N_4(X,Y,Z^W,T) = N(X,Y)N_4(Z,W,X^Y,T).$$

Since N_4 is a polynomial, this shows that

$$f(X,Y,Z,W,T) \in k(Z,W)[X,Y,T] \cap k(X,Y)[Z,W,T] = k[X,Y,Z,W,T]$$

is a polynomial in all five variables.

Let now V be a finite-dimensional complex Jordan pair and let $^-\colon V^\sigma \to V^{-\sigma}$ be a an involution of the second kind, i.e., a complex antilinear map which is an involution (in the sense of [**1**, 1.13]) of the underlying real Jordan pair $({}_{\mathbb{R}}V^+, {}_{\mathbb{R}}V^-)$. We recall here that Jordan pairs with involution of the second kind are in one-to-one correspondence with complex Jordan triple systems whose triple product is antilinear in the middle variable; the triple system associated to $(V, ^-)$ being V^+ with triple product $\{x\bar{y}z\}$. F. Vandebrouck [**2**, Prop. 4] proved for semisimple Jordan triple systems by a case-by case verification that the function

$$|N(x,\bar{y})|^2 N\left((x^{\bar{y}} + y^{\bar{x}}), \bar{z}\right) \tag{5}$$

(where $x, y, z \in V^+$) is a polynomial on $({}_{\mathbb{R}}V^+)^3$. This is now an easy consequence of (4). Indeed, from [**1**, 16.10.1] and the fact that the involution is antilinear, it is easy to see that the generic norm of V satisfies $\overline{N(x,\bar{y})} = N(y,\bar{x})$. Hence $|N(x,\bar{y})|^2 = N(x,\bar{y})N(y,\bar{x})$, and thus $|N(x,\bar{y})|^2 N([x^{\bar{y}} + y^{\bar{x}}], \bar{z}) = f(x,\bar{y},y,\bar{x},\bar{z})$ is a polynomial on $({}_{\mathbb{R}}V^+)^3$.

References

[**1**] O. Loos, *Jordan pairs*, Lecture Notes in Math., vol. 460, Springer-Verlag, 1975.

[**2**] F. Vandebrouck, *The Poisson-Furstenberg kernel of a bounded symmetric domain*, preprint, 1999.