Outer Fractions in Quadratic Jordan Algebras

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Abstract

Using new techniques of Zel'manov, C. Martinez improved on work of Jacobson, McCrimmon, and Parvathi to give a necessary and sufficient Ore-type condition for an arbitrary linear Jordan algebra (with no 2- or 3-torsion) to have an algebra of fractions. In this paper we extend to quadratic algebras the concept of algebras of outer fractions with respect to an Ore monad, and describe necessary and sufficient Ore-type conditions for the imbedding in such an algebra of fractions. The details of the actual imbedding will appear in a subsequent paper.

In the early years of quadratic Jordan algebras it was natural to look for an Ore-like theory of Jordan fractions. In those pre-Zel'manov days it was still possible to hope for a non-Albert exceptional Jordan division algebra. In 1978 Jacobson, McCrimmon, and Parvathi [6] obtained an imbedding of a Jordan algebra J with set S of Ore denominators in an algebra J_S of outer Jordan S-fractions, in the sense that (1) every element of S is invertible in J_S, (2) every element $q \in J_S$ is an *outer S-fraction* $q = U_s^{-1}a$ for some $s \in S$ and $a \in J$. However, to make the product $U_q r$ be quadratic in the variable q they had to impose an unnatural extra condition.

In 2001 Consuelo Martinez [8] discovered a beautiful way to construct an algebra of fractions J_S from a linear Jordan algebra over scalars $\Phi \ni \frac{1}{6}$ by imbedding it in a Tits-Kantor-Koecher Lie algebra via $a \to Ad_a$, constructed as a Lie algebra of germs of derivations (just as in associative ring theory algebras of right quotients as^{-1} are constructed as germs of right R-module homomorphisms $L_{as^{-1}} : sR \to R$).

Since the impetus came from imbedding a Jordan algebra in a division algebra, the map $J \to J_S$ was to be injective, and hence the elements of S were *injective* in the sense that their U-operators were injective. In this paper we will keep the same focus, and not discuss the more general question of localization at an arbitrary S. Much work has been done recently by Anquela, Garcia, Gomez Lozano, Montaner, and others [1, 4, 5] on Jordan algebras of *quotients* with respect to denominators which are dense or essential ideals or inner ideals, not necessarily containing injective elements. Much of this work makes use of the structure theory of Jordan systems, whereas the original work of Jacobson, McCrimmon, Parvathi and Martinez gave intrinsic constructions of algebras of fractions.

The extension to quadratic Jordan algebras over arbitrary rings of scalars (for example, Jordan rings over $\mathbb{Z}$) of Martinez's results is much more involved technically. In this paper, the first of a two-part program, we will develop the general theory of Jordan algebras of outer fractions. We will define what it means for an element q of an over-algebra $Q \supset J$ to be an *outer S-fraction* from J,

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a fraction $U_s^{-1}a$ with *numerator* $a \in J$ and *outer denominator* $s \in S$ for a suitable denominator set S. We discuss various possible "denominators". One of the crucial steps in Martinez' work [8] is to demand that a denominator s do more than just map q into J ($U_s q = a \in J$ as in [6]), it should *annihilate* q mod J in the sense of the Zelmanov annihilator ($\{s, q, \widehat{J}\} \subseteq J$), in particular $\{s, q\} = \{s, q, 1\} \in J$. We will use a quadratic version of the annihilator. §1 recalls the notation for the rest of the paper, §2 introduces inner ideals of denominators. §3 discusses generic denominators. §4 discusses Ore monads S and the Ore Condition on J relative to S, and in §5 we relate this to movability of operators, showing that the Ore condition is equivalent to a two-sided Ore condition in the universal multiplication envelope. In §6 we show that the Ore condition (as well as an Unwelcome Condition) must hold if an algebra of fractions is to exist. §7 derives Martinez's lovely result that whenever we can find a home where fractions live, the set of all fractions forms a subalgebra which is an algebra of fractions. We expect these results to have analogues for Jordan triples and pairs, where creation of inverses will be possible only for systems which already imbed in Jordan algebras.

1 Introduction

We begin by recalling some basic concepts for associative algebras, quadratic Jordan algebras, Lie algebras, and Jordan pairs. Throughout the paper we will consider algebras over a fixed unital, commutative associative ring of scalars Φ. Let R be an associative Φ-algebra and let $S \subseteq R$ be an *Ore monoid*, a subset of regular elements (*injective elements* in the sense that their right and left multiplications are injective maps) which is closed under multiplication. An overalgebra $Q \supseteq R$ is called an *algebra of right S-fractions* for R if the following two conditions hold: (i) each $s \in S$ becomes invertible in Q, (ii) each $q \in Q$ has the form of a *right S-fraction* $q = as^{-1}$ for some $a \in R, s \in S$. Ore found a necessary and sufficient condition on R, S (now known as the the *right Ore condition*) for the existence of such an algebra of right fractions [14]: for each $a \in R$ and $s \in S$ there exists $a' \in R$ and $s' \in S$ such that $as' = sa'$. Intuitively, this says that every "left S-fraction" can be rewritten as a right S-fraction, $s^{-1}a = a's'^{-1}$.

A *unital quadratic Jordan algebra* $J = (J, U, 1)$ over an arbitrary ring of scalars Φ, is a Φ-module J containing a distinguished element 1, and having a quadratic map $U : J \to \mathrm{End}_\Phi(J)$ such that

$$\text{(QJ1) } U_1 = Id, \quad \text{(QJ2) } U_x V_{y,x} = V_{x,y} U_x = U_{U_x y, x}, \quad \text{(QJ3) } U_{U_x y} = U_x U_y U_x$$

hold *strictly* in the sense that they continue to hold in all scalar extensions, equivalently, if all their linearizations hold in J. Here $U_{x,z} := U_{x+z} - U_x - U_z$ is the linearization of the U-operator, and $V_{x,y}(z) := \{x, y, z\} := U_{x,z}(y)$. A *quadratic Jordan algebra* is just a subspace $J = (J, U,^2)$ of some unital quadratic Jordan algebra, closed under the U-operator and *squaring* $x^2 := U_x 1$, equivalently, the *free unital hull* $\widehat{J} := \Phi 1 \oplus J$ becomes a unital quadratic Jordan algebra under $U_{\alpha 1 \oplus x}(\beta 1 \oplus y) := \alpha^2\beta \oplus \big(\alpha^2 y + 2\alpha\beta x + \alpha\{x, y\} + \beta x^2 + U_x y\big)$ where $V_x y := \{x, y\} := U_{x,y}1$ denotes the linearization of the square. The operator $V_{x,y} = V_x V_y - U_{x,y}$ is generated by the U- and V-operators U_a, V_a. Unless otherwise stated, we will deal with non-unital quadratic Jordan algebras, but we will always make use of the unital hull. The linear map $U_x : y \mapsto U_x y$ is **outer multiplication** by x, and is a quadratic function of x; turning this on its head, the quadratic map $\cap_x : y \mapsto U_y x$ is **inner multiplication** by x, and is a linear function of x. Outer and inner multiplications play the role in Jordan theory that left and right multiplications do in associative theory. The archetypal example of a Jordan algebra is A^+ for an associative algebra, with $U_x y = xyx,\ V_{x,y} z = \{x, y, z\} = xyz + zyx,\ x^2 = xx,\ V_x y = \{x, y\} = xy + yx$; an algebra is *special* if it can be imbedded in an algebra A^+.

We will use [9] as reference for all formulas (especially the Fundamental Formulas FF on p. 202); remember that for a Jordan operator identity in two variables it suffices by Macdonald's Principle [9, p. 466] to verify it in associative algebras. The following are used frequently enough in the paper for us to display them:

(1.1.1) $V_{x,y}U_z + U_zV_{y,x} = U_{\{x,y,z\},z}, \qquad [V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}},$
(1.1.2) $V_{U_xy} = V_{x,\{y,x\}} - V_{x^2,y}, \qquad V_{x^2} = V_{x,x}, \qquad V_{\{x,y\}} = V_{x,y} + V_{y,x},$
(1.1.3) $U_{\{x,y\}} - \{U_x, U_y\} = V_{x,y}V_{y,x} - V_{U_xy^2} = V_xU_yV_x - U_{U_xy,y},$
(1.1.4) $U_{\{x,y,z\}} = U_xU_yU_z - U_zU_yU_x - U_{U_xU_yz,z} + U_{\{x,y,z\},z}V_{y,x} - U_zV_{y,U_xy},$
(1.1.5) $V_{x,U_yx} = V_{U_xy,y}, \quad V_{x,U_yz} = V_{\{x,y,z\},y} - V_{z,U_yx} = V_{x,y}V_{z,y} - U_{x,z}U_y,$
(1.1.6) $V_{U_yz,x} = V_{y,\{z,y,x\}} - V_{U_yx,z} = V_{y,z}V_{y,x} - U_yU_{z,x},$
(1.1.7) $U_{U_xy,z} = U_{x,z}V_{y,x} - V_{z,y}U_x = V_{x,y}U_x - U_xV_{y,z},$
(1.1.8) $U_{\alpha x+U_xy} = B_{\alpha,x,y}U_x = U_xB_{\alpha,y,x}\ (B_{\alpha,x,y} := \alpha^2\mathbf{1} + \alpha V_{x,y} + U_xU_y),$
(1.1.9) $U_xB_{\alpha,y,U_xz} = B_{\alpha,U_xy,z}U_x, \quad U_xV_{y,U_xz} = V_{U_xy,z}U_x.$

For a subalgebra $K \subseteq J$, the unital multiplication algebra of K on J is denoted by $\mathcal{M}(K|J)$; it is generated over Φ by the identity operator $\mathbf{1}$ and all operators of the form V_a, U_a for $a \in K$; when $K = J$ we get the full multiplication algebra $\mathcal{M}(J)$. The universal gadget for multiplications is the **universal multiplication envelope** $\mathcal{UME}(J)$ [12, 10] generated by 1 and all $\widetilde{\widetilde{V}}_a, \widetilde{\widetilde{U}}_a$ for $a \in J$; if K is a subalgebra of J, we denote by $\mathcal{UME}(K|J)$ the subalgebra generated by 1 and all $\widetilde{\widetilde{V}}_a, \widetilde{\widetilde{U}}_a$ for $a \in K$. We will consistently use $\approx$ to denote generic multiplication operators. The universal multiplication envelope satisfies all Jordan operator identities, in particular, (QJ1-3) and (1.1.1-9). In $\mathcal{UME}(K|J)$ we have

$$(1.2) \qquad \widetilde{\widetilde{V}}_{U_K\widehat{K}} \subseteq \widetilde{\widetilde{V}}_{K,K}, \quad [\widetilde{\widetilde{V}}_K, \widetilde{\widetilde{V}}_{K,K}] \subseteq \widetilde{\widetilde{V}}_{K,K}, \quad \widetilde{\widetilde{V}}_{U_s\widehat{J},\widehat{J}}\widetilde{\widetilde{U}}_s = \widetilde{\widetilde{U}}_s\widetilde{\widetilde{V}}_{\widehat{J},U_s\widehat{J}}.$$

since for $z, w \in I$ all of [by (1.1.2)] $\widetilde{\widetilde{V}}_{U_zw} = \widetilde{\widetilde{V}}_{z,\{w,z\}} - \widetilde{\widetilde{V}}_{z^2,w}$, $\widetilde{\widetilde{V}}_{z^2} = \widetilde{\widetilde{V}}_{z,z}$, $\widetilde{\widetilde{V}}_{\{z,w\}} = \widetilde{\widetilde{V}}_{z,w} + \widetilde{\widetilde{V}}_{w,z}$, [by (1.1.1)] $[\widetilde{\widetilde{V}}_x, \widetilde{\widetilde{V}}_{z,w}] = \widetilde{\widetilde{V}}_{\{x,z\},w} - \widetilde{\widetilde{V}}_{z,\{x,w\}}$ lie in $\widetilde{\widetilde{V}}_{K,K}$, [by (1.1.9)] $\widetilde{\widetilde{V}}_{U_s\hat{x},\hat{y}}\widetilde{\widetilde{U}}_s = \widetilde{\widetilde{U}}_s\widetilde{\widetilde{V}}_{\hat{x},U_s\hat{y}}$.

Whenever $J \subseteq Q$ (for example, when $Q = \widehat{J}$ is the unital hull), we have a natural epimorphism $\mathcal{UME}(J) \longrightarrow \mathcal{M}(J|Q)$ sending $1, \widetilde{\widetilde{V}}_a, \widetilde{\widetilde{U}}_a, \widetilde{\widetilde{V}}_{a,b} \to \mathbf{1}, V_a, U_a, V_{a,b} \in \mathrm{End}(Q)$. In particular, Q becomes a left $\mathcal{UME}(J)$-module, and we can form $\widetilde{\widetilde{M}}(q)$ for any $\widetilde{\widetilde{M}} \in \mathcal{UME}(J)$ and any $q \in Q$.

A crucial feature of the universal envelope is the *universal reversal involution* $\widetilde{\widetilde{M}} \to \widetilde{\widetilde{M}}^*$ determined on the generators by $\widetilde{\widetilde{V}}_a, \widetilde{\widetilde{U}}_a, \widetilde{\widetilde{V}}_{a,b} \to \widetilde{\widetilde{V}}_a, \widetilde{\widetilde{U}}_a, \widetilde{\widetilde{V}}_{b,a}$. This provides duality among Jordan operator identities: if $\widetilde{\widetilde{M}}$ vanishes identically in all Jordan algebras, so does its dual $\widetilde{\widetilde{M}}^*$. [Caution: this does not induce an involution on each $\mathcal{M}(\widehat{J}|Q)$: we can have $\widetilde{\widetilde{M}} \to W = 0$ but $\widetilde{\widetilde{M}}^* \to W^* \neq 0$.]

2 Denominators

We begin by describing various denominators for fractions. Throughout this section, let $K \subseteq J \subseteq Q$ be a subalgebra of a quadratic Jordan algebra which is in turn a subalgebra of Q, with $x \in K, q \in Q$.

Denominator Definition 2.1 *A K-**prenominator** (= pre-denominator) for $q \in Q$ is an element $x \in K$ whose V and U operators push q into K,*

$$\text{(Pren1)} \quad V_xq \in K, \qquad \text{(Pren2)} \quad U_xq \in K.$$

The set of K-prenominators for q will be denoted by $Pren_K(q)$. Note that U-operator pushing holds automatically when $q \in J$ by innerness of K.

*An element $x \in K$ will be called a K-**denominator** for q if*

$$\begin{array}{llllll} \text{(D1)} & U_xq \in K, & \text{(DS2)} & U_xU_q\widehat{K} \subseteq K, & \text{(D3)} & \{x,q,\widehat{K}\} \subseteq K, \\ \text{(D1)}' & U_qx \in K, & \text{(DS2)}' & U_qU_x\widehat{K} \subseteq K, & \text{(D3)}' & \{q,x,\widehat{K}\} \subseteq K. \end{array}$$

*An element $x \in K$ will be called a **strong** K-**denominator** for q if*

$$\begin{array}{llllll} \text{(SD1)} & U_xq \in U_K\widehat{K}, & \text{(SD2)} & U_xU_q\widehat{K} \subseteq U_K\widehat{K}, & \text{(SD3)} & \{x,q,\widehat{K}\} \subseteq U_K\widehat{K}, \\ \text{(SD1)}' & U_qx \in U_K\widehat{K}, & \text{(SD2)}' & U_qU_x\widehat{K} \subseteq U_K\widehat{K}, & \text{(SD3)}' & \{q,x,\widehat{K}\} \subseteq U_K\widehat{K}. \end{array}$$

The set of K-denominators and strong denominators for q will be denoted by $Den_K(q), SDen_K(q)$ respectively.

More generally, let $L \lhd K$ be an ideal of K, and $LDen_K(q)$ denote the L, K-denominators for q, the elements $x \in K$ which satisfy the product conditions

$$\begin{array}{llllll} \text{(LD1)} & U_x q \in L, & \text{(LD2)} & U_x U_q \widehat{K} \subseteq L, & \text{(LD3)} & \{x, q, \widehat{K}\} \subseteq L, \\ \text{(LD1)}' & U_q x \in L, & \text{(LD2)}' & U_q U_x \widehat{K} \subseteq L, & \text{(LD3)}' & \{q, x, \widehat{K}\} \subseteq L. \end{array}$$

where (LD3) *implies the analogue of* (Pren1),

$$\text{(LD0)} \qquad \{x, q\} \in L.$$

$LDen_K(q)$ is just the quadratic Zelmanov annihilator $Zann_{K,L}(q)$ of q from K into L. Here $L = K$ corresponds to ordinary K-denominators (D)*, $L = U_K \widehat{K}$ corresponds to strong K-denominators* (SD)*.* ■

We remark that (LD3) $\Longleftrightarrow$ (LD3)$'$, and in the presence of either of these (LD2) $\Longleftrightarrow$ (LD2)$'$. Indeed, (3) $\Longleftrightarrow$ (3)$'$ since $\{q, x, \widehat{K}\} + \{x, q, \widehat{K}\} = \{\{q, x\}, \widehat{K}\} \subseteq \{L, \widehat{K}\} \subseteq L$ by (1.1.2) since both (3) and (3)$'$ imply $\{q, x\} = \{q, x, \hat{1}\} = \{x, q, \hat{1}\}$ fall in L. When these hold, (2) $\Longleftrightarrow$ (2)$'$ because by (1.1.3) we have $(U_q U_x + U_x U_q - U_{\{x,q\}})\widehat{K} = \big(-V_{U_x U_q 1} + V_{x,q} V_{q,x}\big)\widehat{K} = \big(-V_{U_q U_x 1} + V_{q,x} V_{x,q}\big)\widehat{K}$ where $\{x, q\} \in L$ [by (0)]; $U_x U_q 1 \in U_x U_q \widehat{K}$ or $U_q U_x 1 \in U_q U_x \widehat{K}$ falls in L [by (2) or (2)$'$]; $V_{x,q} V_{q,x} \widehat{K} + V_{q,x} V_{x,q} \widehat{K} \subseteq L$ [by (3),(3)$'$]. Thus a plain, strong, or L-denominator needs only to satisfy 4 conditions (1), (1)$'$, (2), (3), not all 6.

At times we will need to find more and stronger denominators. The following result shows how to get more denominators of the same kind.

Innerness Lemma 2.2 *For any ideal L in the subalgebra K, the denominators $LDen_K(q) = Zann_{K,L}(q)$ always form an inner ideal in K (not of J!).*

Proof: It is well known that the Zelmanov Annhilator produces inner ideals in contexts where annihilation is modulo an ideal (the original archetype being the case $L = 0$). We first must show that $LDen_K(q)$ is a linear subspace of K. Consider $x, y \in LDen_K(q)$; clearly $\alpha x \in LDen_K(q)$ for $\alpha \in \Phi$. We claim that $x + y \in LDen_K(q)$ because it satisfies (LD1), (LD1)$'$, (LD2), (LD3). For (LD1) we have $U_{x+y}(q) = U_x q + U_y q + U_{x,y}(q) \in L + L + \{x, q, K\} \subseteq L$ [by (LD1) for x, y and (LD3) for x]. For (LD1)$'$ we have $U_q(x + y) = U_q x + U_q y \in K$ [by (LD1)$'$ for x, y]. For (LD2) we see from (1.1.5) that $U_{x+y} U_q \widehat{K} = U_{x,y} U_q \widehat{K} + U_x U_q \widehat{K} + U_y U_q \widehat{K} = V_{x,q} V_{y,q} \widehat{K} - V_{x, U_q y} \widehat{K} + U_x U_q \widehat{K} + U_y U_q \widehat{K}$, where all 4 terms fall in L: (1) $V_{x,q} V_{y,q} \widehat{K} \subseteq V_{x,q} K \subseteq L$ [by (LD3) for y, x], (2) $V_{x, U_q y} \widehat{K} \subseteq V_{K,L} \widehat{K}$ [by (LD1)$'$ for y] $\subseteq L$ [by $L \lhd K$], (3,4) $U_x U_q \widehat{K} + U_y U_q \widehat{K} \subseteq L$ [by (LD2) for x, y]. For (LD3) we have $\{x + y, q, \widehat{K}\} = \{x, q, \widehat{K}\} + \{y, q, \widehat{K}\} \subseteq L + L \subseteq L$ [by (LD3) for x, y]. Thus $x + y \in LDen_K(q)$ as desired, and $LDen_K(q)$ is indeed a linear subspace.

To show innerness of $LDen_K(q)$ in K, we will show that $y := U_x a \in LDen_K(q)$ for any $x \in LDen_K(q), a \in \widehat{K}$ by showing that it satisfies (LD1), (LD1)$'$, (LD2), (LD3). It lies in the subalgebra K by $U_K \widehat{K} \subseteq K$. It satisfies (LD1) since $U_y(q) = U_x U_a (U_x q) \in L$ [because $U_x q \in L$ by (LD1) and $x \in K, \hat{a} \in \widehat{K}$]. It satisfies (LD1)$'$ since $U_q y = U_q U_x a \in L$ [by (LD2)$'$ for x]. It satisfies (LD2) since $U_y U_q \widehat{K} = U_x U_a U_x (U_q \widehat{K}) \subseteq U_x U_a L$ [by (LD2) for x] $\subseteq L$ [by $x, a \in \widehat{K}$]. It satisfies (LD3) since from (1.1.6) $\{y, q, \widehat{K}\} = V_{U_x a, q} \widehat{K} = V_{x, \{q,x,a\}} \widehat{K} - V_{U_x q, a} \widehat{K} \subseteq L$ [because $x, a \in \widehat{K}$ and $\{q, x, a\}, U_x q \in L$ by (LD3)$'$, (LD1) for x]. ■

It seems quite difficult to produce denominator inner ideals *in* J when K itself is inner. Notice that in all these definitions, an element will remain a denominator or prenominator for any larger subalgebra $K' \supseteq K$. The following result shows how denominators can be made stronger by powers instead of steroids.

Strengthening Lemma 2.3 (I) *If $x \in K$ is a K-prenominator for $q \in Q$, then x^4 will be a K-denominator in $Den_K(q)$.*

(II) *If $x \in K$ is a K-denominator for $q \in Q$, then x^3 will be a strong K-denominator in $SDen_K(q)$.*

Proof: (I) We must show that $x^4 \in Den_K(q)$ by showing that it satisfies conditions (D1), (D1)′, (D2), (D3). Using $x, \{x,q\}, U_x q \in K$ repeatedly, we compute (D1): $U_{x^4} q = U_{x^3}(U_x q) \in U_{x^3}(K) \subseteq K$; (D1)′: $U_q x^4 = (U_x q)^2 - \{\{x,q\}, x, U_x q\} + U_{\{x,q\}} x^2$ [by Macdonald's Theorem] $\in K$; (D3): $\{x^4, q, \widehat{K}\} = \{x^3, \{x,q\}, \widehat{K}\} - \{x^2, U_x q, \widehat{K}\}$ [by Macdonald] $\subseteq K$; (D2): taking the identity

$$U_{x^2} U_q z = \big(U_{\{x^2,q\}} - U_q U_{x^2} + 2U_{U_x q} - V_{x,\{x,q\}} V_{\{x,q\},x} + V_{U_x(\{x,q\})^2} + U_{\{x^2,q\},U_x q}\big) z$$

[by Macdonald] for $z \in \widehat{K}$ gives $U_{x^2} U_q z \in K - U_q U_{x^2} z$ [noting $\{x^2, q\} = \{x, \{x,q\}\} - 2U_x q \in K$], so multiplying by U_x gives $U_{x^4} U_q z \in U_K K - U_x(U_{U_x q}) U_x z \subseteq U_K K - U_K(U_K) U_K \widehat{K} \subseteq U_K K$.

(II) We are given that $x \in K$ satisfies (D1)-(D3)′ and we must show that x^3 satisfies (SD1), (SD1)′, (SD2), (SD3). We again compute (SD1): $U_{x^3} q = U_x U_x (U_x q) \in U_K U_K K \subseteq U_k \widehat{K}$; (SD1)′: $U_q x^3 = -U_x U_q x - V_x U_q U_x 1 + U_{\{x,q\}} x$ [by Macdonald] $\in -U_K K - \{K, K\} + U_K K$ [by (D1)′, (D2)′, (D3) for x] $\subseteq U_K K$; (SD2): $U_{x^3} U_q \widehat{K} = U_x U_x (U_x U_q \widehat{K}) \subseteq U_K U_K K \subseteq U_K \widehat{K}$ [by (D2) for x]; (SD3): $\{x^3, q, \widehat{K}\} = \{x, \{x, x, q\}, \widehat{K}\} - \{U_x q, x, \widehat{K}\} \subseteq \{K, K, \widehat{K}\}$ [by (D3)′, (D1) for x] $\subseteq U_K K + \{K, K\} \subseteq U_K \widehat{K}$. ■

3 Generic Denominators

In this section we discuss the "ultimate" K-denominators for principal inner ideals $K = I_s$, which remain denominators for q in any larger algebra $\widetilde{Q} \supseteq Q$. We speak of s-denominators instead of I_s-denominators, denoting $Pren_{I_s}$ and Den_{I_s} simply by $Pren_s$ and Den_s.

Genominator Definition 3.1 *If $q \in Q \supseteq J$, then an element $x \in I_s$ is called an s-**genominator** (= generic denominator) if it generically pushes multiplications by q into s-multiplications,*

$$\begin{array}{llllll} \text{(G1)} & U_x q = U_s w_1 \in I_s, & \text{(G1)}' & U_q x = U_s w_2 \in I_s, & \text{(G1)}'' & U_q U_x q = U_s w_3 \in I_s, \\ \text{(G2)} & \tilde{\tilde{V}}_{x,q} = \tilde{\tilde{S}} \in \tilde{\tilde{V}}_{\widehat{I}_s, I_s}, & \text{(G3)} & \tilde{\tilde{U}}_x \tilde{\tilde{U}}_q = \tilde{\tilde{U}}_s \tilde{\tilde{N}}_o \tilde{\tilde{U}}_s, & \text{(G4)} & \tilde{\tilde{U}}_x \tilde{\tilde{V}}_q = \tilde{\tilde{U}}_s \tilde{\tilde{M}}_o \tilde{\tilde{U}}_s \end{array}$$

for some elements $w_1, w_2, w_3 \in J$ and generic multiplications $\tilde{\tilde{S}}, \tilde{\tilde{M}}_o, \tilde{\tilde{N}}_o \in \mathcal{UME}(J|Q)$. Any s-genominator x is automatically a prenominator,

$$\text{(G0)} \quad \{x, q\} = V_{x,q} \hat{1} = U_s w_0, \ \{x, q, s\} = V_{x,q} s = U_s v_0 \in V_{\widehat{I}_s, I_s} \widehat{I}_s \subseteq I_s.$$

Due to the reversal involution and the strong assumption of two-sided factors s in (G1-4)*, these relations automatically imply dually that the genominator absorbs multiplications by q into multiplications by I_s,*

$$\text{(G0)}^* \{q, x, s\} = U_s v_0^*, \ \text{(G2)}^* \ \tilde{\tilde{V}}_{q,x} = \tilde{\tilde{S}}^* \in \tilde{\tilde{V}}_{\widehat{I}_s, I_s}, \ \text{(G3)}^* \ \tilde{\tilde{U}}_q \tilde{\tilde{U}}_x = \tilde{\tilde{U}}_s \tilde{\tilde{N}}_o^* \tilde{\tilde{U}}_s, \ \text{(G4)}^* \ \tilde{\tilde{V}}_q \tilde{\tilde{U}}_x = \tilde{\tilde{U}}_s \tilde{\tilde{M}}_o^* \tilde{\tilde{U}}_s.$$

We denote the s-genominators of q by $Gen_s(q)$. ■

Hull Genominator Proposition 3.2 *If $x \in I_s$ is an s-genominator for $a \in J$, it is also an s-genominator for every $\hat{a} := \alpha\hat{1} + a \in Q = \widehat{J}$.*

Proof: The 6 genominator conditions hold for $x, \hat{a}$ in $\tilde{\tilde{\mathcal{U}}} = \mathcal{UME}(J|\widehat{J})$: **(G1)** $U_x(\alpha\hat{1} + a) \in U_{I_s} \widehat{J} \subseteq I_s$ [by innerness]; **(G1)′** $U_{\alpha\hat{1}+a} x = \alpha^2 x + \alpha\{x, a\} + U_a x \in I_s$ [by $x \in I_s$, (G0),(G1)′ for

$x,a]$; **(G1)″** $U_{\alpha\hat{1}+a}U_x\hat{a} = \alpha^2 U_x\hat{a} + \alpha\tilde{\tilde{V}}_a\tilde{\tilde{U}}_x\hat{a} + \tilde{\tilde{U}}_a\tilde{\tilde{U}}_x\hat{a} \in I_s + \alpha\tilde{\tilde{U}}_s\tilde{\tilde{M}}^*_o\tilde{\tilde{U}}_s\hat{a} + \tilde{\tilde{U}}_s\tilde{\tilde{N}}^*_o\tilde{\tilde{U}}_s\hat{a}$ [by (G1) above, (G4)*,(G3)* for $x,a] \subseteq I_s + U_sJ = I_s$; **(G2)** $\tilde{\tilde{V}}_{x,\alpha\hat{1}+a} = \alpha\tilde{\tilde{V}}_x + \tilde{\tilde{V}}_{x,a} \subseteq \tilde{\tilde{V}}_{\widehat{I_s},I_s}$ [by $x \in I_s$, (G2) for $x,a]$; **(G3)** $\tilde{\tilde{U}}_x\tilde{\tilde{U}}_{\alpha\hat{1}+a} = \alpha^2\tilde{\tilde{U}}_x + \alpha\tilde{\tilde{U}}_x\tilde{\tilde{V}}_a + \tilde{\tilde{U}}_x\tilde{\tilde{U}}_a \in \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s$ [by $x \in U_sJ$, (G4), (G3) for $x,a]$; **(G4)** $\tilde{\tilde{U}}_x\tilde{\tilde{V}}_{\alpha\hat{1}+a} = 2\alpha\tilde{\tilde{U}}_x + \tilde{\tilde{U}}_x\tilde{\tilde{V}}_a \in \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s$ [by (QJ3) and $x \in I_s = U_sJ$, (G4) for $x,a]$. ■

Unlike the denominators previously discussed, these genominators do not seem to form an inner ideal: they are closed under scaling and principal inner multiplication, but not under addition.

Genominator Innerness 3.3 *The s-genominators are closed under principal inner multiplication by* I_s *:*

$$x \in Gen_s(q) \Longrightarrow K_{x,I_s} := \Phi x + U_x\widehat{I_s} \subseteq Gen_s(q).$$

Proof: We must show that for $I := I_s$ and any $x = U_sb \in Gen_s(q)$, $\alpha \in \Phi$, $\hat{a} \in \widehat{I}$, the element $y := \alpha x + U_x\hat{a}$ satisfies the G-conditions. This y still lies in the inner ideal I. It satisfies **(G1)** since $U_y(q) = B_{\alpha,x,\hat{a}}U_xq$ [by (1.1.8)] $= (\alpha^2\mathbf{1} + \alpha\tilde{\tilde{V}}_{x,\hat{a}} + \tilde{\tilde{U}}_x\tilde{\tilde{U}}_{\hat{a}})I$ [by (G1) for x] $\subseteq I + V_{I,\widehat{I}}I + U_IJ \subseteq I$. It satisfies **(G1)′** since $U_qy = \alpha U_qx + U_qU_x\hat{a} \in I + U_sJ \subseteq I$ [by (G1)′, (G3)* for x]. It satisfies **(G1)″** since $U_qU_yq = \alpha^2U_qU_xq + \alpha U_qU_{x,U_x\hat{a}}q + U_q(U_xU_{\hat{a}}U_x)q \in \alpha^2I + \alpha(U_qU_x)\{\widehat{I},x,q\} + (U_qU_x)U_{\widehat{I}}(U_xq)$ [by (G1)″ for x] $\subseteq I$ [by (G3)*, (G2)*, (G1) for x]. It satisfies **(G2)** since we have $\tilde{\tilde{V}}_{y,q} = \alpha\tilde{\tilde{V}}_{x,q} + \tilde{\tilde{V}}_{U_xa,q} = \alpha\tilde{\tilde{V}}_{x,q} + \tilde{\tilde{V}}_{x,\{q,x,a\}} - \tilde{\tilde{V}}_{U_xq,a}$ [by (1.1.6)] $\in \alpha\tilde{\tilde{V}}_{\widehat{I},I} + \tilde{\tilde{V}}_{I,\tilde{\tilde{V}}_{\widehat{I},I}(a)} - \tilde{\tilde{V}}_{I,a}$ [by (G2),(G2)*,(G1) for $x \in I$] $\subseteq \tilde{\tilde{V}}_{\widehat{I},I}$ [by $a \in I$]. It satisfies **(G3)** since $\tilde{\tilde{U}}_y\tilde{\tilde{U}}_q = \tilde{\tilde{B}}_{\alpha,x,\hat{a}}\tilde{\tilde{U}}_x\tilde{\tilde{U}}_q$ [by (1.1.8)] $= \tilde{\tilde{B}}_{\alpha,U_sb,\hat{a}}\tilde{\tilde{U}}_s\tilde{\tilde{M}}\tilde{\tilde{U}}_s$ [by (G3) for x] $= \tilde{\tilde{U}}_s(\tilde{\tilde{B}}_{\alpha,b,U_s\hat{a}}\tilde{\tilde{M}})\tilde{\tilde{U}}_s$ [by (1.1.9)]. Similarly, it satisfies **(G4)** since $\tilde{\tilde{U}}_y\tilde{\tilde{V}}_q = \tilde{\tilde{B}}_{\alpha,x,\hat{a}}\tilde{\tilde{U}}_x\tilde{\tilde{V}}_q = \tilde{\tilde{U}}_s(\tilde{\tilde{B}}_{\alpha,b,U_s\hat{a}}\tilde{\tilde{N}})\tilde{\tilde{U}}_s$ [by (G4) for x]. ■

It will be important that prenominators become genominators by empowerment.

Generic Strengthening Lemma 3.4 *If* $x \in Pren_s(q)$ *is an s-prenominator for* $q \in Q$*, then* $x^n \in Gen_s(q)$ *is an s-genominator for* q *for all* $n \geq 4$.

Proof: For $I = I_s$ as usual (still allowing $s = \hat{1}, I = J$), we will show that the 6 G-conditions needed for an s-genominator are satisfied by sufficiently high powers of a prenominator x satisfying only $\{x,q\}, U_xq \in I$ as in (Pren1-2). In the proofs we make constant use of Macdonald's Principle to create operator-identities in $\tilde{\tilde{\mathcal{U}}} = \mathcal{UME}(J|Q)$ involving only x and q. Clearly any x^n lies in I, and for **(G1)** $U_{x^n}q = U_{x^{n-1}}(U_xq) \in I$ for $n \geq 1$. For **(G1)′**, $U_qx^n \in I$ for $n \geq 4$ because then $U_qx^n = U_{\{x,q\}}x^{n-2} + U_{U_xq}x^{n-4} - \{\{x,q\},x^{n-3},U_xq\} \in U_I\widehat{I} - \{I,I,I\} \subseteq I$. Similarly, for **(G1)″** $U_qU_{x^n}q \in I$ for $n \geq 4$ because $U_qU_{x^n}q = U_{\{q,x\}}U_{x^{n-2}}U_xq - U_{U_xq}U_{x^{n-3}}U_xq - \{\{q,x^2\},U_{x^{n-3}}U_xq,U_xq\} + \{U_xU_{U_xq}x^{n-4},x^{n-4},U_xq\}$ [with $\{q,x^2\} = \{x,\{x,q\}\} - 2U_xq \in I$]. For **(G2)**, $\tilde{\tilde{V}}_{x^n,q} \in \tilde{\tilde{V}}_{\widehat{I},I}$ for $n \geq 2$ by $\tilde{\tilde{V}}_{x^n,q} = \tilde{\tilde{V}}_{x^{n-1},\{x,q\}} - \tilde{\tilde{V}}_{x^{n-2},U_xq}$. For **(G3)**, $\tilde{\tilde{U}}_{x^n}\tilde{\tilde{U}}_q \in \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s$ for $n \geq 4$ by $\tilde{\tilde{U}}_{x^n}\tilde{\tilde{U}}_q = \tilde{\tilde{U}}_{x^{n-1}}\tilde{\tilde{U}}_{\{x,q\}} + \tilde{\tilde{U}}_{x^{n-2}}\tilde{\tilde{U}}_{U_xq} + \tilde{\tilde{U}}_{x^{n-3}}(\tilde{\tilde{V}}_{x^2,(U_xq)^2} + \tilde{\tilde{V}}_{U_x(U_xq),U_xq} - \tilde{\tilde{V}}_{x^2,\{x,q\}}\tilde{\tilde{V}}_{x,U_xq}) \in \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s + \tilde{\tilde{U}}_s\tilde{\tilde{U}}_J\tilde{\tilde{U}}_s(\tilde{\tilde{V}}_{I,I} + \tilde{\tilde{V}}_{I,I}\tilde{\tilde{V}}_{I,I})$ [since then $x, x^{n-1}, x^{n-2}, x^{n-3}, \{x,q\}, U_xq \in I = U_sJ$ where $\tilde{\tilde{U}}_I \subseteq \tilde{\tilde{U}}_s\tilde{\tilde{U}}_J\tilde{\tilde{U}}_s$ by (QJ3)] $\subseteq \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}(\mathbf{1} + \tilde{\tilde{V}}_{I,J} + \tilde{\tilde{V}}_{I,J}\tilde{\tilde{V}}_{I,J})\tilde{\tilde{U}}_s$ [since $\tilde{\tilde{U}}_s\tilde{\tilde{V}}_{J,I} \subseteq \tilde{\tilde{V}}_{I,J}\tilde{\tilde{U}}_s$ by (1.1.9) with $x \to s$]. Similarly, for **(G4)** $\tilde{\tilde{U}}_{x^n}\tilde{\tilde{V}}_q \in \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s$ for $n \geq 2$ since $\tilde{\tilde{U}}_{x^n}\tilde{\tilde{V}}_q = \tilde{\tilde{U}}_{x^{n-2}}\tilde{\tilde{U}}_x(\tilde{\tilde{U}}_{\{x,q\},x} - \tilde{\tilde{V}}_q\tilde{\tilde{U}}_x)$ [by (1.1.1)] $= \tilde{\tilde{U}}_{x^{n-2}}(\tilde{\tilde{U}}_x\tilde{\tilde{U}}_{\{x,q\},x} - \tilde{\tilde{U}}_{U_xq,x^2})$ [by (QJ3)] $\in \widehat{\tilde{\tilde{U}}_I}(\tilde{\tilde{U}}_I\tilde{\tilde{U}}_{I,I} - \tilde{\tilde{U}}_{I,I}) \subseteq \tilde{\tilde{U}}_s\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_s$ [by (QJ3) for $I = U_sJ$]. ■

Martinez's work with linear Jordan algebras uses prenominators (Pren), which we may think of as *linear denominators*; for the quadratic case (as with the case of Zel'manov annihilators), it is important to have the full strength of the axioms (G), and the Generic Strengthening Lemma shows that without loss of generality we may adopt the *generic quadratic denominators* (G) as the basic concept.

4 Ore Monads and Denominators

Since our denominators s are destined to become invertible in Q, they certainly must be *injective* on J to begin with (in the sense that their U-operators are injective transformations). In order to be able to add fractions we need to be able to find common denominators, which leads us to the concept of an Ore monad.

Ore Monad Definition 4.1 (1) *A* **Jordan monad** *of a quadratic Jordan algebra J is a nonempty subset S of injective elements of J closed under products,*

$$(4.1.1) \qquad s,t\in S \implies s^2, U_s t \in S \quad (i.e., \ U_s\widehat{S}\subseteq S \text{ for } \widehat{S}:=S\cup\{\hat{1}\}).$$

(2) *An* **Ore monad** *of J is a Jordan monad S with the* **Common Inner Multiple Property (CIMP)***:*

$$(4.1.2) \qquad s,t\in S \implies U_s(S)\cap U_t(S)\neq\emptyset.$$

The set $Inn_S(J):=\{K\lhd_{in} J \mid K\cap S\neq\emptyset\}$ of S-**inner ideals**, those which contain an element of S, form a downward-directed family of inner ideals. Since $s\in K\cap S \ \Leftrightarrow \ I_{s^3}\subseteq K_{s^3}\subseteq I_s\subseteq J_s\subseteq K_s\subseteq K$, the principal inner ideals I_s or J_s or K_s form cofinal subsets of $Inn_S(J)$.

Throughout this section, let s be a fixed element in an Ore monoid S generating a principal inner ideal in a quadratic Jordan algebra J, and q an element of a larger algebra Q :

$$I:=I_s=U_sJ\lhd J\subseteq Q\ni q.$$

We allow $s=\hat{1}\in\widehat{S}$ to be an honorary member of S, in which case I is just J. We will be mostly interested in genominators which fall in the Ore monad S. When $s=\hat{1}$ we write $Gen_J(q)$ in place of $Gen_{\hat{1}}(q)$, and call the elements J-**genominators**. A genominator for $a\in J$ is also a genominator for every $\hat{a}\in\Phi\hat{1}+a$ in the unital hull.

Ore Condition 4.2 *J is said to satisfy the* **Ore condition** *with respect to an Ore monad S if for all $s\in S, a\in J$ there exists $s'\in U_sS$ such that*

$$(4.2.1) \qquad \{s',a\}\in I_s.$$

Since automatically $U_{s'}a\in U_{I_s}a\subseteq I_s$, s' is an s-prenominator for a according to definition 2.1 (Pren)*. Thus the Ore condition is equivalent to*

$$(4.2.2) \qquad s\in S, a\in J \Longrightarrow \exists s'\in U_sS\cap Pren_s(a),$$

and hence by Generic Strengthening 3.4 *to*

$$(4.2.3) \qquad s\in S, a\in J \Longrightarrow \exists s''\in U_sS\cap Gen_s(a).$$

The CIMP allows us to choose our denominators from any principal inner ideal we wish.

Choice Corollary 4.3 *If $q\in Q\supseteq J$ has an s-prenominator s' in an Ore monad $S\subseteq J$, then* any *element in $U_{s'^5}S\cap U_{s_1}S\cap\cdots\cap U_{s_n}S$ for $s_1,\ldots,s_n\in S$ will be an s-genominator for q lying in each $U_{s_i}S\subseteq I_{s_i}$. If s' is already an s-genominator, we may replace s'^5 by s'^2.*

Proof: If q has an s-prenominator s' it has s-genominator $s'' := s'^4 \in S \cap Gen_s(q)$ by Generic Strengthening 3.4. Any element of $U_{s'^5}S \cap U_{s_1}S \cap \cdots \cap U_{s_n}S \neq \emptyset$ [by the CIMP] will lie in $U_{s'^4}(U_{s'}S) \subseteq U_{s'^4}(I_s) \subseteq Gen_s(q)$ [by Genominator Innerness 3.3] and again be an s-denominator for q. If s' is already an s-genominator then already $U_{s'^2}S \cap U_{s_1}S \cap \cdots \cap U_{s_n}S \subseteq U_{s'}(U_{s'}S) \subseteq U_{s'}I \subseteq Gen_s(q)$. [Note that 3.3 does not guarantee $U_{s''}S$ falls in $Gen_s(q)$, only that $U_{s''}(I \cap S)$ does.] ■

The Ore condition is an intrinsic condition relating J and S, but it has far-reaching consequences for denominators of elements outside J.

Genominator Inheritance 4.4 *If $q \in Q \supseteq J$ has a J-prenominator in an Ore monad $S \subseteq J$, and J satisfies the Ore condition relative to S, then q has s-genominators in U_sS for each $s \in S$.*

Proof: Fix an arbitrary $s \in S$, and set $I := I_s = U_sJ$. By Choice Corollary 4.3 for $s = \hat{1}$, q has a J-denominator $s_0 = U_st \in U_sS \cap Gen_J(q) \subseteq I$. Then $a_1 := \{s_0, q\}$, $a_2 := U_{s_0}q$, $a_3 := \{s_0, a_1\}$ lie in J, and by the Ore condition a_3 has an s-genominator s_3. We claim that $s' := U_{s_0}^2 s_3$ is an s-prenominator for q. It certainly lies in $U_{s_0}S = U_sU_tU_sS \subseteq U_sS$ and by (QJ3) $U_{s'}q = U_{s_0}^2 U_{s_3}U_{s_0}(U_{s_0}q) = U_{s_0}^2 U_{s_3}U_{s_0}(a_2) \in U_sJ$. As always, the crux is the V-operator: $V_{s'}q = V_qU_{s_0}(U_{s_0}s_3) = \big(U_{\{q,s_0\},s_0} - U_{s_0}V_q\big)(U_{s_0}s_3)$ [by (1.1.1)] $= U_{a_1,s_0}U_{s_0}s_3 - U_{s_0}(\{q, U_{s_0}s_3\}) \in \big(U_{\{a_1,s_0\},s_0^2} - U_{s_0}U_{a_1,s_0}\big)s_3 - U_{s_0}J$ [by linearized (QJ3) and $U_{s_0}s_3 \in U_{Gen_J(q)}J \subseteq Gen_J(q)$ by Genominator Innerness 3.3] $\subseteq \{a_3, s_3, s_0^2\} - U_{s_0}\{a_1, s_3, s_0\} - U_{s_0}J \subseteq V_{\hat{I},I}I - U_IJ \subseteq I$ [by 3.1 (G2)* for s_3 as an s-genominator for a_3]. By the Generic Strengthening Lemma 3.4, once q has an s-prenominator $s' \in U_sS$ it has an s-genominator $s'^4 \in U_sS$. ■

The Ore condition implies that q will have a J-genominator as soon as it is an outer S-fraction, $q = U_{s_0}^{-1}n_0$ (i.e., some $U_{s_0}q \in J$): though the other prenominator condition $\{s_0, q\} \in J$ may not hold for s_0, the Ore condition inside J guarantees that there is another $s \in S$ with $q = U_s^{-1}n$ where $\{s, q\} = \{n, s^{-1}\} \in J$.

Archetypal Example 4.5 *Suppose $q = U_{s_0}^{-1}n_0 \in Q \supseteq J$ for $s_0, n_0 \in J$ with s_0 invertible in Q, and suppose further that n_0 has an invertible s_0-genominator $t_0 \in I_{s_0}$. Then we have an alternate formulation $q = U_s^{-1}n$ of the element q, where $s := U_{s_0}t_0$ is a J-genominator for q which dominates the numerator $n = U_{s_0}U_{t_0}n_0$ as in* (1.4)(G1-4) *in Q :*

$$(4.5.1)\qquad \begin{gathered} \{q,s\} = w_0, \quad U_sq = n, \quad U_qs = w_2, \quad U_qU_sq = w_3, \\ \overset{\approx}{V}_{s,q} = \overset{\approx}{S}, \quad \overset{\approx}{U}_s\overset{\approx}{U}_q = \overset{\approx}{N}, \quad \overset{\approx}{U}_s\overset{\approx}{V}_q = \overset{\approx}{M} = \overset{\approx}{S}\overset{\approx}{V}_s - \overset{\approx}{V}_n, \\ \overset{\approx}{U}_n = \overset{\approx}{N}\overset{\approx}{U}_s = \overset{\approx}{U}_s\overset{\approx}{N}^*, \qquad \overset{\approx}{U}_{n,s} = \overset{\approx}{S}\overset{\approx}{U}_s = \overset{\approx}{U}_s\overset{\approx}{S}^* \end{gathered}$$

for some $w_0, w_2, w_3 \in J$, $\overset{\approx}{S}, \overset{\approx}{M}, \overset{\approx}{N} \in \mathcal{UME}(J|Q)$. Moreover, these relations imply that the element n is also a genominator for q,

$$(4.5.2)\qquad \begin{gathered} \{q,n\} = \{w_2, s\},\ U_nq = U_sw_3,\ U_qn = w_3,\ U_qU_nq = U_{w_2}n, \\ \overset{\approx}{V}_{n,q} = \overset{\approx}{V}_{s,w_2},\ \overset{\approx}{U}_n\overset{\approx}{U}_q = \overset{\approx}{U}_s\overset{\approx}{U}_{w_2} = \overset{\approx}{N}^2,\ \overset{\approx}{U}_n\overset{\approx}{V}_q = \overset{\approx}{N}\overset{\approx}{M},\ U_ns^{-1} = U_sw_2. \end{gathered}$$

In addition we have the structural relations

$$(4.5.3)\qquad \begin{gathered} \overset{\approx}{N}\overset{\approx}{U}_n = \overset{\approx}{U}_n\overset{\approx}{N} = \overset{\approx}{N}\overset{\approx}{U}_s\overset{\approx}{N}^* = \overset{\approx}{U}_{U_sw_2,s}, \quad \overset{\approx}{N}\overset{\approx}{U}_n\overset{\approx}{N}^* = \overset{\approx}{U}_{U_sw_3}, \\ \overset{\approx}{U}_{w_2}\overset{\approx}{N} = \overset{\approx}{N}^*\overset{\approx}{U}_{w_2} = \overset{\approx}{U}_{w_3}, \quad \overset{\approx}{S}\overset{\approx}{U}_a\overset{\approx}{S} + \overset{\approx}{N}\overset{\approx}{U}_a + \overset{\approx}{U}_a\overset{\approx}{N}^* = \overset{\approx}{U}_{\overset{\approx}{S}(a)} + \overset{\approx}{U}_{\overset{\approx}{N}(a),a}. \end{gathered}$$

We say that q is an **Ore fraction** *if it can be written $q = U_s^{-1}n$ for $s \in S, n \in J$ satisfying* (4.5.1).

Proof: By definition 3.1 of s_0-genominator, in J and $\overset{\approx}{\mathcal{U}} := \mathcal{UME}(J|Q)$ we have

$$(4.5.4)\qquad \begin{gathered} U_{n_0}t_0 = U_{s_0}w_2,\ U_{n_0}U_{t_0}n_0 = U_{s_0}w_3,\ \{n_0,t_0,s_0\} = U_{s_0}v_0^*,\ \{n_0,U_{t_0}n_0,s_0\} = U_{s_0}v_2, \\ \tilde{\tilde{V}}_{n_0,t_0} \in \tilde{\tilde{V}}_{\hat{I_0},I_0},\quad \tilde{\tilde{U}}_{n_0}\tilde{\tilde{U}}_{t_0} = \tilde{\tilde{U}}_{s_0}\tilde{\tilde{N}}_o\tilde{\tilde{U}}_{s_0},\quad \tilde{\tilde{V}}_{n_0}\tilde{\tilde{U}}_{t_0} = \tilde{\tilde{U}}_{s_0}\tilde{\tilde{M}}_o\tilde{\tilde{U}}_{s_0} \end{gathered}$$

for some $\tilde{\tilde{N}}_o, \tilde{\tilde{M}}_o \in \tilde{\tilde{\mathcal{U}}}$, noting that by (1.1.5) $\{n_0, U_{t_0}n_0, s_0\} = \{U_{n_0}t_0, t_0, s_0\} = \{U_{s_0}w_2, t_0, s_0\} = U_{s_0}v_{20}$ for $v_2 := \{w_2, s_0, t_0\}$.

For (1), since s_0^{-1} exists in Q, we know that $\tilde{\tilde{U}}_{s_0}$ is generically invertible in $\mathcal{UME}(Q)$. To establish (G1-4) for $s, \hat{1}$ replacing x, s in 3.1, we compute (G1): $U_s q = U_{s_0}U_{t_0}U_{s_0}(U_{s_0}^{-1}n_0) = U_{s_0}U_{t_0}n_0 = n \in J$ [by definition]; (G1)$'$: $U_q s = U_{s_0}^{-1}U_{n_0}U_{s_0}^{-1}(U_{s_0}t_0) = U_{s_0}^{-1}(U_{n_0}t_0) = U_{s_0}^{-1}(U_{s_0}w_2)$ [by (4)] $= w_2 \in J$; (G1)$''$: $U_qU_sq = U_{s_0}^{-1}U_{n_0}U_{s_0}^{-1}(U_{s_0}U_{t_0}U_{s_0})q = U_{s_0}^{-1}(U_{n_0}U_{t_0}n_0) = U_{s_0}^{-1}(U_{s_0}w_3)$ [by (4)] $= w_3 \in J$; (G2): $\tilde{\tilde{V}}_{s,q} = \tilde{\tilde{V}}_{U_{s_0}t_0,q} = -\tilde{\tilde{V}}_{U_{s_0}q,t_0} + \tilde{\tilde{V}}_{s_0,\{q,s_0,t_0\}} = -\tilde{\tilde{V}}_{n_0,t_0} + \tilde{\tilde{V}}_{s_0,v_0}$ [from (1.1.6), cancelling U_{s_0} from $U_{s_0}\{q,s_0,t_0\} = \{U_{s_0}q,t_0,s_0\} = \{n_0,t_0,s_0\} = U_{s_0}v_0$ by (4)] $=: \tilde{\tilde{S}} \in \tilde{\tilde{V}}_{J,J}$; [hence (G0): $\{s,q\} = V_{s,q}\hat{1} = S(\hat{1}) = -\{n_0,t_0\} + jts_0, v_0 = m_0 \in J$]; (G3): $\tilde{\tilde{U}}_s\tilde{\tilde{U}}_q = (\tilde{\tilde{U}}_{s_0}\tilde{\tilde{U}}_{t_0}\tilde{\tilde{U}}_{s_0})(\tilde{\tilde{U}}_{s_0}^{-1}\tilde{\tilde{U}}_{n_0}\tilde{\tilde{U}}_{s_0^{-1}}) = \tilde{\tilde{U}}_{s_0}(\tilde{\tilde{U}}_{t_0}\tilde{\tilde{U}}_{n_0})\tilde{\tilde{U}}_{s_0}^{-1} = \tilde{\tilde{U}}_{s_0}(\tilde{\tilde{U}}_{s_0}\tilde{\tilde{N}}_o\tilde{\tilde{U}}_{s_0})\tilde{\tilde{U}}_{s_0}^{-1} = \tilde{\tilde{U}}_{s_0}^2\tilde{\tilde{N}}_o$ [by (4)] $=: \tilde{\tilde{N}} \in \tilde{\tilde{U}}_1\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_1$; and finally (G4): $\tilde{\tilde{U}}_s\tilde{\tilde{V}}_q = \tilde{\tilde{V}}_{s,q}\tilde{\tilde{V}}_s - \tilde{\tilde{V}}_{U_sq}$ [by (1.1.6) with $x = 1$] $= \tilde{\tilde{S}}\tilde{\tilde{V}}_s - \tilde{\tilde{V}}_n =: \tilde{\tilde{M}} \in \tilde{\tilde{U}}_1\tilde{\tilde{\mathcal{U}}}\tilde{\tilde{U}}_1$. Thus q has J-genominator s as in (1).

For domination of n by s as in (1.4), we have $\tilde{\tilde{U}}_n = \tilde{\tilde{U}}_s\tilde{\tilde{U}}_q\tilde{\tilde{U}}_s = \tilde{\tilde{N}}\tilde{\tilde{U}}_s$ (hence also $= \tilde{\tilde{U}}_s\tilde{\tilde{N}}^*$), and $\tilde{\tilde{U}}_{n,s} = \tilde{\tilde{U}}_{U_sq,s} = \tilde{\tilde{V}}_{s,q}\tilde{\tilde{U}}_s = \tilde{\tilde{S}}\tilde{\tilde{U}}_s$ (hence also $= \tilde{\tilde{U}}_s\tilde{\tilde{S}}$). Therefore $K_{s\succ n}$ is an inner ideal.

For (2) for n, we note that s is invertible in Q since s_0, t_0 are with $U_s^{-1}n = (U_{s_0}U_{t_0}U_{s_0})^{-1}U_{s_0}U_{t_0}n_0 = U_{s_0}^{-1}n_0 = q$, and compute directly from (1) using (QJ3) that $U_ns^{-1} = U_sU_s^{-1}U_nU_s^{-1}s = U_sU_qs = U_sw_2$, $\{q,n\} = \{U_s^{-1}n, n\} = \{s^{-1}, U_ns^{-1}\}$ [by (1.1.5)] $= \{s^{-1}, U_sw_2\} = \{w_2, s\}$, $U_qn = U_qU_sq = w_3$, $U_nq = U_sU_qU_sq = U_sw_3$, $U_qU_nq = U_qU_sU_qU_sq = U_{U_qs}n = U_{w_2}n$, $\tilde{\tilde{V}}_{n,q} = \tilde{\tilde{V}}_{n,U_s^{-1}n} = \tilde{\tilde{V}}_{U_ns^{-1},s^{-1}}$ [by (1.1.5) again] $= \tilde{\tilde{V}}_{U_sw_2,s^{-1}} = \tilde{\tilde{V}}_{s,w_2}$, $\tilde{\tilde{U}}_n\tilde{\tilde{U}}_q = \tilde{\tilde{U}}_s\tilde{\tilde{U}}_q\tilde{\tilde{U}}_s\tilde{\tilde{U}}_q = \tilde{\tilde{N}}^2 = \tilde{\tilde{U}}_s\tilde{\tilde{U}}_{U_qs} = \tilde{\tilde{U}}_s\tilde{\tilde{U}}_{w_2}$, $\tilde{\tilde{U}}_n\tilde{\tilde{V}}_q = (\tilde{\tilde{U}}_s\tilde{\tilde{U}}_q)(\tilde{\tilde{U}}_s\tilde{\tilde{V}}_q) = \tilde{\tilde{N}}\tilde{\tilde{M}}$.

From (2) we derive (3), $\tilde{\tilde{U}}_{U_sw_2} = \tilde{\tilde{U}}_n\tilde{\tilde{U}}_s^{-1}\tilde{\tilde{U}}_n = \tilde{\tilde{N}}\tilde{\tilde{U}}_n = \tilde{\tilde{N}}\tilde{\tilde{U}}_s\tilde{\tilde{N}}^*$ and dually, $\tilde{\tilde{U}}_{U_sw_3} = \tilde{\tilde{U}}_{U_nq} = \tilde{\tilde{U}}_n\tilde{\tilde{U}}_q\tilde{\tilde{U}}_n = (\tilde{\tilde{N}}\tilde{\tilde{U}}_s)\tilde{\tilde{U}}_q(\tilde{\tilde{U}}_s\tilde{\tilde{N}}^*) = \tilde{\tilde{N}}\tilde{\tilde{U}}_n\tilde{\tilde{N}}^*$, $\tilde{\tilde{U}}_{w_2}\tilde{\tilde{N}} = (\tilde{\tilde{U}}_q\tilde{\tilde{U}}_s\tilde{\tilde{U}}_q)(\tilde{\tilde{U}}_s\tilde{\tilde{U}}_q) = \tilde{\tilde{U}}_{U_qU_sq} = \tilde{\tilde{U}}_{w_3}$ and dually, while for $p = U_s^{-1}a \in Q$ we have $\tilde{\tilde{S}}\tilde{\tilde{U}}_s = \tilde{\tilde{V}}_{s,q}\tilde{\tilde{U}}_s = \tilde{\tilde{U}}_{U_sq,s} = \tilde{\tilde{U}}_{n,s}$ and dually, therefore $\tilde{\tilde{S}}\tilde{\tilde{U}}_a\tilde{\tilde{S}} + \tilde{\tilde{N}}\tilde{\tilde{U}}_a + \tilde{\tilde{U}}_a\tilde{\tilde{N}}^* - \tilde{\tilde{U}}_{\tilde{\tilde{S}}(a)} - \tilde{\tilde{U}}_{\tilde{\tilde{N}}(a),a} = \tilde{\tilde{S}}\tilde{\tilde{U}}_s\tilde{\tilde{U}}_p\tilde{\tilde{U}}_s\tilde{\tilde{S}}^* + \tilde{\tilde{N}}\tilde{\tilde{U}}_s\tilde{\tilde{U}}_p\tilde{\tilde{U}}_s + \tilde{\tilde{U}}_s\tilde{\tilde{U}}_p\tilde{\tilde{U}}_s\tilde{\tilde{N}}^* - \tilde{\tilde{U}}_{\tilde{\tilde{S}}(U_sp)} - \tilde{\tilde{U}}_{\tilde{\tilde{N}}(U_sp),U_sp} = \tilde{\tilde{U}}_{n,s}\tilde{\tilde{U}}_p\tilde{\tilde{U}}_{n,s} + \tilde{\tilde{U}}_n\tilde{\tilde{U}}_p\tilde{\tilde{U}}_s + \tilde{\tilde{U}}_s\tilde{\tilde{U}}_p\tilde{\tilde{U}}_n - \tilde{\tilde{U}}_{U_{n,s}p} - \tilde{\tilde{U}}_{U_np,U_sp} = 0$ by linearized (QJ3). ■

The above example shows it is important to choose the right denominator and numerator for an outer fraction q: the original $q = U_{s_0}^{-1}n_0$ has degree -2 in s, whereas $U_s^{-1}n$ has degree -1 in s since n "contains an s". If $J \subseteq H(Q,*)$ is special $n \approx sw = w^*s$ and $q \approx ws^{-1} = s^{-1}w^*$ is a standard "associative fraction" of degree -1 in s. Here $\tilde{\tilde{N}}(a) \approx waw^*$, $\tilde{\tilde{N}}^*(a) \approx w^*aw$, $\tilde{\tilde{S}}(a) \approx wa + aw^*$, $n = sw = w^*s$, $w_0 = w + w^*$, $w_2 \approx ws^{-1}w^*$, $w_3 \approx w_2w^* = ww_2$. In fact, $w_2 \in J$ shows w "contains $s^{1/2}$", so $n \approx sw$ "contains $s^{3/2}$. Indeed, we can choose denominators s so that n contains $s^{2-1/k}$ arbitrarily close to s^2!! The reason is easily seen by considering $J = \Phi[t]$, $Q = \Phi(t)$, $q = t^{-1} = U_s^{-1}n$ for $s = t^k$, $n = t^{2k-1} = t^{k(2-1/k)} = s^{2-1/k}$.

5 Moving Multiplications

In a direct construction [6] of an algebra of fractions, a crucial step is to show that any element $WU_s^{-1}a$ for $W \in \mathcal{M}(J|Q)$ can be rewritten as an outer S-fraction $U_{s'}^{-1}a'$, and in fact of the form $U_{s'}^{-1}W'a$ independently of a: for every W, s we can find $W' \in \mathcal{M}(J), s' \in S$ with $WU_s^{-1} = U_{s'}^{-1}W'$.

Thus we move W to the *right* past U_s^{-1}. This may be formulated without inverses as $U_{s'}W = W'U_s$, where a pile of s's attack W from the left and manage to move a single U_s to the right across W. It turns out that **left-movability** $WU_{s'} = U_sW'$ is even more important. This movability will hold *universally* on *all* $Q \supseteq J$ (equivalently, on all J-bimodules M) if it holds in the universal multiplication envelope $\mathcal{UME}(J)$. The concept of movability is strictly internal to J, making no reference to any extension Q.

Movable Definition 5.1 *An ordinary multiplication $W \in \mathcal{M}(J)$ is* **left-s-movable** *with* **left-s-mover** $s' \in U_sS$ *if there exists $W' \in \mathcal{M}(J)$ such that $WU_{s'} = U_sW'$. A universal multiplication $\tilde{\tilde{M}} \in \mathcal{UME}(J)$ is* **left-s-movable** *with* **left-s-mover** $s' \in U_sS$ *if there exist $\tilde{\tilde{M}}' \in \mathcal{UME}(J)$ such that*

$$\text{(5.1.1)} \qquad \tilde{\tilde{M}}\tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_s\tilde{\tilde{M}}', \quad (\textit{hence also } \tilde{\tilde{U}}_{s'}\tilde{\tilde{M}}^* = \tilde{\tilde{M}}'^*\tilde{\tilde{U}}_s).$$

A universal multiplication $\tilde{\tilde{M}}$ is **s-movable** *with* **s-mover** $s' \in U_sS$ *if there exist $\tilde{\tilde{M}}' \in \mathcal{UME}(J)$ and $w' \in J$ such that*

$$\text{(5.1.2)} \qquad \tilde{\tilde{M}}\tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_s\tilde{\tilde{M}}'\tilde{\tilde{U}}_s, \qquad \tilde{\tilde{M}}(s') = U_sw' \quad (\textit{hence also } \tilde{\tilde{U}}_{s'}\tilde{\tilde{M}}^* = \tilde{\tilde{U}}_s\tilde{\tilde{M}}'^*\tilde{\tilde{U}}_s).$$

Notice that if both $\tilde{\tilde{M}}$ and $\tilde{\tilde{M}}^*$ are s-movable (in particular, if all *$\tilde{\tilde{M}}$ are movable), then we have right-movability $\tilde{\tilde{U}}_{s'}\tilde{\tilde{M}} = \tilde{\tilde{U}}_s\tilde{\tilde{M}}''\tilde{\tilde{U}}_s$ for $\tilde{\tilde{M}}'' = \tilde{\tilde{M}}'^*$.*

We say that $W \in \mathcal{M}(J)$ is **left-S-movable** *if it is left-s-movable for each $s \in S$, and $\tilde{\tilde{M}} \in \mathcal{UME}(J)$ is* **S-movable** *if it is s-movable for each $s \in S$.*

In fact, any left-mover $s' \in U_sS$ can be turned into a mover,

$$\text{(5.1.3)} \qquad \textit{if } \tilde{\tilde{M}} \textit{ has left-}s\textit{-mover } s' \in U_sS, \textit{ it has } s\textit{-mover any } s'' \in U_{s'}\widehat{S},$$

because $s'' = U_{s'}\hat{t}$ has $\tilde{\tilde{M}}\tilde{\tilde{U}}_{s''} = \tilde{\tilde{M}}\tilde{\tilde{U}}_{s'}\tilde{\tilde{U}}_{\hat{t}}\tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_s\big(\tilde{\tilde{M}}'\tilde{\tilde{U}}_{\hat{t}}\tilde{\tilde{U}}_{s'}\big) = \tilde{\tilde{U}}_s\tilde{\tilde{M}}''\tilde{\tilde{U}}_s$, and $\tilde{\tilde{M}}(s'') = (\tilde{\tilde{M}}\tilde{\tilde{U}}_{s'})\hat{t} = (\tilde{\tilde{U}}_s\tilde{\tilde{M}}')\hat{t} = U_s(w'')$.

In fact, left S-movability of V-operators is the crucial condition, implying movability of all multiplication operators.

Ore Equivalence Theorem 5.2 *The following are equivalent for an Ore monad S in a quadratic Jordan algebra J and any algebra $Q \supseteq J$:*

- *(i) Each $\tilde{\tilde{M}} \in \mathcal{UME}(J)$ is S-movable in $\mathcal{UME}(J)$.*
- *(ii) Each $\tilde{\tilde{M}} \in \mathcal{UME}(J)$ is left S-movable in $\mathcal{UME}(J)$).*
- *(iii) For each $a \in J$, $\tilde{\tilde{U}}_a$ and $\tilde{\tilde{V}}_a$ are S-movable in $\mathcal{UME}(J)$.*
- *(iv) For each $a \in J$, V_a is left S-movable in $\mathcal{M}(J)$.*
- *(v) Each $a \in J$ has an s-prenominator in U_sS for each $s \in S$.*
- *(vi) Each $a \in J$ has an s-genominator in U_sS for each $s \in S$.*

By (v), these are equivalent ways of saying that J satisfies the Ore condition (4.2.2-3) *with respect to S.*

Proof: Clearly $(i) \Rightarrow (ii), (iii)$ and both (ii) and $(iii) \Rightarrow (iv)$ since if $\tilde{\tilde{V}}_a\tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_s\tilde{\tilde{W}}'$ in $\mathcal{UME}(J)$ then under the canonical homomorphism $\mathcal{UME}(J) \xrightarrow{\pi} \mathcal{M}(J)$ we have $V_aU_{s'} = U_s\pi(\tilde{\tilde{W}}') = U_sW'$.

$(iv) \Rightarrow (v)$: For $a \in J$, $s \in S$ we know by (iv) that V_a is left s-movable using some $s' \in U_sS$: $V_aU_{s'} = U_sW'$. Then $s'' = U_{s'}s \in U_sS$ has $V_{s''}a = V_as'' = V_aU_{s'}s = U_sW's \in I_s$. Since $U_{s''}a \in I_s$ trivially, this gives (v).

$(v) \iff (vi)$ follows by (4.1.2-3) from Generic Strengthening. Now that we have followed implications down the tower, we turn around and go back up.

$(vi) \Rightarrow (iii)$: For $a \in J$, $s \in S$ by (vi) there exists an s-denominator $s' = U_s t \in U_s S$ such that $\{s', a\} = U_s b$, $U_a s' = U_s c$ for $b, c \in J$. Then the V-operator is universally s-left-movable by s' since $\tilde{\tilde{V}}_a \tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_{\{s',a\},s'} - \tilde{\tilde{U}}_{s'} \tilde{\tilde{V}}_a$ [by (1.1.1) with $y = 1$] $= \tilde{\tilde{U}}_{U_s b, U_s t} - \tilde{\tilde{U}}_{U_s t} \tilde{\tilde{V}}_a = \tilde{\tilde{U}}_s \big(\tilde{\tilde{U}}_{b,t} \tilde{\tilde{U}}_s - \tilde{\tilde{U}}_t \tilde{\tilde{U}}_s \tilde{\tilde{V}}_a \big)$ [by (QJ3)] $:= \tilde{\tilde{U}}_s \tilde{\tilde{V}}'_a$, while the U-operator is universally left-s-movable since $\tilde{\tilde{U}}_a \tilde{\tilde{U}}_{s'} = \tilde{\tilde{U}}_{\{s',a\}} + \tilde{\tilde{U}}_{U_a s', s'} - \tilde{\tilde{U}}_{s'} \tilde{\tilde{U}}_a - \tilde{\tilde{V}}_a \tilde{\tilde{U}}_{s'} \tilde{\tilde{V}}_a$ [by (1.1.3)] $= \tilde{\tilde{U}}_s \big(\tilde{\tilde{U}}_b \tilde{\tilde{U}}_s + \tilde{\tilde{U}}_{c,t} \tilde{\tilde{U}}_s - \tilde{\tilde{U}}_t \tilde{\tilde{U}}_s \tilde{\tilde{U}}_a - \tilde{\tilde{V}}'_a \tilde{\tilde{V}}_a \big)$ [by linearized (QJ3) and above] $=: \tilde{\tilde{U}}_s \tilde{\tilde{U}}'_a$. But then $s'^2 = U_{s'} \hat{1} \in U_{s'} \hat{S}$ is a s-mover by (5.1.3). This holds for any $s \in S$, so $\tilde{\tilde{V}}_a$ and $\tilde{\tilde{U}}_a$ are S-movable.

$(iii) \Rightarrow (i)$: Let $\mathcal{W} \subseteq \tilde{\tilde{\mathcal{U}}} := \mathcal{UME}(J)$ denote the set of all "multiplications" (elements in the associative algebra $\mathcal{UME}(J)$) which are S-movable, which by (5.1.3) is equivalent to being just left-s-movable for each s. We will show that $\mathcal{W}$ is a unital subalgebra of $\tilde{\tilde{\mathcal{U}}}$: for any $\tilde{\tilde{M}}_1, \tilde{\tilde{M}}_2 \in \mathcal{W}$, any $\alpha \in \Phi$, and any $s \in S$, we must show $\mathbf{1}, \tilde{\tilde{M}}_1 + \tilde{\tilde{M}}_2, \alpha \tilde{\tilde{M}}_1, \tilde{\tilde{M}}_1 \tilde{\tilde{M}}_2$ are left-s-movable. Now $\tilde{\tilde{M}}_1, \tilde{\tilde{M}}_2$ are left-s-movable using movers $s_1, s_2 \in S$, and since S is an Ore monad, we can find a common mover $s_{12} \in U_{s_1} S \cap U_{s_2} S \neq \emptyset$ $[\tilde{\tilde{M}}_i \tilde{\tilde{U}}_{s_{12}} = \tilde{\tilde{U}}_s \tilde{\tilde{M}}'_i]$ by (5.1.3). (1) $\mathbf{1}$ is trivially left-s-movable by s. (2) The sum $\tilde{\tilde{M}}_1 + \tilde{\tilde{M}}_2$ is left-s-movable using s_{12} since $(\tilde{\tilde{M}}_1 + \tilde{\tilde{M}}_2) \tilde{\tilde{U}}_{s_{12}} = \tilde{\tilde{M}}_1 \tilde{\tilde{U}}_{s_{12}} + \tilde{\tilde{M}}_2 \tilde{\tilde{U}}_{s_{12}} = \tilde{\tilde{U}}_s \big(\tilde{\tilde{M}}'_1 + \tilde{\tilde{M}}'_2 \big)$. (3) The scalar multiple $\alpha \tilde{\tilde{M}}_1$ is left-s-movable using s_1: $\big(\alpha \tilde{\tilde{M}}_1 \big) \tilde{\tilde{U}}_{s_1} = \alpha \tilde{\tilde{U}}_s \tilde{\tilde{M}}'_1 = \tilde{\tilde{U}}_s \big(\alpha \tilde{\tilde{M}}'_1 \big)$. (4) The product $\tilde{\tilde{M}}_1 \tilde{\tilde{M}}_2$ is left-s-movable since $\tilde{\tilde{M}}_1$ is left-s-movable using s_1 and *since* $\tilde{\tilde{M}}_2$ *is* S*-movable* it is also s_1-movable using some $s_{11} \in U_s S$ $[\tilde{\tilde{M}}_2 \tilde{\tilde{U}}_{s_{11}} = \tilde{\tilde{U}}_{s_1} \tilde{\tilde{M}}'_2]$, so $(\tilde{\tilde{M}}_1 \tilde{\tilde{M}}_2) \tilde{\tilde{U}}_{s_{11}} = \tilde{\tilde{M}}_1 (\tilde{\tilde{U}}_{s_1} \tilde{\tilde{M}}'_2) = \tilde{\tilde{U}}_s \big(\tilde{\tilde{M}}'_1 \tilde{\tilde{M}}'_2 \big)$. Together these facts give us that $\mathcal{W}$ is a subalgebra of $\tilde{\tilde{\mathcal{U}}}$, containing the identity and all the generators $\tilde{\tilde{U}}_a, \tilde{\tilde{V}}_a$ by condition (iii), so it must be all of $\tilde{\tilde{\mathcal{U}}}$ as in (i). ■

Ore Remark 5.3 *An associative algebra A has the* right-Ore condition *for a monad S if for all $a \in A, s \in S$ we have $aS \cap sA \neq \emptyset$, equivalently a is* left-s-movable (*we can move an s to the left over a*)*: $\exists s' \in S, a' \in A \ni as' = sa'$ (so $s^{-1}a$ can become a right fraction $a's'^{-1}$). Professor Dorfmeister has pointed out that the Jordan Ore condition holds with respect to S in J iff the right Ore condition holds with respect to $\tilde{\tilde{S}} := \tilde{\tilde{U}}_S$ in $\tilde{\tilde{A}} := \tilde{\tilde{\mathcal{U}}}$: for each $\tilde{\tilde{a}} = \tilde{\tilde{M}}$, $\tilde{\tilde{s}} = \tilde{\tilde{U}}_s$ there is $\tilde{\tilde{a}}' = \tilde{\tilde{M}}'$, $\tilde{\tilde{s}}' = \tilde{\tilde{U}}_{s'}$ with $\tilde{\tilde{a}} \tilde{\tilde{s}}' = \tilde{\tilde{s}} \tilde{\tilde{a}}'$. However, it is not known whether the $\tilde{\tilde{U}}_s \in \tilde{\tilde{S}}$ are universally injective in $\tilde{\tilde{A}}$: does $\tilde{\tilde{U}}_s \tilde{\tilde{M}} = 0$ imply $\tilde{\tilde{M}} = 0$? What can one say about the universal $\tilde{\tilde{M}}$ that are killed on the left by $\tilde{\tilde{U}}_s$? It is also unknown whether there can be nonzero $W \in \mathcal{M}(J)$ with $W(U_s J) = 0$. How is the associative Ore quotient algebra $Q_{\tilde{\tilde{U}}_S}(\tilde{\tilde{\mathcal{U}}})$ related to the universal multiplication envelope $\mathcal{UME}(Q_S(J))$ of a Jordan algebra of outer S-fractions $Q_S(J)$?* ■

While we are on the subject of moving, let us observe two consequences for later use.

Movement Consequences 5.4 *Let $s \in S$, $x, x', y \in J$ be such that $\{y, t, U_s x\} = U_s \hat{x}, \{y, t, U_s x'\} = U_s \hat{x'}$ for some $\hat{x}, \hat{x'} \in \widehat{J}$, and such that $\tilde{\tilde{U}}_y, \tilde{\tilde{U}}_{y,s}$ are left s-movable with left mover $t \in S$:*

$$\tilde{\tilde{U}}_y \tilde{\tilde{U}}_t = \tilde{\tilde{U}}_s \tilde{\tilde{M}}_y, \qquad \tilde{\tilde{U}}_{y,s} \tilde{\tilde{U}}_t = \tilde{\tilde{U}}_s \tilde{\tilde{M}}_{y,s}$$

for linear operators $\tilde{\tilde{M}}_y, \tilde{\tilde{M}}_{y,s} \in \mathcal{UME}(J)$. Then

$$\big(\tilde{\tilde{W}}_{x,y} \tilde{\tilde{W}}_{x',y} - (\tilde{\tilde{U}}_{U_s t} \tilde{\tilde{U}}_{x,x'} \tilde{\tilde{U}}_s) \tilde{\tilde{M}}^*_y - \tilde{\tilde{V}}_{U_{U_s t} x, W_y U_s x'} \big) \tilde{\tilde{U}}_s = 0 \tag{5.4.1}$$

where $\tilde{\tilde{W}}_{x,y} := \tilde{\tilde{V}}_{U_s t, \hat{x}} - \tilde{\tilde{V}}_{U_s U_t y, x}$ (analogously for $\tilde{\tilde{W}}_{x',y}$), and

$$\big(\tilde{\tilde{V}}_{t, U_s x} \tilde{\tilde{M}}^*_y + \tilde{\tilde{M}}^*_y \tilde{\tilde{V}}_{U_s t, x} - \tilde{\tilde{M}}^*_{s,y} \tilde{\tilde{V}}_{s,\hat{x}} + \tilde{\tilde{U}}_t \tilde{\tilde{V}}_{y,\hat{x}} \big) \tilde{\tilde{U}}_s = 0. \tag{5.4.2}$$

If s becomes invertible in some $Q \supseteq J$, then these operators in parentheses vanish on Q (not just $U_s Q$).

Proof: (1) Start by observing that if we introduce abbreviations $a := U_s x, b := U_t a$ and dually $a' := U_s x', b' := U_t a'$, then

$$\overset{\approx}{W}_{x,y}\overset{\approx}{U}_s = \overset{\approx}{U}_s\overset{\approx}{V}_{b,y}, \quad \overset{\approx}{W}_{x',y}\overset{\approx}{U}_s = \overset{\approx}{U}_s\overset{\approx}{V}_{b',y}$$

since $\big(\overset{\approx}{V}_{U_s t,\hat{x}} - \overset{\approx}{V}_{U_s U_t y,x}\big)\overset{\approx}{U}_s = \overset{\approx}{U}_s\big(\overset{\approx}{V}_{t,U_s\hat{x}} - \overset{\approx}{V}_{U_t y,U_s x}\big)$ [by (1.1.9)] $= \overset{\approx}{U}_s\big(\overset{\approx}{V}_{t,\{y,t,U_s x\}} - \overset{\approx}{V}_{U_t y,U_s x}\big)$ [by hypothesis on $\hat{x}$] $= \overset{\approx}{U}_s\overset{\approx}{V}_{U_t a,y}$ [by linearized (1.1.5)], and dually for x'. Then we use $\overset{\approx}{M}{}^*_y\overset{\approx}{U}_s = \overset{\approx}{U}_t\overset{\approx}{U}_y$ and compute

$$\begin{aligned}
&\big(\overset{\approx}{W}_{x,y}\overset{\approx}{W}_{z,y} - \overset{\approx}{U}_{U_s t}\overset{\approx}{U}_{x,x'}\overset{\approx}{U}_s\overset{\approx}{M}{}^*_y - \overset{\approx}{V}_{U_{U_s t}x,W_y U_s x'}\big)\overset{\approx}{U}_s \\
&\quad = \overset{\approx}{W}_{x,y}\overset{\approx}{U}_s\overset{\approx}{V}_{b',y} - \overset{\approx}{U}_s\overset{\approx}{U}_t\overset{\approx}{U}_s\overset{\approx}{U}_{x,x'}\overset{\approx}{U}_s\overset{\approx}{U}_t\overset{\approx}{U}_y - \overset{\approx}{V}_{U_s U_t U_s x,W_y a'}\overset{\approx}{U}_s && \text{[by above, (QJ3) twice, $*$ on hypothesis]} \\
&\quad = \overset{\approx}{U}_s\big(\overset{\approx}{V}_{b,y}\overset{\approx}{V}_{b',y} - \overset{\approx}{U}_{U_t U_s x,U_t U_s x'}\overset{\approx}{U}_y - \overset{\approx}{V}_{b,U_s W_y a'}\big) && \text{[by above, (QJ3) twice]} \\
&\quad = \overset{\approx}{U}_s\big(\overset{\approx}{V}_{b,y}\overset{\approx}{V}_{b',y} - \overset{\approx}{U}_{b,b'}\overset{\approx}{U}_y - \overset{\approx}{V}_{b,U_y b'}\big)
\end{aligned}$$

which vanishes by (1.1.5) with $x \to b, z \to b'$.

(2) Here from $\overset{\approx}{M}{}^*_y\overset{\approx}{U}_s = \overset{\approx}{U}_t\overset{\approx}{U}_y, \overset{\approx}{M}{}^*_{y,x}\overset{\approx}{U}_s = \overset{\approx}{U}_t\overset{\approx}{U}_{y,s}$ we have

$$\begin{aligned}
&\big(\overset{\approx}{V}_{t,U_s x}\overset{\approx}{M}{}^*_y + \overset{\approx}{M}{}^*_y\overset{\approx}{V}_{U_s t,x} - \overset{\approx}{M}{}^*_{s,y}\overset{\approx}{V}_{s,\hat{x}} + \overset{\approx}{U}_t\overset{\approx}{V}_{y,\hat{x}}\big)\overset{\approx}{U}_s \\
&\quad = \overset{\approx}{V}_{t,U_s x}\overset{\approx}{U}_t\overset{\approx}{U}_y + \overset{\approx}{M}{}^*_y\overset{\approx}{U}s\overset{\approx}{V}_{t,U_s x} - \overset{\approx}{M}{}^*_{s,y}\overset{\approx}{U}_s\overset{\approx}{V}_{\hat{x},s} + \overset{\approx}{U}_t\overset{\approx}{V}_{y,\hat{x}}\overset{\approx}{U}_s && \text{[by $*$ on hypothesis, (1.1.9), (QJ2)]} \\
&\quad = \overset{\approx}{U}_t\big(\overset{\approx}{V}_{U_s x,t}\overset{\approx}{U}_y + \overset{\approx}{U}_y\overset{\approx}{V}_{t,U_s x} - \overset{\approx}{U}_{s,y}\overset{\approx}{V}_{\hat{x},s} + \overset{\approx}{V}_{y,\hat{x}}\overset{\approx}{U}_s\big) && \text{[by (1.1.9), $*$ on hypotheses twice]} \\
&\quad = \overset{\approx}{U}_t\big(\overset{\approx}{U}_{\{U_s x,t,y\},y} - \overset{\approx}{U}_{s,y}\overset{\approx}{V}_{\hat{x},s} + \overset{\approx}{V}_{y,\hat{x}}\overset{\approx}{U}_s) && \text{[by (1.1.1)]} \\
&\quad = \overset{\approx}{U}_t\big(\overset{\approx}{U}_{U_s\hat{x},y} - \overset{\approx}{U}_{s,y}\overset{\approx}{V}_{\hat{x},s} + \overset{\approx}{V}_{y,\hat{x}}\overset{\approx}{U}_s\big) && \text{[by hypothesis on $\hat{x}$]}
\end{aligned}$$

which vanishes by (1.1.9). ■

6 Algebras of Outer Fractions

Outer Fraction Definition 6.1 *We say that a quadratic Jordan algebra $Q \supseteq J$ is an* **algebra of outer S-fractions** *of J for an Ore monad $S \subseteq J$ if*

(OFI) *Every $s \in S$ is invertible in Q,*
(OFII) *Every $q \in Q$ has a J-prenominator $s \in S : \{s, q\}, U_s q \in J$.*

Notice that invertibility (OFI) *of s in Q implies injectivity of s in J. Condition* (OFII) *that $U_s q = n \in J$ guarantees that $q = U_s^{-1} n$ has the form of an outer-fraction with numerator $n \in J$ and denominator $s \in S$. A crucial step in Martinez's approach is the added condition in* (OFII) *that $\{s, q\} = w_0 \in J$.*

As the associative case tells us, it is not easy for an algebra J to have an algebra of S-fractions. An immediate consequence is that everyone has s-denominators, not just J-denominators.

Ore Necessity Proposition 6.2 *Let J be a Jordan algebra with Ore monad S. If J has an algebra of S-fractions, then J must satisfy the Ore condition with respect to S: every element of J must have an s-denominator for every $s \in S$. Indeed, as soon as the element $q := U_s^{-1} a \in Q$ (or $q' := \{s, a, s^{-1}\}$) has a J-prenominator in S, then a must have an s-denominator.*

Proof: If $q := U_s^{-1} a \in Q$ has a *J-prenominator*, by the Choice Corollary 4.3 it has a *J-denominator* $s' = U_s U_s t \in U_{s^2} S$. Then $s'' := (U_s t)^2$ is an *s-prenominator* for a, since in Q we compute $\{(U_st)^2, a\} = \{U_st, U_st, U_sq\} = U_s\big(\{t, U_s(U_st), q\}\big)$ [by linearized (QJ3)] $\in U_s\{J, s', q\} \in U_sJ$ [by (G2)* for $s' \in Gen_J(q)$], so s'' is an s-prenominator for a by 2.1 (Pren).

We could avoid the denominator strengthening in 4.3 by supposing only that $q' := \{s^{-1}, a, s\}$ has a *J-prenominator* of the form $s' = U_st$: $\{s', q'\} = a' \in J$. Then $s'' := U_{s^2}t \in U_sS$ has $\{s'', a\} := \{U_{s^2}t, U_sq\} = U_s\{U_st, s^2, q\} = U_s\big(\{U_st, \{s^2, q\}\}-\{U_st, q, U_s1\}\big)$ [by (1.1.2)] $= U_s\big(\{s', q'\}-U_sU_{t,1}U_sq\big)$ [note that $q' := U_{s^{-1},s}a = U_{s^{-1},s}U_sq = \{s^2, q\}$ by cancelling U_s from $U_sU_{s^{-1},s}U_s = U_{s,s^3} = U_sU_{1,s^2}$ (which follows by linearized (QJ3) and Power-Associativity [9, 5.3.1(2),p.201] $= U_s\big(a' - \{t, a\}\big) \in U_sJ$, so s'' is an s-prenominator for a. ■

The following condition on elements of J is necessary for a ring of fractions to exist, but (unfortunately) does not seem to be a consequence of the Ore axioms alone.

Unwelcome Condition 6.3 *If $s, x, y \in J$ are such that the operators $\overset{\approx}{U}_y, \overset{\approx}{U}_{y,s}, \overset{\approx}{U}_{y,x}$ are left-s-movable by $t \in S$, so there are $\overset{\approx}{M}$ in $\mathcal{UME}(J)$ with $\overset{\approx}{U}_y\overset{\approx}{U}_t = \overset{\approx}{U}_s\overset{\approx}{M}_y$, $\overset{\approx}{U}_{y,s}\overset{\approx}{U}_t = \overset{\approx}{U}_s\overset{\approx}{M}_{y,s}$, $\overset{\approx}{U}_{y,x}\overset{\approx}{U}_t = \overset{\approx}{U}_s\overset{\approx}{M}_{y,x}$, then whenever s is invertible in some Jordan algebra $Q \supseteq J$ we have*

$$\big(\overset{\approx}{M}{}^*_{y,s}\overset{\approx}{U}_{x,s} - \overset{\approx}{M}{}^*_{y,x}\overset{\approx}{U}_s\big)\overset{\approx}{M}_y = \overset{\approx}{U}_{W^*_yx,U_ty} = \overset{\approx}{M}{}^*_y\big(\overset{\approx}{U}_{s,x}\overset{\approx}{M}_{y,s} - \overset{\approx}{U}_s\overset{\approx}{M}_{y,x}\big) \tag{6.3.1}$$

in the envelope $\mathcal{UME}(J|Q)$.

Proof: In $\mathcal{UME}(Q)$ we have a $\overset{\approx}{U}\overset{\approx}{V}$-inverse identity

$$\overset{\approx}{U}_{z,s}\overset{\approx}{U}{}_s^{-1} = \overset{\approx}{V}_{z,s^{-1}}$$

because $\overset{\approx}{U}_{s^{-1}}(\overset{\approx}{V}_{z,s^{-1}} - \overset{\approx}{U}_{z,s}\overset{\approx}{U}_{s^{-1}}) = \overset{\approx}{U}_{U_{s^{-1}}z,s^{-1}} - \overset{\approx}{U}_{U_{s^{-1}}z,U_{s^{-1}}s} = 0$ [by (QJ2) and linearized (QJ3)] and dually. Our hypothesis $\overset{\approx}{U}_y\overset{\approx}{U}_t = \overset{\approx}{U}_s\overset{\approx}{M}$ yields $\overset{\approx}{M}_y = \overset{\approx}{U}{}_s^{-1}\overset{\approx}{U}_y\overset{\approx}{U}_t$ and (via the involution) $\overset{\approx}{M}{}^*_{y,x}\overset{\approx}{U}_s = \overset{\approx}{U}_t\overset{\approx}{U}_{y,x}$, $\overset{\approx}{M}{}^*_{y,s} = \overset{\approx}{U}_t\overset{\approx}{U}_{y,s}\overset{\approx}{U}{}_s^{-1} = \overset{\approx}{U}_t\overset{\approx}{V}_{y,s^{-1}}$ [by $\overset{\approx}{U}\overset{\approx}{V}$-inverse]. Abreviating $q := U_s^{-1}x$ we have

$$\begin{aligned}
&\big(\overset{\approx}{M}{}^*_{y,s}\overset{\approx}{U}_{x,s} - \overset{\approx}{M}{}^*_{y,x}\overset{\approx}{U}_s\big)\overset{\approx}{M}_y - \overset{\approx}{U}_{W^*_yx,U_ty} \\
&\quad = \big((\overset{\approx}{U}_t\overset{\approx}{U}_{y,s}\overset{\approx}{U}{}_s^{-1})\overset{\approx}{U}_{x,s} - (\overset{\approx}{U}_t\overset{\approx}{U}_{y,x}\overset{\approx}{U}{}_s^{-1})\overset{\approx}{U}_s\big)(\overset{\approx}{U}{}_s^{-1}\overset{\approx}{U}_y\overset{\approx}{U}_t) - \overset{\approx}{U}_{(U_tU_yU_s^{-1})x,U_ty} \\
&\quad = \overset{\approx}{U}_t\big(\overset{\approx}{V}_{y,s^{-1}}\overset{\approx}{V}_{x,s^{-1}} - \overset{\approx}{U}_{y,x}\overset{\approx}{U}_{s^{-1}}\big)\overset{\approx}{U}_y\overset{\approx}{U}_t - \overset{\approx}{U}_t\overset{\approx}{U}_{U_yq,y}\overset{\approx}{U}_t && \text{[by } \overset{\approx}{U}\overset{\approx}{V}\text{-inverse for } y, x, \text{ (QJ3)]} \\
&\quad = \overset{\approx}{U}_t\big(\overset{\approx}{V}_{y,q}\overset{\approx}{U}_y - \overset{\approx}{U}_{U_yq,y}\big)\overset{\approx}{U}_t && \text{[by (1.1.5)]}
\end{aligned}$$

which vanishes by (QJ2). The second equality follows dually. ■

7 A Home for Fractions

Another important observation of Martinez is the surprising fact that we don't need to construct the algebra of fractions precisely (as in [6]), we only need to find a home where it can live, and the algebra of fractions will materialize as the set of elements having an S-denominator.

Fraction Materialization Theorem 7.1 *If J is a Jordan algebra satisfying the Ore condition with respect to an Ore monad S, and Q is any unital Jordan algebra containing J with the property that the elements of S are invertible in Q, then the set $Q(J)$ of all elements in Q having a J-prenominator in S forms an algebra of outer S-fractions for J.*

Proof: The set $Q(J)$ contains J and the identity element of Q [any $s \in S$ will serve as J-prenominator for $a \in J$ and 1, since $\{s, a\}$, U_sa, $\{s, 1\} = 2s$, $U_s1 = s^2$ lie in J] as well as S^{-1} [every $s^{-1} \in Q$ has J-prenominator s^2, since $\{s^2, s^{-1}\} = 2s$ and $U_{s^2}s^{-1} = s^3$ lie in J]. Every $q \in Q(J)$ has a J-prenominator in S by definition of $Q(J)$, so the two conditions (OFI),(OFII) for an algebra of outer S-fractions are trivially met. The only question is whether the *set* $Q(J)$ is actually an *algebra*. It is child's play to see that $Q(J)$ is a linear subspace: if q_1, q_2 have J-prenominators s_1, s_2 then by Choice Corollary 4.3 they will share a common J-prenominator $s \in U_{s_1^5}S \cap U_{s_2^5}S$, which will remain a J-prenominator for any linear combination because $M_s(\alpha_1 q_1 + \alpha_2 q_2) = \alpha_1 M_s q_1 + \alpha_2 M_s q_2 \in \alpha_1 J + \alpha_2 J \subseteq J$ for $M_s = V_s, U_s$, and therefore $\alpha_1 q_1 + \alpha_2 q_2 \in Q(J)$.

The crux is to prove that $Q(J)$ is closed under quadratic products $U_{q_1}q_2$ (in particular, $U_{q_1}1 = q_1^2$). Since by hypothesis q_1 has a J-prenominator in S, by Generic Strengthening 3.4 it has a J-genominator $s_1 \in S \cap Gen_J(q_1)$; then $t_1 = s_1$ has $I_1 := U_{t_1}J \subseteq U_{G_1}J \subseteq G_1 := Gen_J(q_1)$ by Genominator Innerness 3.3 (with $s = \hat{1}$):

$$(7.1.1) \qquad t_1 \in S \cap G_1, \quad I_1 \subseteq G_1 \subseteq J \qquad (G_1 := Gen_J(q_1)).$$

Next, by hypothesis q_2 also has a J-prenominator in S, so by Genominator Inheritance 4.4 it has a t_1-genominator $s_2 \in S \cap Gen_{t_1}(q_2)$; then any $t_2 \in U_{s_2}U_{t_1}S$ has $I_2 := U_{t_2}J \subseteq U_{s_2}U_{t_1}J$ [by (QJ3)] $\subseteq U_{G_2}I_{t_1} \subseteq G_2 := Gen_{t_1}(q_2)$ [by Generic Innerness 3.3 (with $s = t_1$)]. Choosing one such, we have

$$(7.1.2) \qquad t_2 \in S \cap G_2, \quad I_2 \subseteq G_2 \subseteq I_1 \qquad (G_2 := Gen_{t_1}(q_2)).$$

Once again, by Genominator Inheritance 4.4 q_1 has a t_2-genominator $s_3 \in S \cap Gen_{t_2}(q_1)$, and any $t_3 \in U_{s_3}U_{t_2}S$ has $I_3 := U_{t_3}J \subseteq U_{s_3}U_{t_2}J$ [by (QJ3)] $\subseteq U_{G_3}I_{t_2} \subseteq G_3 := Gen_{t_2}(q_1)$ [by Innerness 3.3 (with $s = t_2$)]. Choosing one such, we have

$$(7.1.3) \qquad t_3 \in S \cap G_3, \quad I_3 \subseteq G_3 \subseteq I_2 \subseteq I_1 \quad (G_3 := Gen_{t_2}(q_1)).$$

We will show that $x = t_3^2$ is a J-prenominator in S for $U_{q_1}q_2$, establishing that $U_{q_1}q_2 \in Q(J)$. For U-pushing in 2.1 (Pren), $U_x\big(U_{q_1}q_2\big) = U_{t_3}\big(U_{t_3}U_{q_1}\big)q_2 \subseteq U_J\big(U_{t_2}\mathcal{M}(J|Q)\ U_{t_2}\big)(q_2)$ [by 3.1 (G3) for $t_3 \in Gen_{t_2}(q_1)$ by (7.1.3)] $\subseteq \mathcal{M}(J|Q)J \subseteq J$ since by definition 3.1 (G1) for $Gen_{t_1}(q_2)$ we have $U_{t_2}q_2 \in I_1 \subseteq J$.

The crux, as always, is V-pushing. We have $\{x, U_{q_1}q_2\} = \{t_3^2, U_{q_1}q_2\} = -\{q_2, U_{q_1}t_3^2\} + \{\{t_3^2, q_1, q_2\}, q_1\}$ [by (1.1.5)] $= -\{U_{q_1}t_3^2, q_2\} + \{\big(\{t_3, \{t_3, q_1\}, q_2\} - \{U_{t_3}q_1, q_2\}\big), q_1\}$. We will show that each of these 3 pieces separately falls in J. For the first piece, $\{U_{q_1}t_3^2, q_2\} = \{U_{q_1}U_{t_3}\hat{1}, q_2\} \in \{U_{t_2}\mathcal{M}(J|Q)U_{t_2}\hat{1}, q_2\}$ [by 3.1 (G3)* for $t_3 \in G_3 = Gen_{t_2}(q_1)$ by (7.1.3)] $\subseteq \{I_{t_2}, q_2\} \subseteq I_1$ [by (G0) for $I_2 \subseteq G_2 = Gen_{t_1}(q_2)$ by (7.1.2)] $\subseteq J$. For the second piece we have $\{t_3, \{t_3, q_1\}, q_2\} \in \{I_2, I_2, q_2\}$ [by 3.1 (G0) for $t_3 \in Gen_{t_2}(q_1) \subseteq I_2$ by (7.1.3)] $\subseteq V_{q_2, G_2}I_2 \subseteq V_{\widehat{I_1}, I_1}I_1$ [by 3.1 (G2)* for $G_2 = Gen_{t_1}(q_2) \subseteq I_1$ by (7.1.2)] $\subseteq I_1$, and similarly the third piece $\{U_{t_3}q_1, q_2\} \in \{I_2, q_2\}$ [by 3.1 (G1) for $t_3 \in Gen_{t_2}(q_1)$] $\subseteq I_1$ [by 3.1 (G0) for $I_2 \subseteq G_2 = Gen_{t_1}(q_2)$ by (7.1.2)], and therefore the two pieces combined yield $\{\big(\{t_3, \{t_3, q_1\}, q_2\} - \{U_{t_3}q_1, q_2\}\big),\ q_1\} \subseteq \{I_1, q_1\} \subseteq J$ [by 3.1 (G0) for $I_1 \subseteq G_1 = Gen_J(q_1)$ by (7.1.1)]. Thus all the pieces fall in place, and we have closure of $Q(J)$. ■

Martinez [8] found a home for *linear* Jordan fractions (over scalars with $\frac{1}{6}$) as germs of derivations of a TKK algebra via $a \mapsto Ad_{a^+}$. Finding a home for *quadratic* Jordan algebras is much more complicated. Especially in characteristic 2, the Lie algebra by itself does not provide a home for the quadratic Jordan structure. For example, if Ω is a commutative associative ring of characteristic 2, the Jordan algebra $J = \Omega^+$ has trivial linear structure $\{J, J\} = \{J, J, J\} = 0$, only the quadratic outer and inner multiplications $U_a(b) = a^2 b = \cap_b(a)$ survive. But then the TKK algebra $\pounds(J) = J_+ \oplus 0 \oplus J_{-1}$ has $[a^+, b_-] = V_{a,b} = 0$, so the Lie bracket vanishes completely and the imbedding $a \to Ad_a$ is identically zero. If Ω is an imperfect field of characteristic 2, then for $\Phi := \Omega^2$ any subalgebra $\Phi \subseteq J \subseteq \Omega$ is an upstanding Jordan division algebra ($x^{-1} = (x^2)^{-1}x$) and yet won't imbed in $\pounds(J)$. Of course, as a division algebra it is not in need of inverses, but the polynomial algebra $\Omega[t]$ is almost as respectable an algebra and is in dire need of inverses, yet they cannot be found in $\pounds(J)$ either. It is also not enough to consider the derivation Ad_a together with its "divided-square" $U_a \approx \frac{1}{2}Ad_a^2$ since if a is nilpotent of index 2 then its outer multiplication too disappears, $U_a = 0$, but *its inner multiplication never vanishes*: if $a \neq 0$ then $\cap_a \neq 0$ since $\cap_a(s) = U_s a \neq 0$ by injectivity of $s \in S$. To obtain an imbedding which includes these extremely radical elements, one would have to further enrich the structure of $\pounds(J)$.

In a subsequent paper [3] the first author will establish an imbedding (injective up to extremely radical elements) of a Jordan algebra as divided-square derivations on a TKK algebra, and in [11] the second author will begin an injective imbedding of the whole Jordan algebra as enriched *Jordan derivations* on a TKK algebra.

8 Afterthoughts

We mention here some additional results which will not play a direct role in our subsequent work. In the setting of the Innerness Lemma 2.2, any K-prenomnator $x \in Pren_K(q)$ can be strengthened to a version of (G2): $y = x^2$ or any $y \in U_{x^2}a$ where $a \in \widehat{J}$ satisfies $U_x a + \{x, a, K\} \subseteq K$ (e.g., if $a \in \widehat{K}$, or $a \in \widehat{J}$ but K is an inner ideal), will be (i) a K-prenominator for q whose multiplications $V_{y,q}, V_{q,y}$ become pure K-multiplications in $V_{K,\widehat{K}} = V_{K,K} + V_K$. Indeed, when $a = 1$ the element $y = x^2$ for Since $\{x, q\}, U_x q \in K$ we have $\{x^2, q\} = \{x, \{x, q\}\} - 2U_x q \in K$ and $U_{x^2}q = U_x U_x q \in U_x K$ contained in K as in (i), and for (ii) $V_{x^2,q} = V_{x,\{x,q\}} - V_{U_x q} \in V_{K,K} - V_K$ and dually $V_{q,x^2} \in V_{K,\widehat{K}}$. When $y = U_{x^2}a$ then $\{y, q\} = V_q U_x(U_x \hat{a}) = \big(U_{\{q,x\},x} + V_{x,\{x,q\}} - V_{U_x q}\big)U_x a - V_x\{U_x q, a, x\}$ [by Macdonald] $\subseteq \big(U_{K,K} + V_{K,K} - V_K\big)U_x a - V_K\{K, a, x\} \subseteq K$ by hypothesis, and $U_{U_{x^2}a}q = U_{x^2}U_a U_{x^2}q = U_x U_{U_x a}(U_x q) \in U_K U_K K \subseteq K$ by (QJ3). Thus $y = U_{x^2}a$ is a K-prenominator for q, and we have by (1.1.5) that $V_{y,q} = V_{U_{x^2}a,q} = V_{U_x(U_x a),q} = -V_{U_x q, U_x a} + V_{x,\{U_x a, x, q\}} \in V_{K,K}$ [note that $\{U_x a, x, q\} = \{U_x a, \{x, q\}\} - V_{x,a}(U_x q) \in \{U_x a, K\} - \{x, a, K\} \subseteq K$ by the hypotheses], and dually $V_{q,y} \in V_{K,K}$.

Extending this idea, a stronger-than-strong K-denominator x in the subalgebra K would turn multiplications with q into explicit K-multiplications, satisfying (SD1), (SD1)$'$, (SSD2) $V_{x,q} \in V_{K,K}$, (SSD3) $U_x U_q \in \mathcal{M}_{2,K}$, [hence automatically (SSD2)$'$ $V_{q,x} \in V_{K,K}$, (SSD3)$'$ $U_q U_x \in \mathcal{M}_{2,K}$], (SD0) $\{x, q\} \in U_K \widehat{K}$], where $\mathcal{M}_{2,K} := \mathcal{M}(K|Q)U_K + \mathcal{M}(K|Q)V_{K,K}$ [with more effort we could replace this with the sharper $U_K + U_K U_K + U_K V_{K,K} + V_{K,K} + V_{K,K}^2$]. This form guarantees these multiplications will push $\widehat{K}$ down into $U_K \widehat{K}$: $\mathcal{M}_{2,K}(\widehat{K}) \subseteq U_K \widehat{K} \triangleleft K$, $\mathcal{M}_{2,K}(K) \subseteq U_K K \triangleleft K$, $V_{U_K \widehat{K}} \subseteq V_{K,K}$ [because for $z, w \in K$ all of $V_{U_z w} = V_{z,\{w,z\}} - V_{z^2,w}$, $V_{z^2} = V_{z,z}$, $V_{\{z,w\}} = V_{z,w} + V_{w,z}$ lie in $V_{K,K}$ by (1.1.2)], where $V_{K,K}\widehat{K} = \{K, K\} + \{K, K, K\} \subseteq U_K \widehat{K}$. A standard argument shows that $SSDen_K(q)$ is an inner ideal in K : the element $y := U_x \hat{a} \in U_K \widehat{K}$ lies in the subalgebra K; for (SD1) $U_y(q) \in U_K U_{\widehat{K}}(U_x q) \subseteq U_K K$ by (QJ3), (SD1) for x; for (SD1)$'$ $U_q y \in \mathcal{M}_{2,K}\widehat{K}$ [by (SSD3)$'$ for x and (2.1.1)]; for (SSD2) when $\hat{a} = a \in K$ by $V_{y,q} = V_{x,\{q,x,a\}} - V_{U_x q, a} \in V_{K,K}$ [by (1.1.6) and (SSD2)$'$, (SD1) for x], and when $\hat{a} = \hat{1}$ we have $V_{y,q} = V_{x,\{x,q\}} - V_{U_x q} \in V_{K,K} + V_{U_K \widehat{K}}$ [by (1.1.2), (SD1) for x, and (2.1.1)]; and for (SSD3) $U_y U_q = U_x U_{\hat{a}} U_x U_q \in U_K U_{\widehat{K}} \mathcal{M}_{2,K}$ [by (SSD3) for x].

Alternately, we could require a K-denominator to satisfy (D1), (D1)$'$, (D2), (D2)$'$ and the stronger conditions (D3-3$'$)* $U_x U_q, U_x U_q \in \mathcal{U}_{2,K}$, (D4)* $W_{x,q} \in \mathcal{U}_{2,K}$ for $\mathcal{U}_{2,K} := U_K U_K + U_K$ spanned by U-operators alone, but these elements don't seem to form an inner ideal in K, since in (D3)* $U_{x,y} U_q = V_{x,q} V_{y,q} - V_{x,U_q y}$ passes out of $\mathcal{U}_{2,K}$ into $\mathcal{M}_{2,K}$. ■

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