A BRIEF NOTE ON LOCAL-GLOBAL PRINCIPLES FOR COMPOSITION ALGEBRAS

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ABSTRACT. We state several local-global principles for composition algebras over function fields of certain projective curves.

Local-global principles for the isotropy of quadratic forms and for central simple algebras over certain function fields were proved by Harbater, Hartmann and Krashen [H-H-K1, 2]. Colliot-Thelene, Parimala and Suresh gave a local-global principle for the isotropy behaviour of quadratic forms with respect to the discrete valuations of the function field [CT-P-S, Theorem 3.1].

In this note, we point out some resulting local-global principles for composition algebras over function fields.

Recall that a unital, not necessarily associative, algebra C over a field F is called a *composition algebra*, if it carries a quadratic form $N\colon C \to F$ such that its associated symmetric bilinear form $N(x,y) = N(x+y) - N(x) - N(y)$ is nondegenerate, i.e. determines an F-vector space isomorphism $C \xrightarrow{\sim} \mathrm{Hom}_F(C,F)$, and such that $N(xy) = N(x)N(y)$ for all $x, y \in C$. Composition algebras of rank 2 are exactly the étale algebras over F. Composition algebras of rank 4 are called *quaternion algebras*, those of rank 8 are called *octonion algebras*. A quaternion algebra C is called *split* over F, if $C \cong \mathrm{Mat}(F)$, an octonion algebra if $C \cong \mathrm{Zor}(F)$.

Throughout, let k be a field of characteristic not 2. Let T be a complete discrete valuation ring with fraction field K and residue field k and $\widehat{X}$ a normal irreducible projective T-curve with function field F and with closed fiber X. We assume that $f : \widehat{X} \to \mathbb{P}^1_T$ is a finite morphism such that $\mathcal{P} := f^{-1}(\infty)$ contains all points at which distinct irreducible components of the closed fiber $X \subset \widehat{X}$ meet. (Such an f exists by [H-H], Proposition 6.6.) Let $\mathcal{U}$ be the collection of irreducible components U of $f^{-1}(\mathbb{A}^1_k)$.

Given an irreducible component X_0 of X with generic point η, consider the local ring of $\widehat{X}$ at η. For a (possibly empty) proper subset U of X_0, let R_U denote the subring of this local ring consisting of the rational functions that are regular at each point of U. In particular, $R_\varnothing$ is the local ring of $\widehat{X}$ at the generic point of the

Date: 27.3.2012.

2000 *Mathematics Subject Classification.* Primary: 17A75.

Key words and phrases. Composition algebras, local-global principle.

component X_0. The t-adic completion of R_U is denoted by $\widehat{R}_U$. If P is a closed point of X, we write R_P for the local ring of $\widehat{X}$ at P, and $\widehat{R}_P$ for its completion at its maximal ideal. The fraction fields of the domains $\widehat{R}_U$ and $\widehat{R}_P$ will be denoted by F_U and F_P. The local-global principle [H-H-K1, Theorem 4.2] states that each nondegenerate quadratic form q over F of dimension unequal to 2 is isotropic iff q_{F_ξ} is isotropic for all $\xi \in \mathcal{P} \cup \mathcal{U}$. Applied to the norm of a composition algebra, it yields:

Theorem 1. *Let C be a composition algebra over F of dimension 4 or 8. Then $C \otimes_F F_\xi$ is split for each $\xi \in \mathcal{P} \cup \mathcal{U}$ if and only if C is split over F.*

Moreover, [H-H-K1, Corollary 4.3] together with known results on the Albert form from [A, Theorem 3], [J, 3.12], respectively [M-PI-P, 2.1] implies for instance:

Theorem 2. *Let C_1 and C_2 be two composition algebras over F of the same dimension $2m \geq 4$.*
(i) If $C_1 \otimes_F F_\xi \cong C_2 \otimes_F F_\xi$ for each $\xi \in \mathcal{P} \cup \mathcal{U}$ then $C_1 \cong C_2$.
(ii) Suppose that $C_1 \otimes_F F_\xi \not\cong C_2 \otimes_F F_\xi$ for each $\xi \in \mathcal{P} \cup \mathcal{U}$, but that $C_1 \otimes_F F_\xi$ and $C_2 \otimes_F F_\xi$ contain an isomorphic composition subalgebra of dimension m for each $\xi \in \mathcal{P} \cup \mathcal{U}$ then $C_1 \not\cong C_2$ and C_1 and C_2 both contain an isomorphic composition subalgebra of dimension m.
(iii) Suppose $2m = 8$ and that there is one ξ_0 such that $C_1 \otimes_F F_{\xi_0}$ and $C_2 \otimes_F F_{\xi_0}$ do not have any isomorphic quadratic étale subalgebras over F_{ξ_0}, then neither do C_1 and C_2 over F.

The local-global principle [H-H-K2, 9.3] expresses the anisotropy behaviour of quadratic forms over F in terms of the points P of the closed fiber X and yields:

Theorem 3. *Let C be a composition algebra over F of dimension 4 or 8. Then C is split over the field F_P for all closed $P \in X$ if and only if C is split over F.*

Theorem 4. *Let C_1 and C_2 be two composition algebras over F of the same dimension $2m \geq 4$.*
(i) If $C_1 \otimes_F F_P \cong C_2 \otimes_F F_P$ for each closed $P \in X$ then $C_1 \cong C_2$.
(ii) Suppose that $C_1 \otimes_F F_P \not\cong C_2 \otimes_F F_P$ for each closed $P \in X$, but that $C_1 \otimes_F F_P$ and $C_2 \otimes_F F_P$ contain an isomorphic composition subalgebra of dimension m for each $P \in X$ then $C_1 \not\cong C_2$ and C_1 and C_2 both contain an isomorphic subalgebra of dimension m.
(iii) Suppose $2m = 8$ and that there is one closed point $P_0 \in X$ such that $C_1 \otimes_F F_{P_0}$ and $C_2 \otimes_F F_{P_0}$ do not have any isomorphic quadratic étale subalgebras over F_{P_0}, then neither do C_1 and C_2 over F.

This last result follows directly from [H-H-K2, 9.3], again together with [A, Theorem 3], [J, 3.12], respectively [M-PI-P, 2.1].

In the following, by a discrete valuation we mean a discrete valuation ring of rank one (and look at valuations with value group $\mathbb{Z}$). For each rank one discrete valuation v of F, let F_v denote the completion of F with respect to v. Furthermore, let $\widehat{X}$ be a smooth, projective, geometrically integral curve over K with function field $F = K(\widehat{X})$.

Theorem 5. *Let C be a composition algebra over F of dimension 4 or 8. Then C is split over the completion F_v of F with respect to each discrete valuation v of F if and only if C is split over F.*

This follows from [CT-P-S, 3.1]. [CT-P-S, 3.1] also yields [H-H-H2, 9.10 (b)] which translates to:

Theorem 6. *Let C_1 and C_2 be two composition algebras over F of the same dimension $2m \geq 4$.*
(i) If $C_1 \otimes_F F_v \cong C_2 \otimes_F F_v$ for each discrete valuation v then $C_1 \cong C_2$.
(ii) Suppose that $C_1 \otimes_F F_v \not\cong C_2 \otimes_F F_v$ for each discrete valuation v, but that $C_1 \otimes_F F_v$ and $C_2 \otimes_F F_v$ contain an isomorphic composition subalgebra of dimension m for each discrete valuation v then $C_1 \not\cong C_2$ and C_1 and C_2 both contain an isomorphic subalgebra of dimension m.
(iii) Suppose $2m = 8$ and that there is one discrete valuation v_0 such that $C_1 \otimes_F F_{v_0}$ and $C_2 \otimes_F F_{v_0}$ do not have any isomorphic quadratic étale subalgebras over F_{v_0}, then neither do C_1 and C_2 over F.

These results nicely complement [P, Theorem 6.4] and [K, 4.3.5] if F is a function field of genus one.

Remark 7. (i) Note that Theorems 1, 3 and 6 do not hold for composition algebras of dimension 2 (see [H-H-K], Example 4.4, [CT-P-S], Remark 4.4). Theorem 6 also holds for quadratic étale algebras if the additional assumption is made that the reduction graph Γ of any regular model of F is a tree. For quaternion algebras, Theorems 1 and 3 were proved in [H-H-K1], Theorem 5.1, [H-H-K2], Theorem 9.11 and Theorem 6 (i) in [H-H-K2], Corollary 9.12.
(ii) Composition algebras are known to behave even 'better' over rational function fields: Each composition algebra C over the rational function field $F = k(x)$ which splits over the completion F_v for each the discrete valuation corresponding to a point P on the affine line $\mathbb{A}^1_k$, must already be split itself [K, 4.3.6].

Moreover, given two composition algebras C_1 and C_2 over $F = k(x)$ of the same dimension, if $C_1 \otimes_F F_v \cong C_2 \otimes_F F_v$ for each discrete valuation v corresponding to a point P on the affine line $\mathbb{A}^1_k$, then $C_1 \cong C_2$ [K, 4.4.8].

Example 8. Take a setup from [H-H]: Let $T = k[[t]]$ be the complete discrete valuation ring with uniformizer t and $\widehat{X} = \mathbb{P}^1_T$ be the projective x-line over T.

$X = \mathbb{P}^1_k$ is the closed fiber of $\widehat{X}$. Here $\widehat{X} = X \times_k T$. The function field $F = k((t))(x)$ of $\widehat{X}$ is the fraction field of the ring of functions $k[[t]][x]$ on the dense open subset $\mathbb{A}^1_T$.

Let C be a composition algebra of dimension 4 or 8 over $F = k((t))(x)$. Then $C \otimes_F F_P$ is split for all closed $P \in \mathbb{P}^1_k$ if and only if C is split over $F = k((t))(x)$ (Theorem 3). Given two composition algebras C_1 and C_2 of the same dimension $n \geq 4$ over $F = k((t))(x)$, $C_1 \cong C_2$ iff $C_1 \otimes_F F_P \cong C_2 \otimes_F F_P$ for all closed $P \in \mathbb{P}^1_k$ (Theorem 4).

Example 9. Let $K = \mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic valuation, T the canonical complete extension of the p-adic valuation and $k = \mathbb{F}_p$. Let F be a finitely generated field extension of transcendence degree one over $\mathbb{Q}_p$ and $\widehat{X}$ a regular projective T-curve with function field F. Let X be the closed fiber of $\widehat{X}$. Then given a composition algebra C over F of dimension 4 or 8, C is split if and only if $C \otimes_F F_P$ is split for each closed $P \in X$ (Theorem 3). Moreover, $C \otimes_F F_v$ is split for all discrete valuations v of F if and only if C is split (Theorem 5).

References

[A] Albert, A., *On the Wedderburn norm condition for cyclic algebras.* Bull. Amer. Math. Soc. 37 (1931), 301 - 312.

[CT-P-S] Colliot-Thelene, J.-L., Parimala, R., Suresh, V., *Patching and local global principles for homogeneous spaces over function fields of p-adic curves.* To appear in Comment. Math. Helv. arXiv:math.AG/0812.3099v3.

[H-H] Harbater, D., Hartmann, J., *Patching over fields.* Israel J. Math. 176 (2010), 61 - 107.

[H-H-K1] Harbater, D., Hartmann, J., Krashen, D., *Applications of patching to quadratic forms and central simple algebras.* Invent. Math. 178 (2) (2009), 231 - 263.

[H-H-K2] Harbater, D., Hartmann, J., Krashen, D., *Local-global principles for torsors over arithmetic curves.* Preprint 2011. arXiv:math.NT/1108.3323v2.

[J] Jacobson, N., *Some applications of Jordan norms to involutorial simple algebras.* Adv. Math. 48 (1983), 149 - 165.

[K] Küting, B., *Kompositionsalgebren über rationalen Funktionenkörpern.* Dissertation. Fernuniversität Hagen, 1987.

[M-PI-P] Morandi, P., Pèrez-Izquierdo, J.M., Pumplün, S., *On the tensor product of composition algebras.* J. Algebra 243 (2001), 41 - 68.

[P] Petersson, H.P., *Composition algebras over algebraic curves of genus zero.* Trans. Amer. Math. Soc. 337(1) (1993), 473-491.

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