# New Examples of Simple Jordan Superalgebras over an Arbitrary Field of Characteristic Zero

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**Abstract:** *An new example of a unital simple special Jordan superalgebra over the field of real numbers was constructed in [10]. It turned out to be a subsuperalgebra of the Jordan superalgebra of vector type $J(\Gamma, D)$, but not isomorphic to a superalgebra of this type. Moreover, its superalgebra of fractions is isomorphic to a Jordan superalgebra of vector type. A similar example of a Jordan superalgebra over a field of characteristic 0 in which the equation $t^2 + 1 = 0$ has no solutions was constructed in [12]. In this article we present an example of a Jordan superalgebra with the same properties over an arbitrary field of characteristic 0. A similar example of a superalgebra is found in the Cheng–Kac superalgebra.*

**Keywords:** Jordan superalgebra, $(-1, 1)$-superalgebra, superalgebra of vector type, differentially simple algebra, polynomial algebra, projective module

Jordan algebras and superalgebras constitute an important class of algebras in ring theory. Simple Jordan superalgebras are studied in [1, 2, 3, 4, 5, 6, 7, 8].

The unital simple special Jordan superalgebras with the associative even part $A$ and the odd part $M$ which is an associative $A$-module were described in [9, 10]. The study in [9] was considerably influenced by [11], which described the simple $(-1, 1)$-superalgebras of characteristic $\neq 2, 3$. In the Jordan case, if a superalgebra is not the superalgebra of a nondegenerate bilinear superform, then its even part $A$ is a differentially simple algebra with respect to some set of derivations, and its odd part $M$ is a finitely generated projective $A$-module of rank 1. Here, as for $(-1, 1)$-superalgebras, we define multiplication in $M$ using fixed finite sets of derivations and elements of $A$. It turns out that every Jordan superalgebra of this type is a subsuperalgebra of the superalgebra of vector type $J(\Gamma, D)$. Under certain restrictions on $A$ the odd part $M$ is a cyclic $A$-module, and consequently, the original Jordan superalgebra is isomorphic to the superalgebra $J(\Gamma, D)$. For instance, if $A$ is a local algebra then by the well-known Kaplansky theorem $M$ is free, and consequently, it is a cyclic $A$-module. If the ground field is of characteristic $p > 2$ then [13] implies that $A$ is a local algebra; thus, $M$ is a cyclic $A$-module. If $A$ is the ring of polynomials in finitely many variables then $M$ is free by [14], and consequently, it is a cyclic $A$-module.

A natural question arose: is the original superalgebra isomorphic to $J(\Gamma, D)$? Equivalently, is the odd part $M$ a cyclic $A$-module? Examples are constructed in [10, 12] of unital simple

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special Jordan superalgebras with certain associative even part and the odd part $M$ which is not free, i.e., not cyclic. In those examples the ground field is either the field of real numbers or an arbitrary field of characteristic 0 in which the equation $t^2+1=0$ has no solutions.

In this article we construct a similar example of a Jordan superalgebra over an arbitrary field of characteristic 0, as well as an example of a simple Jordan superalgebra which is a subsuperalgebra of the Cheng–Kac Jordan superalgebra. Examples of these superalgebras answer a question of Cantarini and Kac [8].

Take a field $F$ of characteristic not equal to 2. A superalgebra $J=J_0+J_1$ is a $Z_2$-graded $F$-algebra:

$$J_0^2\subseteq J_0, J_1^2\subseteq J_0, J_1J_0\subseteq J_1, J_0J_1\subseteq J_1.$$

Put $A=J_0$ and $M=J_1$. The spaces $A$ and $M$ are called the even and odd parts of $J$. The elements of $A\cup M$ are called homogeneous. The expression $p(x)$ with $x\in A\cup M$ means the parity of $x$: $p(x)=0$ for $x\in A$ ($x$ is even) and $p(x)=1$ for $x\in M$ ($x$ is odd).

Given $x$ in $J$ denote by $R_x$ the operator of right multiplication by $x$. A superalgebra $J$ is called a Jordan superalgebra if the homogeneous elements satisfy the operator identities

$$aR_b=(-1)^{p(a)p(b)}bR_a, \tag{1}$$

$$R_{a^2}R_a=R_aR_{a^2}, \tag{2}$$

$$\begin{aligned}R_aR_bR_c+(-1)^{p(a)p(b)+p(a)p(c)+p(b)p(c)}R_cR_bR_a+(-1)^{p(b)p(c)}R_{(ac)b}=\\ R_aR_{bc}+(-1)^{p(a)p(b)}R_bR_{ac}+(-1)^{p(a)p(c)+p(b)p(c)}R_cR_{ab}.\end{aligned} \tag{3}$$

In every Jordan superalgebra, the homogeneous elements satisfy

$$(x,tz,y)=(-1)^{p(x)p(t)}t(x,z,y)+(-1)^{p(y)p(z)}(x,t,y)z, \tag{4}$$

where $(x,z,y)=(xz)y-x(zy)$ is the associator of $x$, $z$, and $y$.

Let us give some examples of Jordan superalgebras.

Take an associative $Z_2$-graded algebra $B=B_0+B_1$ with multiplication $*$. Defining on the space $B$ the supersymmetric product

$$a\circ_s b=\frac{1}{2}(a*b+(-1)^{p(a)p(b)}b*a), \qquad a,b\in B_0\cup B_1,$$

we obtain the Jordan superalgebra $B^{(+)s}$. A Jordan superalgebra $J=A+M$ is called *special* whenever it embeds (as a $Z_2$-graded algebra) in the superalgebra $B^{(+)s}$ for a suitable $Z_2$-graded associative algebra $B$.

**The superalgebra of vector type** $J(\Gamma,D)$**.** Take a commutative associative $F$-algebra $\Gamma$ equipped with a nonzero derivation $D$. Denote by $\overline{\Gamma}$ an isomorphic copy of the linear space $\Gamma$, and a fixed isomorphism, by $a\mapsto\overline{a}$. On the direct sum $J(\Gamma,D)=\Gamma+\overline{\Gamma}$ of linear spaces define a multiplication $(\cdot)$ as

$$a\cdot b=ab, \quad a\cdot\overline{b}=\overline{ab}, \quad \overline{a}\cdot b=\overline{ab}, \quad \overline{a}\cdot\overline{b}=D(a)b-aD(b),$$

where $a,b\in\Gamma$ and $ab$ is the product in $\Gamma$. Then $J(\Gamma,D)$ is a Jordan superalgebra with the even part $A=\Gamma$ and the odd part $M=\overline{\Gamma}$. The superalgebra $J(\Gamma,D)$ is simple if and only if $\Gamma$ is a $D$-simple algebra [15] (i.e., $\Gamma$ contains no proper nonzero $D$-invariant ideals, and $\Gamma^2=\Gamma$).

Consider the associative superalgebra $B = M_2^{1,1}(\operatorname{End}\Gamma)$ with the even part

$$B_0 = \left\{ \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \text{ where } \phi, \psi \in \operatorname{End}\Gamma \right\}$$

and the odd part

$$B_1 = \left\{ \begin{pmatrix} 0 & \phi \\ \psi & 0 \end{pmatrix} \text{ where } \phi, \psi \in \operatorname{End}\Gamma \right\}.$$

It is shown in [16] that the mapping

$$a + \bar{b} \mapsto \begin{pmatrix} R_a & 4R_bD + 2R_{D(b)} \\ -R_b & R_a \end{pmatrix}$$

is an embedding of $J(\Gamma, D)$ into $B^{(+)s}$. Consequently, the Jordan superalgebra $J(\Gamma, D)$ is special.

**The Kantor double** $J(\Gamma, \{\,,\})$**.** Take an associative supercommutative superalgebra $\Gamma = \Gamma_0 + \Gamma_1$ with unit 1 equipped with a super-skew-symmetric bilinear mapping $\{\,,\} : \Gamma \mapsto \Gamma$, which we call the bracket. From $\Gamma$ and $\{\,,\}$ we can construct a superalgebra $J(\Gamma, \{\,,\})$ as follows. Consider the direct sum $J(\Gamma, \{\,,\}) = \Gamma \oplus \Gamma x$ of linear spaces, where $\Gamma x$ is an isomorphic copy of $\Gamma$. Take two homogeneous elements $a$ and $b$ of $\Gamma$. The multiplication $(\cdot)$ on $J(\Gamma, \{\,,\})$ is defined as

$$a \cdot b = ab, \quad a \cdot bx = (ab)x, \quad ax \cdot b = (-1)^{p(b)}(ab)x, \quad ax \cdot bx = (-1)^{p(b)}\{a, b\}.$$

Put $A = \Gamma_0 + \Gamma_1 x$ and $M = \Gamma_1 + \Gamma_0 x$. Then $J(\Gamma, \{\,,\}) = A + M$ is a $Z_2$-graded algebra.

Refer to $\{\,,\}$ as a Jordan bracket if $J(\Gamma, \{\,,\})$ is a Jordan superalgebra. It is known (see [17]) that $\{\,,\}$ is a Jordan bracket if and only if it satisfies

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)}b\{a, c\} - \{a, 1\}bc, \tag{5}$$

$$\begin{aligned}\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{p(a)p(b)}\{b, \{a, c\}\} + \{a, 1\}\{b, c\} + \\ (-1)^{p(a)(p(b)+p(c))}\{b, 1\}\{c, a\} + (-1)^{p(c)(p(a)+p(b))}\{c, 1\}\{a, b\},\end{aligned} \tag{6}$$

$$\{d, \{d, d\}\} = \{d, d\}\{d, 1\}, \tag{7}$$

where $a, b, c \in \Gamma_0 \cup \Gamma_1$, and $d \in \Gamma_1$.

In particular, $J(\Gamma, D)$ is the algebra $J(\Gamma, \{\,,\})$ if

$$\{a, b\} = D(a)b - aD(b).$$

The next theorem is proved in [10].

**Theorem.** *Take a simple special unital Jordan superalgebra $J = A + M$ whose even part $A$ is an associative algebra, and whose odd part $M$ is an associative $A$-module. If $J$ is not the superalgebra of a nondegenerate bilinear superform then there exist $x_1, \ldots, x_n \in M$ such that*

$$M = x_1A + \ldots + x_nA,$$

*and the product in $M$ satisfies*

$$ax_i \cdot bx_j = \gamma_{ij}ab + D_{ij}(a)b - aD_{ji}(b), \quad i, j = 1, \ldots, n, \tag{8}$$

*where* $\gamma_{ij} \in A$*, and* $D_{ij}$ *is a derivation of* $A$*. The algebra* $A$ *is differentially simple with respect to the set of derivations* $\Delta\{D_{ij}|i,j = 1,\dots,n\}$*. The module* $M$ *is a projective* $A$*-module of rank 1. Moreover,* $J$ *is a subalgebra of the superalgebra* $J(\Gamma, D)$*.*

In addition, [10] includes an example of a Jordan superalgebra over the field of real numbers satisfying the hypotheses of the theorem which is not isomorphic to $J(\Gamma, D)$. A similar example of a Jordan superalgebra over a field of characteristic zero in which the equation $t^2+1=0$ has no solutions is constructed in [12]. Let us give another example of this kind of superalgebra over an arbitrary field of characteristic zero.

Fix an arbitrary field $F$ of characteristic 0. Consider the polynomial algebra $F[x,y]$ in two variables $x$ and $y$. Denote by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ the operators of differentiation with respect to $x$ and $y$ on $F[x,y]$. Put $D = 2y^3\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $f(x,y) = x^2+y^4-1$. Then $D$ is a derivation of $F[x,y]$, and $D(f(x,y)) = 0$. Take the quotient algebra $\Gamma = F[x,y]/f(x,y)F[x,y]$ of $F[x,y]$ by the ideal $f(x,y)F[x,y]$. It is clear that $D$ induces a derivation of $\Gamma$, which we denote by $D$ as well. Identify the images of $x$ and $y$ under the canonical homomorphism $F[x,y] \mapsto \Gamma$ with the elements $x$ and $y$. Then $\Gamma = F[y] + xF[y]$, where $F[y]$ is the polynomial ring in $y$.

**Proposition 1.** *The algebra* $\Gamma$ *is differentially simple with respect to* $D$*.*

Proof. Suppose that $I$ is a nonzero $D$-invariant ideal of $\Gamma$. If $f(y) \in F[y]$ and $f(y) \in I$ then $D(f(y)) = -xf'(y) \in I$, where $f'(y)$ is the derivative of $f(y)$ with respect to $y$. Then $(1-y^4)f'(y) \in I$ and $D((1-y^4)f'(y)) \in I$. Thus,

$$-x(-4y^3f'(y) + (1-y^4)f''(y)) \in I.$$

This implies that $(1-y^4)^2f''(y) \in I$. Continuing this process, we deduce that $(1-y^4)^kf^{(k)}(y) \in I$ for all $k$, where $f^{(k)}(y)$ is the order $k$ derivative of $f(y)$. Consequently, $(1-y^4)^k \in I$ for some $k$. Take the smallest $k$ with $z_k = (1-y^4)^k \in I$. Then

$$D(z_k) = 4kxy^3(1-y^4)^{k-1} \in I.$$

Thus,

$$x(1-y^4)^{k-1} = xz_k + \frac{1}{4k}yD(z_k) \in I.$$

Consequently,

$$D(x(1-y^4)^{k-1}) = 2y^3(1-y^4)^{k-1} + (k-1)4y^3(1-y^4)^{k-1}2(2k-1)y^3(1-y^4)^{k-1} \in I.$$

This implies that $y^3(1-y^4)^{k-1} \in I$ and $y^4(1-y^4)^{k-1} \in I$. Then,

$$z_{k-1} = (1-y^4)^k + y^4(1-y^4)^{k-1} \in I.$$

Therefore, we may assume that $F[y] \cap I = 0$.

Suppose that $f(y) + xg(y) \in I$. Then

$$(f(y)+xg(y))(f(y)-xg(y)) = f(y)^2 - (1-y^4)g(y)^2 \in I.$$

By the argument above, $f(y)^2 = (1-y^4)g(y)^2$. Then, $1-y^4 = h(y)^2$ for some $h(y) \in F[y]$, and we arrive at a contradiction.

Consequently, $\Gamma$ is a differentially simple algebra with respect to $D$. □

Consider in $\Gamma$ the subalgebra $A$ generated by 1, $y^2$, and $xy$. Then,

$$D(y^2) = -2xy \in A \text{ and } D(xy) = 3y^4 - 1 \in A.$$

Consequently, $D(A) \subseteq A$. Observe that $1, y^{2i}, xy^{2i-1}$, where $i = 1, 2, \ldots,$ constitute a linear basis for $A$. We can express every element of $A$ as $f(y) + xyg(y)$ with $f(y), g(y) \in F[y^2]$.

**Proposition 2.** *The algebra $A$ is differentially simple with respect to $D$.*

PROOF. Suppose that $I$ is a nonzero $D$-invariant ideal of $A$. If $f(y) \in F[y^2]$ and $f(y) \in I$ then $xf'(y) = -D(f(y)) \in I$. Thus, $(1-y^4)yf'(y) = (xy)(xf'(y)) \in I$. Since

$$D(xf'(y)) = 2y^3 f'(y) - (1-y^4)f''(y) \in I,$$

it follows that $(1-y^4)^2 f''(y) \in I$. An easy induction implies that

$$(1-y^4)^{2k-1} y f^{(2k-1)}(y) \in I \quad \text{and} \quad (1-y^4)^{2k} f^{(2k)}(y) \in I.$$

This yields $(1-y^4)^{2k} \in I$.

Take the smallest $k$ with $(1-y^4)^k \in I$. Then,

$$D((1-y^4)^k) = -4kxy^3(1-y^4)^{k-1} \in I.$$

Consequently,

$$xy(1-y^4)^{k-1} = xy(1-y^4)^k + y^2(xy^3(1-y^4)^{k-1}) \in I.$$

Thus,

$$D(xy(1-y^4)^{k-1}) = (3y^4-1)(1-y^4)^{k-1} + (k-1)4y^4(1-y^4)^{k-1}((4k-1)y^4-1)(1-y^4)^{k-1} \in I.$$

Then,

$$(4k-2)(1-y^4)^{k-1} = (4k-1)(1-y^4)^k + ((4k-1)y^4-1)(1-y^4)^{k-1} \in I.$$

Therefore, we may assume that $F[y^2] \cap I = 0$.

Suppose that $f(y) + xyg(y) \in I$. Then,

$$f(y)^2 - (1-y^4)y^2 g(y)^2 = (f(y) + xyg(y))(f(y) - xyg(y)) \in I.$$

By the argument above, $f(y)^2 - (1-y^4)y^2g(y)^2 = 0$, and we arrive at a contradiction since $\deg f(y)^2 = 4n$ but $\deg(1-y^4)y^2g(y)^2 = 4m+6$.

Therefore, $A$ is a differentially simple algebra with respect to $D$. □

The subspace $M = xA + yA$ of $\Gamma$ is an associative $A$-module.

**Proposition 3.** *The module $M$ is not a cyclic $A$-module.*

PROOF. Assuming the contrary, denote the generator of $M$ by $z$. Then $z = xa + yb$ with $a, b \in A$, $x = zc$, and $y = zd$ for some $c, d \in A$. This implies that

$$xd = yc, \tag{9}$$

$$x = x(ac+bd), y = y(ac+bd). \tag{10}$$

We can write

$$a = f_0 + xyf_1, b = g_0 + xyg_1, c = e_0 + xye_1, d = h_0 + xyh_1,$$

where $f_0, f_1, g_0, g_1, e_0, e_1, h_0, h_1$ are polynomials in $F[y^2]$.

From (9) we deduce that

$$h_0 = y^2 e_1 \quad \text{and} \quad e_0 = (1-y^4)h_1.$$

From (10) we deduce that

$$f_0e_0 + (1-y^4)y^2f_1e_1 + g_0h_0 + (1-y^4)y^2g_1h_1 = 1, \tag{11}$$

$$f_0e_1 + f_1e_0 + g_0h_1 + g_1h_0 = 0. \tag{12}$$

Denote by $(e_1, h_1)$ the greatest common divisor of $e_1$ and $h_1$. Since $h_0 = y^2e_1$ and $e_0 = (1-y^4)h_1$, by (11) we have

$$1 = (1-y^4)f_0h_1 + (1-y^4)y^2f_1e_1 + y^2g_0e_1 + (1-y^4)y^2g_1h_1 =$$

$$(1-y^4)(f_0 + y^2g_1)h_1 + y^2((1-y^4)f_1 + g_0)e_1.$$

Consequently, $(e_1, h_1) = 1$. By (12),

$$(f_0 + y^2g_1)e_1 + ((1-y^4)f_1 + g_0)h_1 = 0.$$

This and $(e_1, h_1) = 1$ imply that $f_0 + y^2g_1 = h_1u$, where $u \in F[y]$. Then,

$$h_1ue_1 + ((1-y^4)f_1 + g_0)h_1 = 0.$$

Thus,

$$ue_1 + ((1-y^4)f_1 + g_0) = 0.$$

By the argument above,

$$1 = (1-y^4)(f_0 + y^2g_1)h_1 + y^2((1-y^4)f_1 + g_0)e_1 = (1-y^4)h_1^2u - y^2e_1^2u.$$

Then, $u \in F$. Consequently,

$$(1-y^4)h_1^2u = 1 + y^2e_1^2u,$$

which is impossible since on the left we have a polynomial of degree $4k+4$, while on the right, of degree $4m+2$.

Therefore, $M$ is not a cyclic $A$-module. □

Put

$$D_{11} = (1-y^4)D, D_{12} = xyD, D_{22} = y^2D.$$

Then $D_{11}, D_{12}, D_{22}$ are derivations of $A$.

**Proposition 4.** *The algebra $A$ is differentially simple with respect to the set of derivations $\Delta = \{D_{11}, D_{12}, D_{22}\}$.*

PROOF. Suppose that $I$ is an ideal of $A$ closed under $\Delta$. Then $y^2D_{22}(I) \subseteq y^2I \subseteq I$. Since

$$D = D_{11} + y^2D_{22},$$

it follows that $D(I) \subseteq I$. By Proposition 2, either $I = 0$ or $I = A$. Consequently, $A$ is a differentially simple algebra with respect to $\Delta = \{D_{11}, D_{12}, D_{22}\}$. □

Consider now the superalgebra $J(\Gamma, D)$. Proposition 1 implies that $J(\Gamma, D)$ is a simple superalgebra. Consider its subspace

$$J(A, \Delta) = A + \overline{M}.$$

Recall that $A$ is the subalgebra of $\Gamma$ generated by 1, $y^2$, and $xy$, while $M = xA + yA$.

Given $a, b \in A$, in $J(\Gamma, D)$ we have

$$\overline{xa} \cdot \overline{xb} = D(xa)xb - D(xb)xa =$$

$$D(x)axb + D(a)x^2b - D(x)xab - D(b)x^2a = D_{11}(a)b - aD_{11}(b) \in A.$$

Similarly,

$$\overline{ya} \cdot \overline{yb} = D(y)ayb + D(a)y^2b - D(y)yab - D(b)y^2a = D_{22}(a)b - aD_{22}(b) \in A,$$

$$\overline{xa} \cdot \overline{yb} = D(x)ayb + D(a)xyb - D(y)xab - D(b)yxa = (1 + y^4)ab + D_{12}(a)b - aD_{12}(b) \in A.$$

Consequently, $J(A, \Delta)$ is a subsuperalgebra of $J(\Gamma, D)$. Thus, $J(A, \Delta)$ is a Jordan superalgebra. Moreover, the odd elements in $J(\Gamma, D)$ multiply according to (8), where $\Delta = \{D_{11}, D_{12}, D_{22}\}$, and $\gamma_{12} = 1 + y^4$. By Proposition 3, $J(A, \Delta)$ is not isomorphic to a superalgebra of type $J(\Gamma_0, D_0)$.

Verify that $J(A, \Delta)$ is a simple superalgebra. Suppose that $I$ is a nonzero $Z_2$-graded ideal of $J(A, \Delta)$. Then $I = I_0 + I_1$, where $I_0$ is an ideal of $A$. Given $r \in I_0$, we have

$$D_{11}(r) = \overline{(xr)} \cdot \overline{x} = (r \cdot \overline{x}) \cdot \overline{x} \in I_0.$$

Similarly, $D_{12}(r), D_{22}(r) \in I_0$. Consequently, $I_0$ is invariant under the set of derivations $\Delta$. By Proposition 4, either $I_0 = A$ or $I_0 = 0$. If $I_0 = A$ then $1 \in I_0 \subseteq I$ and $I = J(A, \Delta)$. If $I_0 = 0$ then $I \subseteq \overline{M}$ and $I \cdot \overline{M} \subseteq I_0 = 0$. It is clear that

$$A = AD_{11}(A) + AD_{12}(A) + AD_{22}(A).$$

Thus,

$$1 = \sum_i (a_{1i}, \overline{x}, \overline{x})b_{1i} + \sum_i (a_{2i}, \overline{x}, \overline{y})b_{2i} + \sum_i (a_{3i}, \overline{y}, \overline{y})b_{3i}$$

for some elements $a_{1i}$, $a_{2i}$, $a_{3i}$, $b_{1i}$, $b_{2i}$, and $b_{3i}$ of $A$. By (4) we deduce that $1 \in (A, \overline{M}, \overline{M})$ and

$$I \cdot (A, \overline{M}, \overline{M}) \subseteq (A, I \cdot \overline{M}, \overline{M}) + (A, I, \overline{M}) \cdot \overline{M} = 0.$$

Then, $I = 0$. Consequently, $J(A, \Delta)$ is a simple superalgebra.

Let us summarize the argument as

**Theorem 1.** *Take an arbitrary field $F$ of characteristic 0. Consider the polynomial algebra $F[x, y]$ in two variables $x$ and $y$. Put $f(x, y) = x^2 + y^4 - 1$ and $D = 2y^3\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Put $\Gamma = F[x, y]/f(x, y)F[x, y]$. Then the derivation $D$ induces a derivation of the algebra $\Gamma$, which we denote by $D$ as well. Identify the images of $x$ and $y$ under the canonical homomorphism $F[x, y] \mapsto \Gamma$ with the elements $x$ and $y$. Suppose that $A$ is a subalgebra of $\Gamma$ generated by 1, $y^2$, and $xy$, while $M = xA + yA$. Put*

$$\Delta = \{D_{11}, D_{12}, D_{22}\}, \text{ where } D_{11} = (1 - y^4)D, D_{12} = xyD, D_{22} = y^2D.$$

*Then the subspace $J(A, \Delta) = A + \overline{M}$ is a subsuperalgebra of $J(\Gamma, D)$, and the multiplication of odd elements in $J(A, \Delta)$ is defined as*

$$\overline{xa} \cdot \overline{xb} = D_{11}(a)b - aD_{11}(b), \quad \overline{ya} \cdot \overline{yb} = D_{22}(a)b - aD_{22}(b),$$

$$\overline{xa} \cdot \overline{yb} = (1 + y^4)ab + D_{12}(a)b - aD_{12}(b).$$

*Moreover, $J(A,\Delta)$ is a simple superalgebra, and $\overline{M}$ is not a cyclic $A$-module; i.e., $J(A,\Delta)$ is not isomorphic to a superalgebra of vector type $J(\Gamma_0, D_0)$.*

**The Superalgebra of Type $JS(\Gamma, D)$.** Take an associative supercommutative superalgebra $\Gamma = \Gamma_0 + \Gamma_1$ equipped with a nonzero odd derivation $D$; i.e., $D(\Gamma_i) \subseteq \Gamma_{(i+1) mod\, 2}$ and

$$D(ab) = D(a)b + (-1)^{p(a)} aD(b)$$

for $a, b \in \Gamma_0 \cup \Gamma_1$.

Put $A = \Gamma_1$, $M = \Gamma_0$, and $JS(\Gamma, D) = A + M$. Define on the space $JS(\Gamma, D)$ the multiplication

$$a \circ b = aD(b) + (-1)^{p(a)} D(a)b.$$

Then $JS(\Gamma, D)$ is a Jordan superalgebra. If $JS(\Gamma, D)$ is a simple superalgebra then $\Gamma$ is a differentially simple superalgebra (see [8]).

**Proposition 5.** *The superalgebra $JS(\Gamma, D)$ is not unital.*

Proof. Suppose that $e$ is the unit of $JS(\Gamma, D)$. Then $e \in A \subseteq \Gamma_1$. Given $a \in JS(\Gamma, D)$, we have

$$a = e \circ a = eD(a) + D(e)a.$$

Since $\Gamma$ is supercommutative and $e \in \Gamma_1$, it follows that $e = 2eD(e)$ and $e^2 = 0$ in $\Gamma$. Consequently, $ea = eD(e)a = \frac{1}{2}ea$. This implies that $e\Gamma = 0$. Then, $e = 2eD(e) = 0$. □

**Corollary 1.** *The superalgebra $J(A, \Delta)$ is not isomorphic to the superalgebra $JS(\Gamma, D)$.*

**The Cheng–Kac superalgebra.** Take an associative commutative $F$-algebra $\Gamma$ equipped with a nonzero derivation $D$. Consider two direct sums

$$J_0 = \Gamma + w_1\Gamma + w_2\Gamma + w_3\Gamma$$

and

$$J_1 = \overline{\Gamma} + x_1\overline{\Gamma} + x_2\overline{\Gamma} + x_3\overline{\Gamma}$$

of linear spaces, where $\overline{\Gamma}$ is an isomorphic copy of $\Gamma$.

For $a, b \in \Gamma$ define a multiplication on the space $J_0$ by putting

$$a \cdot b = ab,\ a \cdot w_i b = w_i ab,\ w_1 a \cdot w_1 b = w_2 a \cdot w_2 b = ab,\ w_3 a \cdot w_3 b = -ab,$$

$$w_i a \cdot w_j b = 0 \text{ for } i \neq j.$$

Put $x_{i\times i} = 0$, $x_{1\times 2} = -x_{2\times 1} = x_3$, $x_{1\times 3} = -x_{3\times 1} = x_2$, and $x_{2\times 3} = -x_{3\times 2} = -x_1$. Define a bimodule action $J_0 \times J_1 \mapsto J_1$ by putting

$$a \cdot \overline{b} = \overline{ab},\ a \cdot x_i\overline{b} = x_i\overline{ab},\ w_i a \cdot \overline{b} = x_i\overline{D(a)b},\ w_i a \cdot x_j\overline{b} = x_{i\times j}\overline{ab}.$$

The bracket on $J_1$ is defined as

$$\overline{a} \cdot \overline{b} = D(a)b - aD(b),\ \overline{a} \cdot x_i\overline{b} = -w_i(ab),\ x_i\overline{a} \cdot \overline{b} = w_i(ab),\ x_i\overline{a} \cdot x_j\overline{b} = 0.$$

Then the space $J = J_0 + J_1$ with the multiplication

$$(a_0 + a_1) \cdot (b_0 + b_1) = (a_0 \cdot b_0 + a_1 \cdot b_1) + (a_0 \cdot b_1 + a_1 \cdot b_0)$$

for $a_0, b_0 \in J_0$ and $a_1, b_1 \in J_1$ is an algebra, which is denoted by $CK(\Gamma, D)$. It is known (see [5, 8]) that $CK(\Gamma, D)$ is a Jordan superalgebra, which is simple if and only if $\Gamma$ is $D$-simple.

Suppose now that $\Gamma = F[x,y]/f(x,y)F[x,y]$, where $f(x,y) = x^2 + y^4 - 1$ and $D = 2y^3\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Consider the Jordan superalgebra $J(A,\Delta) = A + \overline{M}$ constructed above. In $CK(\Gamma, D)$ consider the subspace

$$GCK(A,\Delta) = A + w_1A + w_2A + w_3A + \overline{M} + x_1\overline{M} + x_2\overline{M} + x_3\overline{M}.$$

In $\Gamma$ we have $M^2 \subseteq A$. Thus, $GCK(A,\Delta)$ is a subsuperalgebra of $CK(\Gamma, D)$. Consequently, $GCK(A,\Delta)$ is a Jordan superalgebra with the even part $GCK(A,\Delta)_0 = A + w_1A + w_2A + w_3A$ and the odd part $GCK(A,\Delta)_1 = \overline{M} + x_1\overline{M} + x_2\overline{M} + x_3\overline{M}$.

**Theorem 2.** *For an arbitrary field $F$ of characteristic zero $GCK(A,\Delta)$ is a simple unital Jordan superalgebra.*

Proof. Suppose that $I = I_0 + I_1$ is a nonzero ideal of $GCK(A,\Delta)$. Then $K = A \cap I_0$ is an ideal of $A$, and $(K, \overline{M}, \overline{M}) \subseteq K$. Thus, $K + K \cdot \overline{M}$ is an ideal of $J(A,\Delta)$. If $K \neq 0$ then since $J(A,\Delta)$ is a simple superalgebra, we have $1 \in K$. Consequently, $I = GCK(A,\Delta)$.

Suppose that $A \cap I_0 = 0$ and take $r = a + w_1a_1 + w_2a_2 + w_3a_3 \in I_0$. Then $w_2(w_2(w_1r)) = a_1 \in A \cap I_0$. Consequently, $a_1 = 0$. Similarly, $a_2 = a_3 = 0$. Thus, $I_0 = 0$. This implies that $I \subseteq GCK(A,\Delta)_1$ and $I \cdot GCK(A,\Delta)_1 \subseteq I_0 = 0$. Since $1 \in (A, \overline{M}, \overline{M})$, by (4) we deduce that

$$I \cdot (A, \overline{M}, \overline{M}) \subseteq (A, I \cdot \overline{M}, \overline{M}) + (A, I, \overline{M}) \cdot \overline{M} = 0.$$

Then, $I = 0$. Consequently, $GCK(A,\Delta)$ is a simple superalgebra. □

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