REAL NON-UNITAL DIVISION ALGEBRAS WITH $su(3)$ AS DERIVATION ALGEBRA

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Abstract. We obtain a family of non-unital eight-dimensional division algebras over a field F out of a separable quadratic field extension S of F, a three-dimensional anisotropic hermitian form over S of determinant one and three invertible elements $c, d, e \in S$. The algebras always contain a four-dimensional subalgebra which can be viewed as a generalization of a (nonassociative) quaternion algebra and are studied independently.

Over $\mathbb{R}$, this construction can be used to yield division algebras with derivation algebra isomorphic to $su(3)$, which are the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module. Albert isotopes with derivation algebra isomorphic to $su(3)$ are considered briefly.

Introduction

In Benkart's and Osborn's classification of real division algebras according to the isomorphism type of their derivation algebra [B-O1], one possible case which appears is that the Lie algebra of derivations of an eight-dimensional real division algebra A is isomorphic to $su(3)$. A is either an eight-dimensional irreducible $su(3)$-module or the direct sum of two one-dimensional modules and a six-dimensional irreducible $su(3)$-module [B-O2]. If A is an eight-dimensional irreducible $su(3)$-module, A was shown to be a flexible generalized pseudo-octonion algebra.

For an eight-dimensional real division algebra A with $\mathrm{Der}(A) \cong su(3)$ which is reducible as $su(3)$-module, a multiplication table with 16 different scalars was given [B-O2, (4.2)]. Every real algebra defined by this table admits $su(3)$ as derivation algebra [B-O2, Theorem 4.1]. One family of division algebras which fit into the table was presented as an example [B-O1, Theorem 20, Corollary 21]. Another family was discussed in [J-P], Section 4.3. Dokovich and Zhao [Do-Z2] investigated when an algebra with such a multiplication table is a real division algebra. Some conditions were obtained for enlargements of what the authors call the truncated algebra of strictly pure octonions [Do-Z1]. In [P-I], Pérez-Izquierdo exhibited families of composition division algebras with derivation algebras isomorphic to $su(3)$ over a field of characteristic not 2 or 3. Unital real division algebra with derivation algebras isomorphic to $su(3)$ were constructed in [Pu3].

In this paper, we generalize the method developed in [Pu3] to construct non-unital real eight-dimensional division algebras with derivation algebras isomorphic to $su(3)$. This is done in Section 3.

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These algebras contain four-dimensional subalgebras which can be viewed as generalizations of quaternion or nonassociative quaternion algebras. After the preliminaries in Section 1 we start by studying these four-dimensional algebras in Section 2. The description of all algebras is straightforward and base free and given over an arbitrary base field. In Section 4, we obtain some results on the automorphisms of our algebras. Over the reals, we thus obtain new eight-dimensional division algebras with derivation algebra isomorphic to $su(3)$, whose multiplication fits into table [B-O2, (4.2)], see Section 5. All our real eight-dimensional division algebras contain $\mathbb{C}$ and a four-dimensional subalgebra which corresponds to the subalgebra mentioned in [Do-Z2, Proposition 4.1]. By a procedure which is called 'strictly truncating' in [Do-Z1, 2], we obtain the real algebra $(\mathbb{C}^3,\times)$ with $\times$ a vector product. Our algebras are enlargements of $(\mathbb{C}^3,\times)$ in the sense of [Do-Z1, Definition 4.1] if and only if the elements c,d,e used in their constructions all lie in $\mathbb{R}$.

We then use our family of division algebras to construct Albert isotopes in Section 6, which over $F=\mathbb{R}$ also satisfy the multiplication table [B-O2, (4.2)], hence have a derivation algebra again isomorphic to $su(3)$. We conclude with an outlook how to construct more eight-dimensional division algebras in Section 7.

1. Preliminaries

1.1. **Nonassociative algebras.** Let F be a field. By "F-algebra" we mean a finite dimensional unital nonassociative algebra over F.

A nonassociative algebra $A\neq 0$ is called a *division algebra* if for any $a\in A$, $a\neq 0$, the left multiplication with a, $L_a(x)=ax$, and the right multiplication with a, $R_a(x)=xa$, are bijective. A is a division algebra if and only if A has no zero divisors [Sch, pp. 15, 16].

An anti-automorphism $\sigma: A\to A$ of period 2 is called an *involution* on A. An involution is called *scalar* if all *norms* $\sigma(x)x$ are elements of $F1$. For every scalar involution σ, $N_A(x)=\sigma(x)x$ (resp. the *trace* $T_A(x)=\sigma(x)+x$) is a quadratic (resp. a linear) form on A. A is called *quadratic*, if there exists a quadratic form $N\colon A\to F$ such that $x^2-N(1_A,x)x+N(x)1_A=0$ for all $x\in A$, where $N(x,y)=N(x+y)-N(x)-N(y)$ denotes the symmetric bilinear form induced by N. This automatically implies that $N(1_A)=1$. The form N is uniquely determined and called the *norm* $N=N_A$ of the quadratic algebra A [Pu2]. The existence of a scalar involution on an algebra A implies that A is quadratic [M].

A *quadratic étale* algebra S over F is a separable quadratic F-algebra in the sense of [Knu, p. 4] with canonical involution $\bar{\ }\colon S\to S$ and with nondegenerate norm $N_{S/F}\colon S\to F$, $N_{S/F}(s)=s\bar{s}=\bar{s}s$. That means S is a two-dimensional unital commutative associative algebra over F. With the diagonal action of F, $F\times F$ is a quadratic étale algebra with canonical involution $(x,y)\mapsto(y,x)$.

An F-algebra C is called a *unital composition algebra* or a *Hurwitz algebra* if it has a unit element and carries a quadratic form $N\colon C\to F$ whose induced symmetric bilinear form is nondegenerate and which permits composition in the sense that $N(xy)=N(x)N(y)$ for all $x,y\in C$. Hurwitz algebras are quadratic alternative; the norm of a Hurwitz algebra C is the unique nondegenerate quadratic form on C that permits composition A quadratic alternative algebra is a Hurwitz algebra if and only if its norm is nondegenerate [M, 4.6].

Hurwitz algebras exist only in dimensions 1, 2, 4 or 8. Those of dimension 2 are exactly the quadratic étale F-algebras, those of dimension 4 exactly the well-known quaternion algebras. The ones of dimension 8 are called *octonion algebras*. The conjugation $\overline{x} = T_{C/F}(x)1_C - x$ of a Hurwitz algebra C is a scalar involution, called the *canonical involution* of C, where $T_C\colon C \to F$, $T_{C/F}(x) = N_{C/F}(1_C, x)$, is the *trace* of C.

1.2. **Nonassociative quaternion algebras.** A *nonassociative quaternion algebra* is a four-dimensional unital F-algebra A whose nucleus is a quadratic étale algebra over F. Let S be a quadratic étale algebra over F with canonical involution $^-$. For every invertible $b \in S \setminus F$, the vector space

$$\mathrm{Cay}(S, b) = S \oplus S$$

becomes a nonassociative quaternion algebra over F with unit element $(1, 0)$ and nucleus S under the multiplication

$$(u, v)(u', v') = (uu' + b\overline{v}'v, v'u + v\overline{u}')$$

for $u, u', v, v' \in S$. Given any nonassociative quaternion algebra A over F with nucleus S, there exists an invertible element $b \in S \setminus F$ such that $A \cong \mathrm{Cay}(S, b)$ [As-Pu, Lemma 1]. $\mathrm{Cay}(S, b)$ is a division algebra if and only if S is a separable quadratic field extension of F [W, p. 369]. Two nonassociative quaternion division algebras $\mathrm{Cay}(K, b)$ and $\mathrm{Cay}(L, c)$ can only be isomorphic if $L \cong K$. Moreover,

$$\mathrm{Cay}(K, b) \cong \mathrm{Cay}(K, c) \text{ if and only if } g(b) = N_{K/F}(d)c$$

for some automorphism $g \in \mathrm{Aut}(K)$ and some non-zero $d \in K$ [W, Theorem 2].

1.3. **A construction method for octonion algebras.** (cf. Petersson-Racine [P-R, 3.8] or Thakur [T])

Let S be a quadratic étale F-algebra with canonical involution $^-$. Let (P, h) be a ternary nondegenerate $^-$-hermitian space (P a free projective S-module) such that $\bigwedge^3(P, h) \cong \langle 1\rangle$. Choose an isomorphism $\alpha : \bigwedge^3(P, h) \to \langle 1\rangle$ and define a cross product $\times_\alpha : P \times P \to P$ via

$$h(u \times_\alpha v, w) = \alpha(u \wedge v \wedge w).$$

The F-vector space $\mathrm{Cay}(S, P, h, \times_\alpha) = S \oplus P$ becomes an octonion algebra under the multiplication

$$(a, u)(b, v) = (ab - h(v, u), va + u\bar{b} + u \times_\alpha v)$$

for all $u, v \in P$ and $a, b \in S$, with norm

$$N((a, u)) = n_S(a) + h(u, u).$$

This construction is independent of the choice of the isomorphism α and we may simply write $\mathrm{Cay}(S, P, h)$. Any octonion algebra over F can be constructed like this. For $h = \langle e\rangle \perp h_2$ and $D = \mathrm{Cay}(S, -e)$,

$$\mathrm{Cay}(S, P, \langle e\rangle \perp h_2) \cong \mathrm{Cay}(D, -q_{h_2})$$

with $q_{h_2}(x) = h_2(x, x)$ for all $x \in P_2$.

2. Generalizations of (nonassociative) quaternion algebras

From now on let S be a quadratic étale algebra over F with canonical involution $^-$.

For every $c, d, e \in S^\times$, the vector space

$$\mathrm{Cay}(S,(c,d,e)) = S \oplus S$$

becomes an algebra over F via the multiplication

$$(u,v)(u',v') = (uu' + c\overline{v}'v, v'ud + v\overline{u}'e)$$

for $u, u', v, v' \in S$. S is a subalgebra of $\mathrm{Cay}(S,(c,d,e))$ via the embedding $u \to (u,0)$. We identify $S = (S,0)$.

Lemma 1. *Let $A = \mathrm{Cay}(S,(c,d,e))$.*
(i) If $cd \neq \bar{c}e$ then A is not third power-associative and thus in particular not quadratic.
(ii) A has $(1,0)$ as a left-unit element if and only if $d = 1$, as a right-unit element if and only if $e = 1$ and as unit element if and only if $d = e = 1$.
(iii) For $l = (0,1)$ we have $lx = \bar{x}l$ for all $x = (u,0) \in S$ if and only if $e = d$.

Proof. (i) For $l = (0,1)$ we have $l^2 = (c,0)$ and $ll^2 = (0,\sigma(c)e)$ while $l^2l = (0,cd)$, so $ll^2 = l^2l$ if and only if $cd = \bar{c}e$. Thus A is not third power-associative if $cd \neq \bar{c}e$. Every quadratic unital algebra is clearly power-associative, so A is not quadratic in that case.
(ii), (iii) are trivial. □

Note that $\mathrm{Cay}(S,(c,d,e))$ is a (perhaps nonassociative) quaternion algebra if and only if $c \in S$ and $e = d = 1$.

Theorem 2. $\mathrm{Cay}(S,(c,d,e))$ *is a division algebra if and only if*

$$\frac{\bar{c}e}{d} \notin N_{S/F}(S^\times).$$

Proof. Suppose

$$(0,0) = (u,v)(u',v') = (uu' + c\overline{v}'v, v'ud + v\overline{u}'e)$$

for $u, u', v, v' \in S$. This is equivalent to

$$uu' + c\overline{v}'v = 0, \quad v'ud + v\overline{u}'e = 0.$$

If $v = 0$ it immediately follows that either $(u,v) = (0,0)$ or $(u',v') = (0,0)$. So let $v \neq 0$. The second equation yields $\bar{u}' = -v^{-1}v'ude^{-1}$, therefore

$$u' = -\bar{e}^{-1}\bar{d}\bar{u}\bar{v}'\frac{1}{N_{S/F}(v)}v.$$

This together with the first equation implies

$$-u\bar{e}^{-1}\bar{d}\bar{u}\bar{v}'\frac{1}{N_{S/F}(v)}v + c\overline{v}'v = 0,$$

so

$$(-u\bar{e}^{-1}\bar{d}\bar{u}\frac{1}{N_{S/F}(v)} + c)\overline{v}'v = 0.$$

If $v' = 0$ then $v\bar{u}'e = 0$ yields $u' = 0$. So suppose $v' \neq 0$. Then

$$c = u\bar{e}^{-1}\bar{d}\bar{u}\frac{1}{N_{S/F}(v)} = \frac{N_{S/F}(u)}{N_{S/F}(v)}\bar{e}^{-1}\bar{d}.$$

If $u=0$ then $c=0$, a contradiction, so we need $u\neq 0$ and get $\bar{e}c/\bar{d}\in N_{S/F}(S^\times)$. This implies the assertion. $\square$

Corollary 3. *Let $F=\mathbb{R}$. Then $\mathrm{Cay}(\mathbb{C},(c,d,e))$ is a division algebra if and only if*

$$\frac{\bar{c}e}{d}\in\mathbb{R}_{<0} \text{ or } \frac{\bar{c}e}{d}\in\mathbb{C}\setminus\mathbb{R}.$$

Remark 4. Let us denote the multiplication in S by $\cdot$. Then the algebra $\mathrm{Cay}(S,(c,1,-c))$ is a fused algebra in the sense of [A-H-K], obtained out of the two algebras $(S,\cdot)$ and $(S,\star)$ with $u\star v=-cu\bar{v}$. It is also a φ-algebra S^φ in the sense of [A-H-K] with $\varphi(u)=-c\bar{u}$. By [A-H-K, Corollary 4] this means that for $F=\mathbb{R}$, it is a division algebra for all $c\in\mathbb{C}^\times$ which also follows from the statement in Corollary 3.

Lemma 5. *Let F be a field of characteristic not 2 and $A=\mathrm{Cay}(S,(c,d,e))$. The F-linear map*

$$D_0((u,v))=(0,sv)$$

with $s\in S$ is a derivation of A if and only if $\sigma(s)=-s$. In particular, then $F\hookrightarrow\mathrm{Der}(A)$.

The proof is a straightforward calculation.

The well-known cases here are $d=e=1$ and $c\in F^\times$ in which case $\mathrm{Der}(A)\cong su(2)$, and $d=e=1$ and $c\in S\setminus F$ in which case $\mathrm{Der}(A)\cong F$. For $F=\mathbb{R}$, Lemma 5 implies that $\mathrm{Der}(A)\cong\mathbb{R}$ or $\mathrm{Der}(A)\cong su(2)$ using the classification [B-O1], depending on the choice of $c,d,e\in\mathbb{C}$. It is easy to see that A is an enlargement of $(\mathbb{H}_1,\eta_{\mathbb{R}})$ in the sense of [Do-Z1, 4.1] if and only if $A=\mathbb{H}$.

Theorem 6. *Let $A=\mathrm{Cay}(S,(c,d,e))$ with S a separable quadratic field extension. Suppose that K is a separable quadratic field extension of F contained in A.*
(i) Let F have characteristic not 2 and $K=F(\sqrt{\alpha})$. Then $K=S$ or there are $u\in S$, $v\in S^\times$, such that

$$\alpha=-\frac{e}{d}N_{S/F}(u)+cN_{S/F}(v).$$

(ii) Let F have characteristic 2 and $K=F(x)$ with $x^2+x=\alpha\in F$. Then $K=S$ or there are $u,v\in S^\times$ such that

$$N_{S/F}(u)e-cdN_{S/F}(v)+u+\bar{u}e+1=\alpha d.$$

Proof. (i) Let F have characteristic not 2 and suppose that $K=F(\sqrt{\alpha})$ in A. Then there is an element $X\in A$, $X=(u,v)$ with $u,v\in S$, such that $X^2=\alpha\in F^\times$ which implies

$$u^2+cN_{S/F}(v)=\alpha \text{ and } v(ud+\bar{u}e)=0.$$

If $v=0$, then $u^2=\alpha$ and $X=(u,0)\in S$, thus $K=S$.
If $v\neq 0$ then v is invertible and $u=-\frac{e}{d}\bar{u}$. This implies $-\frac{e}{d}N_{S/F}(u)+cN_{S/F}(v)=\alpha$.
(ii) Let F have characteristic 2. Suppose there is a separable quadratic field extension K in A. Then there is an element $X\in A$, $X=(u,v)$ with $u,v\in S$ such that $X^2+X=\alpha\in F$. This implies

$$u^2+cN_{S/F}(v)+u=\alpha \text{ and } v(ud+\bar{u}e+1)=0.$$

If $v=0$, then $u^2+u=\alpha$ and $X=(u,0)\in S$, thus $K=S$.
If $v\neq 0$ then v is invertible and $ud+\bar{u}e+1=0$, so in particular also $u\neq 0$. This implies $u=-\frac{1}{d}(\bar{u}e+1)$, thus $-\frac{1}{d}(N_{S/F}(u)e+u)+cN_{S/F}(v)-\frac{1}{d}(\bar{u}e+1)=\alpha$ i.e.

$$N_{S/F}(u)e+u-cdN_{S/F}(v)+\bar{u}e+1=\alpha d.$$

□

Corollary 7. *Let F have characteristic not 2 and $A=\mathrm{Cay}(S,(c,d,e))$ with S a separable quadratic field extension.*
(i) If $\frac{e}{d}\in F^\times$ and $c\in S\setminus F$, then $S=(S,0)$ is the only separable quadratic subfield of $\mathrm{Cay}(S,(c,d,e))$.
(ii) Let $\frac{e}{d}\in S\setminus F$ and $c\in F^\times$, let $c,e\in F^\times$ and $d\in S\setminus F$, or let $c,d\in F^\times$ and $e\in S\setminus F$. If K is a separable quadratic field extension in A, then $K=S$ or $K=F(\sqrt{cN_{S/F}(v)})$ for some $v\in S^\times$.
(iii) If $e\in F^\times$, $c,d\in S\setminus F$ and K a separable quadratic field extension in $\mathrm{Cay}(S,(c,d,e))$, then $K=S$ or $K=F(\sqrt{cdN_{S/F}(v)})$ for some $v\in S^\times$.

Proof. Suppose $K=F(\sqrt{\alpha})$.
(i) If $\frac{e}{d}\in F^\times$ and $c\in S\setminus F$, then $\alpha+\frac{e}{d}N_{S/F}(u)\neq cN_{S/F}(v)$ for all $\alpha\in F^\times$ and $u,v\in S$, $v\neq 0$, since the left-hand side lies in F and the right-hand side in $S\setminus F$.
If $\frac{e}{d}\in S\setminus F$, $c\in F^\times$ then $K=S$ or there are $u,v\in S$, $v\neq 0$, such that $\alpha-cN_{S/F}(v)=-\frac{e}{d}N_{S/F}(u)$. The left-hand side lies in F, the right-hand side in S, hence $N_{S/F}(u)=0$ which means $u=0$ and so $\alpha=cN_{S/F}(v)$ and $X=(0,v)\in(0,S)\subset\mathrm{Cay}(S,(c,d,e))$.
(ii) If $c,e\in F^\times$, $d\in S\setminus F$ then $K=S$ or there are $u,v\in S$, $v\neq 0$, such that $\frac{1}{e}(\alpha-cN_{S/F}(v))=-\frac{1}{d}N_{S/F}(u)$. The left-hand side lies in F, the right-hand side in S, hence $N_{S/F}(u)=0$ which means $u=0$ and so again $\alpha=cN_{S/F}(v)$ and $X=(0,v)\in(0,S)\subset\mathrm{Cay}(S,(c,d,e))$.
The last assertion is proved analogously.
(iii) If $e\in F^\times$, $c,d\in S\setminus F$ then $K=S$ or there are $u,v\in S$, $v\neq 0$, such that $cdN_{S/F}(v))-\alpha d=-eN_{S/F}(u)$. The left-hand side lies in F, the right-hand side in S, hence $N_{S/F}(u)=0$ which means $u=0$ and so $\alpha=cdN_{S/F}(v)$ and $X=(0,v)\in(0,S)\subset\mathrm{Cay}(S,(c,d,e))$. □

Lemma 8. *(a) Let $s\in S^\times$.*
(i) $\mathrm{Cay}(S,(c,d,e))\cong\mathrm{Cay}(S,(N_{S/F}(s^{-1})c,d,e))$ via $(u,v)\to(u,sv)$.
(ii) $\mathrm{Cay}(S,(c,d,e))\cong\mathrm{Cay}(S,(N_{S/F}(s^{-1})\bar{c},\bar{d},\bar{e}))$ via $(u,v)\to(\bar{u},s\bar{v})$.
(b) Let S,S' be two separable quadratic field extensions. Let $G:\mathrm{Cay}(S,(c,d,e))\to\mathrm{Cay}(S',(c',d',e'))$ be an isomorphism. Suppose that $S=(S,0)$ and $S'=(S',0)$ are the only separable quadratic field extensions contained in $\mathrm{Cay}(S,(c,d,e))$, respectively $\mathrm{Cay}(S',(c',d',e'))$. Then $S\cong S'$.

The proof of (a) is a straightforward calculation, the one of (b) obvious.

Theorem 9. *Let S be a separable quadratic field extension. Let $G:\mathrm{Cay}(S,(c,d,e))\to\mathrm{Cay}(S,(c',d',e'))$ be an isomorphism. Suppose that $G((S,0))=(S,0)$. Then either $d=d'$, $e=e'$ and there is $y\in S^\times$ such that $c'=N_{S/F}(y^{-1})c$ and*

$$G=id_S\oplus y\cdot id_S,$$

or $\bar{d} = d'$ *and* $\bar{e} = e'$ *and there is* $y \in S^\times$ *such that* $\bar{c}' = N_{S/F}(y^{-1})c$ *and*

$$G = {}^- \oplus y \cdot {}^-.$$

Proof. By assumption, $G((S,0)) = (S,0)$ and so $G((u,0)) = (u,0)$ or $G((u,0)) = (\bar{u},0)$. Let $G((0,1)) = (x,y)$ with $x, y \in S$. Then $G((c,0)) = G((0,1))G((0,1)) = (x^2 + c'\bar{y}y, yxd' + y\bar{x}e')$ implies that

$$c = x^2 + c'\bar{y}y \text{ and } yxd' + y\bar{x}e' = 0$$

in case $G((u,0)) = (u,0)$ and

$$\bar{c} = x^2 + c'\bar{y}y \text{ and } yxd' + y\bar{x}e' = 0$$

in case $G((u,0)) = (\bar{u},0)$. In both cases this means either $y = 0$ or $xd' + \bar{x}e' = 0$. However, if $y = 0$ then $x \neq 0$ and G is not injective since $G((0,1)) = (x,0) = G((x,0))$ (resp., $G((0,1)) = (x,0) = G((\bar{x},0))$), so $y \neq 0$ and $xd' + \bar{x}e' = 0$.
Assume first that $G((u,0)) = (u,0)$ for all $u \in S$. Since G is multiplicative, we have

$$G((0,v)) = G((v\frac{1}{d},0)(0,1)) = G((v\frac{1}{d},0))G((0,1)) = (v\frac{1}{d},0)(x,y) = (\frac{1}{d}vx, yv\frac{d'}{d}),$$

therefore

$$(x,y) = G((0,1)) = (\frac{1}{d}x, y\frac{d'}{d})$$

means $d' = d$ and $x = \frac{1}{d}x$, so that either $d = 1$ or $x = 0$. Moreover, using $d' = d$ we get

$$(\frac{1}{d}x, ye') = (\frac{1}{d}x, y)(1,0) = G((0,1))G((1,0)) = G((0,1)(1,0)) = G((0,e)) = (\frac{1}{d}ex, ye)$$

hence $e' = e$ and $\frac{1}{d}x = \frac{1}{d}ex$, so that either $e = 1$ or $x = 0$. A straightforward calculation now shows that the fact that $xd + \bar{x}e = 0$ from above, together with the second entry of the equation $G((u,dv)(u',dv')) = G((u,dv))G((u',dv'))$ for $v' = -1 = -v$, which gives $xd - \bar{x}e = 0$, implies that $x = 0$. Moreover, $c' = N(y^{-1})c$. Thus $G((0,v)) = G((v\frac{1}{d},0))G((0,1)) = (v\frac{1}{d},0)(0,y) = (0,yv)$ and so

$$G((u,v)) = G((u,0)) + G((0,v)) = (u, yv).$$

Assume next that $G((u,0)) = (\bar{u},0)$ for all $u \in S$. The multiplicativity of G yields

$$G((0,v)) = G((v\frac{1}{d},0)(0,1)) = G((v\frac{1}{d},0))G((0,1)) = (\bar{v}\frac{1}{\bar{d}},0)(x,y) = (\frac{1}{\bar{d}}\bar{v}x, y\bar{v}\frac{d'}{\bar{d}}),$$

therefore

$$(x,y) = G((0,1)) = (\frac{1}{\bar{d}}x, y\frac{d'}{\bar{d}})$$

means $d' = \bar{d}$ and $x = \frac{1}{d}x$, so that either $d = 1$ or $x = 0$. Moreover, using $d' = \bar{d}$ we get

$$(\frac{1}{\bar{d}}x, ye') = (\frac{1}{\bar{d}}x, y)(1,0) = G((0,1))G((1,0)) = G((0,1)(1,0)) = G((0,e)) = (\frac{1}{\bar{d}}\bar{e}x, y\bar{e})$$

hence $e' = \bar{e}$ and $\frac{1}{d}x = \frac{1}{d}\bar{e}x$, so that either $e = 1$ or $x = 0$. A straightforward calculation now shows that the fact that $xd' + \bar{x}e' = 0$ from above, together with the second entry of the equation $G((u,v)(u',v')) = G((u,v))G((u',v'))$ again implies that $x = 0$. Moreover, $\bar{c}' = N(y^{-1})c$. Thus

$$G((u,v)) = G((u,0)) + G((0,v)) = (\bar{u}, y\bar{v}).$$

$\square$

Corollary 10. *Let S be a separable quadratic field extension. Suppose that $S = (S, 0)$ is the only separable quadratic field extensions contained in* $\mathrm{Cay}(S,(c,d,e))$ *and* $\mathrm{Cay}(S,(c',d',e'))$.
(i) $\mathrm{Cay}(S,(c,d,e)) \cong \mathrm{Cay}(S,(c',d',e'))$ *if and only if $d = d'$, $e = e'$ and there is $y \in S^\times$ such that $c' = N_{S/F}(y^{-1})c$ or $\bar{d} = d'$ and $\bar{e} = e'$ and there is $y \in S^\times$ such that $\bar{c}' = N_{S/F}(y^{-1})c$. Every $y \in S^\times$ yields a unique isomorphism $G = id_S \oplus y \cdot id_S$ or $G = {}^- \oplus y \cdot {}^-$, respectively.*
(ii) The maps $G = id_S \oplus y \cdot id_S$ for $y \in S^\times$ with $N_{S/F}(y) = 1$ are automorphisms of $\mathrm{Cay}(S,(c,d,e))$. *They are the only ones except for the case where $\bar{c} = -c$, $-1 = N_{S/F}(y)$ for some $y \in S^\times$ and $d, e \in F^\times$. In that case also the maps $G = {}^- \oplus y \cdot {}^-$ are automorphisms of* $\mathrm{Aut}(\mathrm{Cay}(S,(c,d,e)))$.

Proof. Since $S = (S, 0)$ is the only separable quadratic field extensions contained in $\mathrm{Cay}(S,(c,d,e))$, respectively $\mathrm{Cay}(S,(c',d',e'))$, we have $G((S,0)) = (S,0)$ for every isomorphism $G : \mathrm{Cay}(S,(c,d,e)) \to \mathrm{Cay}(S,(c',d',e'))$. The rest of the assertion is clear now by Theorem 9. $\square$

Example 11. Suppose $F = \mathbb{R}$ and $(\mathbb{C}, 0)$ is the only quadratic field extension contained in the algebra $\mathrm{Cay}(\mathbb{C},(c,d,e))$. Then $\mathrm{Aut}(\mathrm{Cay}(\mathbb{C},(c,d,e))) \cong \mathbb{R}$ and $\mathrm{Der}(\mathrm{Cay}(\mathbb{C},(c,d,e))) \cong \mathbb{R}$.

3. Eight-dimensional algebras

From now on let S be a quadratic étale algebra over F with canonical involution $^-$ and (P, h) a ternary nondegenerate $^-$-hermitian space such that $\bigwedge^3(P,h) \cong \langle 1 \rangle$ as in Section 1.3. Choose an isomorphism $\alpha : \bigwedge^3(P,h) \to \langle 1 \rangle$ and define a cross product $\times_\alpha : P \times P \to P$ via

$$h(u \times_\alpha v, w) = \alpha(u \wedge v \wedge w).$$

For $c, d, e \in S^\times$ consider the eight-dimensional algebra

$$A = \mathrm{Cay}(S, P, ch, d, e, \times_\alpha) = S \oplus P$$

with multiplication

$$(a, u)(b, v) = (ab - ch(v, u), vad + u\bar{b}e + u \times_\alpha v)$$

for all $a, b \in S$, $u, v \in P$. S is a subalgebra of $\mathrm{Cay}(S, P, ch, d, e, \times_\alpha)$ via the embedding $a \to (a, 0)$. We observe that

$$(1,0)(b,v) = (b, vd) \text{ and } (a,u)(1,0) = (a, ue)$$

implying that A is unital if and only if $d = e = 1$. The unitary case was treated in [Pu3].

In the terminology of [Do-Z1, 2], $(P, \times_\alpha)$ is the strictly truncated (anticommutative) algebra (P, μ_S) obtained from $\mathrm{Cay}(S, P, ch, d, e, \times_\alpha)$. Moreover, $SP \subset P$ and $PS \subset P$.

Let $A = \mathrm{Cay}(S, P, ch, d, e, \times)$.

Lemma 12. *(i) Let $h(u,u) \neq 0$ and $u\bar{c}e \neq ucd$ for some $u \in P$, then A is not third power-associative. In particular, if S is a separable quadratic field extension, $h(u,u) \neq 0$ for some $u \in P$ and $\bar{c}e \neq cd$, then A is not third power-associative.*
(ii) If $h = \langle f \rangle \perp b$ then $\mathrm{Cay}(D, (-cf, d, e))$ *is a subalgebra of A.*

Proof. (i) For all $u \in P$ we have $(0,u)^2 = (-ch(u,u),0)$ and so $(0,u)(0,u)^2 = (0,-uh(u,u)\bar{c}e)$ while $(0,u)^2(0,u) = (0,-uch(u,u)d)$. Thus $(0,u)^2(0,u) = (0,u)(0,u)^2$ if and only if $uh(u,u)\bar{c}e = uch(u,u)d$ and assuming $h(u,u) \neq 0$ this is equivalent to $u\bar{c}e = ucd$. If P is torsion free, this is the same as $\bar{c}e = cd$. Hence A is not third power-associative. Every quadratic unital algebra is clearly power-associative.
(ii) is trivial. □

That means $\mathrm{Cay}(S,(-cf,d,e)), \mathrm{Cay}(S,(-cg,d,e))$ and $\mathrm{Cay}(S,(-cfg,d,e))$ are subalgebras of $\mathrm{Cay}(S,S^3,c\langle f,g,fg\rangle,\times)$.

Theorem 13. *(i) Let F have characteristic not 2, S be a separable quadratic field extension and h an anisotropic hermitian form. Suppose that*

$$-N(a)e \neq (ch(u,u)+\alpha)d$$

for all $\alpha \in F^\times$ which are no square in F, all $a \in S^\times$ and all $u \in P$, $u \neq 0$. Then S is the only separable quadratic field extension contained in A.
(ii) Let F have characteristic 2. If

$$\bar{a}e + ad \neq -1$$

for all $a \in S^\times$, then $S = (S,0)$ is the only separable quadratic subfield contained in A.

Proof. (i) Let F have characteristic not 2 and let $K = F(\sqrt{\alpha})$ be a quadratic field extension contained in A. Then there is an element $X \in A$, $X = (a,u)$ with $a \in S$, $u \in P$ such that $X^2 = \alpha \in F^\times$, i.e.,

$$a^2 - ch(u,u) = \alpha \text{ and } u(ad+\bar{a}e) = 0.$$

If $a = 0$ then also $u = 0$, a contradiction, thus always $a \neq 0$.

If $u \neq 0$ then $ad + \bar{a}e = 0$ hence $ad = -\bar{a}e$. From the first equation we obtain $a^2d - ch(u,u)d = \alpha d$ and so $-a\bar{a}e - ch(u,u)d = \alpha d$ which means $-N(a)e - ch(u,u)d = \alpha d$. Hence $-N(a)e = (ch(u,u)+\alpha)d$ contradicting our assumption. Therefore $u = 0$, $a^2 = \alpha$ and $X = (a,0) \in S$ implies $K = S$.
Next let F have characteristic 2 and suppose K is a separable quadratic field extension of F contained in A. Hence there is an element $X = (a,u) \in A$, $a \in S$, $u \in P$ such that $X^2 + X = \alpha \in F^\times$. This implies

$$a^2 - ch(u,u) + a = \alpha \text{ and } u(ad + \bar{a}e + 1) = 0.$$

If $u \neq 0$ then $\bar{a}e + ad + 1 = 0$. This implies $ad = -(\bar{a}e+1)$ and so, using $a^2d - ch(u,u)d + ad = \alpha d$ we obtain $-a(\bar{a}e+1) - ch(u,u)d + ad = \alpha d$, i.e. $-N(a)e - a - ch(u,u)d + ad = \alpha d$ and thus $-N(a)e - a - ch(u,u)d + ad = a^2d - ch(u,u)d + ad$. This leads to $-N(a)e - a = a^2d$, i.e. $-\bar{a}e - 1 = ad$, which implies $-(\bar{a}e + ad) = 1$, so $\bar{a}e + ad = -1$, contradicting our assumption. Hence $u = 0$ which implies $a^2 + a = e$ and $X = (a,0) \in D$. Thus the basis $1, X = a$ for K lies in S and we obtain $S = K$. □

Corollary 14. *Let F have characteristic not 2, S be a separable quadratic field extension and h an anisotropic hermitian form. Let K be a separable quadratic field extension contained in $A = \mathrm{Cay}(S,P,ch,d,e,\times)$. If $c \in F^\times$ and $ed^{-1} \in S \setminus F$, then $K = S$.*

Suppose that $-N(a)e \neq (ch(u,u)+\alpha)d$ for all $u \in P$. Then S is the only separable quadratic field extension contained in A.

Proof. Consider the equation $-N(a)e = (ch(u,u)+\alpha)d$ for $a \in S$, $u \in P$, $u \neq 0$ and $\alpha \in F^\times$ which are no square in F. If $c \in F^\times$ and $ed^{-1} \in S \setminus F$, rewrite it as $-N(a)ed^{-1} = (ch(u,u)+\alpha)$, then the right-hand side lies in F and the left-hand side lies in S, so either $N(a) = 0$ and $ch(u,u)+\alpha = 0$ or we get a contradiction. So $a = 0$ and $-ch(u,u) = \alpha$. However, this means there is an element $X \in A$, $X = (0,u)$ with $u \in P$, $u \neq 0$ such that $X^2 = \alpha \in F^\times$, a contradiction since this means $-ch(u,u) = \alpha$ and $u = 0$. □

Example 15. Let $F = \mathbb{R}$. For $c, d, e \in \mathbb{R}^\times$ such that $c < 0$, $d < 0$ and $e > 0$ or such that $c < 0$, $d > 0$ and $e < 0$, $\mathbb{C} = (\mathbb{C}, 0)$ is the only quadratic field extension contained in $\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)$: Consider the equation $-N(a)e = (ch(u,u)+\alpha)d$ for $u \in \mathbb{C}^3$, $u \neq 0$, $\alpha \in \mathbb{R}_{<0}$. This is equivalent to $\beta e = (c\lambda + \alpha)d$ for some $\beta \in \mathbb{R}_{\leq 0}$, $\lambda \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{<0}$. If $c < 0$, $d < 0$ and $e > 0$ or such that $c < 0$, $d > 0$ and $e < 0$, then $\beta = 0$, thus $a = 0$, a contradiction as in Corollary 14.

Lemma 16. *Let S and S' be two separable quadratic field extensions of F and $A = \mathrm{Cay}(S, P, ch, d, e, \times)$, $A' = \mathrm{Cay}(S', P', c'h', d', e', \times')$. Suppose $S = (S, 0)$ and $S' = (S', 0)$ are the only separable quadratic field extensions contained in A, respectively A' and $A \cong A'$. Then $S \cong S'$.*

Theorem 17. *Let S be a separable quadratic field extension and h an anisotropic hermitian form. For all $c, d, e \in S^\times$ such that for all $a \in S^\times$, $u \in P$ with $u \neq 0$,*

$$N_{S/F}(a)\bar{d} \neq -c\bar{e}h(u,u),$$

$\mathrm{Cay}(S, P, ch, d, e, \times)$ is a division algebra over F.

Proof. We show that $A = \mathrm{Cay}(S, P, ch, d, e, \times)$ has no zero divisors: suppose

$$(0,0) = (a,u)(b,v) = (ab - ch(v,u), vad + u\bar{b}e + u \times_\alpha v)$$

for $a, b \in S$, $u, v \in P$. This is equivalent to

$$ab - ch(v,u) = 0 \text{ and } vad + u\bar{b}e + u \times_\alpha v = 0.$$

If $v = 0$ then $b = 0$ or $a = 0$. So either $(b,v) = (0,0)$ or $b \neq 0$, but then $a = 0$ hence $(a,u) = (0,0)$ using the second equation.
If $v \neq 0$ then $vad = -u \times v - u\bar{b}e$ plugged into the first equation yields

$$ab\bar{a}\bar{d} = -ch(u \times v, u) - ch(u\bar{b}e, u) = -ch(u\bar{b}e, u)$$

since $h(u \times v, u) = 0$. ($C = \mathrm{Cay}(S, P, h)$ is alternative, so $h(v \times u, u) = 0$ for all $u, v \in P$, cf. [Pu1].) Hence either $(a,u) = (0,0)$ or we have $(a,u) \neq (0,0)$ and $N_{S/F}(a)b\bar{d} = -cb\bar{e}h(u,u)$, i.e. $N_{S/F}(a)\bar{d} = -c\bar{e}h(u,u)$, contradicting the assumption. □

Corollary 18. *Let S be a separable quadratic field extension and h an anisotropic hermitian form. Suppose one of the following holds:*
(i) $c, d, e \in F^\times$ and $N_{S/F}(a)d \neq -ceh(u,u)$ for all $a \in S^\times$, $u \in P$ with $u \neq 0$,
(ii) $c \in S \setminus F$ and $d = fe$ for some $f \in F^\times$ (e.g., $d, e \in F^\times$),

(iii) $d \in S \setminus F$ *and* $c = fe$ *for some* $f \in F^\times$ *(e.g.,* $c, e \in F^\times$*),*
(iv) $e \in S \setminus F$ *and* $c = fd$ *for some* $f \in F^\times$ *(e.g.,* $c, d \in F^\times$*),*
(v) $e \in F^\times$ *and* $\frac{\bar d}{c} \in S \setminus F$ *(e.g.,* $c, d \in S \setminus F$ *and* $d \neq fc$ *for all* $f \in F^\times$*).*
(vi) $d \in F^\times$ *and* $c\bar e \in S \setminus F$ *(e.g.,* $c, e \in S \setminus F$ *and* $c\bar e \notin F^\times$*).*
(vi) $c \in F^\times$ *and* $\frac{d}{e} \in S \setminus F$ *(e.g.,* $d, e \in S \setminus F$ *and* $d \neq fe$ *for all* $f \in F^\times$*).*
Then $\mathrm{Cay}(S, P, ch, d, e, \times)$ *is a division algebra.*

Proof. Assume there is $a \in S^\times$, $u \in P$ with $u \neq 0$, such that $N_{S/F}(a)\bar d = -c\bar e h(u,u)$. This implies a contradiction in all cases considered. □

Example 19. Let $F = \mathbb{R}$ and $A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)$. A has zero divisors if and only if there is $a \in \mathbb{C}^\times$, $u \in \mathbb{C}^3$ with $u \neq 0$, such that $N_{\mathbb{C}/\mathbb{R}}(a)\bar d = -c\bar e h(u,u)$. This is equivalent to the existence of $\lambda \in \mathbb{R}_{<0}$ such that $\bar d = c\bar e\lambda$, i.e.

$$\frac{c\bar e}{\bar d} \in \mathbb{R}_{>0} \text{ or } \frac{c\bar e}{\bar d} \in \mathbb{C} \setminus \mathbb{R}.$$

Thus A is a division algebra for $c, d, e \in \mathbb{C}^\times$ such that

$$\bar d \neq -c\bar e\lambda \text{ for all } \lambda \in \mathbb{R}_{>0}.$$

This holds in the following cases:
(i) $c, d, e \in \mathbb{R}^\times$ and $cde > 0$,
(ii) $c \in \mathbb{C} \setminus \mathbb{R}$ and $d = fe$ for some $f \in \mathbb{R}^\times$ (e.g., $d, e \in \mathbb{R}^\times$),
(iii) $d \in \mathbb{C} \setminus \mathbb{R}$ and $c = fe$ for some $f \in \mathbb{R}^\times$ (e.g., $c, e \in \mathbb{R}^\times$),
(iv) $e \in \mathbb{C} \setminus \mathbb{R}$ and $c = fd$ for some $f \in \mathbb{R}^\times$ (e.g., $c, d \in \mathbb{R}^\times$),
(v) $e \in \mathbb{R}^\times$ and $\frac{\bar d}{c} \in \mathbb{C} \setminus \mathbb{R}$ (e.g., $c, d \in \mathbb{C} \setminus \mathbb{R}$ and $d \neq fc$ for all $f \in \mathbb{R}^\times$), or
$e \in \mathbb{R}_{>0}$ and $\frac{\bar d}{c} \in \mathbb{R}_{>0}$ or $e \in \mathbb{R}_{<0}$ and $\frac{\bar d}{c} \in \mathbb{R}_{<0}$.
(vi) $d \in \mathbb{R}^\times$ and $c\bar e \in \mathbb{C} \setminus \mathbb{R}$ (e.g., $c, e \in \mathbb{C} \setminus \mathbb{R}$ and $c\bar e \notin \mathbb{R}^\times$),
or $d \in \mathbb{R}_{>0}$ and $c\bar e \in \mathbb{R}_{>0}$ or $d \in \mathbb{R}_{<0}$ and $c\bar e \in \mathbb{R}_{<0}$.
(vii) $c \in \mathbb{R}^\times$ and $\frac{d}{e} \in \mathbb{C} \setminus \mathbb{R}$ (e.g., $d, e \in \mathbb{C} \setminus \mathbb{R}$ and $d \neq fe$ for all $f \in \mathbb{R}^\times$),
or $c \in \mathbb{R}_{>0}$ and $\frac{d}{e} \in \mathbb{R}_{>0}$ or $c \in \mathbb{R}_{<0}$ and $\frac{d}{e} \in \mathbb{R}_{<0}$.

4. The automorphism group

For a ternary nondegenerate $^-$-hermitian space (P, h) such that $\bigwedge^3(P,h) \cong \langle 1\rangle$ and an isomorphism $\alpha : \bigwedge^3(P,h) \to \langle 1\rangle$, every isometry $f : (P,h) \to (P,h)$ yields an automorphism of $(P, \times_\alpha)$, thus

$$SU(3) \subset \mathrm{Aut}(P, \times_\alpha).$$

For $A = \mathrm{Cay}(S, S^3, c\langle e_1, e_2, e_1e_2\rangle, d, e, \times)$ define $^- : S^3 \to S^3$ via

$$u = (u_1, u_2, u_3) \to \overline{u} = \overline{(u_1, u_2, u_3)} = (\overline{u_1}, \overline{u_2}, \overline{u_3}).$$

Clearly, $^- \in \mathrm{Aut}(S^3, \times)$. However, our algebras are not always enlargements of $(S^3, \times_\alpha)$ in the sense of [Do-Z1, 4.1], because (making free use of their terminology here) as we will see in the next Proposition for $F = \mathbb{R}$, the restriction homomorphism $Z_G(\pi) \to \mathrm{Aut}(P, \mu_S)$ is onto (which is equivalent to saying that $^-$ can be extended to $^- \in \mathrm{Aut}(A)$, $\overline{(a,u)} = (\bar a, \bar u)$), if and only if $c \in F^\times$.

Proposition 20. *Let* $\mathrm{Cay}(S,P,h,\times)$ *and* $\mathrm{Cay}(S,P',h',\times')$ *be two octonion division algebras and* $c,c',d,d',e,e'\in S$. *Let* $G:\mathrm{Cay}(S,P,ch,d,e,\times)\longrightarrow \mathrm{Cay}(S,P',c'h',d',e',\times')$ *be an algebra isomorphism with* $G=f\oplus g$ *where* $f\in \mathrm{Aut}(S)$ *and* $g:P\to P'$ *is an* F*-linear bijection.*
(i) If $G=id_S\oplus g$ *then* $g(v\times u)=g(u)\times' g(v)$ *for all* $u,v\in P$.
If $d=d'$ *and* $e=e'$ *and* $c,c'\in S\setminus F$, *then* $\bar c/c=\bar c'/c'$ *and* $(P,ch)\cong(P',c'h')$ *as* ε*-hermitian forms with isometry* g, *where* $\varepsilon=\bar c/c$.
If $d=d'$, $e=e'$ *and* $c,c'\in F^\times$, *then* $(P,ch)\cong(P',c'h')$ *as hermitian forms with isometry* g.
If $G={}^-\oplus g$, *then* $g(v\times u)=g(u)\times' g(v)$ *for all* $u,v\in P$, $g(vad)=g(v)\bar a\bar d'$ *and* $g(u\bar be)=g(u)b\bar e'$ *for all* $a,b\in S$, $u,v\in P$ *and* $c'=\alpha\bar c$ *for some* $\alpha\in F^\times$.
(ii) Suppose $(P,h)\cong(P',h')$. *Then* $\mathrm{Cay}(S,P,ch,d,e,\times)\cong \mathrm{Cay}(S,P,ch',d,e,\times)$. *In particular,*

$$\mathrm{Cay}(S,S^3,c\langle e_1,e_2,e_1e_2\rangle,d,e,\times)\cong \mathrm{Cay}(S,S^3,cd^2\langle e_1,e_2,e_1e_2\rangle,d,e,\times)$$

for all $e_i,d\in F^\times$.
(iii) If $(P,ch)\cong(P',c'h')$ *as* ε*-hermitian spaces with isometry* g, $\varepsilon=\bar c/c$, *and if* $g(v\times u)=g(v)\times' g(u)$ *for all* $u,v\in P$, *then* $\mathrm{Cay}(S,P,ch,\times_\alpha,d,e,\times)\cong \mathrm{Cay}(S,P,c'h',\times_\alpha,d,e,\times)$.
(iv) $\mathrm{Cay}(S,S^3,c\langle e_1,e_2,e_1e_2\rangle,d,e,\times)\cong \mathrm{Cay}(S,S^3,\bar c\langle e_1,e_2,e_1e_2\rangle,\bar d,\bar e,\times)$ *for all* $e_i\in F^\times$.

Proof. (i) Suppose first that $G=id_S\oplus g$, then $G((a,u)(b,v))=G(a,u)G(b,v)$ is equivalent to

$$G(ab-ch(v,u),vad+u\bar be+u\times v)=(ab-ch(v,u),g(vad)+g(u\bar be)+g(u\times v))$$
$$=(ab-c'h'(g(v),g(u)),g(v)ad'+g(u)\bar be'+g(u)\times' g(v)),$$

i.e. equivalent to $ch(v,u)=c'h(g(v),g(u))$ and

$$g(vad)+g(u\bar be)+g(v\times u)=g(v)ad'+g(u)\bar be'+g(u)\times' g(v)$$

for all $u,v\in P$, $a,b\in S$. For $a=b=0$ we obtain $g(v\times u)=g(u)\times' g(v)$ for all $u,v\in P$. For $u=0$ this implies $g(vad)=g(v)ad'$ for all $v\in P$, $a\in S$, hence $g(vd)=g(v)d'$ for all $v\in P$, for $v=0$ we get $g(u\bar be)=g(u)\bar be'$ for all $u\in P$, $b\in S$, hence $g(ue)=g(u)e'$ for all $u\in P$. Hence if $d=d'$ and $e=e'$ then $(P,ch)\cong(P',c'h')$ as ε-hermitian forms with isometry g, where $\varepsilon=\bar c/c=\bar c'/c'$ in case $c,c'\in S\setminus F$.

Suppose next that $G={}^-\oplus g$. Then $G((a,u)(b,v))=G(a,u)G(b,v)$ is equivalent to

$$G(ab-ch(v,u),vad+u\bar be+v\times u)=(\bar a\bar b-\bar c\overline{h(v,u)},g(vad)+g(u\bar be)+g(u\times v))$$
$$=(\bar a\bar b-c'h'(g(v),g(u)),g(v)\bar a\bar d'+g(u)b\bar e'+g(u)\times' g(v))$$

for all $a,b\in S$, $u,v\in P$ which implies that $\bar c\overline{h(v,u)}=c'h'(g(v),g(u))$ and $g(v\times u)=g(u)\times' g(v)$ for all $u,v\in P$. Moreover, $g(vad)=g(v)\bar a\bar d'$ and $g(u\bar be)=g(u)b\bar e'$ for all $a,b\in S$, $u,v\in P$. Now $\bar ch(u,u)=c'h'(g(u),g(u))$ implies that $c'^{-1}\bar c\in F^\times$, i.e. $c'=\alpha\bar c$ for some $\alpha\in F^\times$ and so $\overline{h(v,u)}=\alpha h'(g(v),g(u))$ for all $u,v\in P$.
(ii) This follows directly from the proof of (i) employing [T, Section 2] which implies that $g(v\times_\alpha u)=g(v)\times_{\alpha'} g(u)$ if $(P,h)\cong(P',h')$ with isometry g. The isomorphism is given by $G=id_S\oplus g$. Use that for all $d\in F^\times$, $\langle e_1,e_2,e_1e_2\rangle\cong d^2\langle e_1,e_2,e_1e_2\rangle$ for the second part of

the assertion.
(iii) is trivial.
(iv) A straightforward calculation as in (i) using that $h(\bar v,\bar u)=\overline{h(v,u)}$ shows that $F={}^-\oplus{}^-$ is an isomorphism. $\square$

Theorem 21. *(i) Let $A=\mathrm{Cay}(S,P,ch,d,e,\times_\alpha)$. Then $SU(3)\subset \mathrm{Aut}(A)$.*
(ii) Let $A=\mathrm{Cay}(S,S^3,ch,\times,d,e)$ with h diagonal. Then ${}^-\in\mathrm{Aut}(S^3,\times)$ extends to ${}^-\in \mathrm{Aut}(A)$, $\overline{(a,u)}=(\bar a,\bar u)$ if and only if $c,d,e\in F^\times$. In this case, $SU(3)Z_2\subset\mathrm{Aut}(A)$ the semidirect product $SU(3)Z_2$ of $SU(3)$ and Z_2, with Z_2 the cyclic group of order 2 acting on $SU(3)$, is contained in $\mathrm{Aut}(A)$.
In particular, if $\mathrm{Aut}(S^3,\times)=SU(3)Z_2$ and A is a division algebra then A is an enlargement of the algebra $(S^3,\times)$ as defined in [Do-Z1, 4.1] if and only if $c,d,e\in F^\times$.

Proof. (i) Every isometry $g:(P,h)\to(P,h)$ yields an F-linear bijection $F=id_S\oplus g$ on A. F is multiplicative if and only if $F((a,u)(b,v))=F(a,u)F(b,v)$ which is equivalent to

$$(ab-ch(v,u),g(v)ad+g(u)\bar be+g(u\times_\alpha v))$$

$$=(ab-ch(g(v),g(u)),g(v)ad+g(u)\bar be+g(u)\times_\alpha g(v)),$$

i.e. to $h(v,u)=h(g(v),g(u))$ and $f(u\times_\alpha v)=g(u)\times_\alpha g(v)$ for all $u,v\in P$. Hence F is multiplicative if and only if $g(u\times_\alpha v)=g(u)\times_\alpha g(v)$ which is satisfied for every isometry g, cf. [T, Section 2].
(ii) A straightforward calculation shows that $\overline{(a,u)}=(\bar a,\bar u)$ yields an automorphism of A if and only if

$$\overline{(ab-ch(v,u),vad+u\bar be+v\times u)}=(\bar a\bar b-\bar c\overline{h(v,u)},\bar v\bar a\bar d+\bar u\bar b\bar e+\overline{u\times v})$$

$$=(\bar a\bar b-\bar ch(\bar v,\bar u),\bar v\bar ad+\bar u\bar be+\bar u\times\bar v)$$

for all $a,b\in S$, $u,v\in P$. Now $ch(\bar v,\bar u)=\bar c\overline{h(v,u)}$ is equivalent to $c\in F^\times$ using that $h(\bar v,\bar u)=\overline{h(v,u)}$. Since ${}^-\in\mathrm{Aut}(S^3,\times)$ we know $\bar u\times\bar v=\overline{u\times v}$. Thus the second condition we need to satisfy is $\bar v\bar a\bar d+\bar u\bar b\bar e=\bar v\bar ad+\bar u\bar be$ for all $a,b\in S$, $u,v\in P$. This is clear for $c,d\in F^\times$. Conversely, for $u=0$, $a=1$ we get $\bar v\bar d=\bar vd$ for all $v\in P$ and for $v=0$, $b=1$ we get $\bar u\bar e=\bar ue$ for all $u\in P$. This implies $d,e\in F^\times$. $\square$

We observe that our previous results easily carry over to the opposite algebra. In particular, A^{op} has the same derivation algebra as A.

5. The case $F=\mathbb{R}$

For $F=\mathbb{R}$, $\mathbb{O}=\mathrm{Cay}(\mathbb{H},-1)=\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,\langle 1,1,1\rangle)$ is up to isomorphism the only octonion division algebra over $\mathbb{R}$. Its well-known standard basis is given by $\{1,i,j,k,l,il,jl,kl\}$ with $i^2=j^2=k^2=l^2=-1$, $jk=i$ and $ki=j$. Using this choice of basis, $\mathbb{O}$ fits into multiplication table (4.2) in [B-O2] which is our Table 2. We will choose the basis $u=1$, $v=i$ and

$$z_1=(0,(1,0,0)),z_2=(0,(0,1,0)),z_3=(0,(i,0,0)),z_4=(0,(0,0,1)),$$

$$z_5=(0,(0,0,i)),z_6=(0,(0,i,0))$$

for $\mathbb{O}$. Using this basis, the argument in [B-O2, p. 278] yields Table 1 (see p. 15) instead of (4.2) in [B-O2]. I.e., by choosing this basis, we have to slightly adjust the 6×6-matrix in the lower right hand corner of multiplication table [B-O2, (4.2)], the rest of the table and the parameters stays the same. An algebra fits into multiplication table (4.2) in [B-O2], i.e. Table 2, if and only if it fits into Table 1: Changing the basis via

$$j\to z_1,\quad k\to z_3,\quad l\to z_2,\quad il\to z_6,\quad jl\to z_4\quad kl\to -z_5$$

gives table [B-O2, (4.2)], see the argument in [Do-Z1].

This table contains 16 parameters which we also call the structure constants of A. For $\mathbb{O}$, the parameters satisfy

$$(*)\quad \eta_2=\eta_3=\theta_1=\theta_4=\sigma_2=\sigma_3=\tau_2=\tau_3=0$$

and

$$(**)\quad \theta_2=\sigma_1,\quad \theta_3=\tau_1,\quad \sigma_4=1,\quad \eta_4=\tau_4=-1.$$

Let $A=\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,ch,d,e,\times)$ with $c,d,e\in\mathbb{C}^\times$ and $h=\langle 1,1,1\rangle$. A fits into Table 1 as follows: Choose $z_1,\dots,z_6$ as above. Let $c=x+iy$ with $x,y\in\mathbb{R}$. The multiplication table now forces the choice of $u=c$ and $v=ic$. Since

$$(0,z_i)(0,z_j)=(0,z_i\times z_j)=(0,z_i)\cdot_{\mathbb{O}}(0,z_j)$$

unless $z_i=sz_j$ for some $s\in\mathbb{C}$, the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $\mathbb{O}$, with the exception of its diagonal entries being $-u$ because of

$$(0,z_i)(0,z_i)=(-ch(z_i,z_i),0)=(-c,0)$$

and the entries of the form

$$(0,z_1)(0,z_3)=(-ch(z_1,z_3),0)=(ic,0)=-(0,z_3)(0,z_1),$$
$$(0,z_2)(0,z_6)=(-ch(z_1,z_3),0)=(ic,0)=-(0,z_6)(0,z_2),$$
$$(0,z_4)(0,z_5)=(-ch(z_1,z_3),0)=(ic,0)=-(0,z_5)(0,z_4).$$

The choice of $u=c$, $v=ic$ gives the following structure constants in Table 1 or [B-O2, (4.2)]:

$$\eta_1=x,\quad \eta_2=-y=\eta_3,\quad \eta_4=-x,\quad \theta_1=y,\quad \theta_2=x=\theta_3,\quad \theta_4=-y,$$
$$\sigma_1=\mathrm{Re}(cd),\quad \sigma_2=\mathrm{Im}(cd),\quad \sigma_3=\mathrm{Re}(icd),\quad \sigma_4=\mathrm{Im}(icd),$$
$$\tau_1=\mathrm{Re}(\bar{c}e),\quad \tau_2=\mathrm{Im}(\bar{c}e),\quad \tau_3=\mathrm{Re}(-i\bar{c}e),\quad \tau_4=\mathrm{Im}(-i\bar{c}e).$$

The algebra generated by u,v,z_1,z_3 is the subalgebra $\mathrm{Cay}(\mathbb{C},(-c,d,e))$.

Table 1

$\cdot$	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u + \theta_1 v$	$\eta_2 u + \theta_2 v$	$\sigma_1 z_1 + \sigma_2 z_3$	$\sigma_1 z_2 + \sigma_2 z_6$	$-\sigma_2 z_1 + \sigma_1 z_3$	$\sigma_1 z_4 + \sigma_2 z_5$	$-\sigma_2 z_4 + \sigma_1 z_5$	$-\sigma_1 z_2 + \sigma_2 z_6$
v	$\eta_3 u + \theta_3 v$	$\eta_4 u + \theta_4 v$	$\sigma_3 z_1 + \sigma_4 z_3$	$\sigma_3 z_2 + \sigma_4 z_6$	$-\sigma_4 z_1 + \sigma_3 z_3$	$\sigma_3 z_4 + \sigma_4 z_5$	$-\sigma_4 z_4 + \sigma_3 z_5$	$-\sigma_4 z_2 + \sigma_3 z_6$
z_1	$\tau_1 z_1 + \tau_2 z_3$	$\tau_3 z_1 + \tau_4 z_3$	$-u$	z_4	v	$-z_2$	$-z_6$	z_5
z_2	$\tau_1 z_2 + \tau_2 z_6$	$\tau_3 z_2 + \tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	z_3	v
z_3	$-\tau_2 z_1 + \tau_1 z_3$	$-\tau_4 z_1 + \tau_3 z_3$	$-v$	$-z_5$	$-u$	$-z_6$	z_2	$-z_4$
z_4	$\tau_1 z_4 + \tau_2 z_5$	$\tau_3 z_4 + \tau_4 z_5$	z_2	$-z_1$	z_6	$-u$	v	$-z_3$
z_5	$-\tau_2 z_4 + \tau_1 z_5$	$-\tau_4 z_4 + \tau_3 z_5$	z_6	$-z_3$	$-z_2$	$-v$	$-u$	z_1
z_6	$-\tau_2 z_2 + \tau_1 z_6$	$-\tau_4 z_2 + \tau_3 z_6$	$-z_5$	$-v$	z_4	z_3	$-z_1$	$-u$

Table 2

$\cdot$	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u + \theta_1 v$	$\eta_2 u + \theta_2 v$	$\sigma_1 z_1 + \sigma_2 z_3$	$\sigma_1 z_2 + \sigma_2 z_6$	$-\sigma_2 z_1 + \sigma_1 z_3$	$\sigma_1 z_4 + \sigma_2 z_5$	$-\sigma_2 z_4 + \sigma_1 z_5$	$-\sigma_1 z_2 + \sigma_2 z_6$
v	$\eta_3 u + \theta_3 v$	$\eta_4 u + \theta_4 v$	$\sigma_3 z_1 + \sigma_4 z_3$	$\sigma_3 z_2 + \sigma_4 z_6$	$-\sigma_4 z_1 + \sigma_3 z_3$	$\sigma_3 z_4 + \sigma_4 z_5$	$-\sigma_4 z_4 + \sigma_3 z_5$	$-\sigma_4 z_2 + \sigma_3 z_6$
z_1	$\tau_1 z_1 + \tau_2 z_3$	$\tau_3 z_1 + \tau_4 z_3$	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	$\tau_1 z_2 + \tau_2 z_6$	$\tau_3 z_2 + \tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$-\tau_2 z_1 + \tau_1 z_3$	$-\tau_4 z_1 + \tau_3 z_3$	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	$\tau_1 z_4 + \tau_2 z_5$	$\tau_3 z_4 + \tau_4 z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$-\tau_2 z_4 + \tau_1 z_5$	$-\tau_4 z_4 + \tau_3 z_5$	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	$-\tau_2 z_2 + \tau_1 z_6$	$-\tau_4 z_2 + \tau_3 z_6$	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

Remark 22. The algebra generated by u, v, z_1, z_3, $\langle u, v, z_1, z_3\rangle \cong \mathrm{Cay}(\mathbb{C}, (-c, d, e)))$ is a division algebra if and only if

$$-\frac{\bar{c}e}{d} \notin N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$$

(Theorem 2). We also know that $A = \mathrm{Cay}(\mathbb{C}, c\langle 1, 1, 1\rangle, d, e, \times)$ is a division algebra if and only if for all $a \in \mathbb{C}^\times$, $u \in \mathbb{C}^3$ with $u \neq 0$,

$$N_{\mathbb{C}/\mathbb{R}}(a) \neq -\frac{c\bar{e}}{\bar{d}} h(u, u)$$

(Theorem 17). Since these two conditions both are equivalent to the statement

$$\frac{\bar{c}e}{d} \notin \mathbb{R}_{<0},$$

this confirms [Do-Z2, Proposition 4.1] for our family of algebras.

Our division algebras fit into the multiplication table (4.2) in [B-O2]. Therefore we obtain for $A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle, d, e, \times)$:

Theorem 23. *For $c, d, e \in \mathbb{C}^\times$,*

$$\mathrm{Der}(A) \cong su(3)$$

and A is the direct sum of two irreducible 1-dimensional modules and the irreducible 6-dimensional module $P = \mathbb{C}^3$. $SU(3)$ is the identity component of $\mathrm{Aut}(A)$.

Proof. The first statement is [B-O2], Proposition 4.1. A is not irreducible as $su(3)$-module, or else our algebras would be generalized pseudo-octonion algebras, which they are not. $SU(3)$ is the identity component of $\mathrm{Aut}(A)$ [Do-Z2, Proof of Proposition 4.3., p. 768]. □

Furthermore, for every $G \in \mathrm{Aut}(A)$, $G = id_{\mathbb{C}} \oplus g$ with $g : \mathbb{C}^3 \to \mathbb{C}^3$, cf. [Do-Z2, Proposition 4.3., p. 768].

Equation $(*)$ holds if and only if $c, d, e \in \mathbb{R}^\times$. For $c, d, e \in \mathbb{R}^\times$ we obtain the simplified multiplication table

$\cdot$	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u$	$\theta_2 v$	$\sigma_1 z_1$	$\sigma_1 z_2$	$\sigma_1 z_3$	$\sigma_1 z_4$	$\sigma_1 z_5$	$\sigma_1 z_6$
v	$\theta_3 v$	$\eta_4 u$	$\sigma_4 z_3$	$\sigma_4 z_6$	$-\sigma_4 z_1$	$\sigma_4 z_5$	$-\sigma_4 z_4$	$-\sigma_4 z_2$
z_1	$\tau_1 z_1$	$\tau_4 z_3$	$-u$	z_4	v	$-z_2$	$-\beta z_6$	$-z_5$
z_2	$\tau_1 z_2$	$\tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$\tau_1 z_3$	$-\tau_4 z_1$	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	$\tau_1 z_4$	$\tau_4 z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$\tau_1 z_5$	$-\tau_4 z_4$	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	$\tau_1 z_6$	$-\tau_4 z_2$	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

also given in [Do-Z1] (which is satisfied for every enlargement of $(\mathbb{C}^3, \times)$), with the special structure constants

$$\eta_1 = \theta_2 = \theta_3 = c = -\eta_4, \quad \sigma_1 = cd = \sigma_4, \quad \tau_1 = ce = -\tau_4.$$

For $c, d, e \in \mathbb{R}^\times$, A is a division algebra if and only if $cde > 0$ (Example 19). This can be also shown using [Do-Z1, Proposition 4.7].

Let $d = d_0 + id_1$ and $e = e_0 + ie_1$. The equations $(**)$ hold if and only if $c = 1 + iy$, $1 - d_0 = -yd_1$ and $e_0 = 1 - ye_1$, e. g., if A is the octonion division algebra $\mathbb{O} = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, \langle 1,1,1\rangle)$.

Theorem 24. *(i) For all $c, d, e \in \mathbb{R}^\times$ such that $cde > 0$, A is an enlargement of the real algebra $(\mathbb{C}^3, \times)$ and*

$$\mathrm{Aut}(A) \cong SU(3)Z_2$$

unless $(c, d, e) = (1, 1, 1)$, in which case

$$\mathrm{Aut}(A) \cong G_2.$$

Two algebras $\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)$ and $\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c'\langle 1,1,1\rangle, d', e')$ with $c, d, e, c', d', e' \in \mathbb{R}^\times$ are isomorphic if and only if they have the same structure constants.
(ii) Suppose that $c, d, e \in \mathbb{C}^\times$ are not all real and A is a division algebra, then

$$\mathrm{Aut}(A) \cong SU(3).$$

Proof. (i) For all $c, d, e \in \mathbb{R}^\times$ such that $cde > 0$, A is a division algebra. [Do-Z2, Proposition 4.3.] implies that $\mathrm{Aut}(A) \cong SU(3)Z_2$. The remaining assertion is proved in [Do-Z2, Proposition 4.4.]
(ii) again follows from [Do-Z2, Proposition 4.3.] since in this case the equations $(**)$ fail. $\square$

Corollary 25. *(i) If $c, d, e \in \mathbb{C}$ are not all real and A is a division algebra, then all automorphisms of A are given by $G = id_S \oplus g$ where g is an isometry of $h = \langle 1,1,1\rangle$.*
(ii) If $c, d, e \in \mathbb{R}$ such that $cde > 0$ then the automorphisms of A are given by $G = id_S \oplus g$ where g is an isometry of $h = \langle 1,1,1\rangle$ and by $F((a, u)) = {}^- \oplus {}^-$. A is an enlargement of $(\mathbb{C}^3, \times)$

We conclude from [Do-Z2, Proposition 4.4]:

Proposition 26. *Let $A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)$ and $A' = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c'\langle 1,1,1\rangle, d', e', \times)$ be division algebras. Then $A \cong A'$ if and only if $d = d'$, $e = e'$ and $c = c'$, or if $c = -c'$, $\mathrm{Im}(cd) = \mathrm{Im}(cd')$ and $\mathrm{Re}(i\bar{c}e) = \mathrm{Re}(i\bar{c}e')$.*

6. Some more families of non-unital division algebras

Following the notation introduced in [P, Section 1], denote the set of possibly non-unital algebra structures on an F-vector space by $\mathrm{Alg}(V)$. Given $A \in \mathrm{Alg}(V)$, we write xAy for the product of $x, y \in V$ in the algebra, if it is not clear from the context which multiplication is used. Let $G = \mathrm{Gl}(V) \times \mathrm{Gl}(V)$ be the direct product of two copies of the full linear group of V. It acts on $\mathrm{Alg}(V)$ by means of principal Albert isotopes: For $f, g \in \mathrm{Gl}(V)$ define the algebra $A^{(f,g)}$ as V together with the new multiplication

$$xA^{(f,g)}y = f(x)g(y) \qquad x, y \in V.$$

This defines a right action of G on $\mathrm{Alg}(V)$ which is compatible with passing to the opposite algebra, i.e., $(A^{(f,g)})^{op} = (A^{op})^{(f,g)}$. If A is a division algebra, so is $A^{(f,g)}$. Regular, thus in particular division algebras, are principal Albert isotopes of unital algebras [P, 1.5].

Given a Hurwitz algebra C over F of dimension ≥ 2 with canonical involution $^-$, the multiplications

$$x \star y = \bar{x}\bar{y}, \quad x \star y = \bar{x}y, \quad x \star y = x\bar{y}$$

for all $x, y \in C$ define the *para-Hurwitz algebra*, resp. the *left-* and *right composition algebra* associated to C. Together with C these are called the *standard composition algebras*.

Standard composition algebras of dimension eight satisfy table (4.2) and have derivation algebra isomorphic to G_2. The automorphism group of the para-octonion algebra is isomorphic to G_2 [P-I].

We briefly look at some principal Albert isotopes of our algebras $A = \mathrm{Cay}(S, P, ch, d, e, \times_\alpha)$ with $c, d, e \in S^\times$. Denote the multiplication in A by $\cdot$ or just juxtaposition as before.

If $V = U \oplus W$ with U the underlying two-dimensional vector space of S, W the underlying six-dimensional vector space of P, for $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathrm{Gl}(V)$ with $f_1, g_1 \in \mathrm{Gl}(U)$, $f_2, g_2 \in \mathrm{Gl}(W)$, the algebra $A^{(f,g)}$ contains the two-dimensional subalgebra $S^{(f,g)} = S^{(f_1,g_1)}$. If $f_1, g_1 \in \mathrm{Gl}(U)$ are isometries of the norm $N_{S/F}$ then $A^{(f,g)}$ contains the two-dimensional composition subalgebra $S^{(f_1,g_1)}$.

Let $\varepsilon \in \{1, -1\}$ and $h \in \mathrm{Gl}(U)$. Define $h_\varepsilon : A \to A$ by

$$h_\varepsilon((a, u)) = (h(a), \varepsilon u)$$

and a new multiplication $\star$ via

$$xA^{(h_\varepsilon, h_\varepsilon)}y = h_\varepsilon(x)h_\varepsilon(y), \quad xA^{(h_\varepsilon, id)}y = h_\varepsilon(x)y, \text{ resp. } xA^{(id, h_\varepsilon)}y = xh_\varepsilon(y),$$

for all $x, y \in A$. In particular, we look at the special case $\sigma_\varepsilon : A \to A$ defined by

$$\sigma_\varepsilon((a, u)) = (\bar{a}, \varepsilon u).$$

Moreover, let $\bar{h}_\varepsilon(a, u) = (h(a), \varepsilon\bar{u})$. We will also investigate the algebras $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$, $A^{(\bar{h}_\varepsilon, id)}$ and $A^{(id, \bar{h}_\varepsilon)}$. From now on let

$$F = \mathbb{R} \text{ and } A = \mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1, 1, 1\rangle, d, e, \times)$$

with $c, d, e \in \mathbb{C}^\times$, i.e. $h = \langle 1, 1, 1\rangle$.

Lemma 27. *For the algebras $A^{(h_\varepsilon, h_\varepsilon)}$, $A^{(h_\varepsilon, id)}$ and $A^{(id, h_\varepsilon)}$, $SU(3)$ is contained in their automorphism group.*

Proof. Let $G \in \mathrm{Aut}(A, \cdot)$ such that $G = id_S \oplus g$, g an isometry of h (cf. Corollary 25). Then $G((h(a), \varepsilon u)) = (h(a), \varepsilon g(u)) = h_\varepsilon(G((a, u)))$. Therefore

$$G((a, u)A^{(h_\varepsilon, h_\varepsilon)}(b, v)) = G((h(a), \varepsilon u)(h(b), \varepsilon v)) = G((h(a), \varepsilon u))G((h(b), \varepsilon v))$$

$$= G((a, u))A^{(h_\varepsilon, h_\varepsilon)}G((b, v))$$

and so $G \in \mathrm{Aut}(A^{(h_\varepsilon, h_\varepsilon)})$. The argument is analogous for the other cases. $\square$

In the following, we will show that the division algebras $A^{(h_\varepsilon, h_\varepsilon)}$, $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$, $A^{(h_1, id)}$ and $A^{(id, h_1)}$ fit into multiplication table 1, respectively table (4.2) in [B-O2]. This implies by [B-O2] and [Do-Z2, Proposition 4.3]:

Theorem 28. *Suppose $A = (\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)$ is a division algebra. Then the algebras $A^{(h_\varepsilon, h_\varepsilon)}$, $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$, $A^{(h_1, id)}$ and $A^{(id, h_1)}$ all have a derivation algebra isomorphic to $su(3)$. They are the direct sum of two irreducible 1-dimensional modules and an irreducible 6-dimensional module.*

Unless stated otherwise, define the vectors $u = c$, $v = ic$, $z_1, \dots, z_6$ as above. We fix the following notation: Let $c = x + iy$ with $x, y \in \mathbb{R}$. Let $h \in \mathrm{Gl}(U)$ such that $h(u) = \alpha + i\beta$ and $h(v) = \delta + i\gamma$, $\alpha, \beta, \delta, \gamma \in \mathbb{R}$. Let $c' = x' + iy'$ with $x', y' \in \mathbb{R}$. Let $h' \in \mathrm{Gl}(U)$ such that $h'(c') = \alpha' + i\beta'$ and $h'(v) = \delta' + i\gamma'$, $\alpha', \beta', \delta', \gamma' \in \mathbb{R}$.

6.1. The algebra multiplication of $A^{(h_\varepsilon, h_\varepsilon)}$ fits into Table 1: the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A, \cdot)$. We obtain the following structure constants:

$$\eta_1 = \frac{2\alpha\beta y + (\alpha^2 - \beta^2)x}{x^2 + y^2}, \quad \eta_2 = \eta_3 = \frac{(\alpha\gamma + \beta\delta)y + (\alpha\delta - \beta\gamma)x}{x^2 + y^2}, \quad \eta_4 = \frac{2\delta\gamma y + (\delta^2 - \gamma^2)x}{x^2 + y^2}$$

$$\theta_1 = -\frac{(\alpha^2 - \beta^2)y - 2\alpha\beta x}{x^2 + y^2}, \quad \theta_2 = \theta_3 = -\frac{(\beta\gamma - \alpha\delta)y + (\alpha\gamma + \beta\delta)x}{x^2 + y^2}, \quad \theta_4 = \frac{(\gamma^2 - \delta^2)y + 2\delta\gamma x}{x^2 + y^2},$$

$$\sigma_1 = \varepsilon\mathrm{Re}((\alpha + i\beta)d), \quad \sigma_2 = \varepsilon\mathrm{Im}((\alpha + i\beta)d), \quad \sigma_3 = \varepsilon\mathrm{Re}((\delta + i\gamma)d), \quad \sigma_4 = \varepsilon\mathrm{Im}((\delta + i\gamma)d),$$

$$\tau_1 = \varepsilon\mathrm{Re}((\alpha - i\beta)e), \quad \tau_2 = \varepsilon\mathrm{Im}((\alpha - i\beta)e), \quad \tau_3 = \varepsilon\mathrm{Re}((\delta - i\gamma)e), \quad \tau_4 = \varepsilon\mathrm{Im}((\delta - i\gamma)e).$$

The vectors u, v span the subalgebra $\mathbb{C}^{(h,h)}$. The algebra generated by u, v, z_1, z_3 is the subalgebra $\mathrm{Cay}(\mathbb{C}, (-c, d, e))^{(h_\varepsilon, h_\varepsilon)}$.

Example 29. We obtain the following structure constants for the division algebra $A^{(\sigma_\varepsilon, \sigma_\varepsilon)} = (\mathrm{Cay}(\mathbb{C}, \mathbb{C}^3, c\langle 1,1,1\rangle, d, e, \times)^{(\sigma_\varepsilon, \sigma_\varepsilon)}$:

$$\eta_1 = \frac{x(x^2 - 3y^2)}{x^2 + y^2} = -\eta_4, \quad \eta_2 = \eta_3 = \frac{y(y^2 - 3x^2)}{x^2 + y^2},$$

$$\theta_1 = \frac{y(y^2 - 3x^2)}{x^2 + y^2} = -\theta_4, \quad \theta_2 = \theta_3 = -\frac{x(x^2 - 3y^2)}{x^2 + y^2},$$

$$\sigma_1 = \varepsilon\mathrm{Re}((x - iy)d), \quad \sigma_2 = \varepsilon\mathrm{Im}((x - iy)d), \quad \sigma_3 = \varepsilon\mathrm{Re}((-y - ix)d), \quad \sigma_4 = \varepsilon\mathrm{Im}((-y - ix)d),$$

$$\tau_1 = \varepsilon\mathrm{Re}((x + iy)e), \quad \tau_2 = \varepsilon\mathrm{Im}((x + iy)e), \quad \tau_3 = \varepsilon\mathrm{Re}((-y + ix)e), \quad \tau_4 = \varepsilon\mathrm{Im}((-y + ix)e).$$

The vectors u, v span the para-quadratic subalgebra $\mathbb{C}^{(^-, ^-)}$. The algebra generated by u, v, z_1, z_3 is the subalgebra $\mathrm{Cay}(\mathbb{C}, (-c, d, e))^{(\sigma_\varepsilon, \sigma_\varepsilon)}$.

6.2. The algebra $A^{(\bar{h}_\varepsilon, \bar{h}_\varepsilon)}$ fits into Table 1: We have

$$\overline{z_1} = z_1, \quad \overline{z_2} = z_2, \quad \overline{z_3} = -z_3, \quad \overline{z_4} = z_4, \quad \overline{z_5} = -z_5, \quad \overline{z_6} = -z_6.$$

To assure that the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A, \cdot)$, we change the basis as follows: $u = c$, $v = -ic$, z_1, z_2 and z_4 as before and

$$z_3 \to -z_3, \quad z_5 \to -z_5, \quad z_6 \to -z_6.$$

We obtain the structure constants

$$\eta_1 = \frac{2\alpha\beta y + (\alpha^2 - \beta^2)x}{x^2 + y^2}, \quad \eta_2 = \eta_3 = -\frac{(\alpha\gamma + \beta\delta)y + (\alpha\delta - \beta\gamma)x}{x^2 + y^2}, \quad \eta_4 = \frac{2\delta\gamma y + (\delta^2 - \gamma^2)x}{x^2 + y^2},$$

$$\theta_1=-\frac{(\alpha^2-\beta^2)y-2\alpha\beta x}{x^2+y^2},\quad \theta_2=\theta_3=\frac{(\beta\gamma-\alpha\delta)y+(\alpha\gamma+\beta\delta)x}{x^2+y^2},\quad \theta_4=\frac{(\gamma^2-\delta^2)y+2\delta\gamma x}{x^2+y^2},$$

$$\sigma_1=\varepsilon\mathrm{Re}((\alpha+i\beta)d),\quad \sigma_2=-\varepsilon\mathrm{Im}((\alpha+i\beta)d),\quad \sigma_3=\varepsilon\mathrm{Re}((\delta+i\gamma)d),\quad \sigma_4=-\varepsilon\mathrm{Im}((\delta+i\gamma)d),$$

$$\tau_1=\varepsilon\mathrm{Re}((\alpha-i\beta)e),\quad \tau_2=-\varepsilon\mathrm{Im}((\alpha-i\beta)e),\quad \tau_3=\varepsilon\mathrm{Re}((\delta-i\gamma)e),\quad \tau_4=-\varepsilon\mathrm{Im}((\delta-i\gamma)e).$$

Example 30. Let $\mu: A\to A$ be defined by $\mu_\delta((a,u))=(\bar a,\bar u)$. This yields the following structure constants:

$$\eta_1=\frac{x(x^2-3y^2)}{x^2+y^2}=-\eta_4,\quad \eta_2=\eta_3=-\frac{y(y^2-3x^2)}{x^2+y^2},$$

$$\theta_1=\frac{y(y^2-3x^2)}{x^2+y^2}=-\theta_4,\quad \theta_2=\theta_3=\frac{x(x^2-3y^2)}{x^2+y^2},$$

$$\sigma_1=\varepsilon\mathrm{Re}((x-iy)d),\quad \sigma_2=-\varepsilon\mathrm{Im}((x-iy)d),\quad \sigma_3=\varepsilon\mathrm{Re}((-y-ix)d),\quad \sigma_4=\varepsilon\mathrm{Im}((y+ix)d),$$

$$\tau_1=\varepsilon\mathrm{Re}((x+iy)e),\quad \tau_2=-\varepsilon\mathrm{Im}((x+iy)e),\quad \tau_3=\varepsilon\mathrm{Re}((-y+ix)e),\quad \tau_4=-\varepsilon\mathrm{Im}((-y+ix)e).$$

The vectors u,v span the para-quadratic subalgebra $\mathbb{C}^{(^-,^-)}$.

Remark 31. It is likely that all Albert isotopes $A^{(f,f)}$, where $f=(f_1,f_2)\in \mathrm{Gl}(V)$ with $f_1\in\mathrm{Gl}(U)$, $f_2\in\mathrm{Gl}(W)$, have a multiplicative structure which fits into multiplication table (4.2) in [B-O2]. The subalgebra $\mathbb{C}^{(f_1,f_1)}$ always fits into the upper left 2×2 matrix in the table. Since

$$(0,z_i)A^{(f,f)}(0,z_j)=(0,f(z_i)\times f(z_j))=(0,f(z_i))\cdot_A(0,f(z_j))$$

the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A,\cdot)$, provided we change part of the basis of A from $f(z_1),\dots,f(z_6)$ back to $z_1,\dots,z_6$. However, we do not see at this point how to prove this.

Using [Do-Z2, Proposition 4.4] it is easy to check when two given algebras are isomorphic.

6.3. The multiplication of $A^{(h_1,id)}$ fits into Table 1: Since

$$(0,z_i)A^{(h_1,id)}(0,z_j)=(0,(z_i)\times(z_j))=(0,z_i)\cdot_A(0,z_j)$$

the 6×6-matrix in the lower right hand corner of the multiplication table remains the same as for $(A,\cdot)$. We have the following structure constants:

$$\eta_1=\alpha,\quad \eta_2=-\beta,\quad \eta_3=\delta,\quad \eta_4=-\gamma,\quad \theta_1=\beta,\quad \theta_2=\alpha,\quad \theta_3=\gamma,\quad \theta_4=\delta,$$

$$\sigma_1=\mathrm{Re}((\alpha+i\beta)d),\quad \sigma_2=\mathrm{Im}((\alpha+i\beta)d),\quad \sigma_3=\mathrm{Re}((\delta+i\gamma)d),\quad \sigma_4=\mathrm{Im}((\delta+i\gamma)d),$$

$$\tau_1=\mathrm{Re}((x-iy)e),\quad \tau_2=\mathrm{Im}((x-iy)e),\quad \tau_3=-\mathrm{Re}((y+ix)e),\quad \tau_4=-\mathrm{Im}((y+ix)e).$$

The vectors u,v span the subalgebra $\mathbb{C}^{(h,id)}$. Using [Do-Z2, Proposition 4.4] we obtain: If

$$\mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c\langle 1,1,1\rangle,d,e,\times)^{(h_1,id)}\cong \mathrm{Cay}(\mathbb{C},\mathbb{C}^3,c'\langle 1,1,1\rangle,d',e')^{(h_1',id)}$$

then all the corresponding structure constants are equal, or $(\alpha,\beta,\gamma,\delta)=(\alpha',-\beta',\gamma',-\delta')$.

6.4. The algebra $A^{(id,h_1)}$ fits into Table 1:

$$\eta_1 = \alpha, \quad \eta_2 = \delta, \quad \eta_3 = -\beta, \quad \eta_4 = -\gamma, \quad \theta_1 = \beta, \quad \theta_2 = \gamma, \quad \theta_3 = \alpha, \quad \theta_4 = \delta,$$

$$\sigma_1 = \mathrm{Re}((x+iy)d), \quad \sigma_2 = \mathrm{Im}((x+iy)d), \quad \sigma_3 = \mathrm{Re}((ix-y)d), \quad \sigma_4 = \mathrm{Im}((ix-y)d),$$

$$\tau_1 = \mathrm{Re}((\alpha-i\beta)e), \quad \tau_2 = \mathrm{Im}((\alpha-i\beta)e), \quad \tau_3 = -\mathrm{Re}((\delta-i\gamma)e), \quad \tau_4 = -\mathrm{Im}((\delta-i\gamma)e).$$

The vectors u, v span the subalgebra $\mathbb{C}^{(id,h)}$. We note that $A^{(h_1,id)} = A^{(id,(h^{-1})_1)}$.

Remark 32. (i) The algebra $A^{(h_{-1},id)}$ does not seem to fit into Table 1 since

$$(0,z_i) \star (0,z_j) = (0, (-z_i) \times (z_j)) = -(0,z_i) \cdot_A (0,z_j).$$

Therefore the 6×6-matrix in the lower right hand corner of the multiplication table does not remain the same as for $(A,\cdot)$. It is not clear if a change of basis might change this. The same observation applies to the algebra $A^{(id,h_{-1})} = A^{((h^{-1})_{-1},id)}$ and to the algebras $A^{(\bar{h}_{-1},id)}$ and $A^{(id,\bar{h}_{-1})}$.
(ii) The multiplications of the algebras $A^{(\bar{h}_1,id)}$ and $A^{(id,\bar{h}_1)}$ fit into Table 1 by changing the basis as in 6.2. We leave it to the reader to compute their structure constants if desired.

7. Another construction method

We conclude the paper with the observation that the construction method used for the four-dimensional algebras treated in Section 2 can be generalized in a straightforward way to define eight-dimensional algebras out of a quaternion algebra D with canonical involution $^-$ and elements $c, d, e \in D^\times$: again the vector space $D \oplus D$ can be made into an algebra over F via the multiplication

$$(u,v)(u',v') = (uu' + b\overline{v}'v, v'ud + v\overline{u}'e)$$

for $u, u', v, v' \in D$. Since D is no longer commutative, we now have more options how to place the elements $c, d, e \in D^\times$ inside the multiplication. Depending on their places, the new algebra is denoted by $\mathrm{Cay}(D,(c,d,e)_{xyz})$ with $x, y, z \in \{l, m, r\}$ depending on whether the element c is placed to the left, in the middle, or on the right-hand side of the corresponding factors inside the product. The algebra defined above will thus be denoted $\mathrm{Cay}(D,(c,d,e)_{lrr})$. Using the multiplication

$$(u,v)(u',v') = (uu' + b\overline{v}'v, dv'u + ev\overline{u}')$$

for $u, u', v, v' \in D$ instead gives the algebra $\mathrm{Cay}(D,(c,d,e)_{lll})$, for instance. D is a subalgebra of $\mathrm{Cay}(D,(c,d,e)_{xyz})$ for all $x, y, z \in \{l, m, r\}$ via the embedding $u \to (u, 0)$. Note that $A = \mathrm{Cay}(D,(c,d,e)_{xyz})$ is an octonion algebra if and only if $c \in F^\times$ and $e = d = 1$ and a Dickson algebra (see [Pu2] for the definition) if and only if $c \in S \setminus F$ and $e = d = 1$. Similar observations as before hold:

Lemma 33. *Let $A = \mathrm{Cay}(D,(c,d,e)_{lrr})$.*
(i) If $cd \neq \bar{c}e$ then A is not third power-associative.
(ii) A has $(1,0)$ as a left-unit element if and only if $d = 1$, as a right-unit element if and only if $e = 1$ and as unit element if and only if $d = e = 1$.
(iii) For $l = (0,1)$ we have $lx = \bar{x}l$ for all $x = (u,0) \in S$ if and only if $e = d$.

Proof. (i) For $l=(0,1)$ we have $l^2=(c,0)$ and $ll^2=(0,\sigma(c)e)$ while $l^2l=(0,cd)$, so $ll^2=l^2l$ if and only if $cd=\bar{c}e$. Thus A is not third power-associative if $cd\neq\bar{c}e$.
(ii), (iii) are trivial. $\square$

The new algebras are division algebras under certain conditions on the three elements involved in their construction:

Theorem 34. *Let D be a quaternion division algebra and $de^{-1}\in F^\times$.*
(i) $\mathrm{Cay}(D,(c,d,e)_{lrr})$ *is a division algebra if and only if*

$$c\bar{e}\bar{d}^{-1}\not\in N_{S/F}(D^\times).$$

(ii) $\mathrm{Cay}(D,(c,d,e)_{lll})$ *is a division algebra if and only if*

$$\bar{d}^{-1}\bar{e}c\not\in N_{S/F}(D^\times).$$

Proof. Suppose $(0,0)=(u,v)(u',v')$ for $u,u',v,v'\in S$. This is equivalent to

$$uu'+c\overline{v}'v=0,\quad v'ud+v\overline{u}'e=0$$

in (i) and to

$$uu'+c\overline{v}'v=0,\quad dv'u+ev\overline{u}'=0$$

in (ii). If $v=0$ then either $(u,v)=(0,0)$ or $(u',v')=(0,0)$. So let $v\neq 0$.
(i) The second equation yields $\bar{u}'=-v^{-1}v'ude^{-1}$, therefore

$$u'=-\bar{e}^{-1}\bar{d}\bar{u}\bar{v}'\frac{1}{N_{S/F}(v)}v.$$

This together with the first equation implies

$$-u\bar{e}^{-1}\bar{d}\bar{u}\bar{v}'\frac{1}{N_{S/F}(v)}v+c\overline{v}'v=0,$$

so

$$(-u\bar{e}^{-1}\bar{d}\bar{u}\frac{1}{N_{S/F}(v)}+c)\bar{v}'v=0.$$

If $v'=0$ then $v\bar{u}'e=0$ yields $u'=0$. So suppose $v'\neq 0$. Then

$$c=u\bar{e}^{-1}\bar{d}\bar{u}\frac{1}{N_{S/F}(v)}=\frac{N_{S/F}(u)}{N_{S/F}(v)}\bar{e}^{-1}\bar{d}.$$

If $u=0$ then $c=0$, a contradiction, so we need $u\neq 0$. Suppose $\bar{e}^{-1}\bar{d}\in F^\times$, i.e. $de^{-1}\in F^\times$ then we get $c\bar{e}\bar{d}^{-1}\in N_{S/F}(S^\times)$. This implies the assertion.
(ii) The second equation yields $\bar{u}'=-v^{-1}e^{-1}dv'u$, therefore

$$u'=-\bar{u}\bar{v}'\bar{d}\bar{e}^{-1}\frac{1}{N_{S/F}(v)}v.$$

This together with the first equation implies

$$-u\bar{u}\bar{v}'\bar{d}\bar{e}^{-1}\frac{1}{N_{S/F}(v)}v+c\overline{v}'v=0,$$

so

$$-\bar{v}'\bar{d}\bar{e}^{-1}\frac{N_{S/F}(u)}{N_{S/F}(v)}+c\bar{v}'=0\text{ hence }c\bar{v}'=\bar{v}'\bar{d}\bar{e}^{-1}\frac{N_{S/F}(u)}{N_{S/F}(v)}.$$

If $v' = 0$ then $v\bar{u}'e = 0$ yields $u' = 0$. So suppose $v' \neq 0$. Then

$$c\bar{v}' = \bar{v}'\bar{d}\bar{e}^{-1}\frac{N_{S/F}(u)}{N_{S/F}(v)}.$$

If $u = 0$ then $c = 0$, a contradiction, so we need $u \neq 0$. Suppose $\bar{d}\bar{e}^{-1} \in F^\times$, i.e. $e^{-1}d \in F^\times$ then we get

$$c = \bar{d}\bar{e}^{-1}\frac{N_{S/F}(u)}{N_{S/F}(v)}.$$

Thus $\bar{d}^{-1}\bar{e}c \in N_{S/F}(S^\times)$. This implies the assertion. □

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