LIE TORI OF TYPE B_2 AND GRADED-SIMPLE JORDAN STRUCTURES COVERED BY A TRIANGLE

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Abstract. We classify two classes of B_2-graded Lie algebras which have a second compatible grading by an abelian group Λ: (a) Λ-graded-simple, Λ torsion-free and (b) division-Λ-graded. Our results describe the centreless cores of a class of affine reflection Lie algebras, hence apply in particular to the centreless cores of extended affine Lie algebras, the so-called Lie tori, for which we recover results of Allison-Gao and Faulkner. Our classification (b) extends a recent result of Benkart-Yoshii.

Both classifications are consequences of a new description of Jordan algebras covered by a triangle, which correspond to these Lie algebras via the Tits-Kantor-Koecher construction. The Jordan algebra classifications follow from our results on graded-triangulated Jordan triple systems. They generalize work of McCrimmon and the first author as well as the Osborn-McCrimmon-Capacity-2-Theorem in the ungraded case.

Introduction

This paper deals with two related algebraic objects, Lie algebras graded by the root system B_2 and Jordan structures (algebras, triple systems and pairs), covered by a triangle of idempotents, respectively tripotents. Its aim is to classify the graded-simple structures in these categories.

On the Lie algebra side, the motivation for this paper comes from the theory of extended affine Lie algebras, which generalize affine Lie algebras and toroidal Lie algebras ([AABGP]), and the even more general affine reflection Lie algebras ([N5]):

$$\text{affine LA} \subset \text{extended affine LA} \subset \text{affine reflection LA}$$

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As explained in [N5, §6], an important ingredient in their structure theory is the centreless core, whose structure is, respectively, as follows:

$$\begin{aligned}&\text{un/twisted loop algebra} \subset \text{centreless Lie torus}\\ &\quad\subset \text{centreless predivision-root-graded Lie algebra}\end{aligned}$$

In fact, in the extended affine case one gets all algebras by a well-defined extension process from centreless Lie tori ([N4]). The predivision-root-graded Lie algebras are naturally viewed as special types of Lie algebras graded by a not necessarily reduced irreducible root system R. Their structure is known in more or less precise terms with the precision increasing with the rank of R, see [N5, 5.10] for a summary. One of the most complicated and mysterious case is $R = \mathrm{B}_2$, studied in this paper.

Let Δ be the root system B_2 and put $R = \Delta \cup \{0\}$ (clarification: $0 \notin \Delta$). Also, let Λ be an abelian group. We consider Lie algebras defined over a ring k with $\frac{1}{2}$ and $\frac{1}{3} \in k$ which have a decomposition

(RG1) $L = \bigoplus_{\alpha\in R,\, \lambda\in\Lambda} L_\alpha^\lambda$ with $[L_\alpha^\lambda,\, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu}$ ($= 0$ if $\alpha + \beta \notin R$), satisfying
(RG2) $L_0 = \sum_{\alpha\in\Delta} [L_\alpha, L_{-\alpha}]$.

We call $0 \neq e \in L_\alpha^\lambda$, $\alpha \in \Delta$, *invertible* if there exists $f \in L_{-\alpha}^{-\lambda}$ such that $h = [e,\, f]$ acts on $x_\beta \in L_\beta^\mu$, $\beta \in R$, as $[h, x_\beta] = \langle \beta, \alpha^\vee\rangle x_\beta$ for $\langle\beta, \alpha^\vee\rangle$ the Cartan integer of $\alpha, \beta \in R$. This perhaps unusual definition is justified by examples, in which invertible elements correspond to invertible elements in coordinate algebras, and by the following definitions. A Lie algebra satisfying (RG1) and (RG2) is called

- B_2-*graded with a compatible* Λ-*grading* if every L_α^0, $\alpha \in \Delta$, contains an invertible element,
- B_2-*graded-simple* if L is R-graded with a compatible Λ-grading and if $\{0\}$ and L are the only Λ-graded ideals of L,
- *predivision*-B_2-*graded* if L is R-graded with a compatible Λ-grading and every $0 \neq L_\alpha^\lambda$, $\alpha \in \Delta$, contains an invertible element,
- *division*-B_2-*graded* if L is R-graded with a compatible Λ-grading and every nonzero element in L_α^λ, $\alpha \in \Delta$, is invertible,
- a *Lie torus of type* B_2 if k is a field, L is division-B_2-graded and $\dim_k L_\alpha^\lambda \leq 1$ for all $\alpha \in \Delta$.

The reader will certainly know at least 3 examples of B_2-graded Lie algebras, say for simplicity over a field of characteristic $\neq 2, 3$:

(I) special linear Lie algebras $\mathfrak{sl}_{2n}(k) = \mathfrak{sl}_2(\mathrm{Mat}_n(k))$,
(II) symplectic Lie algebras $\mathfrak{sp}_n(k) = \mathfrak{sp}_2(\mathrm{Mat}_n(k))$ (here $R = \mathrm{C}_2$ is more natural),
(III) orthogonal Lie algebras $\mathfrak{o}(Q)$ of a nondegenerate quadratic form Q on an odd-dimensional quadratic space with base point and containing a hyperbolic plane.

All of these are simple Lie algebras. Our first classification result says that, up to central extensions and allowing graded-simple coordinates, these are all examples in the graded-simple case with Λ torsion-free:

Theorem A (Th. 7.12) *Let Λ be torsion-free and let L be a Lie algebra over a ring k containing $\frac{1}{2}$ and $\frac{1}{3}$. Then L is centerless* B_2*-graded-simple if and only if L is graded isomorphic to*

(I) $\mathfrak{sl}_4(A)/Z\big(\mathfrak{sl}_4(A)\big)$ *where* $\mathfrak{sl}_4(A) = \{X \in \mathfrak{gl}_4(A) : \operatorname{tr}(X) \in [A, A]\}$, *and A is a graded-simple associative k-algebra,*

(II) $\mathfrak{sp}_2(A, \pi)/Z(\mathfrak{sp}_2(A, \pi))$ *for A as in* (I) *with involution π,*

(III) *an elementary orthogonal Lie algebra $\mathfrak{eo}(Q)$ where Q is a graded-nondegenerate quadratic form with base point and containing a hyperbolic plane over a graded-field.*

We point out that even the case $\Lambda = \{0\}$ was not explicitly known before, although it could have been derived from [MN]. Also, in the application to affine reflection Lie algebras and their centreless core our assumption on Λ is fulfilled.

Our second classification result (Th. 7.13) is parallel to Th. A: It allows an arbitrary Λ but assumes that the Lie algebra L is centreless and division-B_2-graded. In this setting, case (I) disappears, the algebra A in (II) is division-graded and Q in (III) is graded-anisotropic. In characteristic 0 this result has also been obtained by Benkart-Yoshii [BY, Th. 4.3] using different methods and giving a less precise description of the Lie algebras. We can easily derive from our results a classification of centreless Lie tori of type B_2.

Corollary (Cor. 7.14) *A Lie algebra L is a centreless Lie torus of type* B_2 *if and only if L is graded isomorphic to one of the following:*

(I) *a symplectic Lie algebra $\mathfrak{sp}_2(A, \pi)$ for A a noncommutative associative torus with involution π, or to*

(II) *an elementary orthogonal Lie algebra $\mathfrak{eo}(Q)$ for Q a graded-anisotropic quadratic form over an associative torus with the same properties as Q in* (III) *of* Th. A.

Again in characteristic 0 this has also been obtained by Benkart-Yoshii [BY, Th. 5.9]. For $k = \mathbb{C}$ and $\Lambda = \mathbb{Z}^n$, the Lie tori classification is due to Allison-Gao [AG]. A different approach to this (again for $\Lambda = \mathbb{Z}^n$) has recently been worked out by Faulkner [F] in the context of his classification of BC_2 Lie tori. We obtain the Lie algebra results as a consequence of our results on so-called triangulated Jordan structures, using the Tits-Kantor-Koecher construction.

This brings us to the second goal for this paper, the classification of graded-simple-triangulated Jordan structures, which seems to be a timely undertaking in the light of the recent growing interest in graded Jordan structures. We restrict ourselves here to Jordan algebras, for which the results are easier to state. A quadratic unital Jordan algebra J is called *graded-triangulated* if $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$ is graded by some abelian group Λ and contains two supplementary orthogonal idempotents $e_1, e_2 \in J^0$ strongly connected by some $u \in J_2(e_1) \cap J_2(e_2) \cap J^0$. Of course, *graded-simple-triangulated* means graded-simple and graded-triangulated.

To put our results into perspective, let us point out that already the case $\Lambda = \{0\}$ is nontrivial: The "Osborn-McCrimmon-Capacity-2-Theorem" ([J, 6.3], [M2, 22.2]), which classifies simple Jordan algebras of capacity 2 (= simple triangulated Jordan algebras with division diagonal Peirce spaces) is the most complicated piece of the classification of simple Jordan algebras with capacity, and a cornerstone of the classification of simple Jordan algebras. The well-known result is that there are three types of simple triangulated Jordan algebras, namely the Jordan algebra analogues of the Lie algebras (I)-(III) above: (I) Matrix algebras $\mathrm{Mat}_2(A)$ for A simple associative, (II) hermitian matrix algebras $\mathrm{H}_2(A, A_0, \pi)$ for A as in (I) and (III) Jordan algebras associated to a nondegenerate quadratic form, nowadays called Clifford Jordan algebras. In complete analogy to Th. A we show that for a torsion-free Λ this remains true in the graded-simple setting if one replaces the simple coordinates by graded-simple coordinates:

Theorem B (Th. 6.3.b) *A triangulated quadratic Jordan algebra J which is graded-simple with respect to a grading by a torsion-free abelian group Λ is graded isomorphic to one of the following Jordan algebras:*

(I) *full matrix algebra* $\mathrm{Mat}_2(B)$ *for a noncommutative graded-simple associative unital B;*
(II) *hermitian matrices* $\mathrm{H}_2(A, A_0, \pi)$ *for a graded-simple noncommutative A with ample subspace A_0 and graded involution π;*
(III) *Clifford Jordan algebra* $\mathrm{AC}_{\mathrm{alg}}(q, F, F_0)$ *for a graded-nondegenerate q over a graded-field F with Clifford-ample subspace F_0.*

Conversely, all Jordan algebras in (I)–(III) *are graded-simple-triangulated.*

As for Th. A, we prove a second classification result (Cor. 6.5) in which the assumption on Λ is replaced by the condition that elements in the Peirce space J_{12} are sums of invertible elements. And of course, there are also corollaries for triangulated Jordan algebra tori, formulated in Cor. 6.6 for $\Lambda = \mathbb{Z}^n$.

We have mentioned that the Lie algebra results follow from our results on Jordan structures. So how do we prove, say, the Jordan algebra result? In fact, we first prove the results for graded-simple-triangulated Jordan triple systems by adapting the approach of [MN] to the graded-simple setting. This paper deals with the case $\Lambda = \{0\}$ and so generalizes the Osborn-McCrimmon-Capacity-2-Theorem to Jordan triple systems. Once the triple case has been established, we can derive the results for Jordan algebras and Jordan pairs (Th 6.12, Cor. 6.14 and Cor. 6.15) by standard techniques.

The paper is divided into seven sections. In the first two sections we establish the terminology, identities and general results about graded Jordan triple systems and graded-triangulated Jordan triple systems, respectively, that will be needed throughout the paper. Proofs in the first two sections are mainly left to the reader since they are easy generalizations of the corresponding ungraded cases. In sections §3 and §4 we present our two basic models for graded-triangulated Jordan triple systems, the hermitian matrix and the Clifford systems and prove Coordinatization Theorems for both of them. Section §5 is devoted to classifying graded-simple-triangulated Jordan triple systems. As a corollary we obtain a classification of division-triangulated Jordan triple systems (Cor. 5.12) and triangulated Jordan triple tori (Cor. 5.13). These classification theorems are extended to Jordan algebras and Jordan pairs in §6. Finally, in the last section, we apply our results to Lie algebras.

Unless specified otherwise, all algebraic structures are defined over an arbitrary ring of scalars, denoted k, and are assumed to be graded by an abelian group Λ, written additively. We will use Loos' Lecture Notes [L] as our basic reference for Jordan triple systems and Jordan pairs.

1. Graded Jordan triple systems

This section introduces some basic notions of graded Jordan triple systems. For example we establish in Th. 1.4 that Peirce-2- and Peirce-0-spaces of a degree 0 tripotent inherit graded-simplicity.

A k-module M is *graded by* Λ if $M = \bigoplus_{\lambda\in\Lambda} M^\lambda$ where $(M^\lambda : \lambda \in \Lambda)$ is a family of k-submodules of M. In this case, we call M Λ-*graded* if the *support set* $\operatorname{supp}_\Lambda\{\lambda \in \Lambda : M^\lambda \neq 0\}$ generates Λ as an abelian group. Of course, if M is graded by Λ, it is Ξ-graded for Ξ the subgroup generated by $\operatorname{supp}_\Lambda M$. But it is usually more convenient to just consider graded modules (and triple systems) as opposed to Λ-graded ones. We say that M *is graded* if M is graded by some (unimportant) abelian group, which for simplicity we assume to be Λ. A *homogeneous element* of a graded M is an element of $\bigcup_{\lambda\in\Lambda} M^\lambda$. If $M = \bigoplus_{\lambda\in\Lambda} M^\lambda$ and $N = \bigoplus_{\lambda\in\Lambda} N^\lambda$ are graded modules, a k-linear map $\varphi : M \to N$ is said to be *homogeneous of degree* $\gamma \in \Lambda$ if $\varphi(M^\lambda) \subseteq N^{\lambda+\gamma}$ for all $\lambda \in \Lambda$.

A Jordan triple system J with quadratic operator P and triple product $\{.,.,.\}$ is *graded by* Λ if the underlying module is so, say $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$, and the family $(J^\lambda : \lambda \in \Lambda)$ satisfies $P(J^\lambda)J^\mu \subseteq J^{2\lambda+\mu}$ and $\{J^\lambda, J^\mu, J^\nu\} \subseteq J^{\lambda+\mu+\nu}$ for all $\lambda, \mu, \nu \in \Lambda$. We will say that J is Λ-*graded* if J is graded by Λ and the underlying module is Λ-graded. As for modules, we will simply speak of a *graded Jordan triple system* if the grading group Λ is not important.

If J and J' are graded Jordan triple systems, a homomorphism $\varphi : J \to J'$ is said to be *graded* if it is homogeneous of degree 0. Correspondingly, a *graded isomorphism* is a bijective graded homomorphism, and we say that

J and J' are *graded isomorphic*, written as $J \cong_\Lambda J'$, if there exists a graded isomorphism between J and J'.

Let J be a graded Jordan triple system. A subsystem M of J is called *graded* if $M = \bigoplus_{\lambda\in\Lambda}(M \cap J^\lambda)$. If M is an arbitrary subsystem of J, the greatest graded subsystem of J contained in M is $M^{\rm gr} = \bigoplus_{\lambda\in\Lambda}(M \cap J^\lambda)$. If M is an ideal of J, then so is $M^{\rm gr}$. We also note that the quotient of J by a graded ideal is again graded with respect to the canonical quotient grading. We call J *graded-simple* if $P(J)J \neq 0$ and every graded ideal is either 0 or equal to J. We say that J is *graded-prime* if $P(I)K = 0$ for graded ideals I, K of J implies $I = 0$ or $K = 0$, and *graded-semiprime* if $P(I)I = 0$ for a graded ideal I implies $I = 0$. We denote by $\mathcal{T}(J) = \{x \in J : P(x)J = 0\}$ the set of *trivial elements* of J, and put $\mathcal{T}^\Lambda(J) := \bigcup_{\lambda\in\Lambda} \mathcal{T}^\lambda(J)$, where $\mathcal{T}^\lambda(J) = \mathcal{T}(J) \cap J^\lambda$. We say that J is *graded-nondegenerate* if $\mathcal{T}^\Lambda(J) = 0$. We note that *if J is graded-nondegenerate, it is also graded-semiprime.* Finally, we say that J is *graded-strongly prime* if it is graded-prime and graded-nondegenerate, and *division-graded* if it is nonzero and every nonzero homogeneous element is invertible in J.

Recall that the *McCrimmon radical* $\mathcal{M}(J)$ of a Jordan triple system J is the smallest ideal of J such that the quotient $J/\mathcal{M}(J)$ is nondegenerate [L, §4]. It can be constructed as follows: $\mathcal{M}(J) := \bigcup_\alpha \mathcal{M}_\alpha(J)$, where $\mathcal{M}_0(J) = 0$, $\mathcal{M}_1(J)$ is the ideal of J generated by the set of trivial elements $\mathcal{T}(J)$ of J and, by using transfinite induction, the ideals $\mathcal{M}_\alpha(J)$ are defined by $\mathcal{M}_\alpha(J)/\mathcal{M}_{\alpha-1}(J) = \mathcal{M}_1\big(J/\mathcal{M}_{\alpha-1}(J)\big)$ if α is a non-limit ordinal and $\mathcal{M}_\alpha(J) = \bigcup_{\beta<\alpha} \mathcal{M}_\beta(J)$ for a limit ordinal α.

Definition 1.1. Let J be a graded Jordan triple system and let $\mathcal{M}(J)$ be its McCrimmon radical. We define the *graded-McCrimmon radical* of J, denoted ${\rm gr}\mathcal{M}(J)$, as the greatest graded ideal contained in $\mathcal{M}(J)$, i.e.,

$$\mathrm{gr}\mathcal{M}(J) := \mathcal{M}(J)^{\rm gr} = \textstyle\bigoplus_{\lambda\in\Lambda} \big(J^\lambda \cap \mathcal{M}(J)\big).$$

Thus ${\rm gr}\mathcal{M}(J) = \bigoplus_{\lambda\in\Lambda} {\rm gr}\mathcal{M}^\lambda(J)$ for ${\rm gr}\mathcal{M}^\lambda(J) = J^\lambda \cap \mathcal{M}(J)$. The following characterization is immediate from the definition, see [L, §4] for the ungraded case.

Proposition 1.2. *Let J be a graded Jordan triple system. Then the homogenous spaces ${\rm gr}\mathcal{M}^\lambda(J)$ of ${\rm gr}\mathcal{M}(J)$ are ${\rm gr}\mathcal{M}^\lambda(J) = \bigcup_\alpha(\mathcal{M}_\alpha(J) \cap J^\lambda)$ for $\mathcal{M}_\alpha(J)$ as defined above. The graded-McCrimmon radical is the smallest graded ideal of J such that the quotient $J/{\rm gr}\mathcal{M}(J)$ is graded-nondegenerate.*

We will need the following result.

Proposition 1.3. (see [A], [KZ] for $\Lambda = 0$) *If J is a graded-simple Jordan triple system, then J is graded-nondegenerate.*

Proof. We can suppose ${\rm gr}\mathcal{M}(J) = J$. Hence ${\rm gr}\mathcal{M}(J) = \mathcal{M}(J) = J$. By [A], [KZ] J is then *locally nilpotent*, i.e., every finitely generated subalgebra of J is nilpotent. But this immediately leads to a contradiction. □

Let e be a tripotent in a Jordan triple system J, i.e., $P(e)e = e$. We thus have the Peirce decomposition of J with respect to e, written as $J = J_2(e) \oplus J_1(e) \oplus J_0(e)$. If, in addition, J is graded and $e \in J^0$, it is immediate that that the Peirce spaces $J_i(e)$, $i = 0, 1, 2$, are graded: $J_i(e) = \bigoplus_{\lambda\in\Lambda} J_i^\lambda(e)$ where $J_i^\lambda(e) = J_i(e) \cap J^\lambda$.

Theorem 1.4. *Let J be a graded-simple Jordan triple system with a tripotent $0 \neq e \in J^0$. Then the Peirce subsystem $J_2(e)$ is graded-simple and if $J_0(e) \neq 0$, then $J_0(e)$ is also graded-simple.*

Proof. This can be proven in the same way as the ungraded result [M1, 3.8]. □

2. Graded-triangulated Jordan triple systems

In this section we begin our study of graded-triangulated Jordan triple systems. We define the basic notations used throughout, present a list of multiplication rules, and discuss graded-nondegeneracy and graded-simplicity (Prop. 2.23). Throughout J is a Jordan triple system, assumed to be graded from Def. 2.11 on.

A triple of nonzero tripotents $(u; e_1, e_2)$ is called a *triangle* if $e_i \in J_0(e_j)$, $i \neq j$, $e_i \in J_2(u)$, $i = 1, 2$, $u \in J_1(e_1) \cap J_1(e_2)$, and the following multiplication rules hold: $P(u)e_i = e_j$, $i \neq j$, and $P(e_1, e_2)u = u$. In this case, $e := e_1 + e_2$ is a tripotent such that e and u have the same Peirce spaces. The verification that $(u; e_1, e_2)$ is a triangle is simplified by the *Triangle Criterion* [N2, I.2.5], which says that *as soon as u and e_1 are tripotents satisfying $u \in J_1(e_1)$ and $e_1 \in J_2(u)$ then $(u; e_1, P(u)e_1)$ is a triangle.*

A Jordan triple system with a triangle $(u; e_1, e_2)$ is said to be *triangulated* if $J = J_2(e_1) \oplus \big(J_1(e_1) \cap J_1(e_2)\big) \oplus J_2(e_2)$ which is equivalent to $J = J_2(e)$. In this case, we will use the notation $J_i = J_2(e_i)$ and $M = J_1(e_1) \cap J_1(e_2)$. Hence

$$J = J_1 \oplus M \oplus J_2.$$

For such a J the index i will always vary in $\{1, 2\}$, in which case $j \in \{1, 2\}$ is given by $j = 3 - i$. An arbitrary product $P(x)y$ in J has the form $P(x_1 + m + x_2)(y_1 + n + y_2) = z_1 + r + z_2$, where

$$\begin{aligned} z_i &= P(x_i)y_i + P(m)y_j + \{x_i, n, m\}, \text{ and} \\ r &= P(m)n + \{x_1, y_1, m\} + \{x_2, y_2, m\} + \{x_1, n, x_2\}. \end{aligned}$$

Using Peirce multiplication rules and standard Jordan identities, most of these products can be written in terms of the quadratic operators

$$Q_i : M \to J_i : m \mapsto Q_i(m) := P(m)e_j,$$

with linearizations $Q_i(m, n) = P(m, n)e_j$, the automorphism $^- : J \to J$

$$x \mapsto \overline{x} := P(e)x, \quad e = e_1 + e_2,$$

and the bilinear maps $J_i \times M \to M$ defined by

2.1. $J_i \times M \to M : (x_i, m) \mapsto x_i \cdot m = L(x_i)m := \{x_i, e_i, m\}$.

Indeed we have [MN, 1.3.2-1.3.6]:

2.2. $P(m)n = Q_i(m, \overline{n}) \cdot m - Q_j(m) \cdot \overline{n}$,

2.3. $\{m, n, x_i\} = Q_i(m, x_i \cdot \overline{n}) = \{m, \overline{x_i} \cdot n, e_i\}$,

2.4. $\{m, x_i, n\} = Q_j(m, \overline{x_i} \cdot n) = Q_j(n, \overline{x_i} \cdot m)$,

2.5. $\{x_i, y_i, m\} = x_i \cdot (\overline{y_i} \cdot m)$,

2.6. $\{x_i, m, y_j\} = x_i \cdot (y_j \cdot \overline{m}) = y_j \cdot (x_i \cdot \overline{m})$,

while $P(x_i)y_i \in J_i$ and $P(m)y_i \in J_j$ cannot be reduced. Also [MN, 1.3.7]:

2.7. $e_i \cdot m = m$ and $P(x_i)y_i \cdot m = x_i \cdot \big(\overline{y_i} \cdot (x_i \cdot m)\big)$,

Note that $^-$ has period 2 with $\overline{e_i} = e_i$, stabilizes the Peirce subspaces J_i and M and reduces to $P(e_i)$ on J_i and $P(e_1, e_2)$ on M. We will also consider the *square* of elements $x_i \in J_i$ defined as

2.8. $x_i^2 := P(x_i)e_i$.

Because of 2.7 we have $L(x_i)^2 = L(x_i^2)$. We say that J is *faithfully triangulated* or that u is *faithful* if any $x_1 \in J_1$ with $x_1 \cdot u = 0$ vanishes. We will also need the traces

$$T_i(m) := Q_i(u, m) = \{u\, e_j\, m\},$$

and the map

$$^* := P(e)P(u) = P(u)P(e),$$

which is an automorphism of J of period 2 such that $u^* = u$, $e_i^* = e_j$, and so $J_i^* = J_j$. The following lemma is shown in the proof of [MN, 1.15] and will be used later.

Lemma 2.9. *Let $J = J_1 \oplus M \oplus J_2$ be a triangulated Jordan triple system. If $z = z_1 + m + z_2 \in \mathcal{T}(J)$, then $z_i \in \mathcal{T}(J_i)$ and $m \in Rad\, Q_i$ for $i = 1, 2$ where $Rad\, Q_i = \{m \in M : Q_i(m) = 0 = Q_i(m, M)\}$. Conversely, if $m \in Rad\, Q_1 \cup Rad\, Q_2$, then $P(m)M = 0 = P(P(m)J_i)J$.*

Definition 2.10. If $M = \bigoplus_{\lambda\in\Lambda} M^\lambda$ and $N = \bigoplus_{\lambda\in\Lambda} N^\lambda$ are graded modules and $Q : M \to N$ is a quadratic map, we call Q *graded* if $Q(M^\lambda) \subseteq N^{2\lambda}$ and $Q(M^\lambda, M^\nu) \subseteq N^{\lambda+\nu}$, where $Q(.,.)$ is the bilinear form associated to Q. Recall that the *radical* of Q is $Rad\, Q = \{m \in M : Q(m) = 0 = Q(m, M)\}$. In our situation the *graded-radical* of Q, defined as

$$\mathrm{gr}Rad\, Q := \textstyle\bigoplus_{\lambda\in\Lambda} \{m \in M^\lambda : Q(m) = 0 = Q(m, M)\},$$

will be more important. Naturally, we say that Q is *graded-nondegenerate* if $\mathrm{gr}Rad\, Q = 0$. It is easily seen that the submodule $\{m \in M : Q(m, M) = 0\}$ is graded. For any $m = \sum_{\lambda\in\Lambda} m^\lambda \in M$ satisfying $Q(m, M) = 0$, we have $Q(m) = \sum_{\lambda\in\Lambda} Q(m^\lambda)$, where $Q(m^\lambda) \in N^{2\lambda}$. Hence $\mathrm{gr}Rad\, Q = Rad\, Q$ if $\frac{1}{2} \in k$ or Λ does not have 2-torsion. In general, $\mathrm{gr}Rad\, Q$ is the greatest graded submodule of $Rad\, Q$.

Definition 2.11. A Jordan triple system J is said to be *graded-triangulated* if J is graded by some abelian group Λ and triangulated by $(u; e_1, e_2) \subseteq J^0$. We call a graded-triangulated J Λ-*triangulated* if $\mathrm{supp}_\Lambda J = \{\lambda \in \Lambda : J_i^\lambda \neq 0 \text{ or } M^\lambda \neq 0\}$ generates Λ as abelian group. We call J *graded-simple-triangulated* if J is graded-simple and graded-triangulated.

Let J be a graded-triangulated Jordan triple system. The grading group of a graded-triangulated Jordan triple system will usually be denoted by Λ. Observe that the Λ-grading is *compatible* with the Peirce decomposition: The subsystems J_i and M are graded: $J_i = \bigoplus_{\lambda\in\Lambda} J_i^\lambda$ and $M = \bigoplus_{\lambda\in\Lambda} M^\lambda$, the quadratic operators Q_i and the automorphisms * and $^-$ are graded.

As before we let $(u; e_1, e_2)$ be the triangle inducing the triangulation. The product formulas 2.2-2.6 show that a graded linear subspace $K = K_1 \oplus N \oplus K_2$ with $K_i \subseteq J_i$, $N \subseteq M$, is a graded subsystem if

2.12. $\overline{K_i} = K_i$, $\overline{N} = N$, $P(K_i)K_i \subseteq K_i$, $P(N)K_i \subseteq K_j$ *and* $K_i \cdot N \subseteq N$.

As in [MN], we denote by C the subalgebra of $\mathrm{End}_k(M)$ generated by

$$C_0 = L(J_1) = \textstyle\bigoplus_{\lambda\in\Lambda} L(J_1^\lambda),$$

and we say that u is C-*faithful* if $cu = 0$ implies $c = 0$. It is easily seen that $\mathrm{End}_k^\Lambda(M) := \bigoplus_{\lambda\in\Lambda} \mathrm{End}_k^\lambda(M)$, where $\mathrm{End}_k^\lambda(M) = \{\varphi \in \mathrm{End}_k(M) : \varphi(M^\gamma) \subseteq M^{\gamma+\lambda} \text{ for all } \gamma \in \Lambda\}$, is a subalgebra of $\mathrm{End}_k M$ which is graded by Λ. Note that $L(J_i^\lambda) \in \mathrm{End}_k^\lambda(M)$. Hence C_0 is a graded submodule, and this implies that C is a graded subalgebra of $\mathrm{End}_k^\Lambda(M)$, i.e., $C = \bigoplus_{\lambda\in\Lambda} C^\lambda$ where $C^\lambda = C \cap \mathrm{End}_k^\lambda(M)$. We have that $\overline{L(x_1)} := L(\overline{x_1}) = P(e)L(x_1)P(e) \in C_0$. Therefore $c \mapsto \overline{c} = P(e)cP(e)|_M$ defines an automorphism on C of period 2, which is graded. Moreover, $L : J_1 \to C_0$, $x_1 \mapsto L(x_1)$ is a nonzero graded specialization with respect to $P(c_0)d_0 = c_0\overline{d_0}c_0 \in C_0$, for $c_0, d_0 \in C_0$. By [MN, 1.6.6], C has a *reversal involution* π, i.e.,

2.13. $\big(L(x_1)\cdots L(x_n)\big)^\pi = L(x_n)\cdots L(x_1)$.

It easily follows from 2.4 that

2.14. $Q_2(cm, n) = Q_2(m, c^\pi n)$.

It is clear from 2.13 that π is homogeneous of degree 0 and commutes with the automorphism $^-$ of C. Moreover, $C_0 = \overline{C_0} \subseteq H(C, \pi)$ is an *ample subspace* of (C, π), i.e., $1 \in C_0$ and $cC_0c^\pi \subseteq C_0$ for all $c \in C$. Indeed, for $c \in C$, $x_1 \in J_1$, we have by [MN, 1.6.9]

2.15. $cL(x_1)c^\pi = L(P(cu)P(u)x_1)$.

Besides the formulas already mentioned we will use the following identities proven in [MN, 1.6.8, 1.6.11, 1.6.2, 1.6.3, 1.6.10, 1.6.12, 1.6.14]. For $c \in C$, $m \in M$ and $x_i \in J_i$ we have

2.16. $c + c^\pi = L\big(T_1(cu)\big)$,

2.17. $Q_2(cu, m) = T_2(c^\pi m)$,

2.18. $T_i(m)^* = T_j(m^*) = T_j(m)$ *and* $Q_i(m)^* = Q_j(m^*)$,

2.19. $m^* = T_i(m) \cdot u - m$,

2.20. $(cu)^* = c^* u = c^\pi u$ and hence $cc^* u = c^* cu$, where $c^* = P(u)cP(u)$. Note that C^* is the subalgebra of $\mathrm{End}_k(M)$ generated by $L(J_2)$.

2.21. $(x_i^* - x_i) \cdot m = \big(T_i(m) \cdot x_i - T_i(x_i \cdot m)\big) \cdot u = -\Gamma_i(x_i; m)u$,

where $\Gamma_i(x_i; m) := L\big(T_i(x_i \cdot m)\big) - L\big(T_i(m)\big)L(x_i)$. Observe that $\Gamma_i(x_i; m)$ is linear in the two variables.

2.22.

$$\begin{aligned}\Gamma_1(x_1; m)\Gamma_1(x_1; m)^\pi m &= L\big(Q_2(m)\big)[L(x_1), \Gamma_1(x_1; m)]u \\ &\quad + L(x_1)[L\big(Q_1(m)\big), L(x_1)]m + [L(x_1), L(P(m)P(u)x_1)]m \in Cu.\end{aligned}$$

The following proposition is a straightforward generalization of the corresponding result for $\Lambda = 0$ [MN, 1.15]. Its proof, which uses Prop. 1.3, Th. 1.4 and Lem. 2.9, will be left to the reader.

Proposition 2.23. *Let J be a graded-triangulated Jordan triple system. Then*

(i) *J is graded-nondegenerate iff J_1 and Q_1 are graded-nondegenerate, iff J_2 and Q_2 are graded-nondegenerate. In this case, J is faithfully triangulated.*

(ii) *J is graded-simple iff J_1 is graded-simple and Q_1 is graded-nondegenerate, iff J_2 is graded-simple and Q_2 is graded-nondegenerate.*

Definition 2.24. A graded-triangulated Jordan triple system J is called *division-triangulated* if the Jordan triple systems J_i, $i = 1, 2$, are division-graded and if every homogeneous $0 \neq m \in M$ is invertible in M, equivalently in J. We call J *division-Λ-triangulated* if J is division-triangulated and Λ-triangulated. Thus, $\mathrm{supp}_\Lambda J = \{\lambda \in \Lambda : J_i^\lambda \neq 0 \text{ or } M^\lambda \neq 0\}$ generates Λ as abelian group.

Any division-triangulated Jordan triple system is in particular graded-simple. Since $e_1 + e_2$ is invertible, an off-diagonal element $m \in M$ is invertible iff $P(m)(e_1 + e_2) = Q_1(m) \oplus Q_2(m)$ is invertible in J which is equivalent to both $Q_i(m)$ being invertible in J_i.

Let k be a field. A *triangulated Jordan triple torus* is a division-triangulated Jordan triple system $J = J_1 \oplus M \oplus J_2$ for which $\dim_k J_i^\lambda \leq 1$ and $\dim_k M^\lambda \leq 1$ for all $\lambda \in \Lambda$. We call such a Jordan triple system a *Λ-triangulated Jordan triple torus* if J is division-Λ-triangulated.

We will use the same approach to define "tori" and Λ-tori in other categories: associative algebras (3.10), Jordan algebras (6.4), Jordan pairs (6.13) and Lie algebras (7.3), and we will see in §7 the connection between them: Λ-triangulated Jordan structures coordinatize B_2-Lie tori, which is our principal motivation for studying them.

For the next lemma we recall that a subset $S \subset \Lambda$ is called a *pointed reflection subspace* if $0 \in S$ and $2S - S \subset S$, see for example [NY, 2.1], where it is also shown that any pointed reflection subspace is a union of cosets modulo $2\mathbb{Z}[S]$, including the trivial coset $2\mathbb{Z}[S]$. Here $\mathbb{Z}[S]$ denotes the $\mathbb{Z}$-span of S. In particular, a pointed reflection subspace is in general not a subgroup.

Lemma 2.25. *Let J be a division-Λ-triangulated Jordan triple system . Put*

$$\mathcal{L} = \operatorname{supp}_\Lambda J_1 = \operatorname{supp}_\Lambda J_2 \quad \text{and} \quad \mathcal{S} = \operatorname{supp}_\Lambda M.$$

Then $\mathcal{L}$ and $\mathcal{S}$ are pointed reflection subspaces of Λ satisfying

$$\mathcal{L} + 2\mathcal{S} \subset \mathcal{L} \quad \text{and} \quad \mathcal{L} + \mathcal{S} \subset \mathcal{S}. \tag{2.1}$$

Proof. We have $\operatorname{supp}_\Lambda J_1 = \operatorname{supp}_\Lambda J_2$ by applying the invertible operator $P(u)$ to J_i. That $\mathcal{L}$ and $\mathcal{S}$ are pointed reflection spaces is a general fact which is true for any division-graded Jordan triple system $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$ with $J^0 \neq 0$: We have, with obvious notation, $(y^\mu)^{-1} \in J^{-\mu}$, whence $P(x^\lambda)(y^\mu)^{-1} \in J^{2\lambda-\mu}$. The formulas in (2.1) follow from $P(m^\lambda)x_1^\mu \in J_2^{2\lambda+\mu}$ and invertibility of $L(x_1^\lambda)$ on M. □

Remark 2.26. The relations (2.1) are well-known from the theory of extended affine root systems of type B_2 ([AABGP, II]), or more generally, the theory of extension data for affine reflection systems ([LN2]). This is of course no accident in view of the connections between triangulated Jordan structures and B_2-graded Lie algebras, explained in §7.

3. Hermitian matrix systems

In this section we introduce the first of the two basic models for our paper, the hermitian matrix system (Def. 3.1), and we characterize them within the class of all triangulated Jordan triple systems in Th. 3.3. We then describe the graded ideals of a hermitian matrix system (Prop. 3.5), which allows us to describe the graded-(semi)prime and graded-simple hermitian matrix systems (Cor. 3.6 and Prop. 3.9). Finally, in Lem. 3.12 we describe the division-triangulated and tori among the hermitian matrix systems.

Definition 3.1. *Hermitian matrix systems* $\mathrm{H}_2(A, A_0, \pi, ^-)$. Recall [MN, §2] that an (associative) coordinate system $(A, A_0, \pi, ^-)$ consists of a unital associative k-algebra A with involution π and an automorphism $^-$ of period 2 commuting with π, together with a $^-$ stable π-*ample* subspace A_0, i.e., $\overline{A_0} = A_0 \subseteq \mathrm{H}(A, \pi)$, $1 \in A_0$ and $aa_0a^\pi \subseteq A_0$ for all $a \in A$ and $a_0 \in A_0$. We will call such a coordinate system *graded* by Λ if $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ is graded, π and $^-$ are homogeneous of degree 0 and A_0 is a graded submodule: $A_0 = \bigoplus_{\lambda\in\Lambda} A_0^\lambda$ for $A_0^\lambda = A_0 \cap A^\lambda$.

To a graded coordinate system $(A, A_0, \pi, ^-)$ we associate the *hermitian matrix system* $\mathrm{H} = \mathrm{H}_2(A, A_0, \pi, ^-)$ which by definition is the Jordan triple system of 2×2-matrices over A which are hermitian ($X = X^{\pi t}$) and have diagonal entries in A_0, with triple product $P(X)Y = X\overline{Y}^{\pi t}X = X\overline{Y}X$, t = transpose. The Jordan triple system $\mathrm{H}_2(A, A_0, \pi, ^-)$ is spanned by elements

$$a_0^\lambda[ii] = a_0^\lambda E_{ii} \quad \text{and} \quad a^\gamma[12] = a^\gamma E_{12} + (a^\gamma)^\pi E_{21} = (a^\gamma)^\pi[21],$$

$a^\gamma \in A^\gamma, a_0^\lambda \in A_0^\lambda$. Such a system is graded by Λ: $\mathrm{H} = \bigoplus_{\lambda\in\Lambda} \mathrm{H}^\lambda$ where $H^\lambda = \operatorname{span}\{a_0^\lambda[ii], a^\lambda[12] : a_0^\lambda \in A_0^\lambda, a^\lambda \in A^\lambda\}$, and graded-triangulated by $(u = 1[12];\ e_1 = 1[11],\ e_2 = 1[22]) \in \mathrm{H}^0$. Note that the automorphisms $^-$ and * of H defined in §2 are

$$\begin{aligned} \overline{a_0[11] + a[12] + b_0[22]} &= \overline{a_0}[11] + \overline{a}[12] + \overline{b_0}[22], \\ (a_0[11] + a[12] + b_0[22])^* &= b_0[11] + a^\pi[12] + a_0[22]. \end{aligned}$$

We say that H is *diagonal* if the diagonal coordinates A_0 generate all coordinates A. In this case, the involution π is the reversal involution with respect to A_0, i.e., $\pi(a_1 \cdots a_n) = a_n \cdots a_1$ for $a_i \in A_0$.

Example 3.2. As an example, suppose $A = B \boxplus B^{\mathrm{op}}$ is a direct algebra sum of an associative graded algebra B and its opposite algebra B^{op} and that π is the exchange involution $(b_1, b_2) \mapsto (b_2, b_1)$ of A. (Here and in the following $\boxplus$ denotes the direct sum of ideals.) Then necessarily $A_0 = \{(b, b) : b \in B\}$ and $\mathrm{H}_2(A, A_0, \pi, ^-)$ is canonically isomorphic to $\mathrm{Mat}_2(B)$, the 2×2-matrices over B, with Jordan triple product $P_x y = x\bar{y}x$ or $P_x y = x\bar{y}^t x$ depending on the automorphism $^-$ of A. Namely, we have the first case if $\overline{(b_1, b_2)} = (\bar{b}_1, \bar{b}_2)$ where $b \mapsto \bar{b}$ is an automorphism of B of period 2, and we have the second case if $\overline{(b_1, b_2)} = (b_2^\iota, b_1^\iota)$ where $b \mapsto b^\iota$ is an involution of B.

Within triangulated Jordan triple systems, the hermitian matrix systems can be characterized as follows.

Theorem 3.3. TRIANGULATED HERMITIAN COORDINATIZATION THEOREM. ([MN, 2.4] for $\Lambda = 0$) *For any graded Jordan triple system $J = J_1 \oplus M \oplus J_2$ which is faithfully triangulated by $(u; e_1, e_2)$, the graded subsystem*

$$J_h = J_1 \oplus Cu \oplus J_2,$$

where C denotes the subalgebra of $\mathrm{End}_k(M)$ generated by $C_0 = L(J_1)$, is graded-triangulated by $(u; e_1, e_2)$ and graded isomorphic to the diagonal hermitian matrix system $\mathrm{H} = \mathrm{H}_2(A, A_0, \pi, ^-)$ under the map

$$x_1 \oplus cu \oplus x_2 \mapsto \begin{pmatrix} L(x_1) & c \\ c^\pi & L(x_2^*) \end{pmatrix}$$

for $A = C|_{Cu}$, $A_0 = C_0|_{Cu}$, c^π as in 2.13, and $\overline{c} = P(e) \circ c \circ P(e)|_{Cu}$. The above isomorphism maps the triangle $(u; e_1, e_2)$ of J onto the standard triangle $(1[12]; 1[11], 1[22])$ *of* H. *We have $J = J_h$ as graded triple systems if and only if $M = Cu$.*

Proof. If J is a graded-triangulated Jordan triple system, then $(A, A_0, \pi, ^-)$, for $A = C|_{Cu}$, $A_0 = C_0|_{Cu}$, d^π as in 2.13 and $\overline{d} = P(e) \circ d \circ P(e)$, is a graded coordinate system. Since A_0 generates A, $\mathrm{H}_2(A, A_0, \pi, ^-)$ is a diagonal hermitian matrix system. Also, by definition, J_h is graded.

Now it follows from [MN, 2.4] that J_h is a subsystem of J isomorphic to $\mathrm{H}_2(A, A_0, \pi, ^-)$ under the map

$$x_1 \oplus cu \oplus x_2 \mapsto \begin{pmatrix} L(x_1) & c \\ c^\pi & L(x_2^*) \end{pmatrix},$$

which is clearly a graded isomorphism (recall that $(J_i^\lambda)^* = J_j^\lambda$). That the isomorphism preserves the triangles is clear. Also by [MN, 2.4], we have that $J_h = J$ if and only if $M = Cu$. □

Let A be a graded algebra and let $\mathfrak{S}$ be a set of endomorphisms of A preserving the grading. We call I a *graded ideal of* $(A, \mathfrak{S})$ if I is a graded ideal left invariant by all $s \in \mathfrak{S}$. If $\mathcal{P}$ is a property of an algebra defined in terms of ideals we will say that $(A, \mathfrak{S})$ is *graded-$\mathcal{P}$* if $\mathcal{P}$ holds for all graded ideals of $(A, \mathfrak{S})$. We will apply this for $\mathcal{P}$=graded-(semi)prime and $\mathcal{P}$=graded-simple. For $\mathfrak{S} = \{\pi, ^-\}$ as above we will determine the graded-simple $(A, \mathfrak{S})$-structures in Prop. 3.7. Here we only note:

Remark 3.4. *If $\mathfrak{S}$ is a finite semi-group consisting of automorphisms or involutions of a graded associative algebra A, then $(A, \mathfrak{S})$ is graded-semiprime if and only if A is graded-semiprime.* Indeed, if I is a graded ideal of A with $I^2 = 0$ then $\hat{I} = \sum_{s \in \mathfrak{S}} s(I)$ is an $\mathfrak{S}$-invariant graded ideal of A with $\hat{I}^n = 0$ for $n > |\mathfrak{S}|$. Hence $\hat{I} = 0$ and so also $I = 0$.

Proposition 3.5. *Let* $\mathrm{H} = \mathrm{H}_2(A, A_0, \pi, ^-)$ *be a hermitian matrix system. Then the graded ideals of* H *are exactly the submodules*

$$\mathrm{H}_2(B, B_0) = B_0[11] \oplus B[12] \oplus B_0[22]$$

for $(\pi, ^-)$-invariant graded submodules $B_0 \subseteq A_0$ and $B \subseteq A$ such that for $a \in A$, $a_0 \in A_0$, $b \in B$, and $b_0 \in B_0$,

(1) *$ba + b^\pi a^\pi$, $ba_0 b^\pi$, and $ab_0 a^\pi$ lie in B_0,*
(2) *ab_0, $a_0 b$, aba, and bab lie in B.*

In particular,

(i) *if B is a graded ideal of $(A, \pi, ^-)$, then $\mathrm{H}_2(B, B \cap A_0)$ is a graded ideal of $\mathrm{H}_2(A, A_0, \pi, ^-)$, and, conversely,*
(ii) *if $(A, \pi, ^-)$ is graded-semiprime and $\mathrm{H}_2(B, B_0)$ is a nonzero graded ideal of $\mathrm{H}_2(A, A_0, \pi, ^-)$, then there exists a nonzero graded ideal I of $(A, \pi, ^-)$ such that $\mathrm{H}_2(I, I_0) \subseteq \mathrm{H}_2(B, B_0)$, for $I_0 = I \cap B_0$.*

Proof. This easily follows from the case $\Lambda = 0$ which is proven in [MN, 2.7]. □

As a consequence, we have the following corollary whose proof is again omitted since it is based on a standard argument.

Corollary 3.6. ([MN, 2.7(5), 2.11] for $\Lambda = 0$) *Let* $\mathrm{H} = \mathrm{H}_2(A, A_0, \pi, ^-)$ *be a hermitian matrix system. Then*

(i) H *is graded-simple iff* $(A, \pi, ^-)$ *is graded-simple.*
(ii) *The following are equivalent:*
  (a) H *is graded-nondegenerate,*
  (b) H *is graded-semiprime,*
  (c) $(A, \pi, ^-)$ *is graded-semiprime,*
  (d) A *is graded-semiprime.*
(iii) H *is graded-prime iff* $(A, \pi, ^-)$ *is graded-prime.*

Because of Cor. 3.6(i) it is of interest to determine the graded-simple coordinate systems $(A, \pi, ^-)$. This can be done without assuming that A is associative.

Proposition 3.7. *Let A be an arbitrary, not necessarily associative, graded algebra with commuting involution π and automorphism $^-$ of order 2, which are homogeneous of degree 0. Then the graded-simple structures $(A, \pi, ^-)$ are precisely the following:*

(I) *graded-simple A with graded involution π and graded automorphism $^-$;*
(II) $A \cong_\Lambda B \boxplus B^{\mathrm{op}}$ *with exchange involution π for a graded-simple B with graded automorphism* $^- : (b_1, b_2)^\pi = (b_2, b_1), \overline{(b_1, b_2)} = (\overline{b_1}, \overline{b_2})$;
(III) $A \cong_\Lambda B \boxplus B^{\mathrm{op}}$ *with exchange involution π for a graded-simple B with graded involution* $\iota : (b_1, b_2)^\pi = (b_2, b_1), \overline{(b_1, b_2)} = ({b_2}^\iota, {b_1}^\iota)$;
(IV) $A \cong_\Lambda B \boxplus B$ *with exchange automorphism $^-$ for a graded-simple B with graded involution* $\pi : (b_1, b_2)^\pi = ({b_1}^\pi, {b_2}^\pi), \overline{(b_1, b_2)} = (b_2, b_1)$;
(V) $A \cong_\Lambda B \boxplus B^{\mathrm{op}} \boxplus B \boxplus B^{\mathrm{op}}$ *for a graded-simple B with π the exchange involution of $C = B \boxplus B^{\mathrm{op}}$ and $^-$ the exchange automorphisms of $C \boxplus \overline{C}$: $(a_1, a_2, a_3, a_4)^\pi = (a_2, a_1, a_4, a_3)$ and $\overline{(a_1, a_2, a_3, a_4)} = (a_3, a_4, a_1, a_2)$.*

Proof. The proof is again a straightforward generalization of the corresponding result in the ungraded situation, which is [MN, 2.8]. ☐

For later use we note the following special case of Prop. 3.7 for a commutative algebra D and π the identity "involution". We note that D is not assumed to be unital.

Corollary 3.8. *Let D be a commutative graded algebra with a graded automorphism $^-$ of order 2. Then $(D, ^-)$ is graded-simple if and only if either D is graded-simple or $D \cong_\Lambda B \boxplus B$, for a commutative graded-simple B with the exchange automorphism.*

Recall ([L, §1.14]) that a Jordan triple system T is called *polarized* if there exist submodules $T^\pm$ such that $T = T^+ \oplus T^-$ and for $\sigma = \pm$ we have $P(T^\sigma)T^\sigma = 0 = \{T^\sigma, T^\sigma, T^{-\sigma}\}$ and $P(T^\sigma)T^{-\sigma} \subseteq T^\sigma$. In this case, $V = (T^+, T^-)$ is a Jordan pair. Conversely, to any Jordan pair $V = (V^+, V^-)$ we can associate a polarized Jordan triple system $T(V) = V^+ \oplus V^-$ with quadratic map P defined by $P(x)y = Q(x^+)y^- \oplus Q(x^-)y^+$ for $x = x^+ \oplus x^-$

and $y = y^+ \oplus y^-$. In fact, the category of Jordan pairs is equivalent to the category of polarized Jordan triple systems. It is also known that for any Jordan triple system T the pair (T, T) is a Jordan pair ([L, §1.13]). Hence, it has an associated polarized Jordan triple system which we will denote $T \oplus T$. Examples are the cases (IV) and (V) of Prop. 3.9 below.

Proposition 3.9. HERMITIAN GRADED-SIMPLICITY CRITERION. *A graded Jordan triple system is a graded-simple-triangulated hermitian matrix system* $\mathrm{H}_2(A, A_0, \pi, ^-)$ *if and only if it is graded isomorphic to one of the following:*

(I) $\mathrm{H}_2(A, A_0, \pi, ^-)$ *for a graded-simple A;*

(II) $\mathrm{Mat}_2(B)$ *for a graded-simple associative unital B with graded automorphism $^-$, where $\overline{(b_{ij})} = (\overline{b_{ij}})$ for $(b_{ij}) \in \mathrm{Mat}_2(B)$ and $P(x)y = x\overline{y}x$;*

(III) $\mathrm{Mat}_2(B)$ *for a graded-simple associative unital B with graded involution ι, where $\overline{(b_{ij})} = (b^\iota_{ij})$ for $(b_{ij}) \in \mathrm{Mat}_2(B)$ and $P(x)y = x\overline{y}^t x$;*

(IV) *polarized* $\mathrm{H}_2(B, B_0, \pi) \oplus \mathrm{H}_2(B, B_0, \pi)$ *for a graded-simple B with graded involution π;*

(V) *polarized* $\mathrm{Mat}_2(B) \oplus \mathrm{Mat}_2(B)$ *for a graded-simple associative unital B and $P(x)y = xyx$.*

Among the cases (II)–(V)*, the matrix system is diagonal iff B is noncommutative.*

Proof. By definition and Cor. 3.6(i), a graded Jordan triple system is a graded-simple $J = \mathrm{H}_2(A, A_0, \pi, ^-)$ if and only if $(A, A_0, \pi, ^-)$ is a graded coordinate system where $(A, \pi, ^-)$ is graded-simple. Since the graded-simple structures $(A, \pi, ^-)$ have been described in Prop. 3.7, it now suffices to show that the cases (I)–(V) of Prop 3.7 correspond to the cases (I)–(V) above. This is straightforward and will be left to the reader, see [MN, 2.10] for the case in which the grading group $\Lambda = 0$. □

In order to describe division-triangulated hermitian matrix systems we need to introduce some concepts from the theory of division-graded algebras.

Definition 3.10. A unital associative graded algebra $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ is called *predivision-graded* if every nonzero homogeneous space contains an invertible element. The *support* $\mathrm{supp}_\Lambda A = \{\lambda \in \Lambda : A^\lambda \neq 0\}$ of a predivision-graded A is a subgroup of Λ. We will call A *predivision-Λ-graded* if $\mathrm{supp}_\Lambda A = \Lambda$.

After choosing a family of invertible elements $(u_\lambda : \lambda \in \Lambda)$ with $u_\lambda \in A^\lambda$, one can identify a predivision-Λ-graded algebra A with a crossed-product algebra $A = (B, \Lambda, \sigma, \tau)$ in the sense of [P] with $B = A^0$ and the twist τ and the action σ defined by $u_\lambda u_\mu = \tau(\lambda, \mu) u_{\lambda+\mu}$ and $u_\lambda b = \big({}^{\sigma(\lambda)}(b)\big) u_\lambda$ for $b \in B$.

An example of a predivision-Λ-graded algebra is the so-called *twisted group algebra* $B^t[\Lambda]$, i.e., the crossed product algebra $(B, \Lambda, \sigma, \tau)$ with $\sigma(\lambda) = \mathrm{Id}_B$ for all $\lambda \in \Lambda$. An immediate special case of a twisted group algebra is $k[\Lambda]$, the *group algebra* of Λ over k where $\tau(\lambda, \mu) = 1_k$ for all $\lambda, \mu \in k$.

A unital associative graded algebra A is called a *division-graded* if every nonzero homogeneous element is invertible, and such an algebra is called *division-Λ-graded* if $\operatorname{supp}_\Lambda A = \Lambda$. A division-$\Lambda$-graded algebra A is the same as a crossed product algebra $(B, \Lambda, \sigma, \tau)$ with B a division algebra. A division-graded algebra is in particular graded-simple.

A unital associative commutative graded algebra A is graded-simple if and only if it is division-graded. Such algebras will be called *graded-fields*, more precisely Λ-*graded-fields* if $\operatorname{supp}_\Lambda A = \Lambda$. A Λ-graded-field is the same as a twisted group algebra $B^t[\Lambda]$ with B a field. If Λ is free, a Λ-graded field is isomorphic to the group algebra of Λ.

If A is a division-graded algebra defined over a field k and such that $\dim_k A^\lambda \le 1$, then A is said to be an *(associative) torus*. In this case we call A a Λ-*torus* if $\operatorname{supp}_\Lambda A = \Lambda$. From the point of view of crossed product algebras, a Λ-torus is the same as a twisted group algebra $k^t[\Lambda]$ over the field k. We note that in this case τ is a 2-cocycle of Λ with coefficients in k.

Example 3.11. $\mathbb{Z}^n$-*tori.* Let A be a $\mathbb{Z}^n$-torus and choose nonzero $t_i \in A^{\epsilon_i}$, where ϵ_i is the ith-canonical basis vector of $\mathbb{Z}^n$. Then the algebra structure of A is uniquely determined by the rules

$$t_i t_i^{-1} = 1_A = t_i^{-1} t_i,\ 1 \le i \le n \quad \text{and} \quad t_i t_j = q_{ij} t_j t_i,\ 1 \le i, j \le n \tag{3.1}$$

where $q_{ij} \in k$ satisfy

$$q_{ii} = 1 = q_{ij}\, q_{ji} \quad \text{for } 1 \le i, j \le n. \tag{3.2}$$

For example $A^\lambda = kt^\lambda$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ and $t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}$. Conversely, let $q = (q_{ij})$ be a $n \times n$-matrix over the field k whose entries satisfy (3.2), then the associative unital algebra k_q defined by generators t_i, t_i^{-1} and relations (3.1) is a $\mathbb{Z}^n$-torus. It is customary to call k_q a *quantum $\mathbb{Z}^n$-torus* or simply a *quantum torus* if the grading group is not important, since k_q can be viewed as a quantization of the coordinate ring of the n-torus $(k^\times)^n$, i.e. the Laurent polynomial ring in n variables. Observe that k_q is a Laurent polynomial ring iff all $q_{ij} = 1$.

A quantum torus has a graded involution iff all $q_{ij} = \pm 1$ ([AG, §2]). In this case, an example of a well-defined involution is the *reversal involution* $\pi_{\rm rev}$ with respect to the generating set $\{t_1^{\pm 1}, \dots, t_n^{\pm 1}\}$:

$$\pi_{\rm rev}\big(t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}\big) = t_n^{\lambda_n} t_{n-1}^{\lambda_{n-1}} \cdots t_1^{\lambda_1} \tag{3.3}$$

The following lemma is immediate from the definitions above and the multiplication rules of hermitian matrix systems.

Lemma 3.12. *Let* $\mathrm{H} = \mathrm{H}_2(A, A_0, \pi, ^-)$ *be a hermitian matrix system.*

(a) *The following are equivalent:*

(i) *Every homogeneous* $0 \ne m \in M = A[12]$ *is invertible in* H,

(ii) A *is division-graded,*

(iii) H *is division-triangulated.*

(b) *Let k be a field. Then* H *is a Λ-triangulated Jordan triple torus iff A is a Λ-torus.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from the multiplication rules of H. If A is division-graded, a nonzero homogeneous element $a_0 \in A_0$ is invertible in A, say with inverse b_0. We have $b_0 \in \mathrm{H}(A,\pi)$ since $a_0 \in \mathrm{H}(A,\pi)$. But then $b_0 = b_0 a_0 b_0^\pi \in A_0$, whence A_0 is division-graded, proving (iii). The implication (iii) $\Rightarrow$ (i) is immediate, and (b) follows from (a). $\square$

Example 3.13. As a special case of Lem. 3.12 we get: $\mathrm{H}_2(A, A_0, \pi, {}^-)$ is a $\mathbb{Z}^n$-triangulated Jordan triple torus iff A is a quantum $\mathbb{Z}^n$-torus.

4. Clifford systems

In this section we introduce the second model of a graded-triangulated Jordan triple system, the ample Clifford systems (Def. 4.2), and we characterize them within the class of triangulated Jordan triple systems in Th. 4.3. We describe the graded-(semi)prime, graded-strongly prime and graded-simple Clifford systems in Prop. 4.4 and the division-triangulated and tori among the Clifford systems in Cor. 4.5.

Definition 4.1. *Quadratic form triples.* Let $D = \bigoplus_{\lambda\in\Lambda} D^\lambda$ be a graded unital commutative associative k-algebra endowed with an involution ${}^-$ of degree 0, i.e., $\overline{D^\lambda} = D^\lambda$ for all $\lambda \in \Lambda$. If

(i) V is a graded D-module, i.e., $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$ is a decomposition into k-submodules such that $d^\lambda x^\gamma \in V^{\lambda+\gamma}$ for $d^\lambda \in D^\lambda$, $x^\gamma \in V^\gamma$ and all $\lambda, \gamma \in \Lambda$,

(ii) $q : V \to D$ is a graded D-quadratic form (cf. Def. 2.10), and

(iii) $S : V \to V$ is a hermitian isometry of order 2 and degree 0, i.e., $S(dx) = \overline{d}S(x)$ for $d \in D$, $q\big(S(x)\big) = \overline{q(x)}$, $S^2 = \mathrm{Id}$ and $S(V^\lambda) = V^\lambda$,

then V becomes a Jordan triple system, denoted $J(q,S)$ and called a *quadratic form triple*, by defining $P(x)y = q\big(x, S(y)\big)x - q(x)S(y)$ for $x, y \in V$ (see for example [N2, §1, Ex. 1.6]). Clearly $J(q,S)$ is graded by Λ. We note for later use:

$$x \in J(q,S) \textit{ is invertible} \iff q(x) \in D \textit{ is invertible,} \tag{4.1}$$

and then $x^{-1} = q(x)^{-1}S(x)$.

Definition 4.2. *Ample Clifford systems* $\mathrm{AC}(q, S, D_0)$. We consider (M, q, S, u), where (M, q, S) satisfy (i)–(iii) of Def. 4.1 above and in addition

(iv) there exists $u \in M^0$ with $q(u) = 1$ and $S(u) = u$.

We then define $(\tilde{M}, \tilde{q}, \tilde{S})$ as follows:

(i)′ $\tilde{M} := De_1 \oplus M \oplus De_2$, where $De_1 \oplus De_2$ is a free graded D-module with basis (e_1, e_2) of degree 0,

(ii)′ $\tilde{q} : \tilde{M} \to D$ is the quadratic form given by $\tilde{q}(d_1e_1 \oplus m \oplus d_2e_2) = d_1d_2 - q(m)$, whence $De_1 \oplus De_2$ is a hyperbolic plane orthogonal to M, and (iii)′ $\tilde{S} : \tilde{M} \to \tilde{M}$ is the map $d_1e_1 \oplus m \oplus d_2e_2 \mapsto \overline{d_2}e_1 \oplus -S(m) \oplus \overline{d_1}e_2$.

It is then easily checked that $(\tilde{M}, \tilde{q}, \tilde{S})$ also satisfies the conditions (i)–(iii) above, and therefore yields a quadratic form triple, called *full Clifford system* and denoted $\mathrm{FC}(q, S)$. Its multiplication is given by

$$P(c_1e_1 \oplus m \oplus c_2e_2)\,(b_1e_1 \oplus n \oplus b_2e_2) = d_1e_1 \oplus p \oplus d_2e_2, \tag{4.2}$$

where

$$\begin{aligned} d_i &= c_i^2\,\overline{b_i} + c_i\,q\big(m, S(n)\big) + \overline{b_j}\,q(m) \\ p &= [c_1\overline{b_1} + c_2\overline{b_2} + q\big(m, S(n)\big)]m + [c_1c_2 - q(m)]S(n) \end{aligned}$$

and

$$\begin{aligned} \{c_1e_1 \oplus m \oplus c_2e_2\,,\, b_1e_1 \oplus n \oplus b_2e_2\,,\, c_1'e_1 \oplus m' \oplus c_2'e_2\} \\ = d_1e_1 \oplus p \oplus d_2e_2, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} d_i &= q\big(c_im' + c_i'm\,,\, S(n)\big) \;+\; b_j\,q(m, m') + 2c_i\,c_i'\,\overline{b_i} \\ p &= [c_1\overline{b_1} + c_2\overline{b_2} + q\big(m, S(n)\big)]\,m' + [c_1'\overline{b_1} + c_2'\overline{b_2} + q\big(m', S(n)\big)]\,m \\ &\quad + [c_1c_2' + c_1'c_2 - q(m, m')]S(n). \end{aligned}$$

Note that $\mathrm{FC}(q, S)$ is graded-triangulated by $(u; e_1, e_2)$.

As already observed in [MN, 3.5], in general we need not take the full Peirce spaces De_i in order to get a graded-triangulated Jordan triple system. Indeed, let us define a *Clifford-ample subspace of* $(D, {}^-, q)$ as a graded k-submodule D_0 of D, such that $D_0 = \overline{D_0}$, $1 \in D_0$ and $D_0q(M) \subseteq D_0$. Then

$$M_0 := D_0\,e_1 \oplus M \oplus D_0\,e_2$$

is a graded subsystem of the full Clifford system $\mathrm{FC}(q, S)$ containing the triangle $(u; e_1, e_2)$. Hence it is a graded-triangulated Jordan triple system, called an *ample Clifford system* and denoted $\mathrm{AC}(q, S, D_0)$ or $\mathrm{AC}(q, M, S, D, {}^-, D_0)$ if more precision is necessary. Note that $J_0 = \mathrm{AC}(q, S, D_0)$ is an outer ideal of the full Clifford system $J = \mathrm{FC}(q, S)$.

We point out that $(D, \mathrm{Id}, {}^-)$ is a graded associative coordinate system in the sense of Def. 3.1, and that a Clifford-ample subspace D_0 is in particular $(\mathrm{Id}, {}^-)$-ample. Hence ample Clifford systems are full in characteristic $\neq 2$, which here means $D_0 = D$.

Our derived operations of §2 on J_0 are

$$\begin{aligned} \overline{d_0e_1 \oplus m \oplus c_0e_2} &= \overline{d_0}e_1 \oplus S(m) \oplus \overline{c_0}e_2, \\ (d_0e_1 \oplus m \oplus c_0e_2)^* &= c_0e_1 \oplus (q(u, m)u - m) \oplus d_0e_2. \end{aligned}$$

Theorem 4.3. Clifford coordinatization theorem. ([MN, 3.6, 3.10] for $\Lambda = 0$) *Let $J = J_1 \oplus M \oplus J_2$ be a graded Jordan triple system which is faithfully triangulated by $(u; e_1, e_2)$. For $i = 1, 2$, define*

(i) *C_i as the subalgebra of $\mathrm{End}_k(M)$ generated by $L(J_i)$,*

(ii) *$\Gamma_i(x_i; m) := L\big(T_i(x_i \cdot m)\big) - L\big(T_i(m)\big)L(x_i) \in C_i$ for $x_i \in J_i, m \in M$,*

(iii) $\Delta_i(x_i; m) = L(P(m)P(u)x_i) - L\big(Q_i(m)\big)L(x_i) \in C_i$,
(iv) $N_0 = \{n_0 \in M : \Gamma_i(J_i; n_0) = 0,\ i = 1, 2\}$,
(v) $K_i = \{k_i \in J_i : \Gamma_i(k_i; C_i N_0) = 0\}$,
(vi) $N^{\mathrm{gr}} = \bigoplus_{\lambda\in\Lambda}(N \cap M^\lambda)$, *for* $N = \{n \in M : \Delta_i(J_i; n) = \Delta_i(J_i; n, N_0) = \Gamma_i(T_i(n); C_i N_0) = 0,\ i = 1, 2\} \subseteq N_0$, *i.e.,* N^{gr} *is the greatest graded submodule of* N.

Then

$$J_q = K_1 \oplus N^{\mathrm{gr}} \oplus K_2$$

is a graded subsystem of J *which is faithfully triangulated by* $(u; e_1, e_2)$ *and graded isomorphic to the ample Clifford system* $\mathrm{AC}(q, N^{\mathrm{gr}}, S, D, ^-, D_0)$ *under the map*

$$x_1 \oplus n \oplus x_2 \mapsto L(x_1) \oplus n \oplus L(x_2^*),$$

where $D_0 = L(K_1)$, D *is the subalgebra of* $\mathrm{End}_k(N^{\mathrm{gr}})$ *generated by* D_0, $\bar{c} = P(e) \circ c \circ P(e)|_{N^{\mathrm{gr}}}$, $q(n) = L\big(Q_1(n)\big)$, *and* $S(n) = P(e)n$. *The above isomorphism maps the triangle of* J *onto the standard triangle of* $\mathrm{AC}(q, N^{\mathrm{gr}}, S, D, ^-, D_0)$.

Moreover, $J = J_q$ *as graded triple systems if and only if* $\Delta_1(J_1; M) \equiv 0$. *In particular, if* u *is* C_1*-faithful and* $(x_1 - x_1^*) \cdot m = 0$, *for all* $x_1 \in J_1$, $m \in M$, *then* $\Delta_1(J_1; M) \equiv 0$ *and so* $J = J_q$.

Proof. Let J be a graded Jordan triple system faithfully triangulated by $(u; e_1, e_2)$. By [MN, 3.10], $K_1 \oplus N \oplus K_2$ is a subsystem of J faithfully triangulated by $(u; e_1, e_2)$. Since N_0 and K_i, $i = 1, 2$, are graded (all the defining identities are linear), J_q is the greatest graded submodule of $K_1 \oplus N \oplus K_2$. Clearly $(u; e_1, e_2) \in J_q^0$. To see that J_q is also a graded subsystem of J faithfully triangulated by $(u; e_1, e_2)$, it is in view of 2.12 enough to prove $\overline{N^{\mathrm{gr}}} = N^{\mathrm{gr}}$ and $K_i \cdot N^{\mathrm{gr}} \subseteq N^{\mathrm{gr}}$. But this follows directly from $\overline{N} = N$, $K_i \cdot N \subseteq N$, $J_i^\lambda \cdot M^\gamma \subseteq M^{\lambda+\gamma}$ and the fact that $^-$ is homogeneous of degree 0. On the other hand, since $\Delta_1(K_1; N^{\mathrm{gr}}) \equiv 0$ by definition, we have by [MN, 3.6] that J_q is isomorphic to the ample Clifford system $\mathrm{AC}(q, N^{\mathrm{gr}}, S, D, ^-, D_0)$ under the map and data specified in the theorem. Clearly the isomorphism is homogeneous of degree 0 and preserves the triangles.

Recall from [MN, 3.10] that $J = K_1 \oplus N \oplus K_2$ if and only if $\Delta_1(J_1; M) \equiv 0$. In this case $N = M$ is graded whence $J_q = J$. Conversely, if $J_q = J$ then $\Delta_1(J_1; M^\lambda) \equiv 0$, which implies that $\Delta_1(J_1; M) \equiv 0$: $\Delta_1(J_1; m^\lambda + m^\gamma) = \Delta_1(J_1; m^\lambda, m^\gamma) \subseteq \Delta_1(J_1; m^\lambda, N_0) \equiv 0$ since $N \subseteq N_0$. Finally, if u is C_1-faithful and $(x_1 - x_1^*) \cdot m = 0$, then $\Delta_1(J_1; M) \equiv 0$ by [MN, 3.8] and so $J = J_q$. □

Proposition 4.4. *Let* $J = \mathrm{AC}(q, M, S, D, ^-, D_0) = D_0 e_1 \oplus M \oplus D_0 e_2$ *be an ample Clifford system for which* D *acts faithfully on* M.

(i) *If* J *is graded-(semi)prime, then* $(D, ^-)$ *is graded-(semi)prime.*
(ii) *The following are equivalent:*
   (a) J *is graded-nondegenerate,*
   (b) q *is graded-nondegenerate and* J *is graded-semiprime,*

(c) *q is graded-nondegenerate and $(D,^{-})$ is graded-semiprime.*

(iii) *J is graded-strongly prime iff q is graded-nondegenerate and $(D,^{-})$ is graded-prime.*

(iv) ([MN, 3.14, 3.15] for $\Lambda = 0$) *The following are equivalent:*

(a) *J is graded-simple,*

(b) *q is graded-nondegenerate and $(D,^{-})$ is graded-simple,*

(c) *J is graded isomorphic to one of the following:*

(I) $\mathrm{AC}(q, S, F_0)$ *for a graded-nondegenerate q over a graded-field F with Clifford-ample subspace F_0, or*

(II) *a polarized* $\mathrm{AC}(q, S, F_0) \oplus \mathrm{AC}(q, S, F_0)$, *where* $\mathrm{AC}(q, S, F_0)$ *is as in* (I).

In this case, u is D-faithful.

Proof. We will first establish some preliminary results on the structure of graded ideals of J. Thus, let I be a graded ideal of J. We will use the multiplication formulas (4.2), (4.3) to evaluate the possibilities for I. First, invariance of I under $P(e_i)$, $P(e_1, e_2)$ and $P(u)$ shows that

$$I = B_0 e_1 \oplus N \oplus B_0 e_2, \tag{1}$$

for some graded k-submodules $B_0 \subseteq D_0$, $N \subseteq M$. From $P(e_1)I \subseteq I$ we obtain $B_0 = \overline{B_0}$. Furthermore, we claim

$$q \text{ graded-nondegenerate and } B_0 = 0 \Rightarrow I = 0. \tag{2}$$

Indeed, in this case $P(N)e_2 + \{N, e_2, M\} \subseteq B_0$ implies $q(N) + q(N, M) \subseteq B_0 = 0$, whence $N^\lambda \subseteq \mathrm{gr}\mathit{Rad}\, q$ for all $\lambda \in \Lambda$, and then $N = 0$, therefore $I = 0$. Next we claim

$$(D,^{-}) \text{ graded-simple, } q \text{ graded-nondegenerate} \Rightarrow J \text{ graded-simple.} \tag{3}$$

First notice that by Cor. 3.8 either D is a division-graded algebra or is the direct sum of two copies of a division-graded algebra with the exchange automorphism. Let I be a proper graded ideal of J which we write in the form (1). Then $B_0 \neq D$, since $B_0 = D$ would imply $e_1 \in I$, which in turn would force $e_2 = P(u)e_1 \in I$, $e_1 + e_2 \in I$ and then $I = J$. Now, as in the proof of [MN, 3.14], it follows that no $b_0 \in B_0$ is invertible in D: If $b_0^{-1} \in D$, then $\overline{b_0^{-2}} = q(\overline{b_0^{-1}}u) \subseteq D_0$, since $q(M) \subseteq D_0$, and $e_1 = b_0^2 \overline{\overline{b_0^{-2}}} e_1 = P(b_0 e_1)(\overline{b_0^{-2}} e_1) \in P(I)J \subseteq I$. Hence, if D is division-graded, $B_0^\lambda = 0$ for all $\lambda \in \Lambda$, and then $B_0 = 0$. If, otherwise, $D = A \boxplus A$ for a division-graded A with the exchange automorphism, let $b_0 = (a, 0)$ or $(0, a)$ be in B_0 for a homogeneous $a \in A^\lambda$. Then $(a, a) = b_0 + \overline{b_0} \in B_0$, and since (a, a) is not invertible in D, we get $a = 0$ and then $B_0 = 0$, in which case $I = 0$ by (2).

$$(D,^{-}) \text{ graded-(semi)prime and } q \text{ graded-nondegenerate } \Rightarrow \\ J \text{ graded-(semi)prime.} \tag{4}$$

We suppose I, K are graded ideals of J with $I = K$ in the semiprime case, satisfying $P(I)K = 0$. By (1), we can write I, K in the form $I = B_0 e_1 \oplus N \oplus$

B_0e_2 and $K = C_0e_1 \oplus L \oplus C_0e_2$. From $P(B_0e_1)C_0e_1 = 0$ we get $B_0^2C_0 = 0$. If D is graded-(semi)prime (always in the semiprime case by Rem. 3.4), then $B_0 = 0$ or $C_0 = 0$. If D is not graded-prime, then it easily follows that D is a subdirect sum $A \boxplus_{\mathrm{s}} A$ of two copies of a graded-prime algebra A with the exchange automorphism. In this case, let $0 \neq b_0 \in B_0$, $0 \neq c_0 \in C_0$ be homogeneous elements. Then, by graded-primeness of A, $b_0^2c_0 = 0$ implies that $b_0 = (b, 0), c_0 = (0, c)$ or $b_0 = (0, b), c_0 = (c, 0)$ for homogeneous $b, c \in A$, respectively. Without loss of generality, assume $b_0 = (b, 0)$ and $c_0 = (0, c)$. Hence $b_0 + \overline{b_0} = (b, b) \in B_0$ and $0 = (b^2, b^2)(0, c) = (0, b^2c)$, thus $b^2c = 0$. But, again by the graded-primeness of A we have that $b^2 = 0$ or $c = 0$, that is, $b = 0$ or $c = 0$, which is a contradiction. Then $B_0 = 0$ or $C_0 = 0$. Therefore $I = 0$ or $K = 0$ by (2).

For the proof of the other directions we again establish some preliminary results. Let B be a graded ideal of $(D, ^-)$. A straightforward verification using the multiplication rules (4.2), (4.3) shows that then

$$\tilde{B} := (B \cap D_0)e_1 \oplus BM \oplus (B \cap D_0)e_2$$

is a graded ideal of J. Since $BM = 0$ implies $B \subseteq \mathrm{Ann}_D(M) = 0$, it is clear that

$$\tilde{B} = 0 \Leftrightarrow B = 0.$$

On the other hand, if $\tilde{B} = J$, then $1 \in D_0 = B \cap D_0$, hence $B = D$. Then

$$\tilde{B} = J \Leftrightarrow B = D.$$

It now follows easily from the multiplication rules in Def. 4.2 that

$$J \text{ graded-(semi)prime } \Rightarrow (D, ^-) \text{ graded-(semi)prime, and} \tag{5}$$

$$J \text{ graded-simple } \Rightarrow (D, ^-) \text{ graded-simple.} \tag{6}$$

For the proof of (ii) we also need

$$(D, ^-) \text{ graded-semiprime } \Rightarrow J_1 \text{ graded-nondegenerate.} \tag{7}$$

Indeed, if $d_0 \in D_0$ is a homogeneous trivial element of J_1, then $d_0^2D_0 = 0$. In particular, $d_0^2 = 0$. But in a graded-semiprime commutative algebra, all homogeneous nilpotent elements vanish, so $d_0 = 0$.

Finally, by using Prop. 2.23(i) and the fact that $Q_1 = q$ in our situation we have

$$J \text{ graded-nondegenerate } \Leftrightarrow \tag{8}$$
$$q \text{ graded-nondegenerate and } J_1 \text{ graded-nondegenerate.}$$

Now the proof of (i)-(iv) follows easily: (i) is (5). For (ii), the fact that any graded-nondegenerate Jordan triple system is also graded-semiprime together with (i), (7) and (8) yields: J graded-nondegenerate (by (8)) $\Rightarrow$ q graded-nondegenerate and J graded-semiprime (by (i)) $\Rightarrow$ q graded-nondegenerate and $(D, ^-)$ graded-semiprime (by (7)) $\Rightarrow$ q graded-nondegenerate and J_1 graded-nondegenerate (by (8)) $\Rightarrow$ J graded-nondegenerate.

For (iii), we have by definition of graded-strongly primeness and (i) and (ii) that J graded-strongly prime implies q graded-nondegenerate and $(D,^-)$ graded-prime. The converse direction follows from (ii) and (4). Finally (iv) follows from (3), (6) and graded-nondegeneracy of J characterized by (8). Note that then u is D-faithful since $\{d \in D : du = 0\}$ is a proper graded ideal of $(D,^-)$. The remaining statements in (iv) are an immediate application of Cor. 3.8. $\square$

The assumption that D acts faithfully on M in the preceding proposition and the following corollary will be automatic in the application later on.

Corollary 4.5. *Let $J = \mathrm{AC}(q, S, D_0) = D_0e_1 \oplus M \oplus D_0e_2$ be an ample Clifford system with D acting faithfully on M.*

(a) *The following are equivalent:*

(i) *Every nonzero homogeneous element of M is invertible,*

(ii) *D is a graded-field and q is* graded-anisotropic *in the sense that $0 \neq q(m) \in D$ for every nonzero homogeneous $m \in M$,*

(iii) *J is division-triangulated.*

In this case M is a free D-module.

(b) *Let k be a field. Then J is a triangulated Jordan triple torus iff*

(I) *D is a torus, say with $\mathrm{supp}_\Lambda D = \Gamma$, hence $D = k^t[\Gamma]$ is a twisted group algebra, and*

(II) *M is a free D-module with a homogeneous D-basis $\{u_i : i \in I\}$, say $u_i \in M^{\delta_i}$, with $q(u_i) \neq 0$ and $(\delta_i + \Gamma) \neq (\delta_j + \Gamma)$ for $i \neq j$.*

If in this case $D = D_0$, then $\{u_i : i \in I\}$ is an orthogonal basis: $q(u_i, u_j) = 0$ for $i \neq j$.

Proof. (a) If (i) holds, the invertibility criterion (4.1) together with $q(du) = d^2$ implies that D is a graded-field and then that q is graded-anisotropic. Suppose (ii). Then clearly every nonzero homogenous $m \in M$ is invertible. Moreover, it follows as in Lem 3.12 that D_0 is a division-graded triple, whence J is division-triangulated. The implication (iii) $\Rightarrow$ (i) is clear. It is a standard fact that any graded module over a division-graded algebra is free with a homogeneous basis.

(b) Suppose J is a triangulated Jordan triple torus. Then (a) applies. Since $D \to D.u \subset M$ is injective and homogeneous of degree 0, D is a torus. Hence $D = k^t[\Gamma]$ is a twisted group algebra for $\Gamma = \mathrm{supp}_\Lambda D$, a subgroup of Λ. As in (a), M is a free D-module with a homogeneous basis $\{u_i : i \in I\}$. We have $q(u_i) \neq 0$ because $u_i \neq 0$. Since $0 \neq D^\gamma M^\mu$ for $\gamma \in \Gamma$ and $\mu \in \mathrm{supp}_\Lambda M$, the condition $(\delta_i + \Gamma) \neq (\delta_j + \Gamma)$ for $i \neq j$ follows from $\dim_k M^\lambda \leq 1$. The converse is easily verified. Observe that $q(u_i, u_j) \in D^{\delta_i + \delta_j}$, but $\delta_i + \delta_j \notin \Gamma$ if $D = D_0$. Otherwise, $\delta_i = -\delta_j + \gamma = \delta_j + (\gamma - 2\delta_j)$ for some $\gamma \in \Gamma$ and $\gamma - 2\delta_j \in \Gamma$ by (2.1) since $\Gamma = \mathcal{L}$ in the notation of loc. cit. and $\mathcal{S} = -\mathcal{S}$. $\square$

Example 4.6. Let $\Lambda = \mathbb{Z}^n$ and let $J = \mathrm{AC}(q, S, D_0) = D_0 e_1 \oplus M \oplus D_0 e_2$ be a Λ-triangulated ample Clifford system such that D_0 generates D as algebra. For $\mathcal{L} = \mathrm{supp}_\Lambda D_0$ we therefore have $\mathbb{Z}[\mathcal{L}] = \mathrm{supp}_\Lambda D = \Gamma$, a subgroup of Λ, and for $\mathcal{S} = \mathrm{supp}_\Lambda M$ we get from (2.1) that $2\mathcal{S} \subset \mathcal{L} \subset \mathcal{S}$, so $\Lambda = \mathbb{Z}[\mathcal{S}]$ and $2\Lambda \subset \mathbb{Z}[\mathcal{L}] = \Gamma \subset \Lambda$, proving that Λ/Γ is a finite group. It follows that Γ is free of rank n, thus D is isomorphic to a Laurent polynomial ring in n variables. Moreover, from $\mathcal{L} + \mathcal{S} \subset \mathcal{S}$ (or from Cor. 4.5(b)) we get $\Gamma + \mathcal{S} \subset \mathcal{S}$, whence the set of coset $\mathcal{S}/\Gamma$ embeds in the finite group Λ/Γ and is therefore finite. Thus, M is free of finite rank.

5. Graded-simple-triangulated Jordan triple systems

In this section we prove (Th. 5.10) that under some mild additional assumptions the graded-simple hermitian matrix and ample Clifford systems give us in fact all the possibilities for graded-simple-triangulated Jordan triple systems, and we describe them completely in Cor. 5.11. Finally, we describe the division-triangulated and tori, in particular the case $\Lambda = \mathbb{Z}^n$, among the graded-triangulated Jordan triple systems in Corollaries 5.12, 5.13 and 5.14.

Unless specified otherwise, $J = J_1 \oplus M \oplus J_2$ is a Jordan triple system over k triangulated by $(u; e_1, e_2)$. We refer the reader to §2 for unexplained notation. We will not right away assume that J is graded or even graded-simple. Rather, to prove the main result of this section we will perform certain reductions to more specific situations (passing to a completion of J over the Laurent series ring or passing to an isotope) and, unfortunately, graded-simplicity can not always be maintained under these reductions. We will therefore begin this section by presenting these reductions.

Let J be an arbitrary Jordan triple system and let t be an indeterminate over k. We denote by

$$\widehat{J} = J((t)) = \big\{ \textstyle\sum_{i \ge N} x_i t^i : x_i \in J, N \in \mathbb{Z} \big\}$$

the Jordan triple system over k whose Jordan triple product is defined by

$$\begin{aligned} &\widehat{P}\big(\sum_{i\ge N} x_i t^i\big)\big(\sum_{j \ge M} y_j t^j\big) \\ &= \sum_{i \ge N,\, j\ge M} P(x_i) y_j t^{2i+j} + \sum_{i_2 > i_1 \ge N,\, j \ge M} \{x_{i_1},\, y_j,\, x_{i_2}\} t^{i_1+i_2+j}. \end{aligned}$$

Note that this makes sense since in any fixed degree the sum on the right hand side is finite. Observe that $\widehat{J}$ contains $J = J\,t^0$ as a subsystem. It is also easy to check that $\sum_{i\ge N} x_i t^i$ with $x_N \ne 0$ is invertible in $\widehat{J}$ if x_N is invertible in J.

Assumption 5.1. *$J = J_1 \oplus M \oplus J_2$ is a Jordan triple system triangulated by $(u; e_1, e_2)$ for which the k-linear map $L : J_1 \to C_0 : x_1 \mapsto L(x_1)$ defined in* 2.1 *is injective.* Note that then $L : J_2 \to C_0^*$ is also injective because

$L(x_2)c_0^* = (L(x_2^*)c_0)^*$ and $x_2^* \in J_1$. Recall that C denotes the subalgebra of $\mathrm{End}_k(M)$ generated by C_0.

Lemma 5.2. *If J satisfies* Assumption 5.1 *with respect to the triangle $\mathcal{T} = (u; e_1, e_2)$ then so does $\widehat{J}$, also with respect to $\mathcal{T}$. Moreover:*

(i) *The Peirce spaces of $\widehat{J}$ with respect to $\mathcal{T}$ are $\widehat{J}_i = J_i((t))$ and $\widehat{M} = M((t))$.*

(ii) *Let $\widehat{C}$ be the subalgebra of $\mathrm{End}_k M((t))$ generated by $\widehat{C_0} = L(\widehat{J}_1) = C_0((t))$, and let $\hat{\pi}$ be the reversal involution of $\widehat{C}$ with respect to $\widehat{C_0}$* (cf. 2.13)*. Then C is canonically isomorphic to the subalgebra of $\widehat{C}$ preserving degrees. Identifying C with this subalgebra we have $\widehat{\pi}_{|C} = \pi$ and, with obvious meaning, $\widehat{*}_{|C} = *$.*

(iii) *For $m \in M$ we put $\widehat{m} = u + tm$ and note that $\widehat{m}$ is invertible in $J((t))$. Then for all $c \in C \subseteq \widehat{C}$ and $\widehat{Q}_2(.) = \widehat{P}(.)e_1$:*

$$
\begin{aligned}
\widehat{Q}_2(c\widehat{m}) = 0 = \widehat{Q}_2(c\widehat{m}, \widehat{m}) \quad &\Longrightarrow \quad Q_2(cm) = 0 = Q_2(cm, m), \\
\widehat{Q}_2(\widehat{m}, c\,c^*\widehat{m}) = 0 \quad &\Longrightarrow \quad Q_2(m, cc^*m) = 0, \text{ and} \\
\widehat{Q}_2(cM, \widehat{m}) = 0 \quad &\Longrightarrow \quad Q_2(cM, m) = 0.
\end{aligned}
$$

(iv) *If J_i does not contain nonzero elements with trivial square* cf. (2.8)*, then neither does $\widehat{J}_i$.*

Proof. (i) and (ii) are clear. (iii) That $\widehat{m} = u + tm$ is invertible in $\widehat{J}$ follows from the invertibility criterion mentioned above. We have $\widehat{Q}_2(c\widehat{m}) = \widehat{P}(cu + ctm)e_1 = Q_2(cu) + Q_2(cu, cm)t + Q_2(cm)t^2$ and $\widehat{Q}_2(c\widehat{m}, \widehat{m}) = Q_2(cu, u) + \big(Q_2(cu, m) + Q_2(cm, u)\big)t + Q_2(cm, m)t^2$, which implies the first equation. The others follow similarly. □

Our second reduction is passing to an isotope. Recall that for an arbitrary Jordan triple system and an invertible element v of J the isotope $J^{(v)}$ is the Jordan triple system with multiplication $P^{(v)}(x)y = P(x)P(v)y$. The following lemma, whose proof is left to the reader, describes which properties are maintained by passing from J to a special isotope.

Lemma 5.3. *Suppose J is triangulated by $\mathcal{T} = (u; e_1, e_2)$, and let $m \in M$ be an invertible element. Then $v = e_1 + Q_2(m)^{-1}$ is invertible in J and the isotope $\widetilde{J} := J^{(v)}$ with $\widetilde{P} = P^{(v)}$ is triangulated by $\widetilde{\mathcal{T}} = (\widetilde{u}; \widetilde{e}_1, \widetilde{e}_2) = (m; e_1, Q_2(m))$ with Peirce spaces $\widetilde{J}_1 = J_1$, $\widetilde{M} = M$ and $\widetilde{J}_2 = J_2$ as k-modules. Moreover, denoting the data for $\widetilde{J}$ by $\widetilde{L}, \widetilde{C}_0$ etc, we have :*

(i) *$\widetilde{L} = L$ as k-linear maps, hence $\widetilde{C}_0 = C_0$ and $(\widetilde{C}, \widetilde{\pi}) = (C, \pi)$ as algebras with involution. In particular, if J satisfies* Assumption 5.1 *then so does $\widetilde{J}$ with respect to $\widetilde{\mathcal{T}}$.*

(ii) *For $n, n_1 \in M$ we have $\widetilde{Q}_2(n) = Q_2(n)$ and $\widetilde{Q}_2(n, n_1) = Q_2(n, n_1)$.*

(iii) *If J_1 does not contain nonzero $x_1 \in J_1$ with $x_1^2 = 0$ the same holds for $\widetilde{J}_1$.*

(iv) *Suppose $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$ is graded-triangulated by $(u; e_1, e_2) \in J^0$. If $m \in M^\lambda$ is homogeneous, then $\widetilde{J}$ is graded-triangulated with*

$$\widetilde{J}_1^\mu = J_1^\mu, \quad \widetilde{M}^\mu = M^{\mu+\lambda}, \quad \widetilde{J}_2^\mu = J_2^{\mu+2\lambda} \quad (\mu \in \Lambda).$$

We point out that $\widetilde{J}_1$ and J_1 are in general not isomorphic as triple systems, rather we have $\widetilde{P}(x_1)y_1 = P(x_1)\bar{y}_1$.

From now on we will use the notations $\widehat{J}$ and $\widetilde{J}$ to denote the Jordan triple systems of Lemma 5.2 and Lemma 5.3.

Proposition 5.4. *Suppose J satisfies* Assumption 5.1*, and let R be a π-invariant ideal of C satisfying $R \cap C_0 = 0$. Then for all $r \in R$, $x_1 \in J_1$, $c \in C$ and $m \in M$ the following hold:*

(i) *$r + r^\pi = r^2 = rr^\pi = rL(x_1)r^\pi = r(c^\pi - c) = 0$. Also $[r, C] = 0$, so R is a central ideal.*

(ii) *$Q_i(ru) = 0 = T_i(ru)$ for $i = 1, 2$,*

(iii) *$Q_2(rm) = 0 = Q_2(rm, m)$,*

(iv) *$Q_1(rm)^2 = 0$,*

(v) *$T_2(rm)^2 = Q_1(rm)^*$,*

(vi) *If either* (a) *$M = Cu$ or* (b) *J_1 does not contain nonzero $x_1 \in J_1$ with $x_1^2 = 0$* (cf. 2.8)*, then $Q_2(RM, M) = 0$.*

Proof. (i) By 2.15 and 2.16, $r+r^\pi = L\big(T_1(ru)\big)$, $rL(x_1)r^\pi = L(P(ru)P(u)x_1)$ and $rr^\pi = L\big(Q_1(ru)\big)$ all lie in $R \cap C_0$. Hence $r + r^\pi = rr^\pi = rL(x_1)r^\pi = 0$ and, because of injectivity of L, also $Q_1(ru) = 0 = T_1(ru)$. It now follows that $r^2 = -rr^\pi = 0$. Linearizing $cL(x_1)c^\pi \in C_0$, we have

$$cL(x_1)d^\pi + dL(x_1)c^\pi \in C_0. \tag{1}$$

Specializing (1) for $d = r$ and using that $r^\pi = -r$, we get $cL(x_1)r = rL(x_1)c^\pi$. For $c = 1$ we then have $L(x_1)r = rL(x_1)$. Since C_0 generates C as a k-algebra, this forces $[r, C] = 0$. Then $cL(x_1)r = rL(x_1)c^\pi$ evaluated for $x_1 = e_1$ shows $r(c^\pi - c) = 0$.

(ii) We have already shown in the proof of (i) that $Q_1(ru) = 0 = T_1(ru) = Q_1(ru, u)$. By 2.20, $(ru)^* = r^\pi u = -ru$, and then by 2.18, $0 = T_1(ru)^* = T_2\big((ru)^*\big) = -T_2(ru)$ and $0 = Q_1(ru)^* = Q_2\big((ru)^*\big) = Q_2(-ru) = Q_2(ru)$.

(iii) We first prove that it is enough to show (iii) for invertible m by passing to $\widehat{J}$. Indeed, because of (iii) of Lem. 5.2 it is enough to establish (iii) for $\widehat{J}$. But for $\widehat{J}$ we know that for any $m \in M$ the element $\widehat{m} = u + tm$ is invertible in $\widehat{J}$ and that $\widehat{J}$ also satisfies Assumption 5.1. Let $\widehat{R}$ be the ideal of $\widehat{C}$ generated by $R \subseteq C \subseteq \widehat{C}$, that is $\widehat{R} = R((t))$. Then $\widehat{R} \cap \widehat{C}_0 = (R \cap C_0)((t)) = 0$ follows. Thus, without loss of generality we can assume that m is invertible.

We then pass to the isotope $\widetilde{J}$, and note that $\widetilde{J}$ satisfies the assumptions of this proposition. Moreover, because of (ii) of Lem. 5.3, it will be sufficient to prove (iii) for $m = \widetilde{u} \in \widetilde{J}$, or equivalently for $u \in J$. But (iii) for $m = u$ is just (ii).

(iv) now follows easily from (iii) since $Q_2(rm) = 0$ implies $Q_1(rm)^2 = P\big(Q_1(rm)\big)e_1 = P(rm)P(e_2)P(rm)e_1 = P(rm)P(e_2)Q_2(rm) = 0$.

(v) We have $T_2(rm)^2 = P\big(T_2(rm)\big)e_2 = P(\{rm, e_1, u\})e_2$, where, by [L, JP21] and because of $P(e_1)u = 0$ and $Q_2(rm) = 0$ by (iii),

$$\begin{aligned} P(\{rm, e_1, u\})e_2 &= P(rm)P(e_1)P(u)e_2 + P(u)P(e_1)P(rm)e_2 \\ &\quad +L(rm, e_1)P(u)L(e_1, rm)e_2 - P(P(rm)P(e_1)u, u)e_2 \\ &= Q_2(rm) + Q_1(rm)^* + Q_2(rm, P(u)\,\overline{rm}) \\ &= Q_1(rm)^* + Q_2(rm, (rm)^*). \end{aligned}$$

By 2.19, $(rm)^* = r^*m^* = r^*(T_1(m)\cdot u - m) = L\big(T_1(m)\big)\,r^*u - r^*m$ where in the last equality we used 2.6 and the fact that C^* is the subalgebra generated by $L(J_2)$, and hence commutes with C. Since $r^\pi = -r$ we then get from 2.14 and 2.20 that

$$\begin{aligned} Q_2(rm, (rm)^*) &= Q_2\big(rm, r^*(T_1(m)\cdot u - m)\big) \\ &= Q_2(m, r^\pi L\big(T_1(m)\big)r^*u) \;-\; Q_2(m, r^\pi r^* m) \\ &= -Q_2(m, rL\big(T_1(m)\big)r^\pi u) + Q_2(m, rr^*m) \\ &= Q_2(m, rr^*m) \end{aligned}$$

since $rL\big(T_1(m)\big)r^\pi = 0$ by (i). Therefore, if we can establish $Q_2(m, rr^*m) = 0$ we are done. As in the proof of (iii) we imbed J into $\widehat{J}$. Then Lem.5.2(iii) shows that it is sufficient to prove this for an invertible m. But for invertible m we have $C^*m \subseteq Cm$, since by [L, JP21] and 2.5

$$\begin{aligned} L(x_2)m &= P(x_2, m)e_2 = P(P(m)P(m)^{-1}x_2, m)e_2 \\ &= L(m, P(m)^{-1}x_2)P(m)e_2 = \{Q_1(m), P(m)^{-1}x_2, m\} \\ &= L\big(Q_1(m)\big)L(\,\overline{P(m)^{-1}x_2}\,)m \in Cm. \end{aligned}$$

Hence $Q_2(m, RC^*m) \subseteq Q_2(m, RCm) \subseteq Q_2(m, Rm) = 0$ by (iii).

(vi) In case (a) we have, using 2.17, $Q_2(RM, M) = Q_2(RCu, Cu) \subseteq Q_2(Ru, Cu) = T_2(R^\pi Cu) = T_2(RCu) \subseteq T_2(Ru)$. Now the claim follows from (ii).

In case (b) first note that neither $J_2 = J_1^*$ contains a nonzero $x_2 \in J_2$ with $x_2^2 = 0$. It then follows from (iv) and (v) that

$$T_2(RM) = 0. \tag{2}$$

Next we show

$$Q_2(RM, m) = 0 \;\text{ for invertible }\; m \in M. \tag{3}$$

Indeed, applying Lem. 5.3, in particular (ii) and (iii), we see that it suffices to prove (3) for $m = u$, in which case it reduces to (2). Finally, we can show that $Q_2(RM, m) = 0$ for arbitrary $m \in M$: By Lem. 5.2 it suffices to show $\widehat{Q}_2(RM, \widehat{m}) = 0$, for $\widehat{m} = u + tm$. But this holds by (3) since $\widehat{J}$ satisfies our assumptions. $\square$

Recall that C_0 is a Jordan triple system with $P(c_0)(d_0) = c_0\overline{d_0}c_0$ and that $L : J_1 \to C_0$, $x_1 \mapsto L(x_1)$ is a nonzero specialization (see 2.7). In particular, $L(x_1^2) = \big(L(x_1)\big)^2$ and if Assumption 5.1 holds then $J_1 \cong_\Lambda C_0$ as graded Jordan triple systems if J is graded by Λ.

Lemma 5.5. *Suppose J is graded-triangulated and satisfies* Assumption 5.1 *with respect to $(u; e_1, e_2) \subseteq J^0$. Let R be a π-invariant graded ideal of C such that $R \cap C_0 = 0$. If C_0 does not contain nonzero homogeneous elements which square to zero, then $Q_2(RM, m) = 0$ for invertible homogeneous $m \in M$.*

Proof. We pass from J to the isotope $\widetilde{J}$ of Lem. 5.3. Since by that lemma all our assumptions are maintained, it follows from (ii) of Lem. 5.3 that it suffices to prove $T_2(RM) = 0$. To do so, let $r \in R$ and $n \in M$ be homogeneous elements. By Prop. 5.4(iv), $L\big(Q_1(rn)\big)^2 = L(Q_1(rn)^2) = 0$, which implies that $Q_1(rn) = 0$ by our assumptions. But then by Prop. 5.4(v), $L\big(T_2(rn)\big)^2 = L(T_2(rn)^2) = L(Q_1(rn)^*) = 0$. Since $C_0 \cong_\Lambda J_1$ does not contain nonzero elements with square 0, the same holds for $C_0^* = L(J_2)$. Hence $T_2(rn) = 0$ follows, and this implies $T_2(RM) = 0$ as desired. $\square$

Lemma 5.6. *Suppose J is graded-triangulated by $(u; e_1, e_2)$. Then $C' := C\{c - c^\pi : c \in C\}C = C[C, C]C$ is a $(\pi, ^-)$-graded ideal of C such that $C'M \subseteq Cu$.*

Proof. The first part of the lemma is straightforward. That $C'M \subseteq Cu$ follows from [MN, 1.6.13]. $\square$

Assumption 5.7. *J is graded-triangulated with grading group Λ and fulfills* Assumption 5.1 *with respect to a triangle $(u; e_1, e_2) \subseteq J^0$.*

Lemma 5.8. *Suppose J fulfills* Assumption 5.7. *If R is a maximal $(\pi, ^-)$-invariant and graded ideal of C with $R \cap C_0 = 0$, then $M = Cu$ or C_0 does not contain nonzero homogeneous elements with trivial square.*

Proof. If R is a maximal $(\pi, ^-)$-invariant and graded ideal of C with $R \cap C_0 = 0$, consider the $(\pi, ^-)$-graded-simple algebra $\breve{C} := C/R$ and let $\varphi : C \to \breve{C}$ be the canonical epimorphism. Since φ is homogeneous of degree 0, the ideal $\varphi(C') = \breve{C}' = \breve{C}[\breve{C}, \breve{C}]\breve{C}$ is graded and invariant under the induced maps $\breve{\pi}$ and $c \mapsto \breve{\bar{c}}$, whence either $\breve{C}' = \breve{C}$ or $\breve{C}' = 0$. If $\breve{C}' = \breve{C}$, then $\breve{1} \in \breve{C}'$ and so $1 = c' + r$ where $c' \in C'$. Now $1^2 = 1 = c'^2 + c'r + rc' + r^2$, but $r^2 = 0$ by Prop. 5.4(i). Hence $1 \in C'$ which by Lem. 5.6 implies $M \subseteq Cu$, so $M = Cu$.

Otherwise $\breve{C}' = 0$. Then $[\breve{C}, \breve{C}] = 0$, that is, $\breve{C}$ is commutative and hence $\breve{\pi}$ is trivial. In this case $(\breve{C}, \breve{^-})$ is graded-simple. By Cor. 3.8, $\breve{C}$ is either division-graded or the direct sum of two copies of a division-graded algebra with the exchange automorphism. In particular, $\breve{C}$ does not contain nonzero homogeneous elements with trivial square. So neither does the subspace $\breve{C}_0 = \varphi(C_0)$ nor C_0 since $R \cap C_0 = 0$. □

Proposition 5.9. *Suppose J is a graded-triangulated Jordan triple system for which J_1 is graded-simple. Then* Assumption 5.7 *holds. If R is a proper $(\pi, ^-)$-invariant graded ideal of C then $R \cap C_0 = 0$ and $Q_2(RM, m) = 0$ for invertible homogeneous $m \in M$.*

Proof. Since J_1 is graded-simple and the specialization $L : J_1 \to C_0$, $x_1 \mapsto L(x_1)$ is nontrivial, it is injective, proving Assumption 5.7. It is then an isomorphism of graded Jordan triple systems, whence C_0 is also graded-simple. Therefore the graded ideal $R \cap C_0$ of the Jordan triple system C_0 must be either 0 or C_0. But if $R \cap C_0 = C_0$, then $C_0 \subseteq R$ which implies $C = R$ contradicting that R is proper. So $R \cap C_0 = 0$. By Lem. 5.8, $M = Cu$ or C_0 does not contain nonzero homogeneous elements of trivial square. If $M = Cu$, then $Q_2(RM, M) = 0$ by Prop. 5.4(vi). Otherwise, it follows from Lem. 5.5 that $Q_2(RM, m) = 0$ for invertible homogeneous $m \in M$. □

Now we are ready to establish our main result.

Theorem 5.10. ([MN, 4.3] for $\Lambda = 0$) *Let J be a graded-simple-triangulated Jordan triple system satisfying one of the following conditions*

(a) *every nonzero $m \in M$ is a linear combination of invertible homogeneous elements, or*
(b) *the grading group Λ is torsion-free.*

Then $(C, \pi, ^-)$ is graded-simple, u is C-faithful, and exactly one of the following two cases holds:

(i) *C is not commutative and $M = Cu$. In this case, $\pi \neq \mathrm{Id}$ and J is graded isomorphic to the graded-simple diagonal hermitian matrix system $\mathrm{H}_2(C, C_0, \pi, ^-)$;*
(ii) *C is commutative. In this case, $\pi = \mathrm{Id}$ and J is graded isomorphic to the graded-simple ample Clifford system $\mathrm{AC}(q, M, S, C, ^-, C_0)$, where $q(m) = L\big(Q_1(m)\big)$ and $S(m) = \overline{m}$.*

In both cases, the triangles are preserved by the isomorphisms.

Remarks. (1) We point out that the assumptions (a) or (b) are only needed to show that $(C, \pi, ^-)$ is graded-simple. Our proof shows that *any graded-triangulated Jordan triple system with a graded-simple $(C, \pi, ^-)$ satisfies* (i) *or* (ii)! We also note that hermitian matrix systems are of course also defined for commutative coordinate algebras C. But in the commutative case they are isomorphic to ample Clifford systems.

(2) This theorem generalizes [MN, Prop. 4.3]: $\Lambda = 0$ is a special case of our assumption (b). Our proof is slightly different from the proof given in [MN] and in fact corrects a small inaccuracy there: The Isotope Trick [MN, 4.1] cannot be applied since $\widetilde{J}$ does not necessarily inherit simplicity from J.

(3) The assumption (a), which, admittedly, looks somewhat funny at first sight, is fulfilled in the most important application of the theorem, the division-triangulated case (Cor. 5.12).

Proof. We will proceed in four steps.

(I) $(C, \pi, ^-)$ *is graded-simple.* Let R be a maximal $(\pi, ^-)$-invariant graded ideal of C. Such an ideal R exists by Zorn. Our claim (I) then means $R = 0$. This will follow if we can show $rm = 0$ for homogeneous $r \in R$ and $m \in M$. Recall that Q_2 is graded-nondegenerate by Prop. 2.23(ii). It is therefore sufficient to prove

$$Q_2(rm) = 0 = Q_2(rm, M) \quad \text{for homogeneous } r \in R \text{ and } m \in M. \qquad (1)$$

Since J_1 is graded-simple by Prop. 2.23(ii), it follows from Prop. 5.9 that J satisfies Assumption 5.7. Since R is in particular proper, it also follows from Prop. 5.9 that $C_0 \cap R = 0$ and $Q_2(rm, n) = 0$ for invertible homogeneous $n \in M$. Also, $Q_2(rm) = 0$ by Prop. 5.4(iii). Thus, (1) holds in case (a).

We also know from Lem. 5.8 that $M = Cu$ or C_0 does not contain nonzero homogeneous elements with trivial square. But if $M = Cu$ then $Q_2(rm, M) = 0$ by Prop. 5.4(vi), hence again (1) follows. We can therefore assume that C_0 does not have nonzero homogeneous elements with trivial square. We will use our assumption (b) to prove (1) in this case. We claim that in fact C_0 does not contain nonzero elements with trivial square: Let $x = \sum x^{\lambda_i} \in C_0$, with $0 \neq x^{\lambda_i} \in C_0^{\lambda_i}$, such that $x^2 = 0$. Since Λ is torsion-free, it can be ordered (as a group) and we can therefore consider $(x^\lambda)^2$ for $\lambda = \max\{\lambda_i\}$. But $x^2 = 0$ implies $(x^\lambda)^2 = 0$, hence $x^\lambda = 0$ by the absence of nonzero homogeneous elements of trivial square, contradiction. Since $(J_1, e_1) \cong_\Lambda (C_0, 1)$ as triple systems with tripotents, the subspace J_1 does not contain nonzero elements with trivial square. But then $Q_2(rm, M) = 0 = Q_2(rm)$ follows from (vi) and (iii) of Prop. 5.4.

(II) u *is* C*-faithful.* By (I), we have that C is $(\pi, ^-)$-graded-simple. Now, $Z = \{z \in C : zu = 0\}$ is obviously a left ideal of C. It is also a right ideal since for $d \in C$ and $z \in Z$ we have $zCu = zC^\pi u$ (by 2.20) $= zC^*u$ (by 2.6) $= C^*zu = 0$. Also, Z is graded since $u \in M^0$, and finally it is $(\pi, ^-)$-invariant since $z^\pi u = (zu)^* = 0$ by 2.20 and $\overline{z}u = \overline{z}\,\overline{u} = \overline{zu} = 0$. Then Z must be C or 0. But note that $Z \neq C$ since $1 \notin Z$. Hence $Z = 0$, that is, u is C-faithful.

We will now distinguish the two cases $\pi \neq \mathrm{Id}$ and $\pi = \mathrm{Id}$.

(III) $\pi \neq \mathrm{Id}$: Then C is noncommutative. Indeed, since $C_0 \subset H(C, \pi)$ generates C as an algebra, C is commutative iff $\pi = \mathrm{Id}$. Also, there exists a homogeneous $c \in C$ such that $c^\pi \neq c$, and then the $(\pi, ^-)$-graded ideal $C' = C\{c - c^\pi : c \in C\}C$ (Lem. 5.6) is nonzero and hence equals C, in particular $1 \in C'$. By Lem. 5.6 again, this implies $M = Cu$. By Th. 3.3 J is then graded isomorphic to the hermitian matrix system $\mathrm{H}_2(C, C_0, \pi, ^-)$ as claimed in (i).

(IV) $\pi = \mathrm{Id}$: Then C is commutative and ${}^-$-graded-simple. We will prove that in this case J is graded isomorphic to an ample Clifford system. By C-faithfulness, this will follow from Th. 4.3 as soon as we have established $(x_1 - x_1^*) \cdot m = 0$ for all $x_1 \in J_1$ and $m \in M$. Now by 2.21 we know $(x_1 - x_1^*) \cdot m = \Gamma_1(x_1; m)u$. By linearity of Γ_1 in x_1 and m, it therefore remains to prove

$$\Gamma_1(x_1; m) = 0 \text{ for homogeneous } x_1 \in J_1 \text{ and } m \in M. \tag{2}$$

Since C is commutative, 2.22 shows $\Gamma_1(x_1; m)^2 m = 0$, so $\Gamma_1(x_1; m)$ is never invertible. If $(C, {}^-)$ is graded-simple, it is a division-graded algebra and so (2) holds. Otherwise, by Cor. 3.8, identify $C = A \boxplus A$ with the direct sum of two copies of a division-graded commutative algebra A and ${}^-$ is the exchange automorphism. Then we have that

$$\Gamma_1(x_1; m) = 0 \text{ for all homogeneous } {}^-\text{-invariant } x_1 \in J_1 \text{ and } m \in M. \tag{3}$$

Let $x_1 \in J_1$ and $m \in M$ be arbitrary homogeneous elements. Since $C = A \boxplus A$ we get $1 = \epsilon + \overline{\epsilon}$ for orthogonal idempotents ϵ and $\overline{\epsilon}$ in C^0, namely $\epsilon = 1_A$. We now claim that

$$(x_1 - x_1^*) \cdot m = \epsilon(y_1 - y_1^*) \cdot n_1 + \overline{\epsilon}(z_1 - z_1^*) \cdot n_2 \tag{4}$$

for some homogeneous $y_1 = \overline{y_1}$ and $z_1 = \overline{z_1}$ in J_1 and $n_i = \overline{n_i} \in M$. The proof of (4) given in the proof of [MN, 4.4] for $\Lambda = 0$ also works in our setting. But (4) together with (3) implies (2), finishing the proof of the theorem. □

From the previous Theorem 5.10 together with Prop. 3.9 and Prop. 4.4(iv) we get the following classification.

Corollary 5.11. ([MN, 4.4] for $\Lambda = 0$) *A graded-simple-triangulated Jordan triple system satisfying* (a) *or* (b) *of* Th. 5.10 *is graded isomorphic to one of the following triple systems:*
non-polarized

- (I) *diagonal* $\mathrm{H}_2(A, A_0, \pi, {}^-)$ *for a graded-simple noncommutative A with graded involution π and automorphism ${}^-$;*
- (II) $\mathrm{Mat}_2(B)$ *with* $P(x)y = x\overline{y}x$ *for a noncommutative graded-simple associative unital B with graded automorphism ${}^-$ and* $\overline{(y_{ij})} = (\overline{y_{ij}})$ *for* $(y_{ij}) \in \mathrm{Mat}_2(B)$*;*
- (III) $\mathrm{Mat}_2(B)$ *with* $P(x)y = x\overline{y}^t x$ *for a noncommutative graded-simple associative unital B with graded involution ι and* $\overline{(y_{ij})} = (y_{ij}^\iota)$ *for* $(y_{ij}) \in \mathrm{Mat}_2(B)$*;*
- (IV) $\mathrm{AC}(q, S, F_0)$ *for a graded-nondegenerate q over a graded-field F with Clifford-ample subspace F_0;*

or polarized

- (V) $\mathrm{H}_2(B, B_0, \pi) \oplus \mathrm{H}_2(B, B_0, \pi)$ *for a diagonal hermitian matrix system* $\mathrm{H}_2(B, B_0, \pi)$ *with graded-simple noncommutative B;*

(VI) $\mathrm{Mat}_2(B) \oplus \mathrm{Mat}_2(B)$ *for a noncommutative graded-simple associative unital* B *with* $P(x)y = xyx$*;*

(VII) $\mathrm{AC}(q, S, F_0) \oplus \mathrm{AC}(q, S, F_0)$ *for* $\mathrm{AC}(q, S, F_0)$ *as in* (IV)*.*

Conversely, the Jordan triple systems in (I)–(VII) *are graded-simple-triangulated.*

Corollary 5.12. *For a graded-triangulated Jordan triple system* J *the following are equivalent:*

(i) J *is graded-simple and every homogeneous* $0 \neq m \in M$ *is invertible,*
(ii) J *is division-triangulated,*
(iii) J *is graded isomorphic to one of the following:*
  (I) *diagonal hermitian matrix system* $\mathrm{H}_2(A, A_0, \pi, ^-)$ *for a noncommutative division-graded* A*.*
  (II) $\mathrm{AC}(q, S, F_0)$ *for a graded-anisotropic* q *over a graded-field* F *with Clifford-ample subspace* F_0*.*

Proof. If (i) holds we can apply Th. 5.10: J is graded isomorphic to a hermitian matrix system or to an ample Clifford system, and C is u-faithful. The assumption on M together with Cor. 3.12 and Cor. 4.5 then show that J is graded isomorphic to one of the two cases in (iii) and that J is division-triangulated. The implication (ii) $\Rightarrow$ (i) is trivial, and (iii) $\Rightarrow$ (i) follows from the quoted corollaries. □

Corollary 5.13. *A graded Jordan triple system* J *over a field* k *is a triangulated Jordan triple torus iff* J *is graded isomorphic to*

(I) *a diagonal hermitian matrix system* $\mathrm{H}_2(A, A_0, \pi, ^-)$ *for a noncommutative torus* A*, or to*
(II) *an ample Clifford system* $\mathrm{AC}(D, q, M)$ *with* D, M *as described in* Cor. 4.5(b)*.*

Proof. This follows from Cor. 5.12 and the description of tori in Lemma 3.12 and Cor. 4.5. □

Corollary 5.14. J *is a* $\mathbb{Z}^n$*-triangulated Jordan triple torus iff* J *is graded isomorphic to*

(I) *a diagonal hermitian matrix system* $\mathrm{H}_2(A, A_0, \pi, ^-)$ *where* A *is a quantum* $\mathbb{Z}^n$*-torus, see* Ex. 3.13 *and* $\pi = \pi_{\mathrm{rev}}$ *is the reversal involution, or to*
(II) *an ample Clifford system as described in* Ex. 4.6*.*

Proof. All statements follow from the quoted references. Note that by construction D_0 generates D in the Clifford case, so that we are indeed in the setting of Ex. 4.6. □

6. Graded-simple-triangulated Jordan algebras and Jordan pairs

In this section we specialize our results on graded-triangulated Jordan triple systems to Jordan algebras and Jordan pairs: We classify graded-simple-triangulated Jordan algebras (Th. 6.3) and Jordan pairs (Th. 6.12), and we deduce from these theorems the classifications of division-triangulated Jordan algebras and pairs (Cor. 6.5, Cor. 6.14). As an example, we describe the classification of $\mathbb{Z}^n$-triangulated Jordan algebra tori (Cor. 6.6) and Jordan pair tori (Cor. 6.15).

In this paper all Jordan algebras are assumed to be unital, with unit element denoted 1 or 1_J if we need to be more precise, and with Jordan product written as $U_x y$. A homomorphism of Jordan algebras is a k-linear map $f : J \to J'$ satisfying $f(U_x y) = U_{f(x)} f(y)$ and $f(1_J) = 1_{J'}$.

In order to apply our results we will view Jordan algebras as Jordan triple systems with identity elements. Thus, to a Jordan algebra J we associate the Jordan triple system $T(J)$ defined on the k-module J with Jordan triple product $P_x y = U_x y$. The element $1_J \in J$ satisfies $P(1_J) = \mathrm{Id}$. Conversely, every Jordan triple system T containing an element $1 \in T$ with $P(1) = \mathrm{Id}$ is a Jordan algebra with unit element 1 and multiplication $U_x y = P_x y$.

For many concepts there is no or not a big difference between J and $T(J)$. For example, a Jordan algebra J is *graded by* Λ if and only if $T(J)$ is graded by Λ, in which case $1_J \in J^0$. In this case, a graded ideal of J is the same as a graded ideal of $T(J)$, and we will call J *graded-simple* if $T(J)$ is so. Moreover, if $e \in J$ is an idempotent, i.e., $e^2 = e$, then e is a tripotent of $T(J)$ and the Peirce spaces of J and $T(J)$ with respect to e coincide, i.e., $T\big(J_i(e)\big) = T(J)_i(e)$, $i = 0, 1, 2$. In particular, the Peirce spaces $J_i(e)$ are graded if $e \in J^0$. We thus get the following corollary from Th. 1.4.

Corollary 6.1. *If J is a graded-simple Jordan algebra with an idempotent $0 \neq e \in J^0$, then the Peirce space $J_2(e)$ is a graded-simple Jordan algebra, and if $J_0(e) \neq 0$ then $J_0(e)$ is graded-simple too.*

A graded Jordan algebra J is called *graded-triangulated* by $(u; e_1, e_2)$ if $e_i = e_i^2 \in J^0$, $i = 1, 2$, are supplementary orthogonal idempotents and $u \in J_1(e_1)^0 \cap J_1(e_2)^0$ with $u^2 = 1$ and $u^3 = u$. It is called Λ*-triangulated* if it is graded-triangulated and $\mathrm{supp}_\Lambda J$ generates Λ as a group. Note that any $x \in J$ with $x^2 = 1$ satisfies $2x^3 = x^2 \circ x = 2x$ and $U_{x^3 - x} = U_x U_{x^2 - 1} = 0$ by [J, (1.5.4) and (3.3.4)], whence $x^2 = 1$ implies $x^3 = x$ in case $\frac{1}{2} \in k$ or x is homogeneous and J is graded-nondegenerate. Of course, we also have $x^2 = 1 \Rightarrow x^3 = x$ if J is special. In any case, with our definition of a triangle in a Jordan algebra, J is graded-triangulated by $(u; e_1, e_2)$ iff $T(J)$ is graded-triangulated by $(u; e_1, e_2)$. Also, J is Λ-triangulated iff $T(J)$ is so.

This close relation to graded-triangulated Jordan triple systems also indicates how to get examples of graded-triangulated Jordan algebras: We take a Jordan triple system which is graded-triangulated by $(u; e_1, e_2)$ and require $P(e) = \mathrm{Id}$ for $e = e_1 + e_2$. Doing this for our two basic examples, yields the following examples of graded-triangulated Jordan algebras.

Definition 6.2. (A) *Hermitian matrix algebras*: This is the graded Jordan triple system $\mathrm{H}_2(A, A_0, \pi, ^-)$ of Def. 3.1 with automorphism $^- = \mathrm{Id}$, which we will write as $\mathrm{H}_2(A, A_0, \pi)$. Note that this is a Jordan algebra with product $U(x)y = P(x)y = xyx$ and identity element $1_J = E_{11} + E_{22}$. If for example $A = B \boxplus B^{\mathrm{op}}$ and π is the exchange involution, then $\mathrm{H}_2(A, A_0, \pi) \cong_\Lambda \mathrm{Mat}_2(B)$ where $\mathrm{Mat}_2(B)$ is the Jordan algebra with product $U_x y = xyx$.

(B) *Quadratic form Jordan algebras*: This is the graded ample Clifford system $\mathrm{AC}(q, M, S, D, ^-, D_0)$ of Def. 4.2 with automorphism $^- = \mathrm{Id}$ and $S|_M = \mathrm{Id}$. Since then $P(e) = \mathrm{Id}$ we get indeed a graded-triangulated Jordan algebra denoted $\mathrm{AC}_{\mathrm{alg}}(q, M, D, D_0)$ or just $\mathrm{AC}_{\mathrm{alg}}(q, D, D_0)$ if M is unimportant. Note that this Jordan algebra is defined on $D_0 e_1 \oplus M \oplus D_0 e_2$ and has product $U_x y = q(x, \tilde{y})x - q(x)\tilde{y}$ where $q(d_1 e_1 \oplus m \oplus d_2 e_2) = d_1 d_2 - q(m)$ and $(d_1 e_1 \oplus m \oplus d_2 e_2)\tilde{} = d_2 e_1 \oplus -m \oplus d_1 e_1$. (If $\frac{1}{2} \in k$ it is therefore a reduced spin factor in the sense of [M2, II, §3.4].)

Theorem 6.3. *A graded-simple-triangulated Jordan algebra satisfying*

(a) *every nonzero $m \in M$ is a linear combination of invertible homogeneous elements, or*
(b) *the grading group Λ is torsion-free,*

is graded isomorphic to one of the following Jordan algebras:

(I) *diagonal* $\mathrm{H}_2(A, A_0, \pi)$ *for a graded-simple noncommutative A;*
(II) $\mathrm{Mat}_2(B)$ *for a noncommutative graded-simple associative unital B;*
(III) $\mathrm{AC}_{\mathrm{alg}}(q, F, F_0)$ *for a graded-nondegenerate q over a graded-field F with Clifford-ample subspace F_0.*

Conversely, all Jordan algebras in (I)–(III) *are graded-simple-triangulated.*

Proof. Let J be a graded-simple-triangulated Jordan algebra that satisfies (a) or (b). Then $T(J)$ is a graded-simple-triangulated Jordan triple system with $e = 1_J$ satisfying (a) or (b) of Th. 5.10, hence graded isomorphic as Jordan triple system to $\mathrm{H}_2(C, C_0, \pi)$, where (C, π) is graded-simple, or to $\mathrm{AC}_{\mathrm{alg}}(q, C, C_0)$, where C is division-graded. But because the graded isomorphisms appearing in Th. 5.10 preserve the triangles, they are in fact isomorphisms of Jordan algebras. Therefore J is graded isomorphic to $\mathrm{H}_2(C, C_0, \pi)$, where (C, π) is graded-simple, or to $\mathrm{AC}_{\mathrm{alg}}(q, C, C_0)$, where C is a graded-field. In the second case J is of type (III) of the statement for $F_0 = C_0$ and $F = C$. On the other hand, it follows from Prop. 3.7 that (C, π) is graded-simple iff either C is graded-simple or $C \cong_\Lambda B \boxplus B$ for a graded-simple associative B and π is the exchange involution. Hence $\mathrm{H}_2(C, C_0, \pi)$, where (C, π) is graded-simple, is as in (I) or (II) of the statement. The converse follows from Cor. 5.11. □

Definition 6.4. As in the Jordan triple system case, a graded Jordan algebra J is called *division-graded* if every nonzero homogeneous element is invertible in J.

We say that J is *division-triangulated* if it is graded-triangulated, the Jordan algebras J_i, $i = 1, 2$, are division-graded and every homogeneous $0 \neq m \in M$ is invertible in J. It is called *division-Λ-triangulated* if it is division-triangulated as well as Λ-graded.

A division-(Λ)-triangulated Jordan algebra J is called a *(Λ)-triangulated Jordan algebra torus* if J is defined over a field k and $\dim_k J_i^\lambda \leq 1$ and $\dim_k M^\lambda \leq 1$.

Thus, J is a division-(Λ)-triangulated Jordan algebra iff $T(J)$ is a division-(Λ)-triangulated Jordan triple system. We therefore get the Jordan algebra versions of the Corollaries 5.12–5.14. We formulate the first and last of them, and leave the translation of the second, Cor. 5.13, to the reader.

Corollary 6.5. *For a graded-triangulated Jordan algebra J the following are equivalent:*

(i) *J is graded-simple and every homogeneous $0 \neq m \in M$ is invertible,*
(ii) *J is division-triangulated,*
(iii) *J is graded isomorphic to one of the following:*
 (I) *a diagonal hermitian matrix algebra $\mathrm{H}_2(A, A_0, \pi)$ for a noncommutative division-graded A;*
 (II) *a quadratic form Jordan algebra $\mathrm{AC}_{\mathrm{alg}}(q, F, F_0)$ for a graded-anisotropic q over a graded-field F with Clifford-ample subspace F_0.*

Corollary 6.6. *A graded Jordan algebra J over a field k is a $\mathbb{Z}^n$-triangulated Jordan algebra torus iff J is graded isomorphic to*

(I) *a diagonal hermitian matrix algebra $\mathrm{H}_2(A, A_0, \pi)$ where A is a noncommutative quantum $\mathbb{Z}^n$-torus and π_{rev} is the reversal involution, see* Ex. 3.13*, or to*
(II) *a quadratic form Jordan algebra $\mathrm{AC}_{\mathrm{alg}}(q, M, D, D_0) = D_0 e_1 \oplus M \oplus De_2$, where*
 (a) *$D = k[\Gamma]$ is the group algebra of a subgroup $\Gamma \subset \mathbb{Z}^n$ with $2\mathbb{Z}^n \subset \Gamma$, hence Γ is free of rank n and D is isomorphic to a Laurent polynomial ring in n variables,*
 (b) *D_0 is a Clifford-ample subspace, hence $D_0 = D$ if* $\mathrm{char}(k) \neq 2$*,*
 (c) *M is a $\mathbb{Z}^n$-graded D-module which is free of finite rank, with a homogeneous basis, say $\{u_0, \ldots, u_l\}$, and the u_i have degree $\delta_i \in \mathbb{Z}^n$ with $\delta_0 = 0$ and $\delta_i + \Gamma \neq \delta_j + \Gamma$ for $i \neq i$,*
 (d) *the D-quadratic form $q : M \to D$ satisfies $q(u_0) = 1$, $0 \neq q(u_i) \in D^{2\delta_i}$ and $q(u_i, u_j) = 0$ for $i \neq j$.*

Remark 6.7. For $\Lambda = \mathbb{Z}^n$ Cor. 6.6 is proven in [AG, Prop. 4.53 and Prop. 4.80] and in an equivalent form (structurable algebras instead of Jordan algebras) in [F, §3, Th. 9], assuming $\mathrm{char}\, k \neq 2, 3$ ([F]) or $k = \mathbb{C}$ in [AG].

Let now $V = (V^+, V^-)$ be a Jordan pair. A *grading of V by Λ* is a decomposition $V^\sigma = \bigoplus_{\lambda \in \Lambda} V^\sigma[\lambda]$, $\sigma = \pm$, such that the associated polarized Jordan triple system $T(V)$ is graded with homogeneous spaces $T(V)^\lambda =$

$V^+[\lambda] \oplus V^-[\lambda]$. The properties of graded Jordan triple systems we have considered so far in this paper make sense for graded Jordan pairs too. For example, a graded Jordan pair V is *graded-nondegenerate* if every homogeneous absolute zero divisor of V^+ or V^- vanishes. It is immediate that V is graded-nondegenerate if and only if $T(V)$ is graded-nondegenerate. As usual, V is said to be *graded-simple* if $Q(V^\sigma)V^{-\sigma} \neq 0$ and every graded ideal is either trivial or equal to V; V is *graded-prime* if it does not contain nonzero graded ideals I and K such that $Q(I^\sigma)K^{-\sigma} = 0$ and *graded-semiprime* if $Q(I^\sigma)I^{-\sigma} = 0$ implies $I = 0$. For properties defined in terms of ideals we have the following lemma, whose proof is again a straightforward adaptation of the proof in the ungraded situation.

Lemma 6.8. ([N1, 1.5] for $\Lambda = 0$) *Let V be a graded Jordan pair.*

(i) *If $I = (I^+, I^-)$ is a graded ideal of V, then $I^+ \oplus I^-$ is a graded ideal of $T(V)$.*

(ii) *Let $\pi^\sigma : T(V) \to V^\sigma$, $^\sigma = \pm$, be the canonical projection. If J is a graded ideal of $T(V)$, then*

$$\underset{\sim}{J} := (J \cap V^+, J \cap V^-) \quad \text{and} \quad \tilde{J} := \big(\pi^+(J), \pi^-(J)\big),$$

are graded ideals of V satisfying $\underset{\sim}{J}^+ \oplus \underset{\sim}{J}^- \subseteq J \subseteq \tilde{J}^+ \oplus \tilde{J}^-$ and

$$Q(\tilde{J}^\sigma)V^{-\sigma} + Q(V^\sigma)\tilde{J}^{-\sigma} + \{V^\sigma V^{-\sigma} \tilde{J}^\sigma\} \subseteq \underset{\sim}{J}^\sigma$$

(iii) *V is graded-(semi)prime or graded-simple if and only if $T(V)$ is, respectively, graded-(semi)prime or graded-simple.*

Idempotents in a Jordan pair V and tripotents in $T(V)$ correspond to each other naturally: Any idempotent $e = (e^+, e^-)$ of V, i.e., $Q(e^\sigma)e^{-\sigma} = e^\sigma$, gives rise to the tripotent $e^+ \oplus e^-$ of $T(V)$, and conversely any tripotent of $T(V)$ arises in this way. Moreover, we have the following obvious though fundamental fact. If $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$ is the Peirce decomposition of V with respect to an idempotent $e = (e^+, e^-)$, then the Peirce spaces of $T(V)$ with respect to $e^+ \oplus e^-$ are $T(V)_i(e^+ \oplus e^-) = T\big(V_i(e)\big)$, $\quad i = 0, 1, 2$. The following corollary is a consequence of Lem. 6.8(iii) and Th. 1.4.

Corollary 6.9. *If $0 \neq e \in V[0]$ is an idempotent of a graded-simple Jordan pair V, then the Peirce space $V_2(e)$ is graded-simple and if $V_0(e) \neq 0$, then $V_0(e)$ is also graded-simple.*

Definition 6.10. Recall [N3] that a triple of nonzero idempotents $(u; e_1, e_2)$ of a Jordan pair V is a *triangle* if $e_i \in V_0(e_j)$, $i \neq j$, $e_i \in V_2(u)$, $i = 1, 2$, $u \in V_1(e_1) \cap V_1(e_2)$, and the following multiplication rules hold for $\sigma = \pm$: $Q(u^\sigma)e_i^{-\sigma} = e_j^\sigma$, $i \neq j$, and $Q(e_1^\sigma, e_2^\sigma)u^{-\sigma} = u^\sigma$.

A graded Jordan pair V is said to be *graded-triangulated* if V contains a triangle $(u; e_1, e_2)$ in $V[0]$ and $V = V_1 \oplus M \oplus V_2$, where $V_i = V_2(e_i)$, $i = 1, 2$, and $M = V_1(e_1) \cap V_1(e_2)$. It is then immediate that V is graded-triangulated if and only if $T(V)$ is so. Naturally, we call V Λ-*triangulated* if $T(V)$ is so. However, rather than applying the Jordan triple classification to graded-(Λ)-triangulated Jordan pairs, we will use the Jordan algebra Classification Th. 6.3. Namely, it is well known [N3, p.470] that V is covered by a triangle $(u; e_1, e_2)$ if and only if $V \cong (J, J)$ where J is the homotope algebra $J = V^{+(e^-)}$, $e = e_1 + e_2$, with multiplication $U_x y = Q(x)Q(e^-)y$ and unit element e^+, which is covered by the triangle $(u^+; e_1^+, e_2^+)$. If V is graded-(Λ)-triangulated then so is J, and $V \cong_\Lambda (J, J)$, since the isomorphism $V \cong (J, J)$ is given by $(\mathrm{Id}, Q(e^-))$. Conversely, if J is a graded-(Λ)-triangulated Jordan algebra (or Jordan triple system), then the associated Jordan pair (J, J) is graded-(Λ)-triangulated. We therefore obtain the following two types of examples of graded-triangulated Jordan pairs.

Example 6.11. *(A′) Hermitian matrix pairs:* $V = (J, J)$ where $J = \mathrm{H}_2(A, A_0, \pi)$ is a hermitian matrix algebra as in example (A) of Def. 6.2.

(B′) *Quadratic form pairs*: $V = (J, J)$ where $J = \mathrm{AC}_{\mathrm{alg}}(q, D, D_0)$ is a quadratic form Jordan algebra as in example (B) of Def. 6.2. We note that $V \cong_\Lambda \big(\mathrm{AC}(q, \mathrm{Id}, D_0), \mathrm{AC}(q, \mathrm{Id}, D_0)\big)$ where $\mathrm{AC}(q, \mathrm{Id}, D_0)$ is the corresponding ample Clifford system.

Theorem 6.12. *A graded-simple-triangulated Jordan pair satisfying*

(a) *every nonzero $m \in M^\sigma$ is a linear combination of invertible homogeneous elements, or*

(b) *the grading group Λ is torsion-free,*

is graded isomorphic to a Jordan pair (J, J) where J is a graded-simple-triangulated Jordan algebra as described in Th. 6.3*. Conversely, all these Jordan pairs (J, J) are graded-simple-triangulated.*

Proof. Let V be graded-simple-triangulated by $(u; e_1, e_2)$. Then V is graded isomorphic to the Jordan pair of the unital Jordan algebra $J = V^{+(e^-)}$. The algebra J is then graded-simple, with $J^\lambda = V^+[\lambda]$, and graded-triangulated by $(u^+; e_1^+, e_2^+)$. Thus J is graded isomorphic to an algebra described in Th. 6.3. Conversely, let $V = (J, J)$ be the Jordan pair for J as in (I)–(III). It follows from Th. 5.11 that the associated Jordan triple system $T(V)$ is graded-simple-triangulated, hence V is graded-simple-triangulated by Lem. 6.8(iii). □

Definition 6.13. A graded Jordan pair V is said to be *division-graded* if it is nonzero and every nonzero element in $V^\sigma[\lambda]$ is invertible in $V[\lambda]$. A graded-triangulated Jordan pair V will be called *division-triangulated*, respectively *division-Λ-triangulated*, if the associated Jordan triple system $T(V) = V^+ \oplus V^-$ is so. Similarly, a division-triangulated Jordan pair defined over a field k is a *triangulated Jordan pair torus* if $\dim_k V_i^\sigma[\lambda] \le 1$, $i = 1, 2$, and $\dim_k M^\sigma[\lambda] \le 1$ for $\sigma = \pm$. The notion of a *Λ-triangulated Jordan pair torus* is the obvious one.

Since invertibility in V is equivalent to invertibility in the unital Jordan algebra associated to V, we get the following corollaries.

Corollary 6.14. *For a graded-triangulated Jordan pair V the following are equivalent:*

(i) *V is graded-simple and every homogeneous $0 \neq m \in M^\sigma$, $\sigma = \pm$, is invertible,*
(ii) *V is division-triangulated,*
(iii) *V is graded isomorphic to the Jordan pair (J, J) where J is one of the Jordan algebras of* Cor. 6.5.

Corollary 6.15. *V is a triangulated Jordan pair torus iff V is graded isomorphic to a Jordan pair (J, J) where J is one of the following:*

(I) *$J = \mathrm{H}_2(A, A_0, \pi)$ is a diagonal hermitian matrix algebra of a noncommutative torus A;*
(II) *$J = \mathrm{AC}_{\mathrm{alg}}(q, F, F_0)$ for a graded-anisotropic q over a graded-field F with Clifford-ample subspace F_0, with $F = D$ and M as described in* Cor. 4.5(b).

7. Graded-simple Lie algebras of type B_2

In this section we apply our results on graded-simple-triangulated Jordan algebras and pairs from the previous section 6 and obtain a classification of (B_2, Λ)-graded-simple and centreless division-(B_2, Λ)-graded Lie algebras in Th. 7.12 and Th. 7.13. In particular, we classify centreless Lie tori of type (B_2, Λ) in Cor. 7.14.

We begin by recalling the relevant definitions from the theory of root-graded Lie algebras (Def. 7.1 and Def. 7.3). The link to triangulated Jordan structures is given by the Tits-Kantor-Koecher construction, reviewed in 7.2 in general and then in Prop. 7.4 for the particular types of Lie algebras studied in this section.

Since we realize B_2-graded Lie algebras as central extensions of Tits-Kantor-Koecher algebras of triangulated Jordan pairs, *we assume in this section that all Lie algebras, Jordan pairs and related algebraic structures are defined over a ring k in which $2 \cdot 1_k$ and $3 \cdot 1_k$ are invertible.*

Definition 7.1. Let R be a finite reduced root system (R could even only be locally finite in the sense of [LN1]). We suppose that $0 \in R$, and denote by $\mathcal{Q}(R)$ the root lattice of R. A Lie algebra L over k is called (R, Λ)-*graded* if

(1) L has a compatible $\mathcal{Q}(R)$- and Λ-gradings, i.e., $L = \oplus_{\lambda \in \Lambda} L^\lambda$ and $L = \oplus_{\alpha \in \mathcal{Q}(R)} L_\alpha$ such that for $L^\lambda_\alpha = L^\lambda \cap L_\alpha$ we have

$$L_\alpha = \oplus_{\lambda \in \Lambda} L^\lambda_\alpha, \quad L^\lambda = \oplus_{\alpha \in \mathcal{Q}(R)} L^\lambda_\alpha, \quad \text{and} \quad [L^\lambda_\alpha, L^\kappa_\beta] \subseteq L^{\lambda+\kappa}_{\alpha+\beta},$$

for $\lambda, \kappa \in \Lambda$, $\alpha, \beta \in \mathcal{Q}(R)$.
(2) $\{\alpha \in \mathcal{Q}(R) : L_\alpha \neq 0\} \subseteq R$.
(3) for every $0 \neq \alpha \in R$ the homogeneous space L^0_α contains an element $e \neq 0$ that is *invertible* in the following sense: There exists $f \in L^0_{-\alpha}$

such that $h = [e, f]$ acts on L_β, $\beta \in R$, by

$$[h, x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta, \quad x_\beta \in L_\beta \tag{7.1}$$

where $\langle \alpha, \beta^\vee \rangle$ denotes the Cartan integer of the two roots $\alpha, \beta \in R$.

(4) $L_0 = \sum_{0 \neq \alpha \in R} [L_\alpha, L_{-\alpha}]$, and $\{\lambda \in \Lambda : L_\alpha^\lambda \neq 0 \text{ for some } \alpha \in R\}$ spans Λ as abelian group.

It follows that $\{\alpha \in \mathcal{Q}(R) : L_\alpha \neq 0\} = R$. Also, any invertible element e generates an $\mathfrak{sl}_2$-triple (e, h, f). If Λ is not spanned by the support, we will simply speak of an *R-graded Lie algebra with a compatible Λ-grading*. An (R, Λ)-graded or R-graded Lie algebra with a compatible Λ-grading is *graded-simple* if it is graded-simple with respect to the Λ-grading. A Lie algebra is *(R, Λ)-graded-simple*, or *R-graded-simple* if it is (R, Λ)-graded and graded-simple, respectively R-graded with a compatible graded-simple Λ-grading.

The definition of a root-graded Lie algebra is taken from [N3]. Originally, root-graded Lie algebras were defined over fields of characteristic 0 by a different system of axioms, [BM] and [BZ]. As explained in [N3, Remark 2.1.2], an R-graded Lie algebra in the sense of [BZ] and [BM] is the same as an R-graded Lie algebra as defined above. A lot is known about the structure of root-graded Lie algebras, see [N5, 5.10] for a summary of results. We will use here that L is a Lie algebra graded by a 3-graded root system R iff L is a central covering of the Tits-Kantor-Koecher algebra $\mathrm{TKK}(V)$ of a Jordan pair V covered by a grid whose associated root system is R ([N3, 2.7]).

7.2. Review of TKK-algebras. Recall, see e.g. [N3, 1.5], that the Tits-Kantor-Koecher algebra $\mathrm{TKK}(V)$ of a Jordan pair V, in short the *TKK-algebra of* $V = (V^+, V^-)$, is a $\mathbb{Z}$-graded Lie algebra defined on the k-module

$$\mathrm{TKK}(V) = V^- \oplus \delta(V^+, V^-) \oplus V^+$$

where $\delta(V^+, V^-)$ is the span of all inner derivations $\delta(x, y) = (D(x, y), -D(y, x))$, $(x, y) \in V$, of V. The $\mathbb{Z}$-grading $\mathrm{TKK}(V) = \bigoplus_{i \in \mathbb{Z}} \mathrm{TKK}(V)_{(i)}$ is a 3-*grading* in the sense that it has support $\{0, \pm 1\}$, namely $\mathrm{TKK}(V)_{(\pm 1)} = V^\pm$ and $\mathrm{TKK}(V)_{(0)} = \delta(V^+, V^-)$. The Lie algebra product is determined by $[x^+, y^-] = \delta(x^+, y^-)$ and by the natural action of $\delta(V^+, V^-)$ on $V^\pm$: $[\delta(x^+, y^-), u^+] = \{x^+,\, y^-,\, u^+\}$ and $[\delta(x^+, y^-), v^-] = -\{y^-,\, x^+,\, v^-\}$. It is important for the connection between Jordan theory and Lie algebras that, conversely, for any 3-graded Lie algebra $L = L_{(1)} \oplus L_{(0)} \oplus L_{(-1)}$ the "wings" $(L_{(1)}, L_{(-1)})$ form a Jordan pair V_L with Jordan triple product

$$\{x^\sigma,\, y^{-\sigma},\, z^\sigma\} = [[x^\sigma,\, y^{-\sigma}],\, z^\sigma] \tag{7.2}$$

for $x^\sigma, z^\sigma \in V^\sigma = L_{(\sigma 1)}$ and $y^{-\sigma} \in V^{-\sigma} = L_{(-\sigma 1)}$, $\sigma = \pm$. Moreover, the ideal $L' = L_{(-1)} \oplus [L_{(-1)},\, L_{(1)}] \oplus L_{(1)}$ of L is a central extension of the TKK-algebra $\mathrm{TKK}(V_L)$, namely $L'/C \cong \mathrm{TKK}(V_L)$ for

$$C = \{d \in [L_{(-1)},\, L_{(1)}] : [d, L_{\pm(1)}] = 0\}.$$

We note that because of $\frac{1}{2}, \frac{1}{3} \in k$, a Jordan pair can be defined by the Jordan triple products $\{.,.,.\}$. The formula (7.2) is crucial: It allows one to transfer properties between the Jordan pair and the associated Lie algebras. An example is Prop. 7.4 below.

A grading of V by Λ extends to a grading of $\mathrm{TKK}(V)$ by Λ using the canonical grading of $\delta(V^+, V^-)$. The gradings of $\mathrm{TKK}(V)$ used in the following will all be induced in this way from gradings of V. We point out that $\mathrm{supp}_\Lambda V \subseteq \mathrm{supp}_\Lambda \mathrm{TKK}(V)$, but both span the same subgroup of Λ.

Definition 7.3. To define special cases of root-graded Lie algebras, we extend the definition of an *invertible element* to any $e \in L^\lambda_\alpha$, $\alpha \neq 0$, requiring the *inverse* $f \in L^{-\lambda}_{-\alpha}$ and the equation (7.1) for $h = [e, f]$. Then, an (R, Λ)-graded or R-graded Lie algebra L with a compatible Λ-grading is *division-graded* if every nonzero element in L^λ_α, $\alpha \neq 0$, is invertible, and a *Lie torus* if it is division-graded, k is a field and $\dim_k L^\lambda_\alpha \leq 1$ for all $0 \neq \alpha \in R$. As usual, in this case we will speak of a *division-(R, Λ)-graded* Lie algebra and a *Lie torus of type* (R, Λ), or *division-R-graded* Lie algebras and *Lie tori of type* R if Λ is not necessarily spanned by the Λ-support.

We now specialize $R = \mathrm{B}_2 = \mathrm{C}_2 = \{0\} \cup \{\pm\varepsilon_1 \pm \varepsilon_2\} \cup \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ and observe that R is 3-graded with 1-part $R_1 = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, 2\varepsilon_2\}$ (an isomorphic realization of this 3-graded root system will be used in Ex. 7.11.

Proposition 7.4. (a) *The TKK-algebra* $\mathrm{TKK}(V)$ *of a graded-triangulated Jordan pair* $V = V_1 \oplus M \oplus V_2$ *is* B_2*-graded with compatible* Λ*-grading. Its homogeneous spaces are*

$$\begin{aligned}
\mathrm{TKK}(V)^\lambda_{\pm 2\varepsilon_i} &= V_i^\pm[\lambda], \\
\mathrm{TKK}(V)^\lambda_{\pm(\varepsilon_1+\varepsilon_2)} &= M^\pm[\lambda], \\
\mathrm{TKK}(V)^\lambda_{\varepsilon_i-\varepsilon_j} &= \delta(e_i^+, M^-[\lambda]), \quad (i,j) \in \{(1,2),(2,1)\}, \\
\mathrm{TKK}(V)^\lambda_0 &= \sum_{i=1,2} \delta(e_i^+, V_i^-[\lambda]) + \delta(u^+, M^-[\lambda]).
\end{aligned}$$

Conversely, if L *is a* B_2*-graded Lie algebra with compatible* Λ*-grading, then its centre* $Z(L)$ *is contained in* L_0*, namely* $Z(L) = \{x \in L_0 : [x, L_\alpha] = 0 \text{ for } \alpha \in (\pm R_1)\}$*, and* $L/Z(L)$ *is graded isomorphic to the TKK-algebra of the graded-triangulated Jordan pair* $V = (V^+, V^-)$ *given by*

$$V_i^\pm[\lambda] = L^\lambda_{\pm 2\varepsilon_i} \quad \text{and} \quad M^\pm[\lambda] = L^\lambda_{\pm(\varepsilon_1+\varepsilon_2)}.$$

(b) *Let* L *be a* B_2*-graded Lie algebra with compatible* Λ*-grading and let* V *be the associated graded-triangulated Jordan pair defined in* (a). *Then* L *is*

(i) *graded-simple if and only if* $L = \mathrm{TKK}(V)$ *and* V *is graded-simple;*
(ii) *division-graded if and only if* V *is division-triangulated;*
(iii) *a Lie torus if and only if* V *is a triangulated Jordan pair torus.*

(iv) (B_2,Λ)*-graded iff* V *is* Λ*-triangulated. In particular,* L *is division-*(B_2,Λ)*-graded iff* V *is division-*Λ*-triangulated, and a Lie torus of type* (B_2,Λ) *iff* V *is a* Λ*-triangulated Jordan pair torus.*

Proof. (a) is the graded version of [N3, 2.3–2.6]. The generalization from $\Lambda = \{0\}$ to arbitrary Λ is immediate. We note that for a root-graded Lie algebra L we have L/L' and the centre of L coincides with the central subspace C defined above.

In (b) the equivalence of graded-simplicity is a general fact ([GN, 2.5]), and (iii) is immediate from (ii) and the formulas in (a). For the proof of (ii) one shows that $e \in L_\alpha$, $\alpha \in R_{\pm 1}$, is invertible in the sense of Def. 7.1, say with inverse $f \in L_{-\alpha}$, iff e has the appropriate invertibility property in the Jordan pair V, again with inverse f. This proves in particular the implication $\Longrightarrow$. For the other direction, it then suffices to show that for a root-graded Lie algebra L invertibility in the spaces L_α, $\alpha \in R_1$, forces invertibility in L_γ, $\gamma \in R_0$. This can be done by using $L_\gamma = [L_\alpha, L_{-\beta}]$ for appropriate $\alpha, \beta \in R_1$. We leave the details to the reader, in particular in view of Rem. 7.5. □

Remark 7.5. In characteristic 0, a centreless Lie torus of type $(\mathrm{B}_2, \mathbb{Z}^n)$ is the same as the centreless core of a extended affine Lie algebra of type B_2. The latter have been studied in [AG, §4] for $k = \mathbb{C}$. Therefore, in this setting the torus part of the theorem above is implicit in [AG, §4].

One can also define Lie algebras graded by non-reduced root systems. A B_2-graded Lie algebra is then a special type of a BC_2-graded Lie algebra. In the setting of BC_2-graded Lie algebras the torus version of Prop. 7.4 has been proven in [F, Th. 3], where Jordan pairs and Jordan algebras are replaced by structurable algebras, and a triangulated Jordan algebra torus by a so-called quasi-torus.

Remark 7.6. We have formulated Prop. 7.4 in terms of triangulated Jordan pairs, since it is in this setting that the proposition can be generalized to describe division-R-graded Lie algebras and Lie tori of type R for any 3-graded root system R. We will however not need this here.

Remark 7.7. We have seen in Th. 6.12 that a graded-triangulated Jordan pair $V = (V^+, V^-)$ is isomorphic to the Jordan pair (J, J) associated to a graded-triangulated Jordan algebra J. Since isomorphic Jordan pairs lead to isomorphic TKK-algebras, one also has the Jordan algebra version of Prop. 7.4. We leave the formulation to the reader.

Remark 7.8. For easier comparison with the literature ([AG, BY, F]) we indicate how to "find" the Λ-triangulated Jordan algebra J in a B_2-graded Lie algebra L, using the notation of above. As a k-module, $J = \bigoplus_{\lambda\in\Lambda}(L^\lambda_{2\varepsilon_1} \oplus L^\lambda_{\varepsilon_1+\varepsilon_2} \oplus L^\lambda_{2\varepsilon_2})$. For $a, b \in J$, the Jordan algebra product is given by $a \cdot b = \frac{1}{2}[[a, 1^-], b]$ where $1^- = f_1 + f_2$ and $f_i \in L^0_{-2\varepsilon_i}$ is the inverse of the invertible element $e_i \in L_{2\varepsilon_i}$ whose existence is guaranteed by condition (3) in Def. 7.1.

The identity element of J is $1_J = e_1 + e_2$, the elements e_i are idempotents of J, and J is triangulated by $(u; e_1, e_2)$ for u the invertible element in $L^0_{\varepsilon_1+\varepsilon_2}$.

Prop. 7.4 reduces the classification of the various types of B_2-graded Lie algebras to determining the TKK-algebras of the corresponding triangulated Jordan pairs and Jordan algebras, which we have described in the previous section §6. Models for these TKK-algebras were given in [N3] for arbitrary triangulated Jordan pairs over arbitrary rings. A more precise description can be obtained in the graded-simple and division-graded case. To this end, we re-visit the description of the TKK-algebras of the two types of Jordan pairs and algebras that appear in the classification of division-triangulated Jordan pairs and algebras in Cor. 6.14 and Cor. 6.5.

Example 7.9. *The TKK-algebra of the Jordan pair (J, J) for $J = \mathrm{H}_2(A, \pi)$, equivalently, of the Jordan algebra J.* ([N3, 4.2]). Note that the ample subspace A_0 of the hermitian matrix algebra is $A_0 = \mathrm{H}(A, \pi)$ since $\frac{1}{2} \in k$. Let

$$\mathfrak{p}_2(A,\pi) := \left\{ \begin{bmatrix} a & b \\ c & -a^{\pi t} \end{bmatrix} \in \mathrm{Mat}_4(A) : \; a \in \mathrm{Mat}_2(A), b, c \in J \right\}.$$

It is easy to see that $\mathfrak{p}_2(A,\pi)$ is the -1-eigenspace of an involution of the associative algebra $\mathrm{Mat}_4(A)$, hence a subalgebra of the general Lie algebra $\mathfrak{gl}_4(A)$. The natural 3-grading of $\mathfrak{gl}_4(A)$ induces one of $\mathfrak{p}_2 = \mathfrak{p}_2(A,\pi)$: We have $\mathfrak{p}_2 = \mathfrak{p}_{2,(1)} \oplus \mathfrak{p}_{2,(0)} \oplus \mathfrak{p}_{2,(-1)}$, where

$$\begin{aligned} \mathfrak{p}_{2,(1)} &= \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : \; b \in \mathrm{H}_2(A,\pi) \right\}, \\ \mathfrak{p}_{2,(0)} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & -a^{\pi t} \end{bmatrix} : \; a \in \mathrm{Mat}_2(A) \right\}, \\ \mathfrak{p}_{2,(-1)} &= \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : \; c \in \mathrm{H}_2(A,\pi) \right\}. \end{aligned}$$

The Lie algebra $\mathfrak{p}$ has a compatible Λ-grading $\mathfrak{p} = \bigoplus_{\lambda \in \Lambda} \mathfrak{p}^\lambda$ for which $\mathfrak{p}^\lambda$ consists of the matrices with all entries in A^λ. The Jordan pair associated to this 3-graded Lie algebra, see the review 7.2, is $V = (J, J)$. We put

$$\mathfrak{sp}_2(A,\pi) = \mathfrak{p}_{2,(1)} \oplus [\mathfrak{p}_{2,(1)}, \mathfrak{p}_{2,(-1)}] \oplus \mathfrak{p}_{2,(-1)},$$

called the *symplectic Lie algebra associated to* (A, π). The proof of [AABGP, III, Prop. 4.2(a), (b)] also works in our more general setting and yields

$$\mathfrak{sp}_2(A,\pi) = [\mathfrak{p}_2(A,\pi), \mathfrak{p}_2(A,\pi)] = \{X \in \mathfrak{p}_2(A,\pi) : \mathrm{tr}(X) \in [A, A]\}.$$

The Lie algebra $\mathfrak{sp}_2(A,\pi)$ is C_2-graded with root spaces indicated in the following tableau:

$$\left[\begin{array}{cc|cc} 0 & \varepsilon_1 - \varepsilon_2 & 2\varepsilon_1 & \varepsilon_1 + \varepsilon_2 \\ \varepsilon_2 - \varepsilon_1 & 0 & \varepsilon_1 + \varepsilon_2 & 2\varepsilon_2 \\ \hline -2\varepsilon_1 & -\varepsilon_1 - \varepsilon_2 & 0 & \varepsilon_2 - \varepsilon_1 \\ -\varepsilon_1 - \varepsilon_2 & -2\varepsilon_2 & \varepsilon_1 - \varepsilon_2 & 0 \end{array} \right]$$

It follows from Prop. 7.4 that

$$\mathrm{TKK}(V) \cong \mathfrak{sp}_2(A,\pi)/Z\big(\mathfrak{sp}_2(A,\pi)\big).$$

However, one has the following criterion for $\mathfrak{sp}_2(A,\pi)$ to be centreless (again [AABGP, III, Prop. 4.2(d)] works in our more general setting):

Lemma 7.10. *If $A = Z(A) \oplus [A, A]$, e.g. if A is a torus* ([NY, Prop. 3.3]), *then $\mathfrak{sp}_2(A,\pi)$ is centreless and hence is (isomorphic to) the TKK-algebra of the Jordan algebra $J = \mathrm{H}_2(A,\pi)$ and the Jordan (J, J).*

Example 7.11. *The TKK-algebra of the Jordan pair $V = (J, J)$ for $J = \mathrm{AC}_{\mathrm{alg}}(q, D)$.* ([N3, 5.1, 5.3]) As in Ex. 7.9 the Clifford-ample subspace $D_0 = D$ since $\frac{1}{2} \in k$. Thus $J = De_1 \oplus M \oplus De_2$ for a graded commutative associative k-algebra D and $q : M \to D$ is a D-quadratic form.

For a D-quadratic form $q_N : N \to D$ on a D-module N we define the *orthogonal Lie algebra of q_N* as $\mathfrak{o}(q_N) = \{X \in \mathrm{End}_D(N) : q_N(Xn, n) = 0 \text{ for all } n \in N\}$, and the *elementary orthogonal Lie algebra $\mathfrak{eo}(q_N)$* as $\mathfrak{eo}(q_N) = \mathrm{Span}_D\{n_1 n_2^* - n_2 n_1^* : n_1, n_2 \in N\}$ where n_1^* is the D-linear form on N defined by $n_1^*(n) = q_N(n_1, n)$.

To describe the TKK-algebra of V or, equivalently of J, we put $h_1 = e_1$, $h_{-1} = e_2$ and define a D-quadratic form q_∞ on

$$J_\infty = Dh_2 \oplus Dh_1 \oplus M \oplus Dh_{-1} \oplus Dh_{-2}$$

by requiring $q_\infty|M = -q$, $(Dh_2 \oplus Dh_{-2}) \perp (Dh_1 \oplus Dh_{-1}) \perp M$, and $q_\infty(h_i, h_{-i}) = 1$, $q_\infty(h_{\pm i}) = 0$ for $i = 1, 2$. It follows from [N3, (5.3.6)] that

$$\mathrm{TKK}(V) \cong \mathfrak{eo}(q_\infty),$$

in particular, $\mathrm{TKK}(V) \cong \mathfrak{o}(q_\infty)$ if M is free of finite rank.

To obtain a more detailed description of the TKK-algebra, *we assume in the following that M has a homogeneous D-basis $\{u_i : i \in I\}$*, an assumption which by Cor. 4.5 is always fulfilled if J is division-triangulated. Then J_∞ is free too and endomorphisms of J_∞ can be identified with column-finite $(4 + |I|) \times (4 + |I|)$-matrices over D, which we do with respect to the basis $h_2, h_1, (u_i)_{i\in I}, h_{-1}, h_{-2}$. Let G be the $|I| \times |I|$-matrix representing q with respect to the basis $(u_i)_{i\in I}$. Then $X \in \mathfrak{eo}(q_\infty) \iff$

$$X = \left[\begin{array}{cc|c|cc} a & b & -m_2^t G & -s & 0 \\ c & d & -m_1^t G & 0 & s \\ \hline & & & & \\ n_1 & n_2 & X_M & m_1 & m_2 \\ & & & & \\ \hline t & 0 & -n_1^t G & d & b \\ 0 & -t & -n_2^t G & c & a \end{array}\right]$$

where $a, b, c, d, s, t \in D$, $m_1, m_2, n_1, n_2 \in D^{(I)} \cong M$ and $X_M \in \mathfrak{eo}(q)$ (if M has finite rank the latter condition is equivalent to $GX_M + X_M^t G = 0$). The Lie algebra $\mathfrak{eo}(q_\infty)$ has a B_2-grading for $\mathrm{B}_2 = \{0\} \cup \{\pm\varepsilon_i, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}$

whose homogeneous spaces $\mathfrak{o}(q_\infty)_\alpha$, $\alpha \in \mathrm{B}_2$, are symbolically indicated by the matrix below.

$$\left[\begin{array}{cc|c|cc}
0 & \varepsilon_2 - \varepsilon_1 & \varepsilon_2 & \varepsilon_1 + \varepsilon_2 & \cdot \\
\varepsilon_1 - \varepsilon_2 & 0 & \varepsilon_1 & \cdot & \varepsilon_1 + \varepsilon_2 \\
\hline
-\varepsilon_2 & -\varepsilon_1 & 0 & \varepsilon_1 & \varepsilon_2 \\
\hline
-(\varepsilon_1 + \varepsilon_2) & \cdot & -\varepsilon_1 & 0 & \varepsilon_2 - \varepsilon_1 \\
\cdot & -(\varepsilon_1 + \varepsilon_2) & -\varepsilon_2 & \varepsilon_1 - \varepsilon_2 & 0
\end{array}\right]$$

Here 0 is the 0-root space, while $\cdot$ indicates an entry 0 in the matrices in $\mathfrak{o}(q_\infty)$. The isomorphism $\mathrm{TKK}(V) \cong \mathfrak{eo}(q_\infty)$ is given by considering the 3-grading of the root system B$_2$ whose 1-part is $\{\varepsilon_2, \varepsilon_2 \pm \varepsilon_1\}$. Hence

$$V^\pm = \mathfrak{o}(q_\infty)_{\pm(\varepsilon_1+\varepsilon_2)} \oplus \mathfrak{o}(q_\infty)_{\pm\varepsilon_2} \oplus \mathfrak{o}(q_\infty)_{\pm(\varepsilon_2-\varepsilon_1)}$$

are the right respectively left columns of the matrices in $\mathfrak{eo}(q_\infty)$.

Before we state the main results of this section, we remind the reader that we assume $\frac{1}{2}, \frac{1}{3} \in k$ in this section.

Theorem 7.12. *For a torsion-free Λ the following are equivalent:*

(i) *L is a B_2-graded-simple Lie algebra,*
(ii) *L is graded isomorphic to the TKK-algebra of a graded-simple-triangulated Jordan algebra,*
(iii) *L is graded isomorphic to one of the following Lie algebras:*
  (I) *$\mathfrak{sp}_2(A,\pi)/Z(\mathfrak{sp}_2(A,\pi))$ for a graded-simple A with involution π,*
  (II) *$\mathfrak{sl}_4(B)/Z\big(\mathfrak{sl}_4(B)\big)$ where $\mathfrak{sl}_4(B) = \{X \in \mathfrak{gl}_4(B) : \mathrm{tr}(X) \in [B,B]\}$, and B is a noncommutative graded-simple associative algebra,*
  (III) *$\mathfrak{eo}(q_\infty)$ in the notation of* Ex. 7.11 *for $D = F$ a graded-field and $q : M \to F$ a graded-nondegenerate F-quadratic form on a graded F-module M with base point $u \in M^0$.*

Proof. The equivalence of (i) and (ii) follows from Prop. 7.4 and Th. 6.12. If (ii) holds, the cases (I) and (III) of Th. 6.3 correspond to the Lie algebras (I) and (III) above, as follows from Ex. 7.9 and Ex. 7.11. That in case (II) of Th.6.3 one gets case (II) above is shown in [N3, (3.4.3)]. The remaining implication (iii) $\Rightarrow$ (ii) is easy. □

With an analogous proof we obtain the classification of B$_2$-division-graded Lie algebras.

Theorem 7.13. *For a Lie algebra L the following are equivalent:*

(i) *L is a centreless division-(B_2,Λ)-graded Lie algebra,*
(ii) *L is graded isomorphic to the TKK-algebra of a division-Λ-triangulated Jordan algebra,*
(iii) *L is graded isomorphic to one of the following Lie algebras:*

(I) $\mathfrak{sp}_2(A,\pi)/Z(\mathfrak{sp}_2(A,\pi))$ *where* A *is a noncommutative division-*Λ*-graded associative algebra with involution* π *and generated by* $\mathrm{H}(A,\pi)$*,*

(II) $\mathfrak{eo}(q_\infty)$ *in the notation of* Ex. 7.11 *for* $D = F$ *a graded-field,* $q : M \to F$ *a graded-anisotropic quadratic form on a graded* F*-module* M *with base point* $u \in M^0$ *and whose* Λ*-support generates* Λ*.*

In particular, using Lem. 7.10 we get the following corollary.

Corollary 7.14. *A Lie algebra* L *is a centreless Lie torus of type* (B_2,Λ) *iff* L *is graded isomorphic to one of the following:*

(I) *A symplectic Lie algebra* $\mathfrak{sp}_2(A,\pi)$ *as in* Ex. 7.9*, where* A *is a noncommutative* Λ*-torus with involution* π *and generated by* $\mathrm{H}(A,\pi)$*.*

(II) *An elementary orthogonal Lie algebra* $\mathfrak{eo}(q_\infty)$ *as in* Ex. 7.11 *with* D *a torus,* M *as described in* Cor. 4.5(b)*,* $q : M \to D$ *graded-anisotropic and* Λ *spanned by* $\operatorname{supp}_\Lambda M$*.*

Remark 7.15. Centreless division-(B_2,Λ)-graded Lie algebras over fields of characteristic 0 and centreless Lie tori of type (B_2,Λ) are also described in [BY, Th. 4.3 and Th. 5.9], using a different method. Our approach gives a more precise description of these Lie algebras. The special case $\Lambda = \mathbb{Z}^n$ had been established before in [AG, Th. 4.87]. It could also be deduced from the results in [F, Th. 9], see Rem. 6.7.

The data A, D, M and q occurring for $\Lambda = \mathbb{Z}^n$ in Cor. 7.14 are described in detail in Cor. 6.6.

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