Maximal subalgebras of octonions[1]

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1. Introduction. Gagola [1] has recently closed a small gap in Racine's classification [3, Thm. 5] of maximal subalgebras of an arbitrary octonion algebra C over a field F: the algebras missing in Racine's list are precisely the subfields of C that are purely inseparable of characteristic 2, exponent 1 and degree 4 over F. What [1] fails to address, however, is the question of why these subalgebras are indeed maximal.

There are surely many ways of proving this. For example, I have been informed by Gagola himself that, in an unpublished preprint dating back more than five years ago, he derived the necessary conclusion from a general theory of maximal subloops sitting inside the Moufang loop of unit octonions. In what follows I will argue in a different manner by appealing to the non-orthogonal Cayley-Dickson construction due to Garibaldi-Petersson [2, Sec. 4].

2. Proposition. *Suppose F has characteristic 2 and $K \subseteq C$ is an inseparable subfield of degree 4 over F. Then K is a maximal subalgebra of C.*

Proof. Being an F-algebra of degree 2 and an inseparable field extension at the same time, K/F is in fact purely inseparable of exponent 1. From [2, Prop. 4.5] we therefore conclude that, for some linear form $s\colon K \to F$ taking 1 into 1 and for some $\mu \in F$, the natural embedding $K \hookrightarrow C$ extends to an isomorphism from C to the non-orthogonal Cayley-Dickson construction $C' := \mathrm{Cay}(K; \mu, s)$ which lives on the vector space direct sum $K \oplus Kj$ of two copies of K under a multiplication having the following properties: identifying $K \subseteq K \oplus Kj$ through the first summand makes K a unital subalgebra of C', and we have

$$u(vj) \equiv (uv)j \bmod K \qquad (u, v \in K); \tag{1}$$

for a full description of the algebra structure of C', which will not be needed here, see [2, 4.3]. Now suppose $A \subseteq C'$ is a subalgebra properly containing K. Since $A = K \oplus (A \cap Kj)$, we have $A \cap Kj \neq \{0\}$, so let $v \in K$ be non-zero satisfying $vj \in A$. Then (1) implies $Kj \equiv (Kv)j \equiv K(vj) \bmod K$, forcing $Kj \subseteq A$, hence $A = C'$, and the proposition is proved. □

3. Remark. Examples of octonion algebras over a field of characteristic 2 containing purely inseparable subfields of exponent 1 and degree 4 can be easily constructed. In fact, if C is a division algebra, we refer to [2, 1.13] for details, while if C is split, we may generalize [1, Example 2] by using [2, Thm. 4.6 (b), Prop. 5.2] to conclude that *every purely inseparable field extension of F having exponent 1 and degree 4 is isomorphic to a subalgebra of C.*

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References

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