Maximal algebras of quotients of Jordan algebras

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Abstract: We define a Jordan analogue of Lambek and Utumi's associative algebra of quotients and we construct the maximal algebra of quotients for nondegenerate algebras. We apply those results to other classes of algebras of quotients appearing in the literature.

Keywords: Jordan algebras; algebras of quotients; dense inner ideals

Introduction

Associative localization theory is a well developed subject which was inaugurated in the works of Ore and Osano and culminates in the general theory of Gabriel localization. Form the viewpoint of Jordan theory, it is natural to ask for extension of these ideas to the setting of Jordan algebras (or more generally, of general Jordan systems). That line of research originated in the question raised by Jacobson [J1, p. 426] of whether it would be possible to imbed a Jordan domain in a Jordan division algebra in a way similar to Ore's construction in associative theory, in connection with the search for new exceptional Jordan division algebras as algebras of fractions of Jordan domains. From a more structure-theoretic standpoint, and related to the possible extensions of Goldie theory to Jordan algebras, the early results of Montgomery [Mon] and Britten [Br1-3] deal with the problem for lineal Jordan algebras $H(R, *)$. A general and purely combinatorial framework was laid by Jacobson *et al.* in [JMP]. Based on the localization of the monoid of U-operators, they obtained an imbedding of a Jordan algebra with a monoid of denominators in an algebra of outer fractions, but in the words of one of the authors (see [BoM]), they had to impose an unnatural extra condition.

As for the search for a Jordan version of Goldie's theorems, which of course required a construction of an algebra of fractions, a definitive answer for linear Jordan algebras came with the papers [Z1,Z2] of Zelmanov, where he made use of his deep results on structure theory rather than the direct approach of [JMP]. This result has been extended in [FGM] to quadratic algebras by using again Zelmanov's structural approach, and refining Zelmanov's ideas with the introduction of inner ideals of

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denominators, which play a key role in some of the notions of algebras of quotients developed in later works (including the present one).

From a entirely different and more intrinsic approach, Martínez has recently given in her beautiful work [M] necessary and sufficient Ore-like conditions for the existence of algebras of fractions of linear (with $\frac{1}{6} \in \Phi$) Jordan algebras, thus solving Jacobson's original problem. Her strategy consists of using the Kantor-Koecher-Tits construction to embed the algebra of fractions in a Lie algebra through partially defined derivations. This approach has also been followed in the quadratic extension [Bo, BoM] of her theorem by Bowling and McCrimmon, where they use Faulkner's Hopf algebra construction which provides an adequate quadratic substitute of the Kantor-Koecher-Tits Lie algebra. It must be pointed out, however that the quadratic result requires some extra Ore condition, called in [BoM] "unwelcome condition", which makes it less neat than its linear counterpart.

Before describing some of the additional literature on Jordan algebras of quotients, let us pause for a moment to reflect on the two approaches to the problem laid in the above mentioned works of Zelmanov and Martínez. As a general remark, the construction of algebras of quotients of a Jordan algebra J amounts to the construction of an over-algebra $Q \supseteq J$ whose elements have some kind of denominators in J, so that one can see those elements as acting partially on J (where that action should include the linear as well as the quadratic actions derived from the way in which an element of Q multiplies by elements of J). So to some extent we are forced to decide what kind of partially defined or germs of "regular representations" will be adequate for that construction. The two mentioned approaches choose different regular representations: a Lie representation in Martínez's work, and an associative representation (hence a specialization) in Zelmanov's. Of course it is a truism that specializations only work for special Jordan algebras, however, if we impose regularity conditions as nondegeneracy, what is left out is algebras satisfying polynomial identities, and for these we can try to use the results of Jordan PI-theory. On the other hand, while partially defined regular representations of associative algebras on suitable filters of one-sided ideals are the main theme of associative localization theory, and are known to produce associative algebras, the effectiveness of using germs of regular representations (hence derivations) of Lie algebras lies in the important fact discovered in [M] that they also produce new Lie algebras.

Following the Lie approach of Martínez, in [GG] Gómez-Lozano and García defined and studied what they called Martindale-like systems of quotients of Jordan systems (not just algebras) over rings Φ of scalars with $\frac{1}{6} \in \Phi$, paralleling the known construction of associative systems of Martindale quotients. By the general reasons mentioned above, that required the previous work of Siles [S] on Martindale-like Lie algebras of quotients, also inspired in the construction given in [M].

On the other hand, among the works that follow Zelmanov's structural approach we can mention the study [MP] of the Jordan analogue of Johnson's algebra of quotients of an associative algebra. Here, the set of denominators is the filter of essential inner ideals of the Jordan algebra, and as in the associative theory, one must impose some nonsingularity condition, which is called strong nonsingularity in that paper (since there already existed a weaker notion of nonsingularity, introduced in [FGM]). Also within the framework of Zelmanov's structural approach, and following some of the ideas of [MP] (in a preliminary version), Anquela, Gómez-Lozano and García have studied in [AGG] algebras of Martindale-like quotients of strongly prime linear Jordan algebras (thus dropping the condition $\frac{1}{3} \in \Phi$ of [GG], but assuming the additional condition of primeness).

In this paper we address the problem of adapting to nondegenerate Jordan algebras Lambek and Utumi's construction of algebras of quotients (also referred to in the literature as general algebras of quotients), and define and prove the existence of maximal algebras of quotients. The importance of this construction is that it embraces all known types of algebras of quotients in the nondegenerate case. We adopt the above mentioned Zelmanov's structural approach, and follow many of the ideas of [MP].

The paper is organized as follows. After a first section of preliminaries, we define in section 1 dense inner ideals, which are intended to be the Jordan analogues of dense one-sided ideals, basic for the construction of Lambek-Utumi's algebras of quotients. It must be noted that, with that definition, the existence of dense inner ideals in a Jordan algebra implies that the algebra is nondegenerate. As commented before, this is a natural restriction for the structural approach that we will follow.

In section 2, we introduce the notion of Jordan algebra of quotients linked to the notion of dense inner ideal. We show that, for nondegenerate algebras, this definition includes as particular cases all the different types of algebras of quotients mentioned above. We study then some of the properties of algebras of quotients, and introduce the natural notion of maximal algebra of quotients. We close the section by studying how the weak centroid of a Jordan algebra relates with the weak centroids of its algebras of quotients, a result which will be instrumental in the study in section 3 of maximal algebras of quotients of PI algebras. To construct a maximal algebra of quotients for those, we extend Beidar and Mikhalev's nearly classical localization to quadratic Jordan algebras, and show that for a PI-algebra that is in fact the maximal algebra of quotients.

Section 4 is devoted to the construction of maximal algebras of quotients for hermitian algebras, or more precisely, for special algebras for which a particular Zelmanov ideal generates an essential ideal. Here the specialization strategy outlined before is used, and we translate the problem to the associative setting, where it can be solved by using Lanning's symmetric algebra of quotients. The fundamental fact here is the good relationship that exists between dense inner ideals and dense one-sided ideals of a $*$-tight associative $*$-envelope, which also suggests the adequacy of the notion of dense inner ideal.

Finally, in section 5, we put all pieces together, and state and prove the existence of maximal algebras of quotients. We obtain as corollaries the existence of maximal Martindale algebras of quotients, and the sufficiency of Martínez's Ore conditions for the existence of algebras of fractions in nondegenerate Jordan algebras.

0. Preliminaries

0.1. We will work with Jordan algebras over a unital commutative ring of scalars Φ which will be fixed throughout. We refer to [J2, MZ] for notation, terminology, and basic results. In particular, we will make use of the identities proved in [J2], which we will quote with the labels QJn of that reference. In this section we recall some of those basic results and notations, together with some other that will be used in the paper.

0.2. A Jordan algebra has products U_xy and x^2, quadratic in x and linear in y, whose linearizations are $U_{x,z}y = V_{x,y}z = \{x,y,z\} = U_{x+z}y - U_xy - U_zy$, and $x \circ y = (x+y)^2 - x^2 - y^2$ respectively.

We will denote by $\hat{J}$ the free unital hull $\hat{J} = \Phi 1 \oplus J$ with products $U_{\alpha 1+x}(\beta 1+y) = \alpha^2\beta 1 + \alpha^2 y + \alpha x \circ y + 2\alpha\beta x + \beta x^2 + U_xy$ and $(\alpha 1 + x)^2 = \alpha^2 1 + 2\alpha x + x^2$. (We will also use this notation for the corresponding construction for associative algebras: $\hat{R} = \Phi 1 + R$.) A tight unital hull J' of J is a Jordan algebra having J as a subalgebra which is tight over J: any nonzero ideal I of J' hits J, $J \cap I \neq 0$.

0.3. A Φ-submodule K of a Jordan algebra J is an inner ideal if $U_x\hat{J} \subseteq K$ for all $x \in K$, and that an inner ideal $I \subseteq J$ is an ideal if $\{I, J, \hat{J}\} + U_JI \subseteq I$. If I, L are ideals of J, so is their product U_IL, and in particular so is the derived ideal $I^{(1)} = U_II$. An (inner) ideal of J is essential if it has nonzero intersection with any nonzero (inner) ideal of J.

For any subset X of the Jordan algebra J, the annihilator of X in J is the set $\mathrm{Ann}_J(X)$ of all $z \in J$ which satisfy $U_zx = U_xz = 0$ and $U_xU_z\hat{J} = U_zU_x\hat{J} = V_{x,z}\hat{J} = V_{z,x}\hat{J} = 0$ for all $x \in X$. This is always an inner ideal of J, and it is also an ideal if

X is an ideal. If J is nondegenerate and I is an ideal of J, the annihilator of I can be characterized in the following alternative ways (see [Mc2, Mo2]):

$$\mathrm{Ann}_J(I) = \{z \in J \mid U_z I = 0\} = \{z \in J \mid U_I z = 0\}.$$

0.4. The centroid $\Gamma(J)$ of a Jordan algebra J is the set of all Φ-linear mappings $\gamma : J \to J$ that satisfy: $\gamma(U_x y) = U_x\gamma(y)$, $\gamma^2(U_x z) = U_{\gamma(x)}z$, and $\gamma(\{x, y, z\}) = \{\gamma(x), y, z\}$ for all $x, y \in J$ and all $z \in \hat{J}$. If J is nondegenerate, then $\Gamma(J)$ is a reduced unital commutative ring, and if in addition J is strongly prime, then $\Gamma(J)$ is a domain acting faithfully on J. In that case we can localize to define the central closure $\Gamma(J)^{-1}J$ which is an algebra over the field of fractions $\Gamma(J)^{-1}\Gamma(J)$. In a nondegenerate J, $\gamma^n x = 0$ implies $\gamma x = 0$ for any $\gamma \in \gamma(J)$, any positive integer n, and any $x \in J$.

Following [Fu], we define the weak center $C_w(J)$ as the set of all $z \in J$ which have $U_z, V_z \in \Gamma(J)$. We will also consider the notion of extended centroid of a Jordan algebra J which we will denote by $\mathcal{C}(J)$, and for which we refer to [Mo2]. Its attached scalar extension $\mathcal{C}(J)$, the extended central closure was defined and studied in [Mo2].

0.5. It is well known that any associative algebra R gives rise to a Jordan algebra $R^{(+)}$ by taking the products $U_x y = xyx$ and $x^2 = xx$. A Jordan algebra is special if it is isomorphic to a subalgebra of an algebra of the form $R^{(+)}$, and it is called *i*-special if it satisfies all the identities satisfied by all special algebras. An important class of special algebras are algebras of symmetric elements $H(R, *)$ of associative algebras with involution $(R, *)$, and more generally, ample subspaces $H_0(R, *) \subseteq H(R, *)$ of symmetric elements, subspaces that satisfy: $r + r^*$, rr^* and rhr^* belong to $H_0(R, *)$ for all $r \in R$ and all $h \in H_0(R, *)$.

For a special Jordan algebra J we can always find a associative $*$-envelope, an associative algebra R with involution $*$ such that J is a subalgebra of $H(R, *)$, and R is generated (as an associative algebra) by J. An associative $*$-envelope of J is $*$-tight is any nonzero $*$-ideal I of R hits J: $I \cap J \neq 0$. By an easy application of Zorn's lemma, one can always $*$-tighten an associative $*$-envelope R of J by factoring out a $*$-ideal I, maximal among those which miss J: $J \cap I = 0$.

A fundamental fact in Jordan theory with important structural consequences for *i*-special algebras is the existence of hermitian ideals in the free special Jordan algebra $FSJ[X]$, generated by X in the $(+)$-algebra of the free associative algebra $Ass[X]$ (see [MZ]): for any special Jordan algebra $J \subseteq H(R, *)$ and any a in the associative subalgebra $alg_R(\mathcal{H}(J))$ of R generated by the evaluation $\mathcal{H}(J)$ of $\mathcal{H}(X)$ on J, the trace $a + a^*$ belongs to $\mathcal{H}(X)$. An *i*-special Jordan algebra J is of hermitian type if $\mathrm{Ann}_J(\sum_{\mathcal{H}} \mathcal{H}(J)) = 0$, where the sum runs on the set of all hermitian ideals.

Often we will consider the stronger condition that there exists a particular hermitian ideal $\mathcal{H}(X)$ with $\mathrm{Ann}_J(\mathcal{H}(J)) = 0$. It does not seem to be known if this is always the case for a hermitian Jordan algebra J.

0.6. We refer to [St, R2] for basic facts about algebras of quotients for associative algebras. If L is a left ideal of an associative algebra R, and $a \in R$, we denote by $(L : a)$ the set of all $r \in R$ with $Lr \subseteq L$. Recall that the left ideal is dense if $(L : a)b \neq 0$ for any $a \in R$ and any nonzero $b \in R$. We will be interested in algebras of quotients attached to the filters of dense right or left ideals of an associative algebra R, and in particular to the right and left maximal algebra of quotients which we will denote by $Q^r_{max}(R)$ and $Q^l_{max}(R)$ respectively. The associative algebras that naturally arise in Jordan theory are associative envelopes and they carry an involution, so it will be important for us to be able to extend involutions to algebras of quotients. This can not be done in general for the one sided maximal algebras of quotients $Q^l_{max}(R)$ and $Q^r_{max}(R)$, so the adequate substitute is the maximal symmetric algebra of quotients $Q_\sigma(R)$ defined by Lanning [L]. Recall that $Q_\sigma(R)$ is the set of elements $q \in Q^r_{max}(R)$ for which there exists a dense left ideal L of R with $Lq \subseteq R$ (or symmetrically, the set of all $q \in Q^l_{max}(R)$ for which there exists a dense right ideal K with $qK \subseteq R$). If R has an involution, this is the biggest subalgebra of the maximal algebra of left (resp. right) quotients to which the involution extends. Another algebra of quotients to which involutions can be extended, and which plays a fundamental role in Zelmanov's structure theory is the Martindale algebra of symmetric quotients $Q_s(R)$ of a semiprime algebra R (see [MZ]). As it is easy to see, one has $Q_s(R) \subseteq Q_\sigma(R)$, and $Q_\sigma(Q_s(R)) = Q_\sigma(R)$, so if S is a subalgebra of R and $R \subseteq Q_s(S)$, then $Q_\sigma(R) = Q_\sigma(S)$.

1. Dense inner ideals

1.1. Let J be a Jordan algebra, K be an inner ideal of J, and $a \in J$. We will use the following notations

$$(K : a)_L = \{x \in K \mid x \circ a \in K\},$$

$$(K : a) = \{x \in (K : a) \mid U_a x \in K\}.$$

Also, for a finite family of elements $a_1, \ldots, a_n \in J$, we inductively define $(K : a_1 : a_2 : \ldots : a_n) = ((K : a_1 : \ldots : a_{n-1}) : a_n)$.

1.2. Lemma. *Let J be a Jordan algebra, $K \subseteq J$ be an inner ideal of J, and $a \in J$. Then, the sets $(K : a)_L$ and $(K : a)$ are inner ideals of J and they satisfy $U_{(K:a)_L} K \subseteq (K : a)$.*

Proof. for all $x \in (K : a)_L$ and $b \in J$, we have $(U_x b)\circ a = \{x, b, x\circ a\} - U_x(b\circ a) \in K$, hence $U_x b \in (K : a)_L$, and $(K : a)_L$ is inner.

On the other hand, for any $x \in (K : a)$ and $b \in J$, we have $U_a U_x b = U_{a\circ x} b - U_x U_a b - U_x(a \circ b) \circ a + \{x, b, U_a x\} \in K$, hence $U_x b \in (K : a)$ y $(K : a)$ is an inner ideal.

Finally, if $k \in K$ and $x \in (K : a)_L$, we have $U_a U_x k = U_{a\circ x} k - U_x U_a k - \{\{a, x, k\}, a, x\} + k \circ U_x a^2$ and $\{a, x, k\} = (a\circ x)\circ k - \{x, a, k\} \in K$, hence $U_a U_x k \in K$, and since $U_x k \in (K : a)_L$, this yields $U_x k \in (K : a)$. ■

1.3. We will say that an inner ideal K of J is *dense* if $U_c(K : a_1 : \ldots : a_n) \neq 0$ for any finite collection of elements $a_1, a_2, \ldots, a_n \in J$, and any $0 \neq c \in J$.

Our next aim is to get a more manageable characterization of density. To that end we consider the following properties for an inner ideal K of J.

$$\forall a, b, c \in J, U_c((K : a) \cap (K : b)) = 0 \Rightarrow c = 0, \tag{1}$$

$$\forall a, b, c \in J, U_c((K : a)_L \cap (K : b)_L) = 0 \Rightarrow c = 0, \tag{1_L}$$

$$\forall a, b, c \in J, U_c(K : a : b) = 0 \Rightarrow c = 0, \tag{2}$$

$$\forall a, b, c \in J, U_c((K : a)_L : b)_L = 0 \Rightarrow c = 0. \tag{2_L}$$

Since $(K : a) \cap (K : b) \subseteq (K : a)_L \cap (K : b)_L$ and $(K : a : b) \subseteq ((K : a)_L : b)_L$ for any $a, b \in J$, clearly (1) implies (1_L), and (2) implies (2_L).

Note also that if J has an inner ideal which satisfies any one of those conditions, then J is nondegenerate.

We will prove next that all these conditions are equivalent to K being dense.

1.4. Lemma. *Let K be an inner ideal of a Jordan algebra J. IfK satisfies* (1_L) *of 1.3, then K satisfies* (2) *of 1.3.*

Proof. Take $a, b \in J$, $x \in (K : a)_L \cap (K : b)_L$, and $y \in (K : a\circ(b\circ x))_L \cap (K : b)_L$, we claim that $U_x y \in ((K : a) : b)_L$.

By QJ20' and its linearizatión, we have $((U_x y) \circ b) \circ a = \{x \circ a, y, x \circ b\} - \{x, a \circ y, x \circ b\} + \{x, y, a \circ (x \circ b)\} - a \circ U_x(y \circ b)$. Now,

- $\{x \circ a, y, x \circ b\} \in \{(K : a) \circ a, J(K : b) \circ b\} \subseteq \{K, J, K\} \subseteq K$,

- $\{x, a \circ y, x \circ b\} \subseteq \{K, J, (K : b) \circ b\} \subseteq \{K, J, K\} \subseteq K$,
- $\{x, y, a \circ (x \circ b)\} = x \circ (y \circ (a \circ (x \circ b))) - \{x, a \circ (x \circ b), y\} \in K \circ ((K : a \circ (x \circ b) \circ (a \circ (x \circ b) + K \subseteq K \circ K + K \subseteq K$, y
- $a \circ U_x(y \circ b) \in a \circ U_{(K:a)}J \subseteq a \circ (K : a) \subseteq K$.

  And therefore $((U_x y) \circ b) \circ a \in K$ and $U_x y \in ((K : a)_L : b)_L$.

On the other hand, $U_a((U_x y) \circ b) = U_a\{x, y, x \circ b\} - U_a U_x(y \circ b)$ (by QJ20') $= \{x \circ a, y, (x \circ b) \circ a\} - \{x, U_a y, x \circ b\} - \{x, a, \{y, x \circ b, a\}\} - \{x \circ b, a, \{y, x, a\}\} + y \circ \{x, a^2, x \circ b\} - U_a U_x(y \circ b)$ (by QJ16 and the linearization in $b = x \circ b$ of the identity $V_a U_b V_a - V_{U_a b, b} = V_{a,b} V_{b,a} - V_{U_a b^2}$, which follows from Macdonald's theorem [J2, 3.4.16]). Now, we have

- $\{x \circ a, y, (x \circ b) \circ a\} = (x \circ a) \circ (y \circ ((x \circ b) \circ a)) - \{x \circ a, (x \circ b) \circ a, y\} \in ((K : a) \circ a) \circ ((K : (x \circ b) \circ a) \circ ((x \circ b) \circ a)) + \{(K : a) \circ a, J, K\} \subseteq K \circ K + \{K, J, K\} \subseteq K$,
- $\{x, U_a y, x \circ b\} \in \{K, J, (K : a) \circ a\} \subseteq K$,
- $\{x, a, \{y, x \circ b, a\}\} = \{x, a, (y \circ ((x \circ b) \circ a)\} - \{x, a, \{y, a, x \circ b\}\} \in \{K, J, (K : (x \circ b) \circ a) \circ ((x \circ b) \circ a)\} + \{K; J : \{K, J, (K : b) \circ b\}\} \subseteq K$
- $\{x \circ b, a, \{y, x, a\}\} = \{x \circ b, a, y \circ (x \circ a)\} - \{x \circ b, a, \{y, a, x\}\} \in \{(K : b) \circ b, J, K \circ ((K : a) \circ a)\} + \{K, J, \{K, J, K\}\} \subseteq K$,
- $U_a U_x(y \circ b) \in U_a U_{(K:a)_L}((K : b)_L \circ b) \subseteq U_a U_{(K:a)_L} K \subseteq U_a(K : a)$ (by 1.2) $\subseteq K$.

  Therefore $U_a((U_x y) \circ b) \in K$ y $(U_x y) \circ b \in (K : a)$, hence $U_x y \in ((K : a) : b)_L$

Suppose now that K has (1_L) , and let $a, b, c \in J$ be elements satisfying $U_c(K : a : b) = 0$. By what has been proved above, for any $x \in (K : a) \cap (K : b)$ and any $y \in (K : a \circ (b \circ x))_L \cap (K : b)_L$, we have $U_x y \in ((K : a) : b)_L$. Thus, for any $z \in (K : a)$ we get $U_{U_x y} z \in (K : a : b)$ by 1.2. Therefore $U_c U_{U_x y} z = 0$ for any x, y, z in those conditions. In particular, taking $z \in U_s K \subseteq (K : a)$ (by 1.2) for a given $s \in (K : a)_L$, we get $U_c U_{U_x y} U_s K = 0$. So if $w \in U_s U_{U_x y} U_c J$, then $U_w K = 0$, hence $w = 0$, that is $U_s U_{U_x y} U_c J = 0$, and thus, $U_{U_{U_x y} U_c s} J = 0$, which implies $U_{U_x y} U_c s = 0$. Since $s \in (K : a)_L$ is arbitrary, we have $U_{U_x y} U_c (K : a)_L = 0$, hence $U_{U_c U_x y}(K : a)_L = U_c U_{U_x y} U_c (K : a)_L = 0$. Now, the property (1_L) of K implies $U_c U_x y = 0$. And since this holds for any $y \in (K : a \circ (b \circ x))_L \cap (K : b)_L$, we have $U_c U_x((K : a \circ (b \circ x))_L \cap (K : b)_L) = 0$. Thus, $U_w((K : a \circ (b \circ x))_L \cap (K : b)_L) = 0$ for any $w \in U_x U_c J$, hence $w = 0$ by (1_L), so we obtain $U_x U_c J = 0$. Therefore $U_{U_c x} J = U_c U_x U_c J = 0$, which yields $U_c x = 0$. Again, since $x \in (K : a)_L \cap (K : b)_L$ can be arbitrarily chosen, we obtain $U_c((K : a)_L \cap (K : b)_L) = 0$, hence $c = 0$ by (1_L). ∎

Since (1) implies (1_L), and (2) implies (2_L), we obtain as a consequence of this lemma that (1) implies (2), and (1_L) implies (2_L).

1.5. Lemma. *Let L, N be inner ideals of the algebra J, and assume that the following properties hold:*

- *For any $a, c \in J$, $U_c(L : a)_L = 0$ implies $c = 0$,*
- *For any $c \in J$, if $U_c N = 0$, then $c = 0$.*

Then $U_c(N \cap L) \neq 0$ for any $0 \neq c \in J$.

Proof. Take $b \in N$. Then $U_b U_{(K:b)_L} K \subseteq N \cap L$ because $U_{(K:b)_L} K \subseteq (K : b)$. Thus, $U_c(N \cap L) = 0$ implies $U_c U_b U_{(K:b)_L} K = 0$. Then, for any $x \in J$ and $k \in (K : b)_L$ we have $U_{U_k U_b U_c x} K = U_k U_b U_c U_x U_c U_b U_k K = 0$, which yields $U_k U_b U_c J = 0$ by the hypothesis on L. Then $U_{U_c U_b k} J = U_c U_b U_k U_b U_c J = 0$, hence $U_c U_b k = 0$ and thus $U_c U_b (K : b)_L = 0$. Therefore, $U_{U_b U_c x}(K : b)_L = U_b U_c U_x U_c U_b (K : b)_L = 0$ for all $x \in J$, hence $U_b U_c J = 0$ by the hypothesis on K. Now, this implies $U_{U_c b} J = U_c U_b U_c J = 0$, hence $U_c b = 0$ since J is nondegenerate. Therefore $U_c b = 0$ for all $b \in N$, hence $c = 0$. ■

1.6. Lemma. *Let K be an inner ideal of the Jordan algebra J. If K satisfies* 1.3(2_L)*, then K satisfies* 1.3(1)*.*

Proof. We first show that if K satisfies (2_L), then $U_c((K : a) : b)_L \neq 0$ for any $a, b \in J$ and $0 \neq c \in J$. Indeed, if $x \in ((K : a)_L : b)_L$ and $y \in (K : b)_L$, we have $(U_x y) \circ b) = \{x, y, x \circ b\} - U_x(y \circ b) \in \{(K : a)_L, K, ((K : a)_L : b)_L \circ b\} + U_{(K:a)_L}((K : b)_L \circ b) \subseteq \{(K : a), K, (K : a)\} + U_{(K:a)_L} K \subseteq (K : a)$ by 1.1. So if $U_c((K : a) : b)_L = 0$ for all $x \in ((K : a)_L : b)_L$, then $U_c U_x (K : b)_L = 0$ Therefore, $U_{U_x U_c z}(K : b)_L = U_x U_c U_z U_c U_x (K : b)_L = 0$ for any $z \in J$, and we get $U_x U_c J = 0$. This implies $U_{U_c x} J = 0$, hence $U_c x = 0$ by nondegeneracy of J. Thus $U_c((K : a)_L : b)_L = 0$, hence $c = 0$ by (2_L).

Now, it suffices to apply 1.5 with $L = (K : a)$ and $N = (K : b)$. ■

1.7. Lemma. *Let K be an inner ideal of the Jordan algebra J. If K satisfies* 1.3(1)*, then $U_c((K : a_1) \cap \cdots \cap (K : a_n)) = 0$ implies $c = 0$ for any n and any $a_1, a_2, \ldots, a_n, c \in J$.*

Proof. To carry out an induction on n it suffices to apply 1.5 with $L = (K : a_n)$ and $N = (K : a_1) \cap \cdots \cap (K : a_{n-1})$, taking into account that 1.3(1) implies 1.3(2). ■

1.8. Lemma. *Let K be an inner ideal of the Jordan algebra J. If K satisfies* 1.3(1)*, then $(K : a)$ also satisfies* (1) *de* 1.3 *for all $a \in J$.*

Proof. Take $a, b, c, d \in J$ with $U_d((K : a : b) \cap (K : a : c)) = 0$. By the proof of lemma1.4, if $x \in (K : a) \cap (K : b) \cap (K : c)$, $y \in (K : a \circ (b \circ x)) \cap (K : a \circ (c \circ x)) \cap (K :$

$b) \cap (K : c)$, and $z \in (K : a)$, then $U_{U_x y} z \in (K : a : b) \cap (K : a : c)$. Therefore $U_d U_{U_x y} z = 0$ for any x, y, z in those conditions. The we have $U_d U_{U_x y}(K : a) = 0$, hence $U_{U_{U_x y} U_d t}(K : a) = U_{U_x y} U_d U_t U_d U_{U_x y}(K : a) = 0$ for any $t \in J$. It follows then from (1) that $U_{U_x y} U_d J = 0$, and hence $U_{U_d U_{U_x y}} J = 0$.Thus $U_d U_x y = 0$, and since y is arbitrary we get $U_d U_x((K : a \circ (b \circ x)) \cap (K : a \circ (c \circ x)) \cap (K : b) \cap (K : c)) = 0$. Arguing as above we get $U_x U_d J = 0$ by lemma 1.7, hence $U_{U_d x} J = 0$ and $U_d x = 0$ by the nondegeneracy of J. Again, since this holds for any $x \in J$ this yields $U_d((K : a) \cap (K : b) \cap (K : c)) = 0$, hence $d = 0$ by 1.7. ∎

1.9. Proposición. *Let J be a Jordan algebra and K be an inner ideal of J. The following assertions are equivalent:*

(0) *K is dense.*

(1) *For any $a, b, c \in J$, $U_c((K : a) \cap (K : b)) = 0$ implies $c = 0$.*

(1_L) *For any $a, b, c \in J, U_c((K : a)_L \cap (K : b)_L) = 0$ implies $c = 0$.*

(2) *for any $a, b, c \in J, U_c(K : a : b) = 0$ implies $c = 0$.*

(2_L) *For any $a, b, c \in J, U_c((K : a)_L : b)_L = 0$ implies $c = 0$.*

(3) *For any n and any $a_1, a_2, \ldots, a_n, c \in J$, $U_c((K : a_1) \cap \cdots \cap (K : a_n)) = 0$ implies $c = 0$.*

Proof. The equivalence of (1), (1_L), (2), and (2_L) is proved in 1.4 and 1.6. On the other hand, (1) ⇒ (3) is proved in lemma 1.7, and (3) ⇒ (1) is obvious. Finally, (0) ⇒ (2) is also obvious, and (1) ⇒ (0) immediately follows from 1.8. ∎

1.10. Lemma. *Let $K, L \subseteq J$ be inner ideals of a Jordan algebra J. If K and L are dense, then $K \cap L$ is dense.*

Proof. Suppose that $a, b, c \in J$ have $U_c((K \cap L : a) \cap (K \cap L : b)) = 0$. Applying lemma 1.5 with $(K : a) \cap (K : b)$ and $(L : a)$ as N and L of that lemma respectively, gives $U_x((K : a) \cap (K : b) \cap (L : a)) \neq 0$ if $x \neq 0$. Thus, again by 1.5 with $(K : a) \cap (K : b) \cap (L : a)$ as N, and $(L : b)$ as L, we get $U_c((K : a) \cap (K : b) \cap (L : a) \cap (L : b)) \neq 0$ if $c \neq 0$. So it suffices to note that $(K : x) \cap (L : x) = (K \cap L : x)$ for any $x \in J$. ∎

1.11. Let J be a Jordan algebra. Following [MP], we will say that a set $\mathcal{F}$ of inner ideal is a linearly topological filter of inner ideals if it satisfies:

FT I. Any inner ideal of J which contains an element from F belongs to F.

FT II. If $K, L \in \mathcal{F}$, then $K \cap L \in \mathcal{F}$.

FT III. If $K \in \mathcal{F}$ and $a \in J$, then $(K : a) \in \mathcal{F}$.

It is obvious that the set of all dense inner ideals satisfies FT I and FT III. Property FT II is proved in 1.10, and consequently, the set of all dense inner ideals of J is a linearly topological filter. This notion parallels the corresponding one used in associative theory (see[St]). However, a complete Jordan version of the associative localization theory would require the definition of a Jordan analogue of the notion of Gabriel filter, which is far from obvious. Nevertheless, some of the consequences that one should expect from that definition can be obtained when we consider dense inner ideals, as is in particular the case with the fact that the product $U_K L$ of a dense inner ideals K and L contains a dense inner ideal (although it may not be an inner ideal itself). The proof of this fact follows the proof of the corresponding fact for essential inner ideals in strongly nonsingular Jordan algebras [MP].

1.12. Lemma. *For a Jordan algebra J and a Φ-submodule $A \subseteq J$, the set*

$$\mathcal{K}(A) = \{a \in A \mid U_a J + \{a, J, A\} \subseteq A\}$$

is an inner ideal of J.

Proof. This is [MP,1.4] ■

1.13. Lemma. *Let $K \subseteq J$ be a dense inner ideal of the Jordan algebra J. Suppose that $H \subseteq K \subseteq J$ is an inner ideal of J which is also an ideal of K: $H \triangleleft K$. If $U_x H \neq 0$ for any $0 \neq x \in K$, then H is a dense inner ideal of J.*

Proof. For any $a, b \in J$ and $x, y \in H \cap (K : a) \cap (K : b)$, we have $(U_x y) \circ a = \{a, y, x \circ a\} - U_x(y \circ a) \in \{H, H, K\} + U_H K \subseteq L$ (since $H \triangleleft K$). Therefore $U_x y \in (H : a)_L \cap (H : b)_L$. Thus, if $c \in J$ satisfies $U_c((H : a)_L \cap (H : b)_L) = 0$, then $U_c U_x y = 0$ for any $x, y \in H \cap (K : a) \cap (K : b)$.

Now, since $(K : a)$ and $(K : b)$ are dense, so is $L = (K : a) \cap (K : b)$ by lemma 1.10. Then we get that $U_d(L : e) = 0$ implies $d = 0$ for all $d, e \in J$. On the other hand, if $U_d H = 0$, then $U_{U_k U_d z} H = U_k U_d U_z U_d U_k H \subseteq U_k U_d U_z U_d H$ (since H is an ideal of K) $= 0$ for any $k \in K$ y $z \in J$. Then, from $U_k U_d z \in K$, it follows that $U_k U_d z = 0$ by the hypothesis on H. Thus $U_k U_d J = 0$, hence $U_{U_d k} J = 0$ and this yields $U_d k = 0$. Thus $U_d K = 0$, and so $d = 0$. This shows that L and H satisfy the hypothesis of lemma 1.5 on L and N respectively. As a consequence, $U_d(H \cap (K : a) \cap (K : b)) = 0$ implies $d = 0$

Now take $z \in J$, since $U_c U_x y = 0$ for all $x, y \in H \cap (K : a) \cap (K : b)$, we get $U_{U_x U_c z} H \cap (K : a) \cap (K : b) = U_x U_c U_z U_c U_x H \cap (K : a) \cap (K : b) = 0$, hence $U_x U_c z = 0$, and sice z is arbitrary we have $U_x U_c J = 0$. Thus $U_{U_c x} J = U_c U_x U_c J = 0$, hence $U_c x = 0$ by the nondegeneracy of J. Then we get $U_c(H \cap (K : a) \cap (K : b)) = 0$, which implies $c = 0$, so the lemma follows from 1.13. ■

1.14. Remark. Let $H \subseteq K$ be inner ideals of J. If K is dense and $H \triangleleft K$, then H is dense if and only if it is an essential ideal of K. Indeed, if H is dense, then for any nonzero ideal I of K, and any $0 \neq y \in I$, we have $0 \neq U_y(H : y) \subseteq H \cap I$. On the other hand, since K is a nondegenerate algebra, the essentiality of H is equivalent to the condition $U_k H = 0 \Rightarrow k = 0$ (0.3).

1.15. Lemma. *Let K be an inner ideal of the Jordan algebra J.*

(a) *If A is an ideal of K, then $\mathcal{K}(A)$ is also an ideal of K.*

(b) *If in K is a nondegenerate algebra and $A \neq 0$, then $\mathcal{K}(A) \neq 0$.*

(c) *If in K is a nondegenerate algebra and A is an essential ideal of K, then $\mathcal{K}(A)$ is dense.*

Proof. (a) Take $k \in K$, $a \in \mathcal{K}(A)$ and $z \in J$. We have $U_{a\circ k}z = U_aU_kz + U_kU_az + k \circ U_a(k \circ z) - \{a, z, U_ka\}$. Using now that A is an ideal of K we get

$$U_aU_kz \in U_aJ \subseteq A,$$

$$U_kU_az \in U_KU_aJ \subseteq U_KA \subseteq A$$

and

$$\{a, z, U_ka\} \in \{a, J, U_kA\} \subseteq \{a, J, A\} \subseteq A.$$

Hence $U_{a\circ k}z \subseteq A$, and we have $U_{a\circ k}J \subseteq A$.

On the other hand, for any $y \in J$ and $b \in A$ we get

$$\begin{aligned}\{a \circ k, y, b\} &= \{a, k \circ y, b\} - \{a, y, k \circ b\} + k \circ \{a, y, b\} \in \\ &\in \{a, J, A\} + \{a, J, K \circ A\} + K \circ \{a, J, A\} \subseteq \\ &\subseteq A + \{a, J, A\} + K \circ A \subseteq A.\end{aligned}$$

And therefore, $a \circ k \in \mathcal{K}(A)$ hence $\mathcal{K}(A) \circ K \subseteq \mathcal{K}(A)$.

Next, for a and k as before, and any $z \in J$,

$$U_{U_ka}z = U_kU_aU_kz \in U_KU_aJ \subseteq U_KA \subseteq A.$$

Now, we have

$$\{a \circ k, z, b \circ k\} = \{a, U_kz, b\} + U_k\{a, z, b\}+$$

$$+k \circ \{a, z \circ k, b\} - \{a, z, U_kb\} - \{b, z, U_ka\}.$$

Hence

$$\begin{aligned}\{U_k a, z, b\} &= \{a, U_k z, b\} + U_k\{a, z, b\}+\\ &+ k \circ \{a, z \circ k, b\} - \{a, z, U_k b\} - \{a \circ k, z, b \circ k\} \in \\ &\in \{a, J, A\} + U_K\{a, J, A\} + K \circ \{a, J, A\}+\\ &+ \{a, J, U_K A\} + \{a \circ k, J, A \circ K\} \subseteq \\ &\subseteq A + U_K A + K \circ A + \{a, J, A\} + \{k \circ a, J, A\} \subseteq \\ &\subseteq A. \qquad\qquad (\text{since } a \circ k \in \mathcal{K}(A))\end{aligned}$$

Thus $U_K\mathcal{K}(A) \subseteq \mathcal{K}(A)$, and this proves that $\mathcal{K}(A)$ is an ideal of K.

(b) Now suppose that K is a nondegenerate algebra and $A \neq 0$. We claim that $U_{U_a b}c \in \mathcal{K}(A)$ for any $a, b, c \in A$. Indeed we have $U_{U_{U_a b}c}J \subseteq U_a U_b J \subseteq U_A U_K J \subseteq U_A K \subseteq A$, and for any $x \in J$, $d \in A$ we also have, by QJ15 ,

$$\begin{aligned}\{U_{U_a b}c, x, d\} &= \{U_a b, c, \{U_a b, x, d\}\} - U_{U_a b}\{x, c, d\} = \\ &= \{U_a b, c, \{U_a b, x, d\}\} - U_a U_b U_a\{x, c, d\} \in \\ &\in \{A, A, K\} + U_A U_K J \subseteq A.\end{aligned}$$

Thus, if $\mathcal{K}(A) = 0$, then $U_{U_a b}c = 0$ for all $a, b, c \in A$, hence $U_{U_a b}A = 0$ for all $a, b \in A$ and from 0.3and the nondegeneracy of we get $K\ U_a b \in \mathrm{Ann}_K(A) \cap A = 0$ for all $a, b \in A$. So again $U_a A = 0$ for all $a \in A$, hence $A = 0$.

(c) Since K is nondegenerate, by 1.14 and (b) it suffices to show that if $k \in K$ has $U_k\mathcal{K}(A) = 0$, then $k = 0$. Now, by the proof of (b), $U_{U_a b}c \in \mathcal{K}(A)$ for all $a, b, c \in A$, hence $U_k\mathcal{K}(A) = 0$ implies $U_k U_{U_a b}c = 0$ for all $a, b, c \in A$. But if $U_k U_{U_a b}c = 0$ for all $a, b, c \in A$, then $U_{U_a b U_k l}A = U_{U_a b}U_k U_l U_k U_{U_a b}A = 0$, hence $U_{U_a b}U_k K \subseteq \mathrm{Ann}_K(A)$ (by ideals) $= 0$ (by essentiality of A). Thus $U_{U_k U_a b}K = U_k U_{U_a b}U_k K = 0$, hence $U_k U_a b = 0$ for all $a, b \in A$. Arguing as before, we get $U_k a = 0$ for all $a \in A$, and finally $k = 0$. ■

1.16. Lemma. *Let K be an inner ideal of the Jordan algebra J. If K is dense, then $\mathcal{K}(U_K K)$ is also dense. In particular, $U_K K$ contains a dense inner ideal.*

Proof. Since K is dense, it is nondegenerate, hence $U_K K$ is an essential ideal of K and 1.15(c) applies. ■

1.17. Following [MP], a Jordan algebra J is called strongly nonsingular if $U_c K \neq 0$ for any essential inner ideal K of J and any nonzero $c \in J$.

1.18. Lemma. *Let J be a Jordan algebra. Then:*

(a) *Any dense inner ideal of J is essential.*

(b) *J is strongly nonsingular if and only if any essential inner ideal is dense.*

Proof. (a) Let K a dense inner ideal of J and L be a nonzero inner ideal of J. For any $0 \neq x \in L$ we have $U_x J \subseteq L$ and $0 \neq U_x(K : x) \subseteq K \cap U_x J \subseteq K \cap L$, which proves that K is essential.

(b) Suppose first that J is strongly nonsingular, and take $a, b \in J$. If K is essential, then $(K : a)$ and $(K : b)$ are essential by [MP, 1.2], hence $(K : a) \cap (K : b)$ is essential. Thus, $U_c(K : a) \cap (K : b) = 0$ implies $c = 0$ by strong nonsingularity of J, and this proves that K is dense. The reciprocal is obvious. ∎

1.19. We next examine some inheritance properties of dense inner ideals which will be useful later. Recall that if J is a Jordan algebra and $a \in J$, the local algebra J_a of J at a is the quotient of the a-homotope $J^{(a)}$ by the ideal $\operatorname{Ker} a$ of $J^{(a)}$ of all the elements $x \in J$ with $U_a x = U_a U_x a = 0$. If J is nondegenerate, the condition $U_a x = 0$ already implies $a \in \operatorname{Ker} a$. We refer to [DAM] for a throughout study of local algebras.

1.20. Lemma. *Let J be a Jordan algebra, a be an element of J and K be an inner ideal of J. If K is dense, then the inner ideal $\bar{K} = K + \operatorname{Ker} a/\operatorname{Ker} a$ is dense in J_a.*

Proof. Denote with bars the projections on J_a and take $\bar{x} = x + \operatorname{Ker} a \in J_a$. Then, for all $k \in (K : x : a) \cap (K : a + x) = N_x$ we have $\{k, a, x\} = (k \circ a) \circ x - \{a, k, x\}$, and since $N_x \subseteq (K : a) \cap (K : x) \cap (K : a + x)$, we get $U_{x,a} k = U_{x+a} k - U_a k - U_x k \in K$, hence $\{k, a, x\} \in K$, and $\bar{x} \circ \bar{k} \in \bar{K}$. Also, $U_x U_a k \in U_x U_a (K : x : a) \subseteq U_x (K : x) \subseteq K$, hence $U \bar{x} \bar{k} \in \bar{K}$. Therefore $\bar{N}_x \subseteq (\bar{K} : \bar{x})$.

Now, if $\bar{c} \in J_a$ has $U_{\bar{c}}((\bar{K} : \bar{x}) \cap (\bar{K} : \bar{y})) = 0$ for some $\bar{x}, \bar{y} \in J_a$. Using the previous notation, we have $\overline{U_c U_a (N_x \cap N_y)} \subseteq U_{\bar{c}}(\bar{N}_x \cap \bar{N}_y) = 0$, hence $U_{U_a c}(N_x \cap N_y) = U_a U_c U_a (N_x \cap N_y) = 0$. But $N_x \cap N_y$ is dense by 1.10 since N_x and N_y are dense, so we get $U_a c = 0$, hence $\bar{c} = 0$, and $\bar{K}$ is dense. ∎

1.21. Local algebras can also be defined for associative algebras in the same way as for Jordan algebras, and for those, the analogue of lemma 1.20 is also true: If L is a dense left ideal of an associative algebra R, and $a \in R$, then $L + \operatorname{Ker} a/\operatorname{Ker} a$ is a dense left ideal of the local algebra R_a.

1.22. Lemma. *Let J be a nondegenerate Jordan algebra and let I be an ideal of J. If K is a dense inner ideal of J, then $K + \operatorname{Ann}_J(I)/\operatorname{Ann}_J(I)$ is a dense inner ideal of $J/\operatorname{Ann}_J(I)$.*

Proof. We denote with bars the projections in $\bar{J} = J/\operatorname{Ann}_J(I)$. Now suppose that $U_{\bar{c}}((\bar{K} : \bar{a}) \cap (\bar{K} : \bar{b})) = 0$ for some $a, b, c \in J$. Since K is dense, so is $(K : a) \cap (K : b)$, and since we obviously have $\overline{(K : a) \cap (K : b)} \subseteq (\bar{K} : \bar{a}) \cap (\bar{K} : \bar{b})$, it suffices to show that for any dense inner ideal K of J, $U_{\bar{c}}\bar{K} = 0$ implies $\bar{c} = 0$. Now, if $U_{\bar{c}}\bar{K} = 0$, then $U_c K \subseteq \mathrm{Ann}_J(I)$, hence $U_{U_c y} K \subseteq I \cap \mathrm{Ann}_J(I) = 0$ for any $y \in I$, which implies $U_c I = 0$ by the density of K. Then $c \in \mathrm{Ann}_J(I)$ by 0.3, hence $\bar{c} = 0$. ■

1.23. Again, the corresponding result holds for associative algebras: if R is a semiprime associative algebra, I is an ideal of R, and L is a dense left ideal of R, then $L + \mathrm{Ann}_R(I)/\mathrm{Ann}_R(I)$ is a dense left ideal of $R/\mathrm{Ann}_J(I)$.

2. Algebras of quotients

2.1. Let $\tilde{J}$ be a Jordan algebra, let J be a subalgebra of $\tilde{J}$ and let $\tilde{a} \in \tilde{J}$. Recall from [Mo2] that an element $x \in J$ is a J-denominator of $\tilde{a}$ if the following multiplications take $\tilde{a}$ back into J:

(Di)	$U_x \tilde{a}$	(Dii)	$U_{\tilde{a}} x$	(Diii)	$U_{\tilde{a}} U_x \hat{J}$
(Diii')	$U_x U_{\tilde{a}} \hat{J}$	(Div)	$V_{x,\tilde{a}} \hat{J}$	(Div')	$V_{\tilde{a},x} \hat{J}$

We will denote the set of J-denominators of $\tilde{a}$ by $\mathcal{D}_J(\tilde{a})$. It has been proved in [Mo2, 4.2] that $\mathcal{D}_J(\tilde{a})$ is an inner ideal of J. We remark (see [FGM, p.410]) that any $x \in J$ satisfying (Di), (Dii), (Diii) and (Div) belongs to $\mathcal{D}_J(\tilde{a})$.

2.2. Let J be a subalgebra of a Jordan algebra Q. We will say that Q is an *algebra of quotients* of J if the following conditions hold:

(i) $\mathcal{D}_J(q)$ is a dense inner ideal of J for all $q \in Q$.

(ii) $U_q \mathcal{D}_J(q) \neq 0$ for any nonzero $q \in Q$.

Clearly, any nondegenerate algebra J is its own algebra of quotients since its inner ideals of denominators $\mathcal{D}_J(x) = J$ are dense for all $x \in J$, and $U_x \mathcal{D}_J(x) = U_x J \neq 0$ by nondegeneracy of J. Reciprocally, any Jordan algebra having an algebra of quotients is nondegenerate since it contains a dense inner ideal.

2.3. Examples.

1. We have already mentioned that a nondegenerate Jordan algebra J is an algebra of quotients of J itself. More generally, if K is a dense inner ideal of J, then J is an algebra of quotients of K. Indeed, any $x \in J$ has $\mathcal{D}_K(x) = (K : x)$, which is dense in J, hence also in K, and $U_x \mathcal{D}_J(x) = U_x(K : x) \neq 0$. In particular, if J is nondegenerate and I is an essential ideal of J, then I is a dense inner ideal and J is an algebra of quotients of I.

2.- We refer to [GG, AGG] for the notion of Martindale-like algebra of quotients of a linear Jordan algebra, which has been generalized for quadratic Jordan algebras to the notion of Martindale-like cover [ACGG1, ACGG2]. Let J be a Jordan algebra and let $\mathcal{F}$ be a filter of essential ideals of J satisfying the property: for all $I \in \mathcal{F}$, the derived ideal $I^{(1)} = U_I I$ is again in $\mathcal{F}$. We will say that an over-algebra $Q \supseteq J$ is a Martindale algebra of $\mathcal{F}$-quotients if for any $q \in Q$ there exists $I \in \mathcal{F}$ with $I \subseteq \mathcal{D}_J(q)$, and $U_q I \neq 0$ if $q \neq 0$. It is easy to see that when J is nondegenerate and $\mathcal{F}$ is the filter of all essential ideals of J, that is exactly the same as a Martindale-like cover of J.

3. Let J be a nondegenerate Jordan algebra with centroid Γ and let $\Sigma \subseteq \Gamma$ be the set of all elements $\gamma\Gamma$ with $\operatorname{Ker}\gamma = 0$. Then Σ is a multiplicatively closed subset and one can consider the module of quotients $J_\Sigma = \Sigma^{-1}J$, which is a Jordan algebra over the ring of quotients $\Sigma^{-1}\Gamma$. Then J_Σ is an algebra of quotients of J (and, in fact, a Martindale algebra of quotients). Indeed, if $q \in J_\Sigma$, then there is $\gamma \in \Sigma$ with $\gamma q = x \in J$. It is easy to see that $\gamma^2 J \subseteq \mathcal{D}_J(q)$ (cf. [FGM;2.1]), and this is clearly an essential ideal of J (since $\operatorname{Ker}\gamma^2 = 0$). Now, if $U_q\mathcal{D}_J(q) = 0$, then $x = \gamma q \in \operatorname{Ann}_J(\gamma^2 J) = 0$, hence $q = 0$.

3. The extended central closure $\mathcal{C}(J)J$ of a nondegenerate Jordan algebra J is an algebra of quotients of J. Indeed, since for any $x \in \mathcal{C}(J)J$ there is an essential ideal of J contained in $\mathcal{D}_J(x)$ by [Mo2, 4.3(ii)], we get $U_x\mathcal{D}_J(x) \neq 0$ by [FGM, 4.3].

5. Let J be a Jordan algebra. Recall that an element $s \in J$ is said to be injective if the mapping U_s is injective over J. Following [FGM] we denote by $\operatorname{Inj}(J)$ the set of injective elements of J. A set $S \subseteq \operatorname{Inj}(J)$ is a monad if $U_s t, s^2 \in S$ for any $s, t \in S$ (see [Z1, Z2, FGM]). A monad S is said to be an Ore monad if $U_s S \cap U_t S \neq \emptyset$ for any $s, t \in S$. An algebra Q containing J as a subalgebra is an algebra of S-quotients (and J is an S-order of Q) if all elements of S are invertible in Q and for all $q \in Q$, $\mathcal{D}_J(q) \cap S \neq \emptyset$. It has been proved in [M,B] that a necessary condition for such an algebra Q to exist is that S satisfies the Ore condition in J: for any $x \in J$ and any $s \in S$ there exists $t \in U_s S$ such that $t \circ x \in K_s = \Phi s + U_s\hat{J}$. Note that for such an element t, we have $U_x t^2 = (x \circ t)^2 + U_t x^2 - \{x \circ t, x, t\} \in K_s$, hence $t^2 \in S \cap (K_s : x)$. Moreover, if $r \in S \cap (K_s : x)$, then any $t \in U_s S \cap U_r S$ has $t \in U_s S$ and $t \circ x \in K_s$. Thus the Ore condition can be rephrased: for any $x \in J$ and any inner ideal K of J, $K \cap S \neq \emptyset$ implies $(K : x) \cap S \neq \emptyset$.

Let J be a nondegenerate Jordan algebra and $S \subseteq \operatorname{Inj}(J)$ be an Ore monad which satisfies the Ore condition in J. Consider the set $\mathcal{I}_S$ of all inner ideals $K \subseteq J$ with $S \cap K \neq \emptyset$. Then $\mathcal{I}_S$ is a filter of inner ideals since for any $K, L \in \mathcal{I}_S$

there are $s \in S \cap K$ and $t \in S \cap L$, and hence there is $r \in U_s S \cap U_t S$ which then belongs to $S \cap K \cap L$, thus giving $K \cap L \in \mathcal{I}_S$. Also, by what has been proved above, $(K : a) \in \mathcal{I}_S$ for any $K \in \mathcal{I}_S$ and any $a \in J$. Now, if $K \in \mathcal{I}_S$ and $U_c K = 0$, then $U_{U_s c} J = U_s U_c U_s J \subseteq U_s U_c K = 0$ for any $s \in S \cap K$. Thus, $U_s c = 0$ by nondegeneracy of J, and $c = 0$ by injectivity of s. This shows that $\mathcal{I}_S$ consists of dense inner ideals. As a consequence, if Q is an algebra of S-quotients of J (whose existence makes superfluous the assumption that S satisfies the Ore condition in J), then Q is an algebra of quotients of J in the sense of 2.2. Indeed, since $\mathcal{D}_J(q) \in \mathcal{I}_S$ is dense for any $q \in Q$, it only remains to show that $U_q \mathcal{D}_J(q) \neq 0$ if $q \neq 0$. To prove that, take $s \in \mathcal{D}_J(q) \cap S$. Then $U_{U_s q} J \subseteq U_s U_q \mathcal{D}_J(q) = 0$ implies $U_s q = 0$ since J is nondegenerate and $U_s q \in J$, hence $q = 0$ because s is invertible in Q.

2.4. Lemma. *Let Q be an algebra of quotients of the Jordan algebra J. Then:*

(i) *Q is nondegenerate,*

(ii) *For any $q \in Q$, $U_q J \cap J \neq 0$,*

(iii) *Any nonzero inner ideal of Q hits J nontrivially (hence Q is tight over J),*

(iv) *If K is a dense inner ideal of J, then $U_q K \neq 0$ and $U_K q \neq 0$ for any nonzero $q \in Q$,*

(v) *If L is an inner ideal of Q, then L is dense in Q if and only if $L \cap J$ is a dense inner ideal of J.*

Proof. (i) and (ii) follow from the fact that $0 \neq U_q \mathcal{D}_J(q) \subseteq U_q J \cap J \subseteq U_q Q$, and the tightness (iii) readily follows from this.

Now, if $K \subseteq J$ is a dense inner ideal and $U_q K = 0$ for some $q \in Q$, then $U_{U_q x} K \subseteq U_q U_x U_q K = 0$ for any $x \in \mathcal{D}_J(q)$, hence $U_q D_J(q) = 0$ since $U_q D_J(q) \subseteq J$ and K is dense, and thus $q = 0$. Now, if $U_K q = 0$ for some $q \in Q$, then arguing as in [MP, 2.6], we get $q = 0$. This proves (iv).

To prove (v) first assume that L is dense, and note that if $a \in J$, then $(L \cap J : a) = (L : a) \cap J$, so it suffices to show that if $U_c(L \cap J) = 0$ for a dense inner ideal L of Q and $c \in J$, then $c = 0$. Now, for any $q \in L$ we have $U_{U_q c} \mathcal{D}(q)_J = U_q U_c U_q \mathcal{D}(q)_J \subseteq U_q U_c (L \cap J) = 0$, hence $U_q c = 0$ by (iv), and thus $U_L c = 0$, hence $c = 0$, again by (iv). Reciprocally, suppose that $L \cap J$ is dense in J, and that $U_c((L : a) \cap (L : b)) = 0$ for some $a, b, c \in Q$. Take $x \in K = \mathcal{D}_J(a) \cap \mathcal{D}_J(b)$, so that the elements $U_x a, U_x b, x \circ a$ and $x \circ b$ all belong to J. Then, for any $y \in N = (K : U_x a) \cap (K : U_x b) \cap (K : x \circ a) \cap (K : x \circ b)$, we have

$$(U_x y) \circ a = x \circ (y \circ (x \circ a)) - \{x, x \circ a, y\} - (U_x a) \circ y \in K,$$

and similarly $(U_x y)\circ b \in K$. Therefore we have $U_x y \in (L:a)_L \cap (L:b)_L$, and thus $U_c U_x y = 0$ for any x, y chosen in that way. Then, for any $z \in J$ we have $U_{U_x c} N = 0$, and since N is dense by 1.10, we get $U_x c = 0$ for any $x \in K$, hence $U_K c = 0$, which implies $c = 0$ by (iv). ■

2.5. Lemma. *The following identities are satisfied in any Jordan algebra:*

$$\begin{aligned}(1)\qquad \{r, U_{U_x y} z, t\} = {} & \{(\{r,x,y\}\circ x)\circ z, U_x y, t\} - \{\{x,\{r,x,y\},z\}, U_x y, t\} - \\ & - \{\{y, U_x r, z\}, U_x y, t\} - \{z, U_x U_y U_x r, t\},\end{aligned}$$

$$\begin{aligned}(2)\qquad & \{r, \{U_x y, z, \{x,y,w\}\}, t\} = \\ & = \{(\{r,w,y\}\circ x)\circ z, U_x y, t\} + \{(\{r,x,y\}\circ w)\circ z, U_x y, t\} + \\ & + \{(\{r,x,y\}\circ x)\circ z, \{x,y,w\}, t\} - \{\{w,\{r,x,y\},z\}, U_x y, t\} - \\ & - \{\{x,\{r,w,y\},z\}, U_x y, t\} - \{\{x,\{r,x,y\},z\},\{x,y,w\},t\} - \\ & - \{\{y,\{x,r,w\},z\}, U_x y, t\} - \{\{y, U_x, r, z\}, \{x,y,w\}, t\} - \\ & - \{z, \{x, U_y U_x r, w\}, t\} - \{z, U_x U_y \{x,r,w\}, t\}.\end{aligned}$$

$$\begin{aligned}(3)\qquad U_r U_{U_x y} t = {} & U_{\{r,x,y\}} U_x t - U_y U_{U_x r} t - \\ & - \{\{r, U_x y, t\}, U_x r, y\} + \{t, x, U_y U_x U_r x\}\end{aligned}$$

$$\begin{aligned}(4)\qquad U_r\{U_x y, t, \{x,y,z\}\} = {} & \{\{r,x,y\}, U_x t, \{r,z,y\}\} + U_{\{r,x,y\}}\{x,t,z\} \\ & - U_y\{\{x,r,z\}, t, U_x r\} - \{\{r,\{x,y,z\},t\}, U_x r, y\} - \\ & - \{\{r, U_x y, t\}, \{x,r,z\}, y\} + \{t, z, U_y U_x U_r x\} + \\ & + \{t, x, U_y(\{x, U_r x, z\} + U_x U_r z)\}.\end{aligned}$$

Proof. Note that we have

$$\begin{aligned} & \{r, U_{U_x y} z, t\} = \{\{r, U_x y, z\}, U_x y, t\} - \{z, U_{U_x y} r, t\} = && \text{(by QJ21)} \\ & = \{\{\{r,x,y\},x,t\}, U_x y, t\} - \{\{y, U_x r, z\}, U_x y, t\} - \{z, U_x U_y U_x r, z\} = && \text{(by QJ21)} \\ & = \{(\{r,x,y\}\circ x)\circ z, U_x y, z\} - \{\{x,\{r,x,y\},z\}, U_x y, z\} - \\ & - \{\{y, U_x r, z\}, U_x y, t\} - \{z, U_x U_y U_x r, z\}, && \text{(by QJ14)}\end{aligned}$$

which proves identity (1). Identity (2) is its partial linearization in x .

As for (3), it is an application of QJ6 using the identity

$$\begin{aligned} & -\{\{r, U_x y, t\}, U_x r, y\} + \{t, x, U_y U_x U_r x\} = \\ & \qquad = -\{r, U_x\{r, U_x t, y\}, y\} + \{U_z x, U_x t, U_y x\},\end{aligned}$$

which is just the evaluation in the x-homotope of the identity $\{r,\{r,t,y\},y\} + \{r^2,t,y^2\} = -\{\{r,y,t\},r,y\} + t \circ U_y r^2$, which in turn follows from Macdonald's theorem [J2, 3.4.16]. Finally, (4) is the partial linearization in x of (3). ■

2.6. Lemma. *Let $\tilde{J}$ be a Jordan algebra, let J be a subalgebra of $\tilde{J}$ and let $\tilde{a} \in \tilde{J}$. If J there is a dense inner ideal K of J such that $x \circ \tilde{a}$ and $U_x\tilde{a}$ are in J for all $x \in K$, then $\mathcal{D}_J(q)$ is a dense inner ideal of J.*

Proof. Take $x, y \in K$, and set $z = U_x y$. Note that $z \in K$, hence $z \circ \tilde{a}$ and $U_z\tilde{a}$ belong to J. Next, for all $c \in \hat{J}$, we have $\{z,\tilde{a},c\} = \{U_x y,\tilde{a},c\} = \{x, y\circ(x\circ\tilde{a}), c\} - \{x,\{y,\tilde{a},x\},c\} - \{U_x\tilde{a},y,c\} \in J$. Thus, considering $\mathcal{K}(U_K K)$ instead of K, which is again dense by 1.16, we can assume that $\{K,\tilde{a},J\} \subseteq J$.

Next, take $u, w \in K$ and $v \in J$, and set $x = U_u v$, $z = \{u,v,w\}$. Then we have:

$$U_x U_{\tilde{a}} x = U_{U_u v} U_{\tilde{a}} U_u v = U_u U_v U_{U_u \tilde{a}} v \in J, \tag{1}$$

since $U_v q \in U_K q \in J$. Note that this identity holds in the polynomial algebra $\tilde{J}[t]$, so we can partially linearize it by considering its term in degree 1 when evaluated in $u = u + tw$. This partial linearization yields:

$$\{x, U_{\tilde{a}} x, z\} + U_x U_{\tilde{a}} z \in J. \tag{2}$$

Now take $y \in K$, $t \in J$, and x and z as before. Taking $r = \tilde{a}$ in identity 2.5(3), and using (1), we get

$$U_{\tilde{a}} U_{U_x y} t \in J, \tag{3}$$

and from 2.5(4) and (2), also

$$U_{\tilde{a}}\{U_x y, t, \{x,y,z\}\} \in J, \tag{4}$$

which implies $U_{U_x y} t \in \mathcal{D}_J(q)$.

Let $a \in J$ and take elements x, y, t as above. We have

$$(U_{U_x y} t) \circ a = \{U_x y, t, \{x, y, x\circ a\}\} - \{U_x y, t, U_x(y \circ a)\} - U_{U_x y}(t \circ a)$$

and

$$x \circ a = (U_u v) \circ a = \{u, v, u \circ a\} - U_u(v \circ a).$$

Now take $u, v, y \in (K : a)$ and $w = u \circ a \in K$ in the above formulae. Then, from (4) we get $U_{\tilde{a}}\{U_x y, t, \{x, y, x\circ a\}\} \in J$. On the other hand, $\{U_x y, t, U_x(y\circ a)\} =$

$U_{U_x(y+y\circ a)}t - U_{U_xy}t - U_{U_x(y\circ a)}t$, and from (3) it follows $U_{\tilde a}\{U_xy, t, U_x(y\circ a)\} \in J$. Finally, also from (3) we get $U_{\tilde a}U_{U_xy}(t\circ a) \in J$ and hence $U_{\tilde a}((U_{U_xy}t)\circ a) \in J$

Therefore, with that choice for u, v, x, y y t we get that the element $d = U_{U_xy}t$ satisfies

$$d \in \mathcal{D}_J(\tilde a), \quad d\circ a \in K \quad \text{and} \quad U_{\tilde a}(d\circ a) \in J. \tag{5}$$

Note that if $d' \in K$ has $U_{\tilde a}d' \in J$, then

$$U_{\tilde a}\{d', s, r\} = \{\tilde a, d', \{r, s, \tilde a\}\} - \{U_{\tilde a}d', s, r\} \in J$$

for any $r \in K$ and $s \in J$, and hence

$$U_{\tilde a}\{d', J, K\} \in J. \tag{6}$$

Take now d as above, and let $k \in K$. We claim that $U_dk \in (\mathcal{D}_J(\tilde a) : a)_L$. First note that $(U_dk)\circ a = \{d, k, d\circ a\} - U_d(k\circ a)$ and $U_d(k\circ a) = U_{U_xy}U_tU_{U_xy}(k\circ a) = U_{U_xy}t'$ with $t' = U_tU_{U_xy}(k\circ a)$, hence $U_d(k\circ a) \in \mathcal{D}_J(\tilde a)$ by (5). So it suffices to show that $\{d, k, d\circ a\} \in \mathcal{D}_J(\tilde a)$. It is clear that $\{d, k, d\circ a\} \in K$ and $U_{\tilde a}\{d, k, d\circ a\} \in J$ by(6), so it remains to prove that $U_{\tilde a}U_{\{d,k,d\circ a\}}s \in J$ for any $s \in J$.

By QJ6 we have

$$\begin{aligned} U_{\{d,k,d\circ a\}}s = {} & U_dU_kU_{d\circ a}s + U_{d\circ a}U_kU_ds - \\ & - \{U_d\{k, d\circ a, s\}, k, d\circ a\} + \{d, s, U_{d\circ a}U_kd\}. \end{aligned}$$

Now, $U_{\tilde a}U_dU_kU_{d\circ a}s \in J$ by (5). Also

$$U_{\tilde a}\{U_d\{k, d\circ a, s\}, k, d\circ a\} \in J$$

and

$$U_{\tilde a}\{d, s, U_{d\circ a}U_kd\} \in J$$

by (6).

On the other hand, again by QJ6 we have

$$\begin{aligned} U_{\tilde a}U_{d\circ a}U_kU_ds = {} & U_{\{\tilde a, d\circ a, k\}}U_ds - U_kU_{d\circ a}U_{\tilde a}U_ds + \\ & + \{k, d\circ a, U_{\tilde a}\{U_ds, k, d\circ a\}\} - \{U_kU_{d\circ a}\tilde a, U_ds, \tilde a\}. \end{aligned}$$

Since $\{\tilde a, d\circ a, k\} \in \{\tilde a, K, K\} \in J$, we get $U_{\{\tilde a, d\circ a, k\}}U_ds \in J$. Moreover, $U_{\tilde a}U_ds \in U_{\tilde a}\mathcal{D}_J(\tilde a) \subseteq J$, hence $U_kU_{d\circ a}U_{\tilde a}U_ds \in J$. On the other hand, $U_{\tilde a}\{U_ds, k, d\circ$

$a\} \in J$ by (6), hence $\{k, d \circ a, U_{\tilde{a}}\{U_d s, k, d \circ a\}\} \in J$, and $U_{d \circ a}\tilde{a} \in U_K \tilde{a} \in J$, so $\{U_k U_{d\circ a}\tilde{a}, U_d s, \tilde{a}\} \in J$ by (6). Thus we get $U_{\{d,k,d\circ a\}} s \in J$ and hence $\{d, k, d \circ a\} \in \mathcal{D}_J(\tilde{a})$, which proves that $U_d k \in (\mathcal{D}_J(\tilde{a}) : a)_L$.

Suppose now that $a, b, c \in J$ satisfy $U_c((\mathcal{D}_J(\tilde{a}) : a)_L \cap (\mathcal{D}_J(\tilde{a}) : b)_L) = 0$. Taking as before $u, v, y \in (K : a) \cap (K : b)$, $x = U_u v$, $t \in J$, and $k \in K$ we have $d = U_{U_{U_x y} t} k \in (\mathcal{D}_J(\tilde{a}) : a)_L \cap (\mathcal{D}_J(\tilde{a}) : b)_L)$, and therefore $U_c d = 0$. Since $k \in K$ is arbitrary, we get $U_c U_{U_{U_x y} t} K = 0$, hence for any $t' \in J$, the element $c' = U_{U_{U_x y} t} U_c t'$ has $U_{c'} K = 0$, which implies $c' = 0$, and hence $U_{U_{U_x y} t} U_c J = 0$. It then follows that $U_c U_{U_x y} t = 0$, and since $t \in J$ is arbitrary, that $U_c U_{U_x y} J = 0$, hence $U_{U_c U_x y} J = U_c U_{U_x y} U_c J = 0$. Thus $U_c U_x y = 0$, hence $U_c U_x((K : a) \cap (K : b)) = 0$ and $U_{U_x U_c t''}((K : a) \cap (K : b)) = 0$ for any $t'' \in J$. Thus $U_x U_c t'' = 0$ by density of K. Then $U_{U_c x} = U_c U_x U_c J = 0$, hence $U_c x = 0$, that is $U_c U_u v = 0$ for any $u, v \in (K : a) \cap (K : b)$. Arguing as before we get $c = 0$, and this proves the density of $\mathcal{D}_J(\tilde{a})$ by 1.9. ■

2.7. Lemma. *Let Q be an algebra of quotients of a Jordan algebra J and assume that Q is a subalgebra of a Jordan algebra $\tilde{Q}$. If $\tilde{q} \in \tilde{Q}$ has a dense inner ideal of denominators $\mathcal{D}_J(\tilde{q})$, then $\mathcal{D}_Q(\tilde{q})$ is dense in Q. Moreover, if $U_{\tilde{q}}\mathcal{D}_J(\tilde{q}) \neq 0$, then $U_{\tilde{q}}\mathcal{D}_Q(\tilde{q}) \neq 0$.*

Proof. For any $x, y \in \mathcal{D}_J(\tilde{q})$ and any $p \in Q$ we have by QJ15,

$$\{\tilde{q}, U_x y, p\} = \{\{\tilde{q}, x, y\}, x, p\} - \{y, U_x \tilde{q}, p\} \in \{J, J, Q\} \subseteq Q.$$

Moreover, by QJ6,

$$U_{\tilde{q}} U_{U_x y} p = U_{\{\tilde{q},x,y\}} U_x p - U_y U_{U_x \tilde{q}} p - \{\{\tilde{q}, U_x y, p\}, U_x \tilde{q}, y\} + \{p, x, U_y U_x U_{\tilde{q}} x\} \in$$
$$U_J U_J Q + \{\{J, J, Q\}, J, J\} + \{Q, J, J\} \subseteq Q.$$

Therefore, $U_x y \in \mathcal{D}_Q(\tilde{q})$ for any $x, y \in \mathcal{D}_J(\tilde{q})$, hence $K = \mathcal{K}_J(U_{\mathcal{D}_J(\tilde{q})}\mathcal{D}_J(\tilde{q})) \subseteq \mathcal{D}_Q(\tilde{q})$. Since K is dense in J by 1.16, the density of $\mathcal{D}_Q(\tilde{q})$ follows from 2.4(v).

Now, if $U_{\tilde{q}}\mathcal{D}_J(\tilde{q}) = 0$, then, with the previous notation, $U_{\tilde{q}} K = 0$. Hence for any $p \in \mathcal{D}_Q(\tilde{q})$ we have $U_{U_{\tilde{q}} p} K = 0$, and since $U_{\tilde{q}} p \in Q$ and K is a dense inner ideal of J, we get $U_{\tilde{q}} p = 0$ for all $p \in \mathcal{D}_Q(\tilde{q})$ by 2.4(iv), that is $U_{\tilde{q}}\mathcal{D}_Q(\tilde{q}) = 0$. ■

2.8. Proposition. *Let $J_1 \subseteq J_2 \subseteq J_3$ be Jordan algebras, each a subalgebra of the next one. Then J_3 is an algebra of quotients of J_1 if and only if J_3 is an algebra of quotients of J_2 and J_2 is an algebra of quotients of J_1.*

Proof. If J_3 is an algebra of quotients of J_1, it is obvious that J_2 is also an algebra of quotients of J_1, and it follows from 2.7 that J_3 is an algebra of quotients of J_2.

Now assume that J_2 is an algebra of quotients of J_1, and J_3 is an algebra of quotients of J_2. Take $q \in J_3$ and consider as in [MP, 3.10] the set

$$N = \{x \in J_1 \mid x \circ q \in J_1, U_x q \in J_1, \{x, q, J_1\} \subseteq J_1\},$$

which is an inner ideal of J_1.

Now evaluating the identity 2.5(1) in $r = q$, $t \in J_1$, $x \in \mathcal{D}_{J_2}(q) \cap J_1$ (hence $U_x q \in J_2$ and $\{x, q, J_2\} \subseteq J_2$), $y \in \mathcal{D}_{J_1}(U_x q)$, and $z \in \mathcal{D}_{J_1}(\{q, x, y\} \circ x) \cap \mathcal{D}_{J_1}(\{q, x, y\})$, we get $\{q, U_{U_x y} z, t\} \in J_1$. Also with that choice of x, y, z we have $U_{U_{U_x y} z} q \in U_{J_1} U_y U_x q \subseteq U_{J_1} U_{\mathcal{D}(U_x q)} U_x q \subseteq U_{J_1} J_1 \subseteq J_1$. Therefore we have

$$U_{U_x y} z \in N. \tag{1}$$

On the other hand, applying identity 2.5(2) for $r = q$, any $t \in \hat{J}_1$, $x, w \in \mathcal{D}_{J_2} \cap J_1$, $y \in D_{x,w} = \mathcal{D}_{J_1}(U_x q) \cap \mathcal{D}_{J_1}(\{x, q, w\}) \cap \mathcal{D}_{J_1}(U_w q)$ (note $U_x q, U_w q \in J_2$), and $z \in E_{x,w,y} = \mathcal{D}_{J_1}(\{q, x, y\} \circ x) \cap \mathcal{D}_{J_1}(\{q, x, y\}) \cap \mathcal{D}_{J_1}(\{q, w, y\} \circ x) \cap \mathcal{D}_{J_1}(\{q, w, y\}) \cap \mathcal{D}_{J_1}(U_{\{x,y,w\}} q)$ (note $\{q, x, y\}$, $\{q, w, y\}$, and $U_{\{x,y,w\}} q \in J_1$), we get $\{q, \{U_x y, z, \{x, y, w\}\}, J_1\} \subseteq \hat{J}_1$.

Next, by QJ16 we have

$$\begin{aligned} U_{\{U_x y, z, \{x,y,w\}\}} q = {} & U_{U_x y} U_z U_{\{x,y,w\}} q + U_{\{x,y,w\}} U_z U_{U_x y} q - \\ & -\{U_{U_x y} z, q, U_{\{x,y,w\}} z\} + \{U_x y, U_z \{\{x, y, w\}, q, U_x y\}, \{x, y, w\}\}. \end{aligned}$$

Now, we have

- $U_x U_z U_{\{x,y,w\}} q \in U_x U_{\mathcal{D}_{J_1}(U_{\{x,y,w\}} q)} U_{\{x,y,w\}} q \subseteq J_1$,
- $U_{\{x,y,w\}} U_z U_{U_x y} q \in U^2_{J_1} U_x U_y U_x q \subseteq U^3_{J_1} U_{\mathcal{D}_{J_1}(U_x q)} U_x q \subseteq J_1$,
- $\{U_{U_x y} z, q, U_{\{x,y,w\}} z\} \in \{\{U_{U_x y} z, q, J_1\} \subseteq J_1$ (by above),
- $U_z \{\{x, y, w\}, q, U_x y\} = U_z U_x U_y \{w, q, x\} + U_z (w \circ U_y U_x q) \in$ $\in U^2_{J_1} U_{\mathcal{D}_{J_1}(\{x,q,w\})} \{w, q, x\} + U_{J_1} (J_1 \circ U_{\mathcal{D}_{J_1}(U_x q)} U_x q \in J_1$,
- $\{U_x y, U_z \{\{x, y, w\}, q, U_x y\}, \{x, y, w\}\} \in J_1$.

Thus we get $U_{\{U_x y, z, \{x,y,w\}\}} q \in J_1$.

It follows from those computations that if x, y, z are chosen as above, then

$$\{U_x y, z, \{x, y, w\}\} \in N. \tag{2}$$

Let us now see that N is dense, suppose that $U_c((N : a) \cap (N : b)) = 0$ for some $a, b, c \in J_1$, and keeping the previous notations, take $x \in (\mathcal{D}_{J_2}(q) : a) \cap (\mathcal{D}_{J_2}(q) :$

$a) \cap J_1$, $y \in (D_{x,x\circ a} : a) \cap (D_{x,x\circ b} : b)$, and $z \in (E_{x,x\circ a,y} : a) \cap (E_{x,x\circ a,y} : a)$. Then, using QJ15 two times we get

$$(U_{U_x y} z) \circ a = \{U_x y, z, \{x, y, x \circ a\}\} - \{U_x y, z, U_x(y \circ a)\} - U_{U_x y}(z \circ a).$$

Now, $\{U_x y, z, \{x, y, x \circ a\}\} \in N$ by (2), $\{U_x y, z, U_x(y \circ a)\} = U_{U_x(y+y\circ a)} z - U_{U_x y} z - U_{U_x(y\circ a)} z \in N$ by (1), and $U_{U_x y}(z \circ a) \in N$ by (1), hence $(U_{U_x y} z) \circ a \in N$, and $U_{U_x y} z \in (N : a)_L$. Symmetrically $U_{U_x y} z \in (N : b)_L$, hence $U_{U_x y} z \in (N : a)_L \cap (N : b)_L$, and $U_c U_{U_x y} z = 0$ for all x, y, z in the conditions above. Thus we get $U_c U_{U_x y}((E_{x,x\circ a,y} : a) \cap (E_{x,x\circ a,y} : a)) = 0$. Now, $E_{x,x\circ a,y}$ and $E_{x,x\circ a,y}$ are intersections of inner ideals of J_1-denominators of elements of J_2, which are dense since J_2 is an algebra of quotients of J_1, and hence they are dense inner ideals and so is their intersection by 1.10. Since $U_{U_{U_x y} U_c t}((E_{x,x\circ a,y} : a) \cap (E_{x,x\circ a,y} : a)) = 0$ for all $t \in J_1$, from 2.4(iv) we get $U_{U_x y} U_c J_1 = 0$, hence $U_{U_c U_x y} J_1 = 0$ by nondegeneracy of J_1, and $U_c U_x y = 0$ again by nondegeneracy of J_1. Then $U_c U_x(((D_{x,x\circ a} : a) \cap (D_{x,x\circ b} : b)) = 0$, and arguing as before this yields $U_c x = 0$, hence $U_c((\mathcal{D}_{J_2}(q) : a) \cap (\mathcal{D}_{J_2}(q) : a) \cap J_1) = 0$ which implies $c = 0$ since $(\mathcal{D}_{J_2}(q) : a) \cap (\mathcal{D}_{J_2}(q) : a) \cap J_1$ is dense in J_1 by 2.4(v). Therefore N is dense, and $\mathcal{D}_{J_1}(q)$ is dense by lemma 2.6.

Now, if $U_q \mathcal{D}_{J_1}(q) = 0$, then for any $p \in \mathcal{D}_{J_2}(q)$ we have $U_{U_p q} \mathcal{D}_{J_1}(q) = 0$, hence $U_p q = 0$ by 2.4(iv) and the essentiality of $\mathcal{D}_{J_1}(q)$. Thus $U_{\mathcal{D}_{J_2}(q)} q = 0$, hence $q = 0$ by 2.4(iv). This proves that J_3 is an algebra of quotients of J_1. ■

2.9. We will say that an algebra of quotients Q of a Jordan algebra J is a *maximal algebra of quotients* if for any other algebra of quotients $Q' \supseteq J$ there exists a homomorphism $\alpha : Q' \to Q$ whose restriction to J is the identity mapping: $\alpha(x) = x$ for all $x \in J$.

2.10. Remark. If Q and Q' are algebras of quotients of a Jordan algebra J and $\alpha : Q' \to Q$ is a homomorphism which restricts to the identity on J, then α is injective. Indeed, if $q \in Q$ has $\alpha(q) = 0$, then $U_q \mathcal{D}_J(q) = \alpha(U_q \mathcal{D}_J(q))$ (since $U_q \mathcal{D}_J(q) \subseteq J) = U_{\alpha(q)} \alpha(\mathcal{D}_J(q)) = 0$, hence $q = 0$.

2.11. Lemma. *Let Q and Q' be algebras of quotients of a strongly nonsingular Jordan algebra J. If $\alpha, \beta : Q' \to Q$ are homomorphisms whose restriction to J is the identity mapping, then $\alpha = \beta$.*

Proof. The proof of [MP,2.12] works here, using 2.4(iv) instead of [MP, 2.6]. ■

2.12. Lemma. *If Q and Q' are maximal algebras of quotients of a Jordan algebra J, then there exists a unique isomorphism $\alpha : Q \to Q'$ that extends the identity mapping $J \to J$.*

Proof. This is straightforward from 2.11. ■

In view of this result, if a Jordan algebra J has a maximal algebra of quotients, such an algebra is unique up to an isomorphism extending the identity on J. We will then denote this algebra by $Q_{max}(J)$ and will refer to it as *the* maximal algebra of quotients of J.

To close this section we examine the relationship between the weak center of a Jordan algebra and of an algebra of quotients. This result will be fundamental in the study of maximal algebras of quotients of PI algebras that we carry out in the next section. The proof of the main result is the same as in the case studied in [MP], so we will skip it and refer to that paper.

2.13. Proposition. *Let J be a Jordan algebra and let $Q \supseteq J$ be an algebra of quotients of J. Then $C_w(J) = C_w(Q) \cap J$.*

Proof. The proof of [MP, 3.4] works here, using 2.4(iv) instead of [MP, 2.6]. ∎

3. Algebras of quotients of PI algebras

In this section we construct maximal algebras of quotients for nondegenerate PI-algebras. The construction is a wide generalization of the central closure which was first introduced for linear Jordan algebras (and other classes of algebras) by Beidar and Mikhalev [BM].

3.1. Lemma. *Let J be a nondegenerate PI Jordan algebra. Then:*

(a) *Any dense inner ideal of J hits nontrivially the weak center of J.*

(b) *An inner ideal of J is dense if and only if it contains an essential ideal of J.*

Proof. (a) Any dense inner ideal K of J is itself a nondegenerate Jordan algebra, hence $C_w(K) \neq 0$ by [FGM, 3.6] since K is PI. Now, J is an algebra of quotients of K by 2.3.1, hence $C_w(J) \cap K = C_w(K)$ by 2.13.

(b) It is clear that any essential ideal in a nondegenerate Jordan algebra is a dense inner ideal, hence so is any inner ideal that contains one. Suppose then that K is a dense inner ideal and let I be the core of K, the biggest ideal of J contained in K. If I is not essential, then $\mathrm{Ann}_J(I) \neq 0$ by nondegeneracy, hence $\mathrm{Ann}_K(I) = \mathrm{Ann}_J(I) \cap K \neq 0$ by the essentiality of K (1.18). Since K is nondegenerate and PI, there is a nonzero $z \in C_w(K) \cap \mathrm{Ann}_K(K)$. Then $z \in C_w(J) = C_w(K) \cap J$ as in (a). Moreover, $U_z J \subseteq \mathrm{Ann}_J(I)$ hence $U_z J \cap I = 0$, but $U_z J \subseteq K$, hence $U_z J \subseteq I$, which contradicts the nondegeneracy of J.

3.2. In [BM] the authors introduced what was called the nearly classical localization of an algebra, which included the case of linear Jordan algebras. In the case of associative algebras that construction was called in [W, p. 271] the almost classical localization. Our aim now is to extend their construction to quadratic Jordan algebras.

We consider a nondegenerate Jordan algebra J and denote by $\Gamma = \Gamma(J)$ its centroid, which is a reduced associative commutative ring. Then Γ is nonsingular and J is a Γ-module. Denote by $\mathcal{E}(\Gamma)$ the set of essential ideals of Γ. We will assume that the Γ-module J is nonsingular, that is if $\mathfrak{a}\, x = 0$ for some $\mathfrak{a} \in \mathcal{E}(\Gamma)$, and $x \in J$, then $x = 0$. The one can define the localization $\Gamma_{\mathcal{E}(\Gamma)}$, and the localization of the Γ-module J: $J_{\mathcal{E}(\Gamma)} = \varinjlim\{Hom_\Gamma(\mathfrak{a}\,, J) \mid \mathfrak{a} \in \mathcal{E}(\Gamma)\}$, the direct limit of the directed system $\{Hom_\Gamma(\mathfrak{a}\,, J) \mid \mathfrak{a} \in \mathcal{E}(\Gamma)\}$. Its elements can be represented as classes $[f, \mathfrak{a}\,]$ of pairs $(f, \mathfrak{a}\,)$ where $f \in Hom_\Gamma(\mathfrak{a}\,, J)$ for $\mathfrak{a} \in \mathcal{E}(\Gamma)$, modulo the equivalence relation: $(f, \mathfrak{a}\,) \sim (g, \mathfrak{b}\,)$ if f and g agree on $\mathfrak{a} \cap \mathfrak{b}$. It is well known that there is an action of $\Gamma_{\mathcal{E}(\Gamma)}$ on $J_{\mathcal{E}(\Gamma)}$ extending the action of Γ that gives $J_{\mathcal{E}(\Gamma)}$ a $\Gamma_{\mathcal{E}(\Gamma)}$-module structure.

3.3. Remark. We mention here two instances in which the algebra J is automatically a nonsingular Γ-module. On the one hand, that happens if J is strongly prime. Indeed, for any $\mathfrak{a} \in \mathcal{E}(\Gamma)$, the set $\mathfrak{a}\, J$ is easily seen to be a nonzero ideal of J, so if $x \in J$ has $\mathfrak{a}\, x = 0$, then $U_{\mathfrak{a}\, J} x \subseteq \mathfrak{a}\, U_J x = U_J \mathfrak{a}\, x = 0$ implies $x \in \mathrm{Ann}_J(\mathfrak{a}\, J) = 0$.

On the other hand, if J is a nondegenerate PI-algebra and there are $\mathfrak{a} \in \mathcal{E}(\Gamma)$ and $0 \neq x \in J$ with $\mathfrak{a}\, x = 0$, then as before $\mathrm{Ann}_J(\mathfrak{a}\, J) \neq 0$. Since J is PI, by [FGM, 3.6], there is a nonzero $z \in \mathrm{Ann}_J(\mathfrak{a}\, J) \cap C_w(J)$, and then $(\mathfrak{a}\, U_z)J = U_z \mathfrak{a}\, J = 0$, hence $\mathfrak{a}\, U_z = 0$, which contradicts the essentiality of $\mathfrak{a}$.

3.4. Lemma. *Let J and Γ be as in 3.2. If $\mathfrak{a} \in \mathcal{E}(\Gamma)$ and $n \geq 0$, then $\mathfrak{a}^{[n]} = \sum_{\alpha \in \mathfrak{a}} \Gamma \alpha^n \in \mathcal{E}(\Gamma)$*

Proof. If $\beta \in \Gamma$ annihilates $\mathfrak{a}^{[n]}$, then $\alpha^n \beta = 0$ for all $\alpha \in \mathfrak{a}$, hence $(\alpha\beta)^n = 0$ for all $\alpha \in \mathfrak{a}$, and this implies $\alpha\beta = 0$ for all $\alpha \in \mathfrak{a}$ since Γ is reduced. Thus $\mathfrak{a}\, \beta = 0$, hence $\beta = 0$. ■

3.5. We next give $J_{\mathcal{E}(\Gamma)}$ a structure of $\Gamma_{\mathcal{E}(\Gamma)}$-algebra. To do that, take $p = [f, \mathfrak{a}\,]$ and $q = [g, \mathfrak{b}\,]$ in $J_{\mathcal{E}(\Gamma)}$. We set $p^2 = [k, \mathfrak{a}^{[2]}]$, and $U_p q = [h, \mathfrak{a}^{[2]}\mathfrak{b}\,]$, where h and k are defined as follows: $k(\sum_i \lambda_i \alpha_i^2) = \sum_i \lambda_i f(\alpha_i)^2$, and $h(\sum_i \alpha_i^2 \beta_i) = \sum_i U_{f(\alpha_i)} g(\beta_i)$, where $\alpha_i \in \mathfrak{a}$, $\beta_i \in \mathfrak{b}$ and $\lambda_i \in \Gamma$. To see that these operations are well defined suppose that $\sum_i \alpha_i^2 \beta_i = 0$ for some $\alpha_i \in \mathfrak{a}$ and $\beta_i \in \mathfrak{b}$, and set $x = \sum_i U_{f(\alpha_i)} g(\beta_i) \in J$. Then, for any $\alpha \in \mathfrak{a}$, and any $\beta \in \mathfrak{b}$ we have: $\alpha^2 \beta x = \sum_i \alpha^2 \beta U_{f(\alpha_i)} g(\beta_i) = \sum_i \alpha^2 U_{f(\alpha_i)} \beta g(\beta_i) = \sum_i U_{\alpha f(\alpha_i)} \beta g(\beta_i) = \sum_i U_{f(\alpha\alpha_i)} g(\beta\beta_i) = \sum_i U_{\alpha_i f(\alpha)} \beta_i g(\beta) = \sum_i \alpha_i^2 U_{f(\alpha)} \beta_i g(\beta) = \sum_i \alpha_i^2 \beta_i U_{f(\alpha)} g(\beta) = (\sum_i \alpha_i^2 \beta_i)(U_{f(\alpha)} g(\beta)) = 0 U_{f(\alpha)} g(\beta) = 0$, hence $\mathfrak{a}^{[2]}\mathfrak{b}$ annihilates x, and this implies $x = 0$ by the essentiality of $\mathfrak{a}^{[2]}\mathfrak{b}$ and the nonsingularity of the Γ-module J. So it only remains to prove that $U_p q$ is independent of the choice of the representatives of p and q. Suppose then that $p = [f, \mathfrak{a}] = [f_1, \mathfrak{a}_1]$ and $q = [g, \mathfrak{b}] = [g_1, \mathfrak{b}_1]$. Then it is easy to see that the mapping $h_1 \in Hom_\Gamma(\mathfrak{a}_1^{[2]}\mathfrak{b}, J)$ given by $h_1(\sum_i \alpha_i^2\beta_i) = \sum_i U_{f_1(\alpha_i)}g_1(\beta_i)$ for any finite collection of $\alpha_i \in \mathfrak{a}_1$ and $\beta_i \in \mathfrak{b}_1$ agrees with h on the essential ideal $(\mathfrak{a} \cap \mathfrak{a}_1)^{[2]}(\mathfrak{b} \cap \mathfrak{b}_1)$, and hence $[h, \mathfrak{a}^{[2]}\mathfrak{b}] = [h_1, \mathfrak{a}_1^{[2]}\mathfrak{b}_1]$. Thus U_pq is well defined, and similarly p^2 is well defined. It is also a routine checking to show that U_pq and p^2 are quadratic on p, and that U_pq is linear on q.

Recall that the mapping $\mu : J \to J_{\mathcal{E}(\Gamma)}$, $x \mapsto \mu_x = [m_x, \Gamma]$ given by $m_x(\alpha) = \alpha x$ is well defined, and a monomorphism of Γ-modules (the injectivity is a consequence of the nonsingularity of J). We identify J with its image $\mu(J)$ under μ. Recall the following well known fact:

3.6. Lemma. *Let J and $J_{\mathcal{E}(\Gamma)}$ be as before, and $q \in J_{\mathcal{E}(\Gamma)}$. Then:*

(1) *Suppose that q has a representative $(f, \mathfrak{a}) \in q$. If $\alpha \in \mathfrak{a}$, then $\alpha q = f(\alpha)$.*

(2) *If there is $\mathfrak{b} \in \mathcal{E}(\Gamma)$ such that $\mathfrak{b}\, q = 0$, then $q = 0$.*

Proof. (1) Note that $\alpha q = [\alpha f, \mathfrak{a}]$ and take $\beta \in \mathfrak{a}$. Then $(\alpha f)(\beta) = \alpha f(\beta) = f(\alpha\beta) = \beta f(\alpha) = m_{f(\alpha)}(\beta)$, hence $\alpha q = [\alpha f, \mathfrak{a}] = [m_{f(\alpha)}, \mathfrak{a}] = [m_{f(\alpha)}, \Gamma] = \mu_{f(\alpha)} = f(\alpha)$.

(2) This just means that $J_{\mathcal{E}(\Gamma)}$ is nonsingular as a Γ-module, which is well known. ∎

3.7. Lemma. *Let J and $\mathcal{E}(\Gamma)$ be as before. If $\mathfrak{a} \in \mathcal{E}(\Gamma)$, then $\mathfrak{a}\, J$ is an essential ideal of J.*

Proof. We have already noted in 3.3 that $\mathfrak{a}\, J$ is an ideal of J. Now, if $x \in \mathrm{Ann}_J(\mathfrak{a}\, J)$, then $U_x\alpha J \subseteq U_x\mathfrak{a}\, J = 0$ for all $\alpha \in \mathfrak{a}$, hence $0 = \alpha(U_x\alpha J) = \alpha^2 U_x J = U_{\alpha x}J$. Then $\alpha x = 0$ by nondegeneracy of J, and since this holds for any $\alpha \in \mathfrak{a}$, we have $\mathfrak{a}\, x = 0$, hence $x = 0$ since J is a nonsingular Γ-module. ∎

3.8. Lemma. *Let J be a nondegenerate Jordan algebra, and let Γ be its centroid. Assume that J is a nonsingular J-module, and let $\mathcal{E}(\Gamma)$ and $J_{\mathcal{E}(\Gamma)}$ be as before. Then:*

(1) *$J_{\mathcal{E}(\Gamma)}$ is a Jordan algebra with the operations defined in 3.5,*

(2) *J is a subalgebra of $J_{\mathcal{E}(\Gamma)}$ (through the mapping μ of 3.5),*

(3) *For any $q \in J_{\mathcal{E}(\Gamma)}$ there exists $\mathfrak{a} \in \mathcal{E}(\Gamma)$ with $\mathfrak{a}\, J \subseteq \mathcal{D}_J(q)$, and therefore $J_{\mathcal{E}(\Gamma)}$ is an algebra of quotients of J.*

Proof. (1) Let $F(x_1, \ldots, x_n) = 0$ be one of the defining identities of Jordan algebras, where $F \in FQ[X]$, the free quadratic algebra over a countable set of generators X (see [J, 3.1]). Take $q_1, \ldots, q_n \in J_{\mathcal{E}(\Gamma)}$ and set $p = F(q_1, \ldots, q_n)$. Note that the defining identities of Jordan algebras are homogeneous elements of $FQ[X]$, so suppose that F has degree k_i in x_i. Choose representatives $(f_i, \mathfrak{a}_i) \in q_i$. Then, for any $\alpha_i \in \mathfrak{a}_i$, $i = 1, \dots, n$, we have $\alpha_1^{k_1} \cdots \alpha_n^{k_n} p = \alpha_1^{k_1} \cdots \alpha_n^{k_n} F(q_1 \dots, q_n) = F(\alpha_1 q_1, \dots, \alpha_n q_n) = F(f_1(\alpha_1), \dots, f_n(\alpha_n)) = 0$, and therefore, setting $\mathfrak{b} = \mathfrak{a}^{(k_1)} \cdots \mathfrak{a}^{(k_n)}$ we have $\mathfrak{b}\, p = 0$, hence $p = 0$ by 3.6(2).

Assertion (2) is straightforward. For assertion (3), if $q = [g, \mathfrak{b}]$, it suffices to take $\mathfrak{a} = \mathfrak{b}^{[2]}$. ∎

Following [W], we call $J_{\mathcal{E}(\Gamma)}$ the *almost classical algebra of quotients* of J.

3.9. Corollary. *Let J be a nondegenerate Jordan algebra, and let Γ be its centroid. Assume that J is a nonsingular J-module, and let $\mathcal{E}(\Gamma)$ and $J_{\mathcal{E}(\Gamma)}$ be as before. Then $J_{\mathcal{E}(\Gamma)} \supseteq J$ is tight and $J_{\mathcal{E}(\Gamma)}$ is nondegenerate.*

Proof. We can apply 2.4 since $J_{\mathcal{E}(\Gamma)}$ is an algebra of quotients of J by 3.8(3). ∎

3.10. Lemma. *Let J be a nondegenerate PI Jordan algebra, Γ be its centroid, and $\mathcal{E}(\Gamma)$ the set of all essential ideals of Γ. For any ideal I of J denote by $\mathfrak{u}(I)$ the Γ-linear span of the set of all operators U_z for $z \in C_w(J) \cap I$, and by $(I : J)_\Gamma$ the set of all $\gamma \in \Gamma$ such that $\gamma J \subseteq I$. Then $\mathfrak{u}(I) \subseteq (I : J)_\Gamma$, and I is essential if and only if $\mathfrak{u}(I) \in \mathcal{E}(\Gamma)$.*

Proof. If $\mathfrak{u}(I)$ is essential, then $\mathfrak{u}(I)J$ is an essential ideal of J by 3.7, and since $\mathfrak{u}(I)J \subseteq I$, we obtain that I is essential.

Reciprocally, suppose that I is essential, and take a nonzero $\gamma \in \Gamma$ with $\gamma\mathfrak{u}(I) = 0$. let $L = \mathrm{Ann}_J(\mathfrak{u}(I)J)$. Clearly $\gamma J \subseteq L$, hence $L \neq 0$. Then $I \cap L$ is nonzero by the essentiality of I, hence there is a nonzero $z \in L \cap C_w(J)$ by [FGM, 3.6]. Then we have $U_z \in \mathfrak{u}(I)$, hence $U_z J \subseteq L \cap \mathfrak{u}(I)J = 0$, and this contradicts J being nondegenerate. Therefore $L = 0$ and $\gamma J = 0$, hence $\gamma = 0$. ∎

3.11. Theorem. *Let J be a nondegenerate PI Jordan algebra, then J has maximal algebra of quotients $Q_{max}(J) = J_{\mathcal{E}(\Gamma)}$, the almost classical algebra of quotients of J.*

Proof. Let $Q \supseteq J$ be an algebra of quotients of J. We define a mapping $\phi : Q \to J_{\mathcal{E}(\Gamma)}$ in the following way. For any $q \in Q$, the dense inner ideal $\mathcal{D}_J(q)$ contains an essential ideal I of J by 3.1(b). We set $\phi(q) = [f_q, \mathfrak{u}(I)]$, with $\mathfrak{u}(I)$ as in 3.10, where $f_q : \mathfrak{u}(I) \to J$ is given on a typical element $\alpha = \sum_i \lambda_i U_{z_i}$ of $\mathfrak{u}(I)$, with $\lambda_i \in \Gamma$ and $z_i \in C_w(J) \cap I$, by $f_q(\alpha) = \sum_i \lambda_i U_{z_i} q$ (note that $U_{z_i} q \in J$ for all i, since $z_i \in \mathcal{D}_J(q)$). To see that it is well defined, we have to check, on the one hand, that this does not depend on the particular representation of α as a linear combination of U_{z_i}'s, and on the other hand, that the class $[f_q, \mathfrak{u}(I)]$ does not depend on the particular choice of the essential ideal $I \subseteq \mathcal{D}_J(q)$.

For the first question, suppose that $\sum_i \lambda_i U_{z_i} = 0$. Set $a = \sum_i \lambda_i U_{z_i} q$ and take a nonzero $z \in Cw(J) \cap I$. Then $U_z a = \sum_i \lambda_i U_z U_{z_i} q = \sum_i \lambda_i U_{z_i} U_z q$ (since $U_z \in C_w(Q)$ by 2.13) $= (\sum_i \lambda_i U_{z_i}) U_z q$ (since $U_z q \in J$) $= 0$. Since $\mathfrak{u}(I)$ is spanned by the elements U_z when $z \in C_w(J) \cap I$, we get $\mathfrak{u}(I)a = 0$. Now $\mathfrak{u}(I) \in \mathcal{E}(\Gamma)$ by 3.10, hence $a = 0$ since J is a nonsingular Γ-module by 3.3. This shows that f is well defined, and it is clear that it is a homomorphism of Γ-modules.

Now suppose that L is another essential ideal contained in $\mathcal{D}_J(q)$, and let $g : \mathfrak{u}(L) \to J$ be the corresponding homomorphism: $g(\sum_i \lambda_i U_{z_i}) = \sum_i \lambda_i U_{z_i} q$ for $\lambda_i \in \Gamma$ and $z_i \in C_w(J) \cap L$. Then $L \cap I$ is again an essential ideal, and $\mathfrak{u}(L \cap I) \subseteq \mathfrak{u}(L) \cap \mathfrak{u}(I)$ is essential in Γ by 3.10. Clearly f_q and g agree on $\mathfrak{u}(L \cap I)$, hence $[f_q, \mathfrak{u}(I)] = [g, \mathfrak{u}(L)]$, which proves that ϕ is well defined.

Let us now show that ϕ is a homomorphism of algebras. If $p, q \in Q$, then $\mathcal{D}_J(p) \cap \mathcal{D}_J(q)$ is a dense inner ideal of J, hence there exists an essential ideal $I \subseteq \mathcal{D}_J(p) \cap \mathcal{D}_J(q)$ by 3.1(b). Now put $\phi(p) = [f_p, \mathfrak{u}(I)]$ and $\phi(q) = [f_q, \mathfrak{u}(I)]$ defined as above. Then it is easy to see that $I^{(1)} = U_I I \subseteq \mathcal{D}_J(p+q)$, hence $\phi(p+q) = [f_{p+q}, \mathfrak{u}(I^{(1)})]$. Note that $\mathfrak{u}(I^{(1)}) \subseteq \mathfrak{u}(I)$, and the mappings f_{p+q} and $f_p + f_q$ agree on that ideal. Then $\gamma(\phi(p+q)) = f_{p+q}(\gamma) = f_p(\gamma) + f_q(\gamma) = \gamma\phi(p) + \gamma\phi(q) = \gamma(\phi(p) + \phi(q))$ for any $\gamma \in \mathfrak{u}/I^{(1)}$ by 3.6(1), hence $\phi(p+q) = \phi(p) + \phi(q)$ by 3.6(2), and ϕ is linear.

Now take an essential ideal $L \subseteq \mathcal{D}_J(U_p q)$, and $z, w \in I \cap L \cap C_w(J)$. Then $\phi(U_p q) = [f_{U_p q}, \mathfrak{u}(L)]$, and

$$\begin{aligned} U_z^2 U_w \phi(U_p q) &= U_{z^2} U_w \phi(U_p q) = f_{U_p q}(U_{z^2} U_w) = && \text{by } 3.6(1) \\ &= U_{z^2} U_w U_p q = U_{U_z p} U_w q = \\ &= U_{f_p(U_z)} f_q(U_w) = U_{U_z \phi(p)} U_w \phi(q) = && \text{by } 3.6(1) \\ &= U_z^2 U_w U_{\phi(p)} \phi(q). && \text{by 2.13 and } 3.8(3) \end{aligned}$$

Hence $U_z^2 U_w(\phi(U_p q) - U_{\phi(p)}\phi(q)) = 0$. Now $U_z \in C_w(J) \subseteq C_w(J_{\mathcal{E}(\Gamma)})$ by 2.13 and 3.8(3), hence we get $U_z U_w(\phi(U_p q) - U_{\phi(p)}\phi(q)) = 0$, and this implies $\mathfrak{u}(I \cap L)^2(\phi(U_p q) - U_{\phi(p)}\phi(q)) = 0$. Since $\mathfrak{u}(I \cap L)^2$ is essential in Γ we obtain $\phi(U_p q) - U_{\phi(p)}\phi(q) = 0$ from 3.6(2), hence $\phi(U_p q) = U_{\phi(p)}\phi(q)$.

The equality $\phi(q^2) = \phi(q)^2$ is proved analogously, so we obtain that ϕ is a homomorphism, and this proves that $J_{\mathcal{E}(\Gamma)}$ is the maximal algebra of quotients of J. ■

4. Algebras of quotients of algebras of hermitian type

Since algebras of hermitian type are special we can make use of associative envelopes to transfer problems to the associative setting. In the case of algebras of quotients this requires first to have a good relationship between dense inner ideals of the Jordan algebra and dense one sided ideals of its associative envelopes.

4.1. Following [Mo1] we denote by $\mathrm{PI}(J)$ the set of all $a \in J$ such that J_a is a PI-algebra. It is proved in [Mo1] that if J is nondegenerate, $\mathrm{PI}(J)$ is an ideal of J. Similar notions can be defined for associative algebras where we again use the notation $\mathrm{PI}(R)$. Following [FGM] we will say that a Jordan algebra J is PI-less if $\mathrm{PI}(J) = 0$.

4.2. Lemma. *Let R be a semiprime associative algebra with involution $*$, and let $J = H_0(R*)$ be an ample subspace of symmetric elements of R.*

(1) *If I is an essential $*$-ideal of R, then $I \cap J$ is an essential ideal of J.*

(2) *If I is an essential ideal of J, then* $\mathrm{lann}_R(I) = \mathrm{rann}_R(I) = 0$.

Proof. Note first that R is $*$-tight over J by [ACMM,1.1], and that J is a nondegenerate Jordan algebra [Mc3, 2.9]. (1) Set $L = \mathrm{Ann}_J(I \cap J)$, and take $x \in L^{(1)}$ and $a \in R$. Then $ax + xa^*$ and axa^* belong to L (see the proof of [Mc1, Theorem 5]). Now, if $y \in I$ and $x \in L^{(1)}$, then $yx + xy^* \in L \cap I \cap J = 0$, hence $yx = -xy^*$. Thus, $xy^2 = -y^*xy \in L \cap I = 0$, and if $a \in R$, then $xyaxy = x(ya)xy = -a^*y^*x^2y = -a^*x^2y^2 = 0$. Thus $xyRxy = 0$, hence $xy = 0$ since R is semiprime. Therefore we have $L^{(1)}I = 0$, hence $L^{(1)} = 0$ since I is essential, and this implies $L = 0$ since J is nondegenerate, hence semiprime.

(2) Suppose that $Ir = 0$ for some $r \in R$. Now, for all $x \in I$ and $a \in R$, we have $a^*x + xa \in I$, hence $0 = (a^*x + xa)r = xar$. Thus $I^{(1)}Rr = 0$, and $r \in \mathrm{Ann}_R(\hat{R}I^{(1)}\hat{R})$. Now $\mathrm{Ann}_R(\hat{R}I^{(1)}\hat{R}) \cap J \subseteq \mathrm{Ann}_J(I^{(1)}) = 0$ (by [FGM, 1.13]). Therefore $r = 0$. ∎

4.3. Lemma. *Let J be a nondegenerate Jordan algebra. Then the algebra $\bar{J} = J/\mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J))$ is PI-less:* $\mathrm{PI}(\bar{J}) = 0$.

Proof. We denote with bars the projections in $\bar{J}$. Note that $\overline{\mathrm{Ann}_J(\mathrm{PI}(J))}$ is an essential ideal of $\bar{J}$, since if $U_{\bar{z}}\overline{\mathrm{Ann}_J(\mathrm{PI}(J)} = 0$, then $U_z\mathrm{Ann}_J(\mathrm{PI}(J)) \subseteq \mathrm{Ann}_J(\mathrm{PI}(J)) \cap \mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J))) = 0$, hence $z \in \mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J))$, i. e., $\bar{z} = 0$.

Now, if $\mathrm{PI}(\bar{J}) \neq 0$, then there is a nonzero $\bar{x} \in \mathrm{PI}(\bar{J}) \cap \overline{\mathrm{Ann}_J(\mathrm{PI}(J))}$, and we can choose a preimage $x \in \mathrm{Ann}_J(\mathrm{PI}(J))$. Then $\bar{J}_{\bar{x}}$ is PI, and if $f \in FJ[X]$ is an essential polynomial which vanishes on $\bar{J}_{\bar{x}}$, then $\bar{J}$ satisfies $U_{\bar{x}}f(\bar{x}; \bar{J}) = 0$ (where $h(y; x_1, \ldots, x_n)$ is the evaluation in the homotope $FJ[X]^{(y)}$ of the polynomial $h(x_1, \ldots, x_n)$), hence letting $g = f^3$, $\bar{J}$ satisfies $g(\bar{x}; \bar{J}) = 0$. Thus $g(x; J) \subseteq \mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J)))$, but $x \in \mathrm{Ann}_J(\mathrm{PI}(J))$ implies $g(x; J) \subseteq \mathrm{Ann}_J(\mathrm{PI}(J))$, and therefore we get $g(x; J) = 0$, which implies that J_x is PI, hence $x \in \mathrm{PI}(J) \subseteq$

$\mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J)))$, and $\bar{x} = 0$. ∎

4.4. Lemma. *Let J be a nondegenerate Jordan algebra, and let $p \in FJ[X]$ be an essential polynomial. Put $I = id_J(p(J))$, the ideal generated by all the evaluations of p in J. Then $\mathrm{Ann}_J(I) \subseteq \mathrm{PI}(J)$, so if J is PI-less, then I is essential. In particular, J is of hermitian type: $\mathrm{Ann}_J(\mathcal{H}(J)) = 0$ for any hermitian ideal $\mathcal{H}(X)$.*

Proof. Let $a \in \mathrm{Ann}_J(I)$ and consider the algebra $\bar{J} = J/\mathrm{Ann}_J(\mathrm{Ann}_J(I))$, which is nondegenerate by [FGM, 1.15]. Then $\mathrm{Ann}_J(\mathrm{Ann}_J(I)) \subseteq \mathrm{Ker} a$ implies $J_a = \bar{J}_{\bar{a}}$ (where bars denote projections in $\bar{J}$). Note that $\bar{J}$ is PI since $p(\bar{J}) = 0$, and therefore $\mathrm{PI}(\bar{J}) = \bar{J}$ by [Mo1, 2.7]. Thus $\bar{J}_{\bar{a}} = J_a$ is PI, and $a \in \mathrm{PI}(J)$. ∎

4.5. Lemma. *Let J be a PI-less nondegenerate special Jordan algebra, R be a $*$-tight associative $*$-envelope of J, and L be a left ideal of R. Then $L \cap J = 0$ implies $id_R(L) \cap id_R(L)^* = 0$, where $id_R(L)$ is the ideal of R generated by L.*

Proof. Suppose that L is a left ideal of R with $L \cap J = 0$. Let $\mathcal{H}(X)$ be a hermitian ideal, so that $\mathcal{H}(J)$ is essential in J by 4.4. Denote $A = alg_R(\mathcal{H}(J)$, the associative subalgebra of R generated by $\mathcal{H}(J)$, and take $B = L \cap A$. Then, for all $b \in B$ we have $b^*\mathcal{H}(J)b \subseteq \mathcal{H}(J) \cap L = 0$, hence R satisfies the $*$-GPI $b(X + X^*)b = 0$ by [FGM, 6.11]. Thus, if $c = b^*rb \in bRb^*$, we have $c^* = b^*r^*b = -b^*rb = -c$, and for all $x, y \in R$, we have $cxcyc = -cy^*c^*x^*c = cy^*cx^*c = cycac$, hence R_c is commutative and $c \in \mathrm{PI}(R)$. Now $\mathrm{PI}(R) \cap J \subseteq \mathrm{PI}(J)$ (by [Mo1, 4.6(b)]) $= 0$, hence $\mathrm{PI}(R) = 0$ by tightness, and this gives $c = 0$. Therefore $b^*Rb = 0$ for all $b \in B$, hence $id_R(B) \cap id_R(B^*) = 0$ by [FGM, 6.13]. Now,since J generates R, for any $l \in L$ there exists a positive integer $n(l)$ with $\mathcal{H}(J)^{(m)}l \subseteq A \cap L = B$ for any $m \geq n(l)$ by [MZ, 1.5(3)]. Thus, for any $l_1, l_2 \in L$, and any $n \geq n(i_1), n(l_2)$, we have $\mathcal{H}(J)^{(n)}l_1\hat{R}l_2^*\mathcal{H}(J)^{(n)} \subseteq BRB^* \subseteq id_R(B) \cap id_R(B^*) = 0$. Then $l_1\hat{R}l_2^*\mathcal{H}(J)^{(n)} \subseteq \mathrm{rann}_R(\mathcal{H}(J)^{(n)}) = \mathrm{rann}_R(\hat{R}\mathcal{H}^{(n)}) = \mathrm{Ann}_R(\hat{R}\mathcal{H}^{(n)})$. But $\mathrm{Ann}_R(\hat{R}\mathcal{H}^{(n)}) \cap J = \mathrm{Ann}_J(\mathcal{H}(J)^{(n)})$ (by [FGM, 1.15]) $= 0$ (by [FGM, 1.13]), hence $l_1\hat{R}l_2^*\mathcal{H}(J)^{(n)} = 0$. The same argument gives now $l_1\hat{R}l_2^* = 0$, hence $L\hat{L}^* = 0$ which implies $id_R(L) \cap id_R(L)^* = 0$. ∎

4.6. Theorem. *Let J be a nondegenerate special Jordan algebra, let $(R, *)$ be a $*$-tight associative $*$-envelope of J, and let L be a left ideal of R. If there exists a hermitian ideal $\mathcal{H}(X)$ with $\mathrm{Ann}_J(\mathcal{H}(J)) = 0$, then L is a dense left ideal if and only if $L \cap J$ is a dense inner ideal of J.*

Proof. Assume first that L is dense, and set $K = L \cap J$. Then $(K : a) \supseteq (L : a) \cap K$ for all $a \in J$. Indeed, if $x \in (L : a) \cap K$, then $xa, ax \in L$, hence $a \circ x \in L$ and $U_a x = a(xa) \in aL \subseteq L$. Since $x \circ a, U_a x \in J$, we have $x \circ a, U_a x \in K$, hence $x \in (K : a)$. Thus $(K : a) \cap (K : b) = ((L : a) \cap (L : b) \cap L) \cap J$ for any $a, b \in J$ is the intersection with J of a dense left ideal of R, so it suffices to prove that $U_c(L\cap J)=0$ implies $c=0$ for a dense left ideal L of R and any $c\in J$.

Suppose then that $U_c(L\cap J)=0$ for some nonzero $c\in J$, and take $d\in U_c(\mathcal{H}(J)\cap \mathrm{PI}(J))$. We set $A=alg_R(\mathcal{H}(J))$, the associative subalgebra of R generated by $\mathcal{H}(J)$. Then $\mathcal{H}(J)=H_0(A,*)$ is an ample subspace of symmetric elements of A, and its local algebra $\mathcal{H}(J)_d=H_0(A_d,*)$ is also an ample subspace of symmetric elements of the local algebra A_d. We will denote with bars the projections onto the local algebras at d (of $\mathcal{H}(J)$, J, A and R) Since $d\in \mathrm{PI}(J)$, the local algebra $\mathcal{H}(J)_d=H_0(A_d,*)\subseteq J_d$ is also PI, hence A_d is PI by a theorem of Amitsur [R1, 7.4.13]. Now take any $h\in\mathcal{H}(J)$. Then $L_1=(L:dh)$ is a dense left ideal of R and, since R is a ring of quotients of A (see [MZ,]), $L_2=L_1\cap A$ is a dense left ideal of A. Therefore $\bar L_2=L_2+\mathrm{Ker}d/\mathrm{Ker}d$ is a dense left ideal of A_d by 1.21. Since A_d is PI, arguing as in 3.1, it is easy to see that there is an essential ideal $\bar I$ of A_d with $\bar I\subseteq \bar L_2$ (note that the preimage $I\subseteq A$ of $\bar I$ need not be an ideal of A, although it is an ideal of the homotope $A^{(d)}$). Now, for any $\bar y\in \bar I\cap\overline{\mathcal{H}(J)}$ such that $y\in\mathcal{H}(J)$, we have $\bar y=\bar l$ for some $l\in L_2$, hence $U_hU_dy=hdldh\in hdL_2dh\subseteq hdL_1dh\cap A=hd(L:dh)dh\cap A\subseteq L\cap A$, and since $U_hU_dy\in\mathcal{H}(J)\subseteq J$ we get $U_hU_dy\in L\cap J$, hence $U_dU_hU_dy\in U_JU_cU_hU_dy\subseteq U_JU_c(L\cap J)=0$. So we have $U_{\bar h}(\bar I\cap\overline{\mathcal{H}(J)})=0$. Since A is semiprime, A_d is also semiprime, hence $\mathrm{Ann}_{A_d}(\bar I)=0$ which yields $\mathrm{Ann}_{\mathcal{H}(J)_d}(\bar I\cap\bar{\mathcal{H}}(J))=0$ by 4.2, hence $\bar h=0$ for all $h\in\mathcal{H}(J)$. Thus $U_d\mathcal{H}(J)=0$, hence $d\in\mathrm{Ann}_J(\mathcal{H}(J))=0$. So turning back to the choice of d, this implies $U_c(\mathcal{H}(J)\cap\mathrm{PI}(J))=0$, hence $U_c\mathrm{PI}(J)\subseteq\mathrm{Ann}_J(\mathcal{H}(J))=0$, and therefore $c\in\mathrm{Ann}_J(\mathrm{PI}(J))$.

Consider now the algebra $\bar J=J/\mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J)))$, which is PI-less by 4.3. Note that $\bar R=R/\mathrm{Ann}_R(\mathrm{Ann}_J(\mathrm{PI}(J)))$ is a $*$-tight envelope of $\bar J$ by [FGM, 1.15] (in what follows we change our convention and denote with bars the projections into these algebras). Note also that $\bar L$ is a dense left ideal of $\bar R$ by 1.23. Now set $L_1=(L:c)$, which is again a dense left ideal of R. For any $\bar y\in\bar L_1\cap\bar J$ (note that we can choose $y\in J$) there is $l\in L_1$ with $\bar y=\bar l$, hence $y=l+z$ with $z\in\mathrm{Ann}_R(\mathrm{Ann}_J(\mathrm{PI}(J)))$. Thus $czc\in\mathrm{Ann}_R(\mathrm{Ann}_J(\mathrm{PI}(J)))\cap\mathrm{Ann}_R(\mathrm{PI}(J))=0$ (by [FGM, 1.15]), hence $U_cy=U_cl\in J\cap c(L:c)c\subseteq J\cap L$. Thus $U_cy\in U_c(L\cap J)=0$, so we have $U_{\bar c}(\bar J\cap\bar L_1)=0$. We will prove that this situation forces $\bar c=0$.

To alleviate the notation we now omit bars, and we consider a PI-less special Jordan algebra with $*$-tight associative $*$-envelope R, a dense left ideal L of R, and $d\in J$ with $U_d(L\cap J)=0$.

Consider the left ideal Lc of R. If $x\in Lc\cap J$, then $x=ld$ for some $l\in L$, and there is a positive integer n such that $l^*\mathcal{H}(J)^{(n)}l\subseteq J\cap L$ (see [MZ, p. 146]). Thus $U_x\mathcal{H}(J)^{(n)}=x^*\mathcal{H}(J)^{(n)}x\subseteq dl^*\mathcal{H}(J)^{(n)}ld\subseteq U_d(L\cap J)=0$, hence

$x \in \mathrm{Ann}_J(\mathcal{H}(J)^{(n)}) = 0$ by [FGM, 1.13] and the essentiality of $\mathcal{H}(J)$. Thus $Ld \cap J = 0$, hence $id_R(Lc) \cap id_R(cL^*) = 0$ by 4.5, and we get $LcRcL^* = 0$. Then, since L is a dense left ideal (hence L^* is a dense right ideal), we have $cRc = 0$, which implies $c = 0$ by the semiprimeness of R.

Thus, going back to our previous notation, we have proved that $\bar{c} = 0$, hence $c \in \mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J)))$. But since $c \in \mathrm{Ann}_J(\mathrm{PI}(J))$, we get $c = 0$.

Let us now prove the reciprocal. Assume that $L \cap J$ is a dense inner ideal of J. Since $\hat{R}(L \cap J) \subseteq L$, the result will follow if we prove that any dense inner ideal K of J generates a dense left ideal $\hat{R}K$ in R.

First we claim that for any dense inner K of J and any $r \in R$, there exists a dense inner ideal N of J such that $Nr \subseteq \hat{R}K$. Since J generates R, the element r can be written as a sum of products of elements of J, so taking the intersection of the inner ideals corresponding to each of the summands of r, we can assume that $r = a_1 \cdots a_n$ is a product of elements from J. We then carry out an induction on the number n of factors. For the case $n = 1$ note that $(K : a_1)a_1 = (K : a_1) \circ a_1 + a_1(K : a_1) \subseteq \hat{R}K$, hence $N = (K : a_1)$ works since it is dense. Now, if the result holds for products of at most $n-1$ elements from J, the density of $(K : a_n)$ implies that there is a dense inner ideal N in J with $Na_1 \cdots a_n \subseteq \hat{R}(K : a_n)$. Then $Na_1 \cdots a_n \subseteq \hat{R}(K : a_n)a_n \subseteq \hat{R}K$, so we have found the desired inner ideal N. This proves the induction step and therefore the claim.

In view of the fact just proved, for any dense inner ideal K of J and any $r \in R$, the left ideal $(\hat{R}K : r)$ contains a left ideal of the form $\hat{R}N$, for a dense inner ideal N of J. Thus, to prove that a left ideal $\hat{R}K$ generated by a dense inner ideal K of J is dense, it suffices to prove that $Ka \neq 0$ for any dense inner ideal K of J and any $0 \neq a \in R$.

Suppose then that $Ka = 0$ for some dense inner ideal K of J and some $a \in R$, and take $d \in \mathrm{PI}(J) \cap \mathcal{H}(J)$. Then $(K : d)da \subseteq ((K : d) \circ d)a + d(K : d)a \subseteq \hat{R}Ka = 0$. Now set $N = (K : d)$, which is again dense. Denote by A the associative subalgebra $A = alg_R(\mathcal{H}(J))$ generated by $\mathcal{H}(J)$ in R and put $N_1 = N \cap \mathcal{H}(J)$. Then N_1 is a dense inner ideal of $\mathcal{H}(J)$ by 2.4(v), hence $\bar{N}_1 = N_1 + \mathrm{Ker}\, d/\mathrm{Ker}\, d$ is a dense inner ideal of the local algebra $\mathcal{H}(J)_d$ by 1.20. Now, since $d \in \mathrm{PI}(J)$, the algebra $\mathcal{H}(J)_d$ is PI, hence there exits an essential ideal $\bar{I}$ of $\mathcal{H}(J)_d$ contained in $\bar{N}_1$ by 3.1. Now there is n_a with $a\mathcal{H}(J)^{(n)} \subseteq A$ for all $n \geq n_a$, so if $b \in a\mathcal{H}(J)^{(n)}A$, denoting as usual the projections in $\mathcal{H}(J)_d$ and A_d with bars, we have $\bar{I}\,\bar{b} \subseteq \bar{N}_1\,\bar{b} = \overline{N_1 db} \subseteq \overline{((K : d) \cap \mathcal{H}(J))daA} = 0$, hence $\bar{b} \in \mathrm{rann}_{A_d}(\bar{I})$. Note that A_d is semiprime since A is semiprime, and $\mathcal{H}(J)_d = H_0(A_d, *)$ is an ample subspace of symmetric elements of A_d since $\mathcal{H}(J) = H_0(A, *)$ is an ample subspace of symmetric elements of A. Since $\bar{I}$ is essential, 4.2 gives

$\mathrm{rann}_{A_d}(\bar{I}) = 0$, whence $\bar{b} = 0$. Thus we have proved that $da\mathcal{H}(J)^{(n)}Ad = 0$ for any $d \in \mathrm{PI}(J) \cap \mathcal{H}(J)$. Then we have $(da\mathcal{H}(J)^{(n)})A(da\mathcal{H}(J)^{(n)}) = 0$, which implies $da\mathcal{H}(J)^{(n)} = 0$ for all $d \in \mathrm{PI}(J) \cap \mathcal{H}(J)$, hence $(\mathrm{PI}(J) \cap \mathcal{H}(J))a \subseteq \mathrm{lann}_R(\mathcal{H}(J)) = 0$ (since $\mathrm{lann}_R(\mathcal{H}(J)) = \mathrm{lann}_R(id_R(\mathcal{H}(J))) = \mathrm{Ann}_R(id_R(\mathcal{H}(J)) = 0$ by the essentiality of $\mathcal{H}(J)$) and we get $(\mathrm{PI}(J) \cap \mathcal{H}(J))a = 0$. Now, for any $h, g \in \mathcal{H}(J)$ and any $x \in \mathrm{PI}(J)$, we have $(U_h g)xa = h\{g,h,x\}a - ((U_h x)\circ g)a + g(U_h x)a \in \hat{\mathcal{H}}(J)(\mathcal{H}(J) \cap \mathrm{PI}(J))a = 0$. Therefore $\mathcal{H}(J)^{(1)}\mathrm{PI}(J)a = 0$, and $\mathrm{PI}(J)a \subseteq \mathrm{Ann}_R(\mathcal{H}(J)^{(1)}) = 0$, hence $a \in \mathrm{Ann}_R(\mathrm{PI}(J))$.

Consider now the algebra $\bar{J} = J/\mathrm{Ann}_J(\mathrm{Ann}_J(\mathrm{PI}(J))$, and its $*$-tight associative $*$-envelope $\bar{R} = R/\mathrm{Ann}_R(\mathrm{Ann}_J(\mathrm{PI}(J))$ (see [FGM, 1.15]), where, as usual, we denote with bars the images under the projections. We have $\mathrm{PI}(\bar{J}) = 0$ by 4.3, the inner ideal $\bar{K}$ of $\bar{J}$ is dense by 1.20, and $\bar{K}\bar{a} = 0$. Now, for any $\bar{c} \in \bar{J} \cap \bar{a}\bar{R}$, the equality $U_{\bar{c}}\bar{K} = \bar{c}\bar{K}\bar{c} = 0$ implies $\bar{c} = 0$. Thus $J \cap \bar{a}\bar{R} = 0$, and since $\bar{J}$ is PI-less, 4.5 gives $id_{\bar{R}}(\bar{a}\bar{R}) \cap id_{\bar{R}}(\bar{R}\bar{a}^*) = 0$. Set now $V = \bar{a}\hat{\bar{R}} + \bar{a}^*\hat{\bar{R}}$ and take $\bar{x} \in \bar{N} \cap \bar{J}$. Then $\bar{x}\bar{K}\bar{x} = \bar{x}^*\bar{K}\bar{x} \subseteq V^*\bar{K}V \subseteq (\hat{\bar{R}}\bar{a}^*\bar{K} + \hat{\bar{R}}\bar{a}\bar{K})V = (\hat{\bar{R}}(\bar{K}\bar{a})^* + \hat{\bar{R}}\bar{a}\bar{K})V = \hat{\bar{R}}\bar{a}\bar{K}V = \hat{\bar{R}}\bar{a}\bar{K}(\bar{a}\hat{\bar{R}} + \bar{a}^*\hat{\bar{R}}) = \hat{\bar{R}}\bar{a}\bar{K}\bar{a}^*\hat{\bar{R}} \subseteq id_{\bar{R}}(\bar{a}\hat{\bar{R}}) \cap id_{\bar{R}}(\hat{\bar{R}}\bar{a}^*) = 0$, which implies $\bar{x} = 0$ by the density of $\bar{K}$. Thus $V \cap \bar{J} = 0$, hence $VV^* = 0$ by 4.5, and we get $\bar{a}\bar{R}\bar{a} \subseteq VV^* = 0$, hence $a = 0$ by semiprimeness of $\bar{R}$. Therefore we have $a \in \mathrm{Ann}_R(\mathrm{Ann}_J(\mathrm{PI}(J)))$, and since $a \in \mathrm{Ann}_R(\mathrm{PI}(J))$, we obtain $a = 0$. ∎

4.7. Lemma. *Let J be a nondegenerate Jordan algebra, and let Q be an algebra of quotients of J. Assume that Q is special and let A be a $*$-tight associative $*$-envelope of Q. Denote by $T = alg_A(J)$ the associative subalgebra of A generated by J. Then:*

(i) *For any $a \in A$, there exits a dense inner ideal K of J such that $Ka \subseteq T$.*

(ii) *T is a $*$-tight associative envelope of J.*

Proof. (i) Since any element $a \in A$ can be written as a sum of elements from Q, if we prove that for any element of the form $q_1 \cdots q_n$, with $q_i \in Q$, there exists a dense inner ideal K of J with $Kq_1 \ldots q_n \subseteq T$, the result will follow for an arbitrary $a \in A$ by taking the intersection of the inner ideals obtained for each summand that makes up a. We can therefore assume that $a = q_1 \cdots q_n$ is a product of elements from Q and csrry out an induction on the number n of factors.

For $n = 1$, let $K = \mathcal{K}(U_{\mathcal{D}_J(q_1)}\mathcal{D}_J(q_1))$, which is dense by 1.16 since $\mathcal{D}_J(q_1)$ is. If $k \in K$, then there exist $x, y \in \mathcal{D}_J(q_1)$ with $k = U_x y$ and we have $kq_1 = (U_x y)q_1 = x\{y, x, q_1\} - (U_x q_1)y \in JJ \subseteq T$.

So suppose that the result holds for products of at most $n-1$ elements from Q, and let $b = q_2 \cdots q_n$ and $q = q_1$, so that $a = qb$. By induction hypothesis there exists a dense inner ideal N of J with $Nb \subseteq T$. Put $L = \{x \in T \mid xq \in TN\}$. It is clear that L is a left ideal of T, hence $K = L \cap J$ is an inner ideal of J that has $Ka = Kqb \subseteq TNb \subseteq T$. Therefore it suffices to show that $L \cap J$ is dense.

Take $u, v, c \in J$, and assume that $U_c((L \cap J : u)_L \cap (L \cap J : v))_L = 0$. For any $s \in N \cap \mathcal{D}_J(q)$ and $t \in N$ we have $\{t, s, q\} \in J$. Now, if $y \in (N : s \circ \{t, s, q\})$, we have

$$\begin{aligned}(U_x y)q = x\{y, x, q\} - (U_x q)y =\\ x\{y, U_s t, q\} - (U_{U_s t} q)y =\\ = x(y \circ (s \circ \{t, s, q\})) - x\{y, \{t, s, q\}, s\} - (U_s U_t U_s q)y,\end{aligned}$$

but $y \circ (s \circ \{t, s, q\}) \in (N : s \circ \{t, s, q\}) \circ (s \circ \{t, s, q\}) \in N$, $\{y, \{t, s, q\}, s\} \in U_N J \subseteq N$, and $U_s U_t U_s N \in U_s U_t J \subseteq J \subseteq T$, hence $(U_x y)q \in xN + Ty \subseteq TN$ and we get $U_x y \in L \cap J$

Take now $s \in (N \cap \mathcal{D}_J(q) : u) \cap (N \cap \mathcal{D}_J(q) : v)$, $t \in (N : u) \cap (N : v)$. For $d = u$ or v, we set $M_{s,t,d} = (N : s \circ \{t, s \circ d, q\}) \cap (N : (s \circ d) \circ \{t, s, q\}) \cap (N : s \circ \{t \circ d, s, q\}) \cap ((N : s \circ \{t, s, q\} : d)$ (note that if $d = a$ or b, then $s \circ d \in N \cap \mathcal{D}_J(q)$, hence $\{t, s \circ d, q\}, \{t, s, q\}$ and $\{t \circ d, s, q\}$ belong to J). Take now $y \in M_{s,t,u} \cap M_{s,t,v}$ and put $x = U_s t$. Let us see that $U_x y \in (L \cap J : u)_L \cap (L \cap J : v)_L$.

we have:

$$\begin{aligned}(U_x y) \circ u &= \{x, y, x \circ u\} - U_x(y \circ u) =\\ &= \{x, y, \{s, t, s \circ u\}\} - \{x, y, U_s(t \circ u)\} - U_x(y \circ u).\end{aligned}$$

We will show next that each of the terms in this sum belongs to $L \cap J$.

First note that $U_x(y \circ u) \in L \cap J$ and, $\{x, y, U_s(t \circ u)\} = U_{U_s(t + t \circ u)} y - U_{U_s t} y - U_{U_s(t \circ u)} y \in L \cap J$ follow from what was proved above. Now, we have:

$$\{x, y, \{s, t, s \circ u\}\}q = x\{y, \{s, t, s \circ u\}q\} + \{s, t, s \circ u\}\{y, x, q\} - \{x, q, \{s, t \circ u\}\}y$$

where, on the one hand:

$$\begin{aligned}\{y, \{s, t, s \circ u\}q\} &= \{y, s, \{t, s \circ u, q\}\} + \{y, s \circ u, \{t, s, q\}\} =\\ &= y \circ (s \circ \{t, s \circ u, q\}) - \{y, \{t, s \circ u, q\}, s\} +\\ &+ y \circ ((s \circ u) \circ \{t, s, q\}) - \{y, \{t, s, q\}, s \circ u\} \in\\ &\in (N : s \circ \{t, s \circ u, q\}) \circ (s \circ \{t, s \circ u, q\}) + U_N J +\\ &+ (N : (s \circ u) \circ \{t, s, q\}) \circ ((s \circ u) \circ \{t, s, q\}) + U_N J \in N,\end{aligned}$$

hence $x\{y, \{s, t, s \circ u\}q\} \in JN \subseteq TN$.

On the other hand

$$\begin{aligned}\{s,t,s\circ u\}\{y,x,q\} &= \{s,t,s\circ u\}\{y,U_s t,q\}\\ &= \{s,t,s\circ u\}((y\circ(s\circ\{t,s,q\})) - \{y,\{t,s,q\},s\} + \{y,U_s q,t\} \in\\ &\in J((N : s\circ\{t,s,q\})\circ(s\circ\{t,s,q\})) - U_N J) \subseteq JN \subseteq TN,\end{aligned}$$

And finally,

$$\{x,q,\{s,t\circ u\}\}y \in (U_{\mathcal{D}_J(q)}q)N \subseteq JN \subseteq TN.$$

Therefore $\{x,y,\{s,t,s\circ u\}\} \in L\cap J$ and we obtain $U_x y \in (L\cap J : u)_L$. Analogously, $U_x y \in (L\cap J : v)_L$, hence $U_x y \in (L\cap J : u)_L \cap (L\cap J : v)_L$ and $U_c U_x y = 0$ for any x, y chosen as above. Arguing as in the proof of 2.8 we get $c = 0$, hence $L\cap J$ satisfies 1.9(1_L). This proves the induction step, hence (i).

(ii) is proved as [MP, 4.4(ii)] with the obvious changes. ■

4.8. Proposition. *Let J be a nondegenerate Jordan algebra, and assume that there is a hermitian ideal $\mathcal{H}(X)$ such that $\mathcal{H}(J)$ is essential. If R is a $*$-tight associative $*$-envelope of J, then the set*

$$\mathcal{Q} = \{q \in H(Q_\sigma(R), *) \mid \mathcal{D}_J(q) \textit{ is dense in } J\}$$

is an ample subspace of symmetric elements of the maximal algebra of symmetric quotients $Q_\sigma(R)$ of R.

Proof. Again, the proof of the corresponding result [MP, 4.6] can be easily adapted to the present case. ■

4.9. Remark. In the previous results we have assumed that our Jordan algebras were hermitian in the strong sense that there existed a hermitian ideal whose values in the algebra was an essential ideal. We will now choose a particular hermitian ideal to apply the results of [Mc4] without further comments. Recall that the Zelmanov polynomial in the variables $X^{(i)} = \{x_i, y_i, z_i, w_i\}$ $(i = 1,2,3)$ has the form ([MZ, pp. 192, 195])

$$Z_{48} = [[P_{16}(X^{(1)}), P_{16}(X^{(2)})], P_{16}(X^{(3)}]$$

for $P_{16}(X) = [[[t,[t,z]]^2,[t,w]],[t,w]]$ $(t = [x,y])$, where $[[a,b]c] = \{a,b,c\} - \{b,c,a\}$. Denote by $\mathcal{Z}(X)$ the ideal generated in the free Jordan algebra $FJ[X]$ over a countable set of generators X by all the evaluations $Z_{48}(a_1,\dots,a_{12})$ for $a_i \in FJ[X]$. Then $\mathcal{Z}(X)$ is a hermitian ideal.

4.10. Theorem. *Let J be a special nondegenerate Jordan algebra and assume that $\mathcal{Z}(J)$ is an essential ideal of J. Then the algebra $\mathcal{Q}$ of 4.8 is the maximal algebra of quotients of J.*

Proof. Once more, the proof of [MP, 4.7] works in this situation with minor obvious changes. Note however that since we are using the ideal $\mathcal{Z}(X)$ we do not need here to adapt the results of [Mc4], since the General Zelmanov Extension Theorem [Mc4, 2.1] can be applied directly. ■

5. Main theorem and consequences

In this section we collect the previous results and find maximal algebras of quotients for nondegenerate algebras. This will stem from a (finite) subdirect decomposition of Jordan algebras which will transfer to a subdirect decomposition of their algebras of quotients.

5.1. Lemma. *Let Q be an algebra of quotients of the nondegenerate Jordan algebra J, let L be an ideal of Q, and put $I = \mathrm{Ann}_Q(L)$. Then $I \cap J = \mathrm{Ann}_J(L \cap J)$.*

Proof. Since $I \cap L = 0$ by nondegeneracy 2.4(i) of Q, we have $(I \cap J) \cap (L \cap J) = 0$, and this implies $I \cap J \subseteq \mathrm{Ann}_J(L \cap J)$ since both $I \cap J$ and $L \cap J$ are ideals of J.

Now take $x \in \mathrm{Ann}_J(L \cap J)$ and any $q \in L$, and set $p = U_x q$, $K = (\mathcal{D}_J(q) : x)$. Then $U_p K = U_x U_q U_x K \subseteq U_x U_q \mathcal{D}_J(q) \subseteq U_x(L \cap J) = 0$, hence $p = 0$ by 2.4(iv) since K is dense. Thus we get $U_x L = 0$, hence $x \in \mathrm{Ann}_Q(L) = I$ by 0.3. Therefore $\mathrm{Ann}_J(L \cap J) \subseteq I \cap J$ and this gives the equality. ■

5.2. Lemma. *Let Q be an algebra of quotients of the nondegenerate Jordan algebra J, let L be an ideal of Q, and set $I = \mathrm{Ann}_Q(L)$. Then $\bar{Q} = Q/I$ is an algebra of quotients of $\bar{J} = J/J \cap I (= J + I/I \subseteq \bar{Q})$.*

Proof. First note the obvious containment $\overline{\mathcal{D}_J(q)} \subseteq \mathcal{D}_{\bar{J}}(\bar{q})$ for any $\bar{q} = q + I \in \bar{Q}$. Since $I \cap J = \mathrm{Ann}_J(L \cap J)$ by 5.1, we have $\bar{J} = J/\mathrm{Ann}_J(L \cap J)$, so we can apply 1.22 to conclude that $\overline{\mathcal{D}_J(q)}$ is dense, hence that $\mathcal{D}_{\bar{J}}(\bar{q})$ is dense. Moreover, $U_{\bar{q}} \mathcal{D}_{\bar{J}}(\bar{q}) = 0$ implies $U_q \mathcal{D}_J(q) \subseteq I$, hence for all $p \in L$ we have ${}_{U_q p}\mathcal{D}_J(q) \subseteq U_q U_L \mathrm{Ann}_Q(L) = 0$, and this implies $U_q L = 0$ by 2.4(iv). Thus $q \in \mathrm{Ann}_Q(L) = I$ by 0.3, hence $\bar{q} = 0$. ■

5.3. Lemma. *Let J be a Jordan algebra and let Q be an algebra of quotients of J. If I, L are orthogonal ideals of J: $I \cap L = 0$, then they generate orthogonal ideals in Q: $id_Q(I) \cap id_Q(L) = 0$.*

Proof. Consider first the particular case where Q is PI. For an ideal N of J we denote by $G(N)$ the set of all $q \in Q$ for which there exists a dense inner ideal K of J such that for any $z \in K \cap C_w(J)$ there is a positive integer n with $U_{z^n} q \in N$.

Clearly $N \subseteq G(N)$, and we claim that $G(N)$ is an ideal of Q. (Although this is not important in what follows, we note that $C_w(J) = 0$ would imply $G(N) = Q$, but this is not the case with our nondegenerate PI algebra J by [FGM, 3.6]. Note also that if N is essential, then $G(N) = Q$ since for any $q \in Q$ we can consider the dense inner ideal $N \cap \mathcal{D}_J(q)$, and any $z \in N \cap \mathcal{D}_J(q) \cap C_w(J)$ has $U_{z^2}q = U_zU_zq \subseteq U_NU_{\mathcal{D}_J(q)}q \subseteq U_NJ \subseteq N$.)

Take now $q_1, q_2 \in G(N)$ and dense inner ideals K_i with $U_{z_i^{n_i}}q_i \in N$ for any $z_i \in C_w(J) \cap K_i$ and some $n_i > 0$ (for $i = 1, 2$). Then $K = K_1 \cap K_2$ is dense and for any $z \in C_w(J) \cap K$ there are positive integers n_1, n_2 with $U_{z^{n_i}}q_i \in N$, hence $U_{z^n}(q_1 + q_2) = U_{z^n}q_1 + U_{z^n}q_2 \in N$ for any $n \geq n_1, n_2$. Thus $q_1 + q_2 \in G(N)$ and $G(N)$ is a submodule.

Next, take $q \in G(N)$ and $p \in \hat{Q}$. Note that if $p = \alpha 1 + p'$ with $\alpha \in \Phi$ and $p' \in Q$, then $\mathcal{D}_J(p) = \mathcal{D}_J(p')$ is dense in J. Now choose a dense inner ideal K of J such that for all $z \in C_w(J) \cap K$ there exists a positive integer n with $U_{z^n}q \in N$. Then $K' = \mathcal{D}_J(p) \cap K$ is dense, and for any $z \in C_w(J) \cap K'$ and $n > 0$ with $U_{z^n}q \in N$ (note that $K' \subseteq K$), we have $U_{z^{n+2}}U_pq = U_z^2U_{z^n}U_pq = U_zU_pU_zU_{z^n}q$ (since $z \in C_w(Q)$ by 2.13) $= U_{U_zp}U_{z^n}q \in U_JN \subseteq N$. This proves that $U_{\hat{Q}}G(N) \subseteq G(N)$. On the other hand we have $U_{z^{2n+1}}U_qp = U_{z^n}^2U_zU_qp = U_{z^n}U_qU_{z^n}U_zp$ (since $z \in C_w(Q)$ by 2.13)$= U_{U_{z^n}q}U_zp \subseteq U_NJ \subseteq N$, which proves $U_{G(N)}\hat{Q} \subseteq G(N)$, and therefore the claim.

We now go back to our ideals I, L. By what we have just proved, we have $id_Q(I) \subseteq G(I)$ and $id_Q(L) \subseteq G(L)$. So if $q \in id_Q(I) \cap id_Q(L)$, then there are dense inner ideals K_I and K_L such that for any $z \in C_w(J) \cap K_I \cap K_L$ there is n with $U_{z^n}q \in I$ and $U_{z^n}q \in L$, hence $U_z^nq = U_{z^n}q \in I \cap L = 0$. This implies that $U_zq = 0$ for any $z \in C_w(J) \cap K$ for some dense inner ideal K $(= K_I \cap K_L)$ of J by 0.4. Thus we have $U_z^2U_qQ = U_{U_zq}Q = 0$ (since $z \in C_w(Q)$ by 2.13), hence $0 = U_zU_qQ$ (by 0.4) $= U_qU_zQ$, and since U_zQ is an ideal of Q, we obtain $q \in \mathrm{Ann}_J(U_zQ)$, hence $id_Q(q) \subseteq \mathrm{Ann}_J(U_zQ)$ for any $z \in C_w(J) \cap K$. Now consider the ideal $A = id_Q(q) \cap K$. Then for any $z \in C_w(J) \cap A$, we have $z \in id_Q(q) \subseteq \mathrm{Ann}_J(U_zQ)$ (since $z \in C_w(J) \cap K$), hence $z = 0$. Thus $C_w(J) \cap (A \cap J) = C_w(J) \cap A = 0$, which yields $A \cap J = 0$ by [FGM, 3.6], hence $id_Q(q) \cap J = 0$ since K is dense. Thus $id_Q(q) = 0$ by tightness 2.4(iii) of Q over J, and we get $q = 0$, hence $id_Q(I) \cap id_Q(L) = 0$.

We consider next the case where Q is special. For an $*$-tight associative $*$-envelope A of Q we have $id_Q(I) \subseteq \hat{A}I\hat{A}$ and $id_Q(L) \subseteq \hat{A}L\hat{A}$. Thus, if $q \in id_Q(I) \cap id_Q(L)$, there are $a_i, b_i, c_j, d_j \in \hat{A}$, $y_i \in I$, and $x_j \in L$ with $q = \sum_i a_iy_ib_i = \sum_j c_jx_jd_j$. Now, by 4.7 the associative subalgebra $R = alg_A(J)$ generated by J in A is $*$-tight over J, and there is a dense inner ideal K in J with $Ka_i + Kc_j + b_iK + d_jK \subseteq R$. Therefore $U_kq \in id_R(I) \cap id_R(L)$ for all $k \in K$. Since $id_R(I) \cap id_R(L) = 0$ by

[FGM, 1.15], we get $U_K q = 0$, hence $q = 0$ by 2.4(iv). This proves $id_Q(I) \cap id_Q(L) = 0$.

Finally, the general case is proved exactly as in the analogous lemma [FGM, 7.8] with the obvious changes for the references. ■

5.4. Lemma. *Let Q be an algebra of quotients of a nondegenerate Jordan algebra J, and let I be an ideal of J, then* $\mathrm{Ann}_J(I) = \mathrm{Ann}_Q(id_Q(I)) \cap J$.

Proof. This is proved exactly as [FGM, 7.9(i)]. ■

5.5. We will say that an ideal I of a nondegenerate Jordan algebra is closed if $I = \mathrm{Ann}(\mathrm{Ann}_J(I))$. It is clear that if I is of the form $I = \mathrm{Ann}_J(L)$ for some ideal L, then I is closed. Note that by [FGM, 1.16], the quotient J/I by any closed ideal of a nondegenerate Jordan algebra J is again nondegenerate.

5.6. Lemma. *Let J be a nondegenerate Jordan algebra and let I_1, I_2 be ideals with $I_1 = \mathrm{Ann}_J(I_2)$ and $I_2 = \mathrm{Ann}_J(I_1)$. If $J_i = J/I_i$ has a maximal algebra of quotients Q_i, then J has maximal algebra of quotients $Q_{max}(J) \cong Q_1 \times Q_2$.*

Proof. Denote by $\pi_i : J \to J_i$ the projection onto J_i. We have a natural monomorphism $J \to J_1 \times J_2$ given by $x \mapsto (\pi_1(x), \pi_2(x))$. Composing this with the monomorphism $J_1 \times J_2 \to Q_1 \times Q_2$, we can assume that $J \subseteq Q_1 \times Q_2$. To see that $Q_1 \times Q_2$ is an algebra of quotients of J, note that the projections induce monomorphisms $I_1 \to J_2$ and $I_2 \to J_1$, so we can identify I_1 with the ideal $I_1 + I_2/I_2$ of J_2, and I_2 with the ideal $I_2 + I_1/I_1$ of J_1, and each of these ideals is essential. Thus J_1 is an algebra of quotients of I_2 by 2.3.1, hence Q_1 is an algebra of quotients of I_2 by 2.8, and similarly Q_2 is an algebra of quotients of I_1. Thus it is easy to see that $Q_1 \times Q_2$ is an algebra of quotients of $I = I_1 + I_2 \cong I_2 \times I_1$. Since I is an essential ideal of J, hence J is an algebra of quotients of I, we get from 2.8 that $Q_1 \times Q_2$ is an algebra of quotients of J.

Now, let Q be an algebra of quotients of J. Then $Q/\mathrm{Ann}_Q(id_Q(I_1))$ is an algebra of quotients of $J/J \cap \mathrm{Ann}_Q(id_Q(I_1))$ by 5.2. Now $J \cap \mathrm{Ann}_Q(id_Q(I_1)) = \mathrm{Ann}_J(I_1) = I_2$ by 5.4, hence $Q/\mathrm{Ann}_Q(id_Q(I_1))$ is an algebra of quotients of J_2, and there exists a homomorphism $\phi_2 : Q/\mathrm{Ann}_Q(id_Q(I_1)) \to Q_2$ extending the inclusion $J_2 \subseteq Q_2$ by the maximality of Q_2. Similarly, there exists a homomorphism $\phi_1 : Q/\mathrm{Ann}_Q(id_Q(I_1)) \to Q_1$ extending the inclusion $J_1 \subseteq Q_1$. Note now that $\mathrm{Ann}_Q(id_Q(I_1)) \cap \mathrm{Ann}_Q(id_Q(I_2)) \cap J = I_2 \cap I_1 = 0$, hence $\mathrm{Ann}_Q(id_Q(I_1)) \cap \mathrm{Ann}_Q(id_Q(I_2)) = 0$ by tightness. thus we have a monomorphism $\psi : Q \to Q/\mathrm{Ann}_Q(id_Q(I_1)) \times Q/\mathrm{Ann}_Q(id_Q(I_2))$ made up of the corresponding projections. Then we get a homomorphism $\phi_1 \times \phi_2 : Q/\mathrm{Ann}_Q(id_Q(I_1)) \times Q/\mathrm{Ann}_Q(id_Q(I_2)) \to Q_1 \times Q_2$, whose composition with ψ defines a homomorphism $Q \to Q_1 \times Q_2$, whose restriction to J is the identity mapping. ■

5.7. Remark. Lemma 5.6 can be easily extended to a finite collection of ideals: If J is a nondegenerate Jordan algebra, $I_1, \ldots, I_n$ is a finite collection of closed pairwise orthogonal ideals ($I_i \cap I_j = 0$ if $i \neq j$) whose sum $I = I_1 + \cdots + I_n$ is an essential ideal, and for each $i = 1, \ldots, n$ the algebra $J_i = J/\sum_{j\neq i} I_j$ has a maximal algebra of quotients Q_i, then J has maximal algebra of quotients $Q_1 \times \cdots \times Q_n$.(It is easy to see that $L_1 = \sum_{i\neq 1} I_i$ has $\mathrm{Ann}_J(L_1) = I_1$ and $\mathrm{Ann}_J(I_1) = L_1$. Also, J/I_1 has closed pairwise orthogonal ideals $\bar{I}_2 = I_2 + I_1/I_1, \ldots, \bar{I}_n = I_n + I_1/I_1$, and $\bar{L}_1 = \sum_{j\neq i,1} \bar{I}_j = L_i/I_1$ for all $i \neq 1$, the algebra $(J/I_1)/\bar{L}_i = (J/I_1)/(L_i/I_1) \cong J/L_i$ has a maximal algebra of quotients Q_i. Then it is clear that an induction on n proves the assertion.

5.8. Theorem. *Any nondegenerate Jordan algebra J has maximal algebra of quotients $Q_{max}(J)$.*

Proof. Following the proof of [FGM, 7.8], we consider the T-ideal $\mathcal{A}(X) \subseteq FJ[X]$ satisfied by all Albert algebras. For each strongly prime ideal P of J, either J/P is Albert, hence $\mathcal{A}(J) \subseteq P$, or J/P is special. Denote by B the intersection of all containers (strongly prime ideals P of J such that $\mathcal{A}(J) \subseteq P$), and by C the intersection of all noncontainers (those P not containing $\mathcal{A}(J)$). It is proved in [FGM, 7.8] that C is a closed ideal that satisfies $\mathrm{Ann}_J(B) = C$. Moreover, J/C is nondegenerate, and a subdirect product of special algebras J/P for all containers P, hence it is itself special, and $J/\mathrm{Ann}_J(C)$ is nondegenerate, and PI since $\mathcal{A}(J) \subseteq B \subseteq \mathrm{Ann}_J(C)$.

Thus, lemma 5.6 implies that J will have maximal algebra of quotients $Q_{max}(J)$ as soon as J/C and $J/\mathrm{Ann}_J(C)$ do. Now, since $J/\mathrm{Ann}_J(C)$ is nondegenerate and PI, it has maximal algebra of quotients by 3.11, so it remains to show that J/C has maximal algebra of quotients. In other words, we can assume that J is special.

Under that assumption, consider the hermitian ideal $\mathcal{Z}(X)$ of 4.9, and set $I = \mathrm{Ann}_J(\mathcal{Z}(J))$ and $L = \mathrm{Ann}(I)$. By 5.6 it suffices to prove that J/I and J/L have maximal algebras of quotients.

First note that both J/I and J/L are special algebras by [FGM, 1.5(vi)]. Now, $\mathcal{Z}(J/L) = \mathcal{Z}(J) + L/L = 0$ since $\mathcal{Z}(J) \subseteq \mathrm{Ann}_J(\mathrm{Ann}_J(\mathcal{Z}(J))) = L$, hence J/L is PI and therefore it has maximal algebra of quotients by 3.11. On the other hand, $\mathcal{Z}(J/L) = \mathcal{Z}(J) + L/L$ is essential in J/L by [FGM, 1.13(iii)], hence it has maximal algebra of quotients by 4.10. ∎

5.9. Remark. For any nondegenerate Jordan algebra $Q_{max}(J)$ is unital. Indeed,take a tight unital hull J' of J, and denote by 1 its unit element. Since J is an essential ideal of J, J' is an algebra of quotients of J, hence $Q_{max}(J')$ is an algebra of quotients of J by 2.8, so we have $Q_{max}(J) = Q_{max}(J')$ by the maximality of

$Q_{max}(J)$ and of $Q_{max}(J')$. Now, arguing as in the proof of [MP, 3.2], we conclude that 1 is the unit element of $Q_{max}(J') = Q_{max}(J)$. ∎

Our next aim is to show that the maximal algebra of quotients is independent of the ring Φ of scalars over which J is an algebra.

5.10. Lemma. *Let J be a special nondegenerate Jordan algebra and let R be a ast-tight associative $*$-envelope of R. Then the action of the centroid $\Gamma = \Gamma(J)$ extends to R, so that R is a Γ-algebra.*

Proof. We claim that for any $\gamma \in \Gamma$ and any $x, y \in J$, we have $\gamma(x)y = x\gamma(y)$. Choose $\gamma \in \Gamma$ and for any $x \in J$ set $h(x) = \gamma(x)x - \gamma(x^2)$. Then we have $h(x) = \gamma(x)x - \gamma(x^2) = \gamma(x) \circ x - x\gamma(x) - \gamma(x^2) = \gamma(x \circ x) - x\gamma(x) - \gamma(x^2) = 2\gamma(x^2) - x\gamma(x) - \gamma(x^2) = -x\gamma(x) + \gamma(x^2) = -h(x)^*$. On the other hand, if $z \in J$, then $zh(x) + h(x)z = zh(x) - h(x)^*z = z\gamma(x)x - z\gamma(x^2) + x\gamma(x)z - \gamma(x^2)z = \{z, \gamma(x), x\} - \gamma(x^2) \circ z = \gamma(\{z, x, x\}) - \gamma(x^2 \circ z) = 0$. Therefore $h(x)R = Rh(z)$. Now, $h(x)^2 = -h(x)h(x)^* = \gamma^2(x^4) - \gamma(x^2)\gamma(x)x - x\gamma(x)\gamma(x^2) + \gamma^2(x^4) = 2\gamma^2(x^4) - \{\gamma(x^2)\gamma(x), x\} = 2\gamma^2(x^4) - 2\gamma^2(x^4) = 0$. Thus $h(x)Rh(x) = 0$, and we get $h(x) = 0$ by semiprimeness of R. Linearizing the condition $h(x) = 0$ we get $\gamma(x)y + \gamma(y)x = \gamma(x \circ y)$ and since $\gamma(x \circ y) = x \circ \gamma(y) = \gamma(y)x + x\gamma(y)$, we obtain $\gamma(x)y = x\gamma(y)$.

Now, since J generates J, for any $r \in R$ there are elements $x_{ij} \in J$ with $r = \sum_i x_{i1} \cdots x_{in_i}$. We define $\gamma(r) = \sum_i \gamma(x_{i1}) \cdots x_{in_i}$. To see that this is well defined, suppose that $\sum_i x_{i1} \cdots x_{in_i} = 0$ and set $s = \sum_i \gamma(x_{i1}) \cdots x_{in_i}$. Then, for any $y \in J$ we have $ys = \sum_i y\gamma(x_{i1}) \cdots x_{in_i} = \sum_i \gamma(y)x_{i1} \cdots x_{in_i} = \gamma(y) \sum_i x_{i1} \cdots x_{in_i} = 0$. Hence $Js = 0$, which implies $Rs = 0$, hence $s = 0$ by semiprimeness of R. Therefore $\gamma(r)$ is well defined, and it is clear that the action $r \mapsto \gamma(r)$ for $\gamma \in \Gamma$ extends the action of Γ on J and makes R a Γ-algebra. ∎

5.11. Lemma. *Let J be a nondegenerate Jordan algebra and let $Q \supseteq J$ be an algebra of quotients of J. If Q is a Φ-algebra for some ring of scalars Φ, and J is a Φ-subalgebra, then Q is an algebra of quotients of J over Φ.*

Proof. This is straightforward from the fact that every denominator inner ideal $\mathcal{D}_J(q)$ for $q \in Q$ is a Φ-inner ideal, which is obviously dense as a Φ-inner ideal. ∎

Let J be a nondegenerate Jordan algebra over a ring of scalars Φ, we denote (temporarily) denote by $Q_{max}(J_\Phi)$ its maximal Φ-algebra of quotients. It is clear that for ring of scalars Φ we have $Q_{max}(J_\Gamma) \subseteq Q_{max}(J_\Phi) \subseteq Q_{max}(J_{\mathbb{Z}})$.

5.12. Lemma. *For any nondegenerate Jordan algebra we have $Q_{max}(J_\Gamma) = Q_{max}(J_{\mathbb{Z}})$, and therefore, the maximal algebra of quotients of a nondegenerate Jordan algebra does not depend on the ring of scalars.*

Proof. In view of 5.11, it suffices to prove that if J is a nondegenerate Jordan Φ-algebra, then $Q = Q_{max}(J_{\mathbb{Z}})$ is a Φ-algebra. Following the proof of 5.8, we can find ideals I, L of J with $\mathrm{Ann}_J(I) = L$ and $\mathrm{Ann}_J(L) = I$ such that J/L is PI, and J/I is special and has $\mathcal{Z}(J/I)$ essential. Then I and L are Φ-ideals, hence J/I and J/L are Φ-algebras, and since $Q_{max}(J_{\mathbb{Z}}) = Q_{max}((J/I)_{\mathbb{Z}}) \times Q_{max}((J/L)_{\mathbb{Z}})$, we can assume that either J is PI, or J is special and $\mathcal{Z}(J)$ is essential in J. In the first case, the description 3.11 of the algebra of quotients as the almost classical algebra of quotients of J makes it clear that $Q_{max}(J_{\mathbb{Z}})$ is a $\Gamma(J)$-algebra, hence a Φ-algebra.

On the other hand, if J is special and $\mathcal{Z}(J)$ is essential in J, by 4.10 $Q_{max}(J)$ is the set of all $q \in H(Q_\sigma(R), *)$, for a $*$-tight associative $*$-envelope R of J which have $\mathcal{D}_J(q)$ dense in J. It follows from 5.10 that R is a Φ-algebra, and it is easy to show that then $Q_{max}(J)$ is a Φ-algebra. ■

We next apply 5.8 to other classes of algebras of quotients.

5.13. Lema. *Let $J \subseteq \tilde{J}$ be nondegenerate Jordan algebras and let $\mathcal{F}$ be a filter of essential ideals of J such that $I^{(1)} \in \mathcal{F}$ for all $I \in \mathcal{F}$. Assume that $U_{\tilde{a}} I \neq 0$ for any ideal $I \in \mathcal{F}$ and any $\tilde{a} \in \tilde{J}$. Then, the set $Q = \{\tilde{a} \in \tilde{J} \mid \mathcal{D}_J(\tilde{a})$ contains an ideal of $\mathcal{F}\}$ is a subalgebra of $\tilde{J}$ which is a Martindale algebra of $\mathcal{F}$-quotients of J.*

Proof. Note first that for any $p, q \in Q$ with ideals $I \subseteq \mathcal{D}_J(p)$ and $L \subseteq \mathcal{D}_J(q)$, $I, L \in \mathcal{F}$, we have $N = I \cap L \in \mathcal{F}$, and $\{p, U_x y, q\} = \{\{p, x, y\}, x, q\} \in J$ for any $x, y \in N$. Thus, $U_{p+q} N^{(1)} \subseteq U_p N^{(1)} + U_q N^{(1)} + \{p, N^{(1)}, q\} \subseteq J$, and since $N^{(1)} \circ (p + q) \subseteq J$, we get $N^{(2)} = (N^{(1)})^{(1)} \subseteq \mathcal{D}_J(p + q)$ (see 2.3.2) and $N^{(2)} \in \mathcal{F}$, hence $p + q \in Q$.

Now, keeping the notation, but assuming $q \in \hat{J}$ (and $L = J$ if $q = 1$), if $M \subseteq N$ and $M \in \mathcal{F}$,then $\{z, U_x y, p\} = \{z, x, \{y, x, p\}\} - \{z, U_x p, y\} \in \{\hat{J}, M, J\} + \{\hat{J}, J, M\} \subseteq M$ for any $x, y \in M^{(1)}$ and any $z \in \hat{J}$, i. e.

$$\{\hat{J}, M^{(1)}, p\} \subseteq M \quad \text{and} \quad \{\hat{J}, M^{(1)}, q\} \subseteq M \tag{1}$$

and

$$\{M^{(1)}, \hat{J}, p\} \subseteq M \quad \text{and} \quad \{M^{(1)}, \hat{J}, q\} \subseteq M. \tag{2}$$

Now, $U_p U_x y = U_{p \circ x} y - U_x U_p y - U_p(x \circ y) \circ x + \{p, y, U_x p\} \in U_J M + U_M J + J \circ M + \{p, M^{(1)}, J\} \subseteq M$ for any $x, y \in M^{1})$ using (1). Hence

$$U_p M^{(2)} \subseteq M \quad \text{and} \quad U_q M^{(2)} \subseteq M. \tag{3}$$

Moreover, for any $x, y \in M^{(1)}$ we have $\{U_x y, p, q\} = \{x, \{y, x, p\} q\} - \{U_x p, y, q\} \in \{M^{(1)}, J, q\} + \{J, M^{(1)}, q\} \subseteq M$ by (1) and (2), hence

$$\{M^{(3)}, p, q\} \subseteq M \quad \text{and} \quad \{M^{(3)}, q, p\} \subseteq M. \tag{4}$$

Therefore, $z \circ (U_p q) = \{z, p, q\} \circ p - q \circ U_p z \in \{N^{(3)}, p, q\} \circ p + q \circ U_p N^{(2)} \subseteq N \circ p + q \circ N \subseteq J$ for any $z \in N^{(3)}$ by (3) and (4), i. e.

$$N^{(3)} \circ U_p q \subseteq J. \tag{5}$$

On the other hand, $U_{U_p q} N^{(4)} = U_p U_q U_p N^{(4)} \subseteq U_p U_q N^{(2)} \subseteq U_p N \subseteq J$ by (3). This together with (5) proves that the ideal $N^{(4)} \in \mathcal{F}$ satisfies $N^{(4)} \circ U_p q \subseteq J$ and $U_{U_p q} N^{(4)} \subseteq J$. Now, for any $x, y \in N^{(4)}$ and any $z \in J$, we have $\{U_x y, U_p q, z\} = \{x, \{y, x, U_p q\}, z\} - \{U_x q, y, z\} = \{x, y \circ (x \circ U_p q), z\} - \{x, \{x, U_p q, y\}, z\} - \{U_x q, y, z\} \in J$, hence $\{N^{(5)}, U_p q, J\} \subseteq J$. Now, $U_{U_x y} U_p q \in J$ for any $x, y \in M^{(4)}$ follows easily using QJ16 and the above containments. This yields $N^{(5)} \subseteq \mathcal{D}(U_p q)$, and since $N^{(5)} \in \mathcal{F}$, this establishes $U_p q \in Q$, and proves that Q is a subalgebra of $\tilde{J}$. ■

5.14. Let J be a nondegenerate Jordan algebra and let $\mathcal{F}$ be a filter of essential ideals of J such that $I^{(1)} \in \mathcal{F}$ for any $I \in \mathcal{F}$. A Martindale algebra of $\mathcal{F}$-quotients Q will be said to be maximal if for any other Martindale algebra of $\mathcal{F}$-quotients Q' of J there exits an algebra homomorphism $Q' \to Q$ which extends the inclusion $J \subseteq Q$. Since Martindale algebras of $\mathcal{F}$-quotients are, in particular, algebras of quotients, it follows from 2.11 that there is at most one such extension of the inclusion $J \subseteq Q$, and as in the case of maximal algebras of quotients, that up to isomorphism there exists at most one maximal Martindale algebra of $\mathcal{F}$-quotients.

5.15. Corollary. *Let J be a nondegenerate Jordan algebra and let $\mathcal{F}$ be a filter of essential ideals of J such that $I^{(1)} \in \mathcal{F}$ for any $I \in \mathcal{F}$. Then there exits a maximal Martindale algebra of $\mathcal{F}$-quotients.*

Proof. The set $Q = \{q \in Q_{max}(J) \mid \mathcal{D}_J(q)$ contains an ideal from $\mathcal{F}\}$ is a subalgebra of $Q_{max}(J)$ by 5.13. It is easy to see that this is in fact a Martindale algebra of $\mathcal{F}$-quotients of J, and its maximality readily follows from the maximality of $Q_{max}(J)$ since any Martindale algebra of $\mathcal{F}$-quotients is in particular an algebra of quotients. ■

The next result, proved in [M] for linear algebras, and extended in [Bo] to quadratic algebras, is the analogue of 5.13 for algebras of S-quotients (see 2.3.5).

5.16. Lema. *Let $J \subseteq \tilde{J}$ be Jordan algebras, J a subalgebra of $\tilde{J}$, and let S be an Ore monad in J which satisfies the Ore condition in J. If any element from S is invertible in $\tilde{J}$, then $U_{S^{-1}} J = \{U_{s^{-1}} x \in \tilde{J} \mid s \in S, x \in J\} = \{\tilde{a} \in \tilde{J} \mid \mathcal{D}_J(\tilde{a}) \cap S \neq \emptyset\}$ is a subalgebra of $\tilde{J}$ which is an algebra of S-quotients of J.* ■

5.17. Lemma. *Let J be a nondegenerate Jordan algebra, $I \neq J$ be a closed ideal of J, and $S \subseteq J$ an Ore monad. Then $S/I = \{s + I \mid s \in S\}$ is an Ore monad* *in J/I. Moreover, if S satisfies the Ore condition in J, then S/I satisfies the Ore condition in J/I.*

Proof. Denote with bars the projections in $\bar{J} = J/I$. We first show that $\overline{Inj(J)} \subseteq Inj(\bar{J})$. Indeed, if $U_{\bar{s}}\bar{x} = 0$ for some $s \in S$ and $x \in J$, then $U_s x \in I$ and $U_s U_x U_s \mathrm{Ann}_J(I) = U_{U_s x}\mathrm{Ann}_J(I) \subseteq I \cap \mathrm{Ann}_J(I) = 0$. Therefore $U_x U_s \mathrm{Ann}_J(I) = 0$ since s is injective. Now we have $U_{U_s U_x z}\mathrm{Ann}_J = U_s U_x U_z U_x U_s \mathrm{Ann}_J(I) = 0$ for any $z \in J$, hence $U_s U_x J \subseteq \mathrm{Ann}_J(\mathrm{Ann}_J(I)) = I$ (by 0.3, since I is closed). Then, for any $z \in J$, we get $U_s U_{U_x z}\mathrm{Ann}_J(I) \subseteq U_s U_x J \cap \mathrm{Ann}_J(I) \subseteq I \cap \mathrm{Ann}_J(I) = 0$, and since s is injective, this implies $U_{U_x z}\mathrm{Ann}_J(I) = 0$ for any $z \in J$, hence $U_x J \subseteq \mathrm{Ann}_J(\mathrm{Ann}_J(I)) = I$ (again by 0.3). In particular, $U_x \mathrm{Ann}_J(I) \subseteq I \cap \mathrm{Ann}_J(I) = 0$, hence $x \in \mathrm{Ann}_J(\mathrm{Ann}_J(I)) = I$ by 0.3, and we obtain $\bar{x} = 0$. Now the fact that S/I is an Ore monad is a straightforward verification.

Suppose next that S satisfies the Ore condition in J. as has been proved in 2.3, this means that for any $s \in S$ and any $a \in J$, the inner ideal $K_s = \Phi s + U_s \hat{J}$ has $(K_s : a) \cap S \neq \emptyset$. Now we clearly have $\bar{K}_s = K_{\bar{s}} = \Phi\bar{s} + U_{\bar{s}}\hat{\bar{J}}$ and $\overline{(K_s : a)} \subseteq (\bar{K}_s : \bar{a}) = (K_{\bar{s}} : \bar{a})$, and since $(K_s : a) \cap S \neq \emptyset$, we obtain $(K_{\bar{s}} : \bar{a}) \cap \bar{S} \neq \emptyset$, hence $\bar{S} = S/I$ satisfies the Ore condition in $\bar{J} = J/I$. ∎

5.18. Corollary. *Let J be a nondegenerate Jordan algebra. and let $S \subseteq J$ be an Ore monad of J. If S satisfies the Ore condition in J, then there exists an algebra of S-quotients of J.*

Proof. In view of 5.16, it suffices to find an algebra $\tilde{J} \supseteq J$ in which every element from S becomes invertible. We will show that this is indeed the case for $\tilde{J} = Q_{max}(J)$.

We retrieve here the notation of the proof of 5.8, and consider the intersection C of all noncontainers, which is a closed ideal. Take now ideals I and N of J with $C \subseteq I \cap N$, $I/C = \mathrm{Ann}_{J/C}(\mathcal{Z}(J/C))$ and $N/C = \mathrm{Ann}_{J/C}(\mathrm{Ann}_{J/C}(\mathcal{Z}(J/C)))$, and set $L = N \cap \mathrm{Ann}_J(C)$. Since $\mathcal{Z}(J/C) = \mathcal{Z}(J) + C/C$, it is clear that $I = \mathrm{Ann}_J(\mathcal{Z}(J) + C) + C$ (see [FGM 1.13(iii)]) $= \mathrm{Ann}_J(\mathcal{Z}(J) + C) + \mathrm{Ann}_J(\mathrm{Ann}_J(C)) = \mathrm{Ann}_J[(\mathcal{Z}(J)+C) \cap \mathrm{Ann}_J(C)]$ and $N = \mathrm{Ann}_J(I) + C = \mathrm{Ann}_J(I) + \mathrm{Ann}_J(\mathrm{Ann}_J(C)) = \mathrm{Ann}_J(I \cap \mathrm{Ann}_J(C))$ are closed ideals. Now set $L = \mathrm{Ann}_J(C) \cap N = \mathrm{Ann}_J(C) \cap \mathrm{Ann}_J(I \cap \mathrm{Ann}_J(C)) = \mathrm{Ann}_J(C + I \cap \mathrm{Ann}_J(C))$. Since $C + I \cap \mathrm{Ann}_J(C) \subseteq C + I \subseteq I$, we have $\mathrm{Ann}_J(I) \subseteq L$. On the other hand, if $z \in \mathrm{Ann}_J(C) \cap \mathrm{Ann}_J(I \cap \mathrm{Ann}_J(C)) = L$, then $U_{U_z I}\mathrm{Ann}_J(C) \subseteq U_z(I \cap \mathrm{Ann}_J(C) = 0$, hence $U_z I \subseteq \mathrm{Ann}(\mathrm{Ann}_J(C)) = C$, but since $z \in \mathrm{Ann}_J(C)$ this gives $U_z I = 0$, hence $z \in \mathrm{Ann}_J(I)$. thus we have $L = \mathrm{Ann}_J(I)$. Note also that $J_1 = J/L = J/(\mathrm{Ann}_J(C) \cap N$ is a subdirect product of the PI algebras $J/\mathrm{Ann}_J(C)$ and $J/N \cong (J/C)/\mathrm{Ann}_{J/C}(\mathrm{Ann}_{J/C}(\mathcal{Z}(J/C)))$, and $J_2 = J/I = (J/C)/(I/C) = (J/C)/\mathrm{Ann}_{J/C}(\mathcal{Z}(J/C))$ is special and has essential

$\mathcal{Z}(J/C)$. From 5.6 we get then that $Q_{max}(J) = Q_{max}(J_1) \times Q_{max}(J_2)$ through the inclusion $J \subseteq J_1 \times J_2 \subseteq Q_{max}(J_1) \times Q_{max}(J_2)$. Moreover, an $s \in S$ will be invertible in $Q_{max}(J)$ if and only if $s_1 = s + L$ and $s_2 = s + I$ are invertible in $Q_{max}(J_1)$ and $Q_{max}(J_2)$ respectively.

Note now that by 5.17, with the notation introduced there, S/I and S/L are Ore monads in J/I and J/L respectively, and they satisfy the Ore condition in their respective algebras. Thus, it suffices to show that the theorem holds for J/I and for J/L or, in other words, we can consider separately the case where J is PI, and the case where J is special and the ideal $\mathcal{Z}(J)$ is essential. Before going into the proof of that fact we note that if $s \in J$ is injective, then s is also injective in every algebra of quotients Q of J. Indeed, if $U_s q = 0$ for some $q \in Q$, then $U_s U_q (K_{s^2} \cap \mathcal{D}_J(q)) \subseteq U_s U_q U_s \hat{J} = U_{U_s q} \hat{J} = 0$, hence $U_q(K_{s^2} \cap \mathcal{D}_J(q)) = 0$ by the injectivity of s since $U_q(K_{s^2} \cap \mathcal{D}_J(q)) \subseteq J$. Now $K_{s^2} \cap \mathcal{D}_J(q)$ is a dense inner ideal of J, hence $q = 0$ by 2.4(iv).

Assume first that J is PI. Then $Q_{max}(J) = J_{\mathcal{E}(\Gamma)}$ is the almost classical algebra of quotients of J by 3.11. Take any injective $s \in J$ and note that since $U_s \hat{J} \supseteq K_{s^2}$ is a dense inner ideal of J, there exists an essential ideal N of J with $N \subseteq U_s \hat{J}$ by 3.1. Since s is injective, every element of N can be written in the form $U_s a$ for a unique $a \in \hat{J}$. Now take $z = U_s a \in N \cap C_w(J)$. Then for any $p \in Q_{max}(J)$ and any $w = U_s c \in N \cap C_w(J)$ we have: $U_s U_w U_s U_a p = U_s U_s U_a U_w p$ (since $w \in C_w(Q)$ by 2.13) $= U_s U_s U_a U_s U_c U_s p = U_s U_z U_c U_s p = U_z U_s U_c U_s p = U_z U_w p = U_w U_z p = U_s U_w U_a U_s p$, hence $U_s U_w (U_s U_a p - U_a U_s p) = 0$. Thus $U_w(U_s U_a p - U_a U_s p) = 0$ by the injectivity of s in $Q_{max}(J)$ proved above, and we get $\mathfrak{u}(N)(U_s U_a p - U_a U_s p) = 0$ (with the notations of section 2), hence $U_s U_a p - U_a U_s p = 0$ by 3.6, i. e.

$$(*) \qquad U_s U_a p = U_a U_s p \qquad \text{for all} \quad p \in Q.$$

We define a mapping $f_s : \mathfrak{u}(N) \to J$ by $f_s(\sum_i \lambda_i U_{z_i}) = \sum_i \lambda_i U_{a_i} s^2$, where $\lambda_i \in \Gamma$, and $z_i \in C_w(J) \cap N$ has the form $z_i = U_s a_i$ for a unique $a_i \in \hat{J}$. To see that this is well defined suppose that $\sum_i \lambda_i U_{z_i} = 0$ with λ_i and z_i as before. Then $0 = \sum_i \lambda_i U_{z_i} s^2 = \sum_i \lambda_i U_{a_i} s^4 = \sum_i \lambda_i U_s U_{a_i} s^2$ (by $(*)$) $= U_s \sum_i \lambda_i U_{a_i} s^2$, hence $\sum_i \lambda_i U_{a_i} s^2 = 0$ since s is injective. It is clear that f is a homomorphism of Γ-modules, an thus it defines an element $q = [f, \mathfrak{u}(N)] \in J_{\mathcal{E}(\Gamma)}$.

Now take $r \in Q_{max}(J)$. For any $z = U_s a \in C_w(J) \cap N$ we have:

$$\begin{aligned} U_z^4 U_{U_s U_q s} r = U_z^4 U_s U_q U_s U_q U_s^2 r &= U_s U_{U_z q} U_s U_{U_z q} U_s^2 r = \\ &\text{(since } C_w(J) \subseteq C_w(Q_{max}(J)) \text{ by 2.13)} \\ = U_s U_{f(U_s)} U_s U_{f(U_s)} U_s^2 r &= U_s U_{U_a s^2} U_s U_{U_a s^2} U_s^2 r = \qquad \text{(by 3.6)} \end{aligned}$$

$$
\begin{aligned}
&= U_{U_s a} U_{U_s a} U_a U_s U_{U_s a} U_s r = U_z U_z U_a U_s U_z U_s r = \\
&= U_z^3 U_a U_s^2 r = \\
&\qquad\qquad\qquad\qquad \text{(again since } C_w(J) \subseteq C_w(Q_{max}(J))) \\
&= U_z^3 U_s U_a U_s r = U_z^4 r, \qquad\qquad\qquad\qquad \text{(by } (*))
\end{aligned}
$$

so $U_z^4(U_{U_s U_q s} U_s r - r) = 0$ for all $r \in Q_{max}(J)$, hence $U_{U_s U_q s} U_s r - r = 0$ by 0.3 since $U_z \in \Gamma(Q_{max}(J))$, and $U_{U_s U_q s} U_s r = r$ for all $r \in Q_{max}(J)$, which proves that s is invertible in $Q_{max}(J)$ with inverse $s^{-1} = U_s U_q s$, and the theorem in the case where J is PI.

Assume finally that J is special and $\mathcal{Z}(J)$ is an essential ideal of J. For any $*$-tight associative $*$-envelope R of J, the maximal algebra of quotients of J is $Q_{max}(J) = \{q \in H(Q_\sigma(R), *) \mid \mathcal{D}_J(q) \text{ is dense in } J\}$ by 4.10. Now take $s \in S$, and note that since the inner ideal $K_s = \Phi s + U_s \hat{J}$ is dense in J, the left ideal $\hat{R}s \supseteq K_s$ is dense in R by 4.6, hence $Rs = R\hat{R}s$ is also dense left ideal, and $sR = (Rs)^*$ is a dense right ideal. In particular this implies that the right annihilator of $\hat{R}s$ is zero, hence s is regular in R.

Now note that the mapping $f : Rs \to R$ given by $f(rs) = r$ is well defined since s is regular, and is a homomorphism of left R-modules. Then f_s defines an element $q \in Q^l_{max}(R)$ which satisfies $xsq = x$ for all $x \in R$. Moreover, $ys(qsx - x) = ysqsx - ysx = ysx - ysx = 0$ for any $x, y \in R$, hence $Rs(qsx - x) = 0$, and this implies $qsx = x$ for any $x \in R$, since Rs is dense. It follows then that $qsR \subseteq R$, hence $q \in Q_\sigma(R)$ by the density of sR, and $q = s^{-1}$ is the inverse of s in $Q_\sigma(R)$. Note now that $q = q^*$, and $\mathcal{D}_J(q) \supseteq K_s$ is a dense inner ideal of J, hence $s^{-1} \in q \in Q_{max}(J)$, which proves the theorem for the present case. ■

5.19. Notes.

1.- We have already noted in 1.18(b) that in a strongly nonsingular Jordan algebra, an inner ideal is dense if and only if it is essential, and in this case the algebras of quotients in the sense of [MP] are the same as our algebras of quotients. Therefore Theorem 5.8 generalizes [MP, 4.8]. (It is easy to see that if for a strongly prime Jordan algebra J, the almost classical localization $J_{\mathcal{E}(\Gamma)}$ coincides with the usual central closure $\Gamma^{-1}J$.)

2.- Note that the description 4.10 of the maximal algebra of quotients easily provides the corresponding description of the Martindale algebra of quotients by 5.13: If J is a nondegenerate special Jordan algebra such that $\mathcal{Z}(J)$ is essential, and R is a $*$-tight associative $*$-envelope of J, then the maximal Martindale algebra of quotients of J consists of the set of all $q \in H(Q_\sigma(R), *)$ (or equivalently in this case $q \in H(Q_s(R), *)$) such that $\mathcal{D}_J(q)$ contains an essential ideal of J. This shows that 5.15 generalizes [AGG,4.6] both to quadratic and to nondegenerate algebras.

3.- Corollary 5.18 gives an answer to the quadratic version of Jacobson's original problem [J1, p. 426] of finding rings of fractions of Jordan domains (which are nondegenerate) without the need of the "unwelcome Ore condition" of [BoM]. Of course, since it is shown in [BoM] that this is a necessary condition for general algebras of fraction to exist, there is no point in trying to avoid this extra Ore condition unless it turns out to be a consequence of the monad being Ore in the algebra, which seems unlikely. Corollary 5.18 just shows that a more familiar condition like nondegeneracy is enough. It would be desirable however to have a direct combinatorial proof of this fact (that is, of the fact that together with the usual Ore condition, nondegeneracy implies the "unwelcome Ore condition").

4.- It was proved in [ACGG] that if $Q$is a Martindale-like cover of a nondegenerate Jordan algebra J, then: (a) if J is PI, then Q is PI, and in this case, every homogeneous polynomial p which vanishes on J, also vanishes on Q [ACGG1, 2.5], and (b) if J is special, then Q is special. It is clear that the proof of 3.8(1) adapts to yield the corresponding results for any algebra of quotients Q of a nondegenerate J, which contains the above results as particular cases. Of course, the situation in [ACGG1] is in principle more general, since the authors consider there what they call covers satisfying the condition $C_w(J) \subseteq C_w(Q)$, and the outer absorption property IA1 of [ACGG1, 0.10]: for any $q \in Q$ there exits an essential ideal I of J such that $0 \neq U_I q \in J$. Note however the following

Lemma. *Let Q be a cover of the nondegenerate Jordan algebra J. If J is PI and Q satisfies IA1 of [ACGG1], then Q is a Martindale-like cover of J.*

Proof. We first prove the following claim:

$$(1) \qquad U_L q \neq 0 \quad \text{for any essential ideal } L \text{ of } J, \text{ and any } 0 \neq q \in Q.$$

Indeed, if L is an essential ideal of J and there is a nonzero $q \in Q$ with $U_L q = 0$, the there is a nonzero $p \in Q$ with $U_p L = 0$ by [MP, 2.5]. Now, we can find an essential ideal I of J with $o \neq U_I p \subseteq J$, and for any $y \in I$ we have $U_{U_y p} L \subseteq U_y U_p L = 0$, hence $U_y p \in \mathrm{Ann}_J(L) = 0$, and we get $U_I p = 0$, a contradiction.

Using this fact, the corresponding part of the proof of [MP, 3.4] can be easily adapted to yield:

$$(2) \qquad U_z U_x q = U_x U_z q \qquad \text{and} \qquad U_z\{x, y, q\} = \{x, y, U_z q\}$$

for any $z \in C_w(J)$, $x, y \in \hat{J}$, and $q \in Q$.

Next take an essential ideal L of J, and fix $q \in Q$, we claim:

$$\text{(3)} \qquad \text{if} \quad U_z q = 0 \quad \text{for all} \quad z \in C_w(J) \cap L, \quad \text{then} \quad q = 0$$

Indeed, suppose that there is an essential ideal L with $U_z q = 0$ for all $z \in C_w(J) \cap L$, and take an essential ideal I with $0 \neq U_I q \subseteq J$. Then $U_z U_I q = U_I U_z q$ (by (2)) $= 0$ implies $\mathfrak{u}\,(L) U_I q = 0$ (see 3.10), hence $U_I q = 0$ by 3.6, a contradiction.

Now take $q \in Q$ and $z \in C_w(J)$ and choose an essential ideal I of J with $U_I q \subseteq J$. For any $w \in C_w(I)$ we have $U_w(\{U_z x, y, q\}) = \{U_z x, y, U_w q\}$ (by (2)) $= \{x, y, U_z U_w q\}$ (since $z \in C_w(J)$ and $U_w q \in J$) $= \{x, y, U_w U_z q\}$ (by (2) since $w \in C_w(J)$) $= U_w\{x, y, U_z q\}$ (by (2)). Thus $U_w(\{U_z x, y, q\} - \{x, y, U_z q\}) = 0$ for all $w \in C_w(J) \cap I$, and (3) gives:

$$\text{(4)} \qquad \{U_z x, y, q\} = \{x, U_z y, q\} = \{x, y, U_z q\}$$

for any $z \in C_w(J)$, $x, y \in \hat{J}$, and $q \in Q$.

Now fix $q \in Q$, and an essential ideal I of J with $U_I q + U_I q^2 \subseteq J$. Set $N = \mathfrak{u}\,(I)J$, which is an essential ideal of J by 3.10 and 3.7. Then, for any $z \in C_w(J) \cap I$, $a, b \in \hat{J}$, we have $\{U_z a, q, b\} = ((U_z a) \circ q) \circ b - \{b, U_z a, q\} = (a \circ U_z q) \circ b - \{b, a, U_z q\} \in J$, hence $\{N, q, \hat{J}\} \subseteq J$. Consider now the ideal $\mathfrak{b}$ of Γ generated by all U_z^2 for $z \in C_w(J) \cap N$. This is an essential ideal of Γ for if $\alpha\mathfrak{b} = 0$ for some $\alpha \in \Gamma$, then $\alpha(U_z)^2 = 0$ for all $z \in C_w(J) \cap N$, hence $(\alpha U_z)^2 = 0$, and $\alpha U_z = 0$ for all $z \in C_w(J) \cap N$ since Γ is reduced. Thus $\alpha\mathfrak{u}\,(N) = 0$, hence $\alpha = 0$ by 3.10. Set now $L = \mathfrak{b}\,J$, which is an essential ideal of J by 3.7. Then $L \subseteq N \subseteq I$, hence $U_L q + \{L, q, \hat{J}\} \subseteq J$. Also, for any $z \in C_w(J) \cap N$ and any $a \in J$ we have $U_q U_z^2 a = U_{q \circ z} U_z a - U_{U_z q} U_a - \{\{q, z, U_z a\}, q, z\} + (U_z a) \circ (U_z q^2)$ (apply Macdonald's theorem [J2, 3.4.16]), which belongs to J by the above containments and since $U_N q^2 \subseteq U_I q^2 \subseteq J$. This shows that $U_q L \subseteq L$ and proves the lemma. ■

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