

Local orders in Jordan algebras

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Abstract

We study a notion of order in Jordan algebras based on the version for Jordan algebras of the ideas of Fountain and Gould [FoGo1] as adapted to the Jordan context by Fernández-López and García-Rus [FG1], making use of results on general algebras of quotients of Jordan algebras. In particular, we characterize the set of Lesieur-Croisot elements of a nondegenerate Jordan algebra as those elements of the Jordan algebra lying in the socle of its maximal algebra of quotients, and apply this relationship to extend to quadratic Jordan algebras the results of Fernández-López and García-Rus on local orders in nondegenerate Jordan algebras satisfying the descending chain condition on principal inner ideals and not containing ideals which are nonartinian quadratic factors.

Introduction

Local orders for Jordan algebras were introduced and studied by Fernández-López and García-Rus in [FG1, FG2] inspired by the work of Fountain and Gould [FoGo1, FoGo2, FoGo3], and Áhn and Márki [AM1, AM2] on orders

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of associative rings following ideas on quotients in semigroup theory. The original purpose of that research on associative algebras was to introduce a notion of localization inspired in Ore's construction but without the requirement of having an identity element, so that the "regular elements" were not intended to become invertible, but merely "locally invertible". That turn to locality fits well with some of the ideas of Jordan theory, in particular with that of local algebra at a given element [DM], a remark that made natural the step taken by Fernández-López and García-Rus of extending these notion to the Jordan context.

Fernández-López and García-Rus' work was preceded and inspired by results on Jordan algebras of fractions which originated in the question raised by Jacobson on whether results similar to Ore's construction could be obtained for Jordan algebras [J1, p. 426]. As it is well known, Ore's results were extended by Goldie [G1, G2] to the study of embeddability of associative rings in simple or semisimple artinian rings, which, in turn, motivated associative localization theory. In the realm of Jordan theory, Jacobson's question, or rather the related question on the possibility of extending Goldie's results to the Jordan setting was first answered by Zelmanov [Z1, Z2] (later extended by Fernández-López, García-Rus and Montaner to quadratic Jordan algebras [FGM]). As for Jacobson's original question, a complete answer was given by Martínez [Ma] based on a quite different approach that allowed her to provide necessary and sufficient Ore-like conditions for the existence of algebras of fractions of Jordan algebras (with $\frac{1}{6} \in \Phi$) (later generalized to quadratic algebras by Bowling and McCrimmon [BM].)

Those results opened the way to a sizable area of research on algebras of quotients of Jordan algebras. We refer to [Mo3] for a concise overview of that field, some of whose results will be recalled in the following sections as needed. At this point we limit ourselves to mentioning two notions which were introduced in those developments, and which we will need in order to describe some of the problems to whose solution this paper is devoted.

First of all, whereas Martínez's answer of Jacobson's problem faithfully parallels the associative situation (although through a quite different proof), Fernández-López, García-Rus and Montaner showed in [FGM] that the Jor-

dan version of Goldie theory deviates from its associative counterpart at a significant point that will be pivotal in the present research: the natural Jordan version of the characterization of left Goldie associative algebras (and similarly, right Goldie algebras) as those for which left (resp. right) essential inner ideals are precisely the ones that contain regular elements, namely the characterization of Goldie Jordan algebras as those for which an inner ideal is essential if and only if it contains injective elements, no longer holds for Jordan algebras. However, that missing Goldie-type property has its own interest since, as proved in [FGM], algebras that satisfy it are precisely those which are orders in nondegenerate algebras of finite capacity, which is the natural finiteness condition from the viewpoint of the classical Jordan theory based on the use of idempotents. Those algebras were termed Lesieur-Croisot algebras (LC-algebras for short) in [MoT1]. Jordan algebras having local algebras that are Lesieur-Croisot were studied later by Montaner and Tocón in [MoT1, MoT2].

A second notion belonging to the just mentioned study of algebras of quotients, and that will be central in the present research, is the formulation in the Jordan setting of a more general constructions of algebras of quotients that parallels Lambeck-Utumi's associative algebras of quotients, and that turns out to provide a common environment for most of the previously developed constructions of quotients for nondegenerate Jordan algebras (see [Mo3]).

In spite of its generality, and in contrast to the situation in the associative theory [AM1], local orders as defined by Fernández-López and García-Rus have not been shown to fit in that setting. Since the study of that notion of local orders is the objective of the present paper, that will be one of the issues we will address, although we will restrict that study to the case in which the algebras are local orders in over-algebras with dcc on principal inner ideals.

The organization of the paper, and the questions we will deal with will be the following:

After an initial section of preliminaries, in section 1 we recall basic facts on the just mentioned two classes of algebras of quotients of Jordan algebras,

classical algebras of fractions, including the Jordan analogues of the notions and results of Goldie Theory, and general algebras of quotients which are Jordan analogues of Lambek-Utumi's algebras of quotients in the associative theory.

In addition to recalling the basic notions of these two kinds of algebras of quotients, including that of LC-elements of a Jordan algebra, we prove a central result of the section which will ease computations later on, namely the coincidence of essentiality and density for inner ideals in algebras in which every element is LC.

Sections 2 and 3 are devoted to a deeper study of nondegenerate Jordan algebras having nonzero LC-elements, addressing our first objective, the study of the existing connection between the set of LC-elements of a nondegenerate Jordan algebra and the socle of its maximal algebra of quotients. Drawing inspiration from a well known result of associative theory, we prove in section 2 that the set of LC-elements of a strongly prime Jordan algebra coincides with the intersection of the socle of its maximal algebra of quotients with the original algebra, and extend that result to nondegenerate algebras in Section 3.

In section 4 we introduce the notion of local order of a Jordan algebra to which this paper is devoted. This notion basically coincides with the one introduced by Fernández-López and García-Rus, although it slightly differs from theirs in that it makes use of the quadratic version [FGM] of Goldie's theorems for Jordan algebras, rather than its original linear form [Z1, Z2].

Since, as mentioned in [FG1], the original motivation for the introduction of local orders in associative theory was the introduction of an Ore-like localization in algebras which need not have an identity element thus generalizing Goldie's theorems for Jordan algebras, the regularity condition to be satisfied by the resulting over-algebra is a natural generalization of the artinian property to nonunital algebras, namely having the dcc on principal inner ideals, that is, being equal to its socle. According to that, we give here a version of the definition of local order in algebras that equal their socles, and prove that the result of the previous section on the socle of the algebra of quotients of a Jordan algebra is properly understood through the notion

of local order, since the LC-ideal of a nondegenerate algebra turns out to be a local order in the socle of its algebra of quotients. As a final result in this section, we prove an analogue of a fact proved by Ánh and Márki [AM1], and show that local orders in algebras that equal their socles are indeed orders in the sense of [Mo3] thus showing that this kind of quotients can also be viewed through the framework of that construction.

In Section 5 we revisit the theory of Fernández-López and García-Rus developed in [FG1, FG2], on what they named local Goldie conditions, and their consequences. We add to that study the local LC-condition, and obtain here all those results as a natural application of the theory developed in the previous sections, thus obtaining the characterization of strongly prime and nondegenerate Jordan algebras that are local orders either in local artinian algebras, or in algebras satisfying the dcc on principal inner ideals.

0 Preliminaries

0.1 We will work with Jordan algebras over a unital commutative ring of scalars Φ which will be fixed throughout. We refer to [J2, McZ] for notation, terminology and basic results on Jordan algebras. We will occasionally make use some results on Jordan pairs, mainly obtained from algebras, for which we refer to [Lo1], and we will often rely on an associative background, both as an ingredient of Jordan theory when dealing with special Jordan algebras, and as a source of notions which have been extended to the Jordan algebra setting, among which we will mainly consider those from localization theory, for which we refer to [St].

We will make use of the identities proved in [J2] and [Lo1], which will be quoted with the labellings QJn and JPn respectively. In this section we recall some of those basic results and notations together with some other results that will be used in the paper.

0.2 A Jordan algebra has products $U_x y$ and x^2 , quadratic in x and linear in y , whose linearizations are $U_{x,z} y = V_{x,y} z = \{x, y, z\} = U_{x+zy} - U_x y - U_z y$ and $x \circ y = V_x y = (x + y)^2 - x^2 - y^2$.

We will denote by \widehat{J} the free unital hull $\widehat{J} = \Phi 1 \oplus J$ with products

$$U_{\alpha 1+x}(\beta 1+y) = \alpha^2 \beta 1 + \alpha^2 y + \alpha x \circ y + 2\alpha \beta x + \beta x^2 + U_x y \text{ and } (\alpha 1+x)^2 = \alpha^2 1 + 2\alpha x + x^2.$$

It is well known that any associative algebra A gives rise to a Jordan algebra A^+ with products $U_x y = xyx$ and $x^2 = xx$. A Jordan algebra is special if it is isomorphic to a subalgebra of an algebra A^+ for an associative A . If A has an involution $*$ then $H(A, *) = \{a \in A \mid a = a^*\}$ is a Jordan subalgebra of A^+ and so are ample subspaces $H_0(A, *)$ of symmetric elements of A , subspaces such that $a + a^*$, aa^* and aha^* are in $H_0(A, *)$ for all $a \in A$ and all $h \in H(A, *)$.

0.3 A Φ -submodule K of a Jordan algebra J is an *inner ideal* if $U_k \widehat{J} \subseteq K$ for all $k \in K$. An inner ideal $I \subseteq J$ is an *ideal* if $\{I, J, \widehat{J}\} + U_J I \subseteq I$, a fact that we will denote in the usual way $I \triangleleft J$. If I, L are ideals of J , so is their product $U_I L$ and in particular the *derived ideal* $I^{(1)} = U_I I$ of I . An (inner) ideal of J is *essential* if it has nonzero intersection with any nonzero (inner) ideal of J .

A Jordan algebra J is *nondegenerate* if $U_x J \neq 0$ for any nonzero $x \in J$, and *prime* if $U_I L \neq 0$ for any nonzero ideals I and L of J . The algebra J is said to be *strongly prime* if J is both nondegenerate and prime.

0.4 If $X \subseteq J$ is a subset of a Jordan algebra J , the *annihilator* of X in J is the set $\text{ann}_J(X)$ of all $z \in J$ such that $U_x z = U_z x = 0$ and $U_x U_z \widehat{J} = U_z U_x \widehat{J} = V_{x,z} \widehat{J} = V_{z,x} \widehat{J} = 0$ for all $x \in X$. The annihilator is always an inner ideal of J , and it is an ideal if X is an ideal. If J is a nondegenerate Jordan algebra and I is an ideal of J , then the annihilator of I in J can be characterized as follows ([Mc3, Mo2]):

$$\text{ann}_J(I) = \{z \in J \mid U_z I = 0\} = \{z \in J \mid U_I z = 0\}.$$

0.5 For any element a in a Jordan algebra J , the *local algebra* J_a of J at a is the quotient of the a -homotope $J^{(a)}$, defined over the Φ -module J with operations $U_x^{(a)} y = U_x U_a y$ and $x^{(2,a)} = U_x a$, by the ideal $\text{Ker } a$ of $J^{(a)}$ of all the elements $x \in J$ such that $U_a x = U_x a = 0$. If J is nondegenerate the above conditions on x reduce to $U_a x = 0$. Local algebras at nonzero elements of a nondegenerate (resp. strongly prime) Jordan algebra are nondegenerate (resp. strongly prime) [ACM, Theorem 4.1]. (We recall that

similar definitions can be given for associative algebras, for which we will also use the notation R_x for the local algebra at an element x of the R .)

0.6 A Jordan algebra or triple system J gives rise to its double Jordan pair $V(J) = (J, J)$, with (quadratic) operations obtained from the quadratic operation of J : $Q_xy = U_xy$ or P_xy . Reciprocally, every Jordan pair $V = (V^+, V^-)$ gives rise to a (polarized) triple system $T(V) = V^+ \oplus V^-$. If J is a triple system, it is obvious that if $I \triangleleft J$, then (I, I) is an ideal of $V(J)$, however, not all ideals of $V(J)$ arise in that way from ideals of J . In fact, if $I = (I^+, I^-)$ is an ideal of $V(J) = (J, J)$ we may well have $I^+ \neq I^-$, and even $I^+ \cap I^- = 0$, so that $I^+ \oplus I^-$ is a polarized ideal of J as a triple system. We however have the following result:

0.7 Lemma. *Let J be a nondegenerate Jordan algebra, if $I = I^+ \oplus I^-$ is a polarized ideal of J as a triple system (that is (I^+, I^-) is an ideal of $V(J)$ and $I^+ \cap I^- = 0$), then $I = 0$.*

Proof. Following the proof of [Mo2, Proposition 2.4], the set $I_{alg} = \{y \in I \mid y^2 + y \circ I \subseteq I\}$ is an ideal of J as an algebra which satisfies $U_I J \subseteq I_{alg} \subseteq I$. In particular, I_{alg} is still polarized, and $I_{alg} = 0$ if and only if $I = 0$. Therefore we can assume that $I = I^+ \oplus I^-$ is a polarized algebra ideal.

Take $x^\sigma \in I^\sigma$ for $\sigma = \pm$. Since $U_{I^\sigma} U_{I^\sigma} J = 0$, we have $U_{(x^\sigma)^2} J = U_{x^\sigma}^2 J = 0$, hence $(x^\sigma)^2 = 0$ by nondegeneracy of J . Denoting $a = x^+ \circ x^-$, for any $z \in J$ we have $U_z a = U_z(x^+ \circ x^-) = z \circ \{x^+, x^-, z\} - \{z^2, x^-, x^+\} \in J \circ \{I^+, I^-, J\} + \{J, I^-, I^+\} = 0$, since $\{I^\sigma, I^{-\sigma}, J\} \subseteq I^\sigma \cap I^{-\sigma} = 0$. Then $a \in \text{ann}_J(I)$ by the characterization of annihilators of ideals mentioned in 0.4. Since I is an algebra ideal, $a \in I$, hence $a \in I \cap \text{ann}_J(I) = 0$ by nondegeneracy of J . As a consequence of these equalities, we get $x^2 = (x^+ + x^-)^2 = (x^+)^2 + (x^-)^2 + x^+ \circ x^- = 0$, hence $I^2 = 0$.

Consider now a tight unital hull J' of J (a unital hull $J \triangleleft J' = J + \Phi 1$ which is tight over J , and therefore inherits nondegeneracy from J [McZ, 0.16, 0.17]). Since I is an ideal of J , it is also an ideal of J' , and the equality $I^2 = 0$ can be rewritten in J' as $U_I 1 = 0$, which again by the characterization of the annihilator mentioned in 0.4, implies $1 \in \text{ann}_{J'}(I)$, and thus $I = U_1 I = 0$. \square

0.8 The *socle* $Soc(J)$ of a nondegenerate Jordan algebra J is the sum of all minimal inner ideals of J . The socle of linear Jordan algebras has been studied by Osborn and Racine in [OR]. For Jordan pairs over arbitrary rings of scalars, the socle has been thoroughly studied by Loos in [Lo2]. Our handling of the socle will rely on that reference. In particular, it is proved there that the socle of a nondegenerate Jordan pair V is a direct sum of simple ideals of V , consists of regular elements, and satisfies the dcc on principal inner ideals [Lo4]. Applying these assertions to the equality $V(Soc(J)) = Soc(V(J))$ for a Jordan algebra J , gives that $Soc(J)$ consists of regular elements and satisfies the dcc on principal inner ideals. On the other hand, if $I = (I^+, I^-)$ is a simple ideal of $V(J)$, then $V(I^+ \cap I^-) \subseteq I$ is an ideal of $V(Soc(J))$, hence either $I = V(I^+ \cap I^-)$, which gives $I^+ = I^-$, and $I = V(L)$ for the simple ideal $L = I^+ = I^-$, or $I^+ \cap I^- = 0$. The latter case means that $I^+ + I^-$ is a polarized ideal of J , so it is the zero ideal by 0.7. As a consequence, $Soc(J)$ is a direct sum of simple ideals coinciding with their own socles.

The elements of the socle of a nondegenerate Jordan algebra J are exactly those whose local algebra J_x has finite capacity [Mo1, Lemma 0.7(b)]. From [Lo2] we also obtain that a nondegenerate Jordan algebra J satisfying the dcc on principal inner ideals (equivalently coinciding with its socle) also satisfies acc on the inner ideals $ann_J(x)$ for $x \in J$.

0.9 Let J be a Jordan algebra. We will say that J is *locally artinian* if for any $x \in J$, J_x is artinian. Note that if J is locally artinian, then each local algebra J_x has finite capacity, hence $J = Soc(J)$ by 0.8 (obviously, the analogous definition obtained for nondegenerate algebras by substituting 'artinian' by 'of finite capacity' in the definition of locally artinian gives nothing new, since as mentioned before, those are just the algebras that equal their socle).

It follows from the structure of inner ideals of Jordan algebras having finite capacity [Mc1] that simple Jordan algebras with finite capacity (equivalently with dcc on principal inner ideals) are either artinian or the Jordan algebra of a nondegenerate quadratic form containing an infinite dimensional totally isotropic vector subspace, in short, a *nonartinian quadratic factor*.

Every element in a simple Jordan algebra with dcc on principal inner ideals which is not a nonartinian quadratic factor has finite uniform dimension (in the sense of [FGM, p. 425], which we will explicitly recall later). Moreover such Jordan algebras are locally artinian, hence in particular, for each idempotent e in J , the unital Jordan algebra $U_e J = J_2(e) \cong J_e$ (see [LN, Example 1.12]) is simple and artinian.

0.10 Lemma. *For a nondegenerate Jordan algebra, the following facts are equivalent:*

- (i) J is locally artinian,
- (ii) $J = \text{Soc}(J)$, and J does not contain ideals that are nonartinian quadratic factors.
- (iii) J is a direct sum of simple Jordan algebras coinciding with their socles, none of which is a nonartinian quadratic factor.

Proof. (i) \Rightarrow (ii) As noted before, for an algebra J , being locally artinian implies $J = \text{Soc}(J)$. Moreover, if J contains an ideal I which is a nonartinian quadratic factor, then I is unital, and the local algebra of J at the unit element e of I is the nonartinian algebra $J_e \cong I$.

(ii) \Rightarrow (iii) That J is a direct sum of simple algebras coinciding with their socles is a consequence of the general result on the structure of the socle of a nondegenerate algebra 0.8. That none of these summands is a nonartinian quadratic factor stems directly from the fact that local algebras of the summands are local algebras of J itself, and local algebras at the unit element of a nonartinian quadratic factor are themselves nonartinian quadratic factors.

(iii) \Rightarrow (i) Local algebras of direct sums of simple Jordan algebras coinciding with their socles are Jordan algebras with finite capacity, so either they are artinian or they are nonartinian quadratic factors. This latter case can occur only if the local algebra is taken at an element of a direct summand which is itself a nonartinian quadratic factor, but this is ruled out by condition (iii). \square

1 Algebras of quotients

As mentioned in the introduction, the study of algebras of quotients of Jordan algebras draws its inspiration from associative theory (see [Mo3]). We recall next some basic notation from the latter and refer the reader to [Ro, St] for basic results about algebras of quotients for associative algebras.

1.1 Let L be a left ideal of an associative algebra R . Recall the usual notation (for instance, see [St]) $(L : a)$, with $a \in R$, for the set of all elements $x \in R$ such that $xa \in L$. A left ideal L of R is *dense* if $(L : a)b \neq 0$ for any $a \in R$ and any nonzero $b \in R$.

1.2 The associative algebras naturally arising in Jordan theory are associative envelopes of Jordan algebras, and therefore carry an involution. That makes important to be able to extend involutions to their algebras of quotients. The fact that this is not always possible for the one-sided maximal algebras of quotients $Q_{max}^r(R)$ and $Q_{max}^l(R)$ leads to the use of the maximal symmetric algebra of quotients $Q_\sigma(R)$ (see [L]) as an adequate substitute of that algebra. Recall that $Q_\sigma(R)$ is the set of elements $q \in Q_{max}^r(R)$ for which there exists a dense left ideal L of R such that $Lq \subseteq R$ (which up to a canonical isomorphism can be viewed symmetrically as the set of all $q \in Q_{max}^l(R)$ for which there exists a dense right ideal K of R such that $qK \subseteq R$). If R has an involution, $Q_\sigma(R)$ is the biggest subalgebra of $Q_{max}^r(R)$ and $Q_{max}^l(R)$ to which that involution extends.

1.3 Let J be a Jordan algebra, K be an inner ideal of J and $a \in J$. Following [Mo3, MoP] we set

$$(K : a)_L = \{x \in K \mid x \circ a \in K\},$$

$$(K : a) = \{x \in K \mid U_a x, x \circ a \in K\}.$$

It is straightforward to check that both $(K : a)_L$ and $(K : a)$ are inner ideals of J for all $a \in J$, and that in addition, the containment $U_{(K : a)_L} K \subseteq (K : a)$ holds [Mo3, Lemma 1.2]. Given any finite family of elements $a_1, \dots, a_n \in J$, we inductively define $(K : a_1 : a_2 : \dots : a_n) = ((K : a_1 : \dots : a_{n-1}) : a_n)$.

1.4 An inner ideal K of J is said to be *dense* if $U_c(K : a_1 : a_2 : \dots : a_n) \neq 0$

for any finite collection of elements $a_1, \dots, a_n \in J$ and any $0 \neq c \in J$. Different characterizations of density are given in [Mo3, Proposition 1.9]. Recall that if K is a dense inner ideal of J so are the inner ideals $(K : a)$ for all $a \in J$ [Mo3, Lemma 1.8].

1.5 Let \tilde{J} be a Jordan algebra, J be a subalgebra of \tilde{J} , and $\tilde{a} \in \tilde{J}$. Recall from [Mo2] that an element $x \in J$ is a J -denominator of \tilde{a} if the following multiplications take \tilde{a} back into J :

$$\begin{array}{lll} \text{(Di)} U_x \tilde{a} & \text{(Dii)} U_{\tilde{a}} x & \text{(Diii)} U_{\tilde{a}} U_x \hat{J} \\ \text{(Diii')} U_x U_{\tilde{a}} \hat{J} & \text{(Div)} V_{x, \tilde{a}} \hat{J} & \text{(Div')} V_{\tilde{a}, x} \hat{J} \end{array}$$

We will denote the set of J -denominators of \tilde{a} by $\mathcal{D}_J(\tilde{a})$. It has been proved in [Mo2] that $\mathcal{D}_J(\tilde{a})$ is an inner ideal of J . Recall also from [FGM, p. 410] that any $x \in J$ satisfying (Di), (Dii), (Diii) and (Div) belongs to $\mathcal{D}_J(\tilde{a})$. The following procedure for obtaining denominators is given in [FGM, Lemma 2.2], and has the advantage of being "context free", that is of not depending of the overalgebra \tilde{J} : for any $x \in J$, the containments $x \circ \tilde{a}, U_x \tilde{a} \in J$ imply $x^4 \in \mathcal{D}_J(\tilde{a})$.

1.6 Let J be a subalgebra of a Jordan algebra Q . Following [Mo3], we say that Q is a *general algebra of quotients of J* if the following conditions hold:

(AQ1) $\mathcal{D}_J(q)$ is a dense inner ideal of J for all $q \in Q$.

(AQ2) $U_q \mathcal{D}_J(q) \neq 0$ for any nonzero $q \in Q$.

Note that any nondegenerate Jordan algebra is its own algebra of quotients. Conversely any Jordan algebra having an algebra of quotients is nondegenerate.

A different, though closely related approach to algebras of quotients was carried out in [MoP] by using essential inner ideals as sets of denominators. That second approach, which in fact motivated and inspired the one in [Mo3] as well as some other treatments of algebras of quotients in Jordan algebras (see the references in [Mo3]), has the advantage that checking essentiality is significantly simpler than checking density. However, for that choice to be feasible, the additional condition of strong nonsingularity, introduced in

[MoP], is needed. We will comment on that at the end of the present section.

1.7 An algebra of quotients Q of a Jordan algebra J is said to be a *maximal algebra of quotients* of J if for any other algebra of quotients Q' of J there exists a homomorphism $\alpha : Q' \rightarrow Q$ whose restriction to J is the identity map.

If there exists, a maximal algebra of quotients of a nondegenerate Jordan algebra is easily seen to be unique up to an isomorphism fixing the subalgebra J . The existence of maximal algebras of quotients of nondegenerate Jordan algebras was proved in [Mo3, Theorem 5.8]. We denote by $Q_{max}(J)$ the maximal algebra of quotients of a nondegenerate Jordan algebra J .

1.8 Theorem. *Any nondegenerate Jordan algebra J has a maximal algebra of quotients $Q_{max}(J)$.*

We refer to [Mo3, Theorem 3.11 and Theorem 4.10] for the explicit description of the maximal algebra of quotients of a nondegenerate Jordan algebra. We also recall the straightforward fact that maximal algebras of quotients of nondegenerate Jordan algebras are unital [Mo3, Remark 5.9].

As mentioned in the introduction, and as was the case in the associative setting, the study of general algebras of quotients of Jordan algebras was originated in the study of algebras of fractions. We next recall some of the facts concerning these.

1.9 A nonempty subset $S \subseteq J$ is a *monad* if $U_s t$ and s^2 are in S for all $s, t \in S$. A subalgebra J of a unital Jordan algebra Q is an *S -order in Q* or equivalently Q is a *S -algebra of quotients, or an algebra of fractions (of J relative to S)* if:

(ClQ1) every element $s \in S$ is invertible in Q .

(ClQ2) each $q \in Q$ has a J -denominator in S .

(ClQ3) for all $s, t \in S$, $U_s S \cap U_t S \neq \emptyset$.

1.10 An element s of a Jordan algebra J is said to be *injective* if the mapping U_s is injective over J . Following [FGM] we will denote by $Inj(J)$

the set of injective elements of J .

1.11 A Jordan algebra Q containing J as a subalgebra is a *classical algebra of quotients of J* or an *algebra of fractions of J* (and J is a *classical order in Q*) if all injective elements of J are invertible in Q and for all $q \in Q$, $\mathcal{D}_J(q) \cap \text{Inj}(J) \neq \emptyset$. In other words, classical algebras of quotients are S -algebras of quotients (or algebras of fractions relative to S) for $S = \text{Inj}(J)$. Moreover, they are general algebras of quotients (in the sense of 1.6, as usual) [Mo3, Examples 2.3.5].

1.12 The proximity of a nondegenerate algebra and its algebras of quotients can be expressed through the following notion introduced in [FGM, p. 414], which includes that of classical algebras of quotients as a particular case: Let $J \leq \tilde{J}$ be Jordan algebras. An over-algebra \tilde{J} is said to be an *innerly tight extension* of J if

- $U_{\tilde{a}}J \cap J \neq \emptyset$ for any $0 \neq \tilde{a} \in \tilde{J}$, and
- $\mathcal{D}_J(\tilde{a})$ is an essential inner ideal of J for any $\tilde{a} \in \tilde{J}$.

By [Mo3, Lemma 2.4], an algebra of quotients of a nondegenerate Jordan algebra is an innerly tight extension. As for the reciprocal, a partial result follows from [FGM, Proposition 2.10]: unital innerly tight extensions with finite capacity are classical algebras of quotients.

1.13 Let J be a Jordan algebra. We follow [FGM] for the next definitions that will be used below, in the statement of the Goldie theorem for Jordan algebras.

– For a subset $X \subseteq J$, denote by $[X]_J$ the inner ideal of J generated by X . A family $\{K_i\}_{i \in I}$ of nonzero inner ideals of J forms a *direct sum* if $K_i \cap [\sum_{j \neq i} K_j]_J = 0$ for each $i \in I$. Following [FGM, p. 426], we say that a Jordan algebra J satisfies the *acc*(\oplus) if it does not have infinite families of nonzero inner ideals that form a direct sum. In analogy with the corresponding notion in associative theory (see [Ro, p. 361] or, under the name of finite right rank, [St, II.2]), the *uniform (or Goldie) dimension* $u\dim(J)$ of a Jordan algebra J is defined as the supremum of the $n \geq 1$ such that there are K_1, \dots, K_n nonzero inner ideals of J which form a direct sum

(including the possibility that the set of such numbers n is not bounded, in which case J will be said to have *infinite uniform dimension*).

As for its associative counterpart, and in accordance with the notation used in [FGM, Lemma 5.4], for an associative algebra R , we will denote respectively by $u_l \dim(R)$ and $u_r \dim(R)$ the left and right uniform dimensions of R . If $x \in R$ we put $u_l \dim(x) = u_l \dim(R_x)$ and $u_r \dim(x) = u_r \dim(R_x)$. (Note that $u_l \dim(R_x)$ coincides with the uniform dimension of R_x as a left R -module, and similarly with the "right" instead of the "left" version.)

If a nondegenerate Jordan algebra J satisfies the $\text{acc}(\oplus)$ if and only if it has finite uniform dimension [FGM, Proposition 7.6].

–The *singular set* of a Jordan algebra J is

$$\Theta(J) = \{x \in J \mid \text{ann}_J(x) \text{ is an essential inner ideal of } J\}.$$

If J is nondegenerate then $\Theta(J)$ is an ideal of J [FGM, Theorem 6.1], and J is *nonsingular* if $\Theta(J) = 0$.

– A nonzero element $u \in J$ is *uniform* if $\text{ann}_J(u) = \text{ann}_J(x)$ for any nonzero $0 \neq x \in U_u \widehat{J}$. It is straightforward that $\text{ann}_J(u) \subseteq \text{ann}_J(x)$ for every $x \in U_u \widehat{J}$, hence every nonzero element $u \in J$ with maximal annihilator is uniform. If J satisfies the acc on annihilators of its elements, every nonzero inner ideal of J contains a uniform element [FG2, p.55]. Uniform elements of nondegenerate Jordan algebras can be characterized through their local algebras. Indeed, by [FGM, Proposition 8.4], a nonzero element of a nondegenerate Jordan algebra is uniform if and only if the local algebra at that element is a Jordan domain.

1.14 A nondegenerate Jordan algebra is *Goldie* if it satisfies the acc on annihilators and has no infinite direct sum of inner ideals. Different equivalent characterizations of Goldie Jordan algebras are given in [FGM, Theorem 9.3], among them we select the following:

1.15 Theorem. [FGM, Theorem 9.3] *For a Jordan algebra J the following conditions are equivalent:*

(i) J is a classical order in a nondegenerate artinian Jordan algebra Q .

(ii) J is nondegenerate, satisfies the acc on the annihilators of its elements and has finite uniform dimension.

(iii) J is nondegenerate, any nonzero ideal of J contains a uniform element, and J has finite uniform dimension.

(iv) J is nondegenerate, nonsingular and has finite uniform dimension.

Moreover, Q is simple if and only if J is strongly prime.

The study in [FGM] of Jordan algebras that are classical orders in semisimple artinian algebras (that is nondegenerate Goldie Jordan algebras) extends to the study of a wider class of algebras, those which are classical orders in nondegenerate unital Jordan algebras of finite capacity. The main result on those is the following:

1.16 Theorem. [FGM, Theorem 10.2] *A Jordan algebra J is a classical order in a nondegenerate unital Jordan algebra Q with finite capacity if and only if it is nondegenerate and satisfies the following property: An inner ideal K of J is essential if and only if K contains an injective element. Moreover, Q is simple if and only if J is prime.*

1.17 Following [MoT1, MoT2], a Jordan algebra J satisfying the above equivalent properties will be called a *Lesieur-Croisot Jordan algebra* or an *LC Jordan algebra*, for short.

The set $LC(J)$ of elements $x \in J$ at which the local algebra J_x is LC will be one of our main concerns in our development of a local Goldie theory for Jordan algebras based on the ideas of [FG1] and [FG2]. We recall here that if J is nondegenerate, this set is an ideal of J [MoT1, Theorem 5.13]. The next to sections will be devoted to the study of that ideal.

1.18 We have already mentioned in 1.6 the possibility of developing a version of the theory of algebras of quotients based on essential inner ideals instead of on dense inner ideals following [MoP]. That requires the following version of nonsingularity introduced in [MoP]: a Jordan algebra J is *strongly nonsingular* if for any essential inner ideal K of J , and any $a \in J$, the equality $U_a K = 0$ implies $a = 0$. Following [P], an element $z \in J$ will be

called an *essential zero divisor* if there exists an essential inner ideal K of J such that $U_z K = 0$. Therefore, a Jordan algebra is strongly nonsingular if it does not have nonzero essential zero divisors. In that case, the theories developed in [Mo3] and [MoP] coincide since by [Mo3, Lemma 1.18 (b)], a Jordan algebra J is strongly nonsingular if and only if any essential inner ideal of J is dense.

Essential zero divisors can be gathered in an analogue of the singular ideal 1.13 whose vanishing will imply strong nonsingularity. Again following [P], we denote by $\mathcal{Z}_{ess}(J)$ the linear span of all essential zero divisors of J .

1.19 Proposition. *For any nondegenerate Jordan algebra J , the set $\mathcal{Z}_{ess}(J)$ is an ideal.*

Proof. We first prove that if z is an essential zero divisor, then the inner ideal $(z) = \Phi z + U_z \hat{J}$ generated by z is contained in $\mathcal{Z}_{ess}(J)$. Indeed, if $U_z K = 0$ for some essential inner ideal K , and $u = \alpha z + U_z a \in (z)$, we have $U_u K \subseteq \alpha^2 U_z K + \alpha \{z, K, U_z a\} + U_z U_a U_z K = 0 + \{U_z K, a, z\} + 0 = 0$, hence $u \in \mathcal{Z}_{ess}(J)$.

Recall that a mapping $S : J \rightarrow J$ is a structural transformation if there exists $S^* : J \rightarrow J$ such that $U_{Sx} = S U_x S^*$ for any $x \in J$. For $x \in J$, the mappings U_x are structural transformations, as are the mappings $B_x = Id - V_x + U_x$ (equal to U_{1-x} in a unital J). Since $V_x = Id + B_x + U_x$, and the mapping $a \mapsto \{x, y, a\}$ is $V_x V_y - U_{x \circ y} + U_x + U_y$, [Mo2, Lemma 4.4] (or from [J2, Proposition 4.1.6] applied to the unital \hat{J}) implies that the ideal generated by $\mathcal{Z}_{ess}(J)$ consists of the elements which are sums of elements of the form $W_1 \cdots W_n z$, for $W_i = U_{a_i}$ or B_{a_i} , and an essential zero divisor z .

Now, if z is an essential zero divisor with $U_z K = 0$ for the essential inner ideal K , for any $a \in J$,

$$U_{U_a z}(K : a) = U_a U_z U_a(K : a) \subseteq U_a U_z K = 0$$

and

$$\begin{aligned} U_{B_a z}(K : a) &= B_a U_z B_a(K : a) \subseteq \\ &\subseteq B_a(U_z(K : a) + U_z(a \circ (K : a)) + U_z U_a(K : a)) \\ &\subseteq U_a U_z K = 0. \end{aligned}$$

Therefore both elements $U_a z$ and $B_a z$ are essential zero divisors, and thus any $W_1 \cdots W_n z$ as above is an essential zero divisor, which proves that $\mathcal{Z}_{ess}(J)$ is an ideal of J . \square

1.20 Lemma. *Let J be a nondegenerate Jordan algebra. For any $a \in J$, the following containment holds:*

$$(\mathcal{Z}_{ess}(J) + Ker a) / Ker a \subseteq \mathcal{Z}_{ess}(J_a),$$

and therefore, $\mathcal{Z}_{ess}(J_a) = 0$ implies $a \in ann_J(\mathcal{Z}(J))$.

Proof. Let $z \in J$ be an essential zero divisor, so that $U_z K = 0$ for an essential inner ideal K of J . Set $M = \{x \in J \mid U_a x \in K\}$, which is the preimage of $\bar{K} = (K + Ker a) / Ker a$ by the natural projection $J = J^{(a)} \rightarrow J_a, y \mapsto \bar{y} := y + Ker a$. Clearly, since the set M is the preimage of an inner ideal by that homomorphism, it is an inner ideal of the homotope $J^{(a)}$.

Now, let $\bar{N} = N / Ker a$ be a nonzero ideal of J_a , where $Ker a \subseteq N$ is a nonzero ideal of $J^{(a)}$. Then $U_a N$ is a nonzero inner ideal of J [DM], hence by essentiality of K , there exists a nonzero $k = U_a x \in U_a N \cap K$, for some $0 \neq x \in N$. Therefore $0 \neq \bar{x} \in \bar{N} \cap \bar{M}$, which proves that \bar{M} is essential.

Finally, $U_{\bar{z}} \bar{M} = \overline{U_z^{(a)} M} = \overline{U_z U_a M} \subseteq \overline{U_z K} = 0$, hence $\bar{z} \in \mathcal{Z}_{ess}(J_a)$, and thus $\overline{\mathcal{Z}_{ess}(J)} \subseteq \mathcal{Z}_{ess}(J_a) = 0$, so we get the containment $\overline{\mathcal{Z}_{ess}(J)} \subseteq \mathcal{Z}_{ess}(J_a)$, and then $\overline{\mathcal{Z}_{ess}(J)} = 0$ implies $U_a \mathcal{Z}_{ess}(J) = 0$, hence $a \in ann_J(\mathcal{Z}_{ess}(J))$. \square

1.21 Proposition. *If J is a nondegenerate Jordan algebra, then $LC(J) \subseteq ann_J(\mathcal{Z}_{ess}(J))$. Therefore, if $LC(J)$ is an essential ideal, then J is strongly nonsingular.*

Proof. The second assertion follows directly from the first one since the containment $LC(J) \subseteq ann_J(\mathcal{Z}_{ess}(J))$ implies $\mathcal{Z}_{ess}(J) \subseteq ann_J(LC(J))$. As for the first assertion, it suffices to prove that for any $a \in LC(J)$, J_a is

strongly nonsingular, according to 1.20, and this follows from the fact that if a Jordan algebra J is LC, then it is strongly nonsingular. Indeed, if J is LC, and $z \in J$ has $U_z K = 0$ for an essential inner ideal K , then K contains an injective element s , and $U_{U_s z} J = U_s U_z U_s J \subseteq U_s U_z K = 0$, hence $U_s z = 0$ by nondegeneracy of J , and thus $z = 0$ since s is injective. \square

1.22 Corollary. *Let J be a nondegenerate Jordan algebra. If $J = LC(J)$ then an inner ideal of J is essential if and only if it is dense.*

Proof. This is straightforward from 1.21 and the coincidence of essentiality and density for inner ideals in strongly nonsingular algebras 1.18. \square

2 Strongly prime Jordan algebras with nonzero LC-elements.

In this section we consider nonzero LC-elements of strongly prime Jordan algebras and prove that such elements are exactly those elements of the Jordan algebra lying in the socle of its maximal algebra of quotients.

2.1 Let J be a strongly prime Jordan algebra. If J has nonzero PI-elements (i.e. nonzero elements at which the local algebra is a PI-algebra) then $LC(J) = PI(J)$ [MoT1, Proposition 3.3]. Otherwise, if $PI(J) = 0$ (a situation that from now on, following [FGM], we will refer to by saying that J is *PI-less*), J is special of hermitian type [FGM, Lemma 5.1] and given a $*$ -tight associative envelope R of J , the set of LC-element coincides with the set of elements of J having finite uniform dimension, more precisely, $LC(J) = F(J) = I_l(R) \cap J = I_r(R) \cap J$, where $F(J) = \{x \in J \mid udim(x) < \infty\}$, $I_l(R) = \{x \in R \mid u_l dim(x) < \infty\}$, and $I_r(R) = \{x \in R \mid u_r dim(x) < \infty\}$ [MoT1, Theorem 4.4].

A consequence of the inheritance of density of inner ideals by their projections in local algebras proved in [Mo3], is that algebras of quotients interact well with taking local algebras:

2.2 Lemma. *Let Q be a general algebra of quotients of a Jordan algebra J .*

For any $x \in J$, Q_x is a general algebra of quotients of J_x .

Proof. Let $x \in J$. Then, by 1.6, J is nondegenerate and we have $\text{Ker}_J x = \text{Ker}_{Q_x} x \cap J$. Therefore Q_x contains the subalgebra $(J^{(x)} + \text{Ker}_{Q_x} x) / \text{Ker}_{Q_x} x$ isomorphic to J_x . We denote with bars the projections in both $\overline{Q} = Q_x$ and $\overline{J} = J_x$.

Take $q \in Q$. It follows from 1.4 and [Mo3, Lemma 1.20] that $\overline{(\mathcal{D}_J(q) : x)}$ is a dense inner ideal of \overline{J} . We will prove that $\mathcal{D}_{\overline{J}}(\overline{q})$ is dense in \overline{J} by checking the containment $\overline{(\mathcal{D}_J(q) : x)} \subseteq \mathcal{D}_{\overline{J}}(\overline{q})$.

Take $a \in (\mathcal{D}_J(q) : x)$. By QJ16 we have

$$\begin{aligned} U_a U_x q &= -U_x U_a q - V_a U_x V_a q + U_{a \circ x} q + U_{U_a x, x} q \in \\ &\in U_x U_{\mathcal{D}_J(q)} q + V_a U_x (\mathcal{D}_J(q) \circ q) + U_{\mathcal{D}_J(q)} q + \{\mathcal{D}_J(q), q, J\} \subseteq J, \end{aligned}$$

hence $U_a \overline{q} = \overline{U_a U_x q} \in \overline{J}$. On the other hand

$$\begin{aligned} U_{\overline{q}} \overline{a} &= \overline{U_q U_x a} \in \overline{U_q U_x (\mathcal{D}_J(q) : x)} \subseteq \overline{U_q \mathcal{D}_J(q)} \subseteq \overline{J}, \\ U_{\overline{q}} U_{\overline{a}} \overline{J} &= \overline{U_q U_x U_a U_x J} = \overline{U_q U_{U_a x} J} \subseteq \overline{U_q U_{\mathcal{D}_J(q)} J} \subseteq \overline{J}, \end{aligned}$$

and

$$V_{\overline{q}, \overline{a}} \overline{J} = \{\overline{q}, \overline{a}, \overline{J}\} = \overline{\{q, U_x a, J\}} \subseteq \overline{\{q, U_x (\mathcal{D}_J(q) : x), J\}} \subseteq \overline{\{q, \mathcal{D}_J(q), J\}} \subseteq \overline{J}$$

which implies that $\overline{(\mathcal{D}_J(q) : x)} \subseteq \mathcal{D}_{\overline{J}}(\overline{q})$ by 1.5, and thus $\mathcal{D}_{\overline{J}}(\overline{q})$ is a dense inner ideal of \overline{J} .

Finally, if $U_{\overline{q}} \mathcal{D}_{\overline{J}}(\overline{q}) = \overline{0}$, then $U_x U_q U_x (\mathcal{D}_J(q) : x) = 0$ which implies that $U_x q = 0$ by the density of $\mathcal{D}_J(q)$ in J . Hence $\overline{q} = \overline{0}$ and therefore \overline{Q} is a general algebra of quotients of \overline{J} . \square

As noticed in the previous section classical algebras of quotients are general algebras of quotients. The converse holds for unital algebras with finite capacity.

2.3 Lemma. *Let J be a subalgebra of a unital Jordan algebra Q of finite capacity. Then the following assertions are equivalent:*

- (i) Q is an algebra of quotients of J .

(ii) J is a classical order in Q .

Moreover, under the above conditions both J and Q are nondegenerate.

Proof. Suppose that Q is an algebra of quotients of J . Since dense inner ideals are, in particular, essential [Mo3, Lemma 1.18 a], Q is an innerly tight extension of J (see 1.12) hence, by [FGM, Proposition 2.10], J is a classical order in Q . Conversely, if J is a classical order in Q , then (i) follows from [Mo3, Examples 2.3.5] and the fact that Q is generated by J and the inverses in Q of the elements of $\text{Inj}(J)$ since J is a classical order in Q .

The nondegeneracy of J is a straightforward consequence of the above equivalent conditions, and that of Q follows from [Mo3, Lemma 2.4(1)] if Q is an algebra of quotients of J , and from [FGM, Proposition 2.9(vi)] if J is a classical order in Q . \square

To prove the next proposition we need to recall a fact included in the proof of [MoT1, Theorem 4.4]. Specifically, we recall the following result which first appeared in [Mo1, Theorem 6.5].

2.4 Lemma. [Mo1, Theorem 6.5] *Let J be a PI-less strongly prime Jordan algebra and R a $*$ -tight associative envelope of J . Then, for each $a \in J$, the subalgebra S of J generated by $\mathcal{H}(J)^{(1)}$, where $\mathcal{H}(J) \neq 0$ for some hermitian ideal $\mathcal{H}(X)$ of $FSJ(X)$, and the element a , is strongly prime of hermitian type. Moreover, $A = \text{alg}_R(S)$ is a $*$ -tight associative envelope of S , and $S = H_0(A, *)$ is ample in A .*

2.5 Proposition. *Let J be a PI-less strongly prime Jordan algebra and let R a $*$ -tight associative $*$ -envelope of J . If $LC(J) \neq 0$, then R_a is Goldie for any nonzero $a \in LC(J)$.*

Proof. Let a be a nonzero LC-element of J . By [MoT1, Proposition 4.2], J is nonsingular and J_a has finite uniform dimension, so that by [MoT1, Theorem 4.4], R_a has finite right and left uniform dimension, and thus for R_a to be Goldie it suffices that R_a be right and left nonsingular.

Since J is PI-less, J is special of hermitian type [FGM, Lemma 5.1]. Let $\mathcal{H}(X)$ be an hermitian ideal of $FSJ(X)$ such that $\mathcal{H}(J) \neq 0$ and then

consider the subalgebra S of J generated by the ideal $I = \mathcal{H}(J)^{(1)}$ and the element a . By 2.4 S is a strongly prime Jordan algebra and $A = \text{alg}_R(S)$ is a $*$ -tight associative envelope of S .

Clearly, the local algebra S_a contains a nonzero ideal (hence an essential ideal) which is isomorphic to the nonzero ideal $(I + \text{Ker } J_a)/\text{Ker } J_a$ of the strongly prime Jordan algebra J_a . Since $LC(J) \neq 0$, J is nonsingular [MoT1, Proposition 4.2], and so is the local algebra J_a [MoT1, Lemma 4.1]. Therefore S_a is nonsingular by [FGM, Proposition 6.2]. Moreover we have $\text{udim}(J_a) = \text{udim}(S_a)$, which implies that S_a has finite uniform dimension.

Next we claim that $\text{alg}_{A_a}(S_a)$, the $*$ -subalgebra of A_a generated by S_a , is $*$ -tight over S_a and that $\text{alg}_{A_a}(S_a)$ contains a nonzero $*$ -ideal of A_a . This claim is proved in [MoT1, Theorem 4.4]. Now since $\text{alg}_{A_a}(S_a)$ is $*$ -tight over S_a and S_a has finite uniform dimension, the nonsingularity of $\text{alg}_{A_a}(S_a)$ follows from that of S_a by [FGM, Theorem 7.17]. Finally since $\text{alg}_{A_a}(S_a) \subseteq A_a \subseteq R_a \subseteq Q_s(A_a)$, where $Q_s(A_a)$ denotes the algebra of symmetric Martindale ring of quotients of A_a and $\text{alg}_{A_a}(S_a)$ contains a nonzero $*$ -ideal of A_a , we have that R_a is nonsingular, hence R_a is Goldie. \square

2.6 Lemma. *Let R be a semiprime associative algebra. Let $a \in R$ be such that R_a is Goldie. Then for any $s \in R$ such that $\bar{s} \in \text{Reg}(R_a)$:*

- (i) $L(as) = Ras + l_R(a)$, where $l_R(a)$ denotes the left annihilator of a in R , is a dense left ideal of R .
- (ii) The pair $(L(as), f_s)$, where the map $f_s : L(as) \rightarrow R$ is given by $f_s(xas + z) = xa$, for all $x \in R$ and $z \in l_R(a)$, defines an element in $Q_{max}^l(R)$, the maximal left algebra of quotients of R .

Proof. Let $a \in R$ be such that R_a is Goldie and take $s \in R$ whose projection \bar{s} in R_a is regular in R_a .

(i) $L(as)$ is clearly a left ideal of R . To prove its density, take $b, c \in R$ and assume that $(L(as) : b)c = 0$. Take $\bar{x} \in (R_a\bar{s} : \bar{b})$. Then we have $\overline{xab} = \bar{x}\bar{b} \in R_a\bar{s} = \overline{Ras}$, thus there is $y \in R$ such that $axaba = ayasa$, hence $axab = ayas - z$ for some $z \in l_R(a)$ and $axab \in Ras + l_R(a) = L(as)$. Therefore, for all $\bar{x} \in (R_a\bar{s} : \bar{b})$ we have $axa \in (L(as) : b)$ and thus $(L(as) :$

$b)c = 0$ implies that $(R_a\bar{s} : \bar{b})\bar{c} = \bar{0}$. But since R_a is Goldie, $R_a\bar{s}$ is dense (by the regularity of \bar{s} in R_a) and we get $\bar{c} = \bar{0}$, or equivalently, $aca = 0$.

Now, for any $r, t \in R$. we have $(L(as) : rb)rct \subseteq (L(as) : b)ct = 0$, and as above, we get $arcta = 0$. Since $r, t \in R$ are arbitrary, we get $aRcRa = 0$, hence $(RcRa)R(RcRa) = 0$, and from the semiprimeness of R , it follows that the ideal RcR is contained in $l_R(a)$. Then $RcR \subseteq (L(as) : b)$, and we have $RcRc = 0$ since $(L(as) : b)c = 0$, which again by semiprimeness of R implies $c = 0$. Thus, $L(as)$ is a dense left ideal of R .

(ii) We claim that the pair $(L(as), f_s)$ defines an element of $Q_{max}^l(R)$, so we begin by proving that f_s is a well-defined homomorphism of left R -modules. To that end, we first note that $Ras \cap l_R(a) = 0$. Indeed, if $xas \in l_R(a)$ with $x \in R$, for all $r \in R$ we have $rxasa = 0$, hence $arxasa = 0$, and thus $\overline{(rx)\bar{s}} = \bar{0}$ in R_a . But since \bar{s} is regular, then $\overline{rx} = \bar{0}$. Therefore $aRxa = 0$, and $xa = 0$ by semiprimeness of R , hence $xas = 0$.

Thus, to prove that f_s is well-defined it suffices to check that if $xas = 0$ for some $x \in R$, then $xa = 0$. But this can be proved as before. Thereby, f_s is well-defined, and since it is clearly a homomorphism of left R -modules, from the density of $L(as)$ in R it follows that $(L(as), f_s)$ defines an element of $Q_{max}^l(R)$. \square

2.7 Let us denote by q_s the element of $Q_{max}^l(R)$ determined by the pair $(L(as), f_s)$ described above. It is straightforward to check that q_s does indeed belong to the maximal symmetric algebra of quotients $Q_\sigma(R)$ of R .

2.8 Lemma. *Let R be a semiprime associative algebra and $Q_\sigma(R)$ be its maximal symmetric algebra of quotients. For any $a \in R$ such that R_a is Goldie, and any $s \in R$ with $\bar{s} \in \text{Reg}(R_a)$, the pair $(L(as), f_s)$ defines an element of $Q_\sigma(R)$.*

Proof. Let $K(sa) = saR + r_R(a)$, where $r_R(a)$ is the right annihilator of a in R . Following 2.6 we can prove that $K(sa)$ is a dense right ideal of R . We claim that $q_s K(sa) \subseteq R$.

Take $sax + u \in K(sa)$ with $x \in R$ and $u \in r_R(a)$. Then, for any $y \in R$ and $z \in l_R(a)$, we have $(yas + z)q_s(sax + u) = ya(sax + u) = yasax =$

$(yas + z)ax$, hence $L(as)(q_s(sax + u) - ax) = 0$ which by the density of $L(as)$ implies that $q_s(sax + u) = ax$. Hence $q_sK(sa) \subseteq R$, and it follows from [L] that q_s does indeed belong to $Q_\sigma(R)$. \square

2.9 Proposition. *Let R be a semiprime associative algebra with maximal symmetric algebra of quotients $Q_\sigma(R)$. For any $a \in R$ such that R_a is Goldie we have $a \in \text{Soc}(Q_\sigma(R))$.*

Proof. Take $a \in R$ and assume that R_a is Goldie. Take $s \in R$ with $\bar{s} \in \text{Reg}(R_a)$ and consider $q_s \in Q_\sigma(R)$ as defined in 2.7. Clearly $asq_s = q_ssa = a$ and $l_R(a)q_s = q_sr_R(a) = 0$. Since $\overline{sas} = \bar{s}\bar{s} \in \text{Reg}(R_a)$, the element q_{sas} is also defined. We claim that $u = sq_{sas}s$ satisfies $aua = a$. Indeed since $asasq_{sas} = a$, for all $x \in R$ and $z \in l_R(a)$ we have $(xas + z)(asq_{sas}sa - a) = xasa - xasa = 0$, hence $L(as)(aua - a) = 0$ which implies $aua = a$ by the density of $L(as)$ in R . As a result we get that $Q_\sigma(R)_a$ is unital with unit \bar{u} .

Now since $Q_\sigma(R)_a$ is an algebra of quotients of R_a , we have $Q_\sigma(R)_a \subseteq Q_{\max}(R_a)$ (here $Q_{\max}(R_a)$ stands for any the left or right maximal algebra of quotients of R_a). But R_a being Goldie, $Q_{\max}(R_a)$ is the algebra of $\text{Reg}(R_a)$ -fractions of R_a , hence it is generated by R_a and the inverses of the elements of $\text{Reg}(R_a)$. Thus, to ensure that $Q_\sigma(R)_a = Q_{\max}(R_a)$ it is enough to prove that every element of $\text{Reg}(R_a)$ is invertible in $Q_\sigma(R)_a$. Take $\bar{s} \in \text{Reg}(R_a)$ and set $p = sq_{sas}u$ with $u = sq_{sas}s$. Then $asapa = asasq_{sas}ua = auu$ and so $\bar{s}\bar{p} = \bar{u}$ in $Q_\sigma(R)_a$. Hence $\bar{p} = \bar{s}^{-1}$. Therefore $Q_\sigma(R)_a = Q_{\max}(R_a)$, and since R_a is Goldie, $Q_\sigma(R)_a = Q_{\max}(R_a)$ is artinian (and semiprime) which implies that $a \in \text{Soc}(Q_\sigma(R))$. \square

2.10 Theorem. *Let J be a strongly prime Jordan algebra. Then $LC(J) = \text{Soc}(Q_{\max}(J)) \cap J$.*

Proof. If J has nonzero PI-elements this follows directly from [Mo2, Theorem 5.1] since by [MoT1, Proposition 3.3] we have $LC(J) = PI(J)$, and by [MoP, Lemma 3.1(c), Theorem 3.5] we have $Q_{\max}(J) = \Gamma^{-1}(J)J$. Therefore we can assume that J has no nonzero PI-elements. As a consequence, J is special [FGM, Lemma 5.1], and we can fix a $*$ -tight associative envelope R of J .

Let $a \in LC(J)$. By 2.5, R_a is Goldie, and therefore, by 2.9, $a \in Soc(Q_\sigma(R))$. Now since by [Mo3, Theorem 4.10], $Q_{max}(J) = H_0(Q_\sigma(R), *)$, by [FT, Proposition 4.1(2)] we have $a \in Soc(Q_\sigma(R)) \cap J = Soc(Q_\sigma(R)) \cap Q_{max}(J) \cap J = Soc(Q_\sigma(R)) \cap H_0(Q_\sigma(R), *) \cap J = Soc(H_0(Q_\sigma(R), *)) \cap J = Soc(Q_{max}(J)) \cap J$.

Conversely take $a \in Soc(Q_{max}(J)) \cap J$. Then by 2.2 $Q_{max}(J)_a$ is an algebra of quotients of J_a , which has finite capacity by [Mo1, Lemma 0.7(b)]. Thus J_a is a classical order in $Q_{max}(J)_a$ by 2.3, which implies that $a \in LC(J)$. \square

3 Nondegenerate Jordan algebras with nonzero LC-elements.

3.1 Recall that a *subdirect product* of a collection of Jordan algebras $\{J_\alpha\}$ is any subalgebra J of the full direct product $\prod_\alpha J_\alpha$ of the algebras of that collection such that the restrictions of the canonical projections $\pi_\alpha : J \rightarrow J_\alpha$ are onto. An *essential subdirect product* is a subdirect product which contains an essential ideal of the full direct product. If J is actually contained in the direct sum of the J_α , then J is called an *essential subdirect sum* [FGM, p.448].

3.2 Local algebras at nonzero LC-elements of nondegenerate Jordan algebras are essential subdirect products of finitely many strongly prime Jordan algebras, and therefore they are essential direct sums of the corresponding factors. That is the case for the local algebra J_a at a nonzero LC-element a of a nondegenerate Jordan algebra J . Indeed, by [MoT1, Theorem 5.13], [FGM, 10.3], and [MoT1, Lemma 5.3], the local algebra $J_a \cong \left(J/ann_J(id_J(a)) \right)_{a+ann_J(id_J(a))}$ is an essential subdirect sum of finitely many strongly prime Jordan algebras. These essential subdirect sums arise from the fact that for any LC-element $a \in LC(J)$ the ideal $id_J(a)$ is semi-uniform [MoT1, Proposition 5.10] in the following sense: an ideal I of a nondegenerate Jordan algebra J is *semi-uniform* if there exist prime ideals P_1, \dots, P_n of J such that $P_1 \cap \dots \cap P_n \subseteq ann_J(I)$. For any semi-uniform ideal I of a nondegenerate Jordan algebra J there exists a unique minimal

set of prime ideals $\mathcal{P} = \{P_1, \dots, P_n\}$ with $P_1 \cap \dots \cap P_n = \text{ann}_J(I)$, named (after the similar situation appearing in commutative ring theory) the *set of prime ideals associated to I* .

3.3 Lemma. *Let J be a nondegenerate Jordan algebra. For any essential ideal I of J we have $Q_{\max}(I) = Q_{\max}(J)$.*

Proof. Let I be an essential ideal of J . Then, by [Mo3, Examples 2.3.1] J is an algebra of quotients of I and therefore $I \subseteq J \subseteq Q_{\max}(J)$ is a sequence of algebras of quotients which implies by [Mo3, Proposition 2.8] that $Q_{\max}(J)$ is an algebra of quotients of I . But since I is nondegenerate [Mc3], I has a maximal algebra of quotients $Q_{\max}(I)$ whose maximality implies that $Q_{\max}(J) \subseteq Q_{\max}(I)$ [Mo3, Lemma 2.12].

Now, again by [Mo3, Proposition 2.8] we have that $Q_{\max}(I)$ is an algebra of quotients of J , and thus by the maximality of $Q_{\max}(J)$ [Mo3, Lemma 2.12], it follows that $Q_{\max}(I) \subseteq Q_{\max}(J)$. Hence $Q_{\max}(I) = Q_{\max}(J)$ \square

3.4 Lemma. *Let J be a nondegenerate Jordan algebra, and let I and L be ideals of J such that $J = I \oplus L$. Then $Q_{\max}(J) = Q_{\max}(I) \oplus Q_{\max}(L)$.*

Proof. The nondegeneracy of J implies that $I = \text{ann}_J(L)$ and $L = \text{ann}_J(I)$, and since L is isomorphic to an essential ideal of the nondegenerate Jordan algebra $J/\text{ann}_J(I)$ [FG2, Lemma 1(i)], by 3.3 we have $Q_{\max}(L) = Q_{\max}(J/\text{ann}_J(I))$, and the result follows from [Mo3, Lemma 5.6]. \square

We next extend 2.10 to nondegenerate algebras.

3.5 Theorem. *Let J be a nondegenerate Jordan algebra and $Q_{\max}(J)$ be its maximal algebra of quotients. Then $LC(J) = J \cap \text{Soc}(Q_{\max}(J))$.*

Proof. Assume first that $LC(J) = 0$ and take $a \in J \cap \text{Soc}(Q_{\max}(J))$. By 2.2, $Q_{\max}(J)_a$ is an algebra of quotients of J_a , and since $a \in \text{Soc}(Q_{\max}(J))$, $Q_{\max}(J)_a$ has finite capacity. Thus $a \in LC(J) = 0$ by 1.17.

Assume now that $LC(J) \neq 0$, and let a be nonzero LC-element of J . By [FGM, 10.3] and [MoT1, Proposition 5.6], J_a is semi-uniform, and so is the ideal $\text{id}_J(a)$ generated by a in J by [MoT1, Proposition 5.10 (ii)].

Hence $\bar{J} = J/ann_J(id_J(a))$ is an essential subdirect sum of finitely many strongly prime Jordan algebras J_1, \dots, J_n [MoT1, Proposition 5.4], that is, $M \subseteq_{ess} \bar{J} \cong J_1 \oplus \dots \oplus J_n$, where M is an essential ideal of $J_1 \oplus \dots \oplus J_n$ contained into \bar{J} (see the proof of [MoT1, Theorem 5.13]). By 3.3, we have $Q_{max}(M) \cong Q_{max}(\bar{J}) \cong Q_{max}(J_1 \oplus \dots \oplus J_n)$. Moreover [Mo3, Remark 5.7] applies here implying that $Q_{max}(\bar{J}) \cong Q_{max}(J_1) \oplus \dots \oplus Q_{max}(J_n)$ since $J_i = J/P_i$, where $\{P_1, \dots, P_n\}$ is the set of minimal prime ideals associated to $id_J(a)$ (see also 3.4).

Write $\bar{a} = \bar{a}_1 + \dots + \bar{a}_n$ with $\bar{a}_i \in J_i$. By [MoT1, Proposition 5.12], we have $\bar{a}_i \in LC(J_i)$, and since J_i is strongly prime, $\bar{a}_i \in Soc(Q_{max}(J_i)) \cap J_i$ by 2.10. Therefore we get $\bar{a} = \bar{a}_1 + \dots + \bar{a}_n \in Soc(Q_{max}(J_1)) \oplus \dots \oplus Soc(Q_{max}(J_n)) = Soc(Q_{max}(J_1) \oplus \dots \oplus Q_{max}(J_n)) = Soc(Q_{max}(\bar{J}))$.

On the other hand, the Jordan algebra \bar{J} contains the essential ideal $(id_J(a) + ann_J(id_J(a)))/ann_J(id_J(a))$, isomorphic to $id_J(a)$. Thus we have $Q_{max}(\bar{J}) \cong Q_{max}(id_J(a))$ by 3.3, and therefore, $a \in Soc(Q_{max}(id_J(a))) \cap id_J(a)$. Finally, making use again of 3.3 gives $a \in Soc(Q_{max}(J)) \cap J$.

As for the reverse containment $Soc(Q_{max}(J)) \cap J \subseteq LC(J)$, it suffices to note that the proof of 2.10 still works here. \square

4 Local orders in Jordan algebras

In order to do this section as self-contained as possible we include here the quadratic versions of some of the results proved in [FG1, FG2] for linear Jordan algebras, but not always their complete proofs as they can be easily obtained from those given in [FG1]. However we outline some of those proofs in order to stress the necessary quadratic references.

4.1 An element x of a Jordan algebra J is said to be *locally invertible* if there exists a (necessarily unique) idempotent $e = P(x)$ such that x is invertible in the unital Jordan algebra $U_e J$. We denote by $LocInv(J)$ the set of all locally invertible elements of J . The inverse x^\sharp of x in $U_e J$ is called the *generalized inverse* of x . The following equivalent characterizations of x^\sharp are given in [FGSS] (see also [FG1, p. 1033]):

- (i) $U_x x^\sharp = x$, $U_{x^\sharp} x = x^\sharp$ and $U_{x^\sharp} U_x = U_x U_{x^\sharp}$.
- (ii) $U_x x^\sharp = x$ and $U_x U_{x^\sharp} x^\sharp = x^\sharp$.
- (iii) $U_x x^\sharp = x$ and $(x^\sharp)^2 \circ x = x^\sharp$.

The idempotent $e = P(x)$ determined by the locally invertible element $x \in J$ is given by $e = U_x(x^\sharp)^2 = U_{x^\sharp}x^2$. Recall that $x \in J$ is locally invertible if and only if x is strongly regular (i.e. $x \in U_{x^2}J$) [FG1, p. 1032].

4.2 A subalgebra J of a (non necessarily unital) Jordan algebra Q is a *weak local order* in Q if for each $q \in Q$ there exists $x \in \text{LocInv}(Q) \cap J$ such that $q \in U_x Q$ with $U_x J$ being an order in the unital Jordan algebra $U_x Q = U_e Q$ ($e = P(x)$) relative to some monad S of $U_x J$.

The notion of weak local order was introduced for non-necessarily unital Jordan algebras as substitute of the notion of order relative to a monad in nonunital Jordan algebras. These are indeed particular cases of weak local orders.

4.3 Proposition. *Let J be a Jordan algebra which is an S -order in a unital Jordan algebra Q . For every $s \in S$, $U_s S$ is a monad of $U_s J$, and $U_s J$ is an $U_s S$ -order in Q . In particular J is a weak local order in Q .*

Proof. A proof similar to that of [FG1, Proposition 12] works here, replacing the linear references used there by their quadratic counterparts which can be found for example in [FGM, Lemma 2.2]. \square

4.4 The concept of local order upon which the theory developed in [FG1, FG2] lies is based on the following notion adapted from the associative theory. An element $x \neq 0$ in a Jordan algebra J is called *semiregular* if $\text{ann}_J(x) = \text{ann}_J(x^2)$. The set of all semiregular elements of J is denoted by $\text{SemiReg}(J)$. In the present paper, however, we will adopt a different Jordan analogue of the homonymous arrogant notion that follows the approach of [FGM] to Jordan rings of fractions, and in particular, the use of the notion of injective element instead that of regular element. An element $x \in J$ will be said to be *semi-injective* if for any $y \in J$, $U_x^2 y = 0$ implies $U_x y = 0$, which

implies (and is equivalent, if J is nondegenerate) the condition $\text{Ker } x = \text{Ker } x^2$. We denote by $\text{SemiInj}(J)$ the set of all nonzero semi-injective elements of J .

For a subalgebra J of a Jordan algebra Q , the containments $\text{LocInv}(J) \subseteq J \cap \text{LocInv}(Q) \subseteq \text{SemiInj}(J) \subseteq \text{SemiReg}(J)$ are straightforward. Moreover the equality holds for nondegenerate Jordan algebras satisfying the descending chain condition on principal inner ideals, since [FG1, Proposition 17] remains true in the quadratic setting. The proof is essentially the one given there, taking into account the obvious changes required by the differences in the notions of annihilators in quadratic and linear Jordan algebras (see 0.4).

Semi-injectivity has the following straightforward consequence in terms of local algebras:

4.5 Lemma. *Let J be a Jordan algebra. If $x \in \text{SemiInj}(J)$ then the subalgebra $U_x J$ of J is isomorphic to the local algebra J_{x^2} .*

Proof. Since the mapping $U_x : J \rightarrow U_x J$ obviously defines a surjective homomorphism $J^{(x^2)} \rightarrow U_x J$, it suffices to notice that its kernel is $\text{Ker } x = \text{Ker } x^2$ since x is semi-injective. \square

4.6 A subalgebra J of a Jordan algebra Q will be said to be a *local order* in Q if it satisfies:

(LO1) $\text{SemiInj}(J) \subseteq \text{LocInv}(Q)$, and

(LO2) for every $q \in Q$ there exists $x \in \text{SemiInj}(J)$ such that $q \in U_x Q$, and $U_x J$ is a classical order in $U_e Q = U_x Q$ ($e = P(x)$).

Property (LO2) in the definition of local order includes the assertion that, for some $x \in \text{SemiInj}(J)$, $U_x J$ is a classical order in $U_x Q$. This can be replaced by the following assertion:

4.7 Lemma. *Let J be a subalgebra of an algebra Q . If $x \in \text{SemiInj}(J)$, $U_x J$ is a classical order in $U_x Q$ if and only if J_{x^2} is a classical order in Q_{x^2}*

(with the obvious identification $(J_{x^2} = J + \text{Ker}_Q x^2)/\text{Ker}_Q x^2$.)

Proof. Considering the homomorphism $Q^{(x^2)} \rightarrow U_x Q$ given by $q \mapsto U_x q$ and its restriction to J instead of Q , the assertion follows directly from 4.5 and the fact that $\text{Ker}_Q x \cap J = \text{Ker}_Q x^2 \cap J = \text{Ker}_J x = \text{Ker}_J x^2$. \square

4.8 Lemma. *Let J be a nondegenerate Jordan algebra. If $Q \supseteq J$ is a general algebra of quotients of J then $\text{SemiInj}(J) \subseteq \text{SemiInj}(Q)$*

Proof. Take $x \in \text{SemiInj}(J)$, and suppose that $U_{x^2} q = 0$ for some $q \in Q$. Set $K = (\mathcal{D}_J(q) : x) \cap (\mathcal{D}_J(q) : x^2)$. Since Q is an algebra of quotients of J , $\mathcal{D}_J(q)$ is a dense inner ideal, hence both $(\mathcal{D}_J(q) : x)$ and $(\mathcal{D}_J(q) : x^2)$ are dense, and thus, so is their intersection K (see [Mo3, Lemma 1.10]). Now we have $0 = U_{U_x^2 q} K = U_{x^2} U_q U_{x^2} K$, but $U_q U_{x^2} K \subseteq U_q U_{x^2} (\mathcal{D}_J(q) : x^2) \subseteq U_q \mathcal{D}_J(q) \subseteq J$, which implies that $U_x U_q U_{x^2} K = 0$ because $x \in \text{SemiInj}(J)$. Thus, for an arbitrary $z \in K$ we have $U_{U_{x^2} U_q U_x z} K = U_{x^2} U_q U_x U_z U_x U_q U_{x^2} K = 0$ by the previous equality, hence $U_{x^2} U_q U_x z = 0$ by [Mo3, Lemma 2.4(iv)] for all $z \in K$, that is $U_{x^2} U_q U_x K = 0$; but $U_{x^2} U_q U_x K \subseteq U_{x^2} U_q U_x (\mathcal{D}_J(q) : x)$, and since $U_q U_x (\mathcal{D}_J(q) : x) \subseteq J$, the semi-injectivity of x in J implies $U_x U_q U_x K = 0$. Then $U_{U_x q} K = 0$, which again by [Mo3, Lemma 2.4(iv)], implies $U_x q = 0$, and therefore $x \in \text{SemiInj}(Q)$. \square

Local orders were defined for non-necessarily unital Jordan algebras, but when the involved Jordan algebras are unital, local orders are algebras of quotients in the sense of [Mo3] and [MoP], which coincide in this case. This will be obtained as a consequence of the following:

4.9 Lemma. *Let J be a nondegenerate Jordan algebra. If J is a subalgebra of a Jordan algebra Q and one of the following situations holds:*

- (i) J is a (weak) local order in Q ,
- (ii) Q is an algebra of quotients of J ,

then $U_q J \cap J \neq 0$ for any $0 \neq q \in Q$, and as consequence, Q is nondegenerate, and any nonzero inner ideal of Q hits J nontrivially.

Proof. (i) Since $J \subseteq Q$ is a (weak) local order (that is, $J \subseteq Q$ satisfies (LO2)), given $q \in Q$, there exists an element $x \in \text{LocInv}(Q) \cap J \subseteq \text{SemiInj}(J)$ such that $q \in U_x Q$, and $U_x J$ is a classical order in $U_e Q = U_x Q$ ($e = P(x)$). Now, since J is nondegenerate, 4.7 together with [ACM, Proposition 0.2] implies that $U_x J$ is nondegenerate, and thus [FGM, Proposition 2.9 (iii)] implies that $U_q U_x J \cap U_x J \neq 0$, which obviously implies $U_q J \cap J \neq 0$.

The assertion in case (ii) is just [Mo3, Lemma 2.4(ii)], and the last assertions are obvious. \square

4.10 Proposition. *Let J be a nondegenerate Jordan algebra and a local order in a Jordan algebra Q . Then:*

- (i) *if J is unital, Q is also unital with the same unit as J .*
- (ii) *if Q is unital, J is a classical order in Q and therefore Q is an algebra of quotients of J .*

Proof. The proof of [MoP, Lemma 3.2.(a)] applies here using 4.9(i), the same property as [MoP, Lemma 2.4 (iii)], which is used in [MoP, Lemma 3.2.(a)]. For (ii), the proof of [FG1, Proposition 19] also works here with the obvious changes for the references. The last statement follows from [Mo3, Examples 2.3.5]. \square

We next aim at giving an alternative characterization of local orders in nondegenerate algebras with dcc on inner ideals which suggests a reason of the use of the adjective "local". We first prove a result which, among other uses, will be instrumental to that end.

4.11 Lemma. *Let the Jordan algebra J be a classical order in a nondegenerate Jordan algebra \tilde{J} . If $a \in \text{Soc}(\tilde{J}) \cap J$, then under the natural identification $J_a \subseteq \tilde{J}_a$ induced by the inclusion $J \subseteq \tilde{J}$, \tilde{J}_a is a classical algebra of quotients of J_a .*

Proof. According to 4.10 \tilde{J} is an algebra of quotients of J , hence J_a is an algebra of quotients of \tilde{J}_a . Since $a \in \text{Soc}(\tilde{J})$, \tilde{J}_a has finite capacity by [Mo1, Lemma 0.7(b)], therefore J_a is a classical order in \tilde{J}_a by 2.3. \square

4.12 We introduce a piece of notation that will be used in the sequel. Let J be a subalgebra of a Jordan algebra \tilde{J} , and let $e \in \tilde{J}$ be an idempotent. We denote $J_2(e) = \{a \in J \mid U_e a = a\} = \tilde{J}_2(e) \cap J$ which is clearly an inner ideal of J because $\tilde{J}_2(e)$ is an inner ideal of \tilde{J} .

The following extended version of Litoff's Theorem for quadratic Jordan algebras, whose linear version was proved in [A], will be a key tool in our study of quadratic Jordan algebras which are local orders in nondegenerate Jordan algebras satisfying dcc on principal inner ideals.

4.13 Theorem. *Let J be a nondegenerate Jordan algebra with $J = \text{Soc}(J)$. Then for every finite collection of elements a_1, \dots, a_n in J there exists an idempotent $e \in J$ such that $a_1, \dots, a_n \in U_e J = J_2(e)$.*

Proof. Since J equals its socle, [Lo2, Theorem 2(b)], J is a direct sum of simple ideals (see 0.8), and we can therefore assume that J is simple. Now the result follows directly if J has finite capacity (since then J is unital), so by the Simple Structure Theorem [McZ, Theorem 15.5], we can assume that either $J = A^+$ or $J = H_0(A, *)$ for a simple associative algebra A . Moreover by [FT, Proposition 4.1] we have $A = \text{Soc}(A)$. Now the proof of [A] carries over unchanged to the quadratic setting since the only elements of $H(A, *)$ whose presence in the subalgebra of symmetric elements is assumed in [A] are either norms or traces, and therefore they belong to any ample $H_0(A, *)$. \square

4.14 Lemma. *Let J be a nondegenerate Jordan, and let $\tilde{J} \supseteq J$ be an innerly tight extension of J with $J = \text{Soc}(J)$. Then for any idempotent $e \in \text{Soc}(\tilde{J})$ there exists $x \in \text{SemiInj}(J)$ such that $e = P(x)$ (i.e. x is invertible in $U_e \tilde{J}$).*

Proof. We consider the double pair $V(\tilde{J}) = (\tilde{J}, \tilde{J})$. Since \tilde{J} equals its socle, so does $V(\tilde{J})$. According to [Lo4] we can take a strong frame $\{e_1 = (e_1^+, e_1^-), \dots, e_n = (e_n^+, e_n^-)\}$ of the nondegenerate Jordan pair $(\tilde{J}_2(e), \tilde{J}_2(e))$, where n is the capacity of the Jordan pair [Mo1, Lemma 0.7(b)].

The inner tightness of \tilde{J} over J implies (see 1.12) that there exist nonzero

elements $0 \neq k_i \in U_{e_i^+} \tilde{J} \cap J$ with $k_i = U_{e_i^+} U_{e_i^-} k_i$. Let $x = k_1 + \cdots + k_n \in J$. Then $U_e x = x$, so that $x \in U_e \tilde{J}$ and we have $rk(x) = rk(e) = n$, so that x is invertible in $U_e \tilde{J}$ [Lo5, Proposition 1, Corollary 1]. Hence $x \in J \cap LocInv(\tilde{J}) \subseteq SemiInj(J)$. \square

The following result contains what will be our operating version of local orders since we will be mainly interested in local orders of algebras with dcc on principal inner ideals, that is on algebras that equal their socles. As mentioned in the introduction, in analogy with [AM1, Theorem 1], later on we will prove that these algebras are in fact general algebras of quotients.

4.15 Theorem. *Let J be nondegenerate Jordan algebra, and let $Q \supseteq J$ be an over algebra such that $Q = Soc(Q)$. Then J is a local order in Q if and only if*

(LOS1) $SemiInj(J) = LocInv(Q) \cap J$,

(LOS2) For any $q \in Q$ there exists $x \in SemiInj(J)$ such that $q \in U_x Q$, and

(LOS3) For any $x \in J$, the local algebra J_x is a classical order in Q_x (with the obvious containment).

Proof. Suppose first that J is a local order in Q . As LOS1 and LOS2 are part of the definition of local order, it is clear that it suffices to prove that LOS3 holds. Take $x \in J$. By the definition of local order, there exists an element $s \in SemiInj(J) = LocInv(Q) \cap J$ such that $x \in U_s J$ and $U_s J$ is a classical order in $U_s Q$ (with respect to a monad of $U_s J$ which is easily seen to coincide with $Inj(U_s J)$ since $Q = Soc(Q)$), and therefore $U_s J$ is a classical order in $U_s Q$. Since s is locally invertible in Q , by 4.1 we can consider the idempotent $e = P(s) \in Q$. With the notation introduced above, we have $U_s J \subseteq J_2(e) \subseteq Q_2(e) = U_s Q$, and since $U_s J$ is a classical order in $U_s Q = Q_2(e)$, $J_2(e)$ is also a classical order in $Q_2(e)$ by [FGM, Corollary 2.3].

Clearly $x \in U_s Q = U_e Q = Q_2(e)$ implies $x \in J_2(e)$, and since $J_2(e) \subseteq J$, the containment $J_2(e)_x \subseteq J_x$ is clear. From the obvious containment $Q_2(e)_x \subseteq Q_x$, making use of the identification $Q_2(e) = Q_e$ (see [LN, Example 1.12]) yields $Q_2(e)_x = (Q_e)_{x+Kere} = Q_{U_e x} = Q_x$ (see [Mo1, Lemma 0.5]).

Since as we have mentioned before $J_2(e)$ is a classical order in $Q_2(e)$, by 4.11 $J_2(e)_x$ is a classical order in $Q_2(e)_x$, and since $J_2(e)_x \subseteq J_x \subseteq Q_x = Q_2(e)_x$ (with the identifications mentioned before), from [FGM, Corollary 2.3] we obtain that J_x is a classical order in Q_x , as asserted in LOS3.

We next address the reciprocal, so we assume that $J \subseteq Q$ is a nondegenerate subalgebra of a Jordan algebra $Q = Soc(Q)$, and that properties LOS1, LOS2 and LOS3 hold. It is clear that to prove that J is a local order on Q it suffices to prove that for any $q \in Q$ there exists $x \in SemiInj(J)$ such that $q \in U_x Q$ and $U_x J$ is a classical order in $U_x Q$. As shown in 4.7, this is equivalent to proving that J_{x^2} is a classical order in Q_{x^2} , which obviously follows from LOS3. \square

We have shown in 4.9 that if J is an order in a Jordan algebra with dcc on principal inner ideals (that is, such that $Soc(Q) = Q$), and J is nondegenerate, then Q is also nondegenerate. We next prove the reciprocal of that assertion:

4.16 Lemma. *Let J be a local order in a Jordan algebra Q with $Soc(Q) = Q$. If Q is nondegenerate then J is nondegenerate.*

Proof. Suppose that $z \in J$ is an absolute zero divisor of J . Since J is a local order in $Q = Soc(Q)$, there exists an idempotent $e = P(s) \in Q$ for some $s \in J \cap LocInv(Q)$ such that $z \in U_s Q = Q_2(e)$. Now, since J is a local order in Q , arguing as in 4.15, $J_2(e)$ is a classical order in $Q_2(e)$, and moreover, the algebra $Q_2(e)$ has finite capacity since $Q = Soc(Q)$, and is nondegenerate since so is Q by hypothesis. Now, the part of the proof of (i) \Rightarrow (ii) of [FGM, Theorem 9.3] that asserts that a classical order in a nondegenerate artinian Jordan algebra is itself nondegenerate applies here verbatim to give that $J_2(e)$ is nondegenerate, taking into account that a nondegenerate algebra of finite capacity is a direct sum of a finite number of simple algebras with finite capacity [J2, Theorem 6.4.1]. Now $U_z J_2(e) \subseteq U_z J = 0$, hence z is an absolute divisor in the nondegenerate algebra $J_2(e)$, and therefore $z = 0$. \square

4.17 Lemma. *Let J be a nondegenerate Jordan algebra, and let $Q \supseteq J$ be an algebra of quotients of J . If $Q = Soc(Q)$, then $SemiInj(J) \subseteq LocInv(Q)$.*

In particular, if J is a nondegenerate algebra with dcc on inner ideals, then $\text{SemiInj}(J) = \text{LocInv}(J)$.

Proof. Since $\text{SemiInj}(J) \subseteq \text{SemiInj}(Q)$ by 4.8, we can assume that $J = Q = \text{Soc}(Q)$. Take then $x \in \text{SemiInj}(J)$, and let $x = x_2 + x_0$ with $x_i \in J_i(e)$ be its Fitting decomposition with respect to the corresponding idempotent e , which exists since J has dcc on principal inner ideals (see [Lo3, Theorem 1]). Our aim is to prove that $x_0 = 0$. If, on the contrary $x_0 \neq 0$, the element x_0 being nilpotent implies that there exists $n \geq 2$ such that $x_0^n = 0$ and $x_0^{n-1} \neq 0$. Since J is nondegenerate, so is $J_0(e)$ we can choose $z_0 \in J_0(e)$ such that $U_{x_0^{n-1}} z_0 \neq 0$. An easy induction using the multiplication properties of the Peirce decomposition shows that for any $y_0 \in J_0(e)$ and any $m \geq 0$, $U_x^m y_0 = U_{x_0^m} y_0$. Since $x_0^n = 0$, we have $U_x^2 U_{x_0^{n-2}} y_0 = U_x^n z_0 = U_{x_0^n} y_0$, and since x is semi-injective, this implies that $0 = U_x U_x^{n-2} y_0 = U_x^{n-1} y_0 = U_{x_0^{n-1}} y_0$, which contradicts the choice of y_0 . Therefore $x = x_2$ is invertible in $J_2(e)$, hence $x \in \text{LocInv}(J)$. \square

4.18 Lemma. *Let Q be a nondegenerate Jordan algebra with $Q = \text{Soc}(Q)$. If Q is a local order in a Jordan algebra \tilde{Q} with $\tilde{Q} = \text{Soc}(\tilde{Q})$, then $\tilde{Q} = Q$.*

Proof. Take $\tilde{q} \in \tilde{Q}$. By 4.15, there exists an element $x \in \text{SemiInj}(Q)$ such that $\tilde{q} \in U_x \tilde{Q}$, and $U_x Q$ is a classical order in $U_x \tilde{Q}$. Since $Q = \text{Soc}(Q)$, by 4.5 $U_x Q \cong Q_{x^2}$ is artinian, hence $U_x Q = U_x \tilde{Q}$, and therefore $\tilde{q} \in U_x \tilde{Q} = U_x Q \subseteq Q$. Since this holds for an arbitrary $\tilde{q} \in \tilde{Q}$, we get $\tilde{Q} = Q$. \square

4.19 Lemma. *Let J be a nondegenerate Jordan algebra which is a local order in a Jordan algebra \tilde{J} such that $\text{Soc}(\tilde{J}) = \tilde{J}$. Then*

- (i) $\text{ann}_J(X) = \text{ann}_{\tilde{J}}(X) \cap J$ for any $X \subseteq J$.
- (ii) For any subsets $X, Y \subseteq J$, $\text{ann}_J(X) \subseteq \text{ann}_J(Y)$ if and only if $\text{ann}_{\tilde{J}}(X) \subseteq \text{ann}_{\tilde{J}}(Y)$, and $\text{ann}_J(X) \neq 0$ if and only if $\text{ann}_{\tilde{J}}(X) \neq 0$.
- (iii) J satisfies acc on annihilators of its elements.

Proof. (i) Clearly, we can assume that $X = \{x\}$ has just one element. The containment $\text{ann}_{\tilde{J}}(x) \cap J \subseteq \text{ann}_J(x)$ being obvious, we only have to prove

the reverse containment, which will follow from the containment $ann_J(x) \subseteq ann_{\tilde{J}}(x)$.

Take then $z \in ann_J(x)$. Since $U_z x = U_x z = 0$, according to [Mc2, Remark 1.7], we only need to prove $U_z U_x \tilde{J} = 0 = \{z, x, \tilde{J}\}$. Take an arbitrary $q \in \tilde{J}$. Since $\tilde{J} = Soc(\tilde{J})$ and \tilde{J} is nondegenerate by 4.9, we can apply 4.13 to find an idempotent $e \in \tilde{J}$ such that $x, z, q \in \tilde{J}_2(e)$.

Now, by 4.14, there exists $s \in SemiInj(J)$ such that $e = P(s)$, and since J_{s^2} is a classical order in \tilde{J}_{s^2} by LOS3, arguing as in 4.7, we get that $U_s J$ is a classical order in $U_s \tilde{J}$. Therefore, applying [FGM, Corollary 2.3] to the containment $U_s J \subseteq J_2(e) \subseteq \tilde{J}_2(e) = U_s \tilde{J}$, we obtain that $J_2(e)$ is a classical order in $\tilde{J}_2(e)$. Then $ann_{J_2(e)}(x) = ann_{\tilde{J}_2(e)}(X) \cap J_2(e)$ by [FGM, Proposition 2.8]. Since $x, z, q \in J_2(e)$, this implies that $U_z U_x q = 0 = \{x, z, q\}$, as desired.

(ii) That $ann_{\tilde{J}}(X) \subseteq ann_{\tilde{J}}(Y)$ implies $ann_J(X) \subseteq ann_J(Y)$ readily follows from (i). For the reciprocal, assume that $ann_J(X) \subseteq ann_J(Y)$ and suppose that $ann_{\tilde{J}}(X) \not\subseteq ann_{\tilde{J}}(Y)$. Since \tilde{J} satisfies the dcc on principal inner ideals, we can choose an inner ideal $U_a \tilde{J}$, minimal among the inner ideals generated by the elements of $ann_{\tilde{J}}(X)$ which do not belong to $ann_{\tilde{J}}(Y)$. Since $a \in Soc(\tilde{J}) = \tilde{J}$, we can complete a to an idempotent $e = (e^+, e^-)$ with $a = e^+$ of the pair $V(\tilde{J}) = (\tilde{J}, \tilde{J})$. We then have $U_a \tilde{J} = Q_{e^+} \tilde{J} \supseteq Q_{e^+} Q_{e^-} \tilde{J} = \tilde{J}_2(e) \supseteq U_a \tilde{J}$, and $U_a \tilde{J}$ is nondegenerate and has finite capacity. Then [Lo4] implies that the pair $V(\tilde{J}_2(e))$ contains a strong frame $F = \{e_1, \dots, e_m\}$, hence $f = \sum F = e_1 + \dots + e_m$ is complete in $V(\tilde{J}_2(e))$, and therefore f^+ is invertible in the algebra $V(\tilde{J}_2(e))^+ = \tilde{J}_2(e)$. Then $V(\tilde{J})_2^+(f) = V(\tilde{J})_2^+ = U_a \tilde{J}$. Now, if $e_i^+ \in ann_{\tilde{J}}(Y)$ for all i , then $f^+ \in ann_{\tilde{J}}(U)$, hence $U_a \tilde{J} = V(\tilde{J})_2^+(f) \subseteq ann_{\tilde{J}}(a)$, which contradicts the choice of a . Thus we can assume that $e_1^+ \notin ann_{\tilde{J}}(Y)$, and since $e_1^+ \in U_a \tilde{J}$, the minimality of $U_a \tilde{J}$ implies that $U_a \tilde{J} = U_{e_1^+} \tilde{J}$. Thus, since e_1 is a division idempotent, we can assume that $U_a \tilde{J}$ is a minimal inner ideal. We now apply 4.9(i) to find an element $0 \neq d \in U_a J \cap J \subseteq ann_{\tilde{J}}(X) \cap J = ann_J(X) \subseteq ann_J(Y) = ann_{\tilde{J}}(Y) \cap J \subseteq ann_{\tilde{J}}(Y)$. But this implies that $a \in U_a \tilde{J} = U_d \tilde{J} \subseteq ann_{\tilde{J}}(Y)$, a contradiction.

If $ann_{\tilde{J}}(X) = 0$, then, by (i) $ann_J(X) = ann_{\tilde{J}}(X) \cap J = 0$. The recip-

rocal is straightforward from 4.9.

(iii) Since \tilde{J} satisfies dcc on principal inner ideals, by [FGL] it satisfies acc on annihilators of single elements, and so does J by (ii). \square

4.20 Theorem. *Let the Jordan algebra J be a local order in the algebra Q satisfying $\text{Soc}(Q) = Q$, then Q is a general algebra of quotients of J .*

Proof. Since $Q = \text{Soc}(Q)$, all the local algebras Q_q for $q \in Q$ have finite capacity, and since J_x is a classical order in the algebra Q_x by LOS3, it is LC, hence $x \in LC(J)$, that is $J = LC(J)$. Then dense inner ideals are essential inner ideals, so we only need to prove that for any $q \in Q$ the inner ideal $\mathcal{D}_J(q)$ is essential, and $U_q \mathcal{D}_J(q) \neq 0$. We begin by proving that the first statement implies the second one. Indeed, if L is an essential inner ideal of J and $U_q L = 0$, then any $z \in U_q Q \cap J$ has $U_z L = 0$, and therefore it is an essential zero divisor. Since J is strongly nonsingular, this implies $U_q Q \cap J = 0$, which contradicts 4.9(i).

Now, to prove the first assertion we have to prove that $\mathcal{D}_J(q) \cap U_a J \neq 0$ for any nonzero $a \in J$. By 4.13 and 4.14, there exists an idempotent $e = P(s)$ for a semiinjective element $x \in J$, such that $a, q \in Q_2(e)$. As we have already noticed before, since $U_s J$ is a classical order in $U_s Q$, then $J_2(e)$ is a classical order in $Q_2(e)$ (4.12, 4.15). Then $\mathcal{D}_{J_2(e)}(q)$ is an essential inner ideal of $J_2(e)$, hence there exists an element $d \in \mathcal{D}_{J_2(e)}(q) \cap U_a J_2(e)$. Note that $J_2(e)$ is strongly nonsingular, since it is LC and 1.21 applies, and there exists an element $y \in \mathcal{D}_{J_2(e)}(q)$ such that $0 \neq z = U_d y \in \mathcal{D}_{J_2(e)}(a) \cap U_a J$. Let us see that $z \in \mathcal{D}_J(q)$.

Since $z \in \mathcal{D}_{J_2(e)}(a) \subseteq J_2(e)$, we have $U_z q$ and $U_q z$ belong to $J_2(e) \subseteq J$, so (Di) and (Dii) of 1.5 hold. Also, $z \circ q \in J_2(e) \subseteq J$, and if $c \in J$, we get $\{z, a, c\} = \{U_d y, a, c\} = \{d, \{y, d, a\}, c\} - \{U_d a, y, c\} \in \{J, J_2(e), J\} + \{J_2(e), J, J\} \subseteq J$, so (Div) of 1.5 holds. Finally, $U_a U_z c = U_a U_{U_d y} c = U_a U_d (U_y J) \subseteq U_a U_d J_2(e) \subseteq J_2(e) \subseteq J$ because $U_y J = U_{U_e y} J = U_e U_y U_e J \subseteq Q_2(e)$ gives $U_y J \subseteq Q_2(e) \cap J = J_2(e)$. Therefore $U_a U_z J \subseteq J$, and (Diii) of 1.5 also holds, and this implies that $0 \neq z \in \mathcal{D}_J(q) \cap U_a J$, and thus, the essentiality, hence the density, of $\mathcal{D}_J(q)$. \square

5 Local Goldie-like conditions and local orders.

In this section we give a Goldie-like characterization of quadratic Jordan algebras that satisfy local Goldie conditions extending the results given in [FG1, FG2]. We first turn our attention to the Local Goldie Conditions as introduced in [FG1].

5.1 An element x in a Jordan algebra J has *finite uniform* (or *Goldie*) *dimension* if $U_x J$ does not contain infinite direct sums of inner ideals of J (in the sense of [FGM, p. 425]).

Every element in a simple Jordan algebra with dcc on principal inner ideals which is not a nonartinian quadratic factor has finite uniform dimension.

5.2 Lemma. *Let J be a nondegenerate Jordan algebra. An element x in J has finite uniform dimension if and only if the local algebra J_x has finite uniform dimension.*

Proof. By [DM, Proposition 2.4] it suffices to note that for any element x in a nondegenerate Jordan algebra J there is a bijective order preserving correspondence between the set of inner ideals of the local algebra J_x and the set of inner ideals of J contained in $U_x J$. \square

5.3 Lemma. *Let J be a nondegenerate Jordan algebra and let Q be an algebra of quotients of J . Then any nonzero ideal I of Q is an algebra of quotients of $I \cap J$.*

Proof. Note that, by 4.9, $I \cap J$ is a nonzero ideal of J , and moreover $I \cap J$ is nondegenerate as an algebra [Mc3].

Let $q \in I$. Clearly $\mathcal{D}_J(q) \cap (I \cap J) = \mathcal{D}_J(q) \cap I \subseteq \mathcal{D}_{I \cap J}(q)$, and for any $a \in I \cap J$ we have $(\mathcal{D}_J(q) : a)_L \cap I \subseteq (\mathcal{D}_{I \cap J}(q) : a)_L$. Thus for any $a, b \in I \cap J$ it holds that $(\mathcal{D}_J(q) : a)_L \cap (\mathcal{D}_J(q) : b)_L \cap I \subseteq (\mathcal{D}_{I \cap J}(q) : a)_L \cap (\mathcal{D}_{I \cap J}(q) : b)_L$.

By [Mo3, Proposition 1.9], to prove the density of $\mathcal{D}_{I \cap J}(q)$ in $I \cap J$ it suffices to prove that $U_c((\mathcal{D}_{I \cap J}(q) : a)_L \cap (\mathcal{D}_{I \cap J}(q) : b)_L) = 0$ for any $a, b, c \in I \cap J$ implies $c = 0$.

Let $a, b, c \in I \cap J$ and assume $U_c((\mathcal{D}_{I \cap J}(q) : a)_L \cap (\mathcal{D}_{I \cap J}(q) : b)_L) = 0$. Then we have $U_c((\mathcal{D}_J(q) : a)_L \cap (\mathcal{D}_J(q) : b)_L) = 0$. Let $K = (\mathcal{D}_J(q) : a)_L \cap (\mathcal{D}_J(q) : b)_L$ and take $x \in K$. Then, since K is an inner ideal of J , $U_{U_c x} I = U_c U_x U_c I \subseteq U_c U_K I \subseteq U_c(K \cap I) = 0$. Hence $U_c K \subseteq \text{ann}_Q(I)$. But $c \in I \cap J$, hence $U_c K \subseteq \text{ann}_Q(I) \cap I = 0$. Thus $U_c K = 0$, which implies $c = 0$ by the density of $\mathcal{D}_J(q)$ in J and [Mo3, Proposition 1.9]. Hence $\mathcal{D}_{I \cap J}(q)$ is a dense inner ideal of $I \cap J$.

Finally suppose that $U_q \mathcal{D}_{I \cap J}(q) = 0$. Then we have $U_{U_q \mathcal{D}_J(q)} \mathcal{D}_{I \cap J}(q) = U_q U_{\mathcal{D}_J(q)} U_q \mathcal{D}_{I \cap J}(q) = 0$, and since $U_q \mathcal{D}_J(q) \subseteq I \cap J$, the density of $\mathcal{D}_{I \cap J}(q)$ in $I \cap J$ implies that $U_q \mathcal{D}_J(q) = 0$. Hence $q = 0$ because Q is an algebra of quotients of J . \square

5.4 Theorem. *Let J be a nondegenerate Jordan algebra with maximal algebra of quotients $Q_{\max}(J)$. Then $LC(J)$ is a local order in $\text{Soc}(Q_{\max}(J))$.*

Proof. Let us denote $Q = Q_{\max}(J)$. Since $LC(J) = J \cap \text{Soc}(Q)$ by 3.5, we can assume that $J = LC(J)$ by 5.3, so that J has an algebra of quotients $Q = \text{Soc}(Q)$ with dcc on principal inner ideals. By 4.17, $\text{SemiInj}(J) \subseteq \text{LocInv}(Q)$ hence $\text{SemiInj}(J) = \text{LocInv}(Q) \cap J$ which is LOS1. Take now an element $q \in Q = \text{Soc}(Q)$, using 4.13 we obtain an idempotent $e \in Q$ such that $q \in U_e Q$, and using 4.14, we can find an element $s \in \text{SemiInj}(J)$ such that $e = P(s)$. Thus $q \in U_s Q$, and LOS2 holds. Finally, it suffices to prove LOS3, for any $x \in J$, J_x is a classical order in Q_x , and this follows as usual from [FGM, Proposition 2.10] applied to the fact that Q_x is an algebra of quotients of J_x (2.2), and Q_x has finite capacity since $x \in \text{Soc}(Q)$. \square

As a straightforward application of 5.4 we get:

5.5 Theorem. *Let J be a Jordan algebra. Then the following conditions are equivalent:*

- (i) *J is a local order in a nondegenerate Jordan algebra Q with dcc on principal inner ideals.*
- (ii) *J is nondegenerate, and $J = LC(J)$ (that is J_x is LC for all $x \in J$).*

In this case, J is strongly prime if and only if Q is simple.

Proof. (i) \Rightarrow (ii) By 4.16, J is nondegenerate. Now, let $a \in J$. Since J is a local order in Q , by LOS3 of 4.15, J_a is a classical order in Q_a , and since $a \in \text{Soc}(Q) = Q$, Q_a has finite capacity. Therefore $a \in LC(J)$.

(ii) \Rightarrow (i) Assume now that J is nondegenerate and satisfies $J = LC(J)$. Then by 5.4, J is a local order in $\text{Soc}(Q_{\max}(J))$ which is nondegenerate and has dcc on principal inner ideals.

The last assertion easily follows from the relation between the annihilators of J and Q proved in 4.19, and the relation between annihilator ideals of a nondegenerate algebra and its algebras of quotients [Mo3, Lemma 5.3 and Lemma 5.4]. \square

5.6 Theorem. *For a Jordan algebra J the following conditions are equivalent:*

- (1) J is a local order in a nondegenerate locally artinian Jordan algebra Q .
- (2) J is nondegenerate, satisfies the acc on annihilators of its elements, and every element $x \in J$ has finite uniform dimension.
- (3) J is nondegenerate and J_x is Goldie for all nonzero element $x \in J$.

Moreover, J is strongly prime if and only if Q is simple.

Proof. (1) \Rightarrow (2) Since J is a local order in a nondegenerate Q with dcc on principal inner ideals, the nondegeneracy of J follows from 4.16, while acc on annihilators of its elements is 4.19(iii). Finally by 5.2, the fact that all $x \in J$ have finite uniform dimension is equivalent to each J_x having finite uniform dimension, and since by LOS3 J_x is a classical order in the artinian algebra Q_x , it follows from 1.15 that J_x has finite uniform dimension.

(2) \Rightarrow (3) We already have that J is nondegenerate by hypothesis. We prove that for all $x \in J$, the local algebra J_x is Goldie by appealing to characterization (iii) of 1.15. Since we are already assuming that J_x has finite uniform dimension, it suffices to prove that each nonzero ideal \bar{I} of J_x contains a uniform element. Set $I = \{y \in J \mid y + \text{Ker } x \in \bar{I}\}$, the preimage of \bar{I} by the projection $J \rightarrow J_x$. Since J has the acc on annihilators of elements

every nonzero inner ideal contains a uniform element, in particular we can choose a uniform element $z_0 \in U_x I$ since $U_x I$ is a nonzero inner ideal. Clearly $z_0 = U_x y_0$ for some $y_0 \in I$, and since z_0 is uniform, $(J_x)_{\overline{y_0}} = U_{U_x y_0} = J_{z_0}$ is a Jordan domain, hence $\overline{z_0} \in \overline{I}$ is uniform, and therefore J_x is Goldie.

(3) \Rightarrow (1) Since every local algebra J_x of J is Goldie, J has $J = LC(J)$. From 5.4 we know that J is a local order in $Q = Soc(Q_{max}(J))$, so it suffices to note that Q is locally artinian since Q_x has finite capacity by 5.4, and is a classical algebra of quotients of the Goldie algebra J_x by LOS3 of 4.15.

The last assertion follows directly as a particular case of the last assertion of 5.5. \square

We finally prove that algebras with dcc on principal inner ideals in which a given algebra is a local order are unique.

5.7 Proposition. *Let Q_1, Q_2 be Jordan algebras and let J be a local order in both Q_1 and Q_2 . If J is nondegenerate, $Soc(Q_1) = Q_1$, and $Soc(Q_2) = Q_2$, then there exists a unique isomorphism $\alpha : Q_1 \rightarrow Q_2$ that extends the identity mapping $J \rightarrow J$.*

Proof. By 4.20, both Q_1 and Q_2 are general algebras of quotients of J . Then $Q_{max}(Q_i) = Q_{max}(J)$ for $i = 1, 2$, and by [Mo3, Lemma 2.12], there is a unique isomorphism $\beta : Q_{max}(Q_1) \rightarrow Q_{max}(Q_2)$ which restricts to the identity on J . Clearly $\beta(Soc(Q_{max}(Q_1))) \subseteq Soc(Q_{max}(Q_2))$. Switching the roles of Q_1 and Q_2 we obtain a unique isomorphism $\gamma : Q_{max}(Q_2) \rightarrow Q_{max}(Q_1)$ which restricts to the identity on J , and again it is clear that $\gamma(Soc(Q_{max}(Q_2))) \subseteq Soc(Q_{max}(Q_1))$. Therefore, from the uniqueness of the isomorphisms we have $\gamma = \beta^{-1}$, and β restricts to an isomorphism $Soc(Q_{max}(Q_1)) \cong Soc(Q_{max}(Q_2))$ which in turn, restricts to the identity on J . This isomorphism obviously induces an isomorphism $LC(Soc(Q_{max}(Q_1))) \cong LC(Soc(Q_{max}(Q_2)))$. Now, since $Q_i = Soc(Q_i)$ implies $Q_i = LC(Q_i)$, Q_i is a local order in $LC(Soc(Q_{max}(Q_i)))$ by 5.4, and we get $Q_i = LC(Soc(Q_{max}(Q_i)))$ by 4.18, and thus there is a unique isomorphism $Q_1 \cong Q_2$ which restricts to the identity on J . \square

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