ASSOCIATIVE GEOMETRIES. II: INVOLUTIONS, THE CLASSICAL GROUDS, AND THEIR HOMOTOPES

WOLFGANG BERTRAM AND MICHAEL KINYON

ABSTRACT. For all classical groups (and for their analogs in infinite dimension or over general base fields or rings) we construct certain contractions, called *homotopes*. The construction is geometric, using as ingredient *involutions of associative geometries*. We prove that, under suitable assumptions, the groups and their homotopes have a canonical semigroup completion.

INTRODUCTION: THE CLASSICAL GROUPS REVISITED

The purpose of this work is to explain two remarkable features of classical groups:

(1) every classical group is a member of a "continuous" family interpolating between the group and its "flat" Lie algebra; put differently, there is a geometric construction of "contractions" (in this context also called *homotopes*),
(2) every classical group and all of its homotopes admit a canonical completion to a semigroup; the underlying (compact) space of all of these "semigroup hulls" is the same for all homotopes.

In fact, these results hold much more generally. The key property of classical groups is that they are closely related to *associative algebras*: either they are (quotients of) unit groups of such algebras, or they are (quotients of) $*$-*unitary groups*

$$\mathrm{U}(\mathbb{A}, *) := \{u \in \mathbb{A} \mid uu^* = 1\} \tag{0.1}$$

for some *involutive associative algebra* $(\mathbb{A}, *)$. This way of characterizing classical groups suggests to consider as "classical" also all other groups given by these constructions, including infinite-dimensional groups and groups over general base fields or rings $\mathbb{K}$, obtained from general involutive associative algebras $(\mathbb{A}, *)$ over $\mathbb{K}$.

On an algebraic, or "infinitesimal", level, features (1) and (2) are supported by simple observations on associative algebras: as to (1), associative algebras really are families of products $(x, y) \mapsto xay$ (the homotopes, see below), and as to (2), it is obvious that an associative algebra forms a semigroup and not a group with respect to multiplication. Our task is, then, to "globalize" these simple observations, and at the same time to put them into the form of a geometric theory: we have to free them from choices of base points (such as 0 and the unit 1 in an associative algebra). Just as in classical geometry, this means to proceed from a "linear" to a "projective" formulation, with an "affine" formulation as intermediate piece. For classical groups of the "general linear type" (A_n), this has already been achieved in Part I of this

2000 *Mathematics Subject Classification.* 20N10, 17C37, 16W10.

Key words and phrases. Lie-Jordan algebras, Jordan-Lie algebras, Hermitian projective line, polar decomposition, classical groups, associative triple systems, ternary rings of operators.

work ([BeKi09]). In the present work we look at the remaining families (B_n, C_n, D_n) and their generalizations. They correspond to associative algebras *with involution*, so that the geometric theory of *involutions* will be a central topic of this work. Let us start by describing the "infinitesimal" situation (i.e., the Lie algebra level), before explaining how to "globalize" it.

0.1. **Homotopes of classical Lie algebras.** The concept of *homotopy* is at the base of Part I of this work: an associative algebra $\mathbb{A}$ should be seen rather as a *family* of associative algebras $(\mathbb{A}, (x, y) \mapsto xay)$, parametrized by $a \in \mathbb{A}$. This gives rise to a family of Lie brackets $[x, y]_a = xay - yax$ also called *homotopes*, interpolating between the "usual" Lie bracket ($a = 1$) and the trivial one ($a = 0$). In particular, taking for $\mathbb{A}$ the matrix space $M(n, n; \mathbb{K})$ with Lie bracket $[X, Y]_A$ for $A \in M(n, n; \mathbb{K})$ we get a Lie algebra which will be denoted by $\mathfrak{gl}_n(A; \mathbb{K})$.

For abstract Lie algebras, there is no such construction; however, there is a variant that can be applied to all classical Lie algebras: let us add an *involution* $*$ (antiautomorphism of order 2) as a new structural feature to our associative algebra $\mathbb{A}$, and write

$$\mathbb{A} = \mathrm{Herm}(\mathbb{A}, *) \oplus \mathrm{Aherm}(\mathbb{A}, *) = \{x \in \mathbb{A} \,|\, x^* = x\} \oplus \{x \in \mathbb{A} \,|\, x^* = -x\}$$

for the eigenspace decomposition. If we fix $a \in \mathrm{Herm}(\mathbb{A}, *)$, then $* : \mathbb{A} \to \mathbb{A}$ is an antiautomorphism of the homotope bracket $[\cdot, \cdot]_a$, and therefore $(\mathrm{Aherm}(\mathbb{A}, *), [\cdot, \cdot]_a)$, $a \in \mathrm{Herm}(\mathbb{A}, *)$, is a family of Lie algebra structures on $\mathrm{Aherm}(\mathbb{A}, *)$, again called homotopes. Remarkably, the construction works also the other way round: if we fix $a \in \mathrm{Aherm}(\mathbb{A}, *)$, then $\mathbb{A} \to \mathbb{A}$, $x \mapsto x^*$ is an automorphism of the homotope bracket $[\cdot, \cdot]_a$, and hence $(\mathrm{Herm}(\mathbb{A}, *), [\cdot, \cdot]_a)$, $a \in \mathrm{Aherm}(\mathbb{A}, *)$, is a family of "homotopic" Lie algebra structures on $\mathrm{Herm}(\mathbb{A}, *)$. For instance, taking for $\mathbb{A}$ the matrix algebra $M(n, n; \mathbb{K})$ with involution $X^* := X^t$ (transposed matrix; in this case we write $\mathrm{Sym}(n; \mathbb{K})$ and $\mathrm{Asym}(n; \mathbb{K})$ for the eigenspaces), we get contractions of the *orthogonal Lie algebras*, denoted by $\mathfrak{o}_n(A; \mathbb{K}) := \mathrm{Asym}(n; \mathbb{K})$ with bracket $[X, Y]_A$ for symmetric matrices A. For $A = 1$, we get the usual Lie algebra $\mathfrak{o}(n)$; for $A = I_{p,q}$ (diagonal matrix of signature (p, q) with $p + q = n$) we get the pseudo-orthogonal algebras $\mathfrak{o}(p, q)$, but for $p + q < n$ we get a new kind of Lie algebras: they are *not* form algebras since the Lie algebra of a degenerate form has bigger dimension than the one of a non-degenerate form, whereas our contractions preserve dimension. Likewise, for skew-symmetric A, we get homotopes of "symplectic type" $\mathfrak{sp}_{n/2}(A; \mathbb{K}) := \mathrm{Sym}(n; \mathbb{K})$ with bracket $[X, Y]_A$. If n is even and A invertible, then this algebra is isomorphic to the usual symplectic algebra $\mathfrak{sp}(n, \mathbb{K})$, and if A is not invertible, we get "degenerate" homotopes; as in the orthogonal case, these algebras are not form algebras of a degenerate skewsymmetric form. If n is odd, then the family contains only "degenerate members", which we call *half-symplectic.*

Summing up, looking at homotope Lie brackets on $\mathrm{Aherm}(\mathbb{A}, *)$ not only serves to imbed the usual Lie bracket into a family, but also to restore a remarkable formal duality between $\mathrm{Aherm}(\mathbb{A})$ and $\mathrm{Herm}(\mathbb{A})$ which usually gets lost. An algebraic setting that takes account of this duality from the outset is the one of an *associative pair* (see Appendix A and [BeKi09]). For instance, the square matrix algebras $\mathfrak{gl}_n(A; \mathbb{K})$ are generalized by the *rectangular matrix algebras* $\mathfrak{gl}_{p,q}(A; \mathbb{K}) := M(p, q; \mathbb{K})$ with bracket $[X,Y]_A$ where A now belongs to the "opposite" matrix space $M(q,p;\mathbb{K})$. In the pair setting, a map ϕ is an involution if and only if so is $-\phi$, and hence $\mathrm{Herm}(\phi)$ and $\mathrm{Aherm}(\phi)$ simply change their rôles if we replace ϕ by $-\phi$. It is only the consideration of *unit* or *invertible elements* that may break this symmetry: they may exist in one space but not in the other.

The following table summarizes the definition of classical Lie algebras and their homotopes. In the general linear cases, $\mathbb{K}$ may be any ring (in particular, the quaternions $\mathbb{H}$ are admitted); in the orthogonal and symplectic families $\mathbb{K}$ has to be a commutative ring, and for the unitary families we use an involution of $\mathbb{K}$: if $\mathbb{K}=\mathbb{C}$, we use usual complex conjugation, and for $\mathbb{K}=\mathbb{H}$ we use the following conventions: if nothing else is specified, we use the "usual" conjugation $\lambda\mapsto\overline{\lambda}$ (minus one on the imaginary part $\mathrm{im}\mathbb{H}$ and one on the center $\mathbb{R}\subset\mathbb{H}$). If we consider $\mathbb{H}$ with its "split" involution $\lambda\mapsto\widetilde{\lambda}:=j\overline{\lambda}j^{-1}$, then we write $\widetilde{\mathbb{H}}$. For instance, $\mathrm{Herm}(n;\widetilde{\mathbb{H}})$ is the space of quaternionic matrices such that $\widetilde{X}=X^t$, and $\mathfrak{u}_n(1;\widetilde{\mathbb{H}})$ is the Lie algebra often denoted by $\mathfrak{so}^*(2n)$. In all cases, the Lie bracket is $[X,Y]_A=XAY-YAX$. Note finally that the trace map does not behave well with respect to our contractions, and therefore we do not define homotopes of special linear or special unitary algebras.

family name	label and space	parameter space	Lie bracket
general linear (square)	$\mathfrak{gl}_n(A;\mathbb{K}):=M(n,n;\mathbb{K})$	$A\in M(n,n;\mathbb{K})$	$[X,Y]_A$
general linear (rectan.)	$\mathfrak{gl}_{p,q}(A;\mathbb{K}):=M(p,q;\mathbb{K})$	$A\in M(q,p;\mathbb{K})$	$[X,Y]_A$
orthogonal	$\mathfrak{o}_n(A;\mathbb{K}):=\mathrm{Asym}(n;\mathbb{K})$	$A\in\mathrm{Sym}(n;\mathbb{K})$	$[X,Y]_A$
[half-] symplectic	$\mathfrak{sp}_{n/2}(A;\mathbb{K}):=\mathrm{Sym}(n;\mathbb{K})$	$A\in\mathrm{Asym}(n;\mathbb{K})$	$[X,Y]_A$
$\mathbb{C}$-unitary	$\mathfrak{u}_n(A;\mathbb{C}):=\mathrm{Aherm}(n;\mathbb{C})$	$A\in\mathrm{Herm}(n;\mathbb{C})$	$[X,Y]_A$
$\mathbb{H}$-unitary	$\mathfrak{u}_n(A;\mathbb{H}):=\mathrm{Aherm}(n;\mathbb{H})$	$A\in\mathrm{Herm}(n;\mathbb{H})$	$[X,Y]_A$
$\mathbb{H}$-unitary split	$\mathfrak{u}_n(A;\widetilde{\mathbb{H}}):=\mathrm{Aherm}(n;\widetilde{\mathbb{H}})$	$A\in\mathrm{Herm}(n;\widetilde{\mathbb{H}})$	$[X,Y]_A$

The expert reader will certainly have remarked that everything we have said so far holds, *mutatis mutandis*, for "Lie" replaced by "Jordan": $\mathrm{Herm}(\mathbb{A},*)$ is a *Jordan* algebra, and in the *Jordan pair* setting the rôles of $\mathrm{Herm}(\mathbb{A})$ and $\mathrm{Aherm}(\mathbb{A})$ become more symmetric. Indeed, a conceptual and axiomatic theory will use the Jordan- *and* Lie-aspects of an associative product in a crucial way – see remarks in Chapter 6 and in [Be08c]. In order to keep this paper accessible for a wide readership, no use of Jordan theory will be made in this work.

0.2. **Homotopes of classical groups.** Now let us explain the main ideas serving to "globalize" the Lie algebra situation just described. First of all, for the classical Lie algebras introduced above it is easy to define explicitly a corresponding algebraic group: in the setting of an abstract unital algebra $\mathbb{A}$ with Lie bracket $[x,y]_a=xay-yax$, one defines the set

$$G(\mathbb{A},a):=\{x\in\mathbb{A}|\,1-xa\in\mathbb{A}^\times\}$$

and checks that

$$x\cdot_a y:=x+y-xay$$

is a group law on $G(\mathbb{A},a)$ with neutral element 0 and inverse of x given by

$$j_a(x):=-(1-xa)^{-1}x.$$

It is easily seen (cf. Lemma 1.2) that the Lie algebra of this group is given by the bracket $[x,y]_a$. Next, observe that an involution $*$ of $\mathbb{A}$ induces an isomorphism from $G(\mathbb{A},a)$ onto the opposite group of $G(\mathbb{A},a^*)$. Therefore, if a is Hermitian, $*$ induces a group antiautomorphism of order 2, and we can define the *a-unitary group* as usual to be the subgroup of elements $g \in G(\mathbb{A},a)$ such that $g^* = j_a(g)$. If a is skew-Hermitian, the *a-symplectic group* is defined similarly by the condition $-g^* = j_a(g)$. Specializing to the classical matrix algebras, we get the following list of classical groups:

label	underlying set	parameter space	product
$\mathrm{Gl}_n(A;\mathbb{K})$	$:= \{X \in M(n,n;\mathbb{K}) \vert 1 - AX \text{ invertible}\}$	$A \in M(n,n;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{Gl}_{p,q}(A;\mathbb{K})$	$:= \{X \in M(p,q;\mathbb{K}) \vert 1 - AX \text{ invertible}\}$	$A \in M(q,p;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{O}_n(A;\mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{K}) \vert X + X^t = X^t A X\}$	$A \in \mathrm{Sym}(n;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{Sp}_{n/2}(A;\mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{K}) \vert X - X^t = X^t A X\}$	$A \in \mathrm{Asym}(n;\mathbb{K})$	$X \cdot_A Y$
$\mathrm{U}_n(A;\mathbb{C})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{K}) \vert X + \overline{X}^t = \overline{X}^t A X\}$	$A \in \mathrm{Herm}(n;\mathbb{C})$	$X \cdot_A Y$
$\mathrm{U}_n(A;\mathbb{H})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{H}) \vert X + \overline{X}^t = \overline{X}^t A X\}$	$A \in \mathrm{Herm}(n;\mathbb{H})$	$X \cdot_A Y$
$\mathrm{O}_n(A;\widetilde{\mathbb{H}})$	$:= \{X \in \mathrm{Gl}_n(A,\mathbb{H}) \vert X + \widetilde{X}^t = \widetilde{X}^t A X\}$	$A \in \mathrm{Herm}(n;\widetilde{\mathbb{H}})$	$X \cdot_A Y$

Finally, one may observe that this realization of classical groups has the advantage to lead to a natural "semigroup hull": e.g., if $A^t = A$, a direct computation shows that the set $\hat{\mathrm{O}}_n(A;\mathbb{K}) := \{X \in M(n,n;\mathbb{K}) \vert\, X^t + X = X^t A X\}$ is stable under the product $\cdot_A$, which turns it into a semigroup with unit element 0, and similarly in all other cases.

0.3. **"Projective" theory of classical grouds.** The definition of the classical groups given above is useful for calculating their Lie algebras and for starting to analyze their group structure (and their topological structure if $\mathbb{K}$ is a topological field or ring), but also has several drawbacks: firstly, note that the product $X \cdot_A Y$ is affine in both variables, and hence our groups are realized as subgroups of the affine group of the matrix space $M(n,n;\mathbb{K})$. The corresponding *linear representation* in a space of dimension n^2+1 is not very natural, and one may wish to realize these groups in more natural linear representations. Secondly, whereas the general linear groups are, for all A, realized as (Zariski-dense) parts of a common ambient space ($M(n,n;\mathbb{K})$, resp. $M(p,q;\mathbb{K})$), this is not the case for the other classical groups: the underlying set depends on A, and hence the realization is not adapted to the point of view of deformations or contractions. Finally, and related to the preceding item, one has the impression that the "semigroup hull" $\hat{\mathrm{O}}_n(A;\mathbb{K})$ depends on the realization, and that it should be part of some maximal semigroup hull intrinsically associated to the group $\mathrm{O}_n(A;\mathbb{K})$.

In the present work, we will give another realization of the classical groups (and, much more generally, of the groups attached to abstract involutive algebras) having none of these drawbacks: it is a sort of projective realization, as opposed to the affine picture just given. In a first step, we get rid of base points in groups by considering them as *grouds*, that is, we work with the ternary product $(xyz) := xy^{-1}z$ of a group. By *classical groud* we simply mean the classical groups from the preceding table equipped with this ternary law, i.e., by forgetting their base points. For the general linear family, we have seen in Part I of this work that there is a common realization of all groups $\mathrm{Gl}_{p,q}(A,\mathbb{K})$ inside the *Grassmannian* $\mathcal{X} := \mathrm{Gras}(\mathbb{K}^{p+q})$ in such a way that they are realized as subgroups of the projective group $\mathbb{P}\mathrm{Gl}(p+q,\mathbb{K})$. The parameter space is again the complete space $\mathcal{X}$, and "space" and "parameter" variables are incorporated into a single object (called an *associative geometry*, given by a pentary product map $\Gamma : \mathcal{X}^5 \to \mathcal{X}$) having surprising properties. In the present work we show that, for the other families, there is a more refined construction, relying on the existence of *involutions* (antiautomorphisms of order 2) of associative geometries. For the classical groups, these involutions are orthocomplementation maps, so that the fixed point spaces are *varieties of Lagrangian subspaces.* We will realize all orthogonal groups as (Zariski dense) subsets of the Lagrangian variety of a quadratic form of signature (n,n), and the [half-] symplectic groups in the Lagrangian variety of a symplectic form on $\mathbb{K}^{2n}$. The underlying Lagrangian variety plays the rôle of a "projective compactification" of these groups, and in particular we will show that the group law extends to a semigroup law on the projective compactification, thus defining the intrinsic and maximal (compact) semigroup hull for all classical groups and their homotopes. As in the general linear case, this achieves a realization in which all "deformations" or "contractions" are globally defined on the space level. In contrast to the general linear case, the parameter space now is different from the underlying Lagrangian variety of the group spaces: it is another Lagrangian variety which we call the *dual Lagrangian.* This duality reflects the duality between $\mathrm{Herm}(\mathbb{A},*)$ and $\mathrm{Aherm}(\mathbb{A},*)$ mentioned in Section 0.1.

0.4. **Contents.** The contents of this paper is as follows: in Chapter 1 we recall basic facts on the "general linear construction"; in Chapter 2 we define and construct involutions of associative geometries: in Theorem 2.2 we prove that orthocomplementation maps of non-degenerate forms are involutions; in Chapter 3 we describe the "projective" construction of grouds and groups associated to (restricted) involutions of associative geometries (Lemma 3.1), their tangent objects with respect to various choices of base points (Theorems 3.6 and 3.6) as well as the link with the "affine" realization given above (Theorem 3.3). In Chapter 4 we present the classification of homotopes of classical groups (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the case of general base fields or rings being at least as complicated as the problem of classifying involutive associative algebras, see [KMRS98]). In Chapter 5 we describe the semi-group completion of classical groups (Theorem 5.6); the main difficulty here is to prove that non-degenerate forms induce involutions of geometries in a "strong" sense. This needs some investigations on the linear algebra of linear relations, complementing those from Chapter 2 of Part I of this work, and which may be of interest in their own right. Finally, in Chapter 6 we give some brief comments on a possible axiomatic approach, involving both the Jordan- and the Lie side of the whole structure, and Appendix A contains the relevant definitions on involutions of associative pairs.

0.5. **Related work.** Finally, let us add some words on related literature. It seems to be folklore (see Appendix on "linear symplectic reduction" in [CDW87]) in symplectic geometry that the group law of $\mathrm{Sp}(m,\mathbb{R})$ extends to the whole Lagrangian variety if we interpret it via *composition of linear relations*: the composition of two Lagrangian linear relations is again Lagrangian. In a case-by-case way, Y. Neretin ([Ner96]) has given similar constructions for other families of complex or real Lagrangrian varieties ("categories B, C, D", see loc. cit., p. 85 ff and loc. cit. Appendix A for their real analogs). It would be very interesting to investigate further the relation of our work with Neretin's, in particular in view of applications in harmonic analysis and quantization. Note that Neretin in loc. cit. p. 59 uses a modified composition law of linear relations in order to obtain a jointly continuous operation; since we do not consider topologies here, we leave the [important] topic of joint continuity for later work.

Notation. Throughout this work, $\mathbb{K}$ denotes a commutative unital ring and $\mathbb{B}$ an associative unital $\mathbb{K}$-algebra, and we will consider *right* $\mathbb{B}$-modules $V, W, \ldots$. We think of $\mathbb{B}$ as "base ring", and the letter $\mathbb{A}$ will be reserved for other associative $\mathbb{K}$-algebras such as $\mathrm{End}_{\mathbb{B}}(W)$.

If $V = a \oplus b$ is a direct sum decomposition of a vector space or module, we denote by $P^a_b : V \to V$ the projection with kernel a and image b.

1. The general linear family

1.1. **Groups and grouds living in Grassmannians.** We are going to recall the basic construction from [BeKi09] which realizes groups like $\mathrm{Gl}_n(A, \mathbb{K})$ inside a Grassmannian manifold. Let W be a right $\mathbb{B}$-module and $\mathcal{X} = \mathrm{Gras}(W)$ be the Grassmannian of all right B-submodules of W. A pair $(x, a) \in \mathcal{X}^2$ is called *transversal* (denoted by $a \top x$ or $x \top a$) if $W = x \oplus a$. The set of all complements of a is denoted by C_a, so that

$$C_{ab} := C_a \cap C_b$$

is the set of common complements of a and b. One of the main results of [BeKi09] says that the set C_{ab} carries two canonical groud-structures. More precisely, we define, for $(x, a, b, z) \in \mathcal{X}^4$ such that $a \top x$, $b \top z$, the endomorphism of W

$$M_{xabz} := P^a_x - P^z_b = P^a_x - 1 + P^b_z. \tag{1.1}$$

By a direct calculation (see [BeKi09], Prop. 1.1), one sees that

$$M_{xabz} = M_{zbax}, \quad M_{xabz} = -M_{axzb}, \tag{1.2}$$

and, if $x, z \in U_{ab}$, then M_{xabz} is invertible with inverse

$$(M_{xabz})^{-1} = M_{zabx} = M_{xbaz}. \tag{1.3}$$

Recall (see, e.g., [BeKi09]) that a *groud* is the base point-free version of a group (a set G with a ternary map $G^3 \to G$, $(xyz) \mapsto (xyz)$ such that $(xyy) = x = (yyx)$ and $(xy(zuv)) = ((xyz)uv)$). Then ([BeKi09], Th. 1.2):

Theorem 1.1. *i) For $a, b \in \mathcal{X}$ fixed, C_{ab} with product*

$$(xyz) := \Gamma(x, a, y, b, z) := M_{xabz}(y)$$

is a groud (which will be denoted by U_{ab}). In particular, for a triple (a, y, b) with $y \in C_{ab}$, C_{ab} is a group with unit y and multiplication $xz = \Gamma(x, a, y, b, z)$.

ii) U_{ab} is the opposite groud of U_{ba} (same set with reversed product):

$$\Gamma(x, a, y, b, z) = \Gamma(z, b, y, a, x)$$

In particular, the groud $U_a := U_{aa}$ is commutative.

iii) The commutative groud U_a is the underlying additive groud of an affine space: for any $a \in \mathcal{X}$, U_a is an affine space over $\mathbb{K}$, with additive structure given by

$$x +_y z = \Gamma(x, a, y, a, z),$$

(sum of x and z with respect to the origin y), and action of scalars given by

$$\Pi_s(x, a, y) := sy + (1-s)x = (sP^x_a + P^a_x)(y)$$

(multiplication of y by s with respect to the origin x).

Definition. (The restricted multiplication map) *We call* restricted multiplication map *the map $\Gamma : D_5 \to \mathcal{X}$, defined on the* set of admissible 5-tuples

$$D_5 := \{(x, a, y, b, z) \in \mathcal{X}^5 | \, x, y, z \in C_{ab}\},$$

by the formula from Part i) of the preceding theorem.

Definition. (Base points and tangent spaces) *A* base point *in $\mathcal{X}$ is a fixed transversal pair, usually denoted by (o^+, o^-). The* tangent space at (o^+, o^-) *is the pair*

$$(\mathbb{A}^+, \mathbb{A}^-) := (C_{o^-}, C_{o^+}).$$

Note that $(\mathbb{A}^+, \mathbb{A}^-)$ is a pair of $\mathbb{K}$-modules (with origin $o^\pm$ in $\mathbb{A}^\pm$), isomorphic to

$$\big(\mathrm{Hom}_{\mathbb{B}}(o^+, o^-), \mathrm{Hom}_{\mathbb{B}}(o^-, o^+)\big). \tag{1.4}$$

This tangent space carries the structure of an associative pair *given by trilinear products (see* [BeKi09], *Th. 1.5)*

$$\mathbb{A}^\pm \times \mathbb{A}^\mp \times \mathbb{A}^\pm \to \mathbb{A}^\pm, \quad (u, v, w) \mapsto < u, v, w >^\pm := \Gamma(u, o^+, v, o^-, w). \tag{1.5}$$

Definition. (Transversal triples) *A* transversal triple *is a triple of mutually transverse elements. If we fix such a triple, we usually denote it by (o^+, e, o^-). In this case, $\mathbb{A} := C_{o^-}$ carries the structure of an associative algebra with origin $o := o^+$ and unit e, called the* tangent algebra at o^+ corresponding to the base triple (o^+, e, o^-)*, with product*

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A}, \quad (u, v) \mapsto \Gamma(u, o^+, e, o^-, v). \tag{1.6}$$

In a dual way, C_{o^+} is turned into an algebra with origin o^-. Both algebras are canonically isomorphic via the inversion map $j = M_{eo^+o^-e}$.

1.2. **Lie algebra and structure of the grouds U_{ab}.** We explain the link between the grouds U_{ab} and the groups $\mathrm{Gl}_{p,q}(A; \mathbb{B})$ defined in the Introduction, as well as the computation of their "Lie algebra".

Lemma 1.2. *Choose an origin o^+ in U_{ab} and an element $o^- \top o^+$. Then the Lie algebra (in a sense to be explained in the following proof) of the group (U_{ab}, o^+) is the "tangent space" $\mathbb{A}^+ = \mathrm{Hom}_{\mathbb{B}}(o^+, o^-)$ with Lie bracket*

$$[X, Y] = X(a-b)Y - Y(a-b)X$$

(note that $o^+ \in U_{ab}$ means that $a, b \in C_{o^+} = \mathbb{A}^-$, so that $a - b \in \mathbb{A}^-$). In particular, choosing $o^- = b$, we get the Lie algebra of U_{A0}:

$$[X, Y] = XAY - YAX.$$

Proof. The Lie algebra can be defined in a purely algebraic way, without using ordinary differential calculus, as follows. Let $T\mathbb{K} := \mathbb{K}[\varepsilon] := \mathbb{K}[X]/(X^2)$, $\varepsilon^2 = 0$ be the ring of dual numbers over $\mathbb{K}$ and $TT\mathbb{K} := T(T\mathbb{K}) := (\mathbb{K}[\varepsilon_1])[\varepsilon_2]$ be the "second order tangent ring". Then $(\mathcal{X}, \Gamma)$ admits scalar extensions from $\mathbb{K}$ to $T\mathbb{K}$ and to $TT\mathbb{K}$, and the commutator in the second scalar extension of the group U_{ab} gives rise to the Lie bracket in the way described in [Be08], Chapter V. This construction is intrinsic and does not depend on "charts". Therefore we may choose $o^- := b$ in order to simplify calculations (the first formula from the claim then follows from the second one). Then $U_{ab} = C_a \cap C_{o^-} = C_a \cap \mathbb{A}^+$, and according to [BeKi09], Section 1.4 we have the following "affine picture" of the group (U_{ab}, o): if, under the isomorphism (1.4), a corresponds to the element $A \in \mathbb{A}^- = \mathrm{Hom}_{\mathbb{B}}(o^-, o^+)$, then U_{ab} corresponds to the set

$$U_{A0} = \{X \in \mathrm{Hom}_{\mathbb{B}}(o^+, o^-) | \, 1 - AX \text{ is invertible in } \mathrm{End}_{\mathbb{B}}(o^+)\} \tag{1.7}$$

with group law given by the product $Z \cdot_A X$ defined in the Introduction:

$$X \cdot Z = X + Z - ZAX. \tag{1.8}$$

Since Formulas (1.7) and (1.8) are algebraic, we may now determine explicitly the *tangent group* of U_{0A} via scalar extension by dual numbers: the operator

$$1 - (A + \varepsilon A')(X + \varepsilon X') = 1 - AX + \varepsilon(A'X + AX')$$

is invertible iff so is $1 - AX$, hence the tangent bundle $T(U_{0A})$ is $U_{0A} \times \varepsilon \mathrm{Hom}_{\mathbb{B}}(o^+, o^-)$, with semidirect product group structure

$$(X, \varepsilon X') \cdot (Z, \varepsilon Z') = \big(X + Z - ZAX, \varepsilon(X' + Z' + Z'AX + ZAX')\big). \tag{1.9}$$

Repeating the construction, we obtain the second tangent bundle $TT(U_{0A})$ by scalar extension from $\mathbb{K}$ to the ring $TT\mathbb{K}$. As explained in [Be08], the Lie bracket $[X, Y]$ arises from the commutator in the second tangent group via

$$\varepsilon_1 \varepsilon_2 [X, Y] = (\varepsilon_1 X)(\varepsilon_2 Y)(\varepsilon_1 X)^{-1}(\varepsilon_2 Y)^{-1}.$$

A direct calculation, based on (1.9), yields

$$(\varepsilon_1 X)(\varepsilon_2 Y) = \varepsilon_1 X + \varepsilon_2 Y + \varepsilon_1 \varepsilon_2 YAX,$$

which, after a short calculation using that $(\varepsilon_1 X)^{-1} = \varepsilon_1(-X)$, $(\varepsilon_1 Y)^{-1} = \varepsilon_1(-Y)$, implies the claim. □

As is easily seen from the explicit formulas given above by choosing for A special (idempotent) elements (cf. [Be08b]), the groups U_{ab} and their Lie algebra have a *double fibered structure*. These and related features for symmetric spaces will be investigated in [BeBi].

2. Construction of involutions

2.1. Definition of (restricted) involutions. Whenever in a category we have for each object $\mathcal{X}$ a canonical notion of an "opposite object" $\mathcal{X}^{op}$, there is a natural notion of *involution*. This is the case for groups, grouds or associative geometries.

Definition. *A* restricted involution *of the Grassmannian geometry* $\mathcal{X} = \mathrm{Gras}(W)$ *is a bijection* $f : \mathcal{X} \to \mathcal{X}$ *of order two and such that*

(1) f preserves transversality: for all $a, x \in \mathcal{X}$: $a \top x$ iff $f(a) \top f(x)$,
(2) f is an isomorphism onto the opposite restricted product map: for all 5-tuples (x, a, y, b, z) such that $x, y, z \in U_{ab}$,

$$f\big(\Gamma(x, a, y, b, z)\big) = \Gamma(fx, fb, fy, fa, fz) = \Gamma(fz, fa, fy, fb, fx).$$

(3) f induces affine maps on affine parts: for all 3-tuples (x, a, y) such that $x, y \top a$, and $r \in \mathbb{K}$,

$$f\big(\Pi_r(x, a, y)\big) = \Pi_r\big(fx, fa, fy\big).$$

In other words, by (1), f induces well-defined restrictions $U_{ab} \to U_{f(a),f(b)}$ and $U_a \to U_{f(a)}$, which induce, by (2), anti-isomorphisms of grouds $U_{ab} \to U_{f(a),f(b)}$, and by (3), isomorphisms of affine spaces $U_a \to U_{f(a)}$.

The fixed point space $\mathcal{Y} := \mathcal{X}^\tau$ of an involution τ will be called the Lagrangian type geometry of $(\mathcal{X}, \tau)$ *(if it is not empty).*

In general, nothing guarantees *existence* of restricted involutions. Before turning to the general theory (next chapter), we will show that under certain conditions one can construct them by using bilinear or sesquilinear forms. In these cases, $\mathcal{Y}$ will be indeed realized as a geometry of Lagrangian subspaces.

2.2. **Non-degenerate forms and adjointable pairs.** We assume that our $\mathbb{B}$-module W admits a non-degenerate sesquilinear form

$$\beta : W \times W \to \mathbb{B}.$$

By sesquilinearity we mean $\beta(vr, w) = \overline{r}\beta(v, w)$, $\beta(v, wr) = \beta(v, w)r$ for $v, w \in W$, $r \in \mathbb{B}$, where

$$\mathbb{B} \to \mathbb{B}, \quad z \mapsto \overline{z}$$

is some fixed involution (antiautomorphism of order 2) of $\mathbb{B}$, and non-degeneracy means that $\beta(v, W) = 0$ or $\beta(W, v) = 0$ implies $v = 0$. Of course, for $\mathbb{B} = \mathbb{K}$ and $\overline{z} = z$ we get bilinear forms. Moreover, we assume that β is Hermitian or skew-Hermitian:

$$\forall v, w \in W : \ \beta(v, w) = \overline{\beta(w, v)}, \quad \text{resp.} \quad \forall v, w \in W : \ \beta(v, w) = -\overline{\beta(w, v)}.$$

As usual, the orthogonal complement of a subset $S \subset W$ will be denoted by $S^\perp$. The orthogonal complement of a right submodule is again a right submodule, but, unfortunately, it is in general not true that the orthocomplementation map $\perp : \mathcal{X} \to \mathcal{X}$ satisfies the properties of a (restricted) involution: in general, it does not even preserve transversality, nor is it of order two.

Definition. *A pair $(x, a) \in \mathcal{X} \times \mathcal{X}$ is called* adjointable *if $W = x \oplus a$ and $W = x^\perp \oplus a^\perp$.*

Lemma 2.1. *A pair $(x, a) \in \mathcal{X} \times \mathcal{X}$ is adjointable if and only if the projection $P := P^a_x$ is adjointable; i.e., there exists a linear operator $P^* : W \to W$ such that*

$$\forall v, w \in W : \qquad \beta(v, Pw) = \beta(P^* v, w). \tag{2.1}$$

Moreover, in this case we have $(x^\perp)^\perp = x$ and $(a^\perp)^\perp = a$.

Proof. Assume P^* exists. If two operators f, g are adjointable, then we have $(gf)^* = f^* g^*$, and hence P^* is again idempotent. Moreover, the kernel of P^* is $\ker P^* = (\mathrm{im} P)^\perp = x^\perp$. Now, P is adjointable if and only if so is $Q := 1 - P$, whence $\mathrm{im} P^* = \ker Q^* = a^\perp$, and thus $W = x^\perp \oplus a^\perp$. Moreover, this shows that

$$(P_x^a)^* = P_{a^\perp}^{x^\perp}. \tag{2.2}$$

Reversing these arguments, we see that, if (x, a) is adjointable, equation (2.2) defines an operator P^*, and a direct check shows that then (2.1) holds. Moreover, from $(P^*)^* = P$ the relations $(x^\perp)^\perp = x$ and $(a^\perp)^\perp = a$ follow. □

The lemma shows that, in the general case, we should not work with the full Grassmannian, but only with its adjointable elements. For simplicity, let us first look at a case where the Grassmannian is well-behaved, namely the case $W = \mathbb{B}^n$:

Theorem 2.2. (Construction of involutions: case of $\mathbb{B}^n$) *Let $W = \mathbb{B}^n$ and $\mathcal{X}$ be the Grassmannian of all right submodules that admit some complementary right submodule, and let β be a non-degenerate Hermitian or skew-Hermitian form on $\mathbb{B}$. Then the orthocomplementation map*

$$\perp_\beta : \mathcal{X} \to \mathcal{X}, \quad x \mapsto x^\perp$$

is a restricted involution of $\mathcal{X}$.

Proof. For $W = \mathbb{B}^n$, every non-degenerate sesquilinear form is given by

$$\beta(x, y) = \sum_{i,j=1}^{n} \overline{x}_i b_{ij} y_j$$

with some invertible matrix $B = (b_{ij})$. By assumption, B is Hermitian or skew-Hermitian. As can be checked by a direct matrix calculation, in this case every linear operator $X : W \to W$ is adjointable, with adjoint given by the adjoint matrix X^* of (X_{ij}):

$$X^* = B^{-1} \overline{X}^t B$$

where X^t is the transposed matrix of X. In particular, if x is an arbitrary complemented right-submodule of $\mathbb{B}^n$ with complement a, then $P := P_x^a$ is adjointable. Thus every transversal pair (x, a) is adjointable, and moreover

$$x^\perp = \ker(P)^\perp = \mathrm{im}(P^*).$$

We have thus shown that the orthocomplementation map is of order two and preserves transversalilty. In order to prove the crucial property

$$\Gamma(z^\perp, a^\perp, y^\perp, b^\perp, x^\perp) = \big(\Gamma(x, a, y, b, z)\big)^\perp. \tag{2.3}$$

of a restricted involution, we use the following lemma (which holds for general Grassmannians):

Lemma 2.3. *Let $\mathcal{X} = \mathrm{Gras}(W)$ and $x, z \in C_{ab}$ and $c \top y$. Then the operator M_{xabz} is invertible with inverse operator M_{zabx}, and*

$$M_{xabz} P_y^c (M_{xabz})^{-1} = P_{M_{xabz}(y)}^{M_{xabz}(c)} = P_{\Gamma(x,a,y,b,z)}^{\Gamma(x,a,c,b,z)}. \tag{2.4}$$

Proof. For the first assertion, see (1.3). Conjugating a projection P_y^c by a linear isomorphism f gives a new projection $P_{f(y)}^{f(c)}$, whence the claim. □

Now we prove (2.3). From (2.2) it follows that, for $x, z \in C_{ab}$, the operator M_{xabz} has an adjoint given by

$$(M_{xabz})^* = (P_x^a - P_b^z)^* = -M_{x^\perp a^\perp b^\perp z^\perp}. \tag{2.5}$$

Let $a, b \in \mathcal{X}$ and $x, y, z \in C_{ab}$. Choose any $c \in C_y$. Then, using the lemma,

$$\begin{aligned}(M_{zabx})^*(P_y^c)^*(M_{xabz})^* &= (M_{xabz}P_y^c M_{zabx})^* \\ &= (P_{(\Gamma(x,a,y,b,z))}^{(\Gamma(x,a,c,b,z))})^* \\ &= P_{(\Gamma(x,a,c,b,z))^\perp}^{(\Gamma(x,a,y,b,z))^\perp}\end{aligned}$$

and on the other hand (again using the lemma)

$$\begin{aligned}(M_{zabx})^*(P_y^c)^*(M_{xabz})^* &= M_{z^\perp a^\perp b^\perp x^\perp} P_{c^\perp}^{y^\perp} M_{x^\perp a^\perp b^\perp z^\perp} \\ &= P_{\Gamma(z^\perp,a^\perp,c^\perp,b^\perp,x^\perp)}^{\Gamma(z^\perp,a^\perp,y^\perp,b^\perp,x^\perp)}.\end{aligned}$$

Comparing the images and kernels of these projections yields (2.3). Finally, property (3) of an involution can be proved in the same way as (2.3) (and this property is already known since it depends only on the underlying Jordan structure, see, e.g., [Be04]). □

The cases $n = 1$ and $n = 2$ of the preceding result deserve special interest. For $n = 1$, we work with the form $\beta(u, v) = \overline{u}\, v$, and we consider the Grassmannian of complemented right ideals in $\mathbb{B}$ with involution $\ker e \mapsto \operatorname{im} \overline{e}$ (where $e \in \mathbb{B}$ is an idempotent, $\ker e = (1 - e)\mathbb{B}$, $\operatorname{im} e = e\mathbb{B}$). The case $n = 2$ enters in the proof of Theorem 3.7 (next chapter).

2.3. **The adjointable Grassmannian.** As we will see in Theorem 3.7, the case $n = 2$ is already suitable to treat all seemingly more general cases. Returning thus to the case of a general $\mathbb{B}$-module W with a non-degenerate Hermitian or skew-Hermitian form β, we may proceed as follows: let

$$\mathbb{A} := \{f \in \mathrm{End}_{\mathbb{B}}(W) \mid \exists f^* \in \mathrm{End}_{\mathbb{B}}(W) : \forall v, w \in W : \beta(v, fw) = \beta(f^* v, w)\}$$

the set of all adjointable linear operators. Then $\mathbb{A}$ is a subalgebra of $\mathrm{End}_{\mathbb{B}}(W)$, and $*$ is an involution on $\mathbb{A}$. Now define the *adjointable Grassmannian of* β to be

$$\mathcal{X}_\beta := \{\operatorname{im} P \mid P \in \mathbb{A}, P^2 = P\},$$

the set of all submodules x admitting a complement a such that the projection $P := P_x^a$ is adjointable. (In general, not all submodules have this property – consider e.g. a dense proper subspace x in a Hilbert space.) Let $\tilde{\mathcal{X}} := \{P\mathbb{A} \mid P \in \mathbb{A}, P^2 = P\}$ be the Grassmannian of all complemented right modules in $\mathbb{A}$. Then the map

$$\tilde{\mathcal{X}} \to \mathcal{X}_\beta, \quad P\mathbb{A} \mapsto \operatorname{im} P$$

is well-defined, bijective and compatible with the structure maps Γ. We use it to push down τ to an involution of $\mathcal{X}_\beta$, so that we can carry out all preceding constructions on the adjointable Grassmannian.

3. Groups and grouds associated to involutions

We assume, for all of this chapter, that $\tau : \mathcal{X} \to \mathcal{X}$ is a restricted involution of the Grassmannian geometry $\mathcal{X} = \mathrm{Gras}_{\mathbb{B}}(W)$ and write $\mathcal{Y}$ for its fixed point space. There are two different ways to construct groups and grouds associated to $(\mathcal{X}, \tau)$. Here is the first construction, which simply mimics the usual definition of unitary and orthogonal groups:

Definition. *Fix three points $a, o, b \in \mathcal{Y}$ such that $o \in U_{ab}$, considered as origin in the group (U_{ab}, o), and let $x^{-1} := M_{oabo}(x)$ be inversion in this group. Then τ induces an antiautomorphism of this group:*

$$\begin{aligned} \tau(xy) &= \tau\Gamma(x, a, o, b, y) = \Gamma(\tau(y), \tau(a), \tau(o), \tau(b), \tau(x)) \\ &= \Gamma(\tau(y), a, o, b, \tau(x)) = \tau(y)\tau(x), \end{aligned}$$

and hence

$$\mathrm{U}(\tau; a, o, b) := \{x \in U_{ab} |\, \tau(x) = x^{-1}\}$$

*is a subgroup, called the τ-*unitary group (located at (a, o, b)).

This group is not a subset of the Lagrangien geometry $\mathcal{Y}$, but rather is "tangent" to the "antifixed space of τ": indeed, the differential of inversion at o is the negative of the identity, and hence the tangent space of $\mathrm{U}(\tau; a, o, b)$ at the identity should be the minus one eigenspace of τ. This will be made precise below (Theorem 3.3). Next, we describe a second construction of groups having the advantage that it directly leads to grouds living in the Lagrangian geometry:

Lemma 3.1. *Let $\tau : \mathcal{X} \to \mathcal{X}$ be a restricted involution of the Grassmannian geometry $\mathcal{X} = \mathrm{Gras}_{\mathbb{B}}(W)$ and denote by $\mathcal{Y} := \mathcal{X}^\tau$ its Lagrangian type geometry. Then*

i) for any $a \in \mathcal{X}$, τ induces a groud-automorphism of the groud $U_{a,\tau(a)}$. In particular, the fixed point set

$$\mathcal{G}(\tau; a) := (U_{a,\tau(a)})^\tau = U_{a,\tau(a)} \cap \mathcal{Y}$$

is a subgroud of $U_{a,\tau(a)}$.

ii) As a set, $\mathcal{G}(\tau, a) = U_a \cap \mathcal{Y}$.

iii) $\mathcal{G}(\tau, \tau(a))$ is the opposite groud of $\mathcal{G}(\tau, a)$. If $a \in \mathcal{Y}$, then the groud $\mathcal{G}(\tau, a)$ is abelian, and it is the underlying additive groud of an affine space over $\mathbb{K}$.

Proof. (i) Note first that $x \in C_{a,\tau(a)}$ if and only if $\tau(x) \in C_{\tau(a),\tau^2(a)} = C_{a,\tau(a)}$ since τ preserves transversality and is of order 2. Next we show that τ preserves the groud law $(xyz)_a = \Gamma(x, a, y, \tau(a), z)$ of $U_{a,\tau(a)}$:

$$\begin{aligned} \tau(((xyz)_a) &= \tau(\Gamma(x, a, y, \tau(a), z)) = \Gamma(\tau z, \tau a, \tau y, a, \tau x) \\ &= \Gamma(\tau x, a, \tau y, \tau a, \tau z) = (\tau x\, \tau y\, \tau z)_a. \end{aligned}$$

Clearly, the fixed point space $U_{a,\tau(a)} \cap \mathcal{Y}$ is then a subgroud.

(ii) If $x \in \mathcal{Y}$, i.e., $\tau(x) = x$, then $x \top a$ is equivalent to $x \top \tau(a)$, whence

$$\mathcal{Y} \cap U_a = \mathcal{Y} \cap U_a \cap U_{\tau(a)} = \mathcal{Y} \cap U_{a,\tau(a)}.$$

(iii) $U_{a,\tau(a)}$ is the opposite groud of $U_{\tau(a),a}$. If $a = \tau(a)$, then the arguments given above show that τ is an automorphism of order 2 of the affine space U_a and hence its fixed point space is an affine subspace. □

In order to compare both constructions, we have to to study the behaviour of involutions with respect to basepoints.

3.1. **Basepoints, and the dual involution.** Let us fix a base point (o^+, o^-) in $\mathcal{X}$. Recall from [BeKi09], Th. 1.3, that the middle multiplication operator $M_{o^+o^-o^-o^+}$ is an automorphism of Γ. By (1.3), it is invertible and equal to its own inverse. Moreover,

$$M_{o^+o^-o^-o^+}(o^\pm) = \Gamma(o^+, o^-, o^\pm, o^-, o^+) = o^\pm.$$

Thus $M_{o^+o^-o^-o^+}$ is a base point preserving automorphism of the Grassmannian geometry. Its effect on the additive groups $\mathbb{A}^\pm$ is simply inversion, that is, multiplication by the scalar -1.

Definition. *A (restricted) involution τ of $\mathcal{X}$ is called*

- base point preserving *if $\tau(o^+) = o^+$ and $\tau(o^-) = o^-$, and*
- base point exchanging *if $\tau(o^+) = o^-$ and $\tau(o^-) = o^+$.*

Lemma 3.2. *Assume τ is a base point preserving or base point exchanging involution of $\mathcal{X}$. Then τ commutes with the automorphism $M_{o^+o^-o^-o^+}$, and*

$$\tau' := M_{o^+o^-o^-o^+} \circ \tau = \tau \circ M_{o^+o^-o^-o^+}$$

is again of the same type (base point preserving, resp. exchanging involution) as τ, called the dual involution *(denoted by $-\tau$ in a context where (o^+, o^-) is fixed).*

Proof. Thanks to the symmetry relation $M_{xabz} = M_{axzb}$ we get in either case

$$\tau \circ M_{o^+o^-o^-o^+} \circ \tau = M_{\tau o^+, \tau o^-, \tau o^-, \tau o^+} = M_{o^+o^-o^-o^+}.$$

Therefore τ' is again of order 2, and it is an antiautomorphism having the same effect on $o^\pm$ as τ since $M_{o^+o^-o^-o^+}$ is base point preserving. □

Recall from [BeKi09] that, with respect to a fixed base point (o^+, o^-) and $a \in \mathbb{A}^-$,

$$\tilde{t}_a := M_{o^+ao^-o^+} \circ M_{o^+o^-o^-o^+} = M_{ao^+o^+o^-} \circ M_{o^-o^+o^+o^-} = L_{ao^+o^-o^+}$$

is the (left) translation operator by a in the abelian group $U_{o^+} \cong \mathbb{A}^-$. It acts rationally on $\mathbb{A}^+$ by the so-called *quasi inverse map.*

Theorem 3.3. *Assume τ is a base point preserving involution of $\mathcal{X}$ and let $a \in \mathcal{Y} \cap U_{o^-} = (\mathbb{A}^+)^\tau$. Then the groups $\mathcal{G}(-\tau; a)$ and $\mathrm{U}(\tau; 2a, o^+, o^-)$ are isomorphic (the multiple $2a = a + a$ taken in $\mathbb{A}^+$). An isomorphism is induced by $\tilde{t}_a$.*

Proof. Having fixed the base point, we use the notation $-\mathrm{id} := M_{o^+o^-o^-o^+}$. We have to show that the group $U_{a,-a}$ with its automorphism τ' is conjugate to the group U_{2a,o^-} with its automorphism $i_{2a}\tau$ where $i_{2a} := M_{o^+2a\,o^-o^+}$ is inversion in the group (U_{2a,o^-}, o^+). First of all,

$$\tilde{t}_a(a) = a + a = 2a, \quad \tilde{t}_a(-a) = a + (-a) = o^-$$

(sums in $(\mathbb{A}^-, o^-)$), hence $\tilde{t}_a$ induces a groud isomorphism from $U_{a,-a}$ onto U_{2a,o^-} preserving the base point o^+. Next, observe that

$$i_{2a} \circ (-\mathrm{id}) = M_{o^+2ao^-o^+} \circ M_{o^+o^-o^-o^+} = \tilde{t}_{2a}$$

whence, using that $\tau' \circ \tilde{t}_a = \tilde{t}_{\tau' a} \circ \tau' = \tilde{t}_{-a} \circ \tau'$,

$$\tilde{t}_{-a} \circ i_{2a}\tau \circ \tilde{t}_a = \tilde{t}_{-a} \circ \tilde{t}_{2a} \circ (-\mathrm{id}) \circ \tau \circ \tilde{t}_a = \tilde{t}_{-a} \circ \tilde{t}_{2a}\tau' \circ \tilde{t}_a = \tilde{t}_{-a}\tilde{t}_{2a}\tilde{t}_{-a} \circ \tau' = \tau'$$

where the last equality follows from the relation $\tilde{t}_b \tilde{t}_c = \tilde{t}_{b+c}$. □

In the affine chart $\mathbb{A}^+$, $\tilde{t}_a$ acts as a birational map, transforming the affine realization $\mathrm{U}(\tau; 2a, o^+, o^-)$ to a rational realization that is Zariski-dense in $(\mathbb{A}^+)^{-\tau}$. If 2 is invertible in $\mathbb{K}$, all τ-unitary groups $\mathrm{U}(\tau; b, o, c)$ have such a realization $\mathcal{G}(\tau'; a)$ (just choose the base point $(o^+, o^-) = (o, c)$ and let $a := b/2$). If 2 is not invertible in $\mathbb{K}$, such a realization is not always possible.

Concerning *involutions of associative pairs* and *associative triple systems*, to be used in the following result, see Appendix A.

Theorem 3.4. *Assume τ is a restricted involution of the Grassmannian geometry $\mathcal{X}$, and let $(\mathbb{A}^+, \mathbb{A}^-)$ be the associative pair corresponding to a base point (o^+, o^-).*

i) If $\tau : \mathcal{X} \to \mathcal{X}$ is base-point preserving, then by restriction τ induces $\mathbb{K}$-linear maps $\tau^\pm : \mathbb{A}^\pm \to \mathbb{A}^\pm$ which form a type preserving involution of $(\mathbb{A}^+, \mathbb{A}^-)$.

ii) If $\tau : \mathcal{X} \to \mathcal{X}$ is base-point exchanging, then by restriction τ induces $\mathbb{K}$-linear maps $\tau^\pm : \mathbb{A}^\pm \to \mathbb{A}^\mp$ which form a type exchanging involution of $(\mathbb{A}^+, \mathbb{A}^-)$. In this case $\mathbb{A} := \mathbb{A}^+$ becomes an associative triple system of the second kind when equipped with the product

$$< xyz > := \Gamma(x, o^+, \tau(y), o^-, z).$$

iii) Assume $\tau : \mathcal{X} \to \mathcal{X}$ is base-point preserving, and let $a \in \mathcal{Y}'$ such that $o^+ \top a$ (i.e., $a \in \mathbb{A}^-$ and $\tau(a) = -a$). Then the Lie algebra of the group $(\mathcal{G}(\tau; a), o^+)$ is the space $(\mathbb{A}^+)^{\tau^+}$ with Lie bracket

$$[x, z]_a = 2(< xaz > - < zax >).$$

Proof. (i), (ii): All claims are simple applications of the functoriality of associating an associative pair to an associative geometry with base pair, [BeKi09], Theorem 3.5. For convenience, let us just spell out the computation proving the property of an associative triple system in Part ii):

$$\begin{aligned} < u < xyz > w > &= \Gamma\Big(u, o^+, \tau\big(\Gamma(x, o^+, \tau(y), o^-, z)\big), o^-, w\Big) \\ &= \Gamma\Big(u, o^+, \Gamma(\tau z, \tau o^+, y, \tau o^-, \tau x), o^-, w\Big) \\ &= \Gamma\Big(u, o^+, \Gamma(\tau z, o^-, y, o^+, \tau x), o^-, w\Big) \\ &= \Gamma\Big(\Gamma(u, o^+, \tau z, o^-, y), o^+, \tau x, o^-, w\Big) \; = << uzy > xw > \end{aligned}$$

(If we had used a base point preserving *automorphism* instead of an involution, a similar calculation shows that we would get an associative triple system of the *first* kind, see Appendix A.)

(iii): Using Lemma 1.2, with $b = \tau(a) = -a$ (since the effect of τ on $\mathbb{A}^-$ is multiplication by -1), we get the Lie bracket $[x, z] =< x(2a)z > - < z(2a)x >$. □

Putting the preceding two results together, we obtain an explicit description of the groups $\mathcal{G}(-\tau; b/2) \cong \mathrm{U}(\tau; b, o^+, o^-)$ in terms of the associative pair $(\mathbb{A}^+, \mathbb{A}^-)$:

$$\mathrm{U}(\tau; b, o^+, o^-) = \{x \in \mathbb{A}^+ |\, 1 - xb \,\text{invertible}, \tau(x) = j_b(x)\}$$

with $j_b(x) = -(1-xb)^{-1}x$, so that the condition $-\tau(x) = j_b(x)$ is equivalent to $x + \tau(x) =< xb\tau(x) >$. This formulation is valid for an arbitrary associative pair with base-point preserving involution. In practice, all known examples arise for associative pairs corresponding to *unital* associative algebras, to be discussed next.

3.2. **Base triples, unitary groups, and Cayley transform.** Next let us assume that W admits a *transversal triple* (o^+, e, o^-). Then $W = o^+ \oplus o^-$, and saying that e is transversal to o^+ and o^- amounts saying that e is the graph of a linear isomorphism $o^+ \to o^-$. We may consider this isomorphism as an identification, so that e becomes the diagonal Δ_+ in $W = o^+ \oplus o^- = o^+ \oplus o^+$. Then the element

$$-e := M_{o^+o^-o^-o^+}(e)$$

becomes the antidiagonal Δ_- in $o^+ \oplus o^+$. In this situation, we may let act the group $\mathrm{Gl}(2,\mathbb{K})$ by block-matrices on $W = o^+ \oplus o^+$ in the usual way. Let $G \subset \mathrm{Gl}(2,\mathbb{K})$ by the group generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\lambda \in \mathbb{K}^\times$. The first matrix describes left translation by e,

$$L_{eo^-o^+o^-} := 1 - P^e_{o^-} P^{o^-}_{o^+},$$

the second multiplication by the scalar λ,

$$\delta^\lambda_{o^+o^-} = \lambda P^{o^+}_{o^-} + P^{o^-}_{o^+},$$

and the third describes a map j whose effect on the associative algebra $\mathbb{A}$ is inversion:

$$j := M_{eo^+o^-e} = M_{o^+eeo^-}.$$

All of these operators are (inner) automorphisms of the geometry $(\mathcal{X}, \Gamma)$. From (1.1) it follows that j is an automorphism of order 2, but this time it exchanges the points o^+ and o^-:

$$M_{eo^+o^-e}(o^+) = \Gamma(e, o^+, o^+, o^-, e) = \Gamma(o^+, e, o^+, e, o^-) = o^-.$$

Moreover, $j(e) = M_{eo^+o^-e}(e) = \Gamma(e, o^+, e, o^-, e) = e$.

Definition. *If (o^+, e, o^-) is a transversal triple, we call τ a*

- unital base point preserving involution *if* $\tau(o^+) = o^+$, $\tau(o^-) = o^-$, $\tau(e) = e$,
- unital base point exchanging involution *if* $\tau(o^+) = o^-$, $\tau(o^+) = o^-$, $\tau(e) = e$.

Note that, if τ is of one of these two types, then the dual involution τ' no longer preserves e. Indeed, $M_{o^+o^-o^-o^+}(e) = -e$ is the antidiagonal, which is different from the diagonal (if W has no 2-torsion). Thus the rôles of τ and τ' are no longer completely symmetric in the unital case.

Lemma 3.5. *Assume τ is a unital base point preserving involution of $\mathcal{X}$. Then τ commutes with the automorphism $j = M_{eo^+o^-e}$, and*

$$\tilde{\tau} := j\tau = \tau j$$

is a unital base-point exchanging involution. Moreover, if 2 *is invertible in* $\mathbb{K}$*, there exists an automorphism* $\rho : \mathcal{X} \to \mathcal{X}$ *("the real Cayley transform") such that*

$$\rho \circ \tau \circ \rho^{-1} = \tau, \qquad \rho \circ \tilde{\tau} \circ \rho^{-1} = \tau'.$$

Proof. As in the proof of Lemma 3.2, we see that

$$\tau j \tau = \tau M_{eo^+o^-e} \tau = M_{eo^+o^-e} = j,$$

hence $j\tau$ is of order two, and it exchanges base points and is again an involution.

The automorphism ρ is constructed as follows: let $\rho \in G$ be given by the matrix

$$R := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then ρ commutes with τ: indeed, τ commutes with all generators of the group G mentioned above (since these operators are partial maps of Γ involving only the τ-fixed elements $o^+, o^-, e, -e$ and hence commute with τ), hence τ commutes with R. Since R sends the 4-tuple $(o^-, e, o^+, -e)$ to $(e, o^+, -e, o^-)$, it follows that

$$\rho j \rho = \rho M_{eo^+o^-e} \rho = M_{o^+(-e)eo^+} = M_{o^+o^-o^-o^+}$$

(the last equality follows since $M_{o^+(-a)ao^+} = M_{(-a)o^+o^+a} = M_{o^-o^+o^+o^-}$ is the map $x \mapsto (-a) - x + a = -x$ for all $a \in V^-$). Together, this implies

$$\rho \circ \tilde{\tau} \circ \rho^{-1} = \rho \circ \tau j \circ \rho^{-1} = \tau \rho \circ j \circ \rho^{-1} = \tau \circ M_{o^+o^-o^-o^+} = \tau'.$$

(Note that R is not uniquely determined by the property from the lemma, but the given form corresponds of course to the well-known "real" version of the Cayley transform which enjoys further nice properties.) □

Theorem 3.6. *Assume* τ *is a unital base-point preserving involution of the Grassmannian geometry* $(\mathcal{X}; o^+, e, o^-)$*, and let* $\mathbb{A} = C_{o^-}$ *be the corresponding unital associative algebra with origin* o^+ *and* $\mathbb{A}^- = C_{o^+}$ *the one with origin* o^-*, let* τ' *the dual involution of* τ*,* $\tilde{\tau} = j\tau$*,* $\mathcal{Y} := \mathcal{X}^\tau$ *and* $\mathcal{Y}' := \mathcal{X}^{\tau'}$*. Let* $a \in \mathcal{X}$ *such that* $o^+ \top a$*, i.e.,* $a \in \mathbb{A}^-$*.*

i) By restriction, τ *induces an involutive antiautomorphism of* $\mathbb{A}$*. This defines a functor from the category of unital involutive associative geometries to the category of involutive associative algebras.*

ii) If $a \in \mathcal{Y}'$*, then the Lie algebra of the group* $\mathcal{G}(\tau; a)$ *is the space* $\mathrm{Herm}(\mathbb{A}, \tau) = \mathbb{A}^\tau$ *with Lie bracket* $[x, z]_a = 2(< xaz > - < zax >)$*. Identifying* $\mathbb{A}$ *and* $\mathbb{A}^-$ *via the canonical isomorphism* j*,* a *is identified with the element* $j(a) \in \mathrm{Aherm}(\mathbb{A}, \tau)$ *and the Lie bracket is expressed in terms of* $\mathbb{A}$ *as*

$$[x, z]_a = 2(xaz - zax).$$

iii) If $a \in \mathcal{Y}$*, then the Lie algebra of the group* $\mathcal{G}(\tau'; a)$ *is the space* $\mathrm{Aherm}(\mathbb{A}, \tau) = \mathbb{A}^{\tau'}$ *with Lie bracket* $[x, z]_a = 2(< xaz > - < zax >)$*. With similar identifications as above, this can be rewritten as* $[x, z]_a = 2(xaz - zax)$*.*

If, moreover, a *is invertible in* $\mathbb{A}$*, then the group* $\mathcal{G}(\tau'; a)$ *is isomorphic to the unitary group* $\mathrm{U}(\mathbb{A}_a, *) = \{x \in \mathbb{A} | \, xax^* = 1\}$ *of the involutive algebra* $(\mathbb{A}_a, \tau)$ *with product* $x \cdot_a y = xay$ *and involution* τ*.*

Proof. (i) We show that τ induces an algebra involution:

$$\tau(xz) = \tau\Gamma\big(x, o^+, e, o^-, z\big) = \Gamma\big(\tau z, o^+, e, o^-, \tau x\big) = (\tau z)(\tau x)$$

Functoriality follows from [BeKi09], Theorem 3.4.

(ii) The fixed point space of τ in $\mathbb{A}$ is, by definition, $\mathrm{Herm}(\mathbb{A}, *)$, and by Lemma 3.1, τ is an automorphism of $U_{a\tau(a)}$. The formula from the Lie bracket follows from Theorem 3.4. Finally, in order to relate the associative pair to the algebra formulation, recall from [BeKi09] that, for all $a \in \mathbb{A}^-$ and $x, z \in \mathbb{A}^+$,

$$< xay >^+ = x \cdot j(a) \cdot z,$$

where on the right hand side products are taken in the algebra $\mathbb{A}$. Since the $\mathbb{K}$-linear isomorphism $j : \mathbb{A}^+ \to \mathbb{A}^-$ commutes with τ, the formulas from the claim follow.

(iii) The statement on the Lie algebra is proved in the same way as (ii), with signs changed. Now let a be invertible. Assume first $a = 1$. Note that the condition $xx^* = 1$ is equivalent to $x = (x^*)^{-1} = j\tau(x)$, and hence $\mathrm{U}(\mathbb{A}, *)$ is precisely the fixed point set of $\tilde{\tau}$ in $\mathbb{A}$. Its group structure is induced from $\mathbb{A}^\times = U_{o^+o^-}$. Now, the setting $(\mathbb{A}^\times, \tilde{\tau}) = (U_{o^+o^-}, j\tau)$ is conjugate, via the Cayley transform ρ, to the setting $(U_{e,-e}, \tau') = (U_{e,\tau'(e)}, \tau')$, showing that the Cayley transform ρ induces the desired isomorphism. In these arguments, the fixed element $e \in (\mathcal{Y} \cap U_{o^+o^-})$ may be replaced by any other element a of this set; this simply amounts to replacing $\mathbb{A}$ by its isotope algebra $\mathbb{A}_a$. □

Theorem 3.7. *Consider the following classes of objects:*

IG: *associative geometries with base triple and base triple preserving involutions,*
IA: *involutive unital associative algebras.*

There are maps $F : \mathbf{IG} \to \mathbf{IA}$ *and* $G : \mathbf{IA} \to \mathbf{IG}$ *such that* $G \circ F$ *is the identity.*

Proof. The map F is defined by Part (i) of the preceding theorem. We define the map G: given an involutive associative algebra $(\mathbb{A}, *)$, let $\hat{\mathcal{X}}$ be the Grassmannian of complemented right $\mathbb{A}$-submodules in $\mathbb{A}^2$. We define on $\mathbb{A}^2$ the skew-Hermitian ("symplectic") form

$$\beta(x, y) = \overline{x}_1 y_2 - \overline{x}_2 y_1.$$

and consider the involution τ given by the orthocomplementation map with respect to this form. Let $o^+ = \mathbb{A} \oplus 0$ (first factor), $o^- = 0 \oplus \mathbb{A}$ (second factor) and $e = \Delta$ (diagonal in $\mathbb{A}^2$). Then (o^+, e, o^-) is a transversal triple, preserved by τ. This defines G. The associative algebra C_{o^-} associated to these data is the algebra $\mathbb{A}$ we started with (cf. [BeKi09], Theorem 3.5). It remains to prove that restriction of τ to $C_{o^-} = \mathbb{A}$ gives back the involution $*$ we started with. Let $a \in \mathbb{A}$ and identify it with the graph $\{(v, av) | \, v \in \mathbb{A}\}$. Then the graph of the adjoint operator a^* is the orthogonal complement of this graph with respect to β, whence $\tau(a) = a^*$. □

We have seen above that F is a functor; for G, this is less clear – cf. remarks in [BeKi09], Section 3.4. We will not pursue here further the discussion of functoriality, nor will we state an analog of the theorem for the non-unital case. Constructions are similar in that case, but are more complicated (since one has to use some algebra-imbedding of an associative pair, see [BeKi09]), and practically less relevant than the unital case.

4. The classical grouds

Putting together the results from the preceding two chapters, the "projective" description of the classical groups (Table given in the Introduction) is now straightforward: we just have to restate Theorems 3.3 and 3.6 for involutions given by orthocomplementation (Theorem 2.2). In the following, we list the results, first for the case of bilinear forms, then for sesquilinear forms.

4.1. **Orthogonal and (half-) symplectic groups.** We specialize Theorem 3.6 to the case $\mathbb{B} = \mathbb{K}$, $W = \mathbb{K}^{2n} = \mathbb{K}^n \oplus \mathbb{K}^n$. Let (o^+, e, o^-) be the canonical base triple $(\mathbb{K}^n \oplus 0, \Delta, 0 \oplus \mathbb{K}^n)$ and β the standard symplectic form on $\mathbb{K}^{2n}$. By Theorem 2.2, we have the three (restricted) involutions τ, τ', $\tilde{\tau}$: they are the orthocomplementation maps with respect to the three forms given by the matrices

$$\Omega_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad F_n := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \quad I_{n,n} := \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}. \tag{4.1}$$

Note that o^+, o^- and Δ are maximal isotropic for β, hence τ is a unital base point preserving involution. The involutive algebra corresponding to the unital base point preserving involution τ is $\mathbb{A} = M(n, n; \mathbb{K})$ with involution $X^* = X^t$ (usual transposed). The fixed point spaces of the three involutions are the classical Lagrangian varieties corresponding to the three forms, and the tangent space of $\mathcal{Y}$ at o^+ is $\mathrm{Sym}(n, \mathbb{K})$ and the one of $\mathcal{Y}'$ at o^+ is $\mathrm{Asym}(n, \mathbb{K})$. Note that $\mathrm{Sym}(n, \mathbb{K})$ is imbedded in $\mathcal{Y}$, and $\mathrm{Asym}(n, \mathbb{K})$ in $\mathcal{Y}'$, the subsets of elements of $\mathcal{Y}$ (resp. of $\mathcal{Y}'$) that are transversal to o^-. Therefore the elements a parametrizing the grouds $\mathcal{G}(\tau; a)$ (resp. $\mathcal{G}(\tau'; a)$) will be chosen in these subsets. From Theorem 3.3 we get:

Proposition 4.1. *For $a = A \in \mathrm{Sym}(n, \mathbb{K})$, the group $\mathcal{G}(\tau; a)$ with origin o^+ is isomorphic to the group $\mathrm{O}_n(2A, \mathbb{K})$, and for $a = A \in \mathrm{Asym}(n, \mathbb{K})$, the group $\mathcal{G}(\tau'; a)$ with origin o^+ is isomorphic to the group $\mathrm{Sp}_{n/2}(2A; \mathbb{K})$. If 2 is invertible in $\mathbb{K}$, then these groups are isomorphic to $\mathrm{O}_n(A, \mathbb{K})$, resp. $\mathrm{Sp}_{n/2}(A; \mathbb{K})$.*

Having made the link of the projective grouds $\mathcal{G}(\tau; a)$ with the affine realization of the classical grouds from the Introduction, it is now relatively easy to *classify* them (in finite dimension over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$; the case of general base fields is much more difficult, and for general base rings and arbitrary dimension, classification results can only be expected under rather special assumptions).

Proposition 4.2. *A complete classification of the homotopes of complex or real orthogonal, resp. (half-)symplectic groups is given as follows:*

1. *(half-)symplectic case: for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, all homotopes are isomorphic to one of the groups $\mathrm{Sp}_m(\Omega_r; \mathbb{K})$ for $r = 1, \ldots, m$ (with $n = 2m$ or $n = 2m + 1$), where Ω_r denotes the normal form of a skew-symmetric matrix of rank $2r$,*
2. *orthogonal case: for $\mathbb{K} = \mathbb{C}$, all homotopes are isomorphic to one of the groups $\mathrm{O}_n(1_r; \mathbb{C})$ for $r = 1, \ldots, n$, where 1_r denotes the $n \times n$-diagonal matrix of rank r having first r diagonal elements equal to one,*

   for $\mathbb{K} = \mathbb{R}$, all homotopes are isomorphic to one of the groups $\mathrm{O}_n(I_{r,s}; \mathbb{R})$, where $I_{r,s}$ denotes the $n \times n$-diagonal matrix of rank $r + s$ ($r \leq s$, $r + s \leq n$) having first r diagonal elements equal to one and s diagonal elements equal to minus one.

Proof. One can prove the classification from a "projective" point of view: clearly, if a and b belong to the same $\mathrm{Aut}(\mathcal{X},\tau)$-orbit in $\mathcal{X}$, then $\mathcal{G}(\tau;a)$ and $\mathcal{G}(\tau;b)$ are isomorphic, and it is enough to consider orbits of subspaces $a \subset W$ such that a and $\tau(a)$ have same dimension n (otherwise $U_{a,\tau(a)}$ is empty). Classifying such orbits is done by elementary linear algebra using Witt's theorem: a and b are conjugate iff the restriction of the given forms to a, resp. b are isomorphic. In particular, the totally isotropic subspaces form one orbit (the Lagrangian $\mathcal{Y}$). The list of orbits then gives rise to the given list of homotopes.

Alternatively, an "affine" version of these arguments goes as follows: using the explicit description of the classical groups given in the Introduction, one notices that, e.g., $\mathrm{O}_n(A;\mathbb{K})$ and $\mathrm{O}_n(gAg^t;\mathbb{K})$ are isomorphic for all $g \in \mathrm{Gl}(n;\mathbb{K})$; hence it suffices to to consider the classification of $\mathrm{Gl}(n;\mathbb{K})$-orbits in $\mathrm{Sym}(n;\mathbb{K})$. This leads to the same result (note, however, that different orbits may give rise to isomorphic groups: e.g., $\mathrm{O}_n(\lambda A;\mathbb{K})$ and $\mathrm{O}_n(A;\mathbb{K})$ are isomorphic whenever the scalar λ is invertible, be it a square or not in $\mathbb{K}$). Similarly for the symplectic case. □

4.2. **Unitary groups.** The following classification of real classical grouds associated to involutive algebras of Hermitian type is established in the same way as above:

Proposition 4.3. *Homotopes of complex and quaternionic unitary groups are classified as follows (see Introduction for the notation* $\widetilde{\mathbb{H}}$*):*

	$\mathbb{A} = M(n,n;\mathbb{H})$,	$\tau(X) := \overline{X}^t$	
a)	$\mathbb{A}^\tau = \mathrm{Herm}(n,\mathbb{H})$	$\mathrm{U}_n(i1_r;\widetilde{\mathbb{H}})$ $(r \leq n)$	*homotopes of* $\mathrm{O}^*(2n)$
b)	$\mathbb{A}^{\tau'} = \mathrm{Aherm}(n,\mathbb{H})$	$\mathrm{U}_n(I_{r,s},\mathbb{H})$ $(r \leq s,\ r+s \leq n)$	*homotopes of* $\mathrm{Sp}(p,q)$

	$\mathbb{A} = M(n,n;\mathbb{C})$,	$\tau(X) := \overline{X}^t$	
a)	$\mathbb{A}^\tau = \mathrm{Herm}(n,\mathbb{C})$	$\mathrm{U}_n(iI_{r,s};\mathbb{C})$ $(r \leq s,\ r+s \leq n)$	*homotopes of* $\mathrm{U}(p,q)$
b)	$\mathbb{A}^{\tau'} = i\mathrm{Herm}(n,\mathbb{C})$	$\mathrm{U}_n(I_{r,s};\mathbb{C})$ $(r \leq s,\ r+s \leq n)$	*homotopes of* $\mathrm{U}(p,q)$

Over more general base fields or rings the classification of non-degenerate grouds is essentially equivalent to the classification of involutions of associative algebras – see [KMRS98] for this vast topic.

4.3. **Hilbert Grassmannian.** A fairly straightforward infinite dimensional generalization of the preceding situation is the following: $W = H \oplus H$, where H is a Hilbert space W over $\mathbb{B} = \mathbb{C}$ or $\mathbb{R}$, and β corresponding to the matrix

$$B = \Omega_H = \begin{pmatrix} 0 & 1_H \\ -1_H & 0 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 0 & 1_H \\ 1_H & 0 \end{pmatrix}.$$

In this case we may work with the Grassmannian of all *closed* subspaces of W, and it easily seen that all arguments from the proof of Theorem 2.2 go through, showing that the orthocomplementation map of β defines an involution of this geometry. We get infinite dimensional analogs of the classical groups, imbedded, together with their homotopes, in Hilbert-Lagrangian manifolds. Variants of these constructions can be applied to *restricted Grassmannians* and *restricted unitary groups* in the sense of [PS86].

5. Semigrouds

5.1. **Semigroup completion of general linear groups.** Let W be a right $\mathbb{B}$-module and $\mathcal{X}$ its Grassmannian. In [BeKi09] we have shown that the grouds $U_{ab} \subset \mathcal{X}$ admit a "semigroud completion": the ternary law (xyz) from U_{ab} extends to the whole of $\mathcal{X}$, given by the formula

$$\Gamma(x,a,y,b,z) := \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \omega = \zeta + \alpha = \zeta + \eta + \xi = \xi + \beta \end{array} \right\} . \tag{5.1}$$

This formula defines a pentary "product map" $\Gamma : \mathcal{X}^5 \to \mathcal{X}$ having the following remarkable properties: for any fixed pair (a,b), the partial map $(xyz) := \Gamma(x,a,y,b,z)$ satisfies the *para-associative law*

$$(xy(zuv)) = (x(uzy)v) = ((xyz)uv), \tag{5.2}$$

and it is invariant under the Klein 4-group acting on (x,a,b,z):

$$\Gamma(x,a,y,b,z) = \Gamma(a,x,y,z,b) = \Gamma(z,b,y,a,x). \tag{5.3}$$

We say that, for a,b fixed, $\mathcal{X}$ with $(xyz) = \Gamma(x,a,y,b,z)$ is a *semigroud*, denoted by $\mathcal{X}_{ab}$ (for fixed y, it is in particular a semigroup), and $\mathcal{X}_{ba}$ is its *opposite semigroud.*

Definition. *An* involution *of the Grassmannian geometry* $\mathcal{X} = \mathrm{Gras}(W)$ *is a bijection* $f : \mathcal{X} \to \mathcal{X}$ *of order* 2 *such that,* **for all** $x,a,y,b,z \in \mathcal{X}$, $r \in \mathbb{K}$,

$$\tau(\Gamma(x,a,y,b,z) = \Gamma(\tau(z),\tau(a),\tau(y),\tau(b),\tau(x))$$

and $\tau(\Pi_r(x,a,y)) = \Pi_r(\tau(x),\tau(a),\tau(y))$.

The following lemma is now proved exactly as Lemma 3.1:

Lemma 5.1. *Let* $\tau : \mathcal{X} \to \mathcal{X}$ *be an involution of the Grassmannian geometry* $\mathcal{X} = \mathrm{Gras}_{\mathbb{B}}(W)$*, let* $\mathcal{Y} = \mathcal{X}^\tau$ *and* $a \in \mathcal{X}$*. Then* τ *induces a semigroud-automorphism of* $\mathcal{X}_{a,\tau(a)}$*. In particular, the fixed point set* $\mathcal{Y}$ *is a subsemigroud of* $\mathcal{X}_{a,\tau(a)}$*. If* $a \in \mathcal{Y}$*, then the semigroud* $\mathcal{X}_{a,\tau(a)} \cap \mathcal{Y}$ *is abelian.*

We are going to prove that orthocomplementation maps are (under certain conditions) involutions of $\mathcal{X}$ in the strong sense defined above. The proof will be similar to the one of Theorem 2.2; we have to prepare it by proving an analog of Lemma 2.3 in terms of linear relations.

5.2. **Generalized projections.** Linear operators $f \in \mathrm{End}_{\mathbb{B}}(W)$ are generalized by *linear relations in* W, i.e., submodules $F \subset W \oplus W$. Following standard terminology (see, e.g., [Ner96], [Cr98]), *domain*, *image*, *kernel* and *indefiniteness* of F are the subspaces defined by

$$\mathrm{dom} F := \mathrm{pr}_1(F), \quad \mathrm{im} F := \mathrm{pr}_2 F, \quad \ker F := F \cap (W \times 0), \quad \mathrm{indef} F := F \cap (0 \times W).$$

For any $a,b \in \mathcal{X}$, define the linear relation $P^a_x \subset W \oplus W$, called a *generalized projection*, by

$$P^a_x := \big\{ (\zeta,\omega) \,|\, \omega \in x,\, \omega - \zeta \in a \big\}. \tag{5.4}$$

Note that

$$\mathrm{im} P^a_x = x, \quad \ker P^a_x = a, \quad \mathrm{indef} P^a_x = a \cap x, \quad \mathrm{dom} P^a_x = x + a,$$

and that, if $a \top x$, then P^a_x is the graph of the projection denoted previously by P^a_x, so there should be no confusion with preceding notation. We denote the *space of generalized projections* by

$$\mathcal{P} := \{P^a_x \,|\, x, a \in \mathcal{X}\} \subset \mathrm{Gras}(W \oplus W).$$

The map

$$\mathcal{X} \times \mathcal{X} \to \mathcal{P}, \quad (a, x) \mapsto P^a_x$$

is a bijection with inverse $P \mapsto (\ker P, \mathrm{im} P)$. Transversal pairs (x, a) correspond to "true" operators (single valued and everywhere defined).

Lemma 5.2. *The linear relation P^a_x is idempotent: $P^a_x \circ P^a_x = P^a_x$.*

Proof. By definition of composition,

$$P^a_x \circ P^a_x = \{(u, w) | \exists v \in W : v \in x, u - v \in a, w \in x, v - w \in a\}.$$

Since $w \in x$ and $w - u = (w - v) + (v - u) \in a$, we have $P^a_x \circ P^a_x \subset P^a_x$. For the other inclusion, let $(u', w') \in P^a_x$, so $w' \in x$, $w' - u' \in a$. Let $u := u'$, $w := v := w'$; then $v, w \in x$ and $u - v = u' - w' \in a$, $v - w = 0 \in a$, whence $(u', w') \in P^a_x \circ P^a_x$. □

Lemma 5.3. *The set $\mathcal{P}$ of generalized projections is stable under "conjugation" by linear relations in the following sense: for all linear relations $F \subset W \oplus W$ and all $c, z \in \mathcal{X}$, we have*

$$F \circ P^c_z \circ F^{-1} = P^{F(c)}_{F(z)}.$$

Proof. By definition of composition and inverse,

$$\begin{aligned} F \circ P^c_z \circ F^{-1} &= \{(\alpha, \delta) |\, \exists \beta, \gamma \in W : \, (\alpha, \beta) \in F^{-1}, (\beta, \gamma) \in P^c_z, (\gamma, \delta) \in F\} \\ &= \{(\alpha, \delta) |\, \exists \beta \in W, \gamma \in z : \, (\beta, \alpha) \in F, (\gamma, \delta) \in F, \gamma - \beta \in c\} \end{aligned}$$

These conditions imply that $\delta \in Fz$ and $(\beta, \alpha) - (\gamma, \delta) \in F$; since $(\beta - \gamma) \in c$, this implies also $(\alpha - \delta) \in Fc$. It follows that $(\alpha, \delta) \in P^{F(c)}_{F(z)}$.

Conversely, let $(\alpha, \delta) \in P^{F(c)}_{F(z)}$, i.e., $\delta \in F(z)$, $\alpha - \delta \in F(c)$, so there exists $\gamma \in z$ with $(\gamma, \delta) \in F$ and $\eta \in c$ with $(\eta, \alpha - \delta) \in F$. Let $\beta := \gamma - \eta$, so $\gamma - \beta \in c$ and

$$(\beta, \alpha) = (\gamma, \delta) - (\eta, \delta - \alpha) \in F,$$

whence $(\alpha, \delta) \in F \circ P^c_z \circ F^{-1}$. □

For the next statements, recall ([Ar61], [Cr98]) the following general definitions concerning linear relations. For a linear relation $F \subset W \oplus W$ and $z \in \mathcal{X}$, the *image of z under F* is

$$Fz := F(z) := \{\delta \in W |\, \exists \gamma \in z : (\gamma, \delta) \in F\} = \mathrm{pr}_2(\mathrm{pr}_1)^{-1}(z),$$

and the *sum of linear relations* $F, G \subset \mathrm{Gras}(W \oplus W)$, is

$$F + G := \{(\xi, \omega) | \exists \alpha, \beta \in W : (\xi, \alpha) \in F, (\xi, \beta) \in G, \omega = \alpha + \beta\}$$

Remark: This sum (not to be confused with the usual sum of modules!) can also be written in our language in terms of the associative geometry $(\mathrm{Gras}(W \oplus W), \hat{\Gamma})$, with its usual base points o^+, o^-, as

$$F + G := \hat{\Gamma}(F, o^+, o^-, o^+, G),$$

the sum of F and G in the linear space (C_{o^-}, o^+).

Lemma 5.4. *For all* $a, x \in \mathcal{X}$, $1 - P^a_x = P^x_a$.

Proof. $\omega = u - \omega'$ with $\omega' \in x$, $\omega' - u \in a$ is equivalent to $\omega \in a$ with $\omega - u \in x$. □

Theorem 5.5. *Let* Γ *be the multiplication map of the Grassmann geometry* $\mathcal{X}$.

(1) *For all* $(x, a, y, b, z) \in \mathcal{X}^5$,

$$\Gamma(x, a, y, b, z) = (1 - P^x_a P^b_y)(z) = (P^a_x - P^z_b)(y).$$

In other words, the left multiplication operator L_{xayb} *in the geometry* $(\mathcal{X}, \Gamma)$ *is induced by the linear relation* $1 - P^x_a P^b_y$, *and the middle multiplication operator* M_{xabz} *is induced by the linear relation* $P^a_x - P^z_b$. *Thus we can (and will) define, extending the operator notation from Chapter 1, the linear relations*

$$L_{xayb} := 1 - P^x_a P^b_y, \quad M_{xabz} := P^a_x - P^z_b.$$

(2) *For all* $(x, a, z) \in \mathcal{X}^3$,

$$\Gamma(x, a, a, x, z) = P^a_z(x).$$

(3) *For all* $a, b, x, y \in \mathcal{X}$, *using Notation from Part (1),*

$$L^{-1}_{xayb}(z) = L_{yaxb}(z), \quad M^{-1}_{xabz}(y) = M_{zabx}(y).$$

(4) *For all* $a, b, c, x, y, z \in \mathcal{X}$,

$$L_{xayb} \circ P^c_z \circ (L_{xayb})^{-1} = P^{\Gamma(x,a,y,b,c)}_{\Gamma(x,a,y,b,z)}$$

$$M_{xabz} \circ P^c_y \circ (M_{xabz})^{-1} = P^{\Gamma(x,a,c,b,z)}_{\Gamma(x,a,y,b,z)}.$$

Proof. (1) Note that, under certain transversality conditions ensuring that the linear relations in question are indeed graphs of linear operators, the claim has already been proved in [BeKi09]. Let us prove it now in the general situation.

$$\begin{aligned}
P^x_a \circ P^b_y &= \{(\zeta, \omega) | \exists \eta \in y : \zeta - \eta \in b, \omega - \eta \in x, \omega \in a\}, \\
1 - P^x_a P^b_y &= \{(\zeta, \omega') | \exists \omega \in W : (\zeta, \omega) \in P^x_a P^b_y, \omega' = \zeta - \omega\} \\
&= \{(\zeta, \omega') | \exists \omega \in W, \exists \eta \in y : \zeta - \eta \in b, \omega - \eta \in x, \omega \in a, \omega' = \zeta - \omega\}
\end{aligned}$$

whence

$$\begin{aligned}
(1 - P^x_a P^b_y)(z) &= \{\omega' \in W | \exists \zeta \in z, \exists \alpha \in a, \exists \eta \in y : \zeta - \eta \in b, \alpha - \eta \in x, \omega' = \zeta - \alpha\} \\
&= \{\omega' \in W | \exists \alpha \in a, \exists \eta \in y : \omega' + \alpha - \eta \in b, \alpha - \eta \in x, \omega' + \alpha \in z\}
\end{aligned}$$

According to the "(a, y)-description" from [BeKi09], this set is indeed equal to $\Gamma(x, a, y, b, z)$. Similarly,

$$\begin{aligned}
P^a_x - P^z_b &= \{(\eta, \omega) | \exists u, v : (\eta, u) \in P^a_x, (\eta, v) \in P^z_b, \omega = u - v\} \\
&= \{(\eta, \omega) | \exists u \in x, \exists v \in b : u - \eta \in a, v - \eta \in z, \omega = u - v\}
\end{aligned}$$

so that

$$\begin{aligned}
(P^a_x - P^z_b)(y) &= \{\omega | \exists u \in x, \exists v \in b, \exists \eta \in y : u - \eta \in a, v - \eta \in z, \omega = u - v\} \\
&= \{\omega | \exists \beta \in b, \exists \eta \in y : \beta + \omega - \eta \in a, \beta - \eta \in z, \beta + \omega \in x\}
\end{aligned}$$

Again, by the (y, b)-description, this equals $\Gamma(x, a, y, b, z)$.

(2) Using Lemmas 5.2 and 5.4, $\Gamma(x, a, a, x, z) = (1 - P^x_a P^x_a)(z) = (1 - P^x_a)(z) = P^a_x(z)$.

(3) This is a restatement of Theorem 2.5 from [BeKi09].
(4) This follows from Lemma 5.3 together with Parts (1) and (3). □

5.3. **Orthocomplementation maps and adjoints.** We retain notation from Section 2.2: β is a Hermitian or skew-Hermitian non-degenerate form. Using the preceding subsection, we can generalize the arguments used in the proof of Theorem 2.2 in order to show:

Theorem 5.6. *Assume $\mathbb{K}$ is a field and $\mathbb{B}$ a finite dimensional division algebra over $\mathbb{K}$, and let $W = \mathbb{B}^n$. Let $\mathcal{X}$ be the Grassmannnian of complemented $\mathbb{B}$-right submodules of W and β a non-degenerate Hermitian or skew-Hermitian form on W. Then the orthocomplementation map $\perp: \mathcal{X} \to \mathcal{X}$ is an involution of $\mathcal{X}$.*

Proof. We define the *adjoint relation* of a linear relation $F \subset W \oplus W$ by

$$F^* := \{(v', w') | \forall (v, w) \in F : \beta(v', w) = \beta(w', v)\} \subset W \oplus W.$$

This is the orthocomplement of F with respect to the "symplectic form" on $V \oplus V$ associated to β (see [Ar61], [Cr98], Ch. III). We will use the following facts about adjoints:

Lemma 5.7. *Assume F and G are linear relations in W.*

i)

$$(F + G)^* \supset F^* + G^*,$$

with equality, e.g., if the domains of G and G^* are the whole space W.

ii)

$$(G \circ F)^* \supset F^* \circ G^*,$$

with equality, e.g., for linear relations in finite dimensional vector spaces.

iii) For all $a, x \in \mathcal{X}$,

$$P_{a^\perp}^{x^\perp} \subset (P_x^a)^*,$$

with equality, e.g., in case of finite dimension over a field.

Proof. (i) See [Ar61], Theorem 3.41. (ii) See [Ar61], Lemma 3.5 (note that the condition for equality stated in that Lemma is more general, and it holds in particular for finite-dimensional vector spaces since in this case linear forms can always be extended from subspaces to bigger spaces). (iii): By definition of the adjoint,

$$\begin{aligned} (P_x^a)^* &= \{(v', w') | \forall (v, w) \in P_x^a : \beta(v', w) = \beta(w', v)\} \\ &= \{(v', w') | w \in x, v - w \in a \Rightarrow \beta(v', w) = \beta(w', v)\} \end{aligned}$$

Now assume $(v', w') \in P_{a^\perp}^{x^\perp}$, that is, $w' \perp a$, $w' - v' \perp x$. Then, for all $w \in x$ and v with $v - w \in a$:

$$\beta(v', w) = \beta(v' - w', w) + \beta(w', w) = \beta(w', w) = \beta(w', w - v) + \beta(w', v) = \beta(w', v),$$

whence $(v', w') \in (P_x^a)^*$, proving the desired inclusion.

Let us prove that, if W is finite dimensional over $\mathbb{K}$, both spaces in question have the same dimension and hence equality holds. First of all, for every linear relation F, since $\mathrm{pr}_2|_F$ induces an exact sequence $0 \to \ker F \to F \to \mathrm{im} F \to 0$,

$$\dim F = \dim(\ker F) + \dim(\mathrm{im} F)$$

hence

$$\dim P^a_x = \dim(a) + \dim(x), \quad \dim P^{x^\perp}_{a^\perp} = \dim(a^\perp) + \dim(x^\perp).$$

Since $(P^a_x)^*$ is the orthogonal complement of P^a_x with respect to a non-degenerate form on $W \oplus W$,

$$\dim(P^a_x)^* = \dim(W \oplus W) - \dim P^a_x = \dim W - \dim x + \dim W - \dim a = \dim P^{x^\perp}_{a^\perp},$$

proving the claim. □

Now we prove the theorem. Under the given assumptions, using all three parts of the lemma, and that image and domain of the identity relation 1 and its adjoint 1 is the whole space, we get

$$\begin{aligned} (L_{xayb})^* &= (1 - P^x_a P^b_y)^* = 1 - (P^x_a P^b_y)^* = 1 - (P^b_y)^* (P^x_a)^* \\ &= 1 - P^{a^\perp}_{x^\perp} P^{y^\perp}_{b^\perp} = L_{y^\perp b^\perp x^\perp a^\perp}. \end{aligned} \tag{5.5}$$

Now we prove that (2.3) holds for all $x, a, y, b, z \in \mathcal{X}$. Choose an auxiliary element $c \in \mathcal{X}$. Then, using the lemma and part (4) of Theorem 5.5,

$$\begin{aligned} (L^{-1}_{xayb})^*(P^c_z)^*(L_{xayb})^* &= (L_{xayb} P^c_z L^{-1}_{xayb})^* \\ &= (P^{(\Gamma(x,a,y,b,c))}_{(\Gamma(x,a,y,b,z))})^* \\ &= P^{(\Gamma(x,a,y,b,z))^\perp}_{(\Gamma(x,a,c,b,z))^\perp} \end{aligned}$$

and on the other hand (using again the lemma and Theorem 5.5)

$$\begin{aligned} (L^{-1}_{xayb})^*(P^c_z)^*(L_{xayb})^* &= (L^*_{xayb})^{-1}(P^c_z)^*(L_{xayb})^* \\ &= L^{-1}_{y^\perp b^\perp x^\perp a^\perp} P^{z^\perp}_{c^\perp} L_{y^\perp b^\perp x^\perp a^\perp} \\ &= L_{x^\perp b^\perp y^\perp a^\perp} P^{z^\perp}_{c^\perp} L^{-1}_{x^\perp b^\perp y^\perp a^\perp} \\ &= P^{\Gamma(x^\perp, b^\perp, y^\perp, a^\perp, z^\perp)}_{\Gamma(x^\perp, b^\perp, y^\perp, a^\perp, c^\perp)}. \end{aligned}$$

Comparing the images and kernels of these projections yields (2.3). Remark: we work with (5.5) (left multiplications) instead of (2.5) (middle multiplications); the reason is that we cannot prove that the analog of (2.5) holds for the linear relation M_{xabz}. □

With Lemma 5.1, the theorem implies

Corollary 5.8. *All real classical groups, and all of their homotopes $\mathcal{G}(\tau; a)$, admit a canonical semigroup compactification $\mathcal{X}_{a,\tau a} \cap \mathcal{Y}$.*

Remarks and Problems. 1. Which are the most general assumptions under which the theorem holds? Does it hold, e.g., in case of of a Hilbert Grassmannian? One may conjecture that the following inclusion holds "always" (i.e., very generally):

$$\Gamma(x^\perp, a^\perp, y^\perp, b^\perp, z^\perp) \subset \big(\Gamma(x, b, y, a, z)\big)^\perp. \tag{5.6}$$

2. The classification of semigrouds $\mathcal{X}_{a,\tau a}$ is only slightly more complicated than the one of the grouds $\mathcal{G}(\tau_a)$: it suffices to classify all orbits of $\mathrm{Aut}(\mathcal{X})$ in $\mathcal{X} \times \mathcal{X}$, resp. all $\mathrm{Aut}(\mathcal{Y})$-orbits in $\mathcal{X}$. However, the internal structure of the semigrouds may be very complicated! In other words, the classification of *semigroups* is much more difficult than the one of *semigrouds* (a semigroud contains many semigroups).

6. Towards an axiomatic theory

In a way similar to the intrinsic-axiomatic description of associative geometries from Chapter 3 of Part I, we would like describe axiomatically the Lagrangian geometries $\mathcal{Y}$ with their groud and semigroud structures – so far they are only defined by *construction* and not by intrinsic properties. What are these properties? Certainly, on the one hand, the various group and groud structures seem to be the most salient feature. But, on the other hand, there is an underlying "projective" structure playing an important rôle – indeed, Lagrangian geometries are special instances of *generalized projective geometries* as defined in [Be02]. This is most obvious on the "infinitesimal" level of the corresponding algebraic structures: besides the Lie algebra structures $[x, y]_a$ which correspond to the groups and grouds, there are also *Jordan algebra* structures $x \bullet_a y = (xay + yax)/2$, and Jordan structures correspond precisely to generalized projective geometries. There are purely algebraic concepts combining these two structures ("Jordan-Lie" and "Lie-Jordan algebras"; cf. [E84], [Be08c]), and the Lagrangian geometries considered here should be their geometric counterparts. In particular, the infinite dimensional Hilbert Lagrangian geometry then is the geometric analog of the Jordan-Lie algebra of observables in Quantum Mechanics – see [Be08c] for a discussion of some motivations coming from physics to consider such questions.

Appendix A: Associative pairs and their involutions

Recall (e.g., from [BeKi09], Appendix B, or [Lo75]) that an *associative pair (over* $\mathbb{K}$*)* is a pair $(\mathbb{A}^+, \mathbb{A}^-)$ of $\mathbb{K}$-modules together with two trilinear maps

$$< \cdot, \cdot, \cdot >^{\pm} : \mathbb{A}^{\pm} \times \mathbb{A}^{\mp} \times \mathbb{A}^{\mp} \to \mathbb{A}^{\pm}$$

such that

$$< xy < zuv >^{\pm} >^{\pm} = << xyz >^{\pm} uv >^{\pm} = < x < uzy >^{\mp} v >^{\pm} .$$

Definition. *A* type preserving involution *of* $(\mathbb{A}^+, \mathbb{A}^-)$ *is a pair* $(\tau^+ : \mathbb{A}^+ \to \mathbb{A}^+, \tau^- : \mathbb{A}^- \to \mathbb{A}^-)$ *of* $\mathbb{K}$*-linear mappings such that* $\tau^{\pm}$ *are of order* 2 *and*

$$\tau^{\pm} < uvw >^{\pm} = < \tau^{\pm} w, \tau^{\mp} v, \tau^{\pm} u >^{\pm} .$$

A type exchanging involution *of* $(\mathbb{A}^+, \mathbb{A}^-)$ *is a pair* $(\tau^+ : \mathbb{A}^+ \to \mathbb{A}^-, \tau^- : \mathbb{A}^- \to \mathbb{A}^+)$ *of* $\mathbb{K}$*-linear mappings such that* τ^+ *is the inverse of* τ^- *and*

$$\tau^{\pm} < uvw >^{\pm} = < \tau^{\mp} w, \tau^{\pm} v, \tau^{\mp} u >^{\pm} .$$

In other words, a type preserving involution is an isomorphism onto the opposite pair of $(\mathbb{A}^+, \mathbb{A}^-)$*, and a type exchanging involution is an isomorphism onto the dual of the opposite pair, where the* opposite pair *is obtained by reversing orders in products, and the* dual pair *is obtained by exchanging the rôles of* $\mathbb{A}^+$ *and* $\mathbb{A}^-$*.*

Clearly, for any involution $\tau = (\tau^+, \tau^-)$, the pair $\tau' := (-\tau^+, -\tau^-)$ is again an involution (type preserving, resp. exchanging iff so is τ); we call it the *dual involution*. For a type preserving involution, the pairs of 1-eigenspaces or of -1-eigenspaces form in general no longer associative pairs (but they are *Jordan pairs*, see [Lo75]). For type exchanging involutions, there is an equivalent description in terms of *triple systems*: recall that an *associative triple system of the second* *kind* is a $\mathbb{K}$-module $\mathbb{A}$ together with a trilinear map $\mathbb{A}^3 \to \mathbb{A}$, $(x,y,z) \mapsto < xyz >$ satisfying the preceding identity obtained by omitting superscripts (see [Lo72]), and an *associative triple system of the first kind*, or *ternary ring*, is a $\mathbb{K}$-module $\mathbb{A}$ together with a trilinear map $\mathbb{A}^3 \to \mathbb{A}$, $(x,y,z) \mapsto < xyz >$ satisfying the identity

$$< xy < zuv >> = << xyz > uv > = < x < yzu > v >$$

(see [Li71]). It is easily checked that, if (τ^+, τ^-) is a type exchanging involution, the space $\mathbb{A} := \mathbb{A}^+$ with

$$< x, y, z > := < x, \tau^+ y, z >^+$$

becomes an associative triple system of the second kind. Conversely, from an associative triple system of the second kind we may reconstruct an associative pair with type exchanging involution: $\mathbb{A}^+ := \mathbb{A} =: \mathbb{A}^-$, $< x, \tau^+ y, z >^\pm := < x, y, z >$, $\tau^\pm$ given by the identity map of $\mathbb{A}^\pm \to \mathbb{A}^\mp$.

In the same way, *automorphisms of order two* from $(\mathbb{A}^+, \mathbb{A}^-)$ onto the opposite pair $(\mathbb{A}^-, \mathbb{A}^+)$ correspond to associative triple systems of the first kind.

Examples. 1. Every associative algebra $\mathbb{A}$ with $< xyz >= xyz$ is an associative triple system of the first kind. It is equivalent to the associative pair $(\mathbb{A}, \mathbb{A})$ with the exchange automorphism (which is not an involution, in our terminology).

2. The space of rectangular matrices $M(p,q;\mathbb{K})$ with

$$< XYZ >= XY^t Z$$

forms an associative triple system of the second kind. It is equivalent to the associative pair $(\mathbb{A}^+, \mathbb{A}^-) = (M(p,q;\mathbb{K}), M(q,p;\mathbb{K}))$ with type exchanging involution $X \mapsto X^t$.

3. For any involutive algebra $(\mathbb{A}, *)$, the map $(x,y) \mapsto (x^*, y^*)$ is a type preserving involution of the associative pair $(\mathbb{A}, \mathbb{A})$.

Remark. We do not have an example of an associative pair with a type preserving involution which is *not* obtained via Example 3 above. In finite dimension over a field the existence of such examples seems rather unlikely, but there might exist infinite dimensional examples which are "very close", but not isomorphic, to pairs of the type $(\mathbb{A}, \mathbb{A})$, and admit a type-preserving involution.

References

[Ar61] Arens, R., "Operational calculus of linear relations", Pac. J. Math. **11** (1961), 9 – 23

[Be00] Bertram, W., *The geometry of Jordan- and Lie structures*, Springer Lecture Notes in Mathematics **1754**, Berlin 2000

[Be02] Bertram, W., "Generalized projective geometries: general theory and equivalence with Jordan structures", Adv. Geom. **2** (2002), 329–369 (electronic version: preprint 90 at `http://homepage.uibk.ac.at/~c70202/jordan/index.html`)

[Be04] Bertram, W., "From linear algebra via affine algebra to projective algebra", Linear Algebra and its Applications **378**

[Be08] Bertram, W., *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings.* Memoirs of the AMS **192** , no. 900, 2008. arXiv: math.DG/ 0502168

[Be08b] Bertram, W., "Homotopes and conformal deformations of symmetric spaces." J. Lie Theory **18** (2008), 301 – 333. arXiv: math.RA/0606449

[Be08c] Bertram, W., "On the Hermitian projective line as a home for the geometry of quantum theory." In *Proceedings XXVII Workshop on Geometrical Methods in Physics, Białowieża 2008*; arXiv: `math-ph/0809.0561`.

[BeBi] Bertram, W. and P. Bieliavsky, "Structure variety and homotopes of classical symmetric spaces" (work in progress)

[BeKi09] Bertram, W. and M. Kinyon, "Associative geometries. I: grouds, linear relations and Grassmannians", arXiv : math.RA/0903.5441

[BeNe05] Bertram, W. and K.-H. Neeb, "Projective completions of Jordan pairs. II: Manifold structures and symmetric spaces" Geom. Dedicata **112** (2005), 73 – 113; math.GR/0401236

[BlM04] Blecher, D. P. and C. Le Merdy, *Operator algebras and their modules – an operator space approach*, Clarendon Press, Oxford 2004

[CDW87] Coste, A, P. Dazord and A. Weinstein, *Groupoïdes Symplectiques*, lecture notes, Publication du Département de Mathématiques de Lyon 1987 (available eletronically at: http:// www.math.berkeley.edu/ alanw/cdw.pdf)

[Cr98] Cross, R., *Multivalued Linear Operators*, Marcel Dekker, 1998

[E84] Emch, G., *Mathematical and Conceptual Foundations of 20th-century Physics*, North-Holland Mathematics Studies **100**, Amsterdam 1984

[KMRS98] Knus, M.-A., A. Merkurjew, M. Rost and J.-P. Tignol, *The Book of Involutions*, AMS Coll. Publ. **44**, AMS, Providence 1998

[Li71] Lister, W. G., *Ternary rings*, Trans. Amer. Math. Soc. 154 (1971) 37–55.

[Lo72] Loos, Ottmar, *Assoziative Tripelsysteme.* Manuscripta Math. 7 (1972), 103–112.

[Lo75] Loos, O., *Jordan Pairs*, Springer LNM 460, New York 1975

[Ner96] Neretin, Y., *Categories of Symmetries and Infinite-Dimensional Groups*, Oxford LMS **16**, Oxford 1996

[PS86] Pressley, A. and G. Segal, *Loop Groups*, Oxford University Press, Oxford 1986

Institut Élie Cartan Nancy, Nancy-Université, CNRS, INRIA,, Boulevard des Aiguillettes, B.P. 239,, F-54506 Vandœuvre-lès-Nancy, France

E-mail address: `bertram@iecn.u-nancy.fr`

Department of Mathematics, University of Denver, 2360 S Gaylord St, Denver, Colorado 80208 USA

E-mail address: `mkinyon@math.du.edu`