THE LIE ALGEBRA OF SKEW-SYMMETRIC ELEMENTS AND ITS APPLICATION IN THE THEORY OF JORDAN ALGEBRAS

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ABSTRACT. In this article we prove that the Lie algebra of skew-symmetric elements of a free associative algebra of rank 2 in regard to the standard involution is generated as a module by the elements of type $[a,b]$, $[a,b]^3$, where a,b are Jordan polynomials. Using this result we prove that the Lie algebra of Jordan derivations of a free Jordan algebra of rank 2 is generated as a characteristic F-module by two derivations. It is proved that all the commutator Jordan s-identities are consequence of the Glennie-Shestakov s-identity.

Keywords: skew-symmetric elements, standard involution, Lie algebra, free associative algebra, Jordan derivations, Jordan s-identities.

1. INTRODUCTION.

Let A be an associative algebra over a field F with involution $*$. The set $L(A,*)=\{a\in A \mid a^*=-a\}$ of skew-symmetric elements in regard to * is obviously closed under operation $[a,b]=ab-ba$ and is a subalgebra of $A^{(-)}$. The algebra $L(A,*)$ with the operation $[a,b]$ is called *Lie algebra of skew-symmetric elements* of A.

All algebras in this article are considered over a field F of characteristic 0. Standard notations and definitions are taken from [1]. We will use the left-normed bracketing in non-associative words.

Let $SJ[X_n]$, $Ass[X_n]$ be a free special Jordan, and a free associative algebras of set of generators $X_n=\{x_1,x_2,...,x_n\}$. Let $H[X_n]=H(Ass[X_n],*)$, $L[X_n]=L(Ass[X_n],*)$ be a Jordan algebra of symmetric and a Lie algebra of skew-symmentric elements of the algebra $Ass[X_n]$ for standard involution $*$. Let A be some F-algebra and $B\subseteq A$. We will denote by $(B)_F$ the F-module generated by the set B. In this article we will build a simple system of generators of F-module $L=L(Ass[X_2],*)$. In case of Jordan algebra $H[X_n]$ of symmetric elements such system of generators is known for $n\le 3$. According to the Shirshov-Cohn theorem [1, 2] $H[X_n]=SJ[X_n]$ if $n\le 3$, i.e. $H[X_n]=(a;a\in SJ[X_n])_F$ and $SJ[X_4]\ne H[X_4]$. In the case of Lie algebra $L[X_n]$ of skew-symmetric elements no simple system of generators of F-module $L[X_n]$ has been known.

In this work we will prove that the set of elements $[a,b]$, $[a,b]^3$, where $a,b\in SJ[X_2]$, generates the F-module L, i.e.

$$L=([a,b],[a,b]^3;a,b\in SJ[X_2])_F \qquad (1)$$

Using this result we will describe the Lie algebra of Jordan derivations of a free Jordan algebra of rank 2 and prove that all the commutator Jordan s-identities are consequence of the Glennie-Shestakov s-identity (theorems 3 and 4).

2. PRELIMINARY RESULTS.

Let $SJ = SJ[X_2]$, $Ass = Ass[X_2]$. We will denote by $\circ$ the multiplication in the algebra SJ, i.e. $a \circ b = \frac{1}{2}(ab + ba)$, where $a, b \in SJ$. We will use no symbols between the elements multiplicating them in Ass.

Let $u = z_1^{i_1} = z_k^{i_k} \in Ass$, where $z \in X_2$ and $z_i \neq z_{i+1}, i = 1,...,k-1$. We will call the number $h(u) = k$ the *height* of the monomial u. From now on, $x_1 = x$, $x_2 = y$. To reduce the number of indices and powers in writing of the polynomials of Ass, we will denote by x_i, y_j the monomials of the type x^{i_1}, y^{i_2}, where $i_1, i_2, i, j \in \mathbb{N}$. For example, the monomials $x_1 y_2 x_3, x_2 y_1 x_1, x_2 y_2 x_1$ conditionally designate a monomial of the type $x^{i_1} y^{i_2} x^{i_3}$. Further it will be clear that this notation does not lead to any contradiction, but it considerably simplifies the statements. It is known [1] that the set of all monomials

$$\{u_1 = z_1^{i_1} ... z_k^{i_k} \mid \text{where } z_i \in X_2 \text{ and } z_i \neq z_{i+1}, i = 1,...,k-1, i \in \mathbb{N}\}$$

forms the basis of Ass, the set $\{\{u_i\} = u_i + u_i^*, i \in \mathbb{N}\}$ forms the basis of SJ. Let's fix the monomials $u_i, i \in \mathbb{N}$. Let $a \in Ass$ and $a = \sum_{i=1}^{n} \alpha_i u_i, \alpha_i \neq 0$. Set $h(a) = \max u_i$. Let's denote $[u_i] = u_i - u_i^*, i \in \mathbb{N}$. It is easy to check that $L = ([u_i], i \in \mathbb{N})_F$. We will need the following well-known identities (see [1]):

$$[x^2, y] = 2[x, y \circ x] = 2[x, y] \circ x, \; [x,[y,z]] = 4xD_{y,z}, \qquad (2)$$

where $xD_{y,z} = x \circ y \circ z - x \circ z \circ y$,

$$\{xyz\} = 2yU_{x,z},$$

where $yU_{x,z} = y \circ x \circ z + y \circ z \circ x - y \circ (z \circ x)$.

Define

$$L_k = ([a,b]^k \mid a, b \in SJ[X_2])_F, \text{ where } k = 1,3;$$

$$L_2 = ([a^2 b^2 ab] \mid a, b \in SJ[X_2])_F;$$

$$L(n) = \{a \in L \mid h(a) \leq n\}.$$

We will write

$$a \equiv_K b, \text{ if } a - b \in K, \text{ where } K \subseteq Ass.$$

From the above notations it follows that

$$[u_i] = -[u_n^i], i \in \mathbb{N}; \; L = \sum_{i=1}^{\infty} L(i).$$

We begin with the elementary properties of the linear operator $[\ ,\ ]: Ass \to L$.

Lemma 1. Let $a,b \in SJ$. Then for every $i \in \mathbb{N}$

$$[au_i] \underset{L_1}{\equiv} [u_i a], \tag{3}$$

$$[u_i] \underset{L_1}{\equiv} 0 \text{ , for } h(u_i) \le 3, \tag{4}$$

$$[a,b]^3 \underset{L_1}{\equiv} 3[a^2b^2ab]. \tag{5}$$

Proof. We have

$$[au_i]-[u_i a] = au_i - u_i^* a - u_i a + au_i^* = a\{u_i\} - \{u_i\}a = [a,\{u_i\}].$$

By Shirshov-Cohn theorem [1, 2], $\{u_i\} \in SJ$. Consequently, $[au_i]-[u_i a] \in L_1$ and $[au_i] \underset{L_1}{\equiv} [u_i a]$. Then

$$[x_1 y_1] = [x_1, y_1] \underset{L_1}{\equiv} 0, \quad [x_1 y_1 x_2] \underset{(3) L_1}{\equiv} [y_1 x_2 x_1] = [y_1, (x_1 x_2)] \underset{L_1}{\equiv} 0.$$

Finally,

$$[a,b]^3 = (ab-ba)^3 = [ababab] + [ab^2aba] + [ba^2b^2a] + [baba^2b] \underset{(3) L_1}{\equiv} [ababa, b] + 3[a^2b^2ab] \underset{L_1}{\equiv} 3[a^2b^2ab].$$

The lemma is proved.

The proof of the main result will be based on the following observation. By the lemma 1 for the proof of (1) it sufficient to show that

$$\forall i \in \mathbb{N} \quad [u_i] \in L_1 + ([a^2b^2ab] \mid a,b \in SJ)_F \tag{6}$$

Indeed, in this case

$$\forall i \in \mathbb{N} \quad [u_i] \in L_1 + L_2 \underset{(5)}{=} L_1 + L_3.$$

Therefore, $L \subseteq L_1 + L_3 \subseteq L$ and $L = L_1 + L_3$. The proof of (6) will be performed by induction on the height $h(u_i)$. From (4) we have $h(u_1) \in L_1$ if $h(u_1) \le 3$. The first non-trivial case occurs for $h(u_1) = 4$. In this case

$$[u_i] \underset{(3) L_1}{\equiv} [x_1 y_1 x_2 y_2] = [x^{n_1} y^{m_1} x^{n_2} y^{m_2}].$$

Therefore we need to prove that

$$\forall n_1, n_2, m_1, m_2 \in \mathbb{N} \quad [x^{n_1} y^{m_1} x^{n_2} y^{m_2}] \in L_1 + L_2.$$

For this purpose we will prove a simple, but unexpected lemma.

Let $\mathfrak{M}$ be a variety of power associative algebras over F . We will denote by $F_{\mathfrak{M}}[X]$ the free algebra in the variety $\mathfrak{M}$ with the set of generators $X = \{x_1, ..., x_n, ...\}$.

Let's consider an arbitrary homogenous polynomial $f = f(x_1,...,x_n) \in F_{\mathfrak{M}}[X]$, where $n \geq 2$. For brevity, $f = f(x_1,x_2)$. We will fix the polynomial $f = f(x_1,x_2)$ and denote $M = (f(a,a), f(a,a^2) \mid a \in F_{\mathfrak{M}}[x_{n+1}])_F$. Set $z = x_{n+1}$.

Lemma 2. For any $k,m \in \mathbb{N}$

$$f(z^k, z^m) \in M .$$

Proof. We'll write $g \equiv h$ if $g - h \in M$. It follows from the definition that

$$f(a,b) \equiv -f(b,a) \quad f(a,b\cdot c) + f(b,a\cdot c) + f(c,a\cdot b) \equiv 0 , \tag{7}$$

where $a,b,c \in F_{\mathfrak{M}}[z]$, and $\cdot$ is a multiplication in $F_{\mathfrak{M}}[X]$. We will prove that

$$f(z,z^m) \equiv \frac{1}{k} f(z^k, z^{m+1-k}) ,$$

where $k < m+1, m \geq 2$. The proof is by induction on k.

Since

$$f(z,z^m) \underset{(7)}{\equiv} -f(z^{m-1},z^2) - f(z,z^m) ,$$

then

$$f(z,z^m) \underset{(7)}{\equiv} \frac{1}{2} f(z^2, z^{m-1}) .$$

Assume that

$$f(z,z^m) \equiv \frac{1}{k} f(z^k, z^{m+1-k}) ,$$

for $k < m$. Then

$$f(z,z^m) \equiv \frac{1}{k} f(z^k, z^{m-k}\cdot z) \underset{(7)}{\equiv} -\frac{1}{k} f(z,z^m) - \frac{1}{k} f(z^{m-k}, z^{k+1})$$

and

$$\frac{k+1}{k} f(z,z^m) \equiv -\frac{1}{k} f(z^{k+1}, z^{m-k}) .$$

Hence

$$f(z,z^m) \underset{(7)}{\equiv} \frac{1}{k+1} f(z^{k+1}, z^{m-k}) .$$

Consequently,

$$f(z,z^m) \equiv \frac{1}{m} f(z^m, z) \underset{(7)}{\equiv} -\frac{1}{m} f(z,z^m)$$

and $f(z, z^m) = 0$. Applying (7) as much as required, we get

$$f(z^k, z^m) \equiv \lambda f(z, z^{m+k-1})$$

for some $\lambda \in F$. Therefore, $f(z^k, z^m) \equiv 0$. The lemma is proved.

Lemma 3. For any $n_1, n_2, m_1, m_2 \in \mathbb{N}$

$$w = [x^{n_1} y^{m_1} x^{n_2} y^{n_2}] \in L_1 + L_2. \tag{8}$$

Proof. From the lemma 2 it follows that

$$w \in ([ay^{m_1} ay^{m_2}], [ay^{m_1} a^2 y^{m_2}] \mid a \in SJ[x])_F.$$

Notice that

$$[ay^{m_1} ay^{m_2}] = [ay^{m_1} a, y^{m_2}] \underset{L_1}{\equiv} 0.$$

Therefore,

$$w \in L_1 + ([ay^{m_1} a^2 y^{m_2}] \mid a \in SJ[x])_F.$$

Repeating similar arguments for y, we obtain

$$w \in L_1 + ([ab^2 a^2 b], [aba^2 b^2] \mid a \in SJ[x], b \in SJ[y])_F.$$

But

$$[ab^2 a^2 b] \underset{(3)L_1}{\equiv} [b^2 a^2 ba], \quad [aba^2 b^2] \underset{(3)L_1}{\equiv} [a^2 b^2 ab].$$

Consequently, $w \in L_1 + L_2$. The lemma is proved.

3. MAIN RESULTS.

Consider $u_i = x_1 y_1 x_2 y_2 \ldots x_m y_m$, where $m \geq 3$. It is obvious that $h(u_i) = 2m \geq 6$. Then $u_i = x_1 y_1 x_2 y_2 u_k y_m$, where $u_k = x_3 y_3 \ldots x_m$ and $h(u_k) \geq 1$. Notice that u_k begins and ends with x. Set $g(b,c) = [a^2 bac]$, for $a \in SJ[x], b, c \in SJ[x, y]$. From (3) and from the definition of L_2 it follows that

$$g(b,c) \underset{L_1+L_2}{\equiv} -g(c,b), \quad g(b, c \circ d) + g(c, d \circ b) + g(d, b \circ c) \underset{L_1+L_2}{\equiv} 0, \tag{9}$$

for $b, c, d \in SJ[x, y]$. Notice that $h(g(y_1, \{y_2 u_k y_m\})) = 2m$. In the sequel, $u \equiv w$ means that $u - w \in L_1 + L_2 + L(2m-1)$.

Lemma 4. The function $g(y_1, \{y_2 u_k y_m\})$ is skew-symmetric with respect to y_1, y_2, y_m by modulo of $L_1 + L_2 + L(2m-1)$.

Proof. We have

$$g(y_1, \{y_2 u_k y_m\}) = 2g(y_1, \{u_k y_m\} \circ y_2) - g(y_1, \{u_k (y_m y_2)\}).$$

It is easy to notice that $h(g(y_1,\{u_k(y_m y_2)\}))<2m$, therefore

$$g(y_1,\{y_2u_ky_m\})=2g(y_1,\{u_ky_m\}\circ y_2)\underset{(9)}{\equiv}-2g(y_2,\{u_ky_m\}\circ y_1)-2g(\{u_ky_m\},y_1\circ y_2).$$

As u_k starts with x, then $h(g(\{u_ky_m\},(y_1y_2)))<2m$. Thereby $g(\{u_ky_m\},(y_1y_2))\equiv 0$. Consequently,

$$g(y_1,\{y_2u_ky_m\})=-2_g(y_2,\{u_ky_m\}\circ y_1)\equiv -g(y_2,\{y_1u_ky_m\}).$$

Similarly $g(y_1,\{y_2u_ky_m\})\equiv -g(y_m,\{y_2u_ky_1\})$. The lemma is proved.

Lemma 5. We have

$$[x_1y_1x_2\{y_2u_ky_m\}]\equiv 0. \tag{10}$$

Proof. By the lemma 2,

$$[x_1y_1x_2\{y_2u_ky_m\}]\in L_1+([a^2y_1a\{y_2u_ky_m\}]\mid a\in SJ[x])_F.$$

So, it suffices to prove that for any $n_1,n_2,n_3\in\mathbb{N}, a\in SJ[x]$

$$g(y^{n_1},\{y^{n_2}u_ky^{n_3}\})\equiv 0.$$

We will prove by induction on n_1 that

$$\begin{gathered} g(y^{n_1},\{y^{n_2}u_ky^{n_3}\})\in g(y,\{y^{m_1}u_ky^{m_2}\})_F+L_1+L_2+L(2m-1);\\ \text{for } m_1,m_2\in\mathbb{N}, m_1+m_2+1=n_1+n_2+n_3. \end{gathered} \tag{11}$$

Let $n_1>1$, then

$$\begin{aligned} g(y^{n_1},\{y^{n_2}u_ky^{n_3}\})&=g(y^{n_1-1}\circ y,\{y^{n_2}u_ky^{n_3}\})\\ &\underset{(9)}{\equiv}-g(y\circ\{y^{n_2}u_ky^{n_3}\},y^{n_1-1})-g(y^{n_1-1}\circ\{y^{n_2}u_ky^{n_3}\},y)\\ &\underset{(9)}{\equiv}\frac{1}{2}(g(y^{n_1-1},\{y^{n_2+1}u_ky^{n_3}\})+g(y^{n_1-1},\{y^{n_2}u_ky^{n_3+1}\})\\ &\qquad +g(y,\{y^{n_1+n_2-1}u_ky^{n_3}\})+g(y,\{y^{n_2}u_ky^{n_3+n_1-1}\})). \end{aligned}$$

By induction assumption, we obtain (11). Consequently, it remains to prove that

$$\forall p,q\in\mathbb{N}\quad w=g(y,\{y^pu_ky^q\})\equiv 0.$$

By the lemma 4 w is a skew-symmetric function with respect to y,y^p,y^q. Therefore, $p,q>1$. Now

$$\begin{aligned} w=g(y,\{y^pu_ky^q\})&\equiv -g(y^p,\{yu_ky^q\})=-g(y^{p-1}\circ y,\{yu_ky^q\})\\ &\underset{(9)}{\equiv}g(y\circ\{yu_ky^q\},y^{p-1})+g(y^{p-1}\circ\{yu_ky^q\},y)\\ &\underset{(9)}{\equiv}-\frac{1}{2}(g(y^{p-1},\{y^2u_ky^q\})+g(y^{p-1},\{yu_ky^{q+1}\})\\ &\qquad +g(y,\{y^pu_ky^q\})+g(y,\{yu_ky^{p+q-1}\})). \end{aligned}$$

By the lemma 4 $g(y,\{yu_k y^{p+q-1}\})\equiv 0$ and

$$3w=-g(y^{p-1},\{y^2u_k y^q\})-g(y^{p-1},\{yu_k y^{q+1}\})\equiv g(y^2,\{y^{p-1}u_k y^q\})+g(y,\{y^{p-1}u_k y^{q+1}\})$$
$$\underset{(9)}{\equiv}-2g(y\circ\{y^{p-1}u_k y^q\},y)+g(y,\{y^{p-1}u_k y^{q+1}\})\underset{(9)}{\equiv}g(y,\{y^p u_k y^q\})+2g(y,\{y^{p-1}u_k y^{q+1}\})\,.$$

Consequently,

$$w=g(y,\{y^p u_k y^q\})\equiv g(y,\{y^{p-1}u_k y^{q+1}\})\equiv\ldots\equiv g(y,\{yu_k y^{p+q-1}\})\equiv 0\,.$$

Therefore,

$$[x_1y_1x_2\{y_2u_ky_m\}]\equiv 0\,.$$

The lemma is proved.

Theorem 1. $L=([a,b],[a,b]^3\mid a,b\in SJ[x,y])_F$.

Proof. By (6) it suffices to prove that $[u_i]\in L_1+L_2$ for every $i\in\mathbb{N}$. The proof is by induction on $k=h(u_i)$. If $k\le 4$, then the assumption is valid by (4), (8). Let the assumption be valid for all $[u_j],h(u_j)\in L(k-1),k\ge 5$. Consider $[u_i],h(u_i)=k$. If $k=2m+1$, then

$$[u_i]\underset{L1}{\equiv}[x_1,y_1\ldots x_my_mx_{m+1}]\underset{(3)L_1}{\equiv}[y_1\ldots x_my_m(x_{m+1}x_1)]\in L(k-1)\,.$$

By induction assumption, $[u_i]\in L_1+L_2$. Let $k=2m$, then $k\ge 6,m\ge 3$ and $[u_i]\underset{L_1}{\equiv}[x_1y_1x_2y_2u_ky_m]$, where $h(u_k)\ge 1$ and u_k begins and ends with x. We have

$$[u_i]\underset{L_1}{\equiv}[x_1y_1x_2\{y_2u_ky_m\}]-[x_1y_1x_2y_mu_k^*y_2]\,.$$

By the lemma 5 and by induction assumption, $[x_1y_1x_2\{y_2u_ky_m\}]\in L_1+L_2$. Therefore,

$$[u_i]\underset{L_1+L_2}{\equiv}-[x_1y_1x_2y_mu_k^*y_2]\underset{(3)L_1+L_2}{\equiv}[y_mu_k^*y_2x_1y_1x_2]=[x_2y_1x_1y_2u_ky_m]\,.$$

Consequently, the element $[u_i]$ is symmetric in x_1,x_2 by modulo of L_1+L_2. But

$$[u_i]\underset{L_1}{\equiv}[x_1y_1x_2\ldots x_my_m]\underset{(3)L_1+L_2}{\equiv}[x_2y_2\ldots x_my_mx_1y_1]\equiv\ldots\underset{(3)L_1+L_2}{\equiv}[x_my_mx_1y_1\ldots x_{m-1}y_{m-1}]\,.$$

Therefore, $[u_i]$ is symmetric in $x_1,\ldots,x_m$ by modulo of L_1+L_2.

Repeating similar arguments for y we obtain the symmetry of $[u_i]$ in $y_1,\ldots,y_m$ by modulo of L_1+L_2. Denote

$$v=\frac{1}{m!(m-1)!}\sum_{\substack{\sigma\in S_m,\\ \tau\in S_{m-1}}}(y_{\sigma(1)}x_{\tau(2)}\ldots x_{\tau(m)}y_{\sigma(m)})\,,$$

where S_{m-1} is a symmetric group of permutations of the set $\{2,\ldots,m\}$. It is obvious that $v^*=v$. Therefore, $v\in SJ[x,y]$. Consequently,

$$[u_i] \underset{L_1+L_2}{\equiv} \frac{1}{m!(m-1)!} \sum_{\substack{\sigma\in S_m,\\ \tau\in S_{m-1}}} [x_1 y_{\sigma(1)} x_{\tau(2)} \dots y_{\sigma(m)}) = [x_1 v] = [x_1, v] \in L_1$$

and $[u_i] \in L_1 + L_2$. The theorem is proved.

In conclusion we'll mention one more curious application of the Lemma 2. Let again $f = f(x_1, \dots x_n) \in F_{\mathfrak{M}}[X]$ be some polynomial, $n \geq 2$.

Definition. The algebra $A \in \mathfrak{M}$ is called f *-Jordan* if it satisfies the identities

$$f(x_1, x_1, x_3, \dots x_n) = 0, \quad f(x_1, x_1^2 x_3, \dots, x_n) = 0.$$

Subalgebra B of the algebra A is called f *-strongly associative* if for arbitrary $b_1, b_2 \in B, a_3, \dots, a_n \in A$ we have $f(b_1, b_2, a_3, \dots, a_n) = 0$.

Lemma 2 allows us to prove f -strong associativity of any one-generated subalgebra.

Theorem 2. Let A be an f -Jordan algebra. Any one-generated subalgebra B of A is an f -strongly associative.

4. DERIVATIONS OF THE FREE JORDAN ALGEBRA ON TWO GENERATORS AND COMMUTATOR JORDAN s -IDENTITIES.

Let A be an arbitrary algebra over F with the multiplication "$\cdot$" (the operation is not necessarily Jordan or associative). Let $End_F(A)$ denote the associative F -algebra (in regard to the superposition) of all endomorphisms of F -module A. Henceforth,

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a); a, b \in A.$$

Definition. F -linear mapping $D: A \to A$ is called an *external derivation of* A if for any $a, b \in A$

$$(a \cdot b)D = aD \cdot b + a \cdot bD$$

and an *external Jordan derivation of* A, if for any $a \in A$

$$a^2 D = aD \cdot a + a \cdot aD \tag{12}$$

or, equivalent for any $a, b \in A$

$$(a \circ b)D = aD \circ b + a \circ bD.$$

Consider the set $D(A) = End_F(A)$ of all external Jordan derivations of A, i.e. $D(A) = \{\varphi \in End_F(A) \mid \varphi$ is external Jordan derivation of $A\}$. In general case the F -module $D(A)$ is not closed under superposition of mappings. Let us define the commutator operation on $D(A)$ by $[D_1, D_2] = D_1 D_2 - D_2 D_1$, for $D_1, D_2 \in D(A)$. Then $Outer(A) = (D(A), +, [\ ,\ ])$ is a Lie algebra over F. Indeed, for any $a \in A$,

$$(a^2) D_1 D_2 = 2(aD_1 \circ a) D_2 = 2aD_1 D_2 \circ a + 2aD_1 \circ aD_2.$$

Therefore, $[D_1, D_2] \in D(A)$. And for any $D_1, D_2, D_3 \in D(A)$ we have $J(D_1, D_2, D_3) = 0$, where

$$J(D_1, D_2, D_3) = [[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2]$$

is a Jacobian of the elements D_1, D_2, D_3.

For any element $a \in A$, we will define the endomorphisms $R_a, L_a \in End_F(A)$ by the rule:

$$xR_a = x \cdot a, \quad xL_a = a \cdot x \quad \forall x \in A$$

We will call them the *operators* of *left* and *right* a*-multiplication*.

The subalgebra $M(A)$ of the algebra $End_F(A)$, generated by all $R_a, L_a, a \in A$, is called the *algebra of multiplications of* A. If the algebra A is commutative, then $R_a = L_a$ for all $a \in A$, therefore, $M(A) = R(A)$, where $R(A)$ is a subalgebra of $M(A)$ generated by all $R_a, a \in A$.

Definition. The external derivation T of an algebra A is called the *internal derivation* A if $T \in M(A)$ and it is called the *internal Jordan derivation* A if $T \in M(A) \cap Outer(A)$.

Subulgebra $Outer(A)$ generated by all internal Jordan derivations A will be denoted by $Inder(A)$. It is obvious that $Inder(A)$ is an ideal of the algebra $Outer(A)$.

Let's describe all internal Jordan derivations of a free associative algebra $Ass[X_n]$, where $X = \{x_1, ..., x_n\}, n \geq 1$.

Lemma 6. Let $T \in Inder(Ass[X_n])$. Then there exists $B \in Ass[X_n]$ such that $aT = [a, B] = aB - Ba$ for any $a \in Ass[X_n]$. If moreover $a^* = a, (aT)^* = aT$ then $aT = \frac{1}{2}[z,[B]]$, where $[B] = B - B^*$.

Proof. If $n = 1$ then $Inder(Ass[x]) = 0$. Indeed, $Ass[x]$ is a commutative algebra and $(x^2)T = xTx = 2xTx$, therefore $T = 0$.

Let $n \geq 2$. It suffices to prove the statement of lemma for homogenous operators T from $Ass[X_n]$. Consider a non-zero homogenous operator $T \in Inder(Ass[X_n]), d(T) = k$. Then T can be represented as

$$T = R_B + L_C + \sum_{i=1}^{m} \alpha_i R_{B_i} L_{C_i},$$

where the monomials B, C, B_i, C_i are taken from $Ass[X_n], \alpha_i \in F$, $d(B_i), d(C_i) \geq 1$ for all i, and the set $\{R_{B_i} L_{C_i}, i = 1, ..., m\}$ is linearly independent in $M(Ass[X_n])$. Then for any $a \in Ass[X_n]$ we obtain

$$(a^2)T \underset{(12)}{=} (aT)a + a(aT),$$

$$(a^2)T = a^2B + Ca^2 + \sum_{i=1}^{m} \alpha_i (C_i a^2 B_i) = (aT)a + a(aT)$$

$$= aBa + Ca^2 + \sum_{i=1}^{m} \alpha_i (C_i a B_i a) + a^2 B + aCa + \sum_{i=1}^{m} \alpha_i (aC_i aB_i).$$

Therefore, for any $a \in Ass[X_n]$ we obtain an identity

$$\sum_{i=1}^{m} \alpha_i (C_i a^2 B_i) = a(B+C)a + \sum_{i=1}^{m} \alpha_i (aC_i aB_i) \qquad (13)$$

in $Ass[X_n]$.

Substitute $a = \{x_1 x_2 x_1 x_2^2 ... x_1 x_2^k x_1\}$ in (13). Comparing the beginnings and endings of all associative words of (13) and taking into consideration the linear independence of the operators $\{R_{B_i} L_{C_i}, i = 1,...,m\}$, we come to conclusion that $\alpha_i = 0, i = 1,...,m$ and $B + C = 0$.

Consequently, $T = R_B - L_B$ and

$$aT = aB - Ba = [a, B]$$

for any $a \in Ass[X_n]$.

And if $(aT)^* = aT$, then $(aT)^* = [a, B]^* = -[a, B^*]$. Hence

$$2(aT) = aT + (aT)^* = [a, B] + [a, (-B)^*] = [a, [B] .$$

The lemma is proved.

Definition. The operator $T \in M(Ass[X_n]) \cap Outer(H[X_n]), n \geq 2$ is called H*-derivation* of the algebra $Ass[X_n]$.

From what has already been proved, we obtain a description of H-derivations .

Corollary. Lie algebra of H-derivations of the algebra $Ass[X_n]$ coincides with the algebra $Inder(Ass[X_n])$.

Note that if B is a subalgebra of A, then generally $Inder(A) \subseteq M(A) \cap Outer(B)$ and $M(A) \cap Outer(B) \neq Inder(A)$.

Consider the element $f = [z, [x, y]^3] \in Ass[x, y, z]$. Notice that

$$f^* = ([z, [x, y]^3])^* = -[z, ([x, y]^*)^3] = -[z, (-[x, y]^3)] = f .$$

Consequently, $f \in SJ[x, y, z]$ by Shirshov-Cohn theorem [1,2]. Let us find a representation of $f = [z, [x, y]^3]$ in $SJ[x, y, z]$. Set $k = [x, y]$. By the identities (2), we obtain

$$k^2 = [x, y] \circ [x, y] \underset{(2)}{=} \frac{1}{2}[x, [x, y^2]] - [x, [x, y]] \circ y \underset{(2)}{=} 2xD_{x,y^2} - 4xD_{x,y} \circ y , \qquad (14)$$

$$f = [z, k^3] \underset{(2)}{=} [z, k] \circ k^2 + 2[z, k] \circ k \circ k \underset{(2)}{=} 3[z, k] \circ k^2 + 2kD_{[z,k],k}$$

$$\underset{(2)}{=} 12zD_{x,y} \circ k^2 + \frac{1}{2}[k, [[z, k], k]] = 12zD_{x,y} \circ k^2 - \frac{1}{2}[z, k, k, k]$$

$$\underset{(2),(14)}{=} 24zD_{x,y} \circ (xD_{x,y^2} - 2xD_{x,y} \circ y) - 32zD_{x,y}^3 .$$

Consequently,

$$f = 8(3zD_{x,y} \circ (xD_{x,y^2} - 2xD_{x,y} \circ y) - 4zD_{x,y}^3) . \qquad (15)$$

Definition. The operator $S_{x,y} \in R(J[x,y,z])$, where

$$S_{x,y} = 24D_{x,y}R_{(xD_{x,y^2} - 2xD_{x,y} \cdot y)} - 32D_{x,y}^3 ,$$

is called *Shestakov operator*.

The operator $S_{x,y}$ was first introduced by Ivan P. Shestakov (see [3]). Let's note the important properties of this operator. Let $\varphi : J[x,y,z] \to SJ[x,y,z]$ be a canonic homomorphism and $S_3 = Ker\varphi$ be an ideal of Jordan S-identities from three generators. If w belongs to $J[x,y,z]$ (correspondingly $RJ[x,y,z]$)), then notation $\overline{w}$ means that $\varphi(w)$ belongs to $SJ[x,y,z]$ (correspondingly $RJ[x,y,z]$)).

Proposition 1. There exist the relations

$$\overline{zS_{x,y}} = [z,[x,y]^3] ; \qquad (16)$$

for $\overline{S_{x,y}} \in Inder(SJ[x,y,z])$, i.e.

$$a^2\overline{S_{x,y}} = 2a\overline{S_{x,y}} \circ a , \qquad (17)$$

for all $a \in SJ[x,y,z]$;

$$Sh(x,y,z) = (z^2)S_{x,y} - 2zS_{x,y} \cdot z \neq 0 \text{ in } J[x,y,z], \qquad (18)$$

i.e. $Sh(x,y,z)$ is an s-identity, moreover, $Sh(x,y,z)$ is equivalent to Glennie s-identity G_8.

Proof. The relation (16) follows from (15). The relation (17) follows from (16). It is easy to verify that $Sh(x,y,z) \neq 0$ in $H(C_3)$. From (17) it follows that $Sh(x,y,z) = 0$ in all special Jordan algebras. Consequently, $Sh = Sh(x,y,z)$ is an s-identity, and $d_x(Sh) = d_y(Sh) = 3$, $d_z(Sh) = 2$. By the results of [4], $Sh(x,y,z)$ is equivalent to G_8. The proposition is proved.

Definition. F-module $A, A \subseteq M(J[x,y])$ is called *characteristic* if it is closed under all endomorphisms of $End(J[x,y])$.

We first describe the internal Jordan derivations of the algebra $J[x,y]$.

Theorem 3. Lie algebra $Inder(J[x,y])$ is generated as a characteristic F-module by two derivations $D_{a,b}, \overline{S}_{a,b}$, where $a,b \in J[x,y]$, i.e. for any $T \in Inder(J[x,y])$ there exist $a_i, b_i, c_i, d_i \in J[x,y]$ and $\alpha_i \beta_i \in F$ such that

$$T = \sum_i \alpha_i D_{a_i,b_i} + \sum_i \beta_i \overline{S}_{c_i,d_i} .$$

Proof. Let $T \in Inder(J[x,y])$. By the lemma 6 and it's corollary it follows that there exists $B \in Ass[x,y]$ such that $aT = [a,[B]]$ for any $a \in J[x,y]$. It follows from the theorem 1 that there exist $a_i, b_i, c_i, d_i \in J[x,y]$ and $\alpha_i^{'}\beta_i^{'} \in F$, that

$$[B] = \sum_i \alpha_i [a_i, b_i] + \sum_i \beta_i [c_i, d_i]^3 \ .$$

Therefore, from Shirshov-McDonalds theorem [1], correlations (2) and (15) it follows that

$$T = \sum_i \alpha_i D_{a_i,b_i} + \sum_i \beta_i \overline{S}_{c_i,d_i} \ .$$

The theorem is proved.

Define the operator $\langle\ \rangle : H[x,y,z] \to J[x,y,z]$ as follows. Let $a \in H[x,y,z]$, i.e. $a^* = a$. According to Shirshov-Cohn theorem [1,2] $a \in SJ[x,y,z]$. From the proof of Shirshov-Cohn theorem it is easy to extract the algorithm of unique representation $a = f_a(x,y,z)$ where $f_a(x,y,z)$ is uniquely defined j-polynomial of $SJ[x,y,z]$. Then $\langle a \rangle = f_a(x,y,z)^J$ where $f_a(x,y,z)^J$ is an uniquely defined polynomial of $J[x,y,z]$, obtained from $f_a(x,y,z)$, replacing the operation "$\circ$" by the operation "$\cdot$".

For example,

$$\langle xy + yx \rangle = 2x \cdot y,$$
$$\langle xyz + zyx \rangle = 2yUz, x = 2(y \cdot z \cdot x + y \cdot x \cdot z - y \cdot (x \cdot z)),$$
$$\langle [z,[x,y]^3] \rangle = zS_{x,y}.$$

Definition. s-Identities of the type

$$g_u(x,y,z) = \langle [z^2,[u]] \rangle - 2\langle [z,[u]] \rangle \cdot z,$$

where $u \in Ass[x,y], [u] = u - u^*$ are called *commutator* Jordan s-*identities*.

For example, since $[x,y]^3 \underset{(5),L1}{=} 3[x^2y^2xy]$, we have $Sh(x,y,z) = 3g_{x^2y^2xy}(x,y,z)$

Theorem 4. All commutator Jordan s-*identities* of $J[x,y,z]$ are corollary of Glennie-Shestakov s-*identity* $Sh(x,y,z)$.

Proof. From the theorem 1 and from Shirshov-McDonalds theorem it follows that

$$\langle [z,[u]] \rangle = \sum_i \alpha_i z D_{a_i,b_i} + \sum_j \beta_j z D_{c_j,d_j}$$

for some $a_i, b_i, c_j, d_j \in J[x,y]$, $\alpha_i, \beta_j \in F$. Thereby,

$$g_u(x,y,z) = \sum_i \alpha_i (z^2 D_{a_i b_i} - 2zD_{a_i b_i} z) + \sum_j \beta_j Sh(z, c_j, d_j) = \sum_j \beta_j Sh(z, c_j, d_j) \in T(Sh(x,y,z)).$$

The theorem is proved.

5. SOME PROPERTIES OF CUBIC COMMUTATORS AND OPEN QUESTIONS.

Recall that

$$L_1 = ([a,b] \mid a,b \in SJ[x,y])_F \quad L_3 = ([a,b]^3 \mid a,b \in SJ[x,y])_F$$

Definition. The element $a \in SJ[x,y,z]$ is call *annihilator* of $[x,y]^3$, if

$$\frac{\partial x}{\partial a}[x,y]^3 = [a,y][x,y]^2 + [x,y][a,y][x,y] + [x,y]^2[a,y] \in L_1$$

i.e. if it is a commutator, where $\frac{\partial x}{\partial a}$ is an operator of a differential substitution $x \to a$. It is easy to notice that if a is an annihilator of $[x,y]^3$, then it is an annihilator of s -*identity* $Sh(x,y,z)$, i.e.

$$\frac{\partial x}{\partial a} Sh(x,y,z) = 0.$$

Definition. The element $b \in SJ[x,y,z]$ is called a *tetrad annihilator* if $\{bx_1x_2x_3\} \in SJ[x,y,z,x_1,y_1,z_1]$, i.e. b transfers a tetrad into Jordan polynomial.

For example, $w = [x,[y,k^2]]$ is a tetrad annihilator (see [5]).

Lemma 7. The following statements

1) $L_1 \neq L_3$;
2) $L_1 \cap L_3 \neq 0$;
3) $w = [x,[y,k^2]]$ is annihilator for $[x,y]^3$;

are valid in $SJ[x,y]$.

Proof. 1. Suppose that $[x,y]^3 \in L_1$. Then $[x,y]^3 = \sum_i \alpha_i [a_i b_i]$, where $a_i b_i \in SJ[x,y], \alpha_i \in F$.

Consequently,

$$Sh(x,y,z) = \sum_i \alpha_i (z^2 D_{a_i,b_i} - 2zD_{a_i,b_i} \cdot z) = 0 \qquad [x,y]^3 \notin L_1$$

in the algebra $J[x,y,z]$. Therefore $[x,y]^3 \notin L_1$.

2. Consider the tetrad annihilator $w = [x,[y,k^2]]$. Then

$$u = \frac{\partial x}{\partial w}([x,y]^3) = 2[w,y] \circ [x,y] \circ [x,y] + [x,y]^2 \circ [w,y] \in L_3,$$

$$u \underset{L_1}{\equiv} 3[w,y] \circ k^2 \underset{(2)}{=} 3[y,k^2] \circ [x,[y,k^2]] \underset{(2)}{=} \frac{3}{2}[x,[y,k^2]^2] \equiv 0 \bmod L_1$$

Consequently, $u \in L_1 \cap L_3$. It is obvious that $u \neq 0$ in $Ass[x,y]$.

3. From $u \in L_1$ it follows that u is an annihilator of $[x,y]^3$.

Lemma is proved.

Question 1. Describe the F – module of the annihilators of $[x,y]^3$.

Definition. Derivation $T \in Inder(J[X_k]), k \geq 2$ is called *derivation of the degree* k if $T \notin Inder(J[X_{k+1}])$.

For example, derivation $S_{x,y}$ is derivation of the degree 2.

Question 2. Is there a derivation of the degree $k \geq 3$?

It is easy to notice that in this case $Var(J[X_k]) \neq Var(J[X_{k+1}])$. In particular, the derivation $S_{x,y}$ separates the varieties $Var(J[X_2]) \neq Var(J[X_3])$.

It is obvious that $[x,y] \circ z \notin L_1(x,y,z) + L_3(x,y,z)$. In this connection it is appropriate to raise

Question 3. Find constructive generators of F -module $L[X_3]$.

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