Levels and sublevels of algebras obtained by the Cayley-Dickson process

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Abstract. We generalize the concepts of level and sublevel of a composition algebra to algebras obtained by the Cayley-Dickson process. In 1967, R. B. Brown constructed, for every $t \in \mathbb{N}$, a nonassociative division algebra A_t of dimension 2^t over the power-series field $K\{X_1, X_2, ..., X_t\}$. This gives us the possibility to construct a division algebra of dimension 2^t and prescribed level and sublevel 2^k, k, $t \in \mathbb{N}^*$ and a division algebra of dimension and prescribed level and sublevel $2^k + 1, t, k \in \mathbb{N}$.

Key Words: Cayley-Dickson process; Division algebra; Level and sublevel of an algebra.

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0. Introduction

In [Pf; 65], A. Pfister showed that if a field has a finite level this level is a power of 2 and any power of 2 could be realised as the level of a field. In the noncommutative case, the concept of level has many generalisations. The level of division algebras is defined in the same manner as for fields. In [Lew; 87], D. W. Lewis constructed quaternion division algebras of level 2^k and 2^k+1 for all $k \in \mathbb{N}^*$ and he asked if there exist quaternion division algebras whose levels are not of this form. These values were recovered for the quaternions by Laghribi and Mammone in [La,Ma; 01], and for octonion division algebras by Susanne Pumplün in [Pu; 05], using function field techniques. In [Hoff; 08], D. W. Hoffman showed that there are many other values, not of the form 2^k or $2^k + 1$, which could be realised as level of quaternion division algebras. In [Kr, Wa; 91], M. Küskemper and A. Wadsworth constructed the first example of a quaternion algebra of sublevel 3. Starting from this construction, in [O'

Sh; 07(1)], J. O' Shea proved the existence of an octonion algebra of sublevel 3 and constructed an octonion algebra of sublevel 5 and, in [O' Sh; 07(2)], he gave the first example of octonion division algebras of level 6 and 7. These were the first examples of composition algebras whose level is not of the form 2^k or $2^k + 1$ for some $k \in \mathbb{N}^*$ and remained the only known values for the level and sublevel of quaternion and octonion algebras. The existence of a quaternion algebra of sublevel 5 was still an open question. Starting from mentioned works and using Brown's construction for division algebras, we give an example of quaternion algebra of level and sublevel 5 in Section 4.

1. Preliminaries

In this paper, we assume that K is a field and $charK \neq 2$.

For the basic terminology of quadratic and symmetric bilinear spaces, the reader is referred to [Sch; 85] or [La, Ma; 01]. In this paper, we assume that all the quadratic forms are nondegenerate .

A bilinear space (V, b) *represents* $\alpha \in K$ if there is an element $x \in V, x \neq 0$, with $b(x, x) = \alpha$. The space is called *universal* if (V, b) represents all $\alpha \in K$. Every isotropic bilinear space V, $V \neq \{0\}$, is universal. (See [Sch; 85, Lemma 4.11., p. 14])

A subset P of K is called an *ordering* of K if

$$P + P \subset P, P \cdot P \subset P, -1 \notin P,$$

$$\{x \in K \ / \ x \text{ is a sum of squares in } K\} \subset P, P \cup -P = K, P \cap -P = 0.$$

A *quadratic semi-ordering* (or *q-ordering*) of a field K is a subset P with the following properties:

$$P + P \subset P, K^2 \cdot P \subset P, 1 \in P, P \cup -P = K, P \cap -P = 0.$$

Obviously, every ordering is a q-ordering [Sch; 85].

Remark 1.1. ([Sch; 85], p.133) Let P_0 be a q-preordering, i.e.

$$P_0 + P_0 \subset P_0, K^2 \cdot P_0 \subset P_0, P_0 \cap -P_0 = 0.$$

Then there is a q-ordering P such that $P_0 \subset P$ or $-P_0 \subset P$.

Let V be a vector space over an ordered field K. The quadratic form $q : V \to K$ is called *positive definite* if $q(x) > 0$ for all $x \neq 0$. If $q(x) < 0$ for all $x \neq 0$, it is called *negative definite.* If $\varphi \simeq < \alpha_1, ..., \alpha_n >$, it is called *indefinite* if the elements α_i are not all of the same sign and *totally indefinite* if for each ordering P of K there are α_i and α_j depending on P such that $\alpha_i <_P 0 <_P \alpha_j$.

A quadratic form φ is called *strongly anisotropic* if $m \times \varphi$ is anisotropic for all $m \in \mathbb{N}^*$. If the form φ is not strongly anisotropic it is called *weakly isotropic.*

The field K is a *formally real field* if -1 is not a sum of squares in K. Each formally real field has characteristic zero.

Remark 1.2. ([Sch; 85], p.134) Let K be a formally real field. A quadratic form φ over K is weakly isotropic if and only if φ is indefinite with respect to all q-orderings of K. If φ is strongly anisotropic then the set

$$P_0 = \{\alpha \,/\, \alpha = 0 \text{ or } \alpha \text{ is represented by } n \times \varphi, n \in \mathbb{N}^* \}$$

is a $q-$preordering. It follows that there is a q-ordering P such that $P_0 \subset P$ or $-P_0 \subset P$.

A quadratic form ψ is a *subform* of the form φ if $\varphi \simeq \psi \perp \phi$, for some quadratic form ϕ. We denote $\psi < \varphi$.

Let φ be a $n-$dimensional quadratic form over K, $n \in N, n > 1$, which is not isometric to the hyperbolic plane. We may consider φ as a homogeneous polynomial of degree 2, $\varphi(X) = \varphi(X_1, ...X_n) = \sum a_{ij} X_i X_j$, $a_{ij} \in K^*$. The *functions field of* φ, denoted $K(\varphi)$, is the quotient field of the integral domain

$$K[X_1, ..., X_n] \,/\, (\varphi(X_1, ..., X_n)).$$

Since $(X_1, ..., X_n)$ is a non-trivial zero, φ is isotropic over $K(\varphi)$. We remark that $\varphi(X)$ is irreducible. (See [Sch;85])

Proposition 1.3. [Ro; 05] *Let φ and ψ be two quadratic forms over a field K. The form ψ is isotropic over $K(\varphi)$ if and only if $D_{K'}(\varphi) D_{K'}(\varphi) \subseteq D_{K'}(\psi) D_{K'}(\psi)$, for every extension K' of K, where $D_K(\varphi)$ is the set of elements in K^* which are represented by φ.*

Proposition 1.4. (Cassels-Pfister Theorem) *Let $\varphi, \psi = 1 \perp \psi'$ be two quadratic forms over a field K, $charK \neq 2$. If φ is anisotropic over K*

and $\varphi_{K(\varphi)}$ *is hyperbolic, then* $\alpha\psi < \varphi$ *for each scalar represented by* φ. *In particular,* $\dim\varphi \geq \dim\psi$.[La, Ma;01, p.1823, Theorem 1.3.]

For $n \in \mathbb{N}^*$ a $n-$*fold Pfister form* over K is a quadratic form of the type

$$< 1, a_1 > \otimes ... \otimes < 1, a_n >, \ a_1, ..., a_n \in K^*.$$

A Pfister form is denoted by $\ll a_1, a_2, ..., a_n \gg$.

Remark 1.5. A Pfister form φ can be written as

$$< 1, a_1 > \otimes ... \otimes < 1, a_n > = < 1, a_1, a_2, ..., a_n, a_1a_2, ..., a_1a_2a_3, ..., a_1a_2...a_n > .$$

If $\varphi = < 1 > \perp \varphi'$, then φ' is called *the pure subform* of φ. A Pfister form is hyperbolic if and only if is isotropic. This means that a Pfister form is isotropic if and only if its pure subform is isotropic. (See [Sch; 85])

Proposition 1.6. [Sch, Lemma 1.3.(ii), p. 143] *With the above notations, we have the relations:*
i) $\ll -1, \alpha_2, ..., \alpha_n \gg \simeq < 1, -1, 1, -1, ... > \sim 0$;
ii) $\ll 1, \alpha_2, ..., \alpha_n \gg \simeq 2 \times \ll \alpha_2, ..., \alpha_n \gg$.

We recall some definitions and properties for nonassociative algebras.

Let A be an algebra of dimension n over K and let $f_1, ..., f_n$ be a basis for A over K. The multiplication in the algebra A is given by the relations $f_i f_j = \sum_{k=1}^{n} \alpha_{ijk} f_k$, where $\alpha_{ijk} \in K$ and $i, j = 1, ...n$. If $K \subset F$ is a field extension, the algebra $A_F = F \otimes_K A$ is called the *scalar extension* of A to an algebra over F. The elements of A_F are the forms $\sum_{i=1}^{n} \alpha_i \otimes f_i$ and we denote them $\sum_{i=1}^{n} \alpha_i f_i$, $\alpha_i \in F$.

An algebra A over K is called *quadratic* if A is a unitary algebra and, for all $x \in A$, there are $a, b \in K$ such that $x^2 = ax + b1$, $a, b \in K$. The subset $A_0 = \{x \in A - K \mid x^2 \in K1\}$ is a linear subspace of A and $A = K \cdot 1 \oplus A_0$. This decomposition allows us to define a linear form $t : A \rightarrow K$, a linear map $i : A \rightarrow A_0$ such that $x = t(x) \cdot 1 + i(x)$, for all $x \in A$, a symmetric bilinear form, $(,) : A \times A \rightarrow K$, $(x, y) = -\frac{1}{2} t(xy + yx)$ and a quadratic form $n : A \rightarrow K$, $n(x) = (t(x))^2 + (i(x), i(x))$. The element $\bar{x} = t(x) \cdot 1 - i(x)$ is called the *conjugate* of x. The quadratic form n is called *anisotropic* if $n(x) = 0$ implies $x = 0$. In this case, the algebra A is called also *anisotropic,* otherwise A is *isotropic.*

We can decompose the algebra A as the form $A = Sym(A) \oplus Skew(A)$, where $Sym(A) = \{x \in A \mid x = \bar{x}\}$, $Skew(A) = \{x \in A \mid x = -\bar{x}\}$.

A *composition algebra* is an algebra A with a non-degenerate quadratic form $q : A \to K$, such that q is multiplicative, i.e. $q(xy) = q(x)q(y), \forall x, y \in A$. A unitary composition algebra is called a *Hurwitz algebra.* Hurwitz algebras have dimensions $1, 2, 4, 8$.

Since over fields, the classical Cayley-Dickson process generates all possible Hurwitz algebras, in the following, we recall shortly the *Cayley-Dickson process.*

Let A be a finite dimensional unitary algebra over a field K with a *scalar* involution $^{-} : A \to A, a \to \overline{a}$, where $a + \overline{a}$ and $a\overline{a} \in K \cdot 1$ for all $a \in A$. Since A is unitary, we identify K with $K \cdot 1$ and we consider $K \subseteq A$.

Let $\alpha \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space $A \oplus A$.

$$(a_1, a_2)(b_1, b_2) = \left(a_1 b_1 + \alpha \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \qquad (1.1.)$$

We obtain an algebra structure over $A \oplus A$. This algebra, denoted by (A, α), is called the *algebra obtained from A by the Cayley-Dickson process.* A is canonically isomorphic with a subalgebra of the algebra (A, α) (denote $(1, 0)$ by 1, this is the identity in (A, α)) and $\dim(A, \alpha) = 2 \dim A$. Taking $u = (0, 1) \in A \oplus A$, $u^2 = \alpha \cdot 1$ and $(A, \alpha) = A \oplus Au$.

We remark that $x + \overline{x} = a_1 + \overline{a_1} \in K \cdot 1$ and
$x\overline{x} = a_1\overline{a_1} + \alpha a_2 \overline{a_2} \in K \cdot 1$. The map

$$^{-} : (A, \alpha) \to (A, \alpha), \quad x \to \bar{x},$$

is an involution of the algebra (A, α), extending the involution $^{-}$. If $x, y \in (A, \alpha)$, it follows that $\overline{xy} = \overline{y}\,\overline{x}$.

For $x \in A$, we denote $t(x) \cdot 1 = x + \overline{x} \in K$, $n(x) \cdot 1 = x\overline{x} \in K$, and these are called the *trace*, respectively, the *norm* of the element $x \in A$. If $z \in (A, \alpha)$, $z = x + yu$, then $z + \overline{z} = t(z) \cdot 1$ and $z\overline{z} = \overline{z}z = n(z) \cdot 1$, where $t(z) = t(x)$ and $n(z) = n(x) - \alpha n(y)$. It follows that $(z + \overline{z}) z = z^2 + \overline{z}z = z^2 + n(z) \cdot 1$ and

$$z^2 - t(z) z + n(z) = 0 \forall z \in (A, \alpha),$$

therefore each algebra obtained by the Cayley-Dickson process is quadratic. All algebras A obtained by the Cayley-Dickson process are *flexible* (i.e. $x\left(yx\right)=\left(xy\right)x, \forall x,y\in A$) and *power-associative* (i.e. for each $a\in A$, the subalgebra of A generated by a is associative). Moreover, the following conditions are fulfilled:

$$t\left(xy\right)=t\left(yx\right), t\left(\left(xy\right)z\right)=t\left(x\left(yz\right)\right), \forall x,y,z\in\left(A,\alpha\right). \qquad (1.2.)$$

Remark 1.7. If we take $A=K$ and apply this process t times, $t\geq 1$, we obtain an algebra over K, $A_t=K\{\alpha_1,...,\alpha_t\}$. By induction, in this algebra we find a basis $\{1,f_2,...,f_q\}, q=2^t$, satisfying the properties:

$$f_i^2=\alpha_i 1,\ \alpha_i\in K,\alpha_i\neq 0,\ i=2,...,q.$$

$$f_i f_j=-f_j f_i=\beta_k f_k,\ \beta_k\in K,\ \beta_k\neq 0, i\neq j, i,j=\ 2,...q,$$

β_k and f_k being uniquely determined by f_i and f_j.

As an example, we consider the generalized octonion algebra $O(\alpha,\beta,\gamma)$, with basis $\{1,f_2,...,f_8\}$, having the multiplication table:

$\cdot$	1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	α	f_4	$-\alpha f_3$	f_6	$-\alpha f_5$	$-f_8$	αf_7
f_3	f_3	$-f_4$	β	βf_2	f_7	f_8	$-\beta f_5$	$-\beta f_6$
f_4	f_4	αf_3	$-\beta f_2$	$-\alpha\beta$	f_8	$-\alpha f_7$	βf_6	$-\alpha\beta f_5$
f_5	f_5	$-f_6$	$-f_7$	$-f_8$	γ	γf_2	γf_3	γf_4
f_6	f_6	αf_5	$-f_8$	αf_7	$-\gamma f_2$	$-\alpha\gamma$	$-\gamma f_4$	$\alpha\gamma f_3$
f_7	f_7	f_8	βf_5	$-\beta f_6$	$-\gamma f_3$	γf_4	$-\beta\gamma$	$-\beta\gamma f_2$
f_8	f_8	$-\alpha f_7$	βf_6	$\alpha\beta f_5$	$-\gamma f_4$	$-\alpha\gamma f_3$	$\beta\gamma f_2$	$\alpha\beta\gamma$

If $x\in A_t, x=x_1 1+\sum_{i=2}^{q}x_i f_i$, then $\bar{x}=x_1 1-\sum_{i=2}^{q}x_i f_i$ and $t(x)=2x_1, n\left(x\right)=x_1^2-\sum_{i=2}^{q}\alpha_i x_i^2$. In the above decomposition of x, we call x_1 the *scalar part* of x and $x''=\sum_{i=2}^{q}x_i f_i$ the *pure part* of x. If we compute $x^2=x_1^2+x''^2+2x_1x''=x_1^2+\alpha_1 x_2^2+\alpha_2 x_3^2-\alpha_1\alpha_2 x_4^2+\alpha_3 x_5^2-...-(-1)^t\left(\prod_{i=1}^{t}\alpha_i\right)x_q^2+2x_1x''$, the scalar part of x^2 is represented by the quadratic form

$$T_C =< 1, \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, ..., (-1)^t (\prod_{i=1}^{t}\alpha_i) >=< 1, \beta_2, ..., \beta_q > \qquad (1.3.)$$

and, since $x''^2 = \alpha_1 x_2^2 + \alpha_2 x_3^2 - \alpha_1\alpha_2 x_4^2 + \alpha_3 x_5^2 - ... - (-1)^t (\prod_{i=1}^{t}\alpha_i) x_q^2 \in K$, it is represented by the quadratic form

$$T_P =< \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, ..., (-1)^t (\prod_{i=1}^{t}\alpha_i) >=< \beta_2, ..., \beta_q > . \qquad (1.4.)$$

The quadratic form T_C is called *the trace form*, and T_P *the pure trace form* of the algebra A_t. We remark that $T_C =< 1 >\perp T_P$, and the norm $n = n_C =< 1 >\perp -T_P$, resulting that
$n_C =< 1, -\alpha_1, -\alpha_2, \alpha_1\alpha_2, \alpha_3, ..., (-1)^{t+1} (\prod_{i=1}^{t}\alpha_i) >=< 1, -\beta_2, ..., -\beta_q > .$

The trace form n_C has the form $n_C = < 1, -\alpha_1 > \otimes ... \otimes < 1, -\alpha_t >$ and it is a Pfister form.

Using the above notation, we have that $x^2 = t(x)x - n(x)1 = -n(x)1 + 2x_1(x_1 + x'') = 2x_1^2 - n(x) + 2x_1 x''$. It results that $T_C(x) = 2x_1^2 - n(x)$, then

$T_C(x) = \frac{(t(x))^2}{2} - n_C(x)$. But $(t(x))^2 = t(x^2) + 2n_C(x)$, then

$T_C(x) = \frac{t(x^2)}{2}$.

2. Brown's construction of division algebras

In 1967, R. B. Brown constructed, for every t, a division algebra A_t of dimension 2^t over the power-series field $K\{X_1, X_2, ..., X_t\}$. We briefly demonstrate this construction, using polynomial rings over K and their fields of fractions (the rational functions field) instead of power-series fields over K (as it done by R.B. Brown),.

First of all, we remark that if an algebra A is finite-dimensional, then it is a division algebra if and only if A does not contain zero divisors (See [Sc;66]). For every t we construct a division algebra A_t over a field F_t. Let $X_1, X_2, ..., X_t$

be t algebraically independent indeterminates over the field K and $F_t = K(X_1, X_2, ..., X_t)$ be the rational functions field. For $i = 1, ..., t$, we construct the algebra A_i over the rational functions field $K(X_1, X_2, ..., X_i)$ by setting $\alpha_j = X_j$ for $j = 1, 2, ..., i$. Let $A_0 = K$. By induction over i, assuming that A_{i-1} is a division algebra over the field $F_{i-1} = K(X_1, X_2, ..., X_{i-1})$, we may prove that the algebra A_i is a division algebra over the field $F_i = K(X_1, X_2, ..., X_i)$.

Let $A_{F_i}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$. For $\alpha_i = X_i$ we apply the Cayley-Dickson process to algebra $A_{F_i}^{i-1}$. The obtained algebra, denoted A_i, is an algebra over the field F_i and has dimension 2^i.

Let

$$x = a + bv_i,\ y = c + dv_i,$$

be nonzero elements in A_i such that $xy = 0$, where $v_i^2 = \alpha_i$. Since

$$xy = ac + X_i\bar{d}b + (b\bar{c} + da)\, v_i = 0,$$

we obtain

$$ac + X_i\bar{d}b = 0 \tag{2.1}$$

and

$$b\bar{c} + da = 0. \tag{2.2.}$$

But, the elements $a, b, c, d \in A_{F_i}^{i-1}$ are different from zero. Indeed, we have:

i) If $a = 0$ and $b \neq 0$, then $c = d = 0 \Rightarrow y = 0$, false;
ii) If $b = 0$ and $a \neq 0$, then $d = c = 0 \Rightarrow y = 0$, false;
iii) If $c = 0$ and $d \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false;
iv) If $d = 0$ and $c \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false.

This implies that $b \neq 0, a \neq 0, d \neq 0, c \neq 0$. If $\{1, f_2, ..., f_{2^{i-1}}\}$ is a basis in A_{i-1}, then $a = \sum\limits_{j=1}^{2^{i-1}} g_j(1 \otimes f_j) = \sum\limits_{j=1}^{2^{i-1}} g_j f_j, g_j \in F_i, g_j = \dfrac{g_j'}{g_j''}, g_j', g_j'' \in K[X_1, ..., X_i]$, $g_j'' \neq 0$, $j = 1, 2, ...2^{i-1}$, where $K[X_1, ..., X_t]$ is the polynomial ring. Let a_2 be the less common multiple of $g_1'', ...g_{2^{i-1}}''$, then we can write $a = \dfrac{a_1}{a_2}$, where $a_1 \in A_{F_i}^{i-1}, a_1 \neq 0$. Analogously, $b = \dfrac{b_1}{b_2}, c = \dfrac{c_1}{c_2}, d = \dfrac{d_1}{d_2}, b_1, c_1, d_1 \in A_{F_i}^{i-1} - \{0\}$ and $a_2, b_2, c_2, d_2 \in K[X_1, ..., X_t] - \{0\}$.

If we replace in the relations (2.1.) and (2.2.) , we obtain

$$a_1c_1d_2b_2 + X_i\bar{d}_1b_1a_2c_2 = 0 \tag{2.3.}$$

and

$$b_1\bar{c}_1d_2a_2 + d_1a_1b_2c_2 = 0. \tag{2.4.}$$

If we denote $a_3 = a_1b_2, b_3 = b_1a_2, c_3 = c_1d_2, d_3 = d_1c_2,\ a_3, b_3, c_3, d_3 \in A_{F_i}^{i-1} - \{0\}$, the relations (2.3.) and (2.4.) become

$$a_3c_3 + X_i\bar{d}_3b_3 = 0 \tag{2.5.}$$

and

$$b_3\bar{c}_3 + d_3a_3 = 0. \tag{2.6.}$$

Since the algebra $A_{F_i}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$ is an algebra over F_{i-1} with basis $X^i \otimes f_j$, $i \in \mathbb{N}$ and $j = 1, 2, ...2^{i-1}$, we can write a_3, b_3, c_3, d_3 under the form $a_3 = \sum\limits_{j\geq m} x_jX_i^j, b_3 = \sum\limits_{j\geq n} y_jX_i^j, c_3 = \sum\limits_{j\geq p} z_jX_i^j, d_3 = \sum\limits_{j\geq r} w_jX_i^j$, where $x_j, y_j, z_j, w_j \in A_{i-1}, x_m, y_n, z_p, w_r \neq 0$. Since A_{i-1} is a division algebra, we have $x_mz_p \neq 0, w_ry_n \neq 0$, $y_nz_p \neq 0, w_rx_m \neq 0$. Using relations (2.5.) and (2.6.) , we have that $2m + p + r = 2n + p + r + 1$, which is false. Therefore, the algebra A_i is a division algebra over the field $F_i = K(X_1, X_2, ..., X_i)$ of dimension 2^i.

3. A division algebra of dimension 2^t and prescribed level and sublevel $2^k, t, k \in \mathbb{N}^*$

In his paper [O' Sh; 07(1)], J. O'Shea gives a classification of the levels of quaternion and octonion algebras. Now we extend some of these results to the algebras obtained by the Cayley-Dickson process.

The *level* of the algebra, A denoted by $s(A)$, is the least integer n such that -1 is a sum of n squares in A. The *sublevel* of the algebra A, denoted by $\underline{s}(A)$, is the least integer n such that 0 is a sum of $n+1$ nonzero squares of elements in A. If these numbers do not exist, then the level and sublevel are infinite.

Obviously, $\underline{s}(A) \leq s(A)$. We remark that, if in the Cayley-Dickson process, the quaternion algebra A_2 and the octonion algebra are split, then $s(A_2) = s(A_3) = 1$. (See [Pu, 05, Lemma 2.3.])

Let A be an algebra over a field K obtained by the Cayley-Dickson process, of dimension $q = 2^t, T_C$ and T_P be its trace and pure trace forms.

Proposition 3.1. *If $s(A) \leq n$ then -1 is represented by the quadratic form $n \times T_C$.*

Proof. Let $y \in A, y = x_1+x_2f_2+...+x_qf_q$, $x_i \in K$, for all $i \in \{1, 2, ..., q\}$. Using the notations given in the Introduction, we get $y^2 = x_1^2 + \beta_2 x_2^2 + ... + \beta_q x_q^2 + 2x_1 y''$, where $y'' = x_2 f_2 + ... + x_q f_q$. If -1 is a sum of n squares in A, then $-1 = y_1^2 + ... + y_n^2 = \left(x_{11}^2 + \beta_2 x_{12}^2 + ... + \beta_q x_{1q}^2 + 2x_{11} y_1''\right) + ... + \left(x_{n1}^2 + \beta_2 x_{n2}^2 + ... + \beta_q x_{nq}^2 + 2x_{n1} y_n''\right)$. Then we have

$-1 = \sum\limits_{i=1}^{n} x_{i1}^2 + \beta_2 \sum\limits_{i=1}^{n} x_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} x_{iq}^2$ and

$\sum\limits_{i=1}^{n} x_{i1} x_{i2} = \sum\limits_{i=1}^{n} x_{i1} x_{i3} ... = = \sum\limits_{i=1}^{n} x_{i1} x_{in} = 0$, then $n \times T_C$ represents -1.□

In Proposition 3.1, we remark that the quadratic form $< 1 > \perp n \times T_C$ is isotropic.

Proposition 3.2. *For $n \in \mathbb{N}^*$, if the quadratic form $< 1 > \perp n \times T_P$ is isotropic over K, then $s(A) \leq n$.*

Proof. Case 1. If $-1 \in K^{*2}$, then $s(A) = 1$.

Case 2. $-1 \notin K^{*2}$. Since the quadratic form $< 1 > \perp n \times T_P$ is isotropic then it is universal. It results that $< 1 > \perp n \times T_P$ represent -1. Then, we have the elements $\alpha \in K$ and $p_i \in Skew(A)$, $i = 1, ..., n$, such that $-1 = \alpha^2 + \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$, and not all of them are zero.

i) If $\alpha = 0$, then $-1 = \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$. It results $-1 = \left(\beta_2 p_{12}^2 + ... + \beta_q p_{1q}^2\right) + ... + \left(\beta_2 p_{n2}^2 + ... + \beta_q p_{nq}^2\right)$. Denoting $u_i = p_{i2} f_2 + ... + p_{iq} f_q$, we have that $t(u_i) = 0$ and $u_i^2 = -n(u_i^2) = \beta_2 p_{i2}^2 + ... + \beta_q p_{iq}^2$, for all $i \in \{1, 2, ..., n\}$. We obtain $-1 = u_1^2 + ... + u_n^2$.

ii) If $\alpha \neq 0$, then $1 + \alpha^2 \neq 0$ and $0 = 1 + \alpha^2 + \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$. Multiplying this relation with $1+\alpha^2$, it follows that $0 = (1+\alpha^2)^2 + \beta_2 \sum\limits_{i=1}^{n} r_{i2}^2 +$

$... + \beta_q \sum_{i=1}^{n} r_{iq}^2$. Therefore $-1 = \beta_2 \sum_{i=1}^{n} r_{i2}'^2 + ... + \beta_q \sum_{i=1}^{n} r_{iq}'^2$, where $r'_{ij} = r_{ij}(1 + \alpha)^{-1}, j \in \{2, 3, ..., q\}$ and we apply case i). Therefore $s(A) \leq n.\square$

Lemma 3.3.[Sch; 85, p. 151] *Let* $n = 2^k$, *and* $a_1, ...a_n, b_1, ..., b_n \in K$. *Then there are elements* $c_2, ..., c_n \in K$ *such that*

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) = (a_1 b_1 + ... a_n b_n)^2 + c_2^2 + ... + c_n^2.$$

Now we can state and prove some generalizations of J. O'Shea's results (Lemma 3.9, Proposition 3.2. and Proposition 3.3., Lemma 3.4., Theorem 3.5., Corollary 3.10. and Theorem 3.11. from [O'Sh; 07(1)]):

Proposition 3.4. *If* $n \in \mathbb{N}^*$, $n = 2^k - 1$ *such that* $s(K) \geq 2^k$, *then* $s(A) \leq n$ *if and only if* $< 1 > \perp n \times T_P$ *is isotropic.*

Proof. From Proposition 3.1, supposing that $s(A) \leq n$, we have $-1 = \sum_{i=1}^{n} p_{i1}^2 + \beta_2 \sum_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{n} p_{iq}^2$ such that

$$\sum_{i=1}^{n} p_{i1} p_{i2} = \sum_{i=1}^{n} p_{i1} p_{i3} = ... = \sum_{i=1}^{n} p_{i1} p_{iq} = 0.$$

Since $s(K) \geq 2^k$, it results that $-1 + \sum_{i=1}^{n} p_{i1}^2 \neq 0$. Putting $p_{2^k 1} = 1$ and $p_{2^k 2} = p_{2^k 3} = ... p_{2^k q} = 0$, we have

$$0 = \sum_{i=1}^{n+1} p_{i1}^2 + \beta_2 \sum_{i=1}^{n+1} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} p_{iq}^2 \tag{3.1}$$

and $\sum_{i=1}^{n+1} p_{i1} p_{i2} = \sum_{i=1}^{n+1} p_{i1} p_{i3} = ... = \sum_{i=1}^{n+1} p_{i1} p_{iq} = 0$. Multiplying (3.1.) by $\sum_{i=1}^{n+1} p_{i1}^2$, since $\left(\sum_{i=1}^{n+1} p_{i1}^2 \right)^2$ is a square and using Lemma 3.3. for the products $\sum_{i=1}^{n+1} p_{i2}^2 \sum_{i=1}^{n+1} p_{i1}^2, ..., \sum_{i=1}^{n+1} p_{iq}^2 \sum_{i=1}^{n+1} p_{i1}^2$, we obtain

$$0 = \left(\sum_{i=1}^{n+1} p_{i1}^2 \right)^2 + \beta_2 \sum_{i=1}^{n+1} r_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} r_{iq}^2, \tag{3.2}$$

where $r_{i2},...r_{iq} \in K$, $n+1 = 2^k$, $r_{12} = \sum_{i=1}^{n+1} p_{i1}p_{i2} = 0$, $r_{13} = \sum_{i=1}^{n+1} p_{i1}p_{i3} = 0, ..., r_{1q} = \sum_{i=1}^{n+1} p_{i1}p_{iq} = 0$. Therefore, in the sums $\sum_{i=1}^{n+1} r_{i2}^2$, ..., $\sum_{i=1}^{n+1} r_{iq}^2$ we have n factors. From (3.2) , we get that $< 1 > \perp n \times T_P$ is isotropic.□

Proposition 3.6. *If $s(K) \geq 2^k$, then the quadratic form $2^k \times T_C$ is isotropic if and only if $< 1 > \perp 2^k \times T_P$ is isotropic.*

Proof. Since the form $< 1 > \perp 2^k \times T_P$ is a subform of the form $2^k \times T_C$, if the form $< 1 > \perp 2^k \times T_P$ is isotropic, we have that $2^k \times T_C$ is isotropic.

For the converse, supposing that $2^k \times T_C$ is isotropic, then we get

$$\sum_{i=1}^{2^k} p_i^2 + \beta_2 \sum_{i=1}^{2^k} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0, \tag{3.3}$$

where $p_i, p_{ij} \in K, i = 1, ..., 2^k$, $j \in 2, ..., q$ and some of the elements p_i and p_{ij} are nonzero.

If $p_i = 0, \forall i = 1, ..., 2^k$, then $2^k \times T_P$ is isotropic, therefore $< 1 > \perp 2^k \times T_P$ is isotropic.

If $\sum_{i=1}^{2^k} p_i^2 \neq 0$, then, multiplying relation (3.3) with $\sum_{i=1}^{2^k} p_i^2$ and using Lemma 3.3. for the products $\sum_{i=1}^{2^k} p_{i2}^2 \sum_{i=1}^{2^k} p_i^2$,..., $\sum_{i=1}^{2^k} p_{iq}^2 \sum_{i=1}^{2^k} p_i^2$, we obtain

$$(\sum_{i=1}^{2^k} p_i^2)^2 + \beta_2 \sum_{i=1}^{2^k} r_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} r_{iq}^2 = 0,$$

then $< 1 > \perp 2^k \times T_P$ is isotropic.

Since $s(K) \geq 2^k$, the relation $\sum_{i=1}^{2^k} p_i^2 = 0$, for some $p_i \neq 0$, does not work.

Indeed, supposing that $p_1 \neq 0$, we obtain $-1 = \sum_{i=2}^{2^k} (p_i p_1^{-1})^2$, false.□

Proposition 3.6. *Let $n = 2^k - 1$ and $s(K) \geq 2^k$. Then $\underline{s}(A) \leq n$ if and only if $< 1 > \perp (n \times T_P)$ is isotropic or $(n+1) \times T_P$ is isotropic.*

Proof. Since $\underline{s}(A) \leq s(A)$, if $< 1 > \perp (n \times T_P)$ is isotropic, then, from Proposition 3.4, we have $\underline{s}(A) \leq n$. If $(n+1) \times T_P$ is isotropic, then there are the elements $p_{ij} \in K, i = 1, ..., 2^k, j \in 2, ..., q$, some of them are nonzero, such that $\beta_2 \sum_{i=1}^{2^k} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0$. We obtain $0 = \left(\beta_2 p_{12}^2 + ... + \beta_q p_{1q}^2\right) + ... + \left(\beta_2 p_{n2}^2 + ... + \beta_q p_{nq}^2\right)$. Denoting $u_i = p_{i2} f_2 + ... + p_{iq} f_q$, we have $t(u_i) = 0$ and $u_i^2 = -n(u_i^2) = \beta_2 p_{i2}^2 + ... + \beta_q p_{iq}^2$, for all $i \in \{1, 2, ..., n\}$. Therefore $0 = u_1^2 + ... + u_n^2$. It results that $\underline{s}(A) \leq n$.

Conversely, if $\underline{s}(A) \leq n$, then there are the elements $y_1, ..., y_{n+1} \in A$, some of them must be nonzero, such that $0 = y_1^2 + ... + y_{n+1}^2$. As in the proof of Proposition 3.1., we obtain $0 = \sum_{i=1}^{n+1} x_{i1}^2 + \beta_2 \sum_{i=1}^{n+1} x_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} x_{iq}^2$ and $\sum_{i=1}^{n+1} x_{i1} x_{i2} = \sum_{i=1}^{n+1} x_{i1} x_{i3} ... = \sum_{i=1}^{n+1} x_{i1} x_{in} = 0$. If all $x_{i1} = 0$, then $(n+1) \times T_P$ is isotropic. If $\sum_{i=1}^{n+1} x_{i1}^2 \neq 0$, then $(n+1) \times T_C$ is isotropic, or multiplying the last relation with $\sum_{i=1}^{2^k} x_{i1}^2$ and using Lemma 3.3. for the products $\sum_{i=1}^{2^k} x_{i2}^2 \sum_{i=1}^{2^k} x_{i1}^2, ..., \sum_{i=1}^{2^k} x_{iq}^2 \sum_{i=1}^{2^k} x_{i1}^2$, we obtain that $< 1 > \perp (n \times T_P)$ is isotropic. Since $s(K) \geq 2^k$, the relation $\sum_{i=1}^{n+1} x_{i1}^2 = 0$ for some $x_{i1} \neq 0$ is false. □

Proposition 3.7. *If* $-1 \notin K^{*2}$, *then* $\underline{s}(A) = 1$ *if and only if either* T_C *or* $2 \times T_P$ *is isotropic.*

Proof. We apply Proposition 3.6 for $k = 1$.□

Proposition 3.8. *Let A be an algebra obtained by the Cayley-Dickson process. The following statements are true:*

a) If -1 *is a square in* K, *then* $\underline{s}(A) = s(A) = 1$.

b) If $-1 \notin K^{*2}$, *then* $s(A) = 1$ *if and only if* T_C *is isotropic.*

Proof. a) If $-1 = a^2 \in K \subset A$, then $\underline{s}(A) = s(A) = 1$.

b) If $-1 \notin K^{*2}$ and $s(A) = 1$, then, there is an element $y \in A$ such that $-1 = y^2$, with $y = y_1 + y_2 f_2 + ... + y_q f_q$. Since $y^2 + 1 = 0$, then $y_1 = t(y) = 0$ and so $n(y) = 1$. Since $2T_C(y) = t(y^2) = -2n(y) = -2$, we obtain $T_C(y) = -1$, then

$$y^2 = -1 = \beta_2 y_2^2 + ... + \beta_q y_q^2,$$

therefore $0 = 1 + \beta_2 y_2^2 + ... + \beta_q y_q^2$. It results that T_C is isotropic.

Conversely, if T_C is isotropic, then there is $y \in A$, $y \neq 0$, such that $T_C(y) = 0 = y_1^2 + \beta_2 y_2^2 + ... + \beta_q y_q^2$. If $y_1 = 0$, then $T_C(y) = T_P(y) = 0$, so $y = 0$, which is false. If $y_1 \neq 0$, then $-1 = \left(\left(\frac{y_2}{y_1}\right) f_2 + ... + \left(\frac{y_q}{y_1}\right) f_q\right)^2$, obtaining $s(A) = 1$.□

Remark 3.9. Using the above notations, if the algebra A is an algebra obtained by the Cayley-Dickson process, of dimension greater than 2 and if n_C is isotropic, then $s(A) = \underline{s}(A) = 1$. Indeed, if -1 is a square in K, the statement results from Proposition 3.8.a). If $-1 \notin K^{*2}$, since $n_C =< 1 >\perp -T_P$ and n_C is a Pfister form, we obtain that $-T_P$ is isotropic, therefore T_C is isotropic. Using Proposition 3.8., we have that $s(A) = \underline{s}(A) = 1$.

Proposition 3.10. *Let A be an algebra over a field K obtained by the Cayley-Dickson process, of dimension $q = 2^t, T_C$ and T_P be its trace and pure trace forms. If $t \geq 2$ and $2^k \times T_P$ is isotropic over K, $k \geqslant 0$, then $\left(1 + [\frac{2}{3}2^k]\right) \times T_P$ is isotropic over K.*

Proof. If $2^k \times T_P$ is isotropic then $2^k \times -T_P$ is isotropic. Since $2^k \times n_C = 2^k \times (< 1 > \perp - T_P)$ and n_C is a Pfister form, from Proposition 1.6.(ii), we have $2^k \times n_C$ is a Pfister form. Since $2^k \times -T_P$ is a subform of $2^k \times n_C$, it results that $2^k \times n_C$ is isotropic, then it is hyperbolic. Therefore $2^k \times n_C \simeq< 1, 1, ..., 1, -1, ..., -1 >$(there are 2^{k+t-1} of -1 and 2^{k+t-1} of 1). Multiplying by -1, we have that $2^k \times (< -1 > \perp T_P)$ is hyperbolic, then has a totally isotropic subspace of dimension 2^{k+t-1}. It results that each subform of the form $2^k \times (< -1 > \perp T_P)$ of dimension greater or equal to 2^{k+t-1} is isotropic. Since $(2^t - 1)(1 + [\frac{2}{3}2^k]) > (2^t - 1)(\frac{2}{3}2^k) > 2^{t-1}2^k = 2^{k+t-1}$, then $\left(1 + [\frac{2}{3}2^k]\right) \times T_P$ is isotropic over K.□

Proposition 3.11. *Let A be an algebra over a field K obtained by the Cayley-Dickson process, of dimension $q = 2^t, T_C$ and T_P be its trace and pure trace forms. Let $n = 2^k - 1$, $s(K) \geq 2^k$. If $t \geq 2$ and $k > 1$ then $\underline{s}(A) \leq 2^k - 1$ if and only if $< 1 > \perp (2^k - 1) \times T_P$ is isotropic.*

Proof. We use Proposition 3.6. and we have that $\underline{s}(A) \leq 2^k - 1$ if and only if $< 1 >\perp (n \times T_P)$ is isotropic or $(n+1) \times T_P$ is isotropic. In this case, we prove that $2^k \times T_P$ is isotropic implies $< 1 > \perp (2^k - 1) \times T_P$ is isotropic. If $2^k \times T_P$ isotropic over K then $\left(1 + [\frac{2}{3}2^k]\right) \times T_P$ is isotropic over K, from Proposition 3.10. If $k \geq 2$, then $\left(1 + [\frac{2}{3}2^k]\right) \leq 2^k - 1$ and we have that $\left(1 + [\frac{2}{3}2^k]\right) \times T_P$ is an isotropic subform of the form $< 1 > \perp (2^k - 1) \times T_P$.□

Proposition 3.12. *Let K be a field such that $s(K) \geq 2^k$.*

i) If $k \geq 2$, *then* $\underline{s}(A) \leq 2^k - 1$ *if and only if* $s(A) \leq 2^k - 1$.
ii) If $\underline{s}(A) = n$ *and* $k \geq 2$ *such that* $2^{k-1} \leq n < 2^k$, *then* $s(A) \leq 2^k - 1$.
iii) If $\underline{s}(A) = 1$ *then* $s(A) \leq 2$.

Proof. i) For $k \geq 2$, then $\underline{s}(A) \leq 2^k - 1$ if and only if $< 1 > \perp \left(\left(2^k - 1 \right) \times T_P \right)$ is isotropic. This is equivalent with $s(A) \leq 2^k - 1$.

ii) If $n < 2^k$, $k \geq 2$, it results $n \leq 2^k - 1$, and we apply i).

iii) We have that $\underline{s}(A) = 1$ if and only if $< 1 > \perp T_P = T_C$ is isotropic or $2 \times T_P$ is isotropic. If $2 \times T_P$ is isotropic, then it is universal and represents -1. Therefore $s(A) \leq 2$. If T_C is isotropic, then T_P is isotropic, then is universal and represents -1. We obtain $s(A) = 1$.□

Proposition 3.13. *With the above notations, we have*:
i) For $k \geq 2$, *if* $\underline{s}(A) = 2^k - 1$ *then* $s(A) = 2^k - 1$.
ii) For $k \geq 2$, *if* $s(A) = 2^k$ *then* $\underline{s}(A) = 2^k$.
iii) For $k \geq 1$, *if* $s(A) = 2^k + 1$ *then* $\underline{s}(A) = 2^k + 1$ *or* $\underline{s}(A) = 2^k$.

Proof. i) From Proposition 3.12., if $\underline{s}(A) = 2^k - 1$ then $s(A) \leq 2^k - 1$. Since $\underline{s}(A) \leq s(A)$, therefore $s(A) = 2^k - 1$.

ii) If $\underline{s}(A) \leq 2^k - 1$ we have $s(A) \leq 2^k - 1$, false.

iii) For $k \geq 1$, if $s(A) = 2^k + 1$, since $\underline{s}(A) \leq s(A)$, we obtain that $\underline{s}(A) \leq 2^k + 1$. If $\underline{s}(A) \leq 2^k - 1$, then $s(A) \leq 2^k - 1$, false.□

In the following, we give an example of division algebra of dimension 2^t and prescribed level 2^k.

Theorem 3.14 . *Let* K *be a field such that* $s(K) = 2^k$, X *be an algebraically independent indeterminate over* K, D *be a finite-dimensional division* $K-$*algebra with scalar involution* $^{-}$ *such that* $s(D) = s(K)$, $D_1 = K(X) \otimes_K D$ *and* $B = (D_1, X)$. *Then* B *is a division algebra over* $K(X)$ *such that* $s(B) = s(K)$.

Proof. By straightforward calculations, using the same arguments like in Brown's construction, see Section 2, we obtain that B is a division algebra.

For the second part, since $s(B) \leq s(K) = n = 2^k$, we suppose that $s(B) \leq n - 1$. It results that $-1 = y_1^2 + ... + y_{n-1}^2$, where $y_i \in B$, $y_i = a_{i1} + a_{i2}u$, $u^2 = X$, $a_{i1}, a_{i2} \in D_1$, some of y_i are nonzero. We have $y_i^2 = a_{i1}^2 + X\overline{a}_{i2}a_{i2} + \left(a_{i2}\overline{a}_{i1} + a_{i2}a_{i1} \right) u, i \in \{1, 2, ...n - 1\}$. It follows that $-1 = \sum\limits_{i=1}^{n-1} a_{i1}^2 + X \sum\limits_{i=1}^{n-1} \overline{a}_{i2}a_{i2}$, where $\psi = 1 \otimes \varphi$ is involution in $D_1, \psi(x) = \bar{x}$. We remark that $\overline{a}_{i2}a_{i2} \in K(X)$, $i \in \{1, ..., n-1\}$. If $a_{i1} = \sum_{j=1}^{m} \frac{p_{ji1}(X)}{q_{ji1}(X)} \otimes b_j$,

with $b_j \in D$, $\frac{p_{ji1}(X)}{q_{ji1}(X)} \in K(X), i \in \{1, 2, ..., n-1\}$, $j \in \{1, 2, ..., m\}$.

$a_{i2} = \sum_{j=1}^{m} \frac{r_{ji2}(X)}{w_{ji2}(X)} \otimes d_j$, with $d_j \in D$, $\frac{r_{ji2}(X)}{w_{ji2}(X)} \in K(X)$,
$i \in \{1, 2, ..., n-1\}$, $j \in \{1, 2, ..., m\}$, it results

$$-1 = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} \frac{p_{ji1}(X)}{q_{ji1}(X)} \otimes b_j\right)^2 + X \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} \frac{r_{ji2}(X)}{w_{ji2}(X)} \otimes d_j\right)\left(\sum_{j=1}^{m} \frac{r_{ji2}(X)}{w_{ji2}(X)} \otimes \bar{d}_j\right).$$

After clearing denominators, we obtain

$$-v^2(X) = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} p'_{ji1}(X) \otimes b_j\right)^2 + X \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} r'_{ji2}(X) \otimes d_j\right)\left(\sum_{j=1}^{m} r'_{ji2} \otimes \overline{d}_j\right), \quad (3.4.)$$

where $v(X) = lcm\{q_{ji1}(X), w_{ji2}(X)\}, i \in \{1, 2, ..., n-1\}, j \in \{1, 2, ..., m\}$ and $p'_{ji1}(X) = v(X) p_{ji1}(X), r'_{ji2}(X) = v(X) r_{ji2}(X)$,
$i \in \{1, ..., n-1\}, j \in \{1, 2, ..., m\}$. We can write

$$v(X) = v_q X^q + v_{q+1} X^{q+1} + ..., v_q \in K, v_q \neq 0, \quad (3.5.)$$

$$\sum_{j=1}^{m} p'_{ji1}(X) \otimes b_j = \alpha_{r_i} X^{r_i} + \alpha_{r_i+1} X^{r_i+1} + ..., \ \alpha_{r_i}, \alpha_{r_i+1}, ... \in D, \alpha_{r_i} \neq 0, \quad (3.6.)$$

$$\sum_{j=1}^{m} r'_{ji2}(X) \otimes d_j = \beta_{s_i} X^{s_i} + \beta_{s_i+1} X^{s_i+1} + ..., \beta_{s_i}, \beta_{s_i+1}, ... \in D, \beta_{s_i} \neq 0, \quad (3.7.)$$

$$\sum_{j=1}^{m} r'_{ji2} \otimes \overline{d}_j = \overline{\beta}_{s_i} X^{s_i} + \overline{\beta}_{s_i+1} X^{s_i+1} + ..., \overline{\beta}_{s_i}, \overline{\beta}_{s_i+1}, ... \in D, \overline{\beta}_{s_i} \neq 0. \quad (3.8.)$$

By (3.4.), if $s = \min_{i=\overline{1,n-1}} s_i$, $r = \min_{i=\overline{1,n-1}} r_i$, in the left side the minimum degree is $2q$ (q possible zero) in the right side, the first sum has the minimum degree

$2r$ (r possible zero) and the second term has the minimum degree $2s+1$. It results $q = r$ and $2r < 2s+1$. Replacing the relations $(3.5.)$, $(3.6.)$, $(3.7.)$, $(3.8.)$ in the relation $(3.4.)$, if $r > 0$, we divide relation $(3.4.)$ by X^{2r}, such that, in the new obtained relation the minimum degree in the both sides is zero. Putting $X = 0$ in this new relation, we have

$$-v_q^2 = \sum_{i=1}^{n-1} \alpha_{r_i}^2, \quad \alpha_{r_i} \in D.$$

We obtain

$$-1 = \sum_{i=1}^{n-1} \left(\frac{\alpha_{r_i}}{v_q}\right)^2.$$

It follows that $s(D) \leq n-1$, false.

Corollary 3.15. *Let K be a field such that $s(K) \geq 2^k$, X be an algebraically independent indeterminate over K, D be a finite-dimensional division $K-$algebra with scalar involution $^{-}$ such that $s(D) < \infty$, $D_1 = K(X) \otimes_K D$ and $B = (D_1, X)$. Then:*

i) B is a division algebra.

ii) $s(B) = s(D)$.

iii) $\underline{s}(B) = \underline{s}(D)$.

Proof. i) and ii) result from Theorem 3.14.

iii) We prove that $\underline{s}(B) = \underline{s}(D)$. Since $\underline{s}(B) \leq s(B) = s(D)$, then $\underline{s}(B) \leq \underline{s}(D) = m \leq 2^k$, we suppose that $\underline{s}(B) \leq m-1$.

It results $0 = y_1^2 + ... + y_m^2$, where $y_i \in B$. Using the same notations like in Theorem 3.14, after straightforward calculations, we obtain $0=\sum_{i=1}^{m}(\sum_{j=1}^{l} p'_{ji1}(X) \otimes b_j)^2 + X\sum_{i=1}^{m}(\sum_{j=1}^{l} r'_{ji2}(X) \otimes d_j)(\sum_{j=1}^{l} r'_{ji2} \otimes \overline{d}_j)$. It results $0 = \sum_{i=1}^{m} \alpha_{r_i}^2$, $\alpha_{r_i} \in D$, therefore $\underline{s}(D) \leq m-1$, false. Then $\underline{s}(B) = \underline{s}(D)$.$\square$

Remark 3.16. Using Example 4.2. from [O' Sh; 07(1)], we have that, if K_0 is a formally real field, then the field $F_0 = K_0((2^k+1) \times <1>)$ has the level 2^k. If $D = A_0 = F_0, K = F_0, D_1 = K(X_1) \otimes_K A_0$, from Brown's construction and Theorem 3.14., the $K(X_1)-$algebra B, obtained by application of the Cayley-Dickson process with $\alpha = X_1$ to the $K(X_1)-$algebra D_1, is a division algebra of dimension 2 and level 2^k.

By induction, supposing that $D = A_{t-1}$ is a division algebra of dimension 2^{t-1} and level 2^k over the field $K = F_0(X_1, ..., X_{t-1})$, then, if $D = A_{t-1}$, $D_1 = K(X_t) \otimes_K A_{t-1}$ and B is the $K(X_t)-$algebra obtained by application of the Cayley-Dickson process with $\alpha = X_t$ to the $K(X_t)-$algebra D_1, then B is a division algebra of dimension 2^t and level 2^k.

Looking to the field F_0 like as an F_0-algebra, then the field F_0 has the same level and sublevel. Using above proposition, we have that $s(B) = \underline{s}(B) = 2^k$. This is an example of a division algebra of level and sublevel 2^k and dimension $2^t, t, k \in \mathbb{N}^*$.

4. Algebras of sublevels $2^k + 1, k \in \mathbb{N}^*$ obtained by the Cayley-Dickson process

Let F_0 be a formally real field. In their paper [La, Ma; 01], Laghribi and Mammone proved that the quaternion algebras $Q(m) = \left(\frac{X,Y}{F}\right) \otimes_F K$ are division algebras of level m, where $m = 2^k + 1, k \geq 0$, $F = F_0(X, Y)$, $K = F\left(< 1 > \perp m \times T_P^Q\right)$, $Q = \left(\frac{X,Y}{F}\right)$ is a quaternion algebra over F and T_P^Q its pure trace form over F and in her paper [Pu; 05], S. Pumpluen proved that $O(m) = \left(\frac{X,Y,Z}{F}\right) \otimes_F K$ are octonion division algebras of level m, where $m = 2^k + 1, k \geq 0$, $F = F_0(X, Y, Z)$, $K = F\left(< 1 > \perp m \times T_P^O\right)$, $O = \left(\frac{X,Y,Z}{F}\right)$ is an octonion algebra over F and T_P^O its pure trace form over F.

In his paper [O'Sh; 07(1)], Proposition 4.7., J. O'Shea proved that the octonion algebra $O(5)$ is a division algebra of sublevel 5 and, in [O'Sh; 07(2)], he conjectured that the relations

$$s(Q(m)) = \underline{s}(Q(m)) = m \tag{4.1.}$$

and

$$s(O(m)) = \underline{s}(O(m)) = m. \tag{4.2.}$$

are true, but he proved them only for $m = 2^k$, $k \in \mathbb{N}$ and $m \leq 7$ for the octonions and $m = 2^k$, $k \in \mathbb{N}$ and $m \leq 3$ for the quaternions.

Starting from some ideas given in the mentioned works, especially in the Proposition 4.7. from [O'Sh; 07(1)] and in the proof of Theorem 3.3. from [O'Sh; 07(2)], in the following, we prove that the quaternion algebra $Q(5)$ has sublevel 5., and finally, we prove that, for $n = 2^k + 1, k \in \mathbb{N}$, relations (4.1) and (4.2.) hold.

Proposition 4.1. *Let x be a transcendental element over K, V a vector space over K, $\dim V \geq 3$. Let $q : V \rightarrow K$ be a regular quadratic irreducible form. We have that $K(q)(x) = K(x)(q)$, where $K(q)$ is the functions field of q over K and $K(x)(q)$ is the functions field of q over $K(x)$.*

Proof. Suppose that q has the diagonal representation $< a_1, a_2, ..., a_n >$. The function field of q over K is the quotient field of $K[x_2, ..., x_n]/ (a_1 + a_2x_2^2 + ... + a_nx_n^2)$. This field is $K(x_2, ..., x_{n-1})(\sqrt{-\alpha})$, where $\alpha = a_n^{-1}\left(a_1 + a_2x_2^2 + ... + a_{n-1}x_{n-1}^2\right)$. Since q is irreducible over $K(x)$, its functions field over $K(x)$ is the quotient field of

$$K(x)[x_2, ..., x_n]/ \left(a_1 + a_2x_2^2 + ... + a_nx_n^2\right).$$

This field is $K(x)(x_2, ..., x_{n-1})(\sqrt{-\alpha})$, where $\alpha = a_n^{-1}\left(a_1 + a_2x_2^2 + ... + a_{n-1}x_{n-1}^2\right)$. Since $K(x)(x_2, ..., x_{n-1})(\sqrt{-\alpha}) = K(x_2, ..., x_{n-1})(\sqrt{-\alpha})(x)$, it results that $K(q)(x) = K(x)(q)$.□

Theorem 4.2. *Let $F = F_0(X, Y)$, $K = F\left(< 1 > \perp 5 \times T_P^Q\right)$. With the above notations, let Z be an algebraically independent element over F. Let $Q'_5 = Q(5) \otimes_K K(Z)$ and $O'_5 = (Q'_5, Z)$. Then the following statements are true:*

i) The algebra O'_5 is a division algebra of level and sublevel 5.

ii) The algebra $Q(5)$ is a division algebra of level and sublevel 5.

Proof. i) Denoting $\varphi = < 1 > \perp 5 \times T_P^Q$, where $T_P^Q \simeq < X, Y, -XY >$, from the above proposition, it results that

$$K(Z) = F\left(< 1 > \perp 5 \times T_P^Q\right)(Z) = F(Z)\left(< 1 > \perp 5 \times T_P^Q\right).$$

Therefore $K(Z) = F(\varphi)(Z) = F(Z)(\varphi)$. Since the algebra $Q(5)$ has the level 5, using Proposition 3.16., we obtain that the algebra O'_5 is a division algebra and has level 5. We prove that the algebra O'_5 has sublevel 5. Suppose $\underline{s}(O'_5) \leq 4$. Then

$$\sum_{i=1}^{5} c_i^2 + \sum_{i=1}^{5} p_i^2 = 0 \qquad (4.1.)$$

and

$$\sum_{i=1}^{5} c_i p_i = 0, \qquad (4.2.)$$

where $c_i \in F(Z)(\varphi)$, $p_i \in \mathcal{P}, \mathcal{P}$ the $F(Z)(\varphi)$-vector space spanned by the pure octonions.

Case 1. $c_i = 0$, for all $i \in \{1, 2, ..., 5\}$. From $(4.1.)$, it results that $5 \times T_P^O$ is isotropic over $F(Z)(\varphi)$.

Case 2. If there is at least an element i such that $c_i \neq 0$, relation (4.2.) implies that we get a 4-dimensional $F(Z)(\varphi)$- vector subspace V of $\mathcal{P}$, containing p_1, p_2, p_3, p_4, p_5. Let $\beta : V \to F(Z)(\varphi), \beta(p) = p^2$. Therefore β is a 4 dimensional subform of $T_P^O \simeq < X, Y, -XY, Z, -XZ, -XY, -YZ, XYZ >$ and, from $(4.1.)$, the form $\gamma = 5 \times (< 1 > \perp \beta)$ is isotropic over $F(Z)(\varphi)$. We denote $\delta = < 1 > \perp \beta$. Repeated applications of Springer's Theorem implies that $5 \times T_P^O$, γ and $8 \times (< -1 > \perp T_P^O)$ are anisotropic over $F(Z)$.

To prove that $\underline{s}(\ O_5'\ ) \nleq 4$ it is sufficient to show that $5 \times T_P^O$ and γ are anisotropic over $F(Z)(\varphi)$.

If $5 \times T_P^O$ is isotropic over $F(Z)(\varphi)$, since the form $8 \times (< -1 > \perp T_P^O)$ is a Pfister form, then becomes hyperbolic over $F(Z)(\varphi)$. For each $a \in D_{F(Z)}\left(8 \times (< -1 > \perp T_P^O)\right)$ $a\varphi$ is a subform of $8 \times (< -1 > \perp T_P^O)$, from Cassels-Pfister Theorem. Since $X \in D_{F(Z)}\left(8 \times (< -1 > \perp T_P^O)\right)$, using Lemma 3.7., p.8, from [Sch; 85], it results that

$$X\varphi \simeq < X > \perp 5 \times < 1, XY, -Y, XZ, -Z, -XYZ, YZ >$$

is a subform of $8 \times (< -1 > \perp T_P^O)$, therefore $5 \times < 1 >$ is a subform of $8 \times (< -1 > \perp T_P^O)$, false. Hence $5 \times T_P^O$ is anisotropic over $F(Z)(\varphi)$.

If γ is isotropic over $F(Z)(\varphi)$, from Proposition 1.3., we have

$$D_{F(Z)}(\varphi)\, D_{F(Z)}(\varphi) \subseteq D_{F(Z)}(\gamma)\, D_{F(Z)}(\gamma).$$

Since $T_P^Q \simeq < X, Y, -XY >$, it results that $X, Y, -XY \in D_{F(Z)}(\varphi)$. Therefore $X, Y, -XY \in D_{F(Z)}(\varphi)\, D_{F(Z)}(\varphi)$, hence $X, Y, -XY \in D_{F(Z)}(\gamma)\, D_{F(Z)}(\gamma)$. We prove that $< X >$, $< Y >$, $< -XY >$ are subforms of β. If $< X >$ is not a subform of β, then we find a multiple of X in $D_{F(Z)}(\gamma)\, D_{F(Z)}(\gamma)$ of the form $-X\sum\limits_{i=1}^{m} A_i^2$, $A_i \in F(Z)$.(For example, if $\beta \simeq < Y, -XY, Z, -XZ >$, it results $Y, -XY, Z, -XZ \in D_{F(Z)}(\gamma)$, then $-XY^2 \in D_{F(Z)}(\gamma)\, D_{F(Z)}(\gamma)$). From Springer's Theorem, we have that γ is anisotropic over $F(Z)$. From the same Theorem, it results that $m \times \delta$ is anisotropic over $F(Z)$ for all $m \in \mathbb{N}^*$, therefore δ is strongly anisotropic over $F(Z)$. From Remark 1.1., we have that $D_{F(Z)}(\gamma) \subset P_0 = \{\alpha \,/\, \alpha = 0$ or α is represented by $n \times \delta, n \in \mathbb{N}^*\}$, P_0 is a q-preordering and there is a q-ordering P such that $P_0 \subset P$ or $-P_0 \subset P$.

Therefore $D_{F(Z)}(\gamma) D_{F(Z)}(\gamma) \subset P \cdot P \subset P$. If X is positive, then $-X\sum_{i=1}^{5} A_i^2$ is negative and if X is negative, then $-X\sum_{i=1}^{5} A_i^2$ is positive, false.

In the same way, we prove that $< Y >$, $< -XY >$ are subforms of β, therefore $X, Y, -XY \in D_{F(Z)}(\gamma)$. Since $1 \in D_{F(Z)}(\gamma)$, we have that $1 \in P_0 \subset P$, therefore $X, Y, -XY$ are positive, false. We obtain γ is anisotropic over $F(Z)(\varphi)$. It results that $\underline{s}(\ O'_5\ ) = 5$.

ii) From i), using Corollary 3.15., we have $\underline{s}(\ Q(5)\ ) = \underline{s}(\ O'_5\ ) = 5.\square$

Theorem 4.3. *Let $F = F_0(X, Y)$ and $Q = \left(\frac{X,Y}{F}\right)$ be a quaternion algebra over F. Let $K = F\left(< 1 > \perp m \times T_P^Q\right)$. The quaternion algebra $Q(m) = \left(\frac{X,Y}{F}\right) \otimes_F K$ is a quaternion division algebra of level and sublevel m, where $m = 2^k + 1, k \geq 0$, and T_P^Q its pure trace form over F.*

Proof. Since for $k \leq 1$ the result is proved in [O' Sh; 07(2)] and for $k = 2$ we prove the result in the above proposition, in the following, we suppose that $k \geq 3$. We denote $\varphi =< 1 > \perp \left(2^k + 1\right) \times T_P^Q$, where $T_P^Q \simeq < X, Y, -XY >$. Let $Z_3, ..., Z_{k+1}$ be algebraic independent elements over $F = F_0(X, Y), K = F\left(< 1 > \perp \left(2^k + 1\right) \times T_P^Q\right)$. From Proposition 4.1., it results that

$$\begin{aligned} K(Z_3, ..., Z_{k+1}) &= F\left(< 1 > \perp \left(2^k + 1\right) \times T_P^Q\right)(Z_3, ..., Z_{k+1}) = \\ &= F(Z_3, ..., Z_{k+1})\left(< 1 > \perp \left(2^k + 1\right) \times T_P^Q\right) = \\ &= F_0(Z_1, ..., Z_{k+1})\left(< 1 > \perp \left(2^k + 1\right) \times T_P^Q\right). \end{aligned}$$

Let $Q'_m = Q(m) \otimes_K K(Z_3)$ and $O'_m = (Q'_m, Z_3)$ be an octonion algebra as in Brown's construction. Then the algebra O'_m is a division algebra of dimension 2^3 and of level m. We repeat this construction until we obtain a division algebra A_t of dimension 2^t, $t = k+1$, like in the Brown's construction. Let $T_P^{A_t}$ its pure trace form. From Corollary 3.15., the algebra A_t has level m. We suppose that the sublevel is at most $m - 1 = 2^k$.

Then

$$\sum_{i=1}^{m} c_i^2 + \sum_{i=1}^{m} p_i^2 = 0 \qquad (4.3.)$$

and

$$\sum_{i=1}^{m} c_i p_i = 0, \tag{4.4.}$$

where $c_i \in F_0(Z_1, ..., Z_{k+1})(\varphi), \varphi =< 1 > \perp \left(2^k + 1\right) \times T_P^{A_t}$, $p_i \in \mathcal{P}, \mathcal{P}$ the $F_0(Z_1, ..., Z_{k+1})(\varphi)$-vector space spanned by the pure elements in A_t.

Case 1. $c_i = 0$, for all $i \in \{1, 2, ..., m-1\}$. From (4.3.), it results that $\left(2^k + 1\right) \times T_P^{A_t}$ is isotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$.

Case 2. If there is at least an element i such that $c_i \neq 0$, relation (4.4.) implies that we get a $m-1$-dimensional $F_0(Z_1, ..., Z_{k+1})(\varphi)$- vector subspace V of $\mathcal{P}$, containing $p_1, p_2, ..., p_m$. Let $\beta : V \to F_0(Z_1, ..., Z_{k+1})(\varphi), \beta(p) = p^2$. Therefore β is a $m-1$ dimensional subform of $T_P^{A_t}$ and, from (4.4.), the form $\gamma = \left(2^k + 1\right) \times (< 1 > \perp \beta)$ is isotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$. We denote $\delta =< 1 > \perp \beta$. Repeated applications of Springer's Theorem implies that $\left(2^k + 1\right) \times T_P^{A_t}$, γ and $2^{k+1} \times (< -1 > \perp T_P^{A_t})$ are anisotropic over $F_0(Z_1, ..., Z_{k+1})$.

To prove that $\underline{s}(A_t) \nleq m-1$ it is sufficient to show that $\left(2^k + 1\right) \times T_P^{A_t}$ and γ are anisotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$.

If $\left(2^k + 1\right) \times T_P^{A_t}$ is isotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$, since the form $2^{k+1} \times (< -1 > \perp T_P^{A_t})$ is a Pfister form, then this form becomes hyperbolic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$. For any $a \in D_{F_0(Z_1,...,Z_{k+1})}\left(2^{k+1} \times (< -1 > \perp T_P^{A_t})\right)$ $a\varphi$ is a subform of $2^{k+1} \times (< -1 > \perp T_P^{A_t})$, from Cassels-Pfister Theorem. Since $X = Z_1 \in D_{F_0(Z_1,...,Z_{k+1})}\left(2^{k+1} \times (< -1 > \perp T_P^{A_t})\right)$, it results that

$$X\varphi \simeq < X > \perp \times \left(2^k + 1\right) < 1, ... >$$

is a subform of $2^{k+1} \times (< -1 > \perp T_P^{A_t})$, therefore $\left(2^k + 1\right) \times < 1 >$ is a subform of $2^{k+1} \times (< -1 > \perp T_P^{A_t})$, false. Hence $\left(2^k + 1\right) \times T_P^{A_t}$ is anisotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$.

If γ is isotropic over $F_0(Z_1, ..., Z_{k+1})(\varphi)$, from Proposition 1.3., we have

$$D_{F_0(Z_1,...,Z_{k+1})}(\varphi) D_{F_0(Z_1,...,Z_{k+1})}(\varphi) \subseteq D_{F_0(Z_1,...,Z_{k+1})}(\gamma) D_{F_0(Z_1,...,Z_{k+1})}(\gamma).$$

Since $T_P^Q \simeq < X, Y, -XY >$, it results that

$$X, Y, -XY \in D_{F_0(Z_1,...,Z_{k+1})}(\varphi) D_{F_0(Z_1,...,Z_{k+1})}(\varphi).$$

We prove that $< X >, < Y >, < -XY >$ are subforms of β. If $< X >$ is not a subform of β, then we find a multiple of X in $D_{F_0(Z_1,...,Z_{k+1})}(\gamma) D_{F_0(Z_1,...,Z_{k+1})}(\gamma)$

of the form $-X\sum_{i=1}^{r}A_i^2,\ A_i\in F_0\left(Z_1,...,Z_{k+1}\right)$. From Springer's Theorem, we have that γ is anisotropic over $F_0\left(Z_1,...,Z_{k+1}\right)$. From the same Theorem, it results that $m\times\delta$ is anisotropic over $F_0\left(Z_1,...,Z_{k+1}\right)$ for all $m\in\mathbb{N}^*$, therefore δ is strongly anisotropic over $F_0\left(Z_1,...,Z_{k+1}\right)$. From Remark 1.1., we have that $D_{F_0(Z_1,...,Z_{k+1})}\left(\gamma\right)\subset P_0=\{\alpha\,/\,\alpha=0$ or α is represented by $n\times\delta,n\in\mathbb{N}^*$ $\}, P_0$ is a q-preordering and there is a q-ordering P such that $P_0\subset P$ or $-P_0\subset P$. Therefore $D_{F_0(Z_1,...,Z_{k+1})}\left(\gamma\right)D_{F_0(Z_1,...,Z_{k+1})}\left(\gamma\right)\subset P\cdot P\subset P$. If X is positive, then $-X\sum_{i=1}^{r}A_i^2$ is negative and if X is negative, then $-X\sum_{i=1}^{r}A_i^2$ is positive, false.

In the same way, we prove that $<Y>, <-XY>$ are subforms of β, therefore $X,Y,-XY\in D_{F_0(Z_1,...,Z_{k+1})}\left(\gamma\right)$. Since $1\in D_{F_0(Z_1,...,Z_{k+1})}\left(\gamma\right)$, we have that $P_0\subset P$, therefore $X,Y,-XY$ are positive, false. We obtain γ is anisotropic over $F_0\left(Z_1,...,Z_{k+1}\right)\left(\varphi\right)$.

It results that $\underline{s}(\ A_t\ )=2^k+1$, therefore, using Corollary 3.15., we have $\underline{s}(\ Q(m)\ )=m, m=2^k+1.\Box$

Theorem 4.4. *Let* $F=F_0\left(X,Y,Z\right)$ *and* $O=\left(\frac{X,Y,Z}{F}\right)$ *an octonion algebra over* F.

The octonion algebra $O(m)=\left(\frac{X,Y,Z}{F}\right)\otimes_F K$ *is an octonion division algebras of level and sublevel* m, *where* $m=2^k+1, k\geq 0,$, $K=F\left(<1>\perp m\times T_P^O\right)$, *and* T_P^O *is its pure trace form over* F

Proof. The case $k\leq 2$ is proved in [O' Sh; 07(2)], in the following we suppose that $k\geq 3$. We denote $\varphi=<1>\perp\left(2^k+1\right)\times T_P^O$, where $T_P^O\simeq<X,Y,-XY,Z,-XZ,-YZ,XYZ>$. Let $Z_4,...,Z_{k+1}$be algebraic independent elements over $F=F_0\left(X,Y,Z\right)$, $K=F\left(<1>\perp\left(2^k+1\right)\times T_P^Q\right)$. Proposition 4.1. implies that

$$\begin{aligned} K\left(Z_4,...,Z_{k+1}\right) &= F\left(<1>\perp\left(2^k+1\right)\times T_P^Q\right)\left(Z_3,...,Z_{k+1}\right)=\\ &= F\left(Z_4,...,Z_{k+1}\right)\left(<1>\perp\left(2^k+1\right)\times T_P^Q\right)=\\ &= F_0\left(Z_1,...,Z_{k+1}\right)\left(<1>\perp\left(2^k+1\right)\times T_P^Q\right), \end{aligned}$$

where $X=Z_1,Y=Z_2,Z=Z_3$. Let $O_m'=O_m\otimes_K K\left(Z_4\right)$ and $S_m'=\left(O_m',Z_4\right)$ be the sedenion algebra as in Brown's construction. Then S_m' is a division algebra of dimension 2^4 and of level m. We repeat this construction until we obtain a division algebra A_t of dimension $2^t, t = k+1$, like in Brown's construction. Let $T_P^{A_t}$ be its pure trace form. By Corollary 3.15., this algebra has level m. Using the same arguments like in the above proposition, the sublevel of algebra A_t is $2^k + 1$, therefore $\underline{s}(\ O(m)\ ) = m, m = 2^k + 1.\square$

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