Local Lesieur-Croisot theory of Jordan algebras

Fernando Montaner and Maribel Tocón

Abstract. In this paper we study Jordan algebras having nonzero local algebras that satisfy the property of being Lesieur-Croisot (i.e. being orders in nondegenerate Jordan algebras of finite capacity). We will prove that the set of the elements of a nondegenerate Jordan algebra at which the local algebra is Lesieur-Croisot is an ideal.

1. Introduction

Following the pattern of Goldie's theorem for associative rings ([**Go**], [**Go2**], [**LCr**]), the so-called *Goldie's theorem* for Jordan algebras, which is due to Zelmanov for the linear case [**Z**] and to Fernández López, García Rus and Montaner [**FGMo**] for the general quadratic case, characterize those nondegenerate Jordan algebras that have a nondegenerate artinian algebra of quotients, such algebras are called *Goldie*. An important result in the associative setting is the characterization of left (resp. right) Goldie associative algebras as those for which the essential left (resp. right) ideals are the ones containing regular elements. However, a nondegenerate Jordan algebra that satisfies the

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1991 *Mathematics Subject Classification.* 17C10.

Key words and phrases. Jordan algebra, local algebra, order, uniform dimension.

The first author is partially supported by the Ministerio de Educación y Ciencia and FEDER (MTM2004-081159-C04-02) and by the Diputación General de Aragón (Grupo de Investigación de Álgebra).

The second author thanks Professors A. Fernández López and E. Neher for their encouragement. Support from the MEC and Fondos FEDER MTM2004-03845 and by the Junta de Andalucía FQM264 and the hospitality of the University of Ottawa are also gratefully acknowledged.

natural jordanification of this property, that is, *an inner ideal is essential if, and only if, contains injective elements*, is not in general Goldie, that is, an order in a nondegenerate artinian Jordan algebra. But, from the Jordan theory point of view, the artinian condition is somewhat less natural than the existence of capacity [**Lo2**], and it turns out that nondegenerate Jordan algebras satisfying the above property for inner ideals are precisely the orders in nondegenerate Jordan algebras of finite capacity [**FGMo**](10.2). We have chosen the denomination *Lesieur-Croisot* to designate nondegenerate Jordan algebras satisfying the property that inner ideals are essential if, and only if, they contain injective elements.

On the other hand, local algebras attached to elements of an algebraic system, already introduced by Meyberg [**Me**], have recently become a prominent tool in the study of such systems. Inspired by the local PI-theory for Jordan systems ([**Mo**], [**Mo2**]), and as a first step in the more general project of developing a local Goldie theory for Jordan systems, in this paper we prove that the set of the Lesieur-Croisot elements (i.e. at which local algebras are Lesieur-Croisot) of a nondegenerate Jordan algebra is indeed an ideal.

The paper is organized as follows. After a first section of preliminaries, we consider strongly prime Jordan algebras having nonzero PI-elements, that is, having nonzero PI-ideal, and we prove that in this case, the set of the Lesieur-Croisot elements coincides with the PI-ideal. When the Jordan algebra has no nonzero PI-elements, then not only have the quotient algebras of the local algebras at Lesieur-Croisot elements finite capacity but they also are artinian, which enables us to affirm that the set of the Lesieur-Croisot elements is nothing but the set of the finite uniform dimension elements. But the set of the finite uniform dimension elements of a strongly prime Jordan algebra without PI-elements is actually the intersection of the algebra with the set of the finite uniform dimension elements of a $*$-tight associative envelope. Using this characterization and the fact that the set of the finite uniform dimension elements of a semiprime associative algebra is an ideal under some natural assumptions, we conclude that the set of the Lesieur-Croisot elements of a strongly prime Jordan algebra is an ideal. The last section of this paper is devoted to the study of semi-uniform ideals of a nondegenerate Jordan algebra and the essential subdirect sums arising from them. The properties of these essential subdirect sums and the fact that the set of the Lesieur-Croisot elements of a strongly prime Jordan algebra is an ideal will lead us to prove the main result of this paper that asserts that the set of the Lesieur-Croisot elements of a nondegenerate Jordan algebra is an ideal.

2. Preliminaries

Throughout this paper Φ will be a fixed arbitrary ring of scalars, that is, a commutative, associative and unital ring without restrictions on the characteristic. In this section, we remind the reader of the basic notation and results that will be used in the rest of the paper; as general references we mention [**FGMo**], [**Gd**], [**J**] and [**Lo**].

2.1 A (*quadratic*) *Jordan algebra* $(J, U, (\,)^2)$ over Φ is a Φ-module J with quadratic maps $U : J \to \mathrm{End}_\Phi(J)$ and $(\,)^2 : J \to J$ (squaring) satisfying the following identities in all scalar extensions:

(i) $V_{x,x}y = x^2 \circ y$
(ii) $U_x(x \circ y) = x \circ U_x y$
(iii) $U_x x^2 = (x^2)^2$
(iv) $U_x U_y x^2 = (U_x y)^2$
(v) $U_{x^2} = U_x^2$
(vi) $U_{U_x y} = U_x U_y U_x$,

where $x \circ y := (x + y)^2 - x^2 - y^2$ and $V_{x,y}z := U_{x,z}y := \{x, y, z\} := (U_{x+z} - U_x - U_z)y$.

Jordan algebras over ring of scalars of characteristic not 2 are called *linear* and they can be alternatively defined in terms of a commutative product satisfying the so-called *Jordan identity*. For a complete survey on linear Jordan algebras we refer the reader to [**M4**].

Every associative algebra A gives rise to a Jordan algebra A^+ by taking x^2 as usual and $U_x y := xyx$, for all $x, y \in A$. A Jordan algebra is *special* if it is isomorphic to a Jordan subalgebra of some A^+. Important examples are the *hermitian algebras*

$$H(A, *) := \{x \in A : x^* = x\} \subseteq A^+$$

of self-adjoint elements in an associative algebra with involution $(A, *)$. More generally, we may consider *ample subspaces* $H_0(A, *) \subseteq H(A, *)$: $aH_0a^* \subseteq H_0$ for all $a \in A$, and all traces $a + a^*$ and norms aa^* lie in H_0. If $1/2 \in \Phi$, the only ample subspace is $H_0 = H(A, *)$. Other important examples of special Jordan algebras are the (Jordan) algebras of *Clifford* type $J(Q, 1)$, which lie in the associative Clifford algebras $C(Q, 1)$ for quadratic forms Q with basepoint 1.

Jordan algebras that are not special are called *exceptional*. The basic exceptional Jordan algebras are the 27-dimensional *Albert algebras*: *forms* of the reduced algebra of 3×3 hermitian matrices $H_3(\mathbb{O})$ with octonion entries.

2.2 A Φ-submodule K of a Jordan algebra J is said to be an *inner ideal* of J if $K^2 + U_K J \subseteq K$. Following [**Lo2**], a nondegenerate Jordan algebra J is said to have *finite capacity* if it satisfies the ascending chain condition (acc) and descending chain condition (dcc) on principal inner ideals. If J satisfies dcc on all inner ideals, then J is called *artinian*. It follows from [**Lo**](12.12) that nondegenerate artinian Jordan algebras have finite capacity, but the converse is not true as can be seen by considering the Jordan algebra defined by a nondegenerate quadratic form with basepoint over a vector space having an infinite dimensional totally isotropic subspace [**M**](Corollary of Theorem 6).

An *ideal* I of J, denoted $I \lhd J$, is an inner ideal such that is also *outer*, that is, $K \circ J + U_J K \subseteq J$, and an (inner) ideal I of J is said to be *essential* if it has nonzero intersection with any nonzero (inner) ideal of J. A Jordan algebra J is said to be *semiprime* (*prime*) if $U_B B = 0$ implies $B = 0$ ($U_B C = 0$ implies $B = 0$ or $C = 0$) for all ideals B and C of J. Unlike the associative case, primeness does not imply the absence of *absolute zero divisors*, elements $a \in J$ such that $U_a = 0$. Jordan algebras without nonzero absolute zero divisors are called *nondegenerate*. Prime nondegenerate Jordan algebras are called *strongly prime*. Ideals of nondegenerate or strongly prime Jordan algebras inherit the nondegenerancy or strongly primeness respectively.

2.3 For the classification of strongly prime Jordan algebras, the reader is referred to [**M3**] and [**MZ**]. In this classification, it is of important relevance the notion of *hermitian* ideal, that is, an ideal $H(X)$ of the *free special Jordan algebra* $FSJ(X)$ on a countably infinite set X satisfiying the property that if $p(x_1, \ldots, x_n) \in H(X)$, then $p(\sigma(x_1), \ldots, \sigma(x_n)) \in H(X)$ for all permutations σ of the elements of X, and is closed for *n-tads*, that is, $\{x_1 \ldots x_n\} := x_1 \ldots x_n + x_n \ldots x_1 \in H(X)$, for all $x_1, \ldots, x_n \in H(X)$ and all $n \geq 4$. Properties of these ideals can be found in [**MZ**](Part III), including the important fact that they contain Jordan polynomials which do not vanish in all special Jordan algebras. A strongly prime special Jordan algebra J is said to be of *hermitian* type if $H(J) \neq 0$ for some nonzero hermitian ideal $H(X)$. If J is not of hermitian type, then it is a Clifford Jordan algebra. On the other hand, if J is exceptional, then it is a Jordan algebra of Albert type.

2.4 Following [**Lo**] and [**M2**], the *annihilator* of any set X of a Jordan algebra J is the subset $\mathrm{ann}_J(X) := \{z \in J : U_z x = U_x z = V_{z,x} J' = U_z U_x J' = 0\}$, where J' denotes the *unital hull* of J. In particular, $\mathrm{ann}_J(X)$ is an inner ideal of J for each subset X of J, and if $I \lhd J$ is an ideal of J then so is its annihilator. If J is nondegenerate and $I \lhd J$, then $\mathrm{ann}_J(I) = \{x \in J : U_x I = 0\}$ and I is essential as an inner ideal if $\mathrm{ann}_J(I) = 0$ [**FGMo**](1.16). Therefore, if J is strongly prime, then every nonzero ideal is essential as inner ideal. It is well known that an ideal of a semiprime Jordan algebra has zero intersection with its annihilator.

2.5 Let J be a Jordan algebra. For a subset $X \subseteq J$, we denote by $[X]$ the inner ideal of J generated by X. We say that a family $\{K_i\}_{i\in I}$ of nonzero inner ideals of J forms a *direct sum* if $K_i \cap [\sum_{j\neq i} K_j] = 0$ for each $i \in I$.

As for associative algebras, we define the *uniform dimension* of J, denoted udim(J), as the supremum of the $n \geq 1$ such that there are $K_1, \ldots, K_n$ nonzero inner ideals of J which form a direct sum. Indeed, for a strongly prime Jordan algebra, having finite uniform dimension is equivalent to not having infinite direct sums of inner ideals, denoted acc$(\oplus)$ [**FGMo**](5.12). On the other hand, the *singular set* of a Jordan algebra J is defined as

$$\Theta(J) := \{z \in J : \mathrm{ann}_J(z) \,\text{is essential in}\, J\}.$$

If J is nondegenerate, then $\Theta(J) \triangleleft J$. We say that J is *nonsingular* if $\Theta(J) = 0$.

A Jordan algebra is called *Goldie* if it has the acc on annihilators (equivalently, dcc), and satisfies the acc$(\oplus)$. We have that a nondegenerate Jordan algebra is Goldie if, and only if, it is nonsingular and has finite uniform dimension [**FGMo**](9.3).

2.6 Let J be a special Jordan algebra. An associative algebra with involution $(R, *)$ is called an *associative $*$-envelope* for J if it is generated by $J \subseteq H(R, *)$. An associative $*$-envelope R of J is said to be *$*$-tight* if any nonzero $*$-ideal I of R hits J, that is, $I \cap J \neq 0$. Note that a $*$-tight associative envelope of a semiprime (strongly prime) Jordan algebra is semiprime ($*$-prime). If J is nondegenerate, and R is a $*$-tight associative envelope of J, then $I \cap J$ is an ideal of J for every $*$-ideal I of R. We also have that

$$Z_l(R) \cap J = Z_r(R) \cap J = \Theta(J),$$

where $Z_l(R)$ (resp. $Z_r(R)$) denotes the *left (resp. right) singular* ideal of R [**FGMo**](6.14).

2.7 For a Jordan algebra J and a given $a \in J$, we can define (in fact, the definition makes sense for any element of a Jordan system) new Jordan products by $x^{2(a)} = U_x a$, $U_x^{(a)} = U_x U_a$ to get a new Jordan algebra on the Φ-module J called the *a-homotope* of J, denoted $J^{(a)}$. The set $ker_J(a) := \{x \in J : U_a x = U_a U_x a = 0\}$ is an ideal of $J^{(a)}$ and the quotient algebra $J^{(a)}/ker_J(a)$, which is denoted by J_a, is called the *local algebra* of J at a. In most cases (e.g. if $1/2 \in \Phi$, or if J is nondegenerate or special) $ker_J(a) = \{x \in J : U_a x = 0\}$.

Local algebras of Jordan systems were first introduced by Meyberg [**Me**] and have played a prominent role in the recent structure theory of strongly prime Jordan systems. They also have proved to be very useful in the associative context: Let R be an associative algebra and $a \in R$. The *a-homotope* of R, denoted by $R^{(a)}$, is the associative algebra with the same linear structure as R with the new product $x \cdot_a y := xay$. It is readily seen that the set $ker_R(a) := \{x \in R : axa = 0\}$ is an ideal of $R^{(a)}$, so that we can consider the *local algebra* $R_a := R^{(a)}/ker_J(a)$.

A number of properties of Jordan or associative algebras are inherited by their local algebras and also lifted from local algebras to the whole algebra (see [**AnCMo**], [**DM**], and [**FT**]).

For a particular property P of a Jordan or associative algebra A, an element $a \in A$ is said to be P if the local algebra A_a satisfies the property P. Thus, if the local algebra A_a is PI, i.e. satisfies a polinomial identity, we just say that the element a is PI. The set of the PI-elements of a A will be denoted by PI(A). We have that

2.8 [**Mo**](5.4) If J is a nondegenerate Jordan algebra, then PI(J) is an ideal of J.

If J is strongly prime and PI$(J) = 0$, then J is special of hermitian type (since otherwise, it would be of Clifford or Albert type and in both cases, the central closure $\Gamma^{-1}J$ would be PI. Hence $\mathrm{PI}(J) = \mathrm{PI}(\Gamma^{-1}J) \cap J = \Gamma^{-1}J \cap J = J$ which is a contradiction) and we have the following fundamental result:

2.9 [**FGMo**](5.11) Let J be a strongly prime Jordan algebra such that PI$(J) = 0$ and let R be a $*$-tight associative envelope of J. Then J has finite uniform dimension if, and only if, R has finite left uniform dimension if, and only if, R has finite right uniform dimension. Moreover, in this case, $\mathrm{udim}_l(R) = \mathrm{udim}_r(R) = \mathrm{udim}(J)$ if R is prime.

2.10 Let J be a Jordan algebra and denote by S the set of the *injective* elements of J, i.e. $S = \mathrm{Inj}(J) := \{x \in J \,\text{such that}\, U_x \,\text{is injective}\}$. Following [**FGMo**], J is said to be a *(classical) order* in a unital Jordan algebra Q if (i) every injective element $x \in S$ is invertible in Q, (ii) for all $s, t \in S$, $U_x S \cap U_t S \neq \emptyset$ and (iii) each element $q \in Q$ has a J-denominator in S, i.e. there exists $x \in S$ such that the following six multiplications take q back into J: $U_x q, U_q x, U_x U_q J', U_q U_x J', V_{x,q} J'$ and $V_{q,x} J'$.

3. Lesieur-Croisot elements. The Strongly Prime local PI case

Goldie's theorem for Jordan algebras [**FGMo**](9.3) asserts that a Jordan algebra J is a classical order in a nondegenerate (resp. simple) artinian Jordan algebra Q if, and only if, J is nondegenerate (resp. strongly prime) and Goldie, and provides several conditions for a Jordan algebra that are equivalent to being nondegenerate and Goldie. This result can be viewed as a quite faithful jordanification of the classical Goldie's theorem for associative rings ([**Go**], [**Go2**]). An important result in this setting is the characterization of left (resp. right) Goldie associative algebras as those for which the essential left (resp. right) ideals are the ones containing regular elements. The natural jordan version of this property could be that *an inner ideal is essential if, and only if, it contains injective elements*, and in particular, nondegenerate Goldie Jordan algebras satisfy this property. However, unlike the associative case, a nondegenerate Jordan algebra satisfying the above property for essential inner ideals is not in general Goldie as can be seen by considering the Jordan algebra of Clifford type defined on a vector space having an infinite dimensional totally isotropic subspace. They are indeed characterized as follows:

Theorem 3.1. [**FGMo**](10.2) *A Jordan algebra J is a classical order in a nondegenerate unital Jordan algebra Q with finite capacity if, and only if, it is nondegenerate and satisfies the property: an inner ideal K of J is essential if, and only if, K contains an injective element. Moreover Q is simple if, and only if, J is prime.*

Definition 3.2. A Jordan algebra is called *Lesieur-Croisot*, *LC* for short, if it satisfies the equivalent conditions of Theorem 3.1.

It follows from Goldie's theorem for Jordan algebras and the fact that nondegenerate artinian Jordan algebras have finite capacity that nondegenerate Goldie Jordan algebras are Lesieur-Croisot. Also, every strongly prime Jordan algebra satisfying a polinomial identity is Lesieur-Croisot since its central closure is simple with finite capacity [**Mo2**](0.13).

Inspired by the local PI-theory of Jordan systems ([**Mo**], [**Mo2**]), and as a first step in the more general project of developing a local Goldie theory for Jordan systems, we claim that the set of the Lesieur-Croisot elements of a strongly prime (even nondegenerate) Jordan algebra J is an ideal of J. We denote this set by $\mathrm{LC}(J)$, that is,

$$\mathrm{LC}(J) := \{x \in J : J_x \text{ is LC}\}.$$

In the next proposition, we prove our claim for strongly prime Jordan algebras having nonzero PI-elements, since in this case, LC(J) actually coincides with PI(J). The case when J has no nonzero PI-elements will be discussed in the next section.

PROPOSITION 3.3. *Let J be a strongly prime Jordan algebra having nonzero PI-elements. Then, LC(J)=PI(J) and therefore LC(J) is an ideal of J.*

PROOF. Let $x \in \mathrm{PI}(J)$. Then J_x is PI and is also strongly prime by [**AnCMo**](4.1(iii)), so $x \in \mathrm{LC}(J)$. Suppose now that $0 \neq x \in \mathrm{LC}(J)$. Since $\mathrm{PI}(J)$ is a nonzero ideal of J and J is strongly prime, we have that $\mathrm{ann}_J(\mathrm{PI}(J)) = 0$. Then $U_x(\mathrm{PI}(J)) \neq 0$. Take $z \in \mathrm{PI}(J)$ such that $U_x z \neq 0$. In particular, $U_x z \in \mathrm{PI}(J)$, which implies, by the transitivity of the localization that $(J_x)_{z+ker_J(x)} \cong J_{U_x z}$ is PI, and then $0 \neq z + ker_J(x) \in \mathrm{PI}(J_x)$. Since J_x is strongly prime, we have that $\mathrm{PI}(J_x)$ is an essential inner ideal of J_x, and then, $\mathrm{PI}(J_x)$ contains an injective element because J_x is Lesieur-Croisot, which implies by [**FGMo**](4.4) that J_x is itself PI and then $x \in \mathrm{PI}(J)$. □

4. The Strongly Prime PI-less case

In this section we consider strongly prime Jordan algebras without nonzero PI-elements and describe the set of its Lesieur-Croisot elements in terms of the elements of a $*$-tight associative envelope that have finite uniform dimension. As a corollary we will get that this set is always an ideal.

PROPOSITION 4.1. *Let J be a strongly prime Jordan algebra such that $PI(J) = 0$ and $LC(J) \neq 0$. Then J is nonsingular and $LC(J) = F(J) := \{x \in J : udim(J_x) < \infty\}$.*

PROOF. Note that in this setting, every LC-element of J is indeed a Goldie element: let $0 \neq x \in \mathrm{LC}(J)$. Then J_x is an order in a simple Jordan algebra Q of finite capacity. If Q were not artinian, it would be a Jordan algebra of Clifford type and then PI, in which case, J_x would be also PI, which would imply that $x = 0$. From this, we deduce that $\mathrm{LC}(J)$ coincides with the set of the Goldie elements of J, and therefore $\mathrm{LC}(J) \subseteq \mathrm{F}(J)$. Hence to prove that the equality $\mathrm{LC}(J) = \mathrm{F}(J)$ holds we only need to show that every $x \in \mathrm{F}(J)$ is Goldie, equivalently, that J_x is nonsingular for all $x \in \mathrm{F}(J)$.

Let $0 \neq z \in \mathrm{LC}(J)$. Since J_z is Goldie, it is nonsingular and then $0 \neq z \in \mathrm{ann}_J(\Theta(J))$ [**FGMo**](6.4), which implies that the ideal $\Theta(J) = 0$ since J is strongly prime. Now, since local algebras inherit nonsingularity [**FGMo2**](3.6(i)), in particular we have that $\Theta(J_x) = 0$ for all $x \in \mathrm{F}(J)$, as required. $\square$

In order to prove the main result of this section, we need to make use of a strategy that appears in the proof of [**Mo**](6.5) and that, for the sake of completeness, we state here as a lemma omitting the proof:

LEMMA 4.2. *Let J be a strongly prime Jordan algebra such that PI(J) $= 0$ and R a $*$-tight associative envelove of J. Then, for each $a \in J$, the subalgebra S of J generated by $U_{H(J)}H(J)$, where $H(J) \neq 0$ for some nonzero hermitian ideal $H(X)$, and the element a, is strongly prime of hermitian type. Moreover, $S = H_0(A, *)$, where A is a $*$-tight associative envelope of S.*

THEOREM 4.3. *Let J be a strongly prime Jordan algebra such that PI(J) $= 0$ and R a $*$-tight associative envelove of J. Then,*

$$F(J) = I_l(R) \cap J = I_r(R) \cap J,$$

where $I_l(R) := \{x \in R : udim_l(R_a) < \infty\}$ and $I_r(R) := \{x \in R : udim_r(R_a) < \infty\}$. Moreover, if R is prime, then $udim(J_a) = udim_l(R_a) = udim_r(R_a)$ for all $a \in F(J)$.

PROOF. Let $0 \neq a \in J$. We are going to prove the left version of the statement and by symmetry we will get the right one. Therefore, we have to show that J_a has finite uniform dimension if, and only if, R_a has finite left uniform dimension and that they coincide if R is prime. Note that $J_a \subseteq H(R_a, *)$, where $(x + ker_R(a))^* := x^* + ker_R(a)$. However, $*$-tightness is in general lost when localizing, what makes impossible to apply the general principle (2.9); the idea of this proof is to find a suitable setting that enables us to apply (2.9).

Consider the subalgebra S of J generated by $I := U_{H(J)}H(J)$, where $H(J) \neq 0$ for some nonzero hermitian ideal $H(X)$, and the element a. Then by Lemma 4.2, we have that S is strongly prime of hermitian type and $S = H_0(A, *)$, where A is a $*$-tight associative envelope of S. Note that $\mathrm{udim}(J_a) = \mathrm{udim}(S_a)$ since S_a contains the nonzero essential ideal $I + ker_J(a)/ker_J(a)$ of J_a. On the other hand, $\mathrm{udim}_l(R_a) = \mathrm{udim}_l(A_a)$ since $A_a \subseteq R_a \subseteq Q_s(A_a)$, where A_a is $*$-prime (because $a \in H(A, *)$ and A is $*$-prime) and $Q_s(A_a)$ denotes the symmetric Martindale ring of quotients of A_a. Indeed $R \subseteq Q_s(A)$.

We claim that $\mathrm{PI}(S) = 0$. Otherwise, $\mathrm{PI}(A) \neq 0$ by [**Mo**](6.5), and then A would satisfy a generalized polynomial identity [**Mo**](1.1) implying that $Q_s(A)$ satisfies the same generalized polynomial identity (see [**B**]) and therefore also $R \subseteq Q_s(A)$. Then $\mathrm{PI}(R) \neq 0$, which is a contradiction since $\mathrm{PI}(R) \cap J = \mathrm{PI}(J) = 0$.

Then the proof reduces to showing that S_a has finite uniform dimension if, and only if, A_a has finite left uniform dimension and that they coincide if R is prime, where $S_a = H_0(A_a, *)$, with S_a being strongly prime such that $\mathrm{PI}(S_a) = 0$ by the transitivity of the localization and the fact that $\mathrm{PI}(S) = 0$.

However, this is not still the suitable setting that we need since we cannot assure that A_a is a $*$-associative envelope of S_a. To fix this, consider the $*$-subalgebra of A_a generated by S_a that will be denoted by $\mathrm{alg}_{A_a}(S_a)$. We want to show that $\mathrm{alg}_{A_a}(S_a)$ contains a nonzero $*$-ideal of A_a, in order to conclude that $\mathrm{udim}_l(A_a) = \mathrm{udim}_l(\mathrm{alg}_{A_a}(S_a))$. But by [**H**](Theorem 6.4.2) we only need to check that $x\mathrm{alg}_{A_a}(S_a)x^* \subseteq \mathrm{alg}_{A_a}(S_a)$ for all $x \in A_a$ since A_a is not PI (recall that $\mathrm{PI}(A) = 0$). Note that for $x \in A_a$, $s_i \in S_a$, $i = 1, \cdots, n$, all n, we have

$$\begin{aligned} xs_1 \cdots s_n x^* = [x, s_1][s_2 \cdots s_n, x^*] + xs_1x^*s_2 \cdots s_n \\ + s_1xs_2 \cdots s_nx^* - s_1xx^*s_2 \cdots s_n, \end{aligned}$$

where $[x_1, x_2] := x_1x_2 - x_2x_1$, for $x_1, x_2 \in A_a$. On the other hand, $[s, x] = (sx + (sx)^*) - (x + x^*)s$ for all $s \in S_a$, $x \in A_a$. Then, since $[\,,x]$ is an associative derivation, it is clear that $[\mathrm{alg}_{A_a}(S_a), x] \subseteq \mathrm{alg}_{A_a}(S_a)$. Using this fact and the above formula, we have that showing that $xs_1 \cdots s_nx^* \in \mathrm{alg}_{A_a}(S_a)$ reduces to showing that $xs_2 \cdots s_nx^* \in \mathrm{alg}_{A_a}(S_a)$, and by recurrence, to showing that $xs_nx^* \in \mathrm{alg}_{A_a}(S_a)$, but this is a consequence of the fact that S_a is an ample subspace of $(A_a, *)$. Therefore $\mathrm{alg}_{A_a}(S_a)$ contains a nonzero $*$-ideal of A_a, that we will denote by T, and then $\mathrm{udim}_l(A_a) = \mathrm{udim}_l(\mathrm{alg}_{A_a}(S_a))$.

Finally, we will prove that $\mathrm{alg}_{A_a}(S_a)$ is $*$-tight to S_a. Let K be a $*$-ideal of $\mathrm{alg}_{A_a}(S_a)$. If $K \cap S_a = 0$, then $x + x^* = 0$ and $xx^* = 0$ for all $x \in K$ again by the ampleness of S_a. Thus, $(xy)x = -(xy)^*x = -y^*x^*x = 0$, for all $x \in K$ and all $y \in \mathrm{alg}_{A_a}(S_a)$. In particular, $K \subseteq \mathrm{ann}_{A_a}(T) = 0$ and then $K = 0$. Note also that if R is prime, then so is $\mathrm{alg}_{A_a}(S_a)$, since otherwise there would exist $0 \neq b \in \mathrm{alg}_{A_a}(S_a)$ such that the ideal of $\mathrm{alg}_{A_a}(S_a)$ generated by b and the one generated by b^* would have nonzero intersection, in particular, $bIb^* = 0$ and then R would not be prime by [**FGMo**](5.2(ii)), which is a contradiction.

The proof now follows directly from (2.9) applied to

$$S_a \subseteq H(\mathrm{alg}_{A_a}(S_a), *).$$

$\square$

For an associative ring R, it is well known that if $Z_l(R) = 0$, then $\mathrm{I}_l(R)$ is an ideal of R [**AMa**](Proposition 1). We claim that $Z_l(R) \cap$

$Z_r(R) = 0$ implies that $\mathrm{I}_l(R) \cap \mathrm{I}_r(R)$ is also an ideal of R if R is semiprime. To prove this, we will make use of the following facts:

LEMMA 4.4. *Let R be a semiprime associative algebra and let $a \in R$.*

(i) [**FT**](1.3(iv)) *If $\overline{x} = x + ker_R(a) \in Z_l(R_a)$, then $axa \in Z_l(R)$.*

(ii) [**FGMo**](6.3) $\overline{Z_l(R)} = Z_l(R) + ker_R(a)/ker_R(a) \subseteq Z_l(R_a)$. *In particular, if $Z_l(R_a) = 0$, then $a \in ann_R(Z_l(R))$.*

The same statement holds for the right singular ideals of R and R_a.

PROPOSITION 4.5. *Let R be a semiprime associative algebra. If $Z_l(R) \cap Z_r(R) = 0$, then $I_l(R) \cap I_r(R)$ is an ideal of R contained in $ann_R(Z_l(R) + Z_r(R))$.*

PROOF. Set $I = \mathrm{I}_l(R) \cap \mathrm{I}_r(R)$ and take $a \in I$. By Lemma 4.4(i), for all $x + ker_R(a) \in Z_l(R_a) \cap Z_r(R_a)$, we have $axa \in Z_l(R) \cap Z_r(R) = 0$, hence $x + ker_R(a) = 0$, and $Z_l(R_a) \cap Z_r(R_a) = 0$. Now, since $a \in I$, $\mathrm{udim}_l(R_a)$ and $\mathrm{udim}_r(R_a)$ are both finite, hence $Z_l(R_a) = Z_r(R_a) = 0$ by [**FGMo**](7.16). Therefore $a \in \mathrm{ann}_R(Z_l(R)) \cap \mathrm{ann}_R(Z_r(R)) = \mathrm{ann}_R(Z_l(R) + Z_r(R))$ by Lemma 4.4(ii) and we have $I \subseteq \mathrm{ann}_R(Z_l(R) + Z_r(R))$. Set $N := \mathrm{ann}_R(Z_l(R) + Z_r(R))$, $L := \mathrm{ann}_R(N)$ and $\overline{R} := R/L$, and denote with bars the projections in $\overline{R}$. Now $N \cong \overline{N} = N + L/L$ is (isomorphic to) an essential ideal of $\overline{R}$, and $Z_l(N) = Z_l(R) \cap N \subseteq L \cap N = 0$, hence $Z_l(\overline{R}) = 0$ and similarly $Z_r(\overline{R}) = 0$. Thus $\overline{R}$ is left and right nonsingular. Therefore $\mathrm{I}_l(\overline{R})$ and $\mathrm{I}_r(\overline{R})$ are ideals of $\overline{R}$. Now, since $a \in N = \mathrm{ann}_R(L)$, we have that $\overline{R}_{\overline{a}} \cong R_a$ has finite left and right uniform dimension, hence $\overline{a} \in \mathrm{I}_l(\overline{R}) \cap \mathrm{I}_r(\overline{R})$. Now, take $r \in R$. Then $\overline{ra} = \overline{r}\,\overline{a} \in \mathrm{I}_l(\overline{R}) \cap \mathrm{I}_r(\overline{R})$. Moreover, since $a \in \mathrm{ann}_R(L)$, we also have $ra \in \mathrm{ann}_R(L)$, hence $R_{ra} \cong \overline{R}_{\overline{ra}}$ has finite left and right uniform dimension, hence $ra \in \mathrm{I}_l(R) \cap \mathrm{I}_r(R)$. Similarly $ar \in \mathrm{I}_l(R) \cap \mathrm{I}_r(R)$ and $a + b \in \mathrm{I}_l(R) \cap \mathrm{I}_r(R)$ for $b \in I$, and we get that $\mathrm{I}_l(R) \cap \mathrm{I}_r(R)$ is an ideal. □

COROLLARY 4.6. *Let J be a strongly prime Jordan algebra such that LC(J) $\neq 0$. Then LC(J) is an ideal of J.*

PROOF. If $\mathrm{PI}(J) \neq 0$, then $\mathrm{LC}(J)$ is an ideal of J by Proposition 3.3. If $\mathrm{PI}(J) = 0$ and R is a $*$-tight associative envelove of J, then by Proposition 4.1, we have that $\mathrm{LC}(J) = \mathrm{F}(J)$ and $\Theta(J) = 0$ and hence $(Z_l(R) \cap Z_r(R)) \cap J = 0$. But $Z_l(R) \cap Z_r(R) = Z_l(R) \cap Z_l(R)^*$ is a $*$-ideal of R, so, by $*$-tightness we have that $Z_l(R) \cap Z_l(R)^* = 0$. Then it follows from Proposition 4.5 that $I_l(R) \cap I_r(R) = I_l(R) \cap I_l(R)^*$ is a $*$-ideal of R. Now the proof follows from Theorem 4.3 again by $*$-tightness of R. □

5. Semi-uniform Jordan algebras. The nondegenerate case

We start this section by introducing the notion of *semi-uniform* ideal of a nondegenerate Jordan algebra. It turns out that for a nondegenerate Jordan algebra, being semi-uniform is equivalent to being an essential subdirect sum of finitely many strongly prime Jordan algebras, and that local algebras inherit the semi-uniform property. The properties of semi-uniform ideals and the essential subdirect sums arising from them are studied in this section, including the fact that such a sum is Lesieur-Croisot iff each summand is Lesieur-Croisot. This together with the fact that the set of the Lesieur-Croisot elements of a strongly prime Jordan algebra is an ideal, already shown in the previuos section, will allow us to prove the main result of this paper: *the set of the Lesieur-Croisot elements of a nondegenerate Jordan algebra is an ideal.*

LEMMA 5.1. *Let J be a nondegenerate Jordan algebra and I a nonzero ideal of J such that $P_1 \cap \cdots \cap P_n \subseteq ann_J(I)$ for a minimal set of prime ideals $\mathcal{P} = \{P_1, \ldots, P_n\}$ of J. Then $P_1 \cap \cdots \cap P_n = ann_J(I)$ and $\mathcal{P}$ is unique. In fact, $\mathcal{P}$ is the set of minimal elements of $\{P \triangleleft J : P$ is prime and $ann_J(I) \subseteq P\}$.*

PROOF. Note that if P is a prime ideal of J, then $I \cap \mathrm{ann}_J(I) = 0 \subseteq P$, hence either $I \subseteq P$ or $\mathrm{ann}_J(I) \subseteq P$. Let $P_1, \ldots, P_n$ be a minimal set of prime ideals with $P_1 \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$ such that $\mathrm{ann}_J(I) \subseteq P_i$ for $i = 1, \ldots, l$ and $I \subseteq P_i$ for $i = l+1, \ldots, n$. Note that $l \geq 1$ since otherwise we would have $I \subseteq P_1 \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$, hence $I = 0$. Now set $L := P_1 \cap \cdots \cap P_l$ and $N := P_{l+1} \cap \cdots \cap P_n$. Then $I \subseteq N$ and $I \cap L = (I \cap N) \cap L = I \cap (N \cap L) = I \cap \mathrm{ann}_J(I) = 0$. Thus $L \subseteq \mathrm{ann}_J(I)$ and then $l = n$ by the minimality of n. Therefore $\mathrm{ann}_J(I) \subseteq P_i$ for all i. Hence $P_1 \cap \cdots \cap P_n = \mathrm{ann}_J(I)$.

For the uniqueness, suppose that $P_1 \cap \cdots \cap P_n = Q_1 \cap \cdots \cap Q_m = \mathrm{ann}_J(I)$, where $\{Q_j\}_{j=1}^m$ is a minimal set of prime ideals. Then in particular $Q_j \subseteq P_1$ for some j by the primeness of P_1. Also $P_i \subseteq Q_j$ for some $P_i \in \mathcal{P}$ now by the primeness of Q_j. Hence $P_i \subseteq Q_j \subseteq P_1$. Therefore $P_i = P_1$ by the minimality of $\mathcal{P}$, which implies that $Q_j = P_1$. Repeating the argument with all the P_i and using the minimality of the set $\{Q_j\}_{j=1}^m$ we thus get that $n = m$ and $P_i = Q_i$ for all i.

To prove the last part of the statement, let $P_i \in \mathcal{P}$ and P be a nonzero prime ideal containing $\mathrm{ann}_J(I)$ such that $P \subseteq P_i$. Then $\mathrm{ann}_J(I) \subseteq P_1 \cap \cdots \cap (P \cap P_i) \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$, that is, $P_1 \cap \cdots \cap P \cap \cdots \cap P_n = \mathrm{ann}_J(I)$ which implies by the minimality and uniqueness of $\mathcal{P}$ that $P = P_i$, therefore P_i is a minimal element of the set $\{P \triangleleft J :$

P is prime and $\mathrm{ann}_J(I) \subseteq P\}$. Let now P be a minimal element of $\{P \triangleleft J : P \text{ is prime and } \mathrm{ann}_J(I) \subseteq P\}$. Then $P_1 \cap \cdots \cap P_n \subseteq P$ and by primeness of P we thus have that $P_i \subseteq P$ for some i, now because P is minimal among the prime ideals of J containing $\mathrm{ann}_J(I)$ and $P_i \neq 0$ (by minimality of $\mathcal{P}$) we get that $P_i = P$. □

The second part of the statement of the above lemma is indeed a particular case of a more general result on lattices that asserts that if an ideal I of a semiprime algebraic system can be expressed as an intersection of a finite number of prime ideals that is minimal, then this set of prime ideals is unique and consists of the minimal prime ideals containing I.

Definition 5.2. Let J be a nondegenerate Jordan algebra. We say that an ideal $I \triangleleft J$ is *semi-uniform* if there are prime ideals $P_1, \ldots, P_n$ of J such that $P_1 \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$.

Remark 5.3. If I is a semi-uniform ideal of a nondegenerate Jordan algebra J, and $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a minimal set of prime ideals with $P_1 \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$, then it follows from Lemma 5.1 that $\mathcal{P}$ is unique. Hence, we will say that the set $\mathcal{P}$ is *the set of prime ideals associated to I*.

Note that the case $n = 1$ is well known [**FGMo**](7): if P is prime and $P \subseteq \mathrm{ann}_J(I)$, then either $I = 0$ or I is uniform (i.e. $B \cap C \neq 0$ for any nonzero ideals B and C of J contained in I). Indeed, if L_1, L_2 are ideals of J with $L_1 + L_2 \subseteq I$ and $L_1 \cap L_2 = 0$, then $L_1 \cap L_2 \subseteq P$. Hence $L_i \subseteq P \subseteq \mathrm{ann}_J(I)$ for $i = 1$ or $i = 2$, and thus $L_i \subseteq I \cap \mathrm{ann}_J(I) = 0$ for $i = 1$ or $i = 2$. In this case, $\mathrm{ann}_J(I)$ is itself strongly prime, that is, generates a strongly prime quotient algebra $J/\mathrm{ann}_J(I)$ [**FGMo**](7.1).

Proposition 5.4. *Let J be a nondegenerate Jordan algebra and I a nonzero ideal of J. Then, the following are equivalent:*

(i) *The ideal I is semi-uniform with associated set of prime ideals $\{P_1, \ldots, P_n\}$,*
(ii) *$\mathrm{ann}_J(I) = P_1 \cap \cdots \cap P_n$, for a minimal set of prime ideals $\{P_1, \ldots, P_n\}$,*
(iii) *$\overline{J} := J/\mathrm{ann}_J(I)$ is semi-uniform with associated set of prime ideals $\{\overline{P_1}, \ldots, \overline{P_n}\}$.*

Moreover, if the above conditions are satisfied, then the P_i are closed, that is, $P_i = \mathrm{ann}_J(\mathrm{ann}_J(P_i))$ and strongly prime.

Proof. $(i) \Rightarrow (ii)$ follows from Lemma 5.1.

$(ii) \Rightarrow (iii)$. Let $\mathrm{ann}_J(I) = P_1 \cap \cdots \cap P_n$ for a minimal set of prime ideals P_i. Then $\mathrm{ann}_J(I) \subseteq P_i$ and $\overline{P_1} \cap \cdots \cap \overline{P_n} = 0$, where

$\overline{P_i} = P_i/\mathrm{ann}_J(I)$, so it suffices to note that each $\overline{P_i}$ is a prime ideal of the nondegenerate algebra $\overline{J}$.

$(iii) \Rightarrow (i)$ is obvious.

Now, to see that all P_i are closed, note that $P_2 \cap \cdots \cap P_n \cap I \subseteq \mathrm{ann}_J(P_1)$ and $P_2 \cap \cdots \cap P_n \cap I = 0$ would imply that $P_2 \cap \cdots \cap P_n \subseteq \mathrm{ann}_J(I)$ contradicting the minimality of n. Thus $\mathrm{ann}_J(P_1) \neq 0$ and since $\mathrm{ann}_J(\mathrm{ann}_J(P_1)) \cap \mathrm{ann}_J(P_1) = 0 \subseteq P_1$ and $\mathrm{ann}_J(P_1) \nsubseteq P_1$ (because it is nonzero) we get that $P_1 = \mathrm{ann}_J(\mathrm{ann}_J(P_1))$. Hence P_1 is closed and therefore strongly prime since any annihilator ideal produces a nondegenerate quotient by [**FGMo**](1.16(v)), and the same is true for all the P_i. □

Lemma 5.5. *Let J be a nondegenerate Jordan algebra. Then,*

(i) *if I, L are ideals of J such that L is semi-uniform and $I \subseteq L$, then I is semi-uniform.*
(ii) *Any finite sum of semi-uniform ideals is semi-uniform.*

Proof. For (i) just notice that $I \subseteq L$ implies that $\mathrm{ann}_J(L) \subseteq \mathrm{ann}_J(I)$. Now, (ii) follows from $\mathrm{ann}_J(\sum_{i=1}^n I_i) = \bigcap_{i=1}^n \mathrm{ann}_J(I_i)$ for any collection of ideals $I_1, \ldots, I_n$. □

Following [**FGMo**](7), a *subdirect product* of a collection of Jordan algebras $\{J_\alpha\}$ is any subalgebra of the full direct product of the J_α such that the canonical projections $\pi_\alpha : J \to J_\alpha$ are onto. An *essential subdirect product* is a subdirect product which contains an essential ideal of the full direct product. If J is actually contained in the direct sum of the J_α, then J is called an *essential subdirect sum.*

Let us now examine what it does mean that J is itself semi-uniform.

Proposition 5.6. *For a nondegenerate Jordan algebra J, the following are equivalent:*

(i) *J is semi-uniform,*
(ii) *J has no infinite direct sums of ideals,*
(iii) *J has the acc on annihilator ideals,*
(iv) *J is an essential subdirect sum of finitely many strongly prime Jordan algebras.*

Proof. The equivalence of (ii), (iii) and (iv) was already proved in [**FGMo**](7.3).

(i) $\Rightarrow$ (iv). If J is semi-uniform, let $\{P_1, \cdots, P_n\}$ be the set of prime ideals associated to J. Hence $P_1 \cap \cdots \cap P_n = 0$ and n is minimal. Then each $\mathrm{ann}_J(P_i)$ is uniform, and these are all the maximal uniform ideals. Indeed, if M is uniform, then $\bigcap_{i=1}^n (M \cap P_i) = M \cap \bigcap_{i=1}^n P_i = 0$ implies $M \cap P_i = 0$ for some i, and thus $M \subseteq \mathrm{ann}_J(P_i)$. Now let

N be a nonzero closed ideal of J (i.e. $N = \mathrm{ann}_J(\mathrm{ann}_J(N))$), then for all $i = 1, \cdots, n$, we have $N \cap \mathrm{ann}_J(N) = 0 \subseteq P_i$, hence either $N \subseteq P_i$ or $\mathrm{ann}_J(N) \subseteq P_i$. Now, if $N \subseteq P_i$ for all i, then $N \subseteq P_1 \cap \cdots \cap P_n = 0$, contradicting $N \neq 0$. Thus $\mathrm{ann}_J(N) \subseteq P_i$ for some i. Then, $\mathrm{ann}_J(P_i) \subseteq \mathrm{ann}_J(\mathrm{ann}_J(N)) = N$, and N contains a uniform ideal. The proof now follows directy from [**FG**](4.1).

(iv) $\Rightarrow$ (i). If J is a subdirect sum of strongly prime algebras $J_1, \cdots, J_n$, then each $J_i \cong J/P_i$ for a prime ideal P_i, and $P_1 \cap \cdots \cap P_n = 0 = \mathrm{ann}_J(J)$, hence J is semi-uniform. □

Next, we examine finite essential subdirect sums.

Lemma 5.7. *Let $J \subseteq \tilde{J}$ be nondegenerate Jordan algebras, with J being a subalgebra of $\tilde{J}$. If there is an essential ideal I of $\tilde{J}$ such that $I \subseteq J$, then, for any ideal L of $\tilde{J}$, $ann_J(L \cap J) = ann_{\tilde{J}}(L) \cap J$.*

Proof. If $x \in \mathrm{ann}_J(L \cap J)$ and $z \in L$, then $U_x z \in U_x L$ has $U_{U_x z} I = U_x U_z U_x I \subseteq U_x(L \cap J) = 0$, hence $U_x z \in \mathrm{ann}_{\tilde{J}}(I) = 0$ and we get $U_x L = 0$, hence $x \in \mathrm{ann}_{\tilde{J}}(L)$. Therefore $\mathrm{ann}_J(L \cap J) \subseteq \mathrm{ann}_{\tilde{J}}(L) \cap J$. The other containmment being obvious, we have the equality. □

Lemma 5.8. *Let J be a nondegenerate Jordan algebra, and let $I_1, \cdots, I_n$ be ideals of J. Set $J_i := J/I_i$ and $N_i := \bigcap_{i \neq j} I_j$ (if $n = 1$ we set $N_1 := J$) and $\phi : J \to J_1 \oplus \cdots \oplus J_n$ the mapping $\phi(x) := (x+I_1, \cdots, x+I_n)$. Then the algebras J_i are nondegenerate and ϕ makes J an essential subdirect sum of the J_i if, and only if, $ann_J(N_i) = I_i$ for all i.*

Proof. If ϕ makes J an essential subdirect sum of the nondegenerate algebras $J_1, \cdots, J_n$, set $\tilde{J} := J_1 \oplus \cdots \oplus J_n$ and identify J with its image under ϕ, so that J contains an essential ideal of $\tilde{J}$. Then $\mathrm{ann}_J(N_i) = \mathrm{ann}_J(N_i \cap J) = \mathrm{ann}_{\tilde{J}}(J_i) \cap J$ (by Lemma 5.7) $= (\bigoplus_{i \neq j} J_j) \cap J = I_i$.

Reciprocally, $\phi(N_i) \subseteq J_i$ is an essential ideal of J_i. Indeed, $\phi(N_i) = N_i + I_i/I_i$, and $J_i = J/I_i = J/\mathrm{ann}_J(N_i)$. Thus, $\phi(N_1 \oplus \cdots \oplus N_n)$ is an essential ideal of $J_1 \oplus \cdots \oplus J_n$, and $\phi(N_1 \oplus \cdots \oplus N_n) \subseteq \phi(J)$. Note that J_i are nondegenerate because annihilator ideals produce nondegenerate quotients. □

Lemma 5.9. *Let J be a nondegenerate Jordan algebra and assume that J is an essential subdirect sum of the algebras $J_1, \cdots, J_n$. Let $a \in J$ and put $a = a_1 + \cdots + a_n$ with $a_i \in J_i$, then J_a is an essential subdirect sum of $(J_1)_{a_1}, \cdots, (J_n)_{a_n}$.*

Proof. Obvious □

We next relate the semi-uniform condition to local algebras.

PROPOSITION 5.10. *Let J be a nondegenerate Jordan algebra and let $a \in J$.*

(i) *If J is semi-uniform, then J_a is semi-uniform.*
(ii) *J_a is semi-uniform if, and only if, the ideal $id_J(a)$ generated by a is semi-uniform.*

PROOF. (i). If $a = 0$ this is trivial. If $a \neq 0$, it follows from Proposition 5.6 and Lemma 5.9 and the local inheritance of primeness [**AnCMo**](4.1(iii)).

(ii). If $id_J(a)$ is semi-uniform, then $J/\mathrm{ann}_J(id_J(a))$ is semi-uniform by Proposition 5.4 and nondegenerate since annihilator ideals produce nondegenerate quotients. Now, $J_a \cong (J/\mathrm{ann}_J(id_J(a)))_{a+\mathrm{ann}_J(id_J(a))}$ is semi-uniform by (i). Reciprocally, since the case $a = 0$ is trivial, we can assume that $a \neq 0$. If J_a is semi-uniform, there exist prime ideals $\overline{P_1}, \cdots, \overline{P_n}$ of J_a with $\overline{P_i} = \mathrm{ann}_{J_a}(\bigcap_{i\neq j} \overline{P_j})$ by Proposition 5.6 and Lemma 5.8. We can take ideals $P_i \triangleleft J^{(a)}$ with $ker_J(a) \subseteq P_i$ and $P_i/ker_J(a) = \overline{P_i}$. Take $Q_i := \mathrm{Core}_J(P_i)$ (the biggest ideal of J contained in P_i). Then, it is easy to see that each Q_i is prime. Moreover, $Q_1 \cap \cdots \cap Q_n \subseteq P_1 \cap \cdots \cap P_n = ker_J(a)$, hence $Q_1 \cap \cdots \cap Q_n \subseteq \mathrm{Core}_J(ker_J(a)) = \mathrm{ann}_J(id_J(a))$. Then $id_J(a)$ is semi-uniform. □

For a Jordan algebra J we denote

$$\mathrm{SU}(J) := \{a \in J : J_a \text{ is semi-uniform}\}.$$

PROPOSITION 5.11. *Let J be a nondegenerate Jordan algebra, then $SU(J)$ is an ideal of J.*

PROOF. By Proposition 5.10(ii), we have that $\mathrm{SU}(J) = \{a \in J : id_J(a) \text{ is semi-uniform}\}$. Now if $a, b \in \mathrm{SU}(J)$, we have that $id_J(a+b) \subseteq id_J(a) + id_J(b)$ and, since $id_J(a)$ and $id_J(b)$ are semi-uniform, then $id_J(a) + id_J(b)$ is semi-uniform by Lemma 5.5(ii), hence $id_J(a + b)$ is semi-uniform by Lemma 5.5(i), hence $a + b \in \mathrm{SU}(J)$. Now, for any $c \in id_J(a)$ we have that $id_J(c) \subseteq id_J(a)$, so $id_J(c)$ is semi-uniform again by Lemma 5.5(i), hence $c \in \mathrm{SU}(J)$ which shows that $\mathrm{SU}(J)$ is an ideal of J. □

PROPOSITION 5.12. *Let the Jordan algebra J be an essential subdirect sum of the strongly prime Jordan algebras $J_1, \cdots, J_n$. Then J is LC if, and only if, so is each J_i.*

PROOF. If J is LC, then so is each J_i by [**FGMo**](10.6). Reciprocally, if J is an essential subdirect sum of the J_i, set $\tilde{J} := J_1 \oplus \cdots \oplus J_n$

and let $M \triangleleft \tilde{J}$ be an essential ideal with $M \subseteq J$ and set $M_i := M \cap J_i$, so that $J_i \cong J/\mathrm{ann}_J(M_i)$ and M_i is an essential ideal of J_i.

Then, if K is an essential inner ideal of J then it follows that $K_i = K \cap J_i$ is an essential inner ideal of J_i. Indeed, if $0 \neq x_i \in J_i$, then $U_{x_i}M_i \neq 0$ by the essentiality of M_i. Then $U_{x_i}M_i$ is a nonzero inner ideal of J, hence $0 \neq U_{x_i}M_i \cap K \subseteq U_{x_i}M_i \cap J_i \cap K \subseteq U_{x_i}J_i \cap K_i$. Since J_i is LC, there is an injective element s_i of J_i in K_i. Then $s := s_1 + \cdots + s_n \in \mathrm{Inj}(\tilde{J})$, and since for all i, we have $s_i \in K \cap J_i$, then $s \in K \subseteq J$, hence $s \in \mathrm{Inj}(\tilde{J}) \cap K = \mathrm{Inj}(\tilde{J}) \cap J \cap K \subseteq \mathrm{Inj}(J) \cap K$. Thus K contains an injective element.

Assume now that K is an inner ideal of J and $K \cap \mathrm{Inj}(J) \neq \emptyset$. Put, as before, $K_i = K \cap J_i$, and let us see that $K_i \cap \mathrm{Inj}(J_i) \neq \emptyset$. First, since M_i is a nonzero ideal of J_i, it is an essential inner ideal, hence there is $t_i \in M_i \cap \mathrm{Inj}(J_i)$, since J_i is LC. Now, if $s \in K \cap \mathrm{Inj}(J)$, then $U_s t_i \in U_K M_i \subseteq M_i \cap K \subseteq K_i$. Also, if $a \in J_i$ has $U_{U_s t_i} a = 0$, then $U_s U_{t_i} U_s a = 0$ and $U_{t_i} U_s a \in M_i \subseteq J$ give $U_{t_i} U_s a = 0$ since $s \in \mathrm{Inj}(J)$. Next, since $U_s a \in U_s J_i \subseteq J_i$ and $t_i \in \mathrm{Inj}(J_i)$ we get $U_s a = 0$. Thus $0 = U_{U_s a} M_i = U_s U_a U_x M_i$, hence $U_a U_s M_i = 0$, since $U_a U_s M_i \subseteq M_i \subseteq J$ and $s \in \mathrm{Inj}(J)$. Then, for all $x \in M_i$ we get $U_{U_s U_a x} M_i = U_s U_a U_x U_a U_s M_i = 0$ but since $U_s U_a x \in M_i \subseteq J_i$ and M_i is essential in J_i, we get $U_s U_a x = 0$, that is, $U_s U_a M_i = 0$. Again, since $U_a M_i \subseteq M_i \subseteq J$ and $s \in \mathrm{Inj}(J)$, this implies $U_a M_i = 0$, hence $a = 0$ by the essentiality of M_i. Thus we conclude that $U_s t_i \in K_i \cap \mathrm{Inj}(J_i)$. Then K_i is essential in J_i, since J_i is LC. Now, if $0 \neq a \in J$, then $U_a M = \sum_{i=1}^{n} U_a M_i$ is a nonzero inner ideal of $\tilde{J}$, hence there is i such that $U_a M_i$ is a nonzero inner ideal of J_i and the essentiality of K_i gives $U_a M_i \cap K_i \neq 0$. Finally, since $U_a M_i \cap K_i \subseteq U_a J \cap K$ we obtain that K is essential. Hence J is LC. $\square$

THEOREM 5.13. *If J is a nondegenerate Jordan algebra, then $LC(J)$ is an ideal of J.*

PROOF. We first note that $\mathrm{LC}(J) \subseteq \mathrm{SU}(J)$ by [**FGMo**](10.3) and Proposition 5.6. It suffices to show that for any $a, b \in \mathrm{LC}(J)$ and any $x \in J'$, the elements $a + b, U_x a$ and $U_a x$ lie in $\mathrm{LC}(J)$. To see this, set $c := a + b, U_x a$ or $U_a x$. Then $L := id_J(a) + id_J(b)$ is semi-uniform and, setting $I := \mathrm{ann}_J(L)$, the algebra J/I is semi-uniform again by Proposition 5.4 and has $(J/I)_{c+I} \cong J_c$, $(J/I)_{a+I} \cong J_a$ and $(J/I)_{b+I} \cong J_b$ since $a, b, c \in L$ implies $I \subseteq ker_J(a) \cap ker_J(b) \cap ker_J(c)$.

Now J/I is semi-uniform, hence an essential subdirect sum of finitely many strongly prime Jordan algebras $J_1, \cdots, J_n$ again by Proposition 5.6. For any $z \in J$, put $z + I = z_1 + \cdots + z_n$ with $z_i \in J_i$ (so that z_i is the projection of $z + I$ in J_i, also, if $z = 1$ we set $z_i = 1 \in J_i'$).

Then, for $d := a, b$ or c, $J_d \cong (J/I)_{d+I}$ is an essential subdirect sum of $(J_1)_{d_1}, \cdots, (J_n)_{d_n}$ by Lemma 5.9. Thus, for each $i = 1, \cdots, n$, $a \in \mathrm{LC}(J)$ implies $a + I \in \mathrm{LC}(J/I)$, hence $a_i \in \mathrm{LC}(J_i)$ by Proposition 5.12, and similarly $b_i \in \mathrm{LC}(J_i)$. Now, $\mathrm{LC}(J_i)$ is an ideal of J_i by Corollary 4.6, hence $a_i + b_i, U_{x_i}a_i, U_{a_i}x_i \in \mathrm{LC}(J_i)$, but $c_i = a_i + b_i, U_{x_i}a_i$, or $U_{a_i}x_i$ according to whether $c = a + b, U_x a$ or $U_a x$. Therefore we get $c_i \in \mathrm{LC}(J_i)$. Since $(J/I)_{c+I} \cong J_c$ is an essential subdirect sum of $(J_1)_{c_1}, \cdots, (J_n)_{c_n}$, and these are LC, J_c is itself LC by Proposition 5.12, hence $c \in \mathrm{LC}(J)$ as desired. □

References

[AMa] P.N. Ánh and L. Márki, *Orders in primitive rings with nonzero socle and Posner's theorem*, Comm. Algebra, **24** (1996), 289-294.

[AnCMo] J.A. Anquela, T. Cortés and F. Montaner, *Local inheritance in Jordan algebras*, Arch. Math., **64** (1995), 393-401.

[B] K.I. Beidar *et al*, *Rings with generalized identities*, Marcel Dekker, New York Basel Hong Kong (1976).

[DM] A. D'Amour and K. McCrimmon, *The local algebras of Jordan systems*, J. Algebra, **177** (1995), 199-239.

[FG] A. Fernández López and E. García Rus, *Algebraic lattices and nonassociative structures*, Proc. Amer. Math. Soc., **126** (1998), 3211-3221.

[FGMo] A. Fernández López, E. García Rus and F. Montaner, *Goldie theory for Jordan algebras*, J. Algebra, **248** (2002), 397-471.

[FGMo2] A. Fernández López, E. García Rus and F. Montaner, *Goldie theory for Jordan pairs.*

[FT] A. Fernández López and M. Tocón, *The local algebras of an associative algebra and their applications*, In Applicable Mathematics in the Golden Age, J.C. Misra (Ed.), Narosa Publishing House, New Delhi, (2002).

[Gd] K.R.. Goodearl, *Ring Theory: Nonsingular Rings and Modules*, Pure and Applied Mathematics, Marcel Dekker, New York, (1976).

[Go] A.W. Goldie, *The structure of prime rings under ascending chain condition*, Proc. London Math. Soc. **8** (1958), 589-608.

[Go2] A.W. Goldie, *Semiprime rings with maximum condition*, Proc. London Math. Soc. **10** (1960), 201-220.

[H] I.N. Herstein, *Rings with involution*, Chicago Lectures in Mathematics, The University of Chicago Press, (1976).

[J] N. Jacobson, *Structure theory of Jordan Algebras*, Lectures notes in Mathematics, University of Arkansas **5**, Wiley, New York (1981).

[LCr] L. Lesieur and R. Croisot , *Sur les anneaux premiers Noethériens à gauche*, Ann. Sci. Ecole Norm. Sup. **76** (1959), 161-183.

[Lo] O. Loos, *Jordan pairs*, Lectures Notes in Mathematics**406** Springer-Verlag, Berlin Heidelberg (1975).

[Lo2] O. Loos, *Finiteness conditions in Jordan pairs*, Math. Z. **206** (1991), 577-587.

[M] K. McCrimmon, *Inner ideals in quadratic Jordan algebras*, Trans. Amer. Math. Soc. **159** (1971), 445-468.

[M2] K. McCrimmon, *The Zelmanov annihilator and nilpotence of the nil radical in quadratic Jordan algebras with chain conditions*, J. Algebra, **67** (1980), 230-253.

[M3] K. McCrimmon, *Zelmanov's prime theorem for quadratic Jordan algebras*, J. Algebra, **76** (1982), 297-326.

[M4] K. McCrimmon, *A taste of Jordan algebras*, Springer-Verlag, New York (2004)

[MZ] K. McCrimmon and E. Zelmanov, *The structure of strongly prime quadratic Jordan algebra*, Adv. Math. **69** (1988), 133-222.

[Me] K. Meyberg, *Lectures on Algebras and Triple Systems*, Lectures Notes, University of Virginia, Charlottesville (1972).

[Mo] F. Montaner, *Local PI theory of Jordan systems*, J. Algebra **216** (1999), 302-327.

[Mo2] F. Montaner, *Local PI theory of Jordan systems II*, J. Algebra **241** (2001), 473-514.

[Z] E. Zelmanov, *Goldie's theorem for Jordan algebras; II*, Siberian Math. J.,**28** (1987),44-52; **29** (1988), 68-74.

Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address: `fmontane@posta.unizar.es`

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, Ontario K1N 6N5, Canada

E-mail address: `maribel@uma.es`