LOCALLY EXTENDED AFFINE ROOT SYSTEMS

Yoji Yoshii

Akita National College of Technology
1-1 Iijima Bunkyocho Akita-shi, Japan 011-8511
yoshii@akita-nct.jp

ABSTRACT. We consider a natural generalization of both locally finite irreducible root systems and extended affine root systems defined by Saito. We classify the systems.

INTRODUCTION

Let us recall the definition of a finite irreducible root system in a euclidean space V, i.e., $V \approx \mathbb{R}^n$ with a positive definite form $(\cdot, \cdot)$.

Definition 0. A subset $\mathfrak{R}$ of V is called a *finite irreducible root system* if

(A1) $0 \notin \mathfrak{R}$ and $\mathfrak{R}$ spans V;

(A2) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$;

(A3) $\sigma_\alpha(\beta) \in \mathfrak{R}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$;

(A4) $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$ and $(\mathfrak{R}_1, \mathfrak{R}_2) = 0$ imply $\mathfrak{R}_1 = \emptyset$ or $\mathfrak{R}_2 = \emptyset$. ($\mathfrak{R}$ is irreducible.)

We note that $\mathfrak{R}$ becomes automatically a finite set (see [LN1, 4.2] or [MY1, Prop. 4.2]). Needless to say, these interesting subsets were crucial in the classification of finite-dimensional simple Lie algebras and of finite reflection groups in the 20th century. In 1985, K. Saito introduced the notion of a generalized root system [S]. He changed the frame V from the euclidean space to a general vector space over $\mathbb{R}$ equipped with a symmetric bilinear form, not necessarily a positive definite form, and replaced the axiom (A1) to:

$$(\alpha, \alpha) \neq 0 \text{ for all } \alpha \in \mathfrak{R}, \text{ and } \mathfrak{R} \text{ spans } V.$$

This change is natural since $(\alpha, \alpha) \neq 0$ whenever $\alpha \neq 0$ in a euclidean space. Moreover, Saito added two extra axioms:

(A5) the additive subgroup generated by $\mathfrak{R}$ is a full lattice in V;

(A6) the codimension of the radical of V is finite.

He called such a root system an *extended affine root system* if the form is positive semidefinite. (Later the notion of an extended affine root system was used in a different sense [A-P], but it was proved in [A2] there is a natural correspondence between both notions.) If the dimension of the radical of the positive semidefinite form is 1, the extended affine root systems are irreducible affine root systems in the sense of Macdonald [M]. One of the Saito's

main purposes was to construct a Lie algebra whose anisotropic roots form an extended affine root system having a 2-dimensional radical.

Our interest now is not Saito's root systems but extended affine root systems. We generalize Saito's axioms of extended affine root systems with good reasons. First of all, we make our new concept contain the so-called *locally finite irreducible root systems* (see [LN1]), which are obtained simply by changing the frame V in Definition 0 to an infinite-dimensional euclidean space, i.e., an infinite-dimensional vector space over $\mathbb{R}$ with a positive definite form. (Then $\mathfrak{R}$ becomes automatically a locally finite set, i.e., $|W \cap \mathfrak{R}| < \infty$ for any finite-dimensional subspace W of V.) It turns out that if we simply take off the axiom (A6), locally finite irreducible root systems are contained. Next, we assume the base field to be $\mathbb{Q}$, not $\mathbb{R}$. Notice that in the setting of finite-dimensional simple Lie algebras, root systems naturally appear in vector spaces over $\mathbb{Q}$, and then one gets a euclidean space by simply tensoring with $\mathbb{R}$. Besides, our theory of extended affine Lie algebras also produces a root system in a vector space over $\mathbb{Q}$ (see [MY1]). Once we start with a vector space over $\mathbb{Q}$, the axiom (A5) is equivalent to saying that the abelian group generated by $\mathfrak{R}$, say $\langle \mathfrak{R} \rangle$, is free. Thus, it seems better to have as axiom that

(A7) $\langle \mathfrak{R} \rangle$ is free

in our setup. However, we can say much about the classification without assuming (A7). So we simply take off the axiom (A5) (and we do not assume (A7) either), and we get our definition of a *locally extended affine root system* in Definition 1. As a special case, we call a locally extended affine root system an *extended affine root system* if (A6) and (A7) hold. Thus our extended affine root systems are the same as Saito's if we consider the embedding of ours into the real vector space $\mathbb{R} \otimes_{\mathbb{Q}} V$.

We classify locally extended affine root systems in terms of triples of reflection spaces by the methods from [A-P] (see Theorem 7), which was also done in [LN2] in a more general setting. Also, we show some relations between the isomorphisms of locally extended affine root systems and the similarities of reflection spaces in Theorem 10. Then, when $\dim_{\mathbb{Q}} V^0 = 1$, we get more information by a simple observation about subgroups of $\mathbb{Q}$ in Corollary 13. Finally, we give some interesting examples of Lie algebras whose root systems are locally extended affine root systems.

Basic Concepts

Definition 1. Let V be a vector space over $\mathbb{Q}$ with a positive semidefinite bilinear form $(\cdot, \cdot)$. A subset $\mathfrak{R}$ of V is called a *locally extended affine root system* or a *LEARS* for short if

(A1) $(\alpha, \alpha) \neq 0$ for all $\alpha \in \mathfrak{R}$, and $\mathfrak{R}$ spans V;
(A2) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$;
(A3) $\sigma_\alpha(\beta) \in \mathfrak{R}$ for all $\alpha, \beta \in \mathfrak{R}$, where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$;
(A4) $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$ and $(\mathfrak{R}_1, \mathfrak{R}_2) = 0$ imply $\mathfrak{R}_1 = \emptyset$ or $\mathfrak{R}_2 = \emptyset$. ($\mathfrak{R}$ is irreducible.)

A LEARS $\mathfrak{R}$ is called *reduced* if $2\alpha \notin \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$.

Note that if V is finite-dimensional and $(\cdot, \cdot)$ is positive definite, then $\mathfrak{R}$ is exactly a finite irreducible root system (see [MY1, Prop. 4.2]).

Let
$$V^0 := \{x \in V \mid (x, y) = 0 \text{ for all } y \in V\}$$

be the radical of the form. Note that

$$V^0 = \{x \in V \mid (x,x) = 0\}.$$

We call $\dim_{\mathbb{Q}} V^0$ the *null dimension* of $\mathfrak{R}$, which can be any cardinality.

We denote the additive subgroup of V generated by a subset S of V by $\langle S \rangle$.

We call a LEARS $(\mathfrak{R}, V)$ an *extended affine root system* or an *EARS* for short, if

$$\dim_{\mathbb{Q}} V/V^0 < \infty \text{ and } \langle \mathfrak{R} \rangle \text{ is free.}$$

This coincides with the concept, which was firstly introduced by Saito in 1985 [S]. As we mentioned in the Introduction, the notion of an EARS was also used in a different sense in [A-P], but Azam showed that there is a natural correspondence between the two notions in [A2]. We use here the Saito's one since he is the first person who defined it and his root system naturally generalized Macdonald's *affine root systems* in [M].

In Corollary 5 later, we will see that if the abelian group $\langle \mathfrak{R} \rangle \cap V^0$ is free, then $\langle \mathfrak{R} \rangle$ is free. So the condition that $\langle \mathfrak{R} \rangle$ is free can be replaced by the condition that $\langle \mathfrak{R} \rangle \cap V^0$ is free.

Recall the notion of *rank* for a torsion-free abelian group G, that is, $\operatorname{rank} G = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$. It is easy to check that if G is a subgroup of a $\mathbb{Q}$-vector space W, then $\operatorname{rank} G = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} G$, where $\operatorname{span}_{\mathbb{Q}} G$ is the subspace of W spanned by G over $\mathbb{Q}$.

Thus, in our root system $(\mathfrak{R}, V)$, we have

$$\operatorname{rank}(\langle \mathfrak{R} \rangle \cap V^0) = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}(\langle \mathfrak{R} \rangle \cap V^0) = \dim_{\mathbb{Q}} V^0 = \text{(the null dimension of } \mathfrak{R}).$$

Now, when our torsion-free abelian group $\langle \mathfrak{R} \rangle \cap V^0$ happens to be free, we say that $\mathfrak{R}$ has *nullity*. (We simply want to distinguish the easier case "free".) For example, $\mathfrak{R}$ has nullity 1 means that $\langle \mathfrak{R} \rangle \cap V^0 \cong \mathbb{Z}$, and $\mathfrak{R}$ has null dimension 1 means that $\langle \mathfrak{R} \rangle \cap V^0$ is isomorphic to a nonzero subgroup of $\mathbb{Q}$. Also, by Corollary 5, if an EARS $\mathfrak{R}$ has finite nullity, then $\langle \mathfrak{R} \rangle$ is free of finite rank. Thus we simply say that $\mathfrak{R}$ is an EARS of finite rank when the EARS has finite nullity.

Our LEARS are a natural generalization of the existing concept EARS. In fact, Saito's EARS are the same as our EARS embedded into the real vector space $\mathbb{R} \otimes_{\mathbb{Q}} V$. Similarly, irreducible affine root systems in the sense of Macdonald [M] are our EARS of nullity 1. Note that the reduced irreducible affine root systems are the real roots of affine Kac-Moody Lie algebras. The elliptic root systems defined by Saito [S] are our EARS of nullity 2. Also, the sets of nonisotropic roots of EARS in [A-P] are our reduced EARS of finite rank (see [A2]).

Finally, we call a LEARS of nullity 1 a *locally affine root system* or a *LARS* for short.

Let $(\mathfrak{R}, V)$ be a LEARS, and $(\bar{\mathfrak{R}}, \bar{V})$ the canonical image onto V/V^0. Then $\bar{V}$ admits the induced positive definite form, and thus

$$(\bar{\mathfrak{R}}, \bar{V}) \text{ is a locally finite irreducible root system.}$$

Note that our definition of a locally finite irreducible root system is a LEARS (in Definition 1) so that the form is positive definite, and then one can show that the system is in fact locally finite (see [LN1, 4.2] or [MY1, Prop.4.2]).

3

Locally finite irreducible root systems which are not finite were classified as the reduced types $A_{\mathfrak{J}}$, $B_{\mathfrak{J}}$, $C_{\mathfrak{J}}$, $D_{\mathfrak{J}}$, and the nonreduced type $BC_{\mathfrak{J}}$ for any infinite index set $\mathfrak{J}$ (see [LN1, Ch.8]). More precisely, let $\{\epsilon_i\}_{i \in \mathfrak{J}}$ be an orthonormal basis for an infinite-dimensional euclidean space V (or an infinite-dimensional vector space over $\mathbb{Q}$ with positive definite form), and let

$$
\begin{aligned}
A_{\mathfrak{J}} &= \{\epsilon_i - \epsilon_j \mid i \neq j \in \mathfrak{J}\}, \\
B_{\mathfrak{J}} &= \{\pm\epsilon_i, \ \pm(\epsilon_i \pm \epsilon_j) \mid i \neq j \in \mathfrak{J}\}, \\
C_{\mathfrak{J}} &= \{\pm(\epsilon_i \pm \epsilon_j), \ \pm 2\epsilon_i \mid i \neq j \in \mathfrak{J}\}, \\
D_{\mathfrak{J}} &= \{\pm(\epsilon_i \pm \epsilon_j) \mid i \neq j \in \mathfrak{J}\}, \\
BC_{\mathfrak{J}} &= B_{\mathfrak{J}} \cup C_{\mathfrak{J}}.
\end{aligned}
$$

Note that each root system spans V except $A_{\mathfrak{J}}$. If $|\mathfrak{J}| = \ell$ is finite, then an ordinary notation of the root system is $A_{\ell-1}$ instead of A_ℓ. So it might be better to write something like $A_{\mathfrak{J}-1}$ or $\dot{A}_{\mathfrak{J}}$ instead of just $A_{\mathfrak{J}}$. However, to simplify the notation, we stipulate to write $A_{\mathfrak{J}}$ when $|\mathfrak{J}|$ is infinite.

Let $(\mathfrak{R}, V)$ be a locally finite irreducible root system (including the finite case) and assume it is reduced. A basis Π of V is called a *reflectable base* of $\mathfrak{R}$ if $\Pi \subset \mathfrak{R}$ and for any $\alpha \in \mathfrak{R}$,

$$
\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})
$$

for some $\alpha_1, \dots, \alpha_{k+1} \in \Pi$. (Any root can be obtained by reflecting a root of Π relative to hyperplanes determined by Π.) This is a well-known property which a root base of a reduced finite root system possesses. It is known that a locally finite irreducible root system which is countable possesses a root base, but this is not the case for a locally finite irreducible root system which is uncountable. (See [LN1, §6]. They also prove that there always exists an integral base even in the uncountable case. However, it is easy to see that an integral base is not necessarily a reflectable base.) Thus we need to show the existence of a reflectable base in a reduced locally finite irreducible root system which is uncountable, and we have:

Lemma 2. *There exists a reflectable base in any reduced locally finite irreducible root system [LN2, Lem. 5.1]. In particular, the additive subgroup generated by each locally finite irreducible root system is free (see also [LN1, Thm 7.5]).*

Classification

We devote this section to classifying LEARS. (The argument below is a special case of [LN2, 4.9, 5.2].)

Let $(\mathfrak{R}, V)$ be a LEARS. Let V' be a subspace of V so that $V = V' \oplus V^0$, and

$$
\Delta = \Delta_{V'} := \{\alpha \in V' \mid \alpha + s \in \mathfrak{R} \text{ for some } s \in V^0\}.
$$

We note that Δ is bijectively mapped onto $\bar{\mathfrak{R}}$ by $\bar{\ }$. Moreover, $\bar{\ }$ is a linear isomorphism from V' onto $\bar{V}$ satisfying $(v', w') = (\bar{v}', \bar{w}')$ for all $v', w' \in V'$. Hence, (Δ, V') is a locally finite irreducible root system isomorphic to $(\bar{\mathfrak{R}}, \bar{V})$. We often say that $\mathfrak{R}$ has type Δ. For each $\alpha \in \Delta$, we set

$$
S_\alpha := \{s \in V^0 \mid \alpha + s \in \mathfrak{R}\}.
$$

Then
$$\mathfrak{R} = \bigsqcup_{\alpha \in \Delta} (\alpha + S_\alpha).$$

Since $\mathfrak{R}$ spans V,

(S0) $\qquad\qquad\qquad\qquad\qquad \cup_{\alpha \in \Delta} S_\alpha$ spans V^0.

Also, for any $\alpha + s, \beta + s'$ with $\alpha, \beta \in \Delta$, $s \in S_\alpha$ and $s' \in S_\beta$, we have
$$\sigma_{\alpha+s}(\beta + s') = \beta + s' - \langle \beta + s', \alpha + s\rangle(\alpha + s)$$
$$= \sigma_\alpha(\beta) + s' - \langle \beta, \alpha\rangle s \in \mathfrak{R},$$

and so $s' - \langle \beta, \alpha\rangle s \in S_{\sigma_\alpha(\beta)}$, i.e.,

(S1) $\qquad\qquad\qquad S_\beta - \langle\beta, \alpha\rangle S_\alpha \subset S_{\sigma_\alpha(\beta)} \quad$ for all $\alpha, \beta \in \Delta$.

Conversely, let Δ be a locally finite irreducible root system in a vector space V_1 over $\mathbb{Q}$ with positive definite form, and let $\{S_\alpha\}_{\alpha \in \Delta}$ be a family of nonempty subsets in a vector space V_0 indexed by Δ satisfying (S0) and (S1). Extend the positive definite form on V_1 to $V := V_1 \oplus V_0$ so that V_0 is the radical of the form. Let $\mathfrak{R} := \sqcup_{\alpha \in \Delta} (\alpha + S_\alpha)$. Then $\mathfrak{R}$ satisfies the axioms (A1-4) of a LEARS. In particular (assuming that Corollary 5 holds), if Δ is finite and the abelian group $\langle \cup_{\alpha \in \Delta} S_\alpha \rangle$ is free, then $\mathfrak{R}$ is an EARS.

Proposition 3. *A LEARS is a directed union of EARS of finite rank. Namely, if $\mathfrak{R} = \bigsqcup_{\alpha \in \Delta} (\alpha + S_\alpha)$ is a LEARS in the description above, then*
$$\mathfrak{R} = \bigcup_{\Delta'} \bigcup_{\Lambda_{\Delta'}} \bigsqcup_{\alpha \in \Delta'} \big(\alpha + (\Lambda_{\Delta'} \cap S_\alpha)\big),$$

where $\bigcup_{\Delta'}$ means a directed union over finite irreducible subsystems Δ' of Δ and $\bigcup_{\Lambda_{\Delta'}}$ means a directed union over subgroups $\Lambda_{\Delta'}$ generated by a subset $\cup_{\alpha \in \Delta'} S'_\alpha$, where S'_α is chosen to be any nonempty finite subset of S_α.

Proof. Note that a locally finite irreducible root system is a directed union of finite irreducible subsystems [LN1, Cor.3.15]. Hence Δ is a directed union of finite irreducible subsystems Δ', and so $\mathfrak{R}$ is a directed union of $\bigsqcup_{\alpha \in \Delta'} (\alpha + S_\alpha)$, i.e., $\mathfrak{R} = \bigcup_{\Delta'} \bigsqcup_{\alpha \in \Delta'} (\alpha + S_\alpha)$.

Now, since S_α is a directed union of $\Lambda_{\Delta'} \cap S_\alpha$, say $S_\alpha = \bigcup_{\Lambda_{\Delta'}} (\Lambda_{\Delta'} \cap S_\alpha)$, we have $\bigsqcup_{\alpha \in \Delta'} (\alpha + S_\alpha) = \bigcup_{\Lambda_{\Delta'}} \bigsqcup_{\alpha \in \Delta'} \big(\alpha + (\Lambda_{\Delta'} \cap S_\alpha)\big)$. Moreover, $\langle \bigcup_{\alpha \in \Delta'} (\Lambda_{\Delta'} \cap S_\alpha) \rangle = \Lambda_{\Delta'}$ is free of finite rank. So $\bigsqcup_{\alpha \in \Delta'} \big(\alpha + (\Lambda_{\Delta'} \cap S_\alpha)\big)$ is an EARS of finite rank since Δ' is a finite irreducible root system, $\Lambda_{\Delta'} \cap S_\alpha$ is nonempty for all $\alpha \in \Delta'$, and $\{\Lambda_{\Delta'} \cap S_\alpha\}_{\alpha \in \Delta'}$ satisfies (S1). $\square$

Let us recall that we have chosen a complementary subspace V' of V^0 to get $\{S_\alpha\}_{\alpha \in \Delta}$. To classify LEARS, we now choose a nice complementary subspace. First we define for any LEARS $\mathfrak{R}$,
$$\mathfrak{R}^{\text{red}} := \begin{cases} \mathfrak{R} & \text{if } \mathfrak{R} \text{ is reduced} \\ \{\alpha \in \mathfrak{R} \mid \frac{1}{2}\alpha \notin \mathfrak{R}\} & \text{otherwise.} \end{cases}$$

Now, note that $(\mathfrak{R}^{\text{red}}, \bar{V})$ is a reduced locally finite irreducible root system. Thus there exists a reflectable base Π of $(\bar{\mathfrak{R}}^{\text{red}}, \bar{V})$ (by Lemma 2). We fix a preimage $\alpha \in \mathfrak{R}$ for each $\bar{\alpha} \in \Pi$. Let

$\qquad\qquad V'$ be the subspace of V spanned by $\{\alpha\}_{\bar{\alpha} \in \Pi}$.

We call this complementary subspace a *reflectable subspace* determined by a complete set of representatives of a reflectable base Π of $(\bar{\mathfrak{R}}^{\text{red}}, \bar{V})$. Then the subsets S_α of V^0 defined above satisfy the following as in [A-P, Prop.2.11] (see also [LN2, 4.2, 4.5, 4.10, 5.2]).

Lemma 4. *Let Δ^{red} be the corresponding set to $\bar{\mathfrak{R}}^{\mathrm{red}}$ determined by a reflectable subspace V' as above. Then $\Delta^{\mathrm{red}} \subset \mathfrak{R}$, or in other words,*

$$\text{(S2)} \qquad\qquad\qquad 0 \in S_\alpha \quad \text{for all } \alpha \in \Delta^{\mathrm{red}}.$$

Moreover, if $\mathfrak{R}$ is reduced, then

$$\text{(S3)} \qquad\qquad\qquad S_{2\alpha} \cap 2S_\alpha = \emptyset \quad \text{for all } 2\alpha, \alpha \in \Delta.$$

Proof. For any $\alpha \in \Delta^{\mathrm{red}}$, by Lemma 2 when $\bar{\mathfrak{R}}$ is infinite, or a well-known property for a root base when $\bar{\mathfrak{R}}$ is finite, $\bar{\alpha} = \sigma_{\bar{\alpha}_1} \cdots \sigma_{\bar{\alpha}_k}(\bar{\alpha}_{k+1})$ for some $\bar{\alpha}_1, \dots, \bar{\alpha}_{k+1} \in \Pi$. Then, $\alpha_1, \dots, \alpha_{k+1} \in V' \cap \mathfrak{R}$, by our definition of V'. Hence $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1}) \in V'$, and $\overline{\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})} = \sigma_{\bar{\alpha}_1} \cdots \sigma_{\bar{\alpha}_k}(\bar{\alpha}_{k+1}) = \bar{\alpha}$. So we get $\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}(\alpha_{k+1})$. Therefore, $\alpha \in \mathfrak{R}$ by (A3).

For the second statement, if $2s \in S_{2\alpha} \cap 2S_\alpha$ for some $s \in S_\alpha$, then $2\alpha + 2s \in \mathfrak{R}$ and $\alpha + s \in \mathfrak{R}$, contradiction. $\square$

Let

$$G = \langle \cup_{\alpha \in \Delta} S_\alpha \rangle.$$

Corollary 5. *We have $\langle \mathfrak{R} \rangle = \langle \Delta \rangle \oplus G$. In particular, $\langle \mathfrak{R} \rangle \cap V^0 = G$, and if a LEARS $\mathfrak{R}$ has nullity, then $\langle \mathfrak{R} \rangle$ is free.*

Proof. Since $\langle \Delta \rangle = \langle \Delta^{\mathrm{red}} \rangle \subset \langle \mathfrak{R} \rangle$ (by Lemma 4), we have $\langle \mathfrak{R} \rangle = \langle \Delta \rangle \oplus G$ and $\langle \mathfrak{R} \rangle \cap V^0 = G$. Note that $\langle \Delta \rangle$ is free (by Lemma 2). So if G is free, then $\langle \mathfrak{R} \rangle$ is free. $\square$

Now, for a LEARS $\mathfrak{R}$, we obtain a family $\{S_\alpha\}_{\alpha \in \Delta}$ of nonempty subsets in V^0 satisfying (S0), (S1) and (S2). When Δ is a finite irreducible root system, such a family $\{S_\alpha\}_{\alpha \in \Delta}$ satisfying (S1) and (S2) is called a *root system of type Δ extended by the abelian group G*, and *reduced* if it satisfies (S3) (see [Y]).

Remark 6. For $\bar{\alpha} \in \Pi$ and any $s_\alpha \in S_\alpha$, $\alpha' := \alpha + s_\alpha$ is another preimage of $\bar{\alpha} \in \Pi$. Let W be the subspace of V spanned by $\{\alpha'\}_{\bar{\alpha} \in \Pi}$, i.e., W is another reflectable subspace. Or more generally, let W be a reflectable subspace determined by a different reflectable base. Then we get the corresponding root system $\{T_{\alpha'}\}_{\alpha' \in \Delta_W}$ extended by $G' = \langle \cup_{\alpha' \in \Delta_W} T_{\alpha'} \rangle$ so that

$$\mathfrak{R} = \bigsqcup_{\alpha' \in \Delta_W} (\alpha' + T_{\alpha'}).$$

The relation between $\{S_\alpha\}_{\alpha \in \Delta}$ and $\{T_{\alpha'}\}_{\alpha' \in \Delta_W}$ will be clarified in Lemma 8.

Root systems extended by G were classified in [Y]. (The main idea comes from the classification of EARS in [A-P].) To explain the classification, let us introduce some terminology.

Recall that a finite irreducible root system Δ is one of the following types; $\Delta = A_\ell$ ($\ell \geq 1$), B_ℓ ($\ell \geq 1$, $B_1 = A_1$), C_ℓ ($\ell \geq 2$, $C_2 = B_2$), D_ℓ ($\ell \geq 4$), E_ℓ ($\ell = 6, 7, 8$), F_4, G_2 or BC_ℓ ($\ell \geq 1$). We partition the root system Δ according to length. Roots of Δ of minimal length are called *short*. Roots of Δ which are two times a short root of Δ are called *extra long*. Finally, roots of Δ which are neither short nor extra long are called *long*. We denote the subsets of short, long and extra long roots of Δ by Δ_{sh}, Δ_{lg} and Δ_{ex} respectively. Thus

$$\Delta = \Delta_{sh} \sqcup \Delta_{lg} \sqcup \Delta_{ex}.$$

6

Of course the last two terms in this union may be empty. Indeed,

$$\Delta_{lg} = \emptyset \quad \Longleftrightarrow \quad \Delta \text{ has simply laced type or type BC}_1,$$

and

$$\Delta_{ex} = \emptyset \quad \Longleftrightarrow \quad \Delta = \Delta^{\mathrm{red}}.$$

If $\Delta_{lg} \neq \emptyset$, we use the notation k for the ratio of the long square root length to the short square root length in Δ. Hence,

$$k = \begin{cases} 2 & \text{if } \Delta \text{ has type B}_\ell, \text{ C}_\ell, \text{ F}_4 \text{ or BC}_\ell \text{ for } \ell \geq 2. \\ 3 & \text{if } \Delta \text{ has type G}_2. \end{cases}$$

For any abelian group G,

(i) a subset E of G is called a *reflection space* if $E - 2E \subset E$;
(ii) a reflection space E of G is called *full* if E generates G;
(iii) a reflection space E of G is called a *pointed reflection space* if $0 \in E$.

These notions were introduced in [A-P] when G is a full lattice in a finite-dimensional real vector space as a name *semilattice*, or earlier in a more general setting in [L]. We note that if E is a full reflection space of G, then $2G + E \subset 2\langle E \rangle + E \subset E$ and so $2G + E \subset E$ (see [A-P, p.23]). Hence,

$$E \text{ is a union of cosets of } G \text{ by } 2G.$$

Now we can state the classification of root systems $\{S_\alpha\}_{\alpha \in \Delta}$ of type Δ extended by G [Y, Thm 3.4]:

Set $S_\alpha = S$ for all $\alpha \in \Delta_{sh}$, $S_\alpha = L$ for all $\alpha \in \Delta_{lg}$ and $S_\alpha = E$ for all $\alpha \in \Delta_{ex}$, where S is a full pointed reflection space, L is a pointed reflection space and E is a reflection space satisfying

$$L + kS \subset L, \quad S + L \subset S, \quad E + 4S \subset E,$$
$$S + E \subset S, \quad E + 2L \subset E \quad \text{and} \quad L + E \subset L;$$
$$\text{moreover,} \quad S = G \quad \text{if } \Delta \neq \mathrm{A}_1, \text{ B}_\ell, \text{ BC}_\ell,$$
$$L \text{ is a subgroup if } \Delta = \mathrm{B}_\ell \ (\ell \geq 3), \text{ F}_4, \text{ G}_2, \text{ BC}_\ell \ (\ell \geq 3),$$

and if $\{S_\alpha\}_{\alpha \in \Delta}$ is reduced, then

$$E \cap 2S = \emptyset.$$

Conversely, let S, L and E be as above, and define $S_\alpha = S$ for all $\alpha \in \Delta_{sh}$, $S_\alpha = L$ for all $\alpha \in \Delta_{lg}$ and $S_\alpha = E$ for all $\alpha \in \Delta_{ex}$. Then $\{S_\alpha\}_{\alpha \in \Delta}$ is a root system extended by G, and if $E \cap 2S = \emptyset$, then $\{S_\alpha\}_{\alpha \in \Delta}$ is a reduced root system extended by G. We refer to the root system $\{S_\alpha\}_{\alpha \in \Delta}$ by $\mathfrak{R}(S, L, E)_\Delta$.

For the case where Δ is a locally finite irreducible root system, one can classify $\{S_\alpha\}_{\alpha \in \Delta}$ satisfying (S1) and (S2) in the same way. In fact they were classified in [LN2, 5.9] as extension data of locally finite root systems. One can also obtain the classification from the fact that $\{S_\alpha\}_{\alpha \in \Delta}$ is a directed union $\bigcup_{\Delta'} \{S_\alpha\}_{\alpha \in \Delta'}$, where Δ' is a finite irreducible subsystem of Δ (see Proposition 3). Thus the properties for S_α of each infinite type $\mathrm{A}_{\mathfrak{I}}$, $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{C}_{\mathfrak{I}}$, $\mathrm{D}_{\mathfrak{I}}$, or $\mathrm{BC}_{\mathfrak{I}}$ are the same as of finite type A_2, B_3, C_3, D_4, or BC_3, respectively.

We note that $E \subset L \subset S$ in general, and so S spans V^0 by our extra condition (S0). Moreover, from the relations $L + kS \subset L$ and $E + 4S \subset E$, L or E also spans V^0 if it is not empty. Thus, the following is known:

7

Theorem 7. *Let $\mathfrak{R}$ be a LEARS in $V = V' \oplus V^0$ so that Δ is a locally finite irreducible root system in V', described above. Then $E \subset L \subset S$, $\langle S \rangle = G$, S always spans V^0, and L or E also spans V^0 if it is not empty. Moreover:*

If $\Delta = A_{\mathfrak{I}}$, then $\mathfrak{R} = \Delta + S$, where S is a pointed reflection space of V^0, and if $A_{\mathfrak{I}} \neq A_1$, then $S = G$.

If $\Delta = B_{\mathfrak{I}}$, then $\mathfrak{R} = (\Delta_{sh} + S) \sqcup (\Delta_{lg} + L)$, where S and L are pointed reflection spaces of V^0 satisfying $2S + L \subset L$ and $S + L \subset S$, and if $|\mathfrak{I}| > 2$, then L is a subgroup of V^0.

If $\Delta = C_{\mathfrak{I}}$, then $\mathfrak{R} = (\Delta_{sh} + S) \sqcup (\Delta_{lg} + L)$, where S and L are pointed reflection spaces of V^0 satisfying $2S + L \subset L$ and $S + L \subset S$, and if $|\mathfrak{I}| > 2$, then $S = G$.

If $\Delta = D_{\mathfrak{I}}$, E_6, E_7 or E_8, then $\mathfrak{R} = \Delta + G$.

If $\Delta = BC_{\mathfrak{I}}$ for $|\mathfrak{I}| \geq 2$, then $\mathfrak{R} = (\Delta_{sh} + S) \sqcup (\Delta_{lg} + L) \sqcup (\Delta_{ex} + E)$, where S and L are pointed reflection spaces of V^0 and E is a reflection space of V^0 satisfying $2S + L \subset L$, $S + L \subset S$, $4S + E \subset E$, $S + E \subset S$, $2L + E \subset E$ and $L + E \subset L$, and if $|\mathfrak{I}| > 2$, then L is a subgroup of V^0. Also, if $\mathfrak{R}$ is reduced, then $E \cap 2S = \emptyset$.

If $\Delta = BC_1$, then $\mathfrak{R} = (\Delta_{sh} + S) \sqcup (\Delta_{ex} + E)$, where S is a pointed reflection space of V^0 and E is a reflection space of V^0 satisfying $4S + E \subset E$ and $S + E \subset S$. Also, if $\mathfrak{R}$ is reduced, then $E \cap 2S = \emptyset$.

If $\Delta = F_4$, then $\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{lg} + L)$, where L is a subgroup of V^0 satisfying $2G \subset L$.

If $\Delta = G_2$, then $\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{lg} + L)$, where L is a subgroup of V^0 satisfying $3G \subset L$.

Conversely, each set $\mathfrak{R}$ defined above is a LEARS of the specified type (see the paragraph right before Proposition 3).

The reader should always keep in mind that even if a LEARS $\mathfrak{R}$ is reduced, the corresponding finite root system $\bar{\mathfrak{R}}$ or Δ could be nonreduced.

<div align="center">Isomorphisms</div>

By Theorem 7, the classification of LEARS is reduced to the classification of triples $\{S, L, E\}$ described there. We simply say triples, but they might be $\{S\}$, $\{S, L\}$ or $\{S, E\}$ depending on the types. We treat these cases as special cases of triples, and we do not mention this in the argument below. The reader should ignore the description of L or E if the system does not have L or E, i.e., the case $\Delta_{lg} = \emptyset$ or $\Delta_{ex} = \emptyset$. To investigate when two triples give the same LEARS, we show the following: (There is a similar statement in [A1, p.577] for EARS of reduced type.)

Lemma 8. *In the description of Theorem 7, let $s \in S$ and $l \in L$. Then the triples $\{S, L, E\}$ and $\{S - s, L - l, E - 2s\}$ give the same LEARS (by the same Δ in Theorem 7).*

Conversely, let $\{S_1, L_1, E_1\}$ be another triple obtained from a reflectable subspace W of an arbitrary reflectable base. Then, $S_1 = S - s$, $L_1 = L - l$ and $E_1 = E - 2s$ for some $s \in S$ and $l \in L$.

Proof. Recall from the previous section that for each $\bar{\alpha} \in \Pi$ (a reflectable base of $\bar{\mathfrak{R}}$), we have considered a fixed preimage $\alpha \in \mathfrak{R}$. For each $\bar{\alpha} \in \Pi \cap \bar{\mathfrak{R}}_{sh}$, let $\alpha' := \alpha + s$, and for each $\bar{\alpha} \in \Pi \cap \bar{\mathfrak{R}}_{lg}$, let $\alpha' := \alpha + l$. Let U be the subspace of V spanned by $\{\alpha'\}_{\bar{\alpha} \in \Pi}$. In other words, U is another reflectable subspace. Then the new family $\{T_{\alpha'}\}_{\alpha' \in \Delta_U}$ is a root system extended by G, which gives the same LEARS. In particular, $\alpha + s + T_{\alpha'} = \alpha + S$ and $\alpha + l + T_{\alpha'} = \alpha + L$. Thus $T_{\alpha'} = S - s$ if $\bar{\alpha} \in \Pi \cap \bar{\mathfrak{R}}_{sh}$ and $T_{\alpha'} = L - l$ if $\bar{\alpha} \in \Pi \cap \bar{\mathfrak{R}}_{lg}$. Hence, by Theorem 7, we have $T_{\alpha'} = S - s$ for all $\alpha' \in (\Delta_U)_{sh}$ and $T_{\alpha'} = L - l$ for $\alpha' \in (\Delta_U)_{lg}$.

Finally (the case $\Delta_{ex} \neq \emptyset$), for $\bar{\alpha} \in \Pi \cap \bar{\mathfrak{R}}_{sh}$, we have $\alpha' - \alpha = s$, and so $2\alpha' - 2\alpha = 2s$. Since $2\alpha' + T_{2\alpha'} = 2\alpha + E$, we get $T_{2\alpha'} = 2\alpha - 2\alpha' + E = E - 2s$. Thus, by Theorem 7, $T_{2\alpha'} = E - 2s$ for all $\bar{\alpha}, 2\bar{\alpha} \in \bar{\mathfrak{R}}$.

For the second statement, let us remind the reader that the reflectable subspace U determines another root system $\{T_{\alpha'}\}_{\alpha' \in \Delta_U}$ extended by $G' = \langle \cup_{\alpha' \in \Delta_U} T_{\alpha'} \rangle$, as in Remark 6. Then by Theorem 7, the system $\{T_{\alpha'}\}_{\alpha' \in \Delta_U}$ turns out to be just a triple, that is $\{S_1, L_1, E_1\}$ in our assumption. (In particular, $G' = \langle S_1 \rangle$.) Now, for $\alpha' \in (\Delta_U)_{sh}$, there exists $\alpha \in \Delta_{sh}$ such that $\overline{\alpha'} = \bar{\alpha}$. Thus $\alpha' = \alpha + s$ for some $s \in S$. Hence, $S_1 = T_{\alpha'} = S - s$. By the same argument, we get $L_1 = L - l$ for some $l \in L$. Then, by the same argument above, we obtain $E' = E - 2s$. (It is enough that one of the short α''s satisfies $T_{\alpha'} = S - s$ and one of the long α''s satisfies $T_{\alpha'} = L - l$.) In particular, $G' = G$. $\square$

Two LEARS $(\mathfrak{R}, V)$ and $(\mathfrak{S}, W)$ are called *isomorphic* if there exists a linear isomorphism $\varphi : V \longrightarrow W$ such that $\varphi(\mathfrak{R}) = \mathfrak{S}$.

The argument to show that $\varphi(V^0) = W^0$ in the following lemma is adapted from [AY, Lemma 3.1].

Lemma 9. *Suppose that two LEARS are isomorphic, say $\varphi : (\mathfrak{R}, V) \xrightarrow{\sim} (\mathfrak{S}, W)$. Then $\varphi(V^0) = W^0$ and $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \mathfrak{R}$. Thus φ preserves the form up to nonzero scalar. Also, $\varphi \circ \sigma_\alpha \circ \varphi^{-1} = \sigma_{\varphi(\alpha)}$ for all $\alpha \in \mathfrak{R}$.*

Proof. We first show that $\varphi(V^0) = W^0$. Let S and Δ be as in Theorem 7. Since S spans V^0, it is enough to show that $s \in S \Rightarrow s' := \varphi(s) \in W^0$, or equivalently $(s', s') = 0$. Since $S \pm 2S \subset S$, we have $ns \in S$ for all $n \in \mathbb{Z}$. Let $\alpha \in \Delta_{sh}$ and $\alpha' := \varphi(\alpha)$. By Theorem 7, $\alpha + ns \in \mathfrak{R}$ for all $n \in \mathbb{Z}$ and so $\alpha' + ns' = \varphi(\alpha + ns) \in \varphi(\mathfrak{R}) = \mathfrak{S}$. But then by the axiom (A2) of the definition of a LEARS, we have

$$\langle \alpha', \alpha' + ns' \rangle = \frac{2(\alpha', \alpha' + ns')}{(\alpha' + ns', \alpha' + ns')} = \frac{2(\alpha', \alpha') + 2n(\alpha', s')}{(\alpha', \alpha') + 2n(\alpha', s') + n^2(s', s')} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}$ which implies $(s', s') = 0$ (note that $(\alpha', \alpha') \neq 0$ and let $n \to \infty$). Thus we have shown that $\varphi(V^0) = W^0$. Then φ induces a linear isomorphism $\bar{\varphi} : \bar{V} \xrightarrow{\sim} \bar{W}$ with $\bar{\varphi}(\bar{\mathfrak{R}}) = \bar{\mathfrak{S}}$, and this is what means an isomorphism of locally finite root systems in [LN1]. Thus, by [LN1, Lem.3.7], we have $\langle \bar{\varphi}(\bar{\alpha}), \bar{\varphi}(\bar{\beta}) \rangle = \langle \bar{\alpha}, \bar{\beta} \rangle$ for all $\bar{\alpha}, \bar{\beta} \in \bar{\mathfrak{R}}$. So the second statement is shown since we always have $(\alpha, \beta) = (\bar{\alpha}, \bar{\beta})$. The third statement follows from the equivalence between connectedness and irreducibility in our systems (see [LN2, Lem.2.7] or [MP, Prop.3.4.6]). The last statement is now clear. $\square$

We introduce a notion of similarity for triples following [A-P].

Let (S_1, L_1, E_1) and (S_2, L_2, E_2) be two triples satisfying the properties in Theorem 7 in vector spaces W_1 and W_2, respectively. We say that (S_1, L_1, E_1) and (S_2, L_2, E_2) are *similar*, denoted $(S_1, L_1, E_1) \sim (S_2, L_2, E_2)$, if there exists an isomorphism φ from W_1 onto W_2 such that $\varphi(S_1) = S_2 - s_2$, $\varphi(L_1) = L_2 - l_2$ and $\varphi(E_1) = E_2 - 2s_2$ for some $s_2 \in S_2$ and $l_2 \in L_2$. The similarity is an equivalence relation.

The following theorem says that there is a 1-1 correspondence between the isomorphism classes of LEARS and the similarity classes of triples. The theorem generalizes [A-P, Thm 3.1] and our proof is simpler.

Theorem 10. *Suppose that $\varphi : (\mathfrak{R}_1, V_1; V_1', \Delta_1; S_1, L_1, E_1) \xrightarrow{\sim} (\mathfrak{R}_2, V_2; V_2', \Delta_2; S_2, L_2, E_2)$ is an isomorphism of LEARS. Let*

$$\zeta := (\text{projection onto } V_2') \circ \varphi \mid_{V_1'} \quad \text{and} \quad \psi := (\text{projection onto } V_2^0) \circ \varphi \mid_{V_1'}.$$

Then $\zeta : (\Delta_1, V_1') \xrightarrow{\sim} (\Delta_2, V_2')$ and $\varphi(S_1) = S_2 - s_2$, $\varphi(L_1) = L_2 - l_2$ and $\varphi(E_1) = E_2 - 2s_2$ for some $s_2 \in S_2$ and $l_2 \in L_2$.

Conversely, if $\zeta : (\Delta_1, V_1) \xrightarrow{\sim} (\Delta_2, V_2)$ is an isomorphism of locally finite irreducible root systems, two triples (S_1, L_1, E_1) in a vector space W_1 and (S_2, L_2, E_2) in a vector space W_2 satisfy the conditions in Theorem 7 depending on the type of Δ_1, and φ is an isomorphism from W_1 onto W_2 so that $\varphi(S_1) = S_2 - s_2$, $\varphi(L_1) = L_2 - l_2$ and $\varphi(E_1) = E_2 - 2s_2$ for some $s_2 \in S_2$ and $l_2 \in L_2$, then $(\mathfrak{R}(S_1, L_1, E_1), V_1 \oplus W_1)$ is isomorphic to $(\mathfrak{R}(S_2, L_2, E_2), V_2 \oplus W_2)$.

Proof. We have $\zeta(\Delta_1), \Delta_2 \subset V_2'$ and so $\overline{\zeta(\Delta_1)} = \bar{\mathfrak{R}}_2 = \bar{\Delta}_2$ since $\varphi(V_1^0) = V_2^0$ (Lemma 9). Hence $\zeta(\Delta_1) = \Delta_2$, and so ζ is an isomorphism of the root systems. Also, for a fixed $\alpha \in (\Delta_1)_{sh}$, we have $\varphi(\alpha + S_1) = \zeta(\alpha) + \psi(\alpha) + \varphi(S_1) \subset \mathfrak{R}_2$, and so $\psi(\alpha) + \varphi(S_1) = S_2$ since $\zeta(\alpha) \in (\Delta_2)_{sh}$. Also, $s_2 := \psi(\alpha) \in S_2$ since $0 \in \varphi(S_1)$. Similarly, for a fixed $\beta \in (\Delta_1)_{lg}$, we get $l_2 + \varphi(L_1) = L_2$ for $l_2 := \psi(\beta) \in L_2$. Finally, if $2\alpha \in \Delta_1$, then $\varphi(2\alpha + E_1) = \zeta(2\alpha) + 2\psi(\alpha) + \varphi(E_1)$, and so $2s_2 + \varphi(E_1) = E_2$.

For the second statement, let $\tilde{\varphi} = \zeta \oplus \varphi$. Then

$$\tilde{\varphi} : (\mathfrak{R}(S_1, L_1, E_1), V_1 \oplus W_1) \xrightarrow{\sim} (\mathfrak{R}(\varphi(S_1), \varphi(L_1), \varphi(E_1)), V_2 \oplus W_2)$$
$$= (\mathfrak{R}(S_2 - s_2, L_2 - l_2, E_2 - 2s_2), V_2 \oplus W_2)$$
$$= (\mathfrak{R}(S_2, L_2, E_2), V_2 \oplus W_2) \quad \text{by Lemma 8.} \quad \square$$

Remark 11. If two LEARS $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are isomorphic, then $\langle S_1 \rangle / \langle L_1 \rangle$ and $\langle S_2 \rangle / \langle L_2 \rangle$ are clearly isomorphic as abelian groups. Also the reducibility of LEARS is an isomorphic invariant.

<center>SPECIAL CASE</center>

We consider LEARS of null dimension 1. Then the abelian group G in Theorem 7 is just a subgroup of $\mathbb{Q}$. We first observe special properties for a cyclic group or a subgroup of $\mathbb{Q}$. Let us recall the concept of divisibility for an arbitrary abelian group G. We say that a prime number p *is divisible in G* or G *is divisible by p* if $G = pG$, or equivalently $px = g$ has a solution x in G for any $g \in G$. Any cyclic group of infinite order is not divisible by any prime. The following is a useful exercise ([G, p.8]):

($*$) If $mx = ng$ for $(m, n) = 1$ has a solution x in G, then $my = g$ has a solution y in G.

Lemma 12. *(1) If S is a full reflection space of a cyclic group G, then $S = G$ or $S = 2G + s$ for any $s \in G \setminus 2G$. So if S is a full pointed reflection space of a cyclic group G, then $S = G$.*

(2) Suppose that G is a subgroup of $\mathbb{Q}$. If G is not divisible by a prime p, then $G / p^n G \cong \mathbb{Z}_{p^n}$ for any $n \in \mathbb{N}$. Moreover, if $G / H \cong \mathbb{Z}_{p^n}$ for some subgroup H of G and some $n \in \mathbb{N}$, then G is not divisible by p and $H = p^n G$.

(3) If S is a full reflection space of a subgroup G of $\mathbb{Q}$ divisible by 2, then $S = G$.

(4) The same statement in (1) is true for a subgroup G of $\mathbb{Q}$ not divisible by 2.

Proof. For (1), we have $G = 2G \sqcup (2G + s)$ for any $s \in G \setminus 2G$ if $G \neq 2G$. (Note that G is finite of odd order $\Leftrightarrow G = 2G$.) Since S is full, S is a union of cosets of G by $2G$, and $S \neq 2G$ if $G \neq 2G$. So $S = G$ or $S = 2G + s$. Moreover, $2G \subset S$ if $0 \in S$, and hence (1) is proved.

For (2), by the divisibility, there exists $g \in G \setminus pG$. We claim that $0, g, 2g, \ldots, (p^n - 1)g$ are distinct modulo $p^n G$. (G can be any torsion free group for this claim.) Suppose that

<center>10</center>

two of them are equal. Then $p^r q g = p^n g'$ for some $r < n$, $(p, q) = 1$ and $g' \in G$. Hence $p^{n-r} g' = qg$ (since G is torsion free). Then by $(*)$ above, $py = g$ has a solution y in G, which contradicts our choice of g. Thus we showed the claim, and the order of g in $G/p^n G$ is p^n.

Let $g' \in G$. Since $G' := \langle g, g' \rangle$ is cyclic, we have $G'/p^n G' \cong \mathbb{Z}_{p^n}$. Hence, $G'/p^n G' = \langle g \rangle$, and g' is equal to one of $g, 2g, \dots, (p^n - 1)g$ or $p^n g$ modulo $p^n G'$, and so is in the modulo $p^n G$ since $p^n G' \subset p^n G$. Hence $G/p^n G = \{0, g, 2g, \dots, (p^n - 1)g\}$, which is a cyclic group with $p^n g = 0$. But since the order of g in $G/p^n G$ is p^n, we obtain $G/p^n G \cong \mathbb{Z}_{p^n}$.

For the second statement, if G is divisible by p, then for any $g \in G$, $g = p^n g'$ for some $g' \in G$. So for any $\bar{g} \in G/H$, $\bar{g} = \overline{p^n g'} = \bar{0}$, which means $G/H = 0$, contradiction. Hence, G is not divisible by p. Thus by the first statement, we have $G/p^n G \cong \mathbb{Z}_{p^n}$. Note that $G/H \cong \mathbb{Z}_{p^n}$ implies $p^n G \subset H$. So there is a natural epimorphism π from $G/p^n G$ onto G/H. But the order of both groups is p^n, and hence π is an isomorphism and $p^n G = H$.

For (3), we have $G = G + S = 2G + S \subset S$, and hence $G = S$.

For (4), applying (2) for $p = 2$, we have $G = 2G \sqcup (2G + s)$ for any $s \in G \backslash 2G$. Thus we are done. (We note that since $s \in \langle 2G + s \rangle$, we have $2G \subset \langle 2G + s \rangle$, and hence $\langle 2G + s \rangle = G$.) $\square$

We will use the special cases of Lemma 12(2) later, namely $p = 2$ or $p = 3$ for $n = 1$. Note that this is a special property of subgroups of $\mathbb{Q}$. For example, if $G = \langle \sqrt{2}, \sqrt{3} \rangle \subset \mathbb{R}$, then $G/2G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Corollary 13. *Let $\mathfrak{R}$ be a LEARS of null dimension 1 in $V = V' \oplus V^0$, Δ a locally finite irreducible root system in V', and G a subgroup in $V^0 = \mathbb{Q}$, as described above.*
(1a) For the case where $\frac{1}{2} \notin G$:
If $\Delta = A_{\mathfrak{J}}$, $D_{\mathfrak{J}}$, E_6, E_7 or E_8 then $\mathfrak{R} = \Delta + G$.
If $\Delta = B_{\mathfrak{J}}$, $C_{\mathfrak{J}}$ or F_4, then $\mathfrak{R} = \Delta + G$ or $\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{lg} + 2G)$.
If $\Delta = G_2$, then $\mathfrak{R} = \Delta + G$ or $\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{lg} + 3G)$.
If $\Delta = BC_{\mathfrak{J}}$ for $|\mathfrak{J}| > 1$, then

$$\mathfrak{R} = \Delta + G,$$
$$\mathfrak{R} = ((\Delta_{sh} \sqcup \Delta_{lg}) + G) \sqcup (\Delta_{ex} + 2G),$$
$$\mathfrak{R} = (\Delta_{sh} + G) \sqcup ((\Delta_{lg} \sqcup \Delta_{ex}) + 2G),$$
$$\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{lg} + 2G) \sqcup (\Delta_{ex} + 4G) \quad or$$
$$\mathfrak{R} = ((\Delta_{sh} \sqcup \Delta_{lg}) + G) \sqcup (\Delta_{ex} + 2G + s) \quad for \ any \ s \in G \backslash 2G,$$

and moreover, if $\mathfrak{R}$ is reduced, then only the last case happens.
If $\Delta = BC_1$, then

$$\mathfrak{R} = \Delta + G,$$
$$\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{ex} + 2G),$$
$$\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{ex} + 4G) \quad or$$
$$\mathfrak{R} = (\Delta_{sh} + G) \sqcup (\Delta_{ex} + 2G + s) \quad for \ any \ s \in G \backslash 2G,$$

and moreover, if $\mathfrak{R}$ is reduced, then only the last case happens.
(1b) If G is divisible by 2, then $\mathfrak{R} = \Delta + G$ in any type of Δ.
(2) If $\mathfrak{R}$ is a LARS, then $\mathfrak{R}$ has the same description as in (1a) by changing G into $\mathbb{Z}s$, where $s \in G$ so that $G = \mathbb{Z}s$.

Proof. First of all, note that all the LEARS in the list above are not isomorphic by Remark 11. Also, by Lemma 12, we always have $G = S$ and L is a group since S is a full pointed reflection space of $G \subset \mathbb{Q}$ and L is a full pointed reflection space of $\langle L \rangle \subset \mathbb{Q}$, and hence $2G \subset L$ (or $3G \subset L$ for type G_2) by Theorem 7. But then by Lemma 12, $L = 2G$ or G ($L = 3G$ or G for type G_2). So we are done except for the type $BC_{\mathfrak{I}}$.

Now for $|\mathfrak{I}| > 1$, if $L = G$, then $2G + E \subset E$, and so E is a union of cosets of G by $2G$. Hence, $E = 2G$, G or $2G + s$ for any $s \in G \setminus 2G$.

If $L = 2G$, then $E \subset 2G$. So we have $4G + E \subset E \subset 2G$, and hence E is a union of cosets of $2G$ by $4G$. Hence, $E = 4G$, $2G$ or $4G + g$ for any $g \in 2G \setminus 4G$, and $4G + g = 4G + 2s$ for any $s \in G \setminus 2G$. But $4G + 2s$ is excluded since $(G, 2G, 4G)$ and $(G, 2G, 4G + 2s)$ are similar. Also, $E = 2G + s$ is the only reduced one since others do not satisfy $E \cap 2S = \emptyset$.

For the type BC_1, we only have $4G + E \subset E$, and so E is a union of cosets of G by $4G$. By Lemma 12, we have $G = 4G \sqcup (4G + s) \sqcup (4G + 2s) \sqcup (4G + 3s)$ for any $s \in G \setminus 2G$. Note that $4G + s \subset E \Leftrightarrow 4G + 3s \subset E$ since $2E + E \subset E$. Also, $4G, 4G + s \subset E \Rightarrow E = G$. Hence, $E = 4G$, $4G + s$, $4G + 2s$, $4G \sqcup (4G + 2s) = 2G$ or G. But if $E = 4G + s$, then $0 \notin E$ and $\langle E \rangle = G$, and so $E = 2G + s$. As above, $4G + 2s$ is excluded since $(G, 4G)$ and $(G, 4G + 2s)$ are similar, and $E = 2G + s$ is the only reduced one since others do not satisfy $E \cap 2S = \emptyset$.

(1b): We have $S = G$ by Lemma 12(3). Moreover, by Theorem 7, we have $L \supset 2S + L = 2G + L = G + L = G$ and $E \supset 2L + E = 2G + E = G + E = G$, and hence $G = S = L = E$ (if L or E is empty). This shows (1b).

For (2), we have $\langle \mathfrak{R} \rangle \cap V^0 = \langle S \rangle$ has rank 1, and so there exists $s \in S$ so that $S = \mathbb{Z}s = G$ (see Lemma 12(1)). $\square$

Remark 14. (1) Nonreduced EARS of nullity 1, 2 and 3 were already classified in [AKY].

(2) Note that a free abelian group is not divisible by any p. An example of a subgroup of $\mathbb{Q}$ not divisible by p, which is not free, is the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ by the prime ideal $(p) = p\mathbb{Z}$. Also, $\mathbb{Z}[\frac{1}{q}] = \langle \frac{1}{q^n} \mid n \in \mathbb{N} \rangle$ for any prime q different from p is another example of a subgroup of $\mathbb{Q}$ not divisible by p, which is not free. Note that $\mathbb{Z}[\frac{1}{q}] \subset \mathbb{Z}_{(p)}$ and that $\mathbb{Z}_{(p)}$ and $\mathbb{Z}[\frac{1}{q}]$ are not just subgroups but subrings of $\mathbb{Q}$. There are some examples which are not subrings. For example, $\mathbb{Z}_{(p)} + \langle \frac{1}{p^n} \rangle$ is neither divisible by p nor a subring of $\mathbb{Q}$ (nor free). Note that $\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} + \langle \frac{1}{p} \rangle \subset \mathbb{Z}_{(p)} + \langle \frac{1}{p^2} \rangle \subset \cdots$. Also, $\langle \frac{1}{p_1}, \frac{1}{p_2}, \ldots \rangle$ for any infinite series of distinct primes $p_1, p_2, \ldots$ is an example of a subgroup of $\mathbb{Q}$ not divisible by p and not a subring of $\mathbb{Q}$ (and not free, even if one of the p_i's is equal to p). Note that the torsion-free abelian groups of rank 1 were classified (but not for rank > 1).

We note that there are 14 reduced irreducible affine root systems, i.e., $A_\ell^{(1)}$, $B_\ell^{(1)}$, $B_\ell^{(2)}$, $C_\ell^{(1)}$, $C_\ell^{(2)}$, $D_\ell^{(1)}$, $BC_\ell^{(2)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $F_4^{(2)}$, $G_2^{(1)}$ and $G_2^{(3)}$, by Moody's Label, and correspondingly there are 14 affine Lie algebras. It is worth mentioning that there are 14 reduced LARS from Corollary 13, and they are obtained by just changing ℓ of the first 7 above into an infinite index set $\mathfrak{I}$. For the convenience of the reader, we summarize this remark with the above label, denoting the specific type instead of Δ and identifying $\mathbb{Z}s$ with $\mathbb{Z}$ in Corollary 13:

Corollary 15. *There are only seven reduced LARS of infinite rank. Namely,*

$$A_{\mathfrak{I}}^{(1)} = A_{\mathfrak{I}} + \mathbb{Z},$$

$$B_{\mathfrak{I}}^{(1)} = B_{\mathfrak{I}} + \mathbb{Z},$$

$$C_{\mathfrak{I}}^{(1)} = C_{\mathfrak{I}} + \mathbb{Z},$$

$$D_{\mathfrak{I}}^{(1)} = D_{\mathfrak{I}} + \mathbb{Z},$$

$$B_{\mathfrak{I}}^{(2)} = \left((B_{\mathfrak{I}})_{sh} + \mathbb{Z}\right) \sqcup \left((B_{\mathfrak{I}})_{lg} + 2\mathbb{Z}\right),$$

$$C_{\mathfrak{I}}^{(2)} = \left((C_{\mathfrak{I}})_{sh} + \mathbb{Z}\right) \sqcup \left((C_{\mathfrak{I}})_{lg} + 2\mathbb{Z}\right) \quad and$$

$$BC_{\mathfrak{I}}^{(2)} = \left(\left((BC_{\mathfrak{I}})_{sh} \sqcup (BC_{\mathfrak{I}})_{lg}\right) + \mathbb{Z}\right) \sqcup \left((BC_{\mathfrak{I}})_{ex} + 2\mathbb{Z} + 1\right).$$

<center>LOCALLY (G, τ)-LOOP ALGEBRAS</center>

We give examples of Lie algebras whose root systems are LEARS of null dimension 1. All algebras and tensors are over a field F of characteristic 0. Let $\mathfrak{I}$ be any index set. The locally finite split simple Lie algebra of type $X_{\mathfrak{I}}$ (introduced in [NS]) is defined as a subalgebra of the matrix algebra $gl_{\mathfrak{I}}(F)$, $gl_{2\mathfrak{I}+1}(F)$ or $gl_{2\mathfrak{I}}(F)$ consisting of matrices having only a finite number of nonzero entries: (There is a more general construction in [N].)

Type $A_{\mathfrak{I}}$; $sl_{\mathfrak{I}}(F) = \{x \in gl_{\mathfrak{I}}(F) \mid \mathrm{tr}(x) = 0\}$;

Type $B_{\mathfrak{I}}$; $o_{2\mathfrak{I}+1}(F) = \{x \in gl_{2\mathfrak{I}+1}(F) \mid sx = -x^t s\}$;

Type $C_{\mathfrak{I}}$; $sp_{2\mathfrak{I}}(F) = \{x \in gl_{2\mathfrak{I}}(F) \mid s_- x = -x^t s_-\}$;

Type $D_{\mathfrak{I}}$; $o_{2\mathfrak{I}}(F) = \{x \in gl_{2\mathfrak{I}}(F) \mid s_+ x = -x^t s_+\}$;

where x^t is the transpose of x,

$$s = \begin{pmatrix} 0 & I_{\mathfrak{I}} & 0 \\ I_{\mathfrak{I}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_- = \begin{pmatrix} 0 & I_{\mathfrak{I}} \\ -I_{\mathfrak{I}} & 0 \end{pmatrix} \quad or \quad s_+ = \begin{pmatrix} 0 & I_{\mathfrak{I}} \\ I_{\mathfrak{I}} & 0 \end{pmatrix},$$

and $I_{\mathfrak{I}}$ is the identity matrix of size $\mathfrak{I}$. (Each Lie algebra of type $X_{\mathfrak{I}}$ has the locally finite irreducible root system of type $X_{\mathfrak{I}}$ [NS].)

Let $G = (G, +, 0)$ be an abelian group. Let

$$F^\tau[G] = F^\tau[G, t] = \bigoplus_{g \in G} F t^g$$

be a twisted commutative group algebra of G with symmetric twisting $\tau : G \times G \longrightarrow F^\times$, i.e.,

$$\tau(g, h) = \tau(h, g) \quad and \quad \tau(g + h, k)\tau(g, h) = \tau(g, h + k)\tau(h, k)$$

so that

$$t^g t^h = \tau(g, h) t^{g+h}$$

for all $g, h, k \in G$. We call the following four Lie algebras *locally untwisted (G, τ)-loop algebras*, and *untwisted (G, τ)-loop algebras* if $\mathfrak{I}$ is finite.

Type $A_{\mathfrak{I}}^{(1)}$; $sl_{\mathfrak{I}}(F) \otimes F^\tau[G]$;

Type $B_{\mathfrak{I}}^{(1)}$; $o_{2\mathfrak{I}+1}(F) \otimes F^\tau[G]$;

<center>13</center>

Type $C_{\mathfrak{J}}^{(1)}$; $sp_{2\mathfrak{J}}(F) \otimes F^\tau[G]$;

Type $D_{\mathfrak{J}}^{(1)}$; $o_{2\mathfrak{J}}(F) \otimes F^\tau[G]$.

Also, for each finite-dimensional split simple Lie algebra $\mathfrak{g}$ over F of type E_6, E_7, E_8, F_4 or G_2, we call the Lie algebra $\mathfrak{g} \otimes F^\tau[G]$ an *untwisted (G,τ)-loop algebra* of type $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$ or $G_2^{(1)}$.

If there exists a subgroup G' so that $G/G' \cong \mathbb{Z}_2$, then $G = G' \sqcup (G' + g_1)$ for any $g_1 \in G \setminus G'$, and so $F^\tau[G] = F^\tau[G'] \oplus t^{g_1} F^\tau[G']$. (For example, take any subgroup G of $\mathbb{Q}$ which is not divisible by 2, and $G' := 2G$, by Lemma 12(2).) In this case we call the following three Lie algebras *locally twisted (G,τ)-loop algebras*, and *twisted (G,τ)-loop algebras* if $\mathfrak{J}$ is finite. (There is a way to construct by Kac, using an automorphism of a Lie algebra in [K, Ch.8]. But we chose the following way by [BZ] and [ABG] since this construction can be generalized to nonassociative coordinates and is simpler.)

(1) Type $B_{\mathfrak{J}}^{(2)}$; $(o_{2\mathfrak{J}+1}(F) \otimes F^\tau[G']) \oplus (V \otimes t^{g_1} F^\tau[G'])$,
where $V = F^{(2\mathfrak{J}+1)}$ is the natural $o_{2\mathfrak{J}+1}(F)$-module;

(2) Type $C_{\mathfrak{J}}^{(2)}$; $(sp_{2\mathfrak{J}}(F) \otimes F^\tau[G']) \oplus (\mathfrak{s}_- \otimes t^{g_1} F^\tau[G'])$,
where $\mathfrak{s}_- = \{x \in sl_{2\mathfrak{J}}(F) \mid s_- x = x^t s_-\}$;

(3) Type $BC_{\mathfrak{J}}$; $(o_{2\mathfrak{J}+1}(F) \otimes F^\tau[G']) \oplus (\mathfrak{s} \otimes t^{g_1} F^\tau[G'])$,
where $\mathfrak{s} = \{x \in sl_{2\mathfrak{J}+1}(F) \mid sx = x^t s\}$.

Note that $sl_{2\mathfrak{J}}(F) = sp_{2\mathfrak{J}}(F) \oplus \mathfrak{s}_-$ and $sl_{2\mathfrak{J}+1}(F) = o_{2\mathfrak{J}+1}(F) \oplus \mathfrak{s}$.

The Lie bracket of each untwisted type is natural, i.e., $[x \otimes t^g, y \otimes t^h] = [x,y] \otimes \tau(g,h)t^{g+h}$. The Lie bracket of type $C_{\mathfrak{J}}^{(2)}$ or $BC_{\mathfrak{J}}$ is also natural since

$$[sp_{2\mathfrak{J}}(F), \mathfrak{s}_-] \subset \mathfrak{s}_-, \quad [\mathfrak{s}_-, \mathfrak{s}_-] \subset sp_{2\mathfrak{J}}(F),$$
$$[o_{2\mathfrak{J}+1}(F), \mathfrak{s}] \subset \mathfrak{s} \quad \text{and} \quad [\mathfrak{s}, \mathfrak{s}] \subset o_{2\mathfrak{J}+1}(F).$$

Note that $C_{\mathfrak{J}}^{(2)}$ is a subalgebra of $sl_{2\mathfrak{J}}(F) \otimes F^\tau[G]$, and $BC_{\mathfrak{J}}$ is a subalgebra of $sl_{2\mathfrak{J}+1}(F) \otimes F^\tau[G]$.

For $B_{\mathfrak{J}}^{(2)}$, we have $o_{2\mathfrak{J}+1}(F)V \subset V$, and so we define the bracket of $o_{2\mathfrak{J}+1}(F)$ and V by the natural action, i.e., $[x,v] = xv = -[v,x]$ for $x \in o_{2\mathfrak{J}+1}(F)$ and $v \in V$. However, there is no bracket on V. So we define a bracket on V so that $[V,V] \subset o_{2\mathfrak{J}+1}(F)$ as follows. First, let $(\cdot, \cdot)$ be the bilinear form on V determined by s. Then there is a natural identification

$$o_{2\mathfrak{J}+1}(F) = D_{V,V} := \operatorname{span}_F\{D_{v,v'} \mid v,v' \in V\},$$

where $D_{v,v'} \in \operatorname{End}(V)$ is defined by $D_{v,v'}(v'') = (v',v'')v - (v,v'')v'$ for $v'' \in V$. Thus we define $[v,v'] := D_{v,v'}$. Note that $[v',v] = -[v,v']$. It is easy to check that the bracket

$$[x \otimes t^g + v \otimes t^{g'+g_1}, x' \otimes t^h + v' \otimes t^{h'+g_1}]$$
$$= [x,x'] \otimes \tau(g,h)t^{g+h} + D_{v,v'} \otimes \tau(g'+g_1, h'+g_1)t^{g'+h'+2g_1}$$
$$+ xv' \otimes \tau(g, h'+g_1)t^{g+h'+g_1} - x'v \otimes \tau(g'+g_1, h)t^{g'+h+g_1}$$

defines a Lie bracket for $g, g', h, h' \in G'$, $x, x' \in o_{2\mathfrak{J}+1}(F)$, $v, v' \in V$.

Also, we define two more twisted (G,τ)-loop algebras. (We use the way by Kac [K, Ch.8] for $F_4^{(2)}$ in order to avoid introducing a 27-dimensional exceptional Jordan algebra. But for $G_2^{(3)}$, we again use the way in [BZ] since we do not need to assume the existence of a primitive cubic root of unity in our base field F.)

(4) Type $F_4^{(2)}$: Assume that $F^\tau[G] = F^\tau[G, t] = F^\tau[G'] \oplus t^{g_1} F^\tau[G']$ with $g_1 \in G \backslash G'$ again. Let $\mathfrak{g}$ be the finite-dimensional split simple Lie algebra of type E_6, and σ be the automorphism of $\mathfrak{g}$ of order 2 determined by the diagram automorphism. Define the automorphism $\tilde{\sigma}$ of $E_6^{(1)} = \mathfrak{g} \otimes F^\tau[G]$ by $\tilde{\sigma}(x \otimes t^{g_1}) = -\sigma(x) \otimes t^{g_1}$. The subalgebra $L(F_4^{(2)}, F^\tau[G])$ of $E_6^{(1)}$ fixed by $\tilde{\sigma}$ is called a *twisted (G, τ)-loop algebra* of type $F_4^{(2)}$. We note that the subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ fixed by σ has type F_4, say $\mathfrak{g}' = \bigoplus_{\mu \in F_4 \cup \{0\}} \mathfrak{g}'_\mu$. Let $\mathfrak{s}$ be the (-1)-eigenspace. Then $\mathfrak{s}$ is an irreducible highest weight $\mathfrak{g}'$-module whose highest weight is the highest short root in F_4. Thus $L(F_4^{(2)}, F^\tau[G, t]) = \bigoplus_{\mu \in F_4 \cup \{0\}} (\mathfrak{g}'_\mu \otimes F^\tau[G']) \oplus \bigoplus_{\mu \in (F_4)_{sh}} (\mathfrak{s}_\mu \otimes t^{g_1} F^\tau[G'])$.

(5) Type $G_2^{(3)}$: Assume this time that there exists a subgroup G' so that $G/G' \cong \mathbb{Z}_3$. Then $G = G' \sqcup (G' + g_1) \sqcup (G' + 2g_1)$ for any $g_1 \in G \setminus G'$, and so

$$F^\tau[G] = F^\tau[G'] \oplus t^{g_1} F^\tau[G'] \oplus t^{2g_1} F^\tau[G'].$$

(For example, take any subgroup G of $\mathbb{Q}$ which is not divisible by 3, and $G' := 3G$, by Lemma 12(2).) As in [ABGP] (or in [BZ]), let $\mathbb{O}$ be a split octonion algebra over F, and $t : \mathbb{O} \longrightarrow F$ the normalized trace on $\mathbb{O}$, in which $\mathbb{O} = F1 \oplus \mathbb{O}_0$, where $\mathbb{O}_0 = \{x \in \mathbb{O} \mid t(x) = 0\}$. Moreover, if $x, y \in \mathbb{O}$, we have $xy = t(xy)1 + x * y$ for some unique $x * y \in \mathbb{O}_0$. One can check that $x * y = -y * x$ for $x, y \in \mathbb{O}_0$. Next, let

$$D_{\mathbb{O}, \mathbb{O}} := \operatorname{span}_F \{D_{x,y} \mid x, y \in \mathbb{O}\},$$

where $D_{x,y} = \frac{1}{4}(L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y])$. (Here L_x and R_x denote the left and right multiplication operators by x in $\mathbb{O}$.) Then $D_{\mathbb{O}, \mathbb{O}}$ is the Lie algebra of all derivations of $\mathbb{O}$ and $D_{\mathbb{O}, \mathbb{O}}$ is a split simple Lie algebra of type G_2 over F. Let

$$L(G_2^{(3)}, F^\tau[G, t]) = (D_{\mathbb{O}, \mathbb{O}} \otimes F^\tau[G']) \oplus (\mathbb{O}_0 \otimes t^{g_1} F^\tau[G']) \oplus (\mathbb{O}_0 \otimes t^{2g_1} F^\tau[G']).$$

One can check that the bracket

$$[D \otimes t^g + x \otimes t^{g'+g_1} + x' \otimes t^{g''+2g_1}, D' \otimes t^h + y \otimes t^{h'+g_1} + y' \otimes t^{h''+2g_1}]$$
$$= [D, D'] \otimes \tau(g, h) t^{g+h} + Dy \otimes \tau(g, h' + g_1) t^{g+h'+g_1} + Dy' \otimes \tau(g, h'' + 2g_1) t^{g+h''+2g_1}$$
$$- D'x \otimes \tau(g' + g_1, h) t^{g+h+g_1} + (x * y) \otimes \tau(g' + g_1, h' + g_1) t^{g'+h'+2g_1}$$
$$+ D_{x, y'} \otimes \tau(g' + g_1, h'' + 2g_1) t^{g'+h''+3g_1} - D'x' \otimes \tau(g'' + 2g_1, h) t^{g''+h+2g_1}$$
$$+ D_{x', y} \otimes \tau(g'' + 2g_1, h' + g_1) t^{g''+h'+3g_1} + (x' * y') \otimes \tau(g'' + 2g_1, h'' + 2g_1) t^{g''+h''+4g_1}$$

defines a Lie bracket for $D, D' \in D_{\mathbb{O}, \mathbb{O}}$, $x, x', y, y' \in \mathbb{O}_0$ and $g, g', g'', h, h', h'' \in G'$. In fact, if we define an F-linear map tr on $F^\tau[G, t]$ by

$$\operatorname{tr}(t^g) = \begin{cases} t^g & \text{if } g \in G' \\ 0 & \text{otherwise} \end{cases}$$

(so tr is an $F^\tau[G']$-linear map on the algebra $F^\tau[G, t]$ over $F^\tau[G']$), then one can check that any $x \in F^\tau[G, t]$ satisfies the identity

$$x^3 - 3\operatorname{tr}(x)x^2 + \left(\frac{9}{2}\operatorname{tr}(x)^2 - \frac{3}{2}\operatorname{tr}(x^2)\right)x - \operatorname{tr}(x^3) + \frac{9}{2}\operatorname{tr}(x^2)\operatorname{tr}(x) - \frac{9}{2}\operatorname{tr}(x)^3 = 0.$$

This guarantees that the bracket is a Lie bracket by the recognition theorem [BZ, Thm 3.4.7]. We call the Lie algebra $L(\mathrm{G}_2^{(3)}, F^\tau[G,t])$ a *twisted (G,τ)-loop algebra* of type $\mathrm{G}_2^{(3)}$. We note that if F contains a primitive cubic root of unity, a twisted (G,τ)-loop algebra of type $\mathrm{G}_2^{(3)}$ can be constructed similarly to the case of type $\mathrm{F}_4^{(2)}$. But our $L(\mathrm{G}_2^{(3)}, F^\tau[G,t])$ exists over any F.

We often omit the term 'untwisted' or 'twisted' and simply say a (locally) (G,τ)-loop algebra. When $G \cong \mathbb{Z}$, we have $F^\tau[\mathbb{Z}] \cong F[\mathbb{Z}] = F[t^{\pm 1}]$. So it is natural to call the (locally) $(\mathbb{Z},\tau)$-loop algebras above just *(locally) loop algebras*, and of course the loop algebras are the well-known algebras in Kac-Moody theory. Also, if $\tau \equiv 1$, i.e., $F^\tau[G] = F[G]$ is a group algebra, then a (locally) $(G,1)$-loop algebra is simply called a *(locally) G-loop algebra*.

If G is a subgroup of $\mathbb{Q}$, then G is a directed union of cyclic groups of infinite order, and so any locally (G,τ)-loop algebra is a directed union of loop algebras. Also, if F is algebraically closed, then $F^\tau[G] = F[G]$ by a suitable base change. (G can be any abelian group for this statement, see [P, Lem.2.9] in detail.)

Now, let G be a subgroup of $\mathbb{Q}$. For any two elements $x \otimes t^g$ and $y \otimes t^h$ in any locally (G,τ)-loop algebra $\mathcal{L}$, define the new bracket on a 1-dimensional central extension $\tilde{\mathcal{L}} := \mathcal{L} \oplus Fc$ by
$$[x \otimes t^g, y \otimes t^h] := [x,y] \otimes \tau(g,h)t^{g+h} + (x,y)\tau(g,h)\delta_{g+h,0}gc$$

(note $g \in G \subset \mathbb{Q} \subset F$), where (x,y) is the trace form or the Killing form depending on the type of $\mathcal{L}$, or for type $\mathrm{B}_{\mathfrak{J}}^{(2)}$, the direct sum of the trace form and the bilinear form on V determined by the symmetric matrix s above, or for type $\mathrm{G}_2^{(3)}$, the direct sum of the trace form on $D_{\mathbb{O},\mathbb{O}}$ and the trace form t on $\mathbb{O}_0$ above. Indeed, this gives a central extension since $\tilde{\mathcal{L}}$ is locally an affine Lie algebra, i.e., a 1-dimensional central extension of a loop algebra, and $\mathcal{L}$ is a directed union of loop algebras. One can also show that $\tilde{\mathcal{L}}$ is a universal central extension of $\mathcal{L}$ [MY2]. Let
$$\hat{\mathcal{L}} = \tilde{\mathcal{L}} \oplus Fd,$$

where d is the degree derivation, i.e.,
$$[d, x \otimes t^g] = gx \otimes t^g \quad \text{and} \quad [d,c] = 0.$$

Let
$$\mathcal{H} = \mathfrak{h} \oplus Fc \oplus Fd,$$

where $\mathfrak{h}$ is the subalgebra of $\mathcal{L}$ consisting of diagonal matrices of degree 0 when $\mathfrak{J}$ is infinite or the Cartan subalgebra of each finite-dimensional split simple Lie algebra $\mathfrak{g}$ when $\mathfrak{J}$ is finite. Then $\mathcal{H}$ is a Cartan subalgebra of $\hat{\mathcal{L}}$, and one can check that the set of anisotropic roots relative to $(\hat{\mathcal{L}}, \mathcal{H})$ is a LEARS of null dimension 1. We also note that $\hat{\mathcal{L}}$ is an example of locally extended affine Lie algebra of null dimension 1 in the sense of [MY1]. In particular, if $G = \mathbb{Z}$ and $\mathfrak{J}$ is infinite, then the root system of each $\hat{\mathcal{L}}$ is one of seven reduced LARS listed in Corollary 15, which is very close to an affine Kac-Moody Lie algebra, and we call it a *locally affine Lie algebra*. In [MY2] we classify locally affine Lie algebras.

References

[A1]   S. Azam, *Extended affine Weyl groups*, J. Algebra **214** (1999), 571–624.

[A2]   S. Azam, *Extended affine root systems*, J. Lie Theory **12 n° 2** (2002), 515–527.

[A-P]   B. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, *Extended affine Lie algebras and their root systems*, Memoirs Amer. Math. Soc. **126**, vol. 603, 1997.

[ABG]   B. Allison, G. Benkart, Y. Gao, *Lie Algebras Graded by the Root Systems BC_r, $r \geq 2$*, Memoirs Amer. Math. Soc. **751**, vol. 158, 2002.

[ABGP]   B. Allison, S. Berman, Y. Gao, A. Pianzola, *A characterization of affine Kac-Moody Lie algebras*, Commun. Math. Phys. **185 n° 3** (1997), 671–688.

[AKY]   S. Azam, V. Khalili, M. Yousofzadeh, *Extended affine root systems of type BC*, J. Lie Theory **15(1)** (2005), 145–181.

[AY]   S. Azam, M. Yousofzadeh, *Root systems arising from automorphisms*, preprint.

[BZ]   G. Benkart and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), 1–45.

[G]   P. Griffith, *Infinite abelian group theory*, Chicago Lectures in Mathematics, 1970.

[K]   V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press, 1990.

[L]   O. Loos, *Spiegelungsräume und homogene symmetrische Räume*, Math. Z. **99** (1967), 141–170.

[LN1]   O. Loos, E. Neher, *Locally finite root systems*, Memoirs Amer. Math. Soc. **811**, vol. 171, 2004.

[LN2]   O. Loos, E. Neher, *Reflections systems and partial root systems*, preprint.

[M]   I. Macdonald, *Affine root systems and Dedekind's η-functions*, J. Invent. Math. **15** (1972), 91–143.

[MP]   R.V. Moody, A. Pianzola, *Lie algebras with triangular decompositions*, Can. Math. Soc. Series of Monographs and Advanced Texts, John Wiley, 1995.

[N]   E. Neher, *Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids*, Amer. J. Math. **118** (1996), 439–491.

[NS]   K.-H. Neeb, N. Stumme, *The classification of locally finite split simple Lie algebras*, J. Reine Angew. Math. **533** (2001), 25–53.

[MY1]   J. Morita, Y. Yoshii, *Locally extended affine Lie algebras*, J. Algebra **301** (2006), 59–81.

[MY2]   J. Morita, Y. Yoshii, *Locally loop algebras and locally affine Lie algebras*, in preparation.

[P]   D. Passman, *The algebraic structure of group rings*, Krieger Pub. Co., 1985.

[S]   K. Saito, *Extended affine root systems 1 (Coxeter transformations)*, RIMS., Kyoto Univ. **21 n° 1** (1985), 75–179.

[Y]   Y. Yoshii, *Root systems extended by an abelian group and their Lie algebras*, J. Lie Theory **14(2)** (2004), 371–394.