LOCALLY LOOP ALGEBRAS AND LOCALLY AFFINE LIE ALGEBRAS

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ABSTRACT. We investigate a new class of Lie algebras, which are tame locally extended affine Lie algebras of nullity 1. It is an infinite-rank analog of affine Lie algebras, and their centerless cores are a local version of loop algebras. Such algebras are called locally affine Lie algebras and locally loop algebras. We classify both of them.

Throughout the paper F is a field of characteristic 0. All algebras are assumed to be unital except Lie algebras, and tensor products are over F.

1. INTRODUCTION

In [MY], a **locally extended affine Lie algebra**, a **LEALA** for short, is introduced as a generalization of an extended affine Lie algebra, an EALA for short. A LEALA is a Lie algebra $\mathcal{L}$ with a certain Cartan subalgebra $\mathcal{H}$ and a nondegenerate invariant form $\mathcal{B}$ on $\mathcal{L}$ satisfying certain axioms (see Section 4). We classified LEALAs of nullity 0 in [MY]. The purpose of this paper is to classify the second easiest class, namely, the class of tame LEALAs of nullity 1, called a **locally affine Lie algebra**, a **LALA** for short. It turns out that the centerless core of a LALA is a local version of a loop algebra, which we call a **locally loop algebra**. Thus a LALA is really a local analog of an affine Lie algebra. In fact we show that a locally loop algebra is a direct limit (or a directed union) of loop algebras, and that the core of a LALA is a universal covering of a locally loop algebra. This was also shown by Neeb [N2, Cor. 3.13] in a different way. There are seven new locally loop algebras of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, $\mathrm{B}_{\mathfrak{I}}^{(1)}$, $\mathrm{C}_{\mathfrak{I}}^{(1)}$, $\mathrm{D}_{\mathfrak{I}}^{(1)}$, $\mathrm{B}_{\mathfrak{I}}^{(2)}$, $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$, where $\mathfrak{I}$ is an **infinite** index set. Thus the core of a LALA is a universal covering of one of the seven locally loop algebras. Here, one should note that, in the above seven cases, $X_{\mathfrak{I}}^{(r)}$ always means

$$\lim_{\rightarrow} X_{\mathfrak{I}'}^{(r)} ,$$

where this is a direct limit according to inclusions of all finite subsets $\mathfrak{I}'$ of $\mathfrak{I}$.

For each loop algebra, there exists a unique, up to isomorphisms, affine Lie algebra having it as the core. However, for each locally loop algebra, there are infinitely many isomorphism classes of LALAs having it as the core. We roughly explain here about the LALAs of the type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ and the twisted type $\mathrm{C}_{\mathfrak{I}}^{(2)}$.

First of all, let $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$ be the Lie algebra consisting of trace 0 matrices of infinite size $\mathfrak{I}$ (but only finite entries are nonzero) whose entries are in the algebra $F[t^{\pm 1}]$ of Laurent polynomials. For example, if $\mathfrak{I} = \mathbb{N}$ (natural numbers), then we see

$$\mathrm{sl}_{\mathbb{N}}(F[t^{\pm}]) = \bigcup_{n=2}^{\infty} \mathrm{sl}_n(F[t^{\pm}]) = \bigcup_n \left(\begin{array}{c|c} \mathrm{sl}_n(F[t^{\pm}]) & O \\ \hline O & O \end{array} \right) .$$

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We call $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$ a locally loop algebra of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, which is simply an infinite rank analog of a loop algebra $\mathrm{sl}_{\ell+1}(F[t^{\pm 1}])$ of type $\mathrm{A}_{\ell}^{(1)}$. Here, we use the following convention: $\mathrm{sl}_{\mathfrak{I}}$ is of type $\mathrm{A}_{\mathfrak{I}}$ if $\mathfrak{I}$ is an infinite index set, and $\mathrm{sl}_{\mathfrak{I}} = \mathrm{sl}_{\ell+1}$ is of type A_{ℓ} if $\mathfrak{I}$ is a finite index set with $\ell+1$ elements (also see Remark 7.1). As in the case of $\mathrm{sl}_{\ell+1}(F[t^{\pm 1}])$, there exists a universal covering, $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}]) \oplus Fc$, of $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$, where Fc is the 1-dimensional center. Then one can construct the Lie algebra

$$\mathcal{L}^{ms} := \mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}]) \oplus Fc \oplus Fd^{(0)}, \tag{1}$$

where $d^{(0)} = t\frac{d}{dt}$ is the degree derivation. This $\mathcal{L}^{ms}$ endowed with a Cartan subalgebra

$$\mathfrak{h} \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathrm{sl}_{\mathfrak{I}}(F)$ consisting of diagonal matrices, is a simplest example of a LALA, and is called a **minimal standard LALA** of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ (see Definition 5.1). There are more examples even for the type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, adding diagonal derivations of $\mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}])$. Let us explain these examples briefly without mentioning the defining bilinear form $\mathcal{B}$.

Let $M_{\mathfrak{I}}(F)$ be the vector space of matrices of size $\mathfrak{I}$ and $T_{\mathfrak{I}}$ the subspace of $M_{\mathfrak{I}}(F)$ consisting of diagonal matrices. That is, we have

$$M_{\mathfrak{I}}(F) = \left\{ (a_{ij})_{i,j\in\mathfrak{I}} \mid a_{ij} \in F \right\}\ ,\ T_{\mathfrak{I}} = T_{\mathfrak{I}}(F) = \left\{ (a_{ij}) \in M_{\mathfrak{I}}(F) \mid a_{ij} = 0 \text{ for } i \neq j \right\}\ .$$

In fact, we use here two kinds of sets of matrices, namely $\mathrm{sl}_{\mathfrak{I}}$ consists of matrices whose entries are 0 almost everywhere, but $M_{\mathfrak{I}}$ can contain a matrix whose entries are all nonzero. We note that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}$$

is a Lie algebra with the centre $F\iota$, where $\iota = \iota_{\mathfrak{I}} \in T_{\mathfrak{I}}$ is the diagonal matrix whose diagonal entries are all 1. Let

$$\mathcal{A}_{\mathfrak{I}} := \big(\mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}\big)/F\iota \tag{2}$$

be the quotient Lie algebra. **We always identify the sublalgebra**

$$\overline{\mathrm{sl}_{\mathfrak{I}}(F)} = \big(\mathrm{sl}_{\mathfrak{I}}(F) + F\iota\big)/F\iota$$

of $\mathcal{A}_{\mathfrak{I}}$ **with** $\mathrm{sl}_{\mathfrak{I}}(F)$**, and omit the bar for any element or a subalgebra of** $\mathcal{A}_{\mathfrak{I}}$. That is, $\mathrm{sl}_{\mathfrak{I}}(F)$ can be considered as a subalgebra of $\mathcal{A}_{\mathfrak{I}}$. Also, if we choose a complement $T'_{\mathfrak{I}}$ of $F\iota$ in $T_{\mathfrak{I}}$, we sometimes identify $T_{\mathfrak{I}}/F\iota$ with $T'_{\mathfrak{I}}$. Consider the loop algebra $\mathcal{A}_{\mathfrak{I}} \otimes F[t^{\pm 1}]$, and we construct the Lie algebra

$$\hat{\mathcal{A}}_{\mathfrak{I}} := \mathcal{A}_{\mathfrak{I}} \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)} \tag{3}$$

as in (1), which contains $\mathcal{L}^{ms}$. In fact, to show that $\mathcal{A}_{\mathfrak{I}} \otimes F[t^{\pm 1}] \oplus Fc$ is a Lie algebra, we need to discuss a bilinear form on $\mathcal{A}_{\mathfrak{I}}$ (see the details in Example 5.2). We will show that this $\hat{\mathcal{A}}_{\mathfrak{I}}$ is a **maximal LALA** of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, and that any LALA of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ is a graded subalgebra of $\hat{\mathcal{A}}_{\mathfrak{I}}$ containing

$$\mathcal{L}(p) := \mathrm{sl}_{\mathfrak{I}}(F[t^{\pm 1}]) \oplus Fc \oplus F(p + d^{(0)}) \tag{4}$$

for some

$$p \in T_{\mathfrak{I}}/(\mathfrak{h} \oplus F\iota),$$

where $T_{\mathfrak{I}}/(\mathfrak{h} \oplus F\iota)$ is identified with a complement $T''_{\mathfrak{I}}$ of $\mathfrak{h} \oplus F\iota$ in $T_{\mathfrak{I}}$. We use the notation $\mathcal{L}(p)$ instead of $\mathcal{L}(\bar{p})$ by this identification. This $\mathcal{L}(p)$ is called a **minimal LALA determined by** $p \in T''_{\mathfrak{I}} \approx T_{\mathfrak{I}}/(\mathfrak{h} \oplus F\iota)$. We note here that $\mathcal{L}(p)$ is sometimes isomorphic to the minimal standard LALA $\mathcal{L}^{ms} = \mathcal{L}(0)$, but $\mathcal{L}(p)$ is not always isomorphic to $\mathcal{L}^{ms}$ in general (see Example 7.17).

Next, let $s = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$ be the matrix of size $2\mathfrak{I}$, where $\iota = \iota_{\mathfrak{I}}$ is as above. Define an automorphism σ of period 2 on $\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}$ by

$$\sigma(x) = sx^t s$$

for $x \in \mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}$, where x^t is the transpose of x. Let $\mathrm{sp}_{2\mathfrak{I}}(F)$ be the fixed subalgebra of $\mathrm{sl}_{2\mathfrak{I}}(F)$ by σ, which has type $\mathrm{C}_{\mathfrak{I}}$, and $\mathfrak{s}$ the (-1)-eigenspace of σ so that

$$\mathrm{sl}_{2\mathfrak{I}}(F) = \mathrm{sp}_{2\mathfrak{I}}(F) \oplus \mathfrak{s}.$$

Moreover, we have

$$\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}} = \big(\mathrm{sp}_{2\mathfrak{I}}(F) + T^+\big) \oplus \big(\mathfrak{s} + T^-\big),$$

where T^+ is the 1-eigenspace and T^- is the (-1)-eigenspace for σ acting on $T_{2\mathfrak{I}}$. Note that $T_{2\mathfrak{I}} = T^+ \oplus T^-$ and $F\iota_{2\mathfrak{I}} \subset T^-$. We fix a complement T_1^- of $F\iota$ in T^-, and we identify $T^-/F\iota$ with T_1^-.

Now, let

$$\mathcal{A}_{2\mathfrak{I}} := \big(\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}\big)/F\iota_{2\mathfrak{I}}$$

in the same idea as (2). Since $\sigma(F\iota) = F\iota$, we have the induced automorphism on $\mathcal{A}_{2\mathfrak{I}}$ which we also write σ for simplicity. Thus, omitting bars as above, we also have

$$\mathcal{A}_{2\mathfrak{I}}^{\sigma} = \big(\mathrm{sp}_{2\mathfrak{I}}(F) + T^+\big) \oplus \big(\mathfrak{s} + T_1^-\big).$$

Let

$$\hat{\mathcal{A}}_{2\mathfrak{I}} := \mathcal{A}_{2\mathfrak{I}} \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)},$$

as (3). We extend σ to $\hat{\mathcal{A}}_{2\mathfrak{I}}$ as

$$\hat{\sigma}(x \otimes t^k) := (-1)^k \sigma(x) \otimes t^k,$$

and identically on $Fc \oplus Fd^{(0)}$. Then the fixed algebra $\hat{\mathcal{A}}_{2\mathfrak{I}}^{\hat{\sigma}}$ by $\hat{\sigma}$ is the following.

$$\hat{\mathcal{A}}_{2\mathfrak{I}}^{\hat{\sigma}} = \big((\mathrm{sp}_{2\mathfrak{I}}(F) + T^+) \otimes F[t^{\pm 2}]\big) \oplus \big((\mathfrak{s} + T_1^-) \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)}.$$

Note again that we omit bars, especially for T^- and T_1^-. Note also that $\hat{\mathcal{A}}_{2\mathfrak{I}}^{\hat{\sigma}}$ contains the subalgebra

$$\mathcal{L}^{ms} := \big(\mathrm{sp}_{2\mathfrak{I}}(F) \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus Fd^{(0)},$$

which is called a **minimal standard twisted LALA** of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$. We will show that $\hat{\mathcal{A}}_{2\mathfrak{I}}^{\hat{\sigma}}$ is a **maximal twisted LALA** of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$, and that any LALA of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$ is a graded subalgebra of $\hat{\mathcal{A}}_{2\mathfrak{I}}^{\hat{\sigma}}$ containing

$$\mathcal{L}(p) := \big(\mathrm{sp}_{2\mathfrak{I}}(F) \otimes F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s} \otimes tF[t^{\pm 2}]\big) \oplus Fc \oplus F(p + d^{(0)})$$

for some $p \in T^+/(\mathrm{sp}_{2\mathfrak{I}}(F) \cap T^+)$, where we again identify $T^+/(\mathrm{sp}_{2\mathfrak{I}}(F) \cap T^+)$ with a complement T_1^+ of $\mathrm{sp}_{2\mathfrak{I}}(F) \cap T^+$ in T^+. This $\mathcal{L}(p)$ is called a **minimal twisted LALA** determined by p. As in the type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, $\mathcal{L}(p)$ is not necessarily isomorphic to $\mathcal{L}^{ms} = \mathcal{L}(0)$.

We emphasize that the usual twisted process works for not only the Lie algebra $\mathrm{sl}_{2\mathfrak{I}}(F)$ but also bigger Lie algebras contained in $\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}$ to construct twisted LALAs.

The classification of LALAs proceeds as follows. First we classify the cores of LALAs. We show that the core of a LALA is a locally Lie 1-torus, which is isomorphic to a universal covering of a locally loop algebra. We also show that there is a one to one correspondence between reduced locally affine root systems and the cores of LALAs. (This part was already done in [N2] by a different way.) The second step of the classification is to determine a complement of the core of a LALA $\mathcal{L}$. Let

$$\mathcal{L} = \mathcal{L}_c \oplus D,$$

where $\mathcal{L}_c$ is the core and D is a homogeneous complement of the core. Since $\mathcal{L}_c$ is an ideal and $\mathcal{L}$ is tame, D embeds into $\mathrm{Der}_F \mathcal{L}_c$, the space of derivations of $\mathcal{L}_c$. Moreover, one can show that $\mathrm{Der}_F \mathcal{L}_c$ embeds into $\mathrm{Der}_F L$, where

$$L := \mathcal{L}_c / Z(\mathcal{L}_c)$$

and L is a locally loop algebra. Now we need some information about $\mathrm{Der}_F L$. Derivations of this kind of algegbra were studied in [BM], [B] or [NY]. However, the derivations of a locally loop algebra are not classified yet in general. One can use some results in [A1] for untwisted case since L is a tensor algebra in that case (see Remark 6.4). But we need to figure out the twisted case. So one has to develop a new theory. Of course, we need to use the classification of $\mathrm{Der}_F \mathfrak{g}$ for a locally finite split simple Lie algebra $\mathfrak{g}$ by Neeb in [N1]. Fortunately, we do not need the whole information about $\mathrm{Der}_F L$ to classify D. It turns out that we only need to know the **diagonal derivations of degree** m

$$(\mathrm{Der}_F L)_0^m := \{d \in \mathrm{Der}_F L \mid d(L_\alpha^k) \subset L_\alpha^{k+m} \text{ for all } \alpha \in \Delta \text{ and } k \in \mathbb{Z}\},$$

where L_α^k or L_α^{k+m} is a homogeneous space of the double graded algebra (a locally Lie 1-torus)

$$L = \bigoplus_{\alpha \in \Delta \cup \{0\}} \bigoplus_{k \in \mathbb{Z}} L_\alpha^k.$$

It is crucial to determine $(\mathrm{Der}_F L)_0^0$. Once this is done, $(\mathrm{Der}_F L)_0^m$ can be easily figured out for the untwisted case. However, for the twisted case, the classification is still difficult. First we show the same result as in the untwisted case for an even m. For an odd m, we show that a diagonal derivation commutes with a **shift map**, which is a centroidal element on a $\mathbb{Z}$-graded algebra (see Lemma 7.8). Using this fact, one can extend the derivation on the twisted locally loop algebra to the corresponding untwisted locally loop algebra (see Lemma 7.9). Then using the classification of the untwisted ones, we can classify the diagonal derivations of odd degree (see Theorem 7.10).

Thus our interest $D = \bigoplus_{m \in \mathbb{Z}} D^m$ can be identified with a graded subspace of the known space $\bigoplus_{m \in \mathbb{Z}} (\mathrm{Der}_F L)_0^m$ for both untwisted and twisted cases. Finally, we classify the Lie brackets on D for both cases. First one can show that $[D,D] \subset Fc$. Then since $\mathcal{L}$ is a graded Lie algebra, we get

$$[D^m, D^n] \subset F(\delta_{m+n,0}c),$$

where δ is the Kronecker delta. Then, by the fundamental property (21) of a LEALA in Lemma 4.4, the brackets on D^m and D^{-m} are determined by the defining bilinear form $\mathcal{B}$ of $\mathcal{L}$. The concrete brackets are described in Example 5.2.

To conclude this introduction, we briefly describe the contents of the paper. In Section 2, we define a locally Lie G-torus and a locally Lie 1-torus as a special case. We introduce a locally loop algebra which turns out to be a centerless locally Lie 1-torus in Section 3. Here we classify locally Lie 1-tori in general. We prove that a centerless locally Lie 1-torus is uniquely determined by a root system extended by $\mathbb{Z}$, and it is a locally loop algebra or a universal covering of a locally loop algebra. In Section 4 we recall a LEALA and define a LALA. We prove some general properties of a LEALA or a LALA. We will see that the core of a LALA is a universal covering of a locally loop algebra. In Section 5 we construct many examples of LALAs, which will be all. In Section 6 and 7 we classify untwisted LALAs and twisted LALAs. We show that the examples in Section 5 exhaust all LALAs.

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2. LOCALLY LIE G-TORI

Let Δ be a locally finite irreducible root system, and we denote the Cartan integer

$$\frac{2(\mu,\nu)}{(\nu,\nu)}$$

by $\langle \mu,\nu\rangle$ for $\mu,\nu\in\Delta$, and also let $\langle 0,\nu\rangle := 0$ for all $\nu\in\Delta$. Let

$$\Delta^{\text{red}} := \begin{cases} \Delta & \text{if } \Delta \text{ is reduced} \\ \{\alpha\in\Delta \mid \frac{1}{2}\alpha\notin\Delta\} & \text{otherwise.}\end{cases}$$

For the convenience of description later, we partition the locally finite irreducible root system Δ according to length. Roots of Δ of minimal length are called **short**. Roots of Δ which are two times a short root of Δ are called **extra long**. Finally, roots of Δ which are neither short nor extra long are are called **long**. We denote the subsets of short, long and extra long roots of Δ by Δ_{sh}, Δ_{lg} and Δ_{ex} respectively. Thus

$$\Delta = \Delta_{\text{sh}} \sqcup \Delta_{\text{lg}} \sqcup \Delta_{\text{ex}}.$$

Of course the last two terms in this union may be empty. Indeed,

$$\Delta_{\text{lg}} = \emptyset \iff \Delta \text{ has simply laced type or type } \mathrm{BC}_1,$$

and

$$\Delta_{\text{ex}} = \emptyset \iff \Delta = \Delta^{\text{red}}.$$

Let $G = (G,+,0)$ be an arbitrary abelian group. In general, for a subset S of G, the subgroup generated by S is denoted by $\langle S\rangle$.

Definition 2.1. A Lie algebra $\mathcal{L}$ is called a **locally Lie G-torus of type Δ** if

(LT1) $\mathcal{L}$ has a decomposition into subspaces

$$\mathcal{L} = \bigoplus_{\mu\in\Delta\cup\{0\},g\in G} \mathcal{L}_\mu^g$$

such that $[\mathcal{L}_\mu^g,\mathcal{L}_\nu^h]\subset\mathcal{L}_{\mu+\nu}^{g+h}$ for $\mu,\nu,\mu+\nu\in\Delta\cup\{0\}$ and $g,h\in G$;

(LT2) For every $g\in G$, $\mathcal{L}_0^g = \sum_{\mu\in\Delta,h\in G}[\mathcal{L}_\mu^h,\mathcal{L}_{-\mu}^{g-h}]$;

(LT3) For each nonzero $x\in\mathcal{L}_\mu^g$ ($\mu\in\Delta, g\in G$), there exists a $y\in\mathcal{L}_{-\mu}^{-g}$ so that $t:=[x,y]\in\mathcal{L}_0^0$ satisfies $[t,z]=\langle\nu,\mu\rangle z$ for all $z\in\mathcal{L}_\nu^h$ ($\nu\in\Delta\cup\{0\}, h\in G$);

(LT4) $\dim\mathcal{L}_\mu^g\le 1$ for $\mu\in\Delta$ and $\dim\mathcal{L}_\mu^0 = 1$ if $\mu\in\Delta^{\text{red}}$;

(LT5) $\langle\operatorname{supp}\mathcal{L}\rangle = G$, where $\operatorname{supp}\mathcal{L} = \{g\in G \mid \mathcal{L}_\mu^g\neq 0 \text{ for some } \mu\in\Delta\cup\{0\}\}$.

If Δ is finite, $\mathcal{L}$ is called a **Lie G-torus**. Also, if $G\cong\mathbb{Z}^n$, $\mathcal{L}$ is called a **locally Lie n-torus** or simply a **locally Lie torus**. We call the rank of Δ the **rank** of $\mathcal{L}$.

Remark 2.2. (i) Condition (LT5) is simply a convenience. If it fails to hold, we may replace G by the subgroup generated by $\operatorname{supp}\mathcal{L}$.

(ii) It follows from (LT1) and (LT3) that $\mathcal{L}$ admits a grading by the root lattice $Q(\Delta)$: if

$$\mathcal{L}_\lambda := \oplus_{g\in G}\,\mathcal{L}_\lambda^g \tag{5}$$

for $\lambda\in Q(\Delta)$, where $\mathcal{L}_\lambda^g = 0$ if $\lambda\notin\Delta\cup\{0\}$, then $\mathcal{L} = \oplus_{\lambda\in Q(\Delta)}\mathcal{L}_\lambda$ and $[\mathcal{L}_\lambda,\mathcal{L}_\mu]\subset\mathcal{L}_{\lambda+\mu}$.

(iii) $\mathcal{L}$ is also graded by the group G. Namely, if

$$\mathcal{L}^g := \oplus_{\mu\in\Delta\cup\{0\}} \mathcal{L}^g_\mu, \tag{6}$$

then $\mathcal{L} = \oplus_{g\in G} \mathcal{L}^g$ and $[\mathcal{L}^g, \mathcal{L}^h] \subset \mathcal{L}^{g+h}$. Also, $\operatorname{supp}\mathcal{L} = \{g \in G \mid \mathcal{L}^g \neq 0\}$.

(iv) From (LT3) we see for $\mu \in \Delta^{\mathrm{red}}$ that there exist elements $e_\mu \in \mathcal{L}^0_\mu$, $f_\mu \in \mathcal{L}^0_{-\mu}$, and $\mu^\vee = \mu^\vee_{\mathcal{L}} := [e_\mu, f_\mu]$ so that $[\mu^\vee, z] = \langle \nu, \mu\rangle z$ for all $z \in \mathcal{L}^h_\nu$, $\nu \in \Delta$ and $h \in G$. Thus, the elements $e_\mu, f_\mu, \mu^\vee$ determine a canonical basis for a copy of the Lie algebra $\mathrm{sl}_2(F)$. (Note that $\mu^\vee$ is a unique element in $[\mathcal{L}^0_\mu, \mathcal{L}^0_{-\mu}]$ satisfying the property.) The subalgebra $\mathfrak{g}$ of $\mathcal{L}$ generated by the subspaces $\mathcal{L}^0_\mu$ for $\mu \in \Delta^{\mathrm{red}}$ is a locally finite split simple Lie algebra with the split Cartan subalgebra

$$\mathfrak{h} := \sum_{\mu\in\Delta^{\mathrm{red}}} [\mathcal{L}^0_\mu, \mathcal{L}^0_{-\mu}]$$

and $\mu^\vee$ are the coroots in $\mathfrak{h}$. (One can show this in the same way as the proof of [MY, Prop.8.3], or see [St, Sec.III]). Note that if Δ is finite, then $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra. Also, Δ^{red} may be replaced by Δ in the definition of $\mathfrak{g}$ and $\mathfrak{h}$, since it can be shown in the same way as in [Y1, Thm.5.1] that $\mathcal{L}^0_{2\nu} = 0$ for all $\nu \in \Delta^{\mathrm{red}}$. We say the pair $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}, \mathfrak{h})_{\mathcal{L}}$ the **grading pair** of $\mathcal{L}$.

(v) A Lie G-torus is perfect, and so it has a universal covering.

We define the root systems of locally Lie G-tori. Let $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{g\in G} \mathcal{L}^g_\mu$ be a locally Lie G-torus. For each $\mu \in \Delta$, let

$$S_\mu := \{g \in G \mid \mathcal{L}^g_\mu \neq 0\},$$

and we call

$$\tilde{\Delta} := \{S_\mu\}_{\mu\in\Delta}$$

the **root system** of $\mathcal{L}$. Such a system fits into the system introduced in [Y1]. Let us state the precise definition. A family of subsets S_μ of G indexed by Δ, say $\{S_\mu\}_{\mu\in\Delta}$, is called a **root system extended by** G if

$$\langle \cup_{\mu\in\Delta} S_\mu\rangle = G, \tag{7}$$

$$S_\nu - \langle \nu, \mu\rangle S_\mu \subset S_{\nu-\langle\nu,\mu\rangle\mu} \quad \text{for all } \mu, \nu \in \Delta, \text{ and} \tag{8}$$

$$0 \in S_\mu \quad \text{for all } \mu \in \Delta^{\mathrm{red}}. \tag{9}$$

Moreover, $\{S_\mu\}_{\mu\in\Delta}$ is called **reduced** if

$$S_{2\mu} \cap 2S_\mu = \emptyset \quad \text{for all } 2\mu, \mu \in \Delta. \tag{10}$$

By the same way as in [Y1, Thm 5.1], one can show that the root system $\tilde{\Delta}$ is a reduced root system extended by G, i.e., $\tilde{\Delta}$ satisfies (7), (8), (9) and (10). In particular, the root system of a locally loop algebra is a reduced root system extended by $\mathbb{Z}$. Also, by the same way as in [Y1, Thm 5.1], letting

$$S_0 := \{g \in G \mid \mathcal{L}^g_0 \neq 0\}, \tag{11}$$

one gets

$$S_0 = S_\mu + S_\mu \tag{12}$$

for a short root μ.

Lemma 2.3. *A locally Lie G-torus $\mathcal{L}$ of type Δ is a directed union of Lie G-tori. More precisely, $\mathcal{L} = \bigcup_{\Delta'} \mathcal{L}_{\Delta'}$, where Δ' is a finite irreducible full subsystem of Δ containing a short root and $\mathcal{L}_{\Delta'}$ is the subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_\alpha$ for all $\alpha \in \Delta'$.*

Also, if G is torsion-free, then a locally Lie G-torus $\mathcal{L}$ of type Δ is a directed union of Lie n-tori. More precisely, $\mathcal{L} = \bigcup_{\Delta',G'} \mathcal{L}_{\Delta'}^{G'}$, where G' is a finitely generated subgroups of G and $\mathcal{L}_{\Delta'}^{G'}$ is the subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_{\alpha}^{g}$ for all $\alpha \in \Delta'$ and $g \in G'$.

Proof. Since $S = S_\mu$ for a short root μ generates G, it is easy to check that $\mathcal{L}_{\Delta'}$ is a Lie G-torus. Hence the statement is true since Δ is a directed union of finite irreducible full subsystems containing a short root (see [LN2, 3.15 (b) and the proof]). The second statement follows from the fact that G is a directed union of finitely generated subgroups. $\square$

3. LOCALLY LOOP ALGEBRAS

For any index set $\mathfrak{I}$, let

$$M_{\mathfrak{I}}(F) = \left\{ (a_{ij})_{i,j\in\mathfrak{I}} \,\middle|\, a_{ij} \in F \right\} \approx Map(\mathfrak{I} \times \mathfrak{I}, F)$$

be the set of all matrices of size $\mathfrak{I}$, which is naturally a vector space over F. Let $\mathrm{gl}_{\mathfrak{I}}(F)$ be the subspace of $M_{\mathfrak{I}}(F)$ consisting of matrices having only a finite number of nonzero entries. Then $\mathrm{gl}_{\mathfrak{I}}(F)$ is an associative algebra and a Lie algebra with the usual commutator bracket. Also, one can define the trace of a matrix in $\mathrm{gl}_{\mathfrak{I}}(F)$, and the subalgebra of $\mathrm{gl}_{\mathfrak{I}}(F)$ consisting of trace 0 matrices is denoted by $\mathrm{sl}_{\mathfrak{I}}(F)$:

$$\mathrm{sl}_{\mathfrak{I}}(F) = \{x \in \mathrm{gl}_{\mathfrak{I}}(F) \mid \mathrm{tr}(x) = 0\}$$

We note that $M_{\mathfrak{I}}(F)$ cannot be an algebra, but

$$M_{\mathfrak{I}}^{\mathrm{fin}}(F) := \{x \in M_{\mathfrak{I}}(F) \mid \text{each row and column of } x \text{ have only finite nonzeros}\} \tag{13}$$

is an associative algebra with the identity matrix $\iota = \iota_{\mathfrak{I}}$, and a Lie algebra with the commutator bracket. In fact, this gives the Lie algebra of derivations of $\mathrm{sl}_{\mathfrak{I}}(F)$ by Neeb [N1]. More precisely, we have

$$[M_{\mathfrak{I}}^{\mathrm{fin}}(F), \mathrm{sl}_{\mathfrak{I}}(F)] \subset \mathrm{sl}_{\mathfrak{I}}(F) \quad \text{and} \quad \mathrm{Der}_F(\mathrm{sl}_{\mathfrak{I}}(F)) \simeq \mathrm{ad}(M_{\mathfrak{I}}^{\mathrm{fin}}(F)).$$

The locally finite split simple Lie algebra of type $\mathrm{X}_{\mathfrak{I}}$ is defined as a subalgebra of $\mathrm{sl}_{\mathfrak{I}}(F)$, $\mathrm{sl}_{2\mathfrak{I}+1}(F)$ or $\mathrm{sl}_{2\mathfrak{I}}(F)$ as follows:
Type $\mathrm{A}_{\mathfrak{I}}$: $\mathrm{sl}_{\mathfrak{I}}(F)$;
Type $\mathrm{B}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}+1}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}+1}(F) \mid sx = -x^t s\}$;
Type $\mathrm{C}_{\mathfrak{I}}$: $\mathrm{sp}_{2\mathfrak{I}}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}}(F) \mid sx = x^t s\}$;
Type $\mathrm{D}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}}(F) = \{x \in \mathrm{sl}_{2\mathfrak{I}}(F) \mid sx = -x^t s\}$,
where $\mathfrak{I}$ is supposed to be infinite, x^t is the transpose of x, and

$$s = \begin{pmatrix} 0 & \iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } \mathrm{B}_{\mathfrak{I}},\ s = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} \text{ for } \mathrm{C}_{\mathfrak{I}},\ \text{ or }\ s = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \text{ for } \mathrm{D}_{\mathfrak{I}}. \tag{14}$$

Note that $s \in M_{\mathfrak{I}}^{\mathrm{fin}}(F)$, and $s^2 = \iota_{2\mathfrak{I}+1}$ for $\mathrm{B}_{\mathfrak{I}}$, $s^2 = -\iota_{2\mathfrak{I}}$ for $\mathrm{C}_{\mathfrak{I}}$ or $s^2 = \iota_{2\mathfrak{I}}$ for $\mathrm{D}_{\mathfrak{I}}$. Also, the $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{C}_{\mathfrak{I}}$ or $\mathrm{D}_{\mathfrak{I}}$ is the fixed algebra of $\mathrm{sl}_{2\mathfrak{I}+1}(F)$ or $\mathrm{sl}_{2\mathfrak{I}}(F)$ by an automorphism σ, defined as

$$\sigma(x) = -sx^t x \ \text{ for } \mathrm{B}_{\mathfrak{I}} \text{ or } \mathrm{D}_{\mathfrak{I}}, \text{ and } \sigma(x) = sx^t x \ \text{ for } \mathrm{C}_{\mathfrak{I}}. \tag{15}$$

Neeb and Stumme showed in [NS] that these algebras exhaust locally finite split simple Lie algebras. Also, they are considered as locally Lie 0-tori, which exhaust the infinite-dimensional locally Lie 0-tori since locally finite split simple Lie algebras are centrally closed (see [NS]). Note that Lie 0-tori are exactly the finite-dimensional split simple Lie algebras. Our interest in this paper is the class of locally Lie 1-tori.

Let $F[t^{\pm 1}]$ be the Laurent polynomial algebra over F. We call one of the following four Lie algebras an **untwisted locally loop algebra**:

Type $A^{(1)}_{\mathfrak{I}}$: $sl_{\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$;

Type $B^{(1)}_{\mathfrak{I}}$: $o_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 1}]$;

Type $C^{(1)}_{\mathfrak{I}}$: $sp_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$;

Type $D^{(1)}_{\mathfrak{I}}$: $o_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$.

(It is called an untwisted loop algebras if $\mathfrak{I}$ is finite.) One of the following three Lie algebras are called a **twisted locally loop algebra**:

(1) Type $B^{(2)}_{\mathfrak{I}}$: $(o_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 2}]\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,
where $\mathfrak{s}=F^{(2\mathfrak{I}+1)}$ is the natural $o_{2\mathfrak{I}+1}(F)$-module;

(2) Type $C^{(2)}_{\mathfrak{I}}$: $(sp_{2\mathfrak{I}}(F)\otimes F[t^{\pm 2}])\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,
where $\mathfrak{s}=\{x\in sl_{2\mathfrak{I}}(F)\mid sx=x^t s\}$;

(3) Type $BC^{(2)}_{\mathfrak{I}}$: $(o_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 2}])\oplus(\mathfrak{s}\otimes tF[t^{\pm 2}])$,
where $\mathfrak{s}=\{x\in sl_{2\mathfrak{I}+1}(F)\mid sx=x^t s\}$. (It is called a twisted loop algebra if $\mathfrak{I}$ is finite.)

Note that $sl_{2\mathfrak{I}}(F)=sp_{2\mathfrak{I}}(F)\oplus\mathfrak{s}$ and $sl_{2\mathfrak{I}+1}(F)=o_{2\mathfrak{I}+1}(F)\oplus\mathfrak{s}$. The Lie bracket of each untwisted type is natural, i.e., $[x\otimes t^m, y\otimes t^n]=[x,y]\otimes t^{m+n}$. The Lie bracket of type $C^{(2)}_{\mathfrak{I}}$ or $BC^{(2)}_{\mathfrak{I}}$ is also natural since

$$[sp_{2\mathfrak{I}}(F),\mathfrak{s}]\subset\mathfrak{s},\quad [\mathfrak{s},\mathfrak{s}]\subset sp_{2\mathfrak{I}}(F),$$
$$[o_{2\mathfrak{I}+1}(F),\mathfrak{s}]\subset\mathfrak{s}\quad\text{and}\quad [\mathfrak{s},\mathfrak{s}]\subset o_{2\mathfrak{I}+1}(F).$$

Note that $C^{(2)}_{\mathfrak{I}}$ or $BC^{(2)}_{\mathfrak{I}}$ is the fixed subalgebra of $sl_{2\mathfrak{I}}(F)\otimes F[t^{\pm 1}]$ or $sl_{2\mathfrak{I}+1}(F)\otimes F[t^{\pm 1}]$ by the automorphism $\hat{\sigma}$ defined as

$$\hat{\sigma}(x\otimes t^m):=(-1)^m\sigma(x)\otimes t^m \tag{16}$$

(see (15)). This construction is called a **twisting construction** by an automorphism σ.

For $B^{(2)}_{\mathfrak{I}}$, we have $o_{2\mathfrak{I}+1}(F)\mathfrak{s}\subset\mathfrak{s}$, and so we define the bracket of $o_{2\mathfrak{I}+1}(F)$ and $\mathfrak{s}$ by the natural action, i.e., $[x,v]=xv=-[v,x]$ for $x\in o_{2\mathfrak{I}+1}(F)$ and $v\in\mathfrak{s}$. However, there is no bracket on $\mathfrak{s}$. So we define a bracket on $\mathfrak{s}$ so that $[\mathfrak{s},\mathfrak{s}]\subset o_{2\mathfrak{I}+1}(F)$ as follows. First, let $(\cdot,\cdot)$ be the bilinear form on $\mathfrak{s}$ determined by s. Then there is a natural identification

$$o_{2\mathfrak{I}+1}(F)=D_{\mathfrak{s},\mathfrak{s}}:=\text{span}_F\{D_{v,v'}\mid v,v'\in\mathfrak{s}\},$$

where $D_{v,v'}\in\text{End}(\mathfrak{s})$ is defined by $D_{v,v'}(v'')=(v',v'')v-(v,v'')v'$ for $v''\in\mathfrak{s}$. Thus we define $[v,v']:=D_{v,v'}$. Note that $[v',v]=-[v,v']$. It is easy to check that the bracket

$$\begin{aligned}&[x\otimes t^{2m}+v\otimes t^{2m'+1}, x'\otimes t^{2n}+v'\otimes t^{2n'+1}]\\ =&[x,x']\otimes t^{2(m+n)}+D_{v,v'}\otimes t^{2(m'+n'+1)}+xv'\otimes t^{2(m+n')+1}-x'v\otimes t^{2(m'+n)+1}\end{aligned}$$

defines a Lie bracket for $m,m',n,n'\in\mathbb{Z}$.

There is a twisting construction for $B^{(2)}_{\mathfrak{I}}$ (see [N2]), which we will discuss in Section 7, but we think that the simple description of $B^{(2)}_{\mathfrak{I}}$ here is also important to develop the theory of locally Lie n-tori.

We often omit the term 'untwisted' or 'twisted' and simply say a locally loop algebra.

One can easily check that all locally loop algebras are centerless locally Lie 1-tori. For example, let Δ be the root system of type $BC^{(2)}_{\mathfrak{I}}$, and put $\mathfrak{g}=o_{2\mathfrak{I}+1}(F)$ and $\mathfrak{s}\subset sl_{2\mathfrak{I}+1}(F)$, as defined above. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Then $\mathfrak{h}$ decomposes $\mathfrak{g}$ into the root spaces, say $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\mu\in\Delta^{\text{red}}}\mathfrak{g}_\mu$, and $\mathfrak{s}$ into the weight spaces, say $\mathfrak{s} = \bigoplus_{\mu\in\Delta'} \mathfrak{s}_\mu$, where $\Delta' = \Delta_{ex} \cup \{0\}$ in case of $\Delta = \mathrm{BC}_{\mathfrak{J}}^{(2)}$. So the twisted locally loop algebra $(\mathfrak{g}\otimes F[t^{\pm 2}]) \oplus (\mathfrak{s}\otimes tF[t^{\pm 2}])$ of type $\mathrm{BC}_{\mathfrak{J}}^{(2)}$ is decomposed into

$$\bigoplus_{m\in\mathbb{Z}} \left((\mathfrak{h}\otimes Ft^{2m}) \oplus \bigoplus_{\mu\in\Delta^{\mathrm{red}}} (\mathfrak{g}_\mu \otimes Ft^{2m}) \oplus \bigoplus_{\mu\in\Delta_{ex}\cup\{0\}} (\mathfrak{s}_\mu \otimes Ft^{2m+1}) \right).$$

This gives a natural double grading by the groups $\langle\Delta\rangle$ and $\mathbb{Z}$, and one can check the axioms of a locally Lie torus. Also, the center of a locally Lie torus $\mathcal{L}$ is contained in $\mathcal{L}_0$, but $\mathcal{L}_0 = \mathfrak{h}\otimes F[t^{\pm 2}]$ in this example, and hence the locally loop algebra of type $\mathrm{BC}_{\mathfrak{J}}^{(2)}$ is a centerless locally Lie 1-torus. The grading subalgebra is equal to $\mathfrak{g} = \mathfrak{o}_{2\mathfrak{J}+1}(F)$. We call the $\mathfrak{g}$-module $\mathfrak{s}$ the **grading module**.

Now, let $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathcal{L}_\mu^m$ and $\mathcal{M} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathcal{M}_\mu^m$ be centerless Lie 1-tori whose root systems are the same root system $\tilde{\Delta}$ extended by $\mathbb{Z}$. Then there exists an isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{M}$ such that

$$\varphi(\mu_{\mathcal{L}}^\vee) = \mu_{\mathcal{M}}^\vee \quad \text{and} \quad \varphi(\mathcal{L}_\mu^m) = \mathcal{M}_\mu^m \quad \text{for all } \mu\in\Delta \text{ and } m\in\mathbb{Z}. \tag{17}$$

This can be directly proved using Gabber-Kac Theorem or repeat a similar argument in [ABGP]. (It was proved for the base field $\mathbb{C}$ in [ABGP] though.) Hence, a centerless Lie 1-torus is isomorphic to a loop algebra, and a Lie 1-torus with nontrivial center is isomorphic to a derived affine Lie algebra, which has always a 1-dimensional center. Also, for a Lie 1-torus $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathcal{L}_\mu^m$,

$$\text{the center of } \mathcal{L} \text{ is equal to } [\mathcal{L}_0^m, \mathcal{L}_0^{-m}] \text{ for any } 0\neq m\in\mathbb{Z}. \tag{18}$$

This is easily seen from the loop realization. Also, we have

$$\dim \sum_{m\in\mathbb{Z}} [\mathcal{L}_\mu^m, \mathcal{L}_{-\mu}^{-m}] = \begin{cases} 1 & \text{if } \mathcal{L} \text{ is loop} \\ 2 & \text{if } \mathcal{L} \text{ is derived affine} \end{cases} \tag{19}$$

since

$$\sum_{m\in\mathbb{Z}} [\mathcal{L}_\mu^m, \mathcal{L}_{-\mu}^{-m}] = \begin{cases} F\mu^\vee & \text{if } \mathcal{L} \text{ is loop} \\ F\mu^\vee + Fc & \text{if } \mathcal{L} \text{ is derived affine} \end{cases}$$

for $\mu\in\Delta$ and a central element c.

Lemma 3.1. *The center of a locally Lie 1-torus is at most 1-dimensional. More precisely, for a locally Lie 1-torus* $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\},\, m\in\mathbb{Z}} \mathcal{L}_\mu^m$,

$$\mathcal{L} \textit{ has 1-dimensional center} \iff \mathcal{L} \textit{ is a directed union of derived affine Lie algebras},$$

and

$$\mathcal{L} \textit{ is centerless} \iff \mathcal{L} \textit{ is a directed union of loop algebras}$$

in the following sense:

$$\mathcal{L} = \bigcup_{\Delta'\subset\Delta} \mathcal{L}_{\Delta'},$$

where Δ' is a finite irreducible full subsystem of Δ and $\mathcal{L}_{\Delta'}$ is the homogeneous subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_\mu$ for $\mu\in\Delta'$, and $\mathcal{L}_{\Delta'}$ is a derived affine Lie algebra if the center of $\mathcal{L}$ is 1-dimensional and a loop algebra if $\mathcal{L}$ is centerless. (Derived affine Lie algebras and loop algebras are not mixed!) In particular, $\dim\mathcal{L}_\mu^m \neq 0$ *(so* $\dim\mathcal{L}_\mu^m = 1$*) for all $\mu\in\Delta$ and $m\in\mathbb{Z}$. Also, the properties* (18) *of the center and* (19) *of root vectors above hold in a locally Lie torus too.*

Proof. Most of the statements follow from Lemma 2.3. In fact, Lie 1-tori are either derived affine Lie algebras or loop algebras, and so $\mathfrak{L}$ is a directed union of derived affine Lie algebras or loop algebras. Considering the loop realization of a derived affine Lie algebra, we find $\dim \mathfrak{L}_\mu^m = 1$ for all $\mu \in \Delta$ and $m \in \mathbb{Z}$. Moreover, suppose that C is a 2-dimensional subalgebra contained in the center. Then there exists a derived affine Lie algebra or a loop algebra containing C. But this is impossible since their centers have to be 1-dimensional or zero.

Now, we need to show that derived affine Lie algebras and loop algebras cannot appear simultaneously. If this happens, for example, $\mathfrak{L}'$ is a derived affine subalgebra and $\mathfrak{L}''$ is a loop subalgebra, then there exists a derived affine or a loop algebra containing both $\mathfrak{L}'$ and $\mathfrak{L}''$ as graded subalgebras. Suppose that $\mathfrak{L}', \mathfrak{L}'' \subset \mathfrak{L}'''$ for a loop algebra $\mathfrak{L}'''$. But this is impossible because of the property (18) above. So, suppose that $\mathfrak{L}', \mathfrak{L}'' \subset \mathfrak{L}'''$ for a derived affine Lie algebra $\mathfrak{L}'''$. Then this is also impossible because of the property (19) above. Thus a locally Lie 1-torus is either a directed union of derived affine Lie algebras, say $\mathfrak{L}_{da}$, or a directed union of loop algebras, say $\mathfrak{L}_{lo}$. It is now clear that the center of $\mathfrak{L}_{lo}$ is zero. To show the 1-dimensionality of the center of $\mathfrak{L}_{da}$, let C' be the center (1-dimensional) of a derived affine subalgebra of $\mathfrak{L}_{da}$. For any $\mu \in \Delta$ and $m \in \mathbb{Z}$, there exists a derived affine subalgebra M containing $\mathfrak{L}_\mu^m$ and C'. Considering the loop realization of M, we find that C' is the center of M, and in particular, $[C', \mathfrak{L}_\mu^m] = 0$. Hence C' is contained in the center of $\mathfrak{L}_{da}$, and so C' is the 1-dimensional center of $\mathfrak{L}_{da}$.

Finally, let $\mathfrak{L}$ be a locally Lie 1-torus. Then, (19) is clear. To show (18), let $Z := [\mathfrak{L}_0^k, \mathfrak{L}_0^{-k}]$ for $0 \neq k \in \mathbb{Z}$. For any $z \in Z$, $\mu \in \Delta$ and $m \in \mathbb{Z}$, there exists a derived affine subalgebra or a loop subalgebra containing z and $\mathfrak{L}_\mu^m$, and z is in the center of the subalgebra (by (18) for a Lie torus above). Hence $[z, \mathfrak{L}_\mu^m] = 0$ for all $\mu \in \Delta$ and $m \in \mathbb{Z}$. Therefore, Z is contained in the center of $\mathfrak{L}$. Thus $Z = 0$ or $\dim Z = 1$. If $Z = 0$, then there exists a loop subalgebra, and so $\mathfrak{L} = \mathfrak{L}_{lo}$. Hence $Z = 0$ is the center of $\mathfrak{L}$. If $\dim Z = 1$, then Z is the center of $\mathfrak{L}$ since the center of $\mathfrak{L}$ is at most 1-dimensional. □

For any two elements $x \otimes t^m$ and $y \otimes t^n$ in each locally loop algebra $\mathfrak{L}$, define the new bracket on a 1-dimensional central extension

$$\tilde{\mathfrak{L}} := \mathfrak{L} \oplus Fc$$

by

$$[x \otimes t^m, y \otimes t^n] := [x,y] \otimes t^{m+n} + m(x,y)\delta_{m+n,0}c, \tag{20}$$

where (x,y) is the trace form $\mathrm{tr}(xy)$, or for type $B_{\mathfrak{I}}^{(2)}$, the direct sum of the trace form and the bilinear form on $\mathfrak{s}$ determined by the symmetric matrix s above. Indeed, this gives a central extension since $\mathfrak{L}$ is a directed union of loop algebras and $\tilde{\mathfrak{L}}$ is locally a derived affine Lie algebra, i.e., a 1-dimensional central extension of a loop algebra.

Lemma 3.2. *A universal covering of a locally loop algebra is given by* (20) *above.*

Proof. Suppose that $\hat{\mathfrak{L}}$ is a universal covering of a locally loop algebra $\mathfrak{L}$. We know that $\dim_F Z(\hat{\mathfrak{L}}) \geq 1$ since $\tilde{\mathfrak{L}}$ above is a covering. So if $\dim Z(\hat{\mathfrak{L}}) > 1$, then there exists a covering $\mathfrak{L} \oplus Fc_1 \oplus Fc_2$ of $\mathfrak{L}$. Let $x_1, y_1, \ldots, x_m, y_m, u_1, v_1, \ldots, u_n, v_n \in \mathfrak{L}$ be such that $\sum_{i=1}^m [x_i, y_i] = c_1$ and $\sum_{i=1}^n [u_i, v_i] = c_2$. Let $\mathfrak{L}'$ be a loop subalgebra of $\mathfrak{L}$ containing x_i, y_i, u_j, v_j for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then $\mathfrak{L}' \oplus Fc_1 \oplus Fc_2$ is perfect, and so this is a covering of $\mathfrak{L}'$.

Now, a universal covering of a loop algebra has the 1-dimensional center, and so this is a contradiction. Hence, $\dim Z(\hat{\mathfrak{L}}) = 1$. But then it is clear that $\hat{\mathfrak{L}} \cong \tilde{\mathfrak{L}}$ since the unique morphism from $\hat{\mathfrak{L}}$ onto $\tilde{\mathfrak{L}}$ has to be one to one. □

Remark 3.3. By Lemma 3.1, a locally Lie 1-torus has at most 1-dimensional center. Thus if one shows that $\hat{\mathcal{L}}$ is a locally Lie 1-torus, then we also get a proof of Lemma 3.2. In fact, Neher showed that a universal covering of a locally Lie torus is a locally Lie torus in general (see [Ne3] and [Ne4]).

We now classify locally Lie 1-tori. The method we use here comes from [NS]. Namely, we will show that there is only one locally Lie 1-torus for each reduced root system extended by $\mathbb{Z}$. Root systems extended by $\mathbb{Z}$ are known in [Y1, Cor.10, 12]. (There are more general results in [LN2] .) Here is the list of all reduced root systems extended by $\mathbb{Z}$ of infinite rank, writing $\Delta \times S_\mu$ for $\{S_\mu\}_{\mu\in\Delta}$.

$$\mathrm{A}_{\mathfrak{I}} \times \mathbb{Z}, \quad \mathrm{B}_{\mathfrak{I}} \times \mathbb{Z}, \quad \mathrm{C}_{\mathfrak{I}} \times \mathbb{Z}, \quad \mathrm{D}_{\mathfrak{I}} \times \mathbb{Z},$$
$$\big((\mathrm{B}_{\mathfrak{I}})_{\mathrm{sh}} \times \mathbb{Z}\big) \sqcup \big((\mathrm{B}_{\mathfrak{I}})_{\mathrm{lg}} \times 2\mathbb{Z}\big), \quad \big((\mathrm{C}_{\mathfrak{I}})_{\mathrm{sh}} \times \mathbb{Z}\big) \sqcup \big((\mathrm{C}_{\mathfrak{I}})_{\mathrm{lg}} \times 2\mathbb{Z}\big),$$
$$\Big(\big((\mathrm{BC}^{(2)}_{\mathfrak{I}})_{\mathrm{sh}} \sqcup (\mathrm{BC}^{(2)}_{\mathfrak{I}})_{\mathrm{lg}}\big) \times \mathbb{Z}\Big) \sqcup \big((\mathrm{BC}^{(2)}_{\mathfrak{I}})_{\mathrm{ex}} \times (2\mathbb{Z}+1)\big).$$

Note that these 7 systems are exactly the root systems of locally loop algebras introduced above, and so we label each system by

$$\mathrm{A}^{(1)}_{\mathfrak{I}}, \quad \mathrm{B}^{(1)}_{\mathfrak{I}}, \quad \mathrm{C}^{(1)}_{\mathfrak{I}}, \quad \mathrm{D}^{(1)}_{\mathfrak{I}}, \quad \mathrm{B}^{(2)}_{\mathfrak{I}}, \quad \mathrm{C}^{(2)}_{\mathfrak{I}}, \quad \mathrm{BC}^{(2)}_{\mathfrak{I}}.$$

We also use the label for the root system as the **type** of a locally Lie 1-torus.

Now, first we show the following lemma when Δ is finite. Suppose that $\Pi \subset \Delta$ is an integral base, i.e., $\Delta \subset \langle \Pi \rangle$ and Π is linearly independent in the vector space which defines Δ.

Lemma 3.4. *Let $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\},m\in\mathbb{Z}} \mathcal{L}^m_\mu$ and $\mathcal{M} = \oplus_{\mu\in\Delta\cup\{0\},m\in\mathbb{Z}} \mathcal{M}^m_\mu$ be centerless Lie 1-tori of the same type $\tilde{\Delta}$. Let Π be an integral base of Δ containing a fixed short root $\nu \in \Delta$. Let $0 \neq x_\mu \in \mathcal{L}^0_\mu$ and $0 \neq y_\mu \in \mathcal{M}^0_\mu$ for each $\mu \in \Pi$. (Note $\Pi \subset \Delta^{\mathrm{red}}$.) Also, let $0 \neq x \in \mathcal{L}^1_\nu$ and $0 \neq y \in \mathcal{M}^1_\nu$. ($S_\nu = \mathbb{Z}$ since ν is short.) Then there exists a unique isomorphism ψ from $\mathcal{L}$ onto $\mathcal{M}$ such that $\psi(x) = y$, $\psi(\mu^\vee_{\mathcal{L}}) = \mu^\vee_{\mathcal{M}}$ and $\psi(x_\mu) = y_\mu$ for all $\mu \in \Pi$.*

Proof. By (17) above, there exists an isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{M}$ such that $\varphi(\mu^\vee_{\mathcal{L}}) = \mu^\vee_{\mathcal{M}}$ and $\varphi(\mathcal{L}^m_\mu) = \mathcal{M}^m_\mu$ for all $\mu \in \Delta$ and $m \in \mathbb{Z}$. Hence we have $y = a\varphi(x)$ and $y_\mu = a_\mu \varphi(x_\mu) =$ for some a and $a_\mu \in F^\times$. Let $f : \langle \Pi \rangle_{\mathbb{Z}} \times \mathbb{Z} \longrightarrow F^\times$ be the group homomorphism of the abelian groups defined by $f(\mu,0) = a_\mu$ and $f(0,1) = a$. Let D_f be the (diagonal) linear automorphism on $\mathcal{M}$ defined by $D_f(y) = f(\mu,m)y$ for $y \in \mathcal{M}^m_\mu$. Then D_f is an automorhism of the Lie algebra. Indeed, $D_f([y,y']) = f(\mu+\mu',m+m')[y,y'] = f((\mu,m)+(\mu',m'))[y,y'] = f(\mu,m)f(\mu',m')[y,y'] = [f(\mu,m)y, f(\mu',m')y'] = [D_f(y), D_f(y')]$ for $y \in \mathcal{M}^m_\mu$ and $y' \in \mathcal{M}^{m'}_{\mu'}$. Hence $\psi := D_f \circ \varphi$ is the required isomorphism.

For the uniqueness, note first that such an isomorphism is unique on $\mathcal{L}^{-1}_{-\nu}$ and $\mathcal{L}^0_{-\mu}$ for all $\mu \in \Pi$ since $[\mathcal{L}^1_\nu, \mathcal{L}^{-1}_{-\nu}] = F\nu^\vee$ (since $\mathcal{L}$ is centerless) and $[\mathcal{L}^0_\mu, \mathcal{L}^0_{-\mu}] = F\mu^\vee$. Thus it is enough to show that $\mathcal{L}$ is generated by $\mathcal{L}^1_\nu$, $\mathcal{L}^{-1}_{-\nu}$ and $\mathcal{L}^0_{\pm\mu}$ for all $\mu \in \Pi$. But by a standard argument (or see [St, Prop.9.9]), $\mathcal{L}^0$ (= the finite-dimensional split simple Lie algebra $\mathfrak{g}$) is generated by $\mathcal{L}^0_{\pm\mu}$ for all $\mu \in \Pi$. Then one can choose a root base of Δ so that ν is the negative highest short root. Using the affine realization of $\mathcal{L}$, it is clear that $\mathcal{L}$ is generated by $\mathcal{L}^0 = \mathfrak{g}$, $\mathcal{L}^1_\nu$ and $\mathcal{L}^{-1}_{-\nu}$. □

Now we can prove that there is a one to one correspondence between the class of centerless locally Lie 1-tori and the class of reduced root systems extended by $\mathbb{Z}$, and that locally loop algebras exhaust all centerless locally Lie 1-tori. Note that this method works for any cardinarity of Δ.

Theorem 3.5. *Let $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{m\in\mathbb{Z}} \mathcal{L}_\mu^m$ be a locally Lie 1-torus of type $\tilde{\Delta}$. If $\mathcal{L}$ is centerless, then $\mathcal{L}$ is graded isomorphic to the locally loop algebra of type $\tilde{\Delta}$, and if $\mathcal{L}$ has the nontrivial center, then $\mathcal{L}$ is graded isomorphic to a universal covering of the locally loop algebra of type $\tilde{\Delta}$ given by* (20)*.*

Proof. Note first that we already know this theorem for Lie 1-tori, i.e., the case where Δ is finite. Also, it is enough to show the case where $\mathcal{L}$ is centerless (see Lemma 3.2), and so assume that $\mathcal{L}$ is centerless. Let $\mathcal{M} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{m\in\mathbb{Z}} \mathcal{M}_\mu^m$ be a locally loop algebra of type $\tilde{\Delta}$. Also, let $\tilde{\Delta} = \{S_\mu\}_{\mu\in\Delta}$.

Fix a short root ν, and let $0 \neq x \in \mathcal{L}_\nu^1$ and $0 \neq e_\nu \otimes t \in \mathcal{M}_\nu^1$. ($S_\nu = \mathbb{Z}$ since ν is short.) Let Π be an integral base of Δ containing ν. Let $0 \neq x_\mu \in \mathcal{L}_\mu^0$ and $0 \neq e_\mu \otimes 1 \in \mathcal{M}_\mu^0$ for each $\mu \in \Pi$. (Note $\Pi \subset \Delta^{\mathrm{red}}$.) Then we claim that the map $\psi : \mu_{\mathcal{L}}^\vee \mapsto \mu_{\mathcal{M}}^\vee$ and $x_\mu \mapsto e_\mu \otimes 1$ for all $\mu \in \Pi$, and $x \mapsto e_\nu \otimes t$ extend an isomorphism from $\mathcal{L}$ onto $\mathcal{M}$. Indeed, let $\Gamma \subset \Pi$ be a finite irreducible subset containing ν, then Γ is an integral base of the irreducible root system $\Delta_\Gamma := \Delta \cap \langle\Gamma\rangle$.

Let $\tilde{\Delta}_\Gamma = \{S_\mu\}_{\mu\in\Delta_\Gamma}$ be the root system extended by $\mathbb{Z}$. Let $\mathcal{L}_\Gamma$ be the subalgebra determined by Δ_Γ, i.e., the subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_\mu^m$ for all $\mu \in \Delta_\Gamma$ and $m \in \mathbb{Z}$, which is a centerless Lie 1-torus of type $\tilde{\Delta}_\Gamma$ (see Lemma 3.1). Similarly, let $\mathcal{M}_\Gamma$ be the subalgebra of $\mathcal{M}$ determined by Δ_Γ. Then by Lemma 3.4, there exists a unique graded isomorphism ψ_Γ from $\mathcal{L}_\Gamma$ onto $\mathcal{M}_\Gamma$ such that $\psi_\Gamma(x_\mu) = e_\mu \otimes 1$ for all $\mu \in \Gamma$ and $x \mapsto e_\nu \otimes t$.

Suppose that $\Gamma_1, \Gamma_2 \subset \Pi$ are finite irreducible subsets containing ν so that $\mathcal{L}_{\Gamma_1} \subset \mathcal{L}_{\Gamma_2}$. Then the uniqueness of the isomorphisms ψ_{Γ_1} and ψ_{Γ_2} implies that they agree on $\mathcal{L}_{\Gamma_1}$. Since $\mathcal{L}$ is the directed union of the subalgebras $\mathcal{L}_\Gamma$ ($\Gamma \subset \Pi$ is a finite irreducible subset), one can define an isomorphism $\psi : \mathcal{L} \longrightarrow \mathcal{M}$ be $\psi(x) = \psi_\Gamma(x)$ for $x \in \mathcal{L}_\Gamma$, which has the required properties. □

Note that we defined in (20) the Lie bracket of a universal covering of a locally loop algebra, using a symmetric bilinear form $(\cdot,\cdot)$ on a locally loop algebra. More precisely, one can write $(\cdot,\cdot) = \mathrm{tr}(\cdot,\cdot) \otimes \varepsilon(\cdot,\cdot)$, where $\varepsilon(t^m, t^n) = \delta_{m+n,0}$. In fact, it is easily checked that this form is invariant, graded (as a form of a Lie torus), and nondegenerate. We simply say a **form** for a symmetric invariant graded bilinear form on a Lie G-torus. We will use the following lemma later:

Lemma 3.6. *There exists a nonzero form on a locally Lie* 1*-torus. Also, such a form is unique up to a nonzero scalar. In particular, a form of a locally loop algebra is equal to $c(\cdot,\cdot)$ for some $c \in F$, where $(\cdot,\cdot)$ is used in* (20)*.*

Proof. Only the uniqueness part is not clear. But such a form is unique up to a scalar for a Lie 1-torus (see for example, [Y2]). Thus it follows from a local argument since a locally Lie 1-torus is a directed union of Lie 1-tori. □

4. Locally affine Lie algebras

Let us recall locally extended affine Lie algebras (cf. [MY]). A subalgebra $\mathcal{H}$ of a Lie algebra $\mathcal{L}$ is called ad-diagonalizable if

$$\mathcal{L} = \bigoplus_{\xi\in\mathcal{H}^*} \mathcal{L}_\xi,$$

where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$ and

$$\mathcal{L}_\xi = \{x \in \mathcal{L} \mid [h,x] = \xi(h)x \text{ for all } h \in \mathcal{H}\}.$$

This decomposition is sometimes called the **root space decomposition** (of $\mathcal{L}$ with respect to an ad-diagonalizable subalgebra $\mathcal{H}$). Note that an ad-diagonalizable subalgebra $\mathcal{H}$ is automatically abelian. To confirm it, we need the well-known fact that every submodule of a weight module is also a weight module. One can use a common trick for the proof as for example, in [MP, Prop.2.1], but they assume $\mathcal{H}$ to be abelian. To make sure that this assumption is unnecessary, we prove this here. Namely, in this case, we obtain

$$\mathcal{H} = \oplus_{\xi \in \mathcal{H}^*} \mathcal{H}_\xi,$$

where $\mathcal{H}_\xi = \mathcal{L}_\xi \cap \mathcal{H}$. Let us suppose $\mathcal{H} \neq \oplus_{\xi \in \mathcal{H}^*} \mathcal{H}_\xi$. Then, there exists $x \in \mathcal{H}$ such that x can be written as $x = x_1 + \cdots + x_n$ with $n > 1$ satisfying $x_i \in \mathcal{L}_{\xi_i} \setminus \mathcal{H}$ for all i. Take $x \in \mathcal{H}$ among all such elements satisfying that n is minimal, and choose $h \in \mathcal{H}$ such that $\xi_1(h) \neq \xi_2(h)$. Then, $x' := \text{ad } h(x) - \xi_1(h)x = (\xi_2(h) - \xi_1(h))x_2 + \cdots + (\xi_n(h) - \xi_1(h))x_n \in \mathcal{H}$. This contradicts the minimality of n. Hence, actually we have $\mathcal{H} = \oplus_{\xi \in \mathcal{H}^*} \mathcal{H}_\xi$.

Now, suppose $h \in \mathcal{H}_\xi$ and $h' \in \mathcal{H}_{\xi'}$. Then, $[h,h'] = \xi'(h)h' = -\xi(h')h$. Hence, if h and h' are linearly independent, then $[h,h'] = 0$. Also we see $[h,h'] = 0$ if they are linearly dependent. This means that $\mathcal{H}$ is always abelian. In particular, $\mathcal{H} = \mathcal{H}_0 \subset \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$.

An element of the set

$$R = \{\xi \in \mathcal{H}^* \mid \mathcal{L}_\xi \neq 0\}$$

is called a **root**.

Let $\mathcal{L}$ be a Lie algebra, $\mathcal{H}$ a subalgebra of $\mathcal{L}$, and $\mathcal{B}$ a symmetric invariant bilinear form of $\mathcal{L}$. A triple $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ (or simply $\mathcal{L}$) is called a **locally extended affine Lie algebra** or a **LEALA** for short if it satisfies the following 4 axioms: (We will explain what is $R^\times$ later.)

(A1) $\mathcal{H}$ is ad-diagonalizable and self-centralizing, i.e., $\mathcal{L} = \bigoplus_{\xi \in \mathcal{H}^*} \mathcal{L}_\xi$ and $\mathcal{H} = \mathcal{L}_0 = C_{\mathcal{L}}(\mathcal{H})$;
(A2) $\mathcal{B}$ is nondegenerate;
(A3) $\text{ad}x \in \text{End}_F \mathcal{L}$ is locally nilpotent for all $\xi \in R^\times$ and all $x \in \mathcal{L}_\xi$,
(A4) $R^\times$ is irreducible.

Moreover,

(i) If $\mathcal{H}$ is finite-dimensional, then $\mathcal{L}$ is called an **extended affine Lie algebra** or an **EALA** for short.
(ii) If $R^\times = \emptyset$, then $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ is called a **null LEALA** (or a **null EALA** if $\mathcal{H}$ is finite-dimensional) or simply a **null system**. Note that if $R^\times = \emptyset$, then the axioms (A3) and (A4) are empty statements.

Now, using (A1) and (A2), we find that $\mathcal{B}_{\mathcal{L}_\xi \times \mathcal{L}_{-\xi}}$ is nondegenerate for all $\xi \in R$. In particular,

$$\mathcal{B}_{\mathcal{H} \times \mathcal{H}} \text{ is nondegenerate.}$$

Claim 4.1. *For each $\xi \in R$, there exists a unique $t_\xi \in \mathcal{H}$ such that $\mathcal{B}(h, t_\xi) = \xi(h)$ for all $h \in \mathcal{H}$.*

Proof. By the nondegeneracy of $\mathcal{B}_{\mathcal{L}_\xi \times \mathcal{L}_{-\xi}}$, there exist $x \in \mathcal{L}_\xi$ and $y \in \mathcal{L}_{-\xi}$ such that $\mathcal{B}(x,y) = 1$. Let $t_\xi := [x,y] \in \mathcal{H}$. Then

$$\mathcal{B}(h, t_\xi) = \mathcal{B}(h, [x,y]) = B([h,x], y) = \xi(h)\mathcal{B}(x,y) = \xi(h)$$

for all $h \in \mathcal{H}$. The uniqueness of t_ξ follows from the nondegeneracy of $\mathcal{B}_{\mathcal{H}\times\mathcal{H}}$. $\square$

Thus there exists **the induced form on the vector space spanned by** R **over** F, simply denoted $(\cdot,\cdot)$. Namely,

$$(\xi,\eta) := \mathcal{B}(t_\xi, t_\eta)$$

for $\xi,\eta \in R$.

Now we call an element of

$$R^\times := \{\xi \in R \mid (\xi,\xi) \neq 0\}$$

an **anisotropic root**. Thus the meaning of the axiom (A4) is that $R^\times = R_1 \cup R_2$ and $(R_1,R_2) = 0$ imply $R_1 = \emptyset$ or $R_2 = \emptyset$.

Remark 4.2. We occasionally assume $R^\times \neq \emptyset$ for a LEALA or an EALA (i.e., excluding null systems) without mentioning if it is clear from the context, for example, an occasion when we say an anisotropic root of a LEALA or an EALA. In fact we assume $R^\times \neq \emptyset$ through out the paper.

Remark 4.3. We note that there was one more axiom for a LEALA in [MY], but it turns out that the axiom is unnecessary by Claim 4.1 above.

We say that a triple $(\mathcal{L},\mathcal{H},\mathcal{B})$ is called **admissible** if it satisfies (A1) and (A2). A fundamental property of admissible triples is the following:

Lemma 4.4. *For* $\xi \in R$ *and all* $x \in \mathcal{L}_\xi$ *and* $y \in \mathcal{L}_{-\xi}$*, we have*

$$[x,y] = \mathcal{B}(x,y)t_\xi. \tag{21}$$

Proof. Let $h := [x,y] - \mathcal{B}(x,y)t_\xi \in \mathcal{H}$. Then for all $h' \in \mathcal{H}$, we have $\mathcal{B}(h,h') = \mathcal{B}(x,[y,h']) - \mathcal{B}(x,y)\mathcal{B}(t_\xi,h') = \mathcal{B}(x,y)\xi(h') - \mathcal{B}(x,y)\xi(h') = 0$. Hence, by the nondegeneracy of $\mathcal{B}_{\mathcal{H}\times\mathcal{H}}$, we get $h = 0$. $\square$

We can and do scale the above form $(\cdot,\cdot)$ by a nonzero scalar so that $(\xi,\eta) \in \mathbb{Q}$ for all $\xi,\eta \in R^\times$. Let V be the $\mathbb{Q}$-span of R, say

$$V := \mathrm{span}_{\mathbb{Q}} R.$$

We showed the Kac conjecture in [MY, Thm 3.10] that

$$\text{the scaled form } (\cdot,\cdot) \text{ on } V \text{ is positive semidefinite.} \tag{22}$$

As a corollary, $(W,R^\times)$ becomes a reduced locally extended affine root system, where $W = \mathrm{span}_{\mathbb{Q}} R^\times$ (see [MY, §4] and [Y3]). We simply call R the set of roots, but call $R^\times$ a locally extended affine root system, say a LEARS, because $R^\times$ is a natural generalization of classical (locally finite) irreducible root systems, affine root systems in the sense of Macdonald [Ma] or extended affine root systems in the sense of Saito [S]. We do not recall the definition of LEARS here because we do not need it in this paper. The reader can find the precise definition in [Y3].

The dimension of the radical of V is called the **null dimension** for a LEALA. If the additive subgroup of V generated by

$$R^0 := \{\xi \in R \mid (\xi,\xi) = 0\},$$

the set of **isotropic roots** or **null roots**, is free, we call the rank the **nullity** of a LEALA. Thus we only use the term *nullity* when $\langle R^0\rangle$ is a free abelian group. So if we say that a LEALA $\mathcal{L}$ has nullity, it means that $\langle R^0\rangle$ is a free abelian group. (In [MY], we used the term *null rank* for nullity, and *nullity* for null dimension. But we change the names to be consistent with the notion of nullity in [Ne2].)

The **core** of a LEALA $\mathcal{L}$, denoted by $\mathcal{L}_c$, is the subalgebra of $\mathcal{L}$ generated by the root spaces $\mathcal{L}_\alpha$ for all $\alpha \in R^\times$. ($\mathcal{L}_c$ becomes an ideal of $\mathcal{L}$.) If the centralizer of $\mathcal{L}_c$ in $\mathcal{L}$ is contained in $\mathcal{L}_c$, then $\mathcal{L}$ is called **tame**. Note that there is no core for a null system, and so there is not a concept of tameness for a null system.

Now, since $(W, R^\times)$ is a reduced LEARS, there exist a locally finite irreducible root system Δ and a family of subsets $\{S_\mu\}_{\mu\in\Delta}$ of $\operatorname{rad} W$ (the radical of W) indexed by Δ so that

$$R^\times = \bigcup_{\mu\in\Delta} (\mu + S_\mu)$$

and $\{S_\mu\}_{\mu\in\Delta}$ is a reduced root system extended by $G = \langle \cup_{\mu\in\Delta} S_\mu \rangle$ (see [[MY]). Note that

$$\operatorname{rad} W = (\operatorname{rad} V) \cap W.$$

For each $\mu \in \Delta$ and $g \in G$, if $g \in S_\mu$, let

$$(\mathcal{L}_c)^g_\mu := \mathcal{L}_c \cap \mathcal{L}_{\mu+g},$$

and if $g \notin S_\mu$, let $(\mathcal{L}_c)^g_\mu := 0$. Then one can easily show that

$$\mathcal{L}_c = \bigoplus_{\mu\in\Delta\cup\{0\}} \bigoplus_{g\in G} (\mathcal{L}_c)^g_\mu,$$

where $(\mathcal{L}_c)^g_0 := \sum_{\mu\in\Delta} \sum_{g=h+k} [(\mathcal{L}_c)^h_\mu, (\mathcal{L}_c)^k_{-\mu}]$, and that

$$\mathcal{L}_c \text{ is a locally Lie } G\text{-torus of type } \Delta, \tag{23}$$

or more precisely, type $\{S_\mu\}_{\mu\in\Delta}$.

Also, letting

$$\mathcal{L}^g_c := \bigoplus_{\mu\in\Delta\cup\{0\}} (\mathcal{L}_c)^g_\mu,$$

we get a G-graded Lie algebra

$$\mathcal{L}_c = \bigoplus_{g\in G} \mathcal{L}^g_c.$$

Here we give a couple of definitions about graded algebras in general.

Definition 4.5. Let V be a vector space over $\mathbb{Q}$, and G an additive subgroup of V. Let

$$\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$$

be a G-graded algebra. Define a linear transformation d_i on $\mathcal{A}$ by

$$d_i(a_g) = g_i a_g$$

for $a_g \in \mathcal{A}^g$, where g_i is the i-coordinate of g obtained by a fixed basis of V. Note that d_i depends on a basis of V. Then d_i is a derivation of $\mathcal{A}$, for we have

$$d_i(a_g a_h) = (g_i + h_i) a_g a_h = g_i a_g a_h + h_i a_g a_h = d(a_g) g_h + a_g d(a_h)$$

for $a_h \in \mathcal{A}^h$ and $h \in G$. We call each d_i an ***i*-th coordinate-degree derivation**.

If $\dim_F V = 1$, then d_1 is simply called a **degree derivation**.

Now for a LEALA, we define a special type.

Definition 4.6. If a LEALA $\mathcal{L}$ contains all coordinate-degree derivations acting on the G-graded core, then $\mathcal{L}$ is called **standard**.

We classified LEALAs of nullity 0 in [MY, Thm 8.7]. We describe the tame LEALAs of nullity 0 in a slightly different way from the description in [MY]:

Let $M := M_{\mathfrak{I}}(F)$, $M_{2\mathfrak{I}+1}(F)$ or $M_{2\mathfrak{I}}(F)$ be the space of matrices of an infinite size $\mathfrak{I}$, $2\mathfrak{I}+1$ or $2\mathfrak{I}$, respectively, and $T_{\mathfrak{I}}$, $T_{2\mathfrak{I}+1}$ or $T_{2\mathfrak{I}}$ the subspace of M consisting of diagonal matrices. Let T' be a complement of $F\iota$ in $T_{\mathfrak{I}}$, where $\iota = \iota_{\mathfrak{I}}$ is the identity matrix so that

$$T_{\mathfrak{I}} = T' \oplus F\iota.$$

Then the following is the list of **maximal tame LEALAs of nullity** 0:

- Type $\mathrm{A}_{\mathfrak{I}}$:
$$\mathrm{sl}_{\mathfrak{I}}(F) + T' \text{ with a Cartan subalgebra } T' \tag{24}$$
(note that T' is not unique),
- Type $\mathrm{B}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}+1}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}+1} \mid sx = -xs\},$$
- Type $\mathrm{C}_{\mathfrak{I}}$: $\mathrm{sp}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}} \mid sx = xs\},$$
- Type $\mathrm{D}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra T^+, where
$$T^+ := \{x \in T_{2\mathfrak{I}} \mid sx = -xs\},$$

  and each matrix s is the same s defined in (14).

We note that $F\iota$ is the center of $\mathrm{sl}_{\mathfrak{I}}(F) + T$, and that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T' \cong \big(\mathrm{sl}_{\mathfrak{I}}(F) + T\big)/F\iota$$

for any T'.

As in the case of locally finite split simple Lie algebras, each of type $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{C}_{\mathfrak{I}}$ or $\mathrm{D}_{\mathfrak{I}}$ is the fixed algebra of $\mathrm{sl}_{2\mathfrak{I}+1}(F) + T_{2\mathfrak{I}+1}$ or $\mathrm{sl}_{2\mathfrak{I}}(F) + T_{2\mathfrak{I}}$ by the automorphism σ defined in (15). This is the reason why we wrote T^+ since this is the eigenspace of eigenvalue $+1$ of σ. We will write the eigenspace of eigenvalue -1 of σ as T^-.

Any subalgebra of a maximal tame LEALA of nullity 0 containing each locally finite split simple Lie algebra is a tame LEALA of nullity 0. Namely, let $\mathcal{L}$ be a tame LEALA of nullity 0. Then

- Type $\mathrm{A}_{\mathfrak{I}}$: $\mathrm{sl}_{\mathfrak{I}}(F) \subset \mathcal{L} \subset \mathrm{sl}_{\mathfrak{I}}(F) + T'$ with a Cartan subalgebra $\mathcal{L} \cap T'$,
- Type $\mathrm{B}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}+1}(F) \subset \mathcal{L} \subset \mathrm{o}_{2\mathfrak{I}+1}(F) + T^+$ with a Cartan subalgebra $\mathcal{L} \cap T^+$,
- Type $\mathrm{C}_{\mathfrak{I}}$: $\mathrm{sp}_{2\mathfrak{I}}(F) \subset \mathcal{L} \subset \mathrm{sp}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra $\mathcal{L} \cap T^+$,
- Type $\mathrm{D}_{\mathfrak{I}}$: $\mathrm{o}_{2\mathfrak{I}}(F) \subset \mathcal{L} \subset \mathrm{o}_{2\mathfrak{I}}(F) + T^+$ with a Cartan subalgebra $\mathcal{L} \cap T^+$.

We observe $T_{\mathfrak{I}}$ more carefully. Set

$$T_{\mathfrak{I}}^{as} = \{d \in T_{\mathfrak{I}} \mid d \text{ is almost scalar }\}$$

i.e., d has the same diagonal entries except finite numbers of diagonal entries. Clearly $T_{\mathfrak{I}}^{as}$ is a subspace of $M_{\mathfrak{I}}(F)$.

Lemma 4.7. *Let* $\mathfrak{h}$ *be the diagonal subalgebra of* $\mathrm{sl}_{\mathfrak{I}}(F)$. *Then, we have*

$$T_{\mathfrak{I}}^{as} = \mathfrak{h} \oplus F\iota \oplus Fe_{jj}, \tag{25}$$

where e_{jj} *is the matrix in* $M_{\mathfrak{I}}(F)$ *that the* (j,j)*-entry is* 1 *and all the other entries are* 0 *for any fixed index* $j \in \mathfrak{I}$. *In particular, we have*

$$\mathrm{gl}_{\mathfrak{I}}(F) = \mathrm{sl}_{\mathfrak{I}}(F) \oplus Fe_{jj}$$

for any $j \in \mathfrak{I}$.

Also, let I be any finite subset of $\mathfrak{I}$, and $\iota_I := \sum_{i\in I} e_{ii}$. Then we have

$$T_{\mathfrak{I}}^{as} = \mathfrak{h}_I \oplus F\iota_I \oplus T_{\mathfrak{I}\setminus I}^{as}, \tag{26}$$

where $\mathfrak{h}_I$ is the subspace of $\mathfrak{h}$ such that all (k,k)-components for $k \in \mathfrak{I}\setminus I$ are 0, and $T_{\mathfrak{I}\setminus I}^{as}$ is the subspace of $T_{\mathfrak{I}}^{as}$ such that all (i,i)-components for $i \in I$ are 0.

Moreover, we have

$$T_{\mathfrak{I}} = \mathfrak{h}_I \oplus F\iota_I \oplus T_{\mathfrak{I}\setminus I}, \tag{27}$$

where $T_{\mathfrak{I}\setminus I}$ is the subspace of $T_{\mathfrak{I}}$ such that all (i,i)-components for $i \in I$ are 0.

Proof. It is clear that $T_{\mathfrak{I}}^{as} \supset \mathfrak{h} \oplus F\iota \oplus Fe_{jj}$. For the other inclusion, let $x \in T_{\mathfrak{I}}^{as}$. Then there exists $a \in F$ such that $y := x - a\iota \in T_{\mathfrak{I}} \cap \mathrm{gl}_{\mathfrak{I}}(F)$. Hence $y = y - \mathrm{tr}(y)e_{jj} + \mathrm{tr}(y)e_{jj}$ and note that $h := y - \mathrm{tr}(y)e_{jj} \in \mathfrak{h}$. Thus $x = h + a\iota + \mathrm{tr}(y)e_{jj} \in \mathfrak{h} \oplus F\iota \oplus Fe_{jj}$.

For the second decomposition, we have $T_{\mathfrak{I}}^{as} = T_I \oplus T_{\mathfrak{I}\setminus I}^{as}$, where T_I is a subset of $T_{\mathfrak{I}}^{as}$ such that all (k,k)-components for $k \in \mathfrak{I}\setminus I$ are 0. But then it is easy to see that $T_I = \mathfrak{h}_I \oplus F\iota_I$. The last decomposition is now clear. □

Now, we did not mention about the defining bilinear form $\mathcal{B}$ of $\mathcal{L}$ in general. As it was described in [MY], one can state as follows:

Let $\mathfrak{g}$ be

$$\mathrm{sl}_{\mathfrak{I}}(F),\ \mathrm{o}_{2\mathfrak{I}+1}(F),\ \mathrm{sp}_{2\mathfrak{I}}(F) \quad \text{or} \quad \mathrm{o}_{2\mathfrak{I}}(F),$$

which is the locally finite split simple Lie algebra contained in $\mathcal{L}$, as defined above. The restriction $\mathcal{B}_{\mathcal{L}\times\mathfrak{g}}$ of our $\mathcal{B}$ is a nonzero scalar multiple of the trace form, and the rest of part can be any symmetric bilinear form.

In fact, we did not clearly say the reason in [MY] why the restriction $\mathcal{B}_{\mathcal{L}\times\mathfrak{g}}$ of $\mathcal{B}$ is a nonzero scalar multiple of the trace form. But this follows from the perfectness of $\mathfrak{g}$ and the invariance of $\mathcal{B}$. We summarize this phenomenon in a slightly general setup. Let us call a symmetric invariant bilinear form simply a **form** for convenience.

Lemma 4.8. *Let L be a Lie algebra with a certain form B and let $\mathfrak{g}$ be a perfect ideal of L. If any form of $\mathfrak{g}$ is equal to $B' := B\mid_{\mathfrak{g}\times\mathfrak{g}}$, up to a scalar, then any invariant bilinear form on $L\times\mathfrak{g}$ or on $\mathfrak{g}\times L$ is equal to $B\mid_{L\times\mathfrak{g}}$ or $B\mid_{\mathfrak{g}\times L}$, up to a scalar.*

Proof. Let E be an invariant bilinear form on $L\times\mathfrak{g}$. For $x \in L$ and $y \in \mathfrak{g}$, since $y = \sum_i [u_i, v_i]$ for some $u_i, v_i \in \mathfrak{g}$, we have

$$E(x,y) = E(x, \sum_i [u_i,v_i]) = c\sum_i B'([x,u_i],v_i) = c\sum_i B([x,u_i],v_i) = cB(x,\sum_i [u_i,v_i]) = cB(x,y)$$

for some $c \in F$. One can similarly prove the desired result for $\mathfrak{g}\times L$. □

Recall the Lie algebra

$$M_{\mathfrak{I}}^{\mathrm{fin}}(F) = \{x \in M_{\mathfrak{I}}(F) \mid \text{each row and column of } x \text{ have only finite nonzeros}\},$$

which can be identified with the derivations of $\mathrm{gl}_{\mathfrak{I}}(F)$ (see [N1]). One can extend each automorphism σ on $M_{\mathfrak{K}}^{\mathrm{fin}}(F)$, where $\mathfrak{K} = 2\mathfrak{I}$ or $2\mathfrak{I}+1$. Thus each locally finite split simple Lie algebra $\mathfrak{g} := \mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$ is a perfect ideal of each $M_{\mathfrak{K}}^{\mathrm{fin}}(F)^{\sigma}$.

Lemma 4.9. *Let L be any subalgebra of $M_{\mathfrak{I}}^{\mathrm{fin}}(F)$. Let M be any subalgebra of $\mathrm{gl}_{\mathfrak{I}}(F)$. Then one can define the trace form* tr *on $L\times M$ and $M\times L$, and this* tr *is invariant.*

Hence if L contains $\mathrm{sl}_{\mathfrak{I}}(F)$, then any invariant bilinear form on $L\times\mathrm{sl}_{\mathfrak{I}}(F)$ or on $\mathrm{sl}_{\mathfrak{I}}(F)\times L$ is equal to $c\,\mathrm{tr}$ for some $c \in F$.

Moreover, if L is a subalgebra of $M^{\mathrm{fin}}_{\mathfrak{K}}(F)^\sigma$ containing $\mathfrak{g} = \mathrm{sl}_{\mathfrak{K}}(F)^\sigma$, then any invariant bilinear form on $L \times \mathfrak{g}$ or on $\mathfrak{g} \times L$ is equal to $c\,\mathrm{tr}$ for some $c \in F$.

Proof. Since $xy \in \mathrm{gl}_{\mathfrak{I}}(F)$ for $x \in L$ and $y \in M$, the trace form $\mathrm{tr}(xy)$ is well-defined. Thus we only need to show the invariance, i.e., $\mathrm{tr}([A,B]y) = \mathrm{tr}(A[B,y])$ for $A, B \in L$ and $y \in M$. But it is enough to show this for $y = e_{ij}$ (the matrix unit of (i, j)-component). Let $A = (a_{mn})$, $B = (b_{mn})$ and $C = (c_{mn}) = [A,B]$. Then, $c_{mn} = \sum_k (a_{mk}b_{kn} - b_{mk}a_{kn})$ and $\mathrm{tr}([A,B]y) = \mathrm{tr}((c_{mn})e_{ij}) = c_{ji} = \sum_k (a_{jk}b_{ki} - b_{jk}a_{ki})$ and $\mathrm{tr}(A[B,y]) = \mathrm{tr}((a_{mn})\big(\sum_m b_{mi}e_{mj} - \sum_n b_{jn}e_{in}\big)) = \sum_k (a_{jk}b_{ki} - a_{ki}b_{jk})$. Therefore, the trace form is invariant. One can similarly prove the case for $M \times L$.

By [NS, Lem. II.11], any form on $\mathrm{sl}_{\mathfrak{I}}(F)$ is equal to $c\,\mathrm{tr}$ for some $c \in F^\times$. Thus the second and the last statements follow from Lemma 4.8 since $\mathrm{sl}_{\mathfrak{I}}(F)$ or $\mathfrak{g}$ is a perfect ideal of L. □

Suppose that $\mathcal{B}$ is a symmetric invariant bilinear form on

$$\mathcal{M}_{\mathfrak{I}} := \mathrm{sl}_{\mathfrak{I}}(F) + T_{\mathfrak{I}}.$$

Then, by Lemma 4.8, the restriction of $\mathcal{B}$ to $\mathcal{M}_{\mathfrak{I}} \times \mathrm{sl}_{\mathfrak{I}}(F)$ or $\mathrm{sl}_{\mathfrak{I}}(F) \times \mathcal{M}_{\mathfrak{I}}$ is equal to $c\,\mathrm{tr}$ for some $c \in F$. We claim that such a form $\mathcal{B}$ does exist. For this, we choose any complement T' of $\mathfrak{h}$ in T, i.e.,

$$T = T' \oplus \mathfrak{h}.$$

Let

$$\psi : T' \times T' \longrightarrow F$$

be an arbitrary symmetric bilinear form. We now define a symmetric bilinear form $\mathcal{B}$ on $\mathcal{M}_{\mathfrak{I}}$ as

$$\mathcal{B}(x,y) = \psi(x,y)$$

on T', and $c\,\mathrm{tr}$ on $\mathcal{M}_{\mathfrak{I}} \times \mathrm{sl}_{\mathfrak{I}}(F)$ and $\mathrm{sl}_{\mathfrak{I}}(F) \times \mathcal{M}_{\mathfrak{I}}$. To show that $\mathcal{B}$ is invariant, we prove the following.

Claim 4.10. *Let $x \in T \setminus F\iota$ and $y_k \in \mathrm{sl}_{\mathfrak{I}}(F)$ for $k = 1, 2, \ldots, r$. Then there exist a finite subset I of $\mathfrak{I}$, $0 \neq h \in \mathfrak{h}$ and $g \in T$ such that $y_k \in \mathrm{sl}_I(F)$ for all k, and $h \in \mathfrak{h}_I$,*

$$x = h + g, \quad [x, y_k] = [h, y_k] \quad \text{and} \quad \mathcal{B}(x, y_k) = \mathcal{B}(h, y_k)$$

for all k. Moreover, there exist $y \in \mathrm{sl}_{\mathfrak{I}}(F)$ and $h' \in \mathfrak{h}$ such that $[x,y] \neq 0$ and

$$\mathcal{B}(x, h') \neq 0. \tag{28}$$

Proof. Let I be a finite subset of $\mathfrak{I}$ so that $y_k \in \mathrm{sl}_I(F)$ for all k. Moreover, if the $I \times I$-block submatrix of x is a scalar matrix, then we enlarge I until the $I \times I$-block submatrix of x is not a scalar matrix. For such I, by (27) in Lemma 4.7, there exists $0 \neq h \in \mathfrak{h}_I$ so that $x = h + b\iota_I + x'$ for some $b \in F$ and $x' \in T_{\mathfrak{I}\setminus I}$. Put $g := b\iota_I + x'$. Then clearly $[g, y_k] = 0$. Also, we have $\mathcal{B}(g, y_k) = c\,\mathrm{tr}(gy_k) = cb\,\mathrm{tr}(y_k) = 0$ since $\mathrm{tr}(y_k) = 0$.

To show the second statement, it is enough to choose $y \in \mathrm{sl}_I(F)$ and $h' \in \mathfrak{h}_I$ such that $[h,y] \neq 0$ and $\mathrm{tr}(hh') \neq 0$. □

We now prove that $\mathcal{B}$ is invariant. It is enough to consider the case involving some elements in T'. Since T' is an abelian subalgebra, the case involving three elements in T' is clear.

For the case involving one element in T', let $x \in T'$ and $y, z \in \mathrm{sl}_{\mathfrak{I}}(F)$. Then it is enough to show that

$$\mathcal{B}([x,y],z) = \mathcal{B}(x,[y,z]).$$

If $x \in F\iota$, then both sides are clearly 0. Thus, by Claim 4.10, one can change x into h for y and $[y,z]$ so that $\mathcal{B}([x,y],z) = \mathcal{B}([h,y],z)$ and $\mathcal{B}(x,[y,z]) = \mathcal{B}(h,[y,z])$. Thus it follows from the invariance on $\mathrm{sl}_{\mathfrak{J}}(F)$.

The case involving two elements in T' can be shown similarly. Let $x,y \in T'$ and $z \in \mathrm{sl}_{\mathfrak{J}}(F)$. Then it is enough to show that

$$\mathcal{B}(x,[y,z]) = 0 \quad \text{and} \quad \mathcal{B}([x,z],y) = \mathcal{B}(x,[z,y]).$$

Again, if x or $y \in F\iota$, then both sides of both equations are clearly 0. Thus, by Claim 4.10, the left-hand side of the first equation is equal to $\mathcal{B}(h,[h',z])$ for some $h,h' \in \mathfrak{h}_I$, and this is equal to 0 by the invariance on $\mathrm{sl}_{\mathfrak{J}}(F)$. For the second equation, change x into h for z and $[z,y]$ so that (LHS) $= \mathcal{B}([h,z],y)$ and (RHS) $= \mathcal{B}(h,[z,y])$. But then these are equal by the case involving one element above. Thus we have proved that the symmetric bilinear form $\mathcal{B}$ is invariant.

Moreover, the radical of $\mathcal{B}$ is contained in $F\iota$ whenever the restriction to $\mathrm{sl}_{\mathfrak{J}}(F)$ is not zero. In fact, this follows from [MY, Lem. 8.5] since the center of $\mathcal{M}_{\mathfrak{J}}$ is equal to $F\iota$. Or one can directly show this. Let us first mention the graded structure of $\mathcal{M}_{\mathfrak{J}}$.

Let $\mathfrak{g} := \mathrm{sl}_{\mathfrak{J}}(F)$ and let $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in \mathrm{A}_{\mathfrak{J}} \subset \mathfrak{h}^*} \mathfrak{g}_\mu\right)$ be the root-space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$. We extend each root $\mu \in \mathfrak{h}^*$ to an element in T^* as follows.

Let $\mathrm{A}_{\mathfrak{J}} = \{\pm(\varepsilon_i - \varepsilon_j) \mid i,j \in \mathfrak{J}\}$, where ε_i is the linear form of $\mathrm{gl}_{\mathfrak{J}}(F)$ determined by $e_{kl} \mapsto \delta_{lk}\delta_{ki}$. Since an element $p \in T$ can be written as $p = \mathrm{diag}(a_{ii})_{i \in \mathfrak{J}}$, one can define $\varepsilon_i(p) = a_{ii}$. In this way one can embed $\mathrm{A}_{\mathfrak{J}}$ into T^*. Thus $\mathcal{M} := \mathcal{M}_{\mathfrak{J}}$ has the root-space decomposition

$$\mathcal{M} = \bigoplus_{\mu \in T^*} \mathcal{M}_\mu$$

relative to T, where $\mathcal{M}_\mu = \mathfrak{g}_\mu$ for $\mu \neq 0$ and $\mathcal{M}_0 = T$, and $\mathcal{M}_\mu = 0$ if $\mu \notin \mathrm{A}_{\mathfrak{J}}$. This is an $\langle \mathrm{A}_{\mathfrak{J}} \rangle$-graded Lie algebra, and $\mathcal{B}$ is **graded** in the sense that $\mathcal{B}(\mathcal{M}_\xi, \mathcal{M}_\eta) = 0$ unless $\xi + \eta = 0$ for all $\xi, \eta \in \mathrm{A}_{\mathfrak{J}}$. In general, a symmetric invariant bilinear form on a Lie algebra having the root-space decomposition relative to a subalgebra is graded.

In particular, the radical of $\mathcal{B}$ is graded. Thus one can check the nondegeneracy for each homogeneous element. The elements of degree $\mu \in \mathrm{A}_{\mathfrak{J}}$ cannot be in the radical by Lemma 4.9. For the elements of degree 0, only candidate is an element in $F\iota$ by (28). Therefore, we have:

Lemma 4.11. *The radical of $\mathcal{B}$ is equal to $F\iota$ if $\mathcal{B}(\iota,\iota) = 0$, and $\mathcal{B}$ is nondegenerate if $\mathcal{B}(\iota,\iota) \neq 0$.* □

Thus for any symmetric bilinear form ψ on T' with the radical $F\iota$, the quotient Lie algebra $\mathcal{M}_{\mathfrak{J}}/F\iota$ with the induced form $\bar{B}$ is a LEALA of type $\mathrm{A}_{\mathfrak{J}}$ of nullity 0. Note that $\mathcal{M}_{\mathfrak{J}}/F\iota$ is isomorphic to $\mathcal{M}'_{\mathfrak{J}} := \mathrm{sl}_{\mathfrak{J}}(F) \oplus T''$, where T'' is a complement of $\mathfrak{h} \oplus F\iota$ in $T_{\mathfrak{J}}$. Conversely, if ψ' is any symmetric bilinear form on T'', one can define a symmetric nondegenerate invariant form $\mathcal{B}'$ on $\mathcal{M}'_{\mathfrak{J}}$ as above, and $\mathcal{M}'_{\mathfrak{J}}$ is isomorphic to some $\mathcal{M}_{\mathfrak{J}}/F\iota$ choosing some T' and ψ. By a similar argument, one can say that a LEALA of type $\mathrm{A}_{\mathfrak{J}}$ of nullity 0 is isomorphic to a subalgebra of $\mathcal{M}_{\mathfrak{J}}/F\iota$ containing $\mathrm{sl}_{\mathfrak{J}}(F) = (\mathrm{sl}_{\mathfrak{J}}(F) + F\iota)/F\iota$ with the induced form $\bar{B}$.

Example 4.12. The centerless Lie algebra $\mathrm{gl}_{\mathfrak{J}}(F) = \mathrm{sl}_{\mathfrak{J}}(F) \oplus Fe_{jj}$ is an example of a LEALA of type $\mathrm{A}_{\mathfrak{J}}$ of nullity 0, where e_{jj} is the matrix unit for $j \in \mathfrak{J}$. On the other hand, $\mathrm{gl}_n(F) = \mathrm{sl}_n(F) \oplus Fe_{jj}$ has the center FI, where I is the identity matrix on $\mathrm{gl}_n(F)$, and this is a non-tame EALA of nullity 0.

Suppose that $\mathcal{B}$ is a nondegenerate form on $\mathrm{gl}_{\mathfrak{I}}(F)$. Then $\mathcal{B}$ is a nonzero scalar multiple of the trace form except on $Fe_{jj} \times Fe_{jj}$, by Lemma 4.9. Conversely, one can take any value to $\mathcal{B}(e_{jj}, e_{jj})$, and extend a nondegenerate form $\mathcal{B}$ to $\mathrm{gl}_{\mathfrak{I}}(F)$.

For the finite case $\mathrm{gl}_n(F) = \mathrm{sl}_I(F) \oplus Fe_{jj}$, suppose that $\mathfrak{B}$ is a nondegenerate form on $\mathrm{gl}_n(F)$. Since $\mathrm{rad}\,\mathfrak{B}$ is in the center of $\mathrm{gl}_n(F)$ (e.g. [MY, Lem. 8.5]), we have that $\mathfrak{B}$ is nondegenerate $\Longleftrightarrow \mathfrak{B}(I,I) \neq 0$. We claim that this is equivalent to

$$\mathfrak{B}(e_{jj}, e_{jj}) \neq c\frac{n-1}{n}.$$

In fact, consider the expression $I = I - ne_{jj} + ne_{jj}$, noting that $\mathrm{tr}(I - ne_{jj}) = 0$. Since $x := I - ne_{jj} \in \mathrm{sl}_{\mathfrak{I}}(F)$, we have $\mathfrak{B}(I,I) = \mathfrak{B}(x + ne_{jj}, x + ne_{jj}) = \mathfrak{B}(x,x) + 2n\mathfrak{B}(x, e_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) = c\,\mathrm{tr}(x^2) + 2nc\,\mathrm{tr}(xe_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) = c\,\mathrm{tr}(I - 2ne_{jj} + n^2e_{jj}) + 2nc\,\mathrm{tr}(e_{jj} - ne_{jj}) + n^2\mathfrak{B}(e_{jj}, e_{jj}) = c(n - 2n + n^2) + 2nc(1-n) + n^2\mathfrak{B}(e_{jj}, e_{jj})$. Hence, $\mathfrak{B}(I,I) = 0$ if and only if $n^2\mathfrak{B}(e_{jj}, e_{jj}) = c(n^2 - n)$, and so our claim is proved.

Remark 4.13. In the classification of tame LEALAs of nullity 0 of type $\mathrm{A}_{\mathfrak{I}}$ in [MY], we chose $\mathrm{sl}_{\mathfrak{I}}(F) \oplus T'''$ for a complement T''' of $T_{\mathfrak{I}}^{as}$ in $T_{\mathfrak{I}}$ as a maximal one. But in fact, a subalgebra of a bigger Lie algebra $\mathrm{sl}_{\mathfrak{I}}(F) \oplus T''$ defined above is also a tame LEALA as $\mathrm{gl}_{\mathfrak{I}}(F)$ is so.

Now we observe the forms on the other types $\mathrm{B}_{\mathfrak{I}}$, $\mathrm{D}_{\mathfrak{I}}$ and $\mathrm{C}_{\mathfrak{I}}$. Let $\mathcal{B}$ be a symmetric invariant form on

$$\mathcal{M}_{\mathfrak{K}} = \mathrm{sl}_{\mathfrak{K}}(F) + T_{\mathfrak{K}}$$

so that the restriction to $\mathrm{sl}_{\mathfrak{K}}(F)$ is not zero, where $\mathfrak{K} = 2\mathfrak{I}$ or $2\mathfrak{I}+1$. Let $\mathcal{M}_{\mathfrak{K}}^{\sigma}$ be the fixed algebra by the automorphism σ defined above with the restricted form $\mathcal{B}^{\sigma}$. Then the restricted form is still invariant, and by Lemma 4.9, the restriction to $\mathrm{sl}_{\mathfrak{K}}(F)^{\sigma}$ is equal to $c\,\mathrm{tr}$ for some $c \in F^{\times}$.

Moreover, $\mathcal{B}^{\sigma}$ is nondegenerate. This follows from [MY, Lem. 8.5] since $\mathcal{M}_{\mathfrak{K}}^{\sigma}$ has the trivial center. One can also show this using the following lemma similar to Lemma 4.7. (As we mentioned already, $T^+ = T^{\sigma}$ means the eigenspace of eigenvalue $+1$ of σ, and T^- means the eigenspace of eigenvalue -1 of σ.)

Lemma 4.14. *Let I be any finite subset of $\mathfrak{I}$ and fix some index $i_0 \in I$. Then we have*

$$T_{2\mathfrak{I}}^{+} = \mathfrak{h}_{2I}^{+} \oplus T_{2\mathfrak{I}\setminus 2I}^{+} \quad \textit{and} \quad T_{2\mathfrak{I}}^{-} = \mathfrak{h}_{2I}^{-} \oplus F(e_{i_0 i_0} + e_{\mathfrak{I}+i_0, \mathfrak{I}+i_0}) \oplus T_{2\mathfrak{I}\setminus 2I}^{-},$$

where $\mathfrak{h}_{2I}^{+}$ or $\mathfrak{h}_{2I}^{-}$ is a subset of $\mathfrak{h}^{+}$ or $\mathfrak{h}^{-}$ such that all (k,k) and $(\mathfrak{I}+k, \mathfrak{I}+k)$ components for $k \in \mathfrak{I} \setminus I$ are 0, and $T_{2\mathfrak{I}\setminus 2I}^{+}$ or $T_{2\mathfrak{I}\setminus 2I}^{-}$ is a subset of $T_{2\mathfrak{I}}^{+}$ or $T_{2\mathfrak{I}}^{-}$ such that all (i,i) and $(\mathfrak{I}+i, \mathfrak{I}+i)$ components for $i \in I$ are 0.

Also, we have

$$T_{2\mathfrak{I}+1}^{+} = \mathfrak{h}_{2I+1}^{+} \oplus T_{(2\mathfrak{I}+1)\setminus(2I+1)}^{+} \quad \textit{and} \quad T_{2\mathfrak{I}+1}^{-} = \mathfrak{h}_{2I+1}^{-} \oplus Fe_{2\mathfrak{I}+1, 2\mathfrak{I}+1} \oplus T_{(2\mathfrak{I}+1)\setminus(2I+1)}^{-},$$

where $\mathfrak{h}_{2I+1}^{+}$ or $\mathfrak{h}_{2I+1}^{-}$ is a subset of $\mathfrak{h}^{+}$ or $\mathfrak{h}^{-}$ such that (k,k) and $(\mathfrak{I}+k, \mathfrak{I}+k)$ components for all $k \in \mathfrak{I} \setminus I$ are 0, and $T_{(2\mathfrak{I}+1)\setminus(2I+1)}^{+}$ or $T_{(2\mathfrak{I}+1)\setminus(2I+1)}^{-}$ is a subset of $T_{2\mathfrak{I}+1}^{+}$ or $T_{2\mathfrak{I}+1}^{-}$ such that the $(2\mathfrak{I}+1, 2\mathfrak{I}+1)$ component and (i,i) and $(\mathfrak{I}+i, \mathfrak{I}+i)$ components for all $i \in I$ are 0.

Moreover, we have

$$T_{2\mathfrak{I}}^{-} = \mathfrak{h}_{2I}^{-} \oplus F\iota_{2I} \oplus T_{2\mathfrak{I}\setminus 2I}^{-} \quad \textit{and} \quad T_{2\mathfrak{I}+1}^{-} = \mathfrak{h}_{2I+1}^{-} \oplus F\iota_{2I+1} \oplus T_{(2\mathfrak{I}+1)\setminus(2I+1)}^{-}. \tag{29}$$

Proof. It is enough show (29). For this, we show the two equalities

$$T^-_{2I} = \mathfrak{h}^-_{2I} \oplus F(e_{i_0i_0} + e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0}) \quad \text{and} \quad T^-_{2I+1} = \mathfrak{h}^-_{2I+1} \oplus Fe_{2\mathfrak{I}+1,2\mathfrak{I}+1},$$

where T^-_{2I} is a subset of $T^-_{2\mathfrak{I}}$ such that (i,i) and $(\mathfrak{I}+i,\mathfrak{I}+i)$ components for all $i \in \mathfrak{I} \setminus I$ are 0, and T^-_{2I+1} is a subset of $T^-_{2\mathfrak{I}+1}$ such that (i,i) and $(\mathfrak{I}+i,\mathfrak{I}+i)$ components for all $i \in \mathfrak{I} \setminus I$ are 0. But as in the proof of Lemma 4.7, for $y \in T^-_{2I}$ or T^-_{2I+1}, they follow from the equation

$$y = y - \frac{1}{2}\operatorname{tr}(y)(e_{i_0i_0} + e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0}) + \frac{1}{2}\operatorname{tr}(y)(e_{i_0i_0} + e_{\mathfrak{I}+i_0,\mathfrak{I}+i_0})$$

or

$$y = y - \operatorname{tr}(y)e_{2\mathfrak{I}+1,2\mathfrak{I}+1} + \operatorname{tr}(y)e_{2\mathfrak{I}+1,2\mathfrak{I}+1}.$$

Hence (29) holds. □

Corollary 4.15. *Let $x \in T^+$ or $x \in T^- \setminus F\iota$. Then there exists some $0 \neq h \in \mathfrak{h}^\pm$ such that $\mathcal{B}(x,h) \neq 0$.*

Proof. By (29) in Lemma 4.14, there exist a finite subset $I \subset \mathfrak{I}$ and $0 \neq h' \in \mathfrak{h}^\pm_{2I}$ or $\mathfrak{h}^\pm_{2I+1}$ so that $x = h' + b\iota_{2I} + x'$ or $x = h' + b\iota_{2I+1} + x'$ for some $b \in F$ and $x' \in T_{2\mathfrak{I}\setminus 2I}$ or $T_{2\mathfrak{I}+1\setminus 2I+1}$ (since $x \notin F\iota$). Since the trace form is nondegenerate on $\mathfrak{h}^\pm_{2I}$ or $\mathfrak{h}^\pm_{2I+1}$, one can choose $h \in \mathfrak{h}^\pm_{2I}$ or $\mathfrak{h}^\pm_{2I+1}$ so that $\operatorname{tr}(h'h) \neq 0$. Then we have $\mathcal{B}(x,h) = \operatorname{tr}(h'h) + b\operatorname{tr}(h) + \operatorname{tr}(x'h) = \operatorname{tr}(h'h) \neq 0$ (since $x'h = 0$). □

By Corollary 4.15 about T^+, we also see that $\mathcal{B}^\sigma$ is nondegenerate. (We will use the result about T^- later.) Moreover, the restriction of $\mathcal{B}^\sigma$ to any subalgebra $\mathcal{L}$ of $\mathcal{M}^\sigma_{\mathfrak{K}}$ containing $\mathrm{sl}_{\mathfrak{K}}(F)^\sigma$ is still a nondegenerate form.

Conversely, let U be a complement of $\mathfrak{h}^\sigma$ in $\mathcal{L} \cap T^\sigma$, and φ an arbitrary symmetric bilinear form on U. Then one can extend φ to a nondegenerate form on $\mathcal{L}$, using Lemma 4.14 (or embeds $\mathcal{L}$ into $\mathcal{M}_{\mathfrak{K}}$) and Corollary 4.15 again. Consequently, one can say that a LEALA of type $\mathrm{X}_{\mathfrak{I}} \neq \mathrm{A}_{\mathfrak{I}}$ of nullity 0 is isomorphic to a subalgebra of $\mathcal{M}^\sigma_{\mathfrak{K}}$ containing $\mathrm{sl}_{\mathfrak{K}}(F)^\sigma$.

The next interesting objects are LEALAs of null dimension 1. In fact our purpose of the paper is to classify tame LEALAs of nullity 1.

Definition 4.16. We call a tame LEALA of nullity 1 a **locally affine Lie algebra** or a **LALA** for short.

Before giving examples of LALAs and classifying LALAs, we prove a general property about R^0 for LEALAs. For this purpose, we review some properties (which we need) about $\{S_\mu\}_{\mu\in\Delta}$. First one can show that $S_\mu = S_\nu$ if μ and ν have the same length for $\mu,\nu \in \Delta$. Let $S = S_\mu$ for a short root μ. Then S contains all S_ν, and S satisfies $0 \in S$ and $2S - S \subset S$. Also, S spans $\operatorname{rad} W$ (see [Thm 8, Y2]).

Lemma 4.17. *Let $\mathcal{L}$ be a LEALA. Then $S + S \subset R^0$, and $S + S = R^0$ if $\mathcal{L}$ is tame. If $\mathcal{L}$ has nullity, then (nullity of $\mathcal{L}$) = (null dimension of $\mathcal{L}$).*

Proof. The first statement follows from (12) in §1, but we show this for the convenience of the next statement. Let $s,s' \in S$. Then $\mathcal{L}_{-\mu+s} \neq 0$ and $\mathcal{L}_{\mu+s'} \neq 0$ for $\mu \in \Delta_{sh}$, and $[\mathcal{L}_{-\mu+s},\mathcal{L}_{\mu+s'}] \neq 0$, by sl_2-theory. (Consider the sl_2-subalgebra generated by $\mathcal{L}_{\mu-s}$ and $\mathcal{L}_{-\mu+s}$, and act it on $\mathcal{L}_{\mu+s'}$.) So $0 \neq [\mathcal{L}_{-\mu+s},\mathcal{L}_{\mu+s'}] \subset \mathcal{L}_{s+s'}$ and hence $s+s' \in R^0$. Thus $S + S \subset R^0$.

Suppose that $\mathcal{L}$ is tame. Let $\sigma \in R^0$. If $\alpha+\sigma \notin R$ for all $\alpha \in R^\times$, then $\mathcal{L}_\sigma$ centralizes the core, and so $\mathcal{L}_\sigma$ is in the core. Thus $\mathcal{L}_\sigma = \sum_{\mu\in\Delta,\ s+s'=\sigma}[\mathcal{L}_{\mu+s}, \mathcal{L}_{-\mu+s'}]$, and so $\sigma = s+s'$ for some $s,s' \in S_\mu = S_{-\mu} \subset S$. (We get $\sigma = s+s'$ here which is our need. But this case does not occur as in the following argument.) But then $0 \neq \mathcal{L}_{\mu+s} = \mathcal{L}_{\mu-s'+\sigma}$, and $0 \neq \mathcal{L}_{\mu-s'}$ since $-s' \in S_\mu$. Therefore, $\mu - s' + \sigma \in R$ with $\mu - s' \in R^\times$, contradiction. Thus there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$. (This property is often said that σ is **nonisolated**. So we have shown that any isotropic root is nonisolated if $\mathcal{L}$ is tame.) Note that $\alpha = \mu + s$ for some $\mu \in \Delta$ and $s \in S$. Hence $s+\sigma \in S$, and so $\sigma \in S-S = S+S$. Thus $S+S = R^0$.

For the last statement, it is enough to show that $\operatorname{rad} V \subset V^0 := \operatorname{span}_{\mathbb{Q}} R^0$. (The other inclusion is clear.) Since $V = W + V^0$ (where $W = \operatorname{span}_{\mathbb{Q}} R^\times$), it is enough to show that $(\operatorname{rad} V) \cap W = \operatorname{rad} W \subset V^0$. But this is clear since $\operatorname{rad} W = \operatorname{span}_{\mathbb{Q}} S$. □

Note that if we put

$$R^0_c := \{\delta \in R^0 \mid \mathcal{L}_\delta \cap \mathcal{L}_c \neq 0\},$$

then (12) in §2 means that we always have

$$R^0_c = S+S. \tag{30}$$

Remark 4.18. There are notions of null dimension and nullity for LEARS $(W, R^\times)$. Namely, (null dimension of $R^\times$) := $\dim \operatorname{rad} W$ and (nullity of $R^\times$) := $\operatorname{rank}\langle S\rangle$ if $\langle S\rangle$ is free (see [Y3]). In general, (null dimension of $\mathcal{L}$) $\geq$ (null dimension of $R^\times$). If $\mathcal{L}$ has nullity, so does $R^\times$, and (nullity of $\mathcal{L}$) $\geq$ (nullity of $R^\times$) since any subgroup of a free abelian group is free (see e.g. [G]). If $\mathcal{L}$ is tame, then (null dimension of $\mathcal{L}$) $=$ (null dimension of $R^\times$), and if $\mathcal{L}$ has nullity, then

$$\text{(nullity of } \mathcal{L}) = \text{(null dimension of } \mathcal{L}) = \text{(nullity of } R^\times) = \text{(null dimension of } R^\times)$$

since $S+S = R^0$.

Lemma 4.19. *Let $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ be a LEALA over F with the center $Z(\mathcal{L})$, and R^0 the set of isotropic roots of $\mathcal{L}$. Then:*

(1) We have

$$\sum_{\delta\in R^0} F t_\delta \subset Z(\mathcal{L}) \subset \mathcal{H},$$

where t_δ is a unique element in $\mathcal{H}$ defined at (21) *in Lemma 4.4.*

(2) Let $\mathcal{L}_c$ be the core of $\mathcal{L}$ and $R^0_c = \{\delta \in R^0 \mid \mathcal{L}_\delta \cap \mathcal{L}_c \neq 0\}$. Then for $\delta \in R^0_c$, we have $t_\delta \in \mathcal{L}_c$ and

$$\sum_{\delta\in R^0_c} F t_\delta = Z(\mathcal{L}_c) \cap \mathcal{H} \subset Z(\mathcal{L}).$$

(3) Let $R^\times$ be the set of anisotropic roots of $\mathcal{L}$, which is a LEARS. Let m be the null dimension of $R^\times$, which is equal to the dimension of the radical of the induced form from $\mathcal{B}$ on $\operatorname{span}_{\mathbb{Q}} R^\times$*. Then, $m \geq \dim_F \big(Z(\mathcal{L}_c) \cap \mathcal{H}\big)$, and if $m \geq 1$, then $\dim_F \big(Z(\mathcal{L}_c) \cap \mathcal{H}\big) \geq 1$. Hence $m = 1$ implies that $\dim_F \big(Z(\mathcal{L}_c) \cap \mathcal{H}\big) = 1$ and $\dim_F Z(\mathcal{L}) \geq 1$.*

(4) If $\mathcal{L}$ is tame, then $\sum_{\delta\in R^0} F t_\delta = \sum_{\delta\in R^0_c} F t_\delta = Z(\mathcal{L}_c) \cap \mathcal{H} = Z(\mathcal{L})$.

Also, let n be the null dimension of $\mathcal{L}$, i.e., $n =$ $\operatorname{span}_{\mathbb{Q}} R^0$. Then $m = n \geq \dim_F Z(\mathcal{L})$. Moreover, if $n \geq 1$, then $\dim_F Z(\mathcal{L}) \geq 1$. Hence $n = 1$ implies that $\dim_F Z(\mathcal{L}) = 1$.

Proof. (1): Since each δ is an isotropic root, we have $[t_\delta, x] = 0$ for any root vector $x \in \mathcal{L}_\xi$. In fact, $[t_\delta, x] = \xi(t_\delta)x = (\xi, \delta)x = 0$ since δ is in the radical of the form (see (22)). Hence $[t_\delta, \mathcal{L}] = 0$, i.e., $t_\delta \in Z(\mathcal{L})$. Thus $\sum_{\delta\in R^0} F t_\delta \subset Z(\mathcal{L})$. For the second inclusion, note that $\mathcal{L}$ is an $\mathcal{H}$-weight module, and $Z(\mathcal{L})$ is a submodule. Hence, $Z(\mathcal{L})$ is a weight module (by a general theory of weight modules). Namely, $Z(\mathcal{L})$ is a graded subalgebra in our case. But then, since $\mathcal{H}$ is self-centralizing, we obtain $Z(\mathcal{L}) \subset \mathcal{H}$.

(2): For $\delta \in R_c^0$, let $0 \neq x \in \mathcal{L}_\delta \cap \mathcal{L}_c$. Then $t_\delta = [x,y]$ for some $y \in \mathcal{L}_{-\delta}$, and hence $t_\delta \in \mathcal{L}_c$ since $\mathcal{L}_c$ is an ideal. Thus $\sum_{\delta\in R_c^0} Ft_\delta \subset \mathcal{L}_c \cap \mathcal{H}$, and, by (1), we get $\sum_{\delta\in R_c^0} Ft_\delta \subset \mathcal{L}_c \cap Z(\mathcal{L}) \subset Z(\mathcal{L}_c)$. Therefore, we obtain $\sum_{\delta\in R_c^0} Ft_\delta \subset Z(\mathcal{L}_c)\cap\mathcal{H}$.

For the other inclusion, let $x \in \mathcal{L}_c \cap \mathcal{H}$. Since

$$\mathcal{L}_c \cap \mathcal{H} = \sum_{\xi\in R^\times} [\mathcal{L}_\xi, \mathcal{L}_{-\xi}] + \sum_{\delta\in R_c^0} [\mathcal{L}_\delta, \mathcal{L}_{-\delta}],$$

one can write

$$x = \sum_{\xi\in R^\times} a_\xi t_\xi + \sum_{\delta\in R_c^0} a_\delta t_\delta,$$

where $a_\xi, a_\delta \in F$. Let $\Delta \subset R^\times$ is a locally finite irreducible root system determined by a reflectable section of $\bar{R}^\times$ and S a reflection space for a short root in Δ. Then we know that $R^\times \subset \Delta + S$ and $R_c^0 = S+S$ (see (30)). Thus we get

$$\begin{aligned}
x &= \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\alpha+\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta \\
&= \sum_{\alpha\in\Delta,\delta'\in S} (a_{\alpha+\delta'} t_\alpha + a_{\alpha+\delta'} t_{\delta'}) + \sum_{\delta\in S+S} a_\delta t_\delta \\
&= \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_\alpha + \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta,
\end{aligned}$$

and hence,

$$y := \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_\alpha \in Z(\mathcal{L}).$$

But $y \in \mathfrak{h} \subset \mathfrak{g}$, and since $\mathfrak{g}$ is a locally finite split **simple** Lie algebra, this y has to be 0. Therefore,

$$x = \sum_{\alpha\in\Delta,\delta'\in S} a_{\alpha+\delta'} t_{\delta'} + \sum_{\delta\in S+S} a_\delta t_\delta \in \sum_{\delta\in R_c^0} Ft_\delta,$$

and we obtain $Z(\mathcal{L}) = \sum_{\delta\in R_c^0} Ft_\delta$. The second inclusion follows from (1).

(3): We know $R^\times \subset \Delta + S$ and $m = \dim_{\mathbb{Q}} \operatorname{span} S$. But since $R_c^0 = S+S$, we have $m = \dim_{\mathbb{Q}} \operatorname{span} R_c^0$. Now, there is a one to one correspondence

$$\{\delta \in R_c^0\} \leftrightarrow \{t_\delta\}_{\delta\in R_c^0},$$

and note that for some $\delta,\delta' \in R_c^0$, one may have $\delta+\delta' \notin R_c^0$, but since $\delta+\delta' \in \mathcal{H}^*$, we still have a unique element $t_{\delta+\delta'} \in \mathcal{H}$ through $\mathcal{B}(t_{\delta+\delta'}, h) = (\delta+\delta')(h)$ for all $h \in \mathcal{H}$. Also, one can easily see that $t_{\delta+\delta'} = t_\delta + t_{\delta'}$. Similarly, for any $a \in F$, there exists a unique element $t_{a\delta} \in \mathcal{H}$ such that $\mathcal{B}(t_{a\delta}, h) = (a\delta)(h)$ for all $h \in \mathcal{H}$, and one can check that $t_{a\delta} = at_\delta$. Thus for any subfield F' of F, we have a linear isomorphism between the vector spaces $\operatorname{span}_{F'} R_c^0$ and $\sum_{\delta\in R_c^0} F't_\delta$ over F'. In particular, $m = \dim_{\mathbb{Q}} \sum_{\delta\in R_c^0} \mathbb{Q}t_\delta \geq \dim_F \sum_{\delta\in R_c^0} Ft_\delta = \dim_F \big(Z(\mathcal{L}_c)\cap\mathcal{H}\big)$. Finally, if $m \geq 1$, then there exists $0 \neq \delta \in R_c^0$, and so $t_\delta \neq 0$. Thus $Ft_\delta \neq 0$, and hence we get the last statement.

(4): We have $R^0 = S+S = R_c^0$ since $\mathcal{L}$ is tame (see Lemma 4.17). Hence, $\sum_{\delta\in R^0} Ft_\delta = \sum_{\delta\in R_c^0} Ft_\delta$. Also, by (2), we already have $\sum_{\delta\in R_c^0} Ft_\delta = Z(\mathcal{L}_c)\cap\mathcal{H} \subset Z(\mathcal{L})$. Moreover, for $x \in Z(\mathcal{L})$, we have $x \in Z(\mathcal{L}_c)$ since $\mathcal{L}$ is tame. Hence, $Z(\mathcal{L}_c)\cap\mathcal{H} = Z(\mathcal{L})$. The rest of assertions follow from the fact that $R^0 = R_c^0$ using (3). ☐

Remark 4.20. There are examples of a tame LEALA or EALA whose nullity is ∞ but the center is just 1-dimensional. For example, $\mathcal{L} = \mathrm{sl}_2(\mathbb{C}[t_i^{\pm 1}]_{i\in\mathbb{N}}) \oplus Fc \oplus d$ is a tame EALA over $\mathbb{C}$ of type A_1, where $d = \sum_{i=1}^{\infty} a_i d_i$ with degree derivation $d_i = t_i \frac{\partial}{\partial t_i}$, and $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ is linearly independent over $\mathbb{Q}$. This $\mathcal{L}$ has null rank ∞ but the center is equal to Fc. Note that the Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ is just 3-dimensional. (The details are in [MY, Remark 5.2(2)].)

Corollary 4.21. *Let $(\mathcal{L}, \mathcal{H}, \mathcal{B})$ be a tame LEALA. Then we have a natural embedding*

$$\mathrm{ad}\,\mathcal{L} \hookrightarrow \mathrm{Der}_F\left(\mathcal{L}_c/(Z(\mathcal{L}_c)\cap\mathcal{H})\right).$$

In particular, if $x \in \mathcal{L}$ is in a complement of the core, that is, $x \in \mathcal{M}$ with $\mathcal{L} = \mathcal{L}_c \oplus \mathcal{M}$, then $\mathrm{ad}\,x$ *can be identified with an outer derivation of the loop algebra $\mathcal{L}_c/(Z(\mathcal{L}_c)\cap\mathcal{H})$.*

Proof. Since $\mathcal{L}_c$ is an ideal of $\mathcal{L}$, the restriction $\mathrm{ad}\,x\,|_{\mathcal{L}_c}$ is in $\mathrm{Der}_F\,\mathcal{L}_c$. Since $\mathcal{L}$ is tame, this restriction map is injective. Now by Lemma 4.19, we have $Z(\mathcal{L}_c)\cap\mathcal{H} \subset Z(\mathcal{L})$, and so $\mathrm{ad}\,x\,|_{\mathcal{L}_c}$ can be identified with the induced derivation in $\mathrm{Der}_F\left(\mathcal{L}_c/(Z(\mathcal{L}_c)\cap\mathcal{H})\right)$. Note that $[x,\mathcal{L}_c] \subset Z(\mathcal{L}_c)\cap\mathcal{H} \subset Z(\mathcal{L})$ implies $[x,[y,z]] = [[x,y],z]+[y,[x,z]] = 0$ for $y,z \in \mathcal{L}_c$ and $[x,\mathcal{L}_c] = 0$.

For the second assertion, suppose that $\mathrm{ad}\,x$ is inner in $\mathrm{Der}_F\left(\mathcal{L}_c/Z(\mathcal{L}_c)\cap\mathcal{H}\right)$, i.e., $\mathrm{ad}\,x = \mathrm{ad}\,y$ for some $y \in \mathcal{L}_c$. Then

$$[x-y,\mathcal{L}_c] \subset Z(\mathcal{L}_c)\cap\mathcal{H}.$$

But, since $\mathcal{L}_c$ is perfect, again we have $[x-y,\mathcal{L}_c] = [[x-y,\mathcal{L}_c],\mathcal{L}_c]+[\mathcal{L}_c,[x-y,\mathcal{L}_c]] = 0$ as above. Hence $x-y \in C_{\mathcal{L}}(\mathcal{L}_c) = Z(\mathcal{L}_c)$ by tameness. In particular, $x-y \in \mathcal{L}_c$, but then $x \in \mathcal{L}_c$, and this forces x to be 0. Therefore, $\mathrm{ad}\,x$ is an outer derivation of $\mathcal{L}_c/(Z(\mathcal{L}_c)\cap\mathcal{H})$. □

We know that the core of a LALA is a locally Lie 1-torus (see (23)). Moreover:

Corollary 4.22. *Let $\mathcal{L}$ be a LALA. Then:*

(1) The core $\mathcal{L}_c$ is a universal covering of a locally loop algebra.

(2) There exists a natural embedding $\mathrm{ad}\,\mathcal{L} \hookrightarrow \mathrm{Der}_F\left(\mathcal{L}_c/Z(\mathcal{L}_c)\right)$.

In particular, if $x \in \mathcal{L}$ is in a complement of the core, then $\mathrm{ad}\,x$ *can be identified with an outer derivation of the locally loop algebra $\mathcal{L}_c/Z(\mathcal{L}_c)$.*

Proof. By Lemma 4.19, $\mathcal{L}_c$ has a nontrivial center. Hence, by Theorem 3.5, we see that (1) is true. For (2), we have

$$Z(\mathcal{L}_c)\cap\mathcal{H} = Z(\mathcal{L}_c) = Z(\mathcal{L})$$

for a LALA $\mathcal{L}$, and so the assertions follow from Cororally 4.21. □

5. Examples of LALAs

To finish the classification of LALAs, we need to classify a complement of the core. Before doing this, we give examples of LALAs. Let us first define the minimality of a LEALA in general (see [N2] and also Remark 7.16).

Definition 5.1. A LEALA $\mathcal{L}$ is called **minimal** if $\mathcal{L}$ is the only LEALA containing $\mathcal{L}_c$ and contained in $\mathcal{L}$ (, equivalently saying, if there is no LEALA $\mathcal{L}'$ satisfying $\mathcal{L}_c \subset \mathcal{L}' \subsetneq \mathcal{L}$). Note that if the nullity is positive, then $\mathcal{L}_c$ is never a LEALA. So if $\mathcal{L}_c$ is a hyperplane in $\mathcal{L}$ (, that is, $\dim\mathcal{L}/\mathcal{L}_c = 1$) with positive nullity, then $\mathcal{L}$ is minimal.

Example 5.2. Let $\mathfrak{I}$ be an arbitrary index set. One can construct 14 minimal standard LALAs from 14 locally loop algebras $L(\mathrm{X}_{\mathfrak{I}}^{(i)})$ in Section 3. Namely,

$$\mathcal{L}^{ms} = \mathcal{L}^{ms}(\mathrm{X}_{\mathfrak{I}}^{(i)}) := L(\mathrm{X}_{\mathfrak{I}}^{(i)}) \oplus Fc \oplus Fd^{(0)}$$

is a LALA of type $\mathrm{X}_{\mathfrak{I}}^{(i)}$, where c is central and $d^{(0)}$ is the degree derivation, i.e.,

$$d^{(0)}(t^m) = mt^m$$

with a Cartan subalgebra

$$\mathcal{H} = \mathfrak{h} \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}(\mathrm{X}_{\mathfrak{I}})$ consisting of diagonal matrices if $\mathfrak{I}$ is infinite or any Cartan subalgebra if $\mathfrak{I}$ is finite. Also, a nondegenerate invariant symmetric bilinear form $\mathcal{B}$ on $\mathcal{L}^{ms}$ is an extension of the form defined in Section 3 for loop algebras, using the trace form or the Killing form if $\mathfrak{I}$ is finite, and a nondegenerate symmetric associative bilinear form on $F[t^{\pm 1}]$, and defining $\mathcal{B}(c, d^{(0)}) = 1$. In particular, we define $\mathcal{B}(d^{(0)}, d^{(0)}) = 0$ as usual although $\mathcal{B}(d^{(0)}, d^{(0)})$ can be any number in F. These LALAs are minimal LALAs. Note that any standard LALA contains a minimal standard LALA. Note also that if $\mathfrak{I}$ is finite, then LALAs are automatically minimal standard LALAs, which are the affine (Kac-Moody) Lie algebras.

Now, we give examples of bigger (and biggest) LALAs when $\mathfrak{I}$ is infinite. Note that

$$\mathrm{sl}_{\mathfrak{I}}(F) + T = \mathrm{gl}_{\mathfrak{I}}(F) + T,$$

where $T = T_{\mathfrak{I}}$ is the subspace of all diagonal matrices in the matrix space $M_{\mathfrak{I}}(F)$ of size $\mathfrak{I}$, is a Lie algebra with the split center $F\iota$, where ι is the diagonal matrix whose diagonal entries are all 1. Thus its loop algebra

$$\mathcal{U} = \mathcal{U}_{\mathfrak{I}} := \big(\mathrm{sl}_{\mathfrak{I}}(F) + T\big) \otimes F[t^{\pm 1}] \tag{31}$$

is a Lie algebra with the split center $\iota \otimes F[t^{\pm 1}]$.

Assume that $\mathcal{B}$ be a symmetric invariant bilinear form on $\mathcal{U}$, which is not a zero on $\mathrm{sl}_{\mathfrak{I}}(F)$. Then, by Lemma 3.6 and Lemma 4.8, $\mathcal{B}$ is unique up to a scalar to $\mathrm{tr} \otimes \varepsilon$ on

$$\big(\mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}]\big) \times \mathcal{U} \quad \text{and} \quad \mathcal{U} \times \big(\mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}]\big), \tag{32}$$

i.e., for $x, y \in \mathcal{U}$ and if x or $y \in \mathrm{sl}_{\mathfrak{I}}(F)$, then

$$\mathcal{B}(x \otimes t^m, y \otimes t^n) = a\,\mathrm{tr}(xy)\delta_{n,-m} \tag{33}$$

for some $a \in F^{\times}$. We claim that such a form $\mathcal{B}$ does exist. As in the case of nullity 0, we choose a complement T' of $\mathfrak{h}$ in T, i.e., $T = T' \oplus \mathfrak{h}$. For each $m \in \mathbb{Z}$, let

$$\psi_m : T' \times T' \longrightarrow F$$

be an arbitrary bilinear form. We define a symmetric bilinear form $\mathcal{B}$ on $\mathcal{U}$ as

$$\mathcal{B}(x \otimes t^m, y \otimes t^n) = \psi_m(x, y)\delta_{n,-m}$$

on $T' \otimes F[t^{\pm 1}]$, and (33) on (32). One can similarly prove that $\mathcal{B}$ is invariant to the case of nullity 0 using the following claim (which can also be proved similarly to Claim 4.10).

Claim 5.3. *Let $x \in T \setminus F\iota$ and $y_k \in \mathrm{sl}_{\mathfrak{I}}(F)$ for $k = 1, 2, \dots, r$. Then there exist a finite subset I of $\mathfrak{I}$, $0 \neq h \in \mathfrak{h}$ and $g \in T$ such that $y_k \in \mathrm{sl}_I(F)$ for all k, and $h \in \mathfrak{h}_I$,*

$$x = h + g, \quad [x \otimes t^m, y_k \otimes t^n] = [h \otimes t^m, y_k \otimes t^n] \quad \textit{and} \quad \mathcal{B}(x \otimes t^m, y_k \otimes t^n) = \mathcal{B}(h \otimes t^m, y_k \otimes t^n)$$

for all $m,n \in \mathbb{Z}$ and all k. Moreover, there exist $y \in \mathrm{sl}_{\mathfrak{I}}(F)$ and $h' \in \mathfrak{h}$ such that

$$[x \otimes t^m, y \otimes t^n] \neq 0 \quad \textit{and} \quad \mathcal{B}(x \otimes t^m, h' \otimes t^{-m}) \neq 0. \tag{34}$$

□

Now we can use a general construction, that is, a one-dimensional central extension by the 2-cocycle

$$\varphi(u,v) := \mathcal{B}\big(d^{(0)}(u), v\big)$$

for $u, v \in \mathcal{U}$, where $d^{(0)}$ is the degree derivation on $\mathcal{U}$. This is well-known (see e.g. [AABGP]), but for the convenience of the reader, we show that φ is a 2-cocycle in a slightly more general setup. Note that $d^{(0)}$ is a skew derivation relative to $\mathcal{B}$, i.e.,

$$\mathcal{B}\big(d^{(0)}(u), v\big) = -\mathcal{B}\big(u, d^{(0)}(v)\big).$$

More generally, for a $\mathbb{Z}$-graded algebra $A = \bigoplus_{m\in\mathbb{Z}} A_m$ with a symmetric **graded bilinear form** ψ, the degree derivation $d^{(0)}$ is skew relative to ψ. In fact, for $x = \sum_m x_m$ and $y = \sum_m y_m \in A$, we have $\psi\big(d^{(0)}(x), y\big) = \sum_m m\psi(x_m, y) = \sum_m m\psi(x_m, y_{-m}) = \sum_m m\psi(x, y_{-m}) = -\sum_m m\psi(x, y_m) = -\psi\big(x, d^{(0)}(y)\big)$. Hence $d^{(0)}$ is skew.

In general, on a Lie algebra L with a symmetric invariant bilinear form B, one can define $\varphi(u,v) := B\big(d(u), v\big)$ for any skew derivation d and $u, v \in L$. Then $\varphi(u,v)$ is a 2-cocycle (which is also well-known). In fact, clearly the first condition of cocycle, i.e., $\varphi(u,u) = 0$ for all $u \in L$, holds. For the second condition, we have

$$\begin{aligned}
&\varphi([u,v],w) + \varphi([v,w],u) + \varphi([w,u],v) \\
&= B\big(d([u,v]),w\big) - B\big([v,w],d(u)\big) - B\big([w,u],d(v)\big) \\
&= B\big([d((u),v]),w\big) + B\big([u,d(v)]),w\big) - B\big([v,w],d(u)\big) - B\big([w,u],d(v)\big) \\
&= B\big(d((u),[v,w]\big) - B\big(d(v),[u,w]\big) - B\big([v,w],d(u)\big) - B\big([w,u],d(v)\big) = 0.
\end{aligned}$$

Thus we get a 1-dimensional central extension

$$\tilde{\mathcal{U}} := \mathcal{U} \oplus Fc$$

using the 2-cocyle $\varphi(u,v) = \mathcal{B}\big(d^{(0)}(u), v\big)$ above. Then

$$\hat{\mathcal{U}} = \hat{\mathcal{U}}_{\mathfrak{I}} := \tilde{\mathcal{U}} \oplus Fd^{(0)}$$

is naturally a Lie algebra defining

$$[c, d^{(0)}] = 0,$$

anti-symmetrically. Thus the center of $\hat{\mathcal{U}}$ is equal to $Fc \oplus F\iota$. We also extend the form $\mathcal{B}$ by

$$\mathcal{B}(c, d^{(0)}) = 1 \quad \text{and} \quad \mathcal{B}(\mathcal{U}, d^{(0)}) = 0,$$

symmetrically (and the value of $\mathcal{B}(d^{(0)}, d^{(0)})$ can be any). Then one can check that this extended form is also invariant.

Let $\mathfrak{g} := \mathrm{sl}_{\mathfrak{I}}(F)$ and let $\mathfrak{g} = \mathfrak{h} \oplus \big(\bigoplus_{\mu \in \mathrm{A}_{\mathfrak{I}} \subset \mathfrak{h}^*} \mathfrak{g}_\mu\big)$ be the root-space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let

$$\mathcal{H} := T \oplus Fc \oplus Fd^{(0)}.$$

We extend each root $\mu \in \mathfrak{h}^*$ to an element in $\mathcal{H}^*$. First, one can extend μ to $T' \oplus F\iota$ as we did in the case of nullity 0. Then we define $\mu(Fc \oplus Fd^{(0)}) = 0$. Also, define $\delta \in \mathcal{H}^*$ as $\delta(T \oplus Fc) = 0$ and $\delta(d^{(0)}) = 1$. Then $\hat{\mathcal{U}}$ has the root-space decomposition

$$\hat{\mathcal{U}} = \bigoplus_{\xi \in \mathcal{H}^*} \hat{\mathcal{U}}_\xi$$

relative to $\mathcal{H}$, where $\hat{\mathcal{U}}_{\mu+m\delta} = \mathfrak{g}_\mu \otimes t^m$ for $0 \neq \mu \in \mathrm{A}_{\mathfrak{I}}^{(1)} = \mathrm{A}_{\mathfrak{I}} \cup \mathbb{Z}\delta$, $\hat{\mathcal{U}}_{m\delta} = T \otimes t^m$ for $m \neq 0$ and $\hat{\mathcal{U}}_0 = \mathcal{H}$, and $\hat{\mathcal{U}}_\xi = 0$ if $\xi \notin \mathrm{A}_{\mathfrak{I}}^{(1)}$. This is an $\langle \mathrm{A}_{\mathfrak{I}}^{(1)} \rangle$-graded Lie algebra, and $\mathcal{B}$ is graded in the sense that $\mathcal{B}(\hat{\mathcal{U}}_\xi, \hat{\mathcal{U}}_\eta) = 0$ unless $\xi + \eta = 0$ for all $\xi, \eta \in \mathrm{A}_{\mathfrak{I}}^{(1)}$. In particular, the radical of $\mathcal{B}$ is graded.

Claim 5.4. *The radical of* $\mathcal{B}$ *is contained in* $\iota \otimes F[t^{\pm 1}]$.

Proof. Since the radical of $\mathcal{B}$ is graded, one can check the nondegeneracy for each homogeneous element. It is clear that the elements of degree $\mu + m\delta$ for $\mu \in \mathrm{A}_{\mathfrak{I}}$ cannot be in the radical. For the elements of degree $m\delta$, it follows from (34). $\square$

It is now easy to check that $\hat{\mathcal{U}} = (\hat{\mathcal{U}}, \mathcal{H}, \mathcal{B})$ is a LEALA of nullity 1, defining $\psi_0(\iota, \iota) \neq 0$. Since the center of $\hat{\mathcal{U}}$ is equal to $Fc \oplus F\iota$, this is not tame. However, the subalgebra

$$\mathcal{L}^{max} := \big(\mathrm{sl}_{\mathfrak{I}}(F) \oplus T' \big) \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)}$$

of $\hat{\mathcal{U}}$ is tame, and so $\mathcal{L}^{max}$ is a LALA. Moreover, it is easy to check that a 1-dimensional extension of the core $\mathcal{L}_c^{max} = \mathrm{sl}_{\mathfrak{I}}(F) \otimes F[t^{\pm 1}] \oplus Fc$ of $\mathcal{L}^{max}$, say

$$\mathcal{L}(p) = \mathcal{L}_c^{max} \oplus F(d^{(0)} + p)$$

for some $p \in T'$, is a minimal LALA of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ (which is a subalgebra of $\mathcal{L}^{max}$). Also, one can show that any homogeneous subalgebra of $\mathcal{L}^{max}$ containing some $\mathcal{L}(p)$ is a LALA. We will show in Section 6 that any LALA of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ is a homogeneous subalgebra of some $\mathcal{L}^{max}$ containing some $\mathcal{L}(p)$.

Also, let $\mathcal{B}$ be any form on $\hat{\mathcal{U}}$ with the radical $\iota \otimes F[t^{\pm 1}]$. Then

$$\hat{\mathcal{U}} / \big(\iota \otimes F[t^{\pm 1}] \big) = \big(\hat{\mathcal{U}} / \big(\iota \otimes F[t^{\pm 1}] \big), \bar{T}, \bar{\mathcal{B}} \big),$$

where $\bar{T} = \big(T \otimes 1 + \iota \otimes F[t^{\pm 1}] \big) / \big(\iota \otimes F[t^{\pm 1}] \big)$ and $\bar{\mathcal{B}}$ is the induced form of $\mathcal{B}$, is a LALA isomorphic to $\mathcal{L}^{max}$.

We describe the other untwisted LALAs using $\hat{\mathcal{U}}_{2\mathfrak{I}}$ and $\hat{\mathcal{U}}_{2\mathfrak{I}+1}$ and the automorphism σ again defined in (15). First, one can assume that the defining complement T' of $\hat{\mathcal{U}} = \hat{\mathcal{U}}_{2\mathfrak{I}}$ or $\hat{\mathcal{U}}_{2\mathfrak{I}+1}$ (with ψ) is σ-invariant. Such a complement exists. For example, let

$$T = T^+ \oplus T^-$$

be the decomposition of $T = T_{2\mathfrak{I}}$ or $T_{2\mathfrak{I}+1}$, where T^+ is the space of eigenvalue 1 and T^- is the space of eigenvalue -1. Also, since $\mathfrak{h}$ is σ-invariant, we have

$$\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-, \quad \mathfrak{h}^+ \subset T^+ \quad \text{and} \quad \mathfrak{h}^- \subset T^-.$$

Note that $F\iota \subset T^-$. Choose complements $(T^+)'$ and $(T^-)'$ so that

$$T^+ = \mathfrak{h}^+ \oplus (T^+)' \quad \text{and} \quad T^- = \mathfrak{h}^- \oplus F\iota \oplus (T^-)'.$$

Let

$$T' := (T^+)' \oplus (T^-)'. \tag{35}$$

Then we have $T = \mathfrak{h} \oplus T' \oplus F\iota$ and T' is σ-invariant, and

$$(T')^\sigma = (T')^+ = (T^+)' \quad \text{and} \quad (T')^- = (T^-)'.$$

Let us extend the automorphism on $\hat{\mathfrak{U}} = \hat{\mathfrak{U}}_{2\mathfrak{I}}$ or $\hat{\mathfrak{U}}_{2\mathfrak{I}+1}$ as

$$\hat{\sigma}(x \otimes t^k) := \sigma(x) \otimes t^k, \quad \hat{\sigma}(c) := c \quad \text{and} \quad \hat{\sigma}(d^{(0)}) := d^{(0)}.$$

Then the fixed algebra $\hat{\mathfrak{U}}^{\hat{\sigma}}$ with the restriction of the form $\mathfrak{B}$ is a LALA of type $\mathrm{B}_{\mathfrak{I}}^{(1)}$, $\mathrm{C}_{\mathfrak{I}}^{(1)}$ or $\mathrm{D}_{\mathfrak{I}}^{(1)}$, depending on the type of σ. More precisely,

$$\hat{\mathfrak{U}}^{\hat{\sigma}} = \big(\mathfrak{g} \oplus (T')^\sigma\big) \otimes F[t^{\pm 1}] \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{g} = \mathrm{sl}_{2\mathfrak{I}+1}(F)^\sigma$ or $\mathrm{sl}_{2\mathfrak{I}}(F)^\sigma$ is a locally finite split simple Lie algebra of each type. Note that $T^\sigma = T^+ = \mathfrak{h}^+ \oplus (T')^\sigma$ and $T^- = \mathfrak{h}^- \oplus F\iota \oplus (T^-)'$ have the following forms:

$$\begin{aligned}
T^+ &= \{(a_{kk}) \in T_{2\mathfrak{I}+1} \mid a_{ii} = -a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I}),\ a_{2\mathfrak{I}+1,2\mathfrak{I}+1} = 0\} \\
T^- &= \{(a_{kk}) \in T_{2\mathfrak{I}+1} \mid a_{ii} = a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \quad \text{for } \mathrm{B}_{\mathfrak{I}}^{(1)}, \\
T^+ &= \{(a_{kk}) \in T_{2\mathfrak{I}} \mid a_{ii} = -a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \\
T^- &= \{(a_{kk}) \in T_{2\mathfrak{I}} \mid a_{ii} = a_{\mathfrak{I}+i,\mathfrak{I}+i}\ (\forall i \in \mathfrak{I})\} \quad \text{for } \mathrm{C}_{\mathfrak{I}}^{(1)} \text{ or } \mathrm{D}_{\mathfrak{I}}^{(1)}.
\end{aligned} \tag{36}$$

The nondegeneracy of the restricted form $\mathcal{B}$ follows from the following lemma whose proof is similar to the case in nullity 0.

Lemma 5.5. *Let $0 \neq x \in T^+$ or $x \in T^- \setminus F\iota$. Then there exists some $0 \neq h \in \mathfrak{h}^\pm$ such that*

$$\mathcal{B}(x \otimes t^m, h \otimes t^{-m}) \neq 0$$

for all $m \in \mathbb{Z}$. □

As in the case of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, a 1-dimensional extension of the core $\hat{\mathfrak{U}}_c^{\hat{\sigma}}$, say $\mathcal{L}(p) = \hat{\mathfrak{U}}_c^{\hat{\sigma}} \oplus F(d^{(0)} + p)$ for some $p \in T'^\sigma$, is a minimal LALA of each type. Also, one can check that any homogeneous subalgebra of $\hat{\mathfrak{U}}^{\hat{\sigma}}$ containing some $\mathcal{L}(p)$ is a LALA of each type. We will show in Section 6 that any LALA of each type is a homogeneous subalgebra of $\hat{\mathfrak{U}}^{\hat{\sigma}}$ containing some $\mathcal{L}(p)$.

We can now give examples of twisted LALAs similarly. Namely, we use the automorphism σ again defined in (15) to get the type $\mathrm{C}_{\mathfrak{I}}$ or $\mathrm{B}_{\mathfrak{I}}$, and extend the automorphism on $\hat{\mathfrak{U}} = \hat{\mathfrak{U}}_{2\mathfrak{I}}$ or $\hat{\mathfrak{U}}_{2\mathfrak{I}+1}$ as

$$\hat{\sigma}(x \otimes t^k) := (-1)^k \sigma(x) \otimes t^k, \quad \hat{\sigma}(c) = c \quad \text{and} \quad \hat{\sigma}(d^{(0)}) := d^{(0)}, \tag{37}$$

choosing a good complement T' for each σ as in (35). Then the fixed algebra $\hat{\mathfrak{U}}^{\hat{\sigma}}$ with the restriction of the form $\mathcal{B}$ is a LALA of type $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$, depending on the type of σ. More precisely,

$$\hat{\mathfrak{U}}^{\hat{\sigma}} = (\mathfrak{g} \oplus T'^+) \otimes F[t^{\pm 2}] \oplus (\mathfrak{s} \oplus T'^-) \otimes tF[t^{\pm 2}] \oplus Fc \oplus Fd^{(0)},$$

where $\mathfrak{g} = \mathrm{sp}_{2\mathfrak{I}}(F)$ or $\mathfrak{o}_{2\mathfrak{I}+1}(F)$ and $\mathfrak{s} = \mathrm{sl}_{2\mathfrak{I}}(F)^-$ or $\mathrm{sl}_{2\mathfrak{I}+1}(F)^-$. The nondegeneracy of the restricted form $\mathcal{B}$ follows from Lemma 5.5. As in the untwisted case, a 1-dimensional extension of the core $\hat{\mathfrak{U}}_c^{\hat{\sigma}}$, say $\mathcal{L}(p) = \hat{\mathfrak{U}}_c^{\hat{\sigma}} \oplus F(d^{(0)} + p)$ for some $p \in T'^+$, is a minimal LALA of each type. Also, one can show that any homogeneous subalgebra of $\hat{\mathfrak{U}}^{\hat{\sigma}}$ containing some $\mathcal{L}(p)$ is a LALA of each type. We will show in Section 7 that any LALA of each type is a homogeneous subalgebra of some $\hat{\mathfrak{U}}^{\hat{\sigma}}$ containing some $\mathcal{L}(p)$.

For the type $\mathrm{B}_{\mathfrak{J}}^{(2)}$, as Neeb described in [N2, App.1], we define a different kind of automorphism τ on the untwisted LALA $\mathcal{M} := \hat{\mathfrak{U}}^{\hat{\sigma}}_{2\mathfrak{J}+2}$ of type $\mathrm{D}^{(1)}_{\mathfrak{J}+1}$ defined by $s = \begin{pmatrix} 0 & \iota_{\mathfrak{J}+1} \\ \iota_{\mathfrak{J}+1} & 0 \end{pmatrix}$. For convenience, let $\mathfrak{J}+1 = \{j \mid j \in \mathfrak{J}\} \cup \{j_0\}$ and

$$2\mathfrak{J}+2 = (\mathfrak{J}+1)+(\mathfrak{J}+1) = \big(\{j \mid j \in \mathfrak{J}\} \cup \{j_0\}\big) \cup \big(\{-j \mid j \in \mathfrak{J}\} \cup \{-j_0\}\big).$$

Let

$$g = e_{j_0,-j_0} + e_{-j_0,j_0}$$

be the matrix of exchanging rows or columns, and let τ be an involutive automorphism of $\mathfrak{o}_{2\mathfrak{J}+2}(F)$ defined by

$$\tau(x) = gxg.$$

Then one can see that the fixed algebra $\mathfrak{o}_{2\mathfrak{J}+2}(F)^{\tau} = \mathfrak{o}_{2\mathfrak{J}+1}(F)$ (which has type $\mathrm{B}_{\mathfrak{J}}$) and the minus space $\mathfrak{s} := \mathfrak{o}_{2\mathfrak{J}+2}(F)^{-}$ by τ is isomorphic to $F^{2\mathfrak{J}+1}$ as a natural $\mathfrak{o}_{2\mathfrak{J}+1}(F)$-module with

$$\mathfrak{s}_0 = \mathfrak{o}_{2\mathfrak{J}+2}(F)^{-} \cap \mathfrak{h}^{+} = F(e_{j_0 j_0} - e_{-j_0,-j_0}).$$

We can extend τ on $\mathfrak{o}_{2\mathfrak{J}+2}(F) + T^{\sigma}_{2\mathfrak{J}+2} = \mathfrak{h}^{+} \oplus T'^{\sigma}_{2\mathfrak{J}+2}$ (see (36)). Then $T'^{\sigma}_{2\mathfrak{J}+2}$ is clearly τ-invariant, and we have

$$T'' := \big(T'^{\sigma}_{2\mathfrak{J}+2}\big)^{\tau} \cong T'^{+}_{2\mathfrak{J}+1} \cong T'^{+}_{2\mathfrak{J}}$$

and the minus space $\big(T'^{\sigma}_{2\mathfrak{J}+2}\big)^{-}$ by τ is equal to 0.

We further extend τ on $\mathcal{M}$ as the same way as in (37), i.e.,

$$\hat{\tau}(x \otimes t^k) := (-1)^k \tau(x) \otimes t^k, \quad \hat{\tau}(c) = c \quad \text{and} \quad \hat{\tau}(d^{(0)}) := d^{(0)},$$

and get a LALA $\mathcal{M}^{\hat{\tau}}$ of type $\mathrm{B}^{(2)}_{\mathfrak{J}}$. More precisely, we have

$$\mathcal{M}^{\hat{\tau}} = \big(\mathfrak{o}_{2\mathfrak{J}+1}(F) \oplus T''\big) \otimes F[t^{\pm 2}] \oplus \mathfrak{s} \otimes tF[t^{\pm 2}] \oplus Fc \oplus Fd^{(0)}.$$

(For the odd degree part, no extra matrices, i.e., no elements from T', are involved as in an affine Lie algebra of type $\mathrm{B}^{(2)}_{\ell} = \mathrm{D}^{(2)}_{\ell+1}$.) The nondegeneracy of the restricted form $\mathfrak{B}$ follows from Corollary 5.5. As in the above, a 1-dimensional extension of the core $\mathcal{M}^{\hat{\tau}}_c$, say $\mathcal{L}(p) = \mathcal{M}^{\hat{\tau}}_c \oplus F(d^{(0)} + p)$ for some $p \in T''$, is a minimal LALA of type $\mathrm{B}^{(2)}_{\mathfrak{J}}$. Also, one can show that any homogeneous subalgebra of $\mathcal{M}^{\hat{\tau}}$ containing some $\mathcal{L}(p)$ is a LALA of type $\mathrm{B}^{(2)}_{\mathfrak{J}}$. We will show in Section 7 that any LALA of type $\mathrm{B}^{(2)}_{\mathfrak{J}}$ is a homogeneous subalgebra of some $\mathcal{M}^{\hat{\tau}}$ containing some $\mathcal{L}(p)$.

6. Classification of the untwisted case

Let $\mathcal{L}$ be an untwisted LALA of infinite rank, i.e., the core $\mathcal{L}_c$ is a universal covering of an untwisted locally loop algebra. Choosing a homogeneous complement of the $\mathbb{Z}$-graded core, one can write

$$\mathcal{L} = \mathcal{L}_c \oplus \bigoplus_{m \in \mathbb{Z}} D^m.$$

Note that the complement is assumed to be included in the null space:

$$\bigoplus_{m \in \mathbb{Z}} D^m \subset \bigoplus_{\delta \in R^0} \mathcal{L}_{\delta} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}_{m\delta_1} \quad \text{and} \quad D^m \subset \mathcal{L}_{m\delta_1},$$

where δ_1 is a generator of $\langle R^0 \rangle_{\mathbb{Z}}$. Let

$$\mathcal{L}'_c := \mathcal{L}_c / Z(\mathcal{L}_c)$$

be the centerless core. Also, let $(\mathfrak{g},\mathfrak{h})$ be the grading pair of the Lie 1-torus $\mathcal{L}_c$ so that $\mathfrak{h}$ is the set of diagonal matrices of a locally finite split simple Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha=[\mathcal{L}_c^0,\mathcal{L}_c^0]\subset\mathcal{L}_c=\bigoplus_{m\in\mathbb{Z}}\mathcal{L}_c^m,$$

where

$$\mathcal{L}_c^m=\bigoplus_{\alpha\in\Delta\cup\{0\}}(\mathcal{L}_c)_\alpha^m.$$

We identify the grading pair $(\mathfrak{g},\mathfrak{h})$ of the Lie 1-torus $\mathcal{L}_c'$ and $\mathcal{L}_c$. Moreover, we identify

$$\mathcal{L}_c'=L:=\mathfrak{g}\otimes_F F[t^{\pm1}].$$

Now, we classify the **diagonal derivations** of an untwisted locally loop algebra L in general. Let

$$(\mathrm{Der}_F L)_0^0=\{d\in\mathrm{Der}_F L\mid d(\mathfrak{g}_\alpha\otimes t^m)\subset\mathfrak{g}_\alpha\otimes t^m \text{ for all } \alpha\in\Delta \text{ and } m\in\mathbb{Z}\}.$$

We call such an element a **diagonal derivation of degree** 0 in Introduction. Note that since $\mathfrak{g}_0=\mathfrak{h}=\sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$, we have

$$\begin{aligned}d(\mathfrak{h}\otimes t^m)&=\sum_{\alpha\in\Delta}d([\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\otimes t^m)=\sum_{\alpha\in\Delta}d([\mathfrak{g}_\alpha\otimes t^m,\mathfrak{g}_{-\alpha}\otimes 1])\\&=\sum_{\alpha\in\Delta}\big([d(\mathfrak{g}_\alpha\otimes t^m),\mathfrak{g}_{-\alpha}\otimes 1]+[\mathfrak{g}_\alpha\otimes t^m,d(\mathfrak{g}_{-\alpha}\otimes 1)]\big)\\&\subset\sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha\otimes t^m,\mathfrak{g}_{-\alpha}\otimes 1]=\mathfrak{h}\otimes t^m\end{aligned}$$

for all $d\in(\mathrm{Der}_F L)_0^0$.

Note also that $d\mid_{\mathfrak{g}}$ is a diagonal derivation of $\mathfrak{g}$. Hence, by Neeb [N1], we obtain $d\mid_{\mathfrak{g}}=\mathrm{ad}\,p$ for a certain diagonal matrix p of an infinite size. More precisely, $p\in P$, where

$$P=T_{\mathfrak{J}} \text{ for } \mathrm{A}_{\mathfrak{J}}, \text{ and } T_{2\mathfrak{J}}^+ \text{ or } T_{2\mathfrak{J}+1}^+ \text{ for the other types} \tag{38}$$

defined in Example 5.2. Let

$$d':=d-\mathrm{ad}\,p\in(\mathrm{Der}_F L)_0^0.$$

Then we have

$$d'(\mathfrak{g}\otimes 1)=0.$$

In particular, we have $d'(\mathfrak{h}\otimes 1)=0$. So, for $0\neq x\otimes t\in\mathfrak{g}_\alpha\otimes t$, if

$$d'(x\otimes t)=ax\otimes t \tag{39}$$

for $a\in F$, then

$$d'(y\otimes t^{-1})=-ay\otimes t^{-1} \tag{40}$$

for all $y\in\mathfrak{g}_{-\alpha}$. In fact, since $0\neq[x,y]=h\in\mathfrak{h}$ and $d'(y\otimes t^{-1})=by\otimes t^{-1}$ for some $b\in F$, we have

$$\begin{aligned}0=d'(h\otimes 1)&=d'([x\otimes t,y\otimes t^{-1}])=[d'(x\otimes t),y\otimes t^{-1}]+[x\otimes 1,d'(y\otimes t^{-1})]\\&=(a+b)[x\otimes t,y\otimes t^{-1}]=(a+b)[x,y]\otimes 1.\end{aligned}$$

Hence, $b=-a$.

Lemma 6.1. *Let $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be a locally finite split simple Lie algebra. Then $\mathfrak{U}(\mathfrak{g}).\mathfrak{g}_\beta=\mathfrak{g}$ for any $\beta\in\Delta$, where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.*

Proof. Since $\mathcal{U}(\mathfrak{g}).\mathfrak{g}_\beta$ is a nonzero ideal of $\mathfrak{g}$, it must be equal to $\mathfrak{g}$ by simplicity. □

By Lemma 6.1, three subspaces

$$\mathfrak{g}\otimes 1, \quad \mathfrak{g}_\alpha\otimes t \quad \text{and} \quad \mathfrak{g}_{-\alpha}\otimes t^{-1}$$

generate A as a Lie algebra.

Let

$$d'' := d' - a\cdot d^{(0)},$$

where $d^{(0)} = t\dfrac{d}{dt}$. Then we have $d''(\mathfrak{g}\otimes 1) = d'(\mathfrak{g}\otimes 1) = 0$ and using (39),

$$d''(x\otimes t) = d'(x\otimes t) - ax\otimes t = 0.$$

Similarly, using (40),

$$d''(y\otimes t^{-1}) = d'(y\otimes t^{-1}) + ay\otimes t^{-1} = 0.$$

Thus we get $d''(L) = 0$ and $d'' = 0$. Hence we obtain

$$d = \operatorname{ad} p + a\cdot d^{(0)}. \tag{41}$$

We define the **shift map** s_m for $m\in\mathbb{Z}$ on $L = \mathfrak{g}\otimes_F F[t^{\pm 1}]$ by

$$s_m(x\otimes t^k) := x\otimes t^{k+m}$$

for all $k\in\mathbb{Z}$. (Shift maps were discussed in the classification of affine Lie algebras by Moody in [Mo].) The shift maps clearly have the property

$$s_m([x,y]) = [s_m(x),y] = [x,s_m(y)]$$

for $x,y\in L$. (In other words, the shift maps are in the centroid of L.) Thus $s_m\circ d$ is a derivation for any derivation d of L. In fact, for $x,y\in L$,

$$s_m\circ d([x,y]) = s_m([d(x),y]+[x,d(y)]) = [s_m\circ d(x),y]+[x,s_m\circ d(y)].$$

Now, let

$$d\in(\operatorname{Der}_F L)_0^m = \{d\in\operatorname{Der}_F L \mid d(\mathfrak{g}_\alpha\otimes t^k)\subset\mathfrak{g}_\alpha\otimes t^{k+m} \text{ for all } \alpha\in\Delta \text{ and } k\in\mathbb{Z}\}.$$

Then we have

$$s_{-m}\circ d\in(\operatorname{Der}_F L)_0^0.$$

Hence, by (41), there exist $p = p_d\in P$ and some $a = a_d\in F$ such that

$$s_{-m}\circ d = \operatorname{ad} p + a\cdot d^{(0)},$$

and so

$$d = s_m\circ(\operatorname{ad} p + a\cdot d^{(0)}).$$

Thus we have classified diagonal derivations of the untwisted locally loop algebra. Namely:

Theorem 6.2. *For all $m\in\mathbb{Z}$, we have*

$$(\operatorname{Der}_F L)_0^m = s_m\circ(\operatorname{Der}_F L)_0^0 = s_m\circ(\operatorname{ad} P\oplus F d^{(0)}),$$

where P is defined in (38). □

The following property of diagonal derivations is useful.

Lemma 6.3. *For all $m \in \mathbb{Z}$, let*

$$(\mathrm{Der}'_F L)_0^m := \{d \in (\mathrm{Der}_F L)_0^m \mid s_n \circ d = d \circ s_n \textit{ for some } 0 \neq n \in \mathbb{Z}\}$$

and

$$(\mathrm{Der}''_F L)_0^m := \{d \in (\mathrm{Der}_F L)_0^m \mid s_n \circ d = d \circ s_n \textit{ for all } n \in \mathbb{Z}\}.$$

Then we have

$$(\mathrm{Der}'_F L)_0^m = s_m \circ \operatorname{ad} P = (\mathrm{Der}''_F L)_0^m.$$

Proof. First, it is clear that

$$(\mathrm{Der}'_F L)_0^m \supset (\mathrm{Der}''_F L)_0^m \supset s_m \circ \operatorname{ad} P$$

for all $m \in \mathbb{Z}$. Thus it is enough to show

$$(\mathrm{Der}'_F L)_0^m \subset s_m \circ \operatorname{ad} P. \tag{42}$$

So, let $s_m \circ (\operatorname{ad} p + a \cdot d^{(0)}) \in (\mathrm{Der}'_F L)_0^m \subset (\mathrm{Der}_F L)_0^m$. Then for

$$h \otimes t^k \in \mathfrak{h} \otimes t^k \subset \mathrm{Der}_F L,$$

we have

$$s_n \circ s_m([p + a \cdot d^{(0)}, h \otimes t^k]) = s_n(akh \otimes t^{k+m}) = akh \otimes t^{k+m+n}$$

and

$$[s_m \circ (p + a \cdot d^{(0)}), h \otimes t^{k+n}] = a(k+n)h \otimes t^{k+n+m}$$

for some $n \neq 0$. Hence, $an = 0$, and we get $a = 0$. Therefore, we obtain

$$s_m \circ (\operatorname{ad} p + a \cdot d^{(0)}) = s_m \circ \operatorname{ad} p \in s_m \circ \operatorname{ad} P,$$

that is, (42) has been shown. □

Remark 6.4. One can use some results by Azam about the derivations of tensor algebras (see [A2, Thm 2.8]). But the direct application to our tensor algebra $\mathfrak{g} \otimes_F F[t^{\pm 1}]$ gives an isomorphism that

$$\mathrm{Der}_F(\mathfrak{g} \otimes_F F[t^{\pm 1}]) \cong \mathrm{Der}_F \mathfrak{g} \overleftarrow{\otimes}_F F[t^{\pm 1}] \ \oplus\ C(\mathfrak{g}) \overrightarrow{\otimes}_F \mathrm{Der}_F F[t^{\pm 1}],$$

where $C(\mathfrak{g})$ is the centroid of $\mathfrak{g}$ and $\overleftarrow{\otimes}_F$ and $\overrightarrow{\otimes}_F$ are special types of tensor products (since $\mathfrak{g}$ is infinite-dimensional). Thus we need a little more work to get our desired form above. Since we only need a special type of subspaces, namely, $(\mathrm{Der}_F L)_0^m$, we directly approached them, not using the Azam's result. Besides that, we have to investigate derivations of twisted locally loop algebras later which are not tensor algebras.

Now we go back to classify D^m. Let $d \in D^m$. Then $\operatorname{ad} d \in (\mathrm{Der}_F L)_0^m$, by Corollary 4.22. Hence, by Theorem 6.2, there exist $p = p_d \in P$ (see (38)) and some $a = a_d \in F$ such that

$$\operatorname{ad} d = s_m \circ (\operatorname{ad} p + a \cdot d^{(0)}).$$

We claim that $a = 0$ for all $m \neq 0$. First, note that there exist $h,\ h' \in \mathfrak{h}$ such that $\mathrm{tr}(hh') \neq 0$. Also, we have

$$\mathcal{B}(h \otimes t, h' \otimes t^{-1}) = \mathcal{B}(h \otimes t^m, h' \otimes t^{-m}) = c\,\mathrm{tr}(hh') \neq 0$$

for all $m \in \mathbb{Z}$ and some $0 \neq c \in F$ since $\mathcal{B} = c\,\mathrm{tr} \otimes \varepsilon$ (see Lemma 3.6). Now, using such a pair h and h', we have

$$\begin{aligned}\mathcal{B}([d, h \otimes t], h' \otimes t^{-m-1}) &= a\mathcal{B}(h \otimes t^{m+1}, h' \otimes t^{-m-1}) \\ &= a\mathcal{B}(h \otimes t, h' \otimes t^{-1})\end{aligned}$$

While,

$$\begin{aligned}\mathcal{B}([d,h\otimes t],h'\otimes t^{-m-1}) &= -\mathcal{B}(h\otimes t,[d,h'\otimes t^{-m-1}])\\ &= a(m+1)\mathcal{B}(h\otimes t,h'\otimes t^{-1}).\end{aligned}$$

Hence, $a=a(m+1)$, i.e., $am=0$. Thus $m\neq 0$ implies $a=0$.

Moreover, suppose that $a=a_d=0$ for all $d\in D^0$. Then $\operatorname{ad}D^0\subset \operatorname{ad}P$ (see (38)) and for the Cartan subalgebra $\mathcal{H}$ of the original LALA of $\mathcal{L}$, we have $\mathcal{H}=\mathfrak{h}\oplus Fc\oplus D^0$. But this contradicts the axiom $\mathcal{L}_0=\mathcal{H}$ since $\left[\mathfrak{h}\otimes F[t^{\pm 1}],\mathcal{H}\right]=0$. Hence there exists $p\in P$ such that $\operatorname{ad}p+d^{(0)}\in \operatorname{ad}D^0$. Consequently, we get

$$\operatorname{ad}D^m\subset s_m\circ \operatorname{ad}P$$

for $m\neq 0$, and

$$\operatorname{ad}p+d^{(0)}\in \operatorname{ad}D^0\subset \operatorname{ad}P+Fd^{(0)}$$

for some $p\in P$. Note that it happens that $d^{(0)}\notin \operatorname{ad}D^0$. In other words, a LALA is not always standard. Note also that there exists a nonstandard LALA even if $\dim_F D^0\geq 2$.

Finally, we investigate the bracket on $D:=\bigoplus_{m\in\mathbb{Z}}D^m$. Let $D':=\bigoplus_{m\neq 0}D^m$. First, note that $[D',D']$ acts trivially on L since $[\operatorname{ad}(p\otimes t^m),\operatorname{ad}(p'\otimes t^n)]=\operatorname{ad}[p\otimes t^m,p'\otimes t^n]=0$ in $\operatorname{Der}_F L$. Hence,

$$[D',D']\subset Fc=Ft_{\delta_1}\subset\mathcal{H},$$

by tameness. Thus, for $d_m\in D^m$ ($m\neq 0$) and $d_n\in D^n$ ($n\neq 0$), we have, by the fundamental property (21) of a LEALA (see Lemma 4.4),

$$[d^m,d^n]=\delta_{m,n}\mathcal{B}(d_m,d_n)t_{m\delta_1}=m\delta_{m,n}\mathcal{B}(d_m,d_n)t_{\delta_1}.$$

Note that $B(d_m,d_n)$ can be zero since there exists $h\in\mathfrak{h}$ such that $\operatorname{tr}(d_m,h)\neq 0$ (and so $B(d_m,h)\neq 0$).

Next, since $D^0\subset\mathcal{H}$, we have $[D^0,D^0]=0$. Also, for $d\in D^0$ so that $\operatorname{ad}_L d=\operatorname{ad}_L p\in D^0$, we have $[d,D^m]=0$. For the last case, i.e., for $d\in D^0$ so that $\operatorname{ad}_L d=\operatorname{ad}_L p+a\cdot d^{(0)}\in \operatorname{ad}D^0$ and $d_m\in D^m$, we have

$$[d,d_m]=[a\cdot d^{(0)},d_m]=amd_m.$$

Now, if $p\in\mathfrak{h}\oplus F\iota$ for the type $\mathrm{A}_{\mathfrak{I}}$ or if $p\in\mathfrak{h}^+$ for the other types, then there exists some $h\in\mathfrak{h}$ such that $\operatorname{ad}h=\operatorname{ad}p$. Hence, $h-p\in Z(\mathcal{L})$. But $h-p\notin\mathcal{L}_c$, which contradicts the tameness of $\mathcal{L}$. Thus D^m for $m\in\mathbb{Z}$ is contained in a complement of $\mathfrak{h}\oplus F\iota$ in T for the type $\mathrm{A}_{\mathfrak{I}}$ or a complement of $\mathfrak{h}^+$ in T^+ for the other types. Thus:

Theorem 6.5. *Let $\mathcal{L}$ be an untwisted LALA. Then $\mathcal{L}$ is isomorphic to one in Example 5.2.* □

7. Classification of the twisted case

As we already mentioned, each twisted loop algebra L is a subalgebra of an untwisted loop algebra $\tilde{L}$. More precisely, we have

$$\begin{aligned} L \text{ has type } \mathrm{B}_{\mathfrak{I}}^{(2)} &\implies \tilde{L}\text{ has type } \mathrm{D}_{\mathfrak{I}+1}^{(1)}\\ L \text{ has type } \mathrm{C}_{\mathfrak{I}}^{(2)} &\implies \tilde{L}\text{ has type } \mathrm{A}_{2\mathfrak{I}}^{(1)}\\ L \text{ has type } \mathrm{BC}_{\mathfrak{I}}^{(2)} &\implies \tilde{L}\text{ has type } \mathrm{A}_{2\mathfrak{I}+1}^{(1)}.\end{aligned}$$

Remark 7.1. For the case that $\mathfrak{I}$ is finite, the type $\mathrm{A}_{\mathfrak{I}}$ usually means the Lie algebra $\mathrm{sl}_{\mathfrak{I}+1}(F)$. So it may be better to write

$$L \text{ has type } \mathrm{C}_{\mathfrak{I}}^{(2)} \implies \tilde{L} \text{ has type } \mathrm{A}_{2\mathfrak{I}-1}^{(1)}$$
$$L \text{ has type } \mathrm{BC}_{\mathfrak{I}}^{(2)} \implies \tilde{L} \text{ has type } \mathrm{A}_{2\mathfrak{I}}^{(1)}$$

in order to follow the common notations. But in this paper, we already use the type of the Lie algebra $\mathrm{sl}_{\mathfrak{I}}(F)$ as $\mathrm{A}_{\mathfrak{I}}$ instead of $\mathrm{A}_{\mathfrak{I}+1}$ as long as $\mathfrak{I}$ is an infinite set.

Let us first show basic lemmas for locally twisted loop algebras. Let

$$\tilde{\mathfrak{g}}^{\hat{\sigma}} = \mathfrak{g}^+ \otimes F[t^{\pm 2}] \oplus \mathfrak{g}^- \otimes tF[t^{\pm 2}]$$

be a twisted affine Lie algebra, where $\mathfrak{g}^+$ is the 1-eigenspace and $\mathfrak{g}^-$ is the (-1)-eigenspace by σ, as we already used this notations. Also, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of diagonal matrices has the decomposition $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$, where $\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{g}^+$ and $\mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{g}^-$.

Note that the adjoint of the plus-space fixes $\mathfrak{g}^+$ and $\mathfrak{g}^-$, and the adjoint of the minus-space interchanges $\mathfrak{g}^+$ and $\mathfrak{g}^-$.

Lemma 7.2. *$\mathfrak{g}^-$ is an irreducible $\mathfrak{g}^+$-module.*

Proof. It is enough to show that for any root vectors $v, w \in \mathfrak{g}^-$, there exists $x \in \mathfrak{g}^+$ such that $[x, v] = w$. But this is a local property. Namely, there exists a finite dimensional split simple subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ of the same type so that $v, w \in (\mathfrak{g}')^- \subset \mathfrak{g}^-$ and $(\mathfrak{g}')^+ \subset \mathfrak{g}^+$. It is well-known that the property holds in the finite-dimensional case. Thus we are done. □

Lemma 7.3. *The centralizer $C_{\mathfrak{g}+T}(\mathfrak{g}^+)$ of $\mathfrak{g}^+$ in $\mathfrak{g}+T$ is zero in each type.*

Proof. We can write each Lie algebra as

$$\mathfrak{g} + T = \mathfrak{g}^+ + \mathfrak{g}^- + T^+ + T^-,$$

for example, $T^+ = T_{2\mathfrak{I}}^+$ and $T^- = \mathfrak{h}_-$ for the type $\mathrm{D}_{\mathfrak{I}+1}$. Let

$$x = x_+ + x_- + h_+ + h_- \in \mathfrak{g}^+ + \mathfrak{g}^- + T^+ + T^-$$

be in $C_{\mathfrak{g}+T}(\mathfrak{g}^+)$. Then, for any $y \in \mathfrak{g}^+$, we have

$$0 = [x, y] = [x_+, y] + [x_-, y] + [h_+, y] + [h_-, y].$$

Hence, $[x_+ + h_+, y] = 0$ and $[x_- + h_-, y] = 0$. But $\mathrm{Cent}_{\mathfrak{g}^+ + T^+}(\mathfrak{g}^+) = 0$ since $\mathfrak{g}^+ + T^+$ is tame (cf. Section 4). Hence, $x_+ + h_+ = 0$. Also, since $[h_-, y] \in \mathfrak{g}^- \cap \mathfrak{g}^+ = 0$, we get $[x_-, y] = 0$. But $[x_-, \mathfrak{g}^+] = 0$ implies $x_- = 0$ since $\dim_F \mathfrak{g}^- > 1$ and $\mathfrak{g}^-$ is an irreducible $\mathfrak{g}^+$-module (by Lemma 7.2). Therefore, $x = h_- \in T^-$. But then, if $h_- \neq 0$, then there exists $0 \neq w \in \mathfrak{g}^-$ such that $[\mathfrak{g}, w] = 0$, which contradicts that $\mathfrak{g}^-$ is an irreducible $\mathfrak{g}^+$-module again. □

Lemma 7.4. *Let $h \in T_{2\mathfrak{I}+2}^+ \subset \mathfrak{g} + T_{2\mathfrak{I}+2}^+$, $h \in T_{2\mathfrak{I}+1} \subset \mathfrak{g} + T_{2\mathfrak{I}+1}$, or $h \in T_{2\mathfrak{I}} \subset \mathfrak{g} + T_{2\mathfrak{I}}$. Suppose that $[h, \mathfrak{g}^+] \subset \mathfrak{g}^-$. Then $h \in T^-$, $h \in T_{2\mathfrak{I}+1}^-$, or $h \in T_{2\mathfrak{I}}^-$, respectively.*

Proof. Let $x \in \mathfrak{g}^+$ and $y = [h, x] \in \mathfrak{g}^-$. Then $-y = [\sigma(h), x]$. Hence, $[h + \sigma(h), x] = 0$ for all $x \in \mathfrak{g}^+$. So $h + \sigma(h) \in C_{\mathfrak{g}+T}(\mathfrak{g}^+) = 0$, by Lemma 7.3. Thus $\sigma(h) = -h$, and we get $h \in T^-$, $T_{2\mathfrak{I}+1}^-$ or $T_{2\mathfrak{I}}^-$, respectively. □

Let $\mathfrak{L}$ be a twisted LALA of infinite rank, i.e., the core $\mathfrak{L}_c$ is a universal covering of a twisted locally loop algebra. As in the untwisted case, choosing a homogeneous complement of the $\mathbb{Z}$-graded core, one can write

$$\mathfrak{L} = \mathfrak{L}_c \oplus \bigoplus_{m\in\mathbb{Z}} D^m, \quad \bigoplus_{m\in\mathbb{Z}} D^m \subset \bigoplus_{\delta\in R^0} \mathfrak{L}_\delta = \bigoplus_{m\in\mathbb{Z}} \mathfrak{L}_{m\delta_1} \quad \text{and} \quad D^m \subset \mathfrak{L}_{m\delta_1},$$

where δ_1 is a generator of $\langle R^0\rangle_{\mathbb{Z}}$. Let $\mathfrak{L}_c' := \mathfrak{L}_c/Z(\mathfrak{L}_c)$ be the centerless core and let $(\mathfrak{g},\mathfrak{h})$ be the grading pair of the Lie 1-torus $\mathfrak{L}_c$ so that $\mathfrak{h}$ is the set of diagonal matrices of a locally finite split simple Lie algebra $\mathfrak{g}$ as before.

Now, let

$$\mathfrak{s} = \bigoplus_{\beta\in\Delta'\cup\{0\}} \mathfrak{s}_\beta$$

be the irreducible $\mathfrak{g}$-module defined by type of $\mathfrak{L}_c$, where Δ' is a subset of Δ consisting of short roots or of extra long roots, and we identify

$$\mathfrak{L}_c' = L := \big(\mathfrak{g}\otimes_F F[t^{\pm 2}]\big) \oplus \big(\mathfrak{s}\otimes_F tF[t^{\pm 2}]\big).$$

As in the untwisted case, we classify diagonal derivations of a twisted locally loop algebra L in general.

Let $d \in (\mathrm{Der}_F L)^0_0 := \{d \in \mathrm{Der}_F L \mid d(\mathfrak{g}_\alpha\otimes t^{2m}) \subset \mathfrak{g}_\alpha\otimes t^{2m}$

$$\text{and } d(\mathfrak{s}_\beta\otimes t^{2m+1}) \subset \mathfrak{s}_\beta\otimes t^{2m+1} \text{ for all } \alpha\in\Delta,\ \beta\in\Delta' \text{ and } m\in\mathbb{Z}\}.$$

Then, as before, $d\mid_{\mathfrak{g}}$ is a diagonal derivation of $\mathfrak{g}$, and so, by Neeb [N1], $d\mid_{\mathfrak{g}} = \mathrm{ad}\, p$ for some $p\in P$ depending on the type of $\mathfrak{g}$ (see (38)). Let

$$d' := d - \mathrm{ad}\, p \in (\mathrm{Der}_F L)^0_0.$$

Then we have $d'(\mathfrak{g}\otimes 1) = 0$. In particular, we have $d'(\mathfrak{h}\otimes 1) = 0$. Thus, by the same way as in the untwisted case, one can show that for $0 \neq x\otimes t \in \mathfrak{s}_\beta\otimes t$, if

$$d'(x\otimes t) = ax\otimes t \tag{43}$$

for $a\in F$, then

$$d'(y\otimes t^{-1}) = -ay\otimes t^{-1} \tag{44}$$

for all $y\in\mathfrak{s}_{-\beta}$.

Lemma 7.5. *For the above $\mathfrak{s}$, we have $\mathfrak{U}(\mathfrak{g}).\mathfrak{s}_\beta = \mathfrak{s}$ for any $\beta\in\Delta'$, where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.*

Proof. Since $\mathfrak{U}(\mathfrak{g}).\mathfrak{s}_\beta$ is a nonzero submodule of $\mathfrak{s}$, it must be $\mathfrak{s}$ by the irreducibility of $\mathfrak{s}$. ◻

By Lemma 7.5, three subspaces

$$\mathfrak{g}\otimes 1, \quad \mathfrak{s}_\beta\otimes t \quad \text{and} \quad \mathfrak{s}_{-\beta}\otimes t^{-1}$$

generate A as a Lie algebra. As before, let $d'' := d' - a\cdot d^{(0)}$. Then we have $d''(\mathfrak{g}\otimes 1) = d'(\mathfrak{g}\otimes 1) = 0$ and using (43), $d''(x\otimes t) = d'(x\otimes t) - ax\otimes t = 0$. Similarly, using (44), we have $d''(y\otimes t^{-1}) = d'(y\otimes t^{-1}) + ay\otimes t^{-1} = 0$. Thus we get $d''(L) = 0$ and $d'' = 0$. Hence we obtain

$$d = \mathrm{ad}\, p + a\cdot d^{(0)}. \tag{45}$$

We again define the shift map s_{2m} for $m\in\mathbb{Z}$ on $L = \big(\mathfrak{g}\otimes_F F[t^{\pm 2}]\big)\oplus\big(\mathfrak{s}\otimes_F tF[t^{\pm 2}]\big)$ by

$$s_{2m}(x\otimes t^{2k}) := x\otimes t^{2k+2m} \text{ and } s_{2m}(v\otimes t^{2k+1}) := v\otimes t^{2k+2m+1}$$

for $x \in \mathfrak{g}$ and $v \in \mathfrak{s}$. Let $d_{2m} \in (\mathrm{Der}_F L)_0^{2m} := \{d \in \mathrm{Der}_F L \mid d_{2m}(\mathfrak{g}_\alpha \otimes t^{2k}) \subset \mathfrak{g}_\alpha \otimes t^{2k+2m}$ and $d_{2m}(\mathfrak{s}_\beta \otimes t^{2k+1}) \subset \mathfrak{s}_\beta \otimes t^{2k+2m+1}$ for all $\alpha \in \Delta$, $\beta \in \Delta'$ and $m \in \mathbb{Z}\}$.

Then we have $s_{-2m} \circ d_{2m} \in (\mathrm{Der}_F L)_0^0$. Hence, by (45), there exist some $p = p_{d_{2m}} \in P$ and $a = a_{d_{2m}} \in F$ such that $s_{-2m} \circ d_{2m} = \mathrm{ad}\, p + a \cdot d^{(0)}$, and so

$$d_{2m} = s_{2m} \circ \mathrm{ad}\, p + at^{2m+1} \frac{d}{dt}.$$

Thus as Theorem 6.2, we have:

Lemma 7.6. *For all $m \in \mathbb{Z}$, we have*

$$(\mathrm{Der}_F L)_0^{2m} = s_{2m} \circ (\mathrm{Der}_F L)_0^0 = s_{2m} \circ (\mathrm{ad}\, P \oplus F d^{(0)}),$$

where P is defined in (38). □

Also, as Lemma 6.3, we have:

Lemma 7.7. *For all $m \in \mathbb{Z}$, let*

$$(\mathrm{Der}'_F L)_0^{2m} := \{d \in (\mathrm{Der}_F L)_0^{2m} \mid s_{2n} \circ d = d \circ s_{2n} \text{ for some } 0 \neq n \in \mathbb{Z}\}$$

and

$$(\mathrm{Der}''_F L)_0^{2m} := \{d \in (\mathrm{Der}_F L)_0^{2m} \mid s_{2n} \circ d = d \circ s_{2n} \text{ for all } n \in \mathbb{Z}\}.$$

Then we have

$$(\mathrm{Der}'_F L)_0^{2m} = s_{2m} \circ \mathrm{ad}\, P = (\mathrm{Der}''_F L)_0^{2m}.$$

□

Now, we go back to the classification of D^m. Let $d_{2m} \in D^{2m}$. Then $\mathrm{ad}\, d_{2m} = s_{2m} \circ \mathrm{ad}\, p + at^{2m} d^{(0)}$ for some $p \in P$ and $a \in F$ by Lemma 7.6. Then, as in the untwisted case, one can show that $a = 0$ for all $m \neq 0$, using

$$\mathcal{B}([d, h \otimes t], h' \otimes t^{-m-1}) = -\mathcal{B}(h \otimes t^2, [d_{2m}, h' \otimes t^{-2m-2}])$$

for some $h, h' \in \mathfrak{h}$ so that $\mathrm{tr}(h, h') \neq 0$. Also, as in the untwisted case, there exists some $p \in P$ such that $\mathrm{ad}\, p + d^{(0)} \in \mathrm{ad}\, D^0$. Thus the spaces D^m for even m's coincide with the ones in Example 5.2.

Next we determine $(\mathrm{Der}_F L)_0^{2m+1}$.

Lemma 7.8. *Let $q \in (\mathrm{Der}_F L)_0^{2m+1}$. Then q commutes with a shift map s_{2i} for all $i \in \mathbb{Z}$.*

Proof. First note that Δ' does not contain a long root of Δ, where Δ' is the set of grading roots for $\mathfrak{s}$. Also, if Δ has type $\mathrm{BC}_\mathfrak{J}$, then Δ' does not contain a short root neither. Thus, for a long root vector x in $\mathfrak{g}$ or a short root vector x in $\mathfrak{g}$ for the case $\mathrm{BC}_\mathfrak{J}$, we have

$$q(x \otimes t^{2k}) = 0 \tag{46}$$

for all $k \in \mathbb{Z}$. Next we claim that for a short root $y \in \mathfrak{s}_\beta$ in reduced cases, there exist a long root α and a short root γ such that $y = [x, z]$ for some $x \in \mathfrak{g}_\alpha$ and $z \in \mathfrak{s}_\gamma$. In fact, it is enough to consider a finite-dimensional split simple Lie algebra of type $\mathrm{B}_2 = \mathrm{C}_2$. Since there always exist such roots α and γ satisfying $\beta = \alpha + \gamma$, the claim is clear now.

Also, for any extra long root 2β and $y \in \mathfrak{s}_{2\beta}$, we have $y = [x, z]$ for some $x \in \mathfrak{g}_\beta$ and $z \in \mathfrak{s}_\beta$. Finally, we have, for any long root vector x or a short root vector x for the case $\mathrm{BC}_\mathfrak{J}$, and any $k \in \mathbb{Z}$,

$$q \circ s_{2i}(x \otimes t^{2k}) = q(x \otimes t^{2k+2i}) = 0 = s_{2i} \circ q(x \otimes t^{2k}).$$

Moreover, for any short root vector y in reduced cases or any extra long root vector y, and for any $n \in \mathbb{Z}$, choosing x and y above such that $y = [x, z]$, we have

$$\begin{aligned}
q \circ s_{2i}(y \otimes t^n) &= q \circ s_{2i}([x \otimes t^{2\ell}, z \otimes t^r]) \quad (2\ell + r = n) \\
&= q([x \otimes t^{2\ell+2i}, z \otimes t^r]) \\
&= [q(x \otimes t^{2\ell+2i}), z \otimes t^r] + [x \otimes t^{2\ell+2i}, q(z \otimes t^r)] \\
&= [x \otimes t^{2\ell+2i}, q(z \otimes t^r)] \quad \text{(by (46))} \\
&= s_{2i}\big([x \otimes t^{2\ell}, q(z \otimes t^r)]\big) \\
&= s_{2i}\big([x \otimes t^{2\ell}, q(z \otimes t^r)] + [q(x \otimes t^{2\ell}), z \otimes t^r]\big) \\
&\quad \text{(adding } 0 = [q(x \otimes t^{2\ell}), z \otimes t^r]) \\
&= s_{2i} \circ q\big([x \otimes t^{2\ell}, z \otimes t^r]\big) \\
&= s_{2i} \circ q(y \otimes t^n).
\end{aligned}$$

Hence $q \circ s_{2i} = s_{2i} \circ q$. □

Lemma 7.9. *Let $L = \mathfrak{g} \otimes F[t^{\pm 2}] \oplus \mathfrak{s} \otimes tF[t^{\pm 2}]$ be a twisted loop algebra which is double graded by $\Delta \cup \{0\}$ and $\mathbb{Z}$ as above. Let d be in $(\mathrm{Der}_F L)_0^{2m+1}$ such that $s_2 \circ d = d \circ s_2$. Then there exists a unique derivation $\tilde{d}$ on $\tilde{L}$ so that*

$$\tilde{d}\,|_L = d, \quad \tilde{d}(x \otimes t^{2k+1}) = s_1 \circ d(x \otimes t^{2k}) \quad \text{and} \quad \tilde{d}(v \otimes t^{2k}) = s_{-1} \circ d(v \otimes t^{2k+1})$$

for all $x \in \mathfrak{g}$, $v \in \mathfrak{s}$, and $k \in \mathbb{Z}$. Moreover,

$$\tilde{d} \in (\mathrm{Der}_F \tilde{L})_0^{2m+1} \quad \text{such that} \quad s_k \circ \tilde{d} = \tilde{d} \circ s_k \quad \text{for all } k \in \mathbb{Z}.$$

Proof. The uniqueness is clear since the image of all homogeneous elements are determined. So it is enough to show that $\tilde{d}$ is a derivation. Thus we need to check the following: For $x, y \in \mathfrak{g}$ and $v, w \in \mathfrak{s}$,

(a) $\tilde{d}([x \otimes t^{2k}, y \otimes t^{2\ell+1}]) = [\tilde{d}(x \otimes t^{2k}), y \otimes t^{2\ell+1}] + [x \otimes t^{2k}, \tilde{d}(y \otimes t^{2\ell+1})]$
(b) $\tilde{d}([x \otimes t^{2k}, v \otimes t^{2\ell}]) = [\tilde{d}(x \otimes t^{2k}), v \otimes t^{2\ell}] + [x \otimes t^{2k}, \tilde{d}(v \otimes t^{2\ell})]$
(c) $\tilde{d}([x \otimes t^{2k+1}, y \otimes t^{2\ell+1}]) = [\tilde{d}(x \otimes t^{2k+1}), y \otimes t^{2\ell+1}] + [x \otimes t^{2k+1}, \tilde{d}(y \otimes t^{2\ell+1})]$
(d) $\tilde{d}([x \otimes t^{2k+1}, v \otimes t^{2\ell+1}]) = [\tilde{d}(x \otimes t^{2k+1}), v \otimes t^{2\ell+1}] + [x \otimes t^{2k+1}, \tilde{d}(v \otimes t^{2\ell+1})]$
(e) $\tilde{d}([x \otimes t^{2k+1}, v \otimes t^{2\ell}]) = [\tilde{d}(x \otimes t^{2k+1}), v \otimes t^{2\ell}] + [x \otimes t^{2k+1}, \tilde{d}(v \otimes t^{2\ell})]$
(f) $\tilde{d}([v \otimes t^{2k+1}, w \otimes t^{2\ell}]) = [\tilde{d}(v \otimes t^{2k+1}), w \otimes t^{2\ell}] + [v \otimes t^{2k+1}, \tilde{d}(w \otimes t^{2\ell})]$
(g) $\tilde{d}([v \otimes t^{2k}, w \otimes t^{2\ell}]) = [\tilde{d}(v \otimes t^{2k}), w \otimes t^{2\ell}] + [v \otimes t^{2k}, \tilde{d}(w \otimes t^{2\ell})]$.

All the equations are followed by easy calculation, but let us check for sure.

For (a), we have

$$\begin{aligned}
(LHS) &= \tilde{d}([x, y] \otimes t^{2k+2\ell+1}) = s_1 \circ d([x, y] \otimes t^{2k+2\ell}) = s_1 \circ d([x \otimes t^{2k}, y \otimes t^{2\ell}]) \\
&= s_1 \circ \big([d(x \otimes t^{2k}), y \otimes t^{2\ell}] + [x \otimes t^{2k}, d(y \otimes t^{2\ell})]\big) \\
&= [d(x \otimes t^{2k}), y \otimes t^{2\ell+1}] + [x \otimes t^{2k}, s_1 \circ d(y \otimes t^{2\ell})] = (RHS).
\end{aligned}$$

For (b), we have

$$\begin{aligned}
(LHS) &= \tilde{d}([x, v] \otimes t^{2k+2\ell}) = s_{-1} \circ d([x, v] \otimes t^{2k+2\ell+1}]) \\
&= s_{-1} \circ \big([d(x \otimes t^{2k}), v \otimes t^{2\ell+1}] + [x \otimes t^{2k}, d(v \otimes t^{2\ell+1})]\big) \\
&= [d(x \otimes t^{2k}), v \otimes t^{2\ell}] + [x \otimes t^{2k}, \tilde{d}(v \otimes t^{2\ell})] = (RHS).
\end{aligned}$$

For (c), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,y]\otimes t^{2k+2\ell+2}) = d([x,y]\otimes t^{2k+2\ell+2}])\\ &= d([x\otimes t^{2k}, y\otimes t^{2\ell+2}]) = [d(x\otimes t^{2k}), y\otimes t^{2\ell+2}] + [x\otimes t^{2k}, d(y\otimes t^{2\ell+2})]\\ &= s_1\circ[d(x\otimes t^{2k}), y\otimes t^{2\ell+1}] + [x\otimes t^{2k}, d\circ s_2(y\otimes t^{2\ell})]\\ &= [s_1\circ d(x\otimes t^{2k}), y\otimes t^{2\ell+1}] + s_2\circ[x\otimes t^{2k}, d(y\otimes t^{2\ell})]\\ &\quad \text{(since } s_2 \text{ and } d \text{ commute)}\\ &= [\tilde{d}(x\otimes t^{2k+1}), y\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, s_1\circ d(y\otimes t^{2\ell})] = (RHS).\end{aligned}$$

For (d), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,v]\otimes t^{2k+2\ell+2}) = s_{-1}\circ d([x,v]\otimes t^{2\ell+3}])\\ &= s_{-1}\circ d([x\otimes t^{2k}, v\otimes t^{2\ell+3}])\\ &= s_{-1}\circ\big([d(x\otimes t^{2k}), v\otimes t^{2\ell+3}] + [x\otimes t^{2k}, d(v\otimes t^{2\ell+3})]\big)\\ &= s_1\big([d(x\otimes t^{2k}), v\otimes t^{2\ell+1}]\big) + s_{-2}\big([x\otimes t^{2k+1}, d(v\otimes t^{2\ell+3})]\big)\\ &= [s_1\circ d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, s_{-3}\circ d(v\otimes t^{2\ell+3})]\\ &= (RHS) \quad \text{(since } s_2 \text{ and } d \text{ commute).}\end{aligned}$$

For (e), we have

$$\begin{aligned}(LHS) &= \tilde{d}([x,v]\otimes t^{2k+2\ell+1}) = d([x,v]\otimes t^{2k+2\ell+1}]) = d([x\otimes t^{2k}, v\otimes t^{2\ell+1}])\\ &= [d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k}, d(v\otimes t^{2\ell+1})]\\ &= [s_1\circ d(x\otimes t^{2k}), v\otimes t^{2\ell}] + [x\otimes t^{2k}, \tilde{d}(v\otimes t^{2\ell+1})] = (RHS).\end{aligned}$$

For (f), we have

$$\begin{aligned}(LHS) &= \tilde{d}([v,w]\otimes t^{2k+2\ell+1}]) = s_1\circ d([v,w]\otimes t^{2k+2\ell}])\\ &= s_1\circ d([v\otimes t^{2k-1}, w\otimes t^{2\ell+1}])\\ &= s_1\circ\big([d(v\otimes t^{2k-1}), w\otimes t^{2\ell+1}] + [v\otimes t^{2k-1}, d(w\otimes t^{2\ell+1})]\big)\\ &= [s_1\circ d(v\otimes t^{2k-1}), w\otimes t^{2\ell+1}]\big) + [v\otimes t^{2k}, d(w\otimes t^{2\ell+1})]\\ &= [s_1\circ d(x\otimes t^{2k}), v\otimes t^{2\ell+1}] + [x\otimes t^{2k+1}, s_{-3}\circ d(v\otimes t^{2\ell+3})] = (RHS).\end{aligned}$$

For (g), we have

$$\begin{aligned}(LHS) &= \tilde{d}([v,w]\otimes t^{2k+2\ell}]) = d([v,w]\otimes t^{2k+2\ell}]) = d([v\otimes t^{2k-1}, w\otimes t^{2\ell+1}])\\ &= [d(v\otimes t^{2k-1}), w\otimes t^{2\ell+1}] + [v\otimes t^{2k-1}, d(w\otimes t^{2\ell+1})]\\ &= [s_1\circ d(v\otimes t^{2k-1}), w\otimes t^{2\ell}] + [v\otimes t^{2k}, s_{-1}\circ d(w\otimes t^{2\ell+1})] = (RHS).\end{aligned}$$

For the second assertion, it is clear that $\tilde{d}\in(\mathrm{Der}_F\,\tilde{L})_0^{2m+1}$. Also, since d commutes with s_2, so does $\tilde{d}$. Hence, by Lemma 6.3, $\tilde{d}$ commutes with s_k for all $k\in\mathbb{Z}$. □

Thus together with Lemma 7.6, we have classified the diagonal derivations of twisted locally loop algebras.

Theorem 7.10. *Let L be a twisted loop algebra. Then we have $(\mathrm{Der}_F\,L)_0^0 = \mathrm{ad}\,P\oplus Fd^{(0)}$, where P is defined in* (38)*, and*

$$(\mathrm{Der}_F\,L)_0^{2m} = s_{2m}\circ(\mathrm{Der}_F\,L)_0^0 \quad \textit{and} \quad (\mathrm{Der}_F\,L)_0^{2m+1} = s_{2m+1}\circ\mathrm{ad}\,T^-$$

for all $m \in \mathbb{Z}$, where $T^- = \mathfrak{s}_0$ for $B_{\mathfrak{I}}^{(2)}$, $T^- = T^-_{2\mathfrak{I}}$ for $C_{\mathfrak{I}}^{(2)}$ or $T^- = T^-_{2\mathfrak{I}+1}$ for $BC_{\mathfrak{I}}^{(2)}$ defined in Example 5.2.

Proof. By Lemma 7.8, 7.9, and the classification of untwisted case, if $d \in (\mathrm{Der}_F L)_0^{2m+1}$, then $\tilde{d} \in s_{2m+1} \circ (\mathrm{Der}_F L)_0^0$. Also, by Lemma 7.8 and Lemma 7.7, we get $\tilde{d} \in s_{2m+1} \circ \mathrm{ad} P$. Thus $\mathrm{ad}\, p := s_{-2m-1} \circ d \in \mathrm{ad} P$, and we have $[p, \mathfrak{g}^+] \subset \mathfrak{g}^-$, and hence, by Lemma 7.4, we get $p \in T^-$. Therefore, $d \in s_{2m+1} \circ \mathrm{ad}\, T^-$. □

Remark 7.11. If L is any twisted loop algebra of type $\mathrm{B}_{\mathfrak{I}}^{(2)}$, then $(\mathrm{Der}_F L)_0^{2m+1} = s_{2m+1} \circ \mathrm{ad}\, \mathfrak{s}_0 = \mathrm{ad}(\mathfrak{s}_0 \otimes t^{2m+1})$. So there are no outer derivations of odd degree.

We go back to the classification of twisted LALAs. By Theorem 7.10, if $d \in D^{2m+1}$, then $\mathrm{ad}_L d \in s_{2m+1} \circ \mathrm{ad}_L T^-$. The bracket on $D := \bigoplus_{m \in \mathbb{Z}} D^m$ can be investigated by the same way as in the untwisted case. Thus D^m for $m \in \mathbb{Z}$ is exactly one of the examples for each type described in Example 5.2. Thus we have finished the classification:

Theorem 7.12. *Let $\mathcal{L}$ be a twisted LALA. Then $\mathcal{L}$ is isomorphic to one in Example 5.2.* □

Remark 7.13. One can show that any twisted LALA is the fixed algebra of some untwisted LALA. Moreover, for any untwisted LALA $\mathcal{L}$ of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$ or $\mathrm{D}_{\mathfrak{I}}^{(1)}$, there exists a twisted LALA $\mathcal{L}'$ which is a subalgebra of $\mathcal{L}$ so that $\mathcal{L}'$ is the intersection of $\mathcal{L}$ and the fixed algebra of a maximal untwisted LALA $\mathcal{L}^{max}$ containing $\mathcal{L}$. Note that a maximal twisted LALA is also unique, up to isomorphism, as in case of a maximal untwisted LALA.

Remark 7.14. By Theorem 6.5 and Theorem 7.12, the LALAs in Example 5.2 exhaust all. From this fact, the following is clear, and it will be a useful criterion later.

If a diagonal matrix $p \in T$ whose trace is a nonzero value (e.g. e_{ii} or $e_{ii} + e_{\mathfrak{I}+i,\mathfrak{I}+i}$, etc.) is used in a LALA, then such a LALA has to be of type $\mathrm{A}_{\mathfrak{I}}^{(1)}$, $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$. Moreover, if the type is $\mathrm{C}_{\mathfrak{I}}^{(2)}$ or $\mathrm{BC}_{\mathfrak{I}}^{(2)}$, then such a p has to be used in odd degree.

Corollary 7.15. *Let $\mathcal{L}$ be a LALA (untwisted or twisted) with the center Fc and $\mathcal{L}_c$ its core, which is a locally Lie 1-torus with grading pair $(\mathfrak{g}, \mathfrak{h})$. If there exists $0 \neq d \in \mathcal{L}$ such that $[d, \mathfrak{g}] = 0$ and $\mathcal{B}(d, c) \neq 0$, then d is a nonzero multiple of a degree derivation modulo the center, and hence $\mathcal{L}$ is standard.*

Proof. Let $d = \sum_{\xi \in R} x_\xi$ for $x_\xi \in \mathcal{L}_\xi$. If $\xi \in R^\times$, then $[\mathfrak{h}, x_\xi] \subset \mathcal{L}_\xi$, and so $x_\xi = 0$ since $[d, \mathfrak{g}] = 0$. If $\xi \in R^0 \setminus \{0\}$, then $x_\xi \in \mathfrak{h} \otimes t^{2m}$ or $x_\xi \in \mathfrak{s}_0 \otimes t^{2m+1}$ or $x_\xi \in \mathfrak{d}_m$ for some $0 \neq m \in \mathbb{Z}$, by Theorem 6.5 and 7.12. But for each case, if $x_\xi \neq 0$, then there exists a root vector $y \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ so that $[y, x_\xi] \neq 0$. This is a contradiction. Hence $x_\xi = 0$. Thus $d = x_0 \in \mathcal{L}_0 = \mathcal{H}$. Then, by Theorem 6.5 and 7.12,

$$d = h + p + a \cdot d^{(0)} + b \cdot c$$

for some $h \in \mathfrak{h}$, $p \in T = T \otimes t^0$, $a, b \in F$, and $a \neq 0$ since $\mathcal{B}(d, c) \neq 0$. So we have

$$0 = [d, \mathfrak{g}] = [h + p, \mathfrak{g}] = [h, \mathfrak{g}] + [p, \mathfrak{g}].$$

Suppose that p has infinitely many nonzero entries. Then there exists $x \in \mathfrak{g}$ such that $[p, x] \neq 0$ but $[h, x] = 0$. This is a contradiction. Hence p has only finitely many nonzero entries, and so the type of $\mathcal{L}$ is $\mathrm{A}_{\mathfrak{I}}^{(1)}$ (see Remark 7.14). Let I be a finite subset of $\mathfrak{I}$ so that $h \in \mathfrak{h}_I \subset \mathrm{sl}_I(F) \subset \mathrm{sl}_{\mathfrak{I}}(F) = \mathfrak{g}$ and $p \in \mathrm{gl}_I(F) \subset \mathrm{gl}_{\mathfrak{I}}(F)$, where $\mathfrak{h}_I$ is the Cartan subalgebra consisting of diagonal matrices in $\mathrm{sl}_I(F)$. (We use the property that $\mathrm{sl}_{\mathfrak{I}}(F)$ is a directed union of $\mathrm{sl}_I(F)$ running over finite sets I.) Then $p = h' + s\iota_I$ for some $h' \in \mathfrak{h}_I$ and $s \in F$.

Since $\mathrm{sl}_I(F) \subset \mathfrak{g}$ and $[\iota_I, \mathrm{sl}_I(F)] = 0$, we have $0 = [h, \mathrm{sl}_I(F)] + [p, \mathrm{sl}_I(F)] = [h+h', \mathrm{sl}_I(F)]$. Since $h+h' \in \mathrm{sl}_I(F)$, we get $h+h' = 0$. Hence $d = s\iota_I + a\cdot d^{(0)} + bc$. Take $e_{ij} \in \mathfrak{g}$ for $j \notin I$. Then $[d, e_{ij}] = [s\iota_I, e_{ij}] = se_{ij} = 0$, and hence $s = 0$. Thus we obtain $d = a\cdot d^{(0)} + bc$. □

Remark 7.16. Neeb in [N2, Def.3.6] defined a minimal LALA $\mathcal{L}$ as it is minimal in the sense above and satisfies one more condition:

$$\exists\, d \in \mathcal{H} \text{ such that } \operatorname{span}_{\mathbb{Q}}\{\alpha \in R^\times \mid \alpha(d) = 0\} \text{ is a reflectable section.}$$

Thus $[\mathfrak{g}, d] = 0$, and $\delta(d) \neq 0$, where δ is a generator of $R^0 \cong \mathbb{Z}$. But then, d is a nonzero multiple of a degree derivation modulo the center, by Corollary 7.15. Hence, a minimal LALA in [N2] is a minimal standard LALA in our sense.

Example 7.17. Let $p = \operatorname{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ and put $d = p + d^{(0)}$. Then the minimal LALA $\mathcal{L} = \mathrm{sl}_{\mathbb{N}}(F[t,t^{-1}]) \oplus Fc \oplus Fd$ is not isomorphic to a minimal standard LALA. For, if $\mathcal{L}$ is isomorphic to a minimal standard LALA $\mathcal{L}^{ms}$, there exists an isomorphism

$$\psi : \mathcal{L}^{ms} \longrightarrow \mathcal{L}$$

so that $\psi(d^{(0)}) = x + a\cdot d = x + a\cdot(d^{(0)} + p)$ for some $x \in \mathcal{L}_c = \mathrm{sl}_{\mathbb{N}}(F[t,t^{-1}]) \oplus Fc$ and some nonzero $a \in F$. Then, we have

$$\psi \circ \operatorname{ad} d^{(0)} \circ \psi^{-1} = \operatorname{ad}(\psi(d^{(0)})) = \operatorname{ad}(x + a\cdot d^{(0)} + a\cdot p)$$

in $\mathrm{Der}_F(\mathcal{L})$. Now we can compare the eigenvalues of the same operators $\psi \circ \operatorname{ad} d^{(0)} \circ \psi^{-1}$ and $\operatorname{ad}(x + a\cdot d^{(0)} + a\cdot p)$. Note that the eigenvalues of $\psi \circ \operatorname{ad} d^{(0)} \circ \psi^{-1}$ are all integers. We can choose $h = e_{\ell\ell} - e_{\ell+1,\ell+1} \in \mathrm{sl}_{\mathbb{N}}(F[t,t^{-1}])$ such that

$$[x, h] = 0,$$

taking $\ell >> 0$, where e_{ij} is a matrix unit. Then,

$$[x + a\cdot d^{(0)} + a\cdot p, h\otimes t] = a(h\otimes t),$$

which implies that a is a nonzero integer since a is an eigenvalue of $\operatorname{ad}(x + a\cdot d^{(0)} + a\cdot p)$. On the other hand, we can also choose sufficiently large integers $m, n >> 0$ with $m \neq n$ satisfying

$$[x, e_{mn}] = 0$$

and

$$\frac{a(n-m)}{mn} \notin \mathbb{Z}. \tag{47}$$

For such m and n, we see

$$[x + a\cdot d^{(0)} + a\cdot p, e_{mn}] = a\left(\frac{1}{m} - \frac{1}{n}\right) e_{mn} = \frac{a(n-m)}{mn} e_{mn}.$$

Since

$$\frac{a(n-m)}{mn}$$

is an eigenvalue of $\operatorname{ad}(x + a\cdot d^{(0)} + a\cdot p)$, it must be an integer, which is a contradiction to (47). Hence, $\mathcal{L}^{ms}$ is never isomorphic to $\mathcal{L}$.

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