The Kurosh Problem for Jordan Nil Systems over Arbitrary Rings of Scalars

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Abstract: We show that quadratic Jordan nil algebras of bounded degree, and, more generally, quadratic Jordan nil algebras satisfying a polynomial identity, are locally nilpotent. We also extend these results to Jordan pairs and triple systems.

Keywords: Jordan system, nil radical, McCrimmon radical, nil, local nilpotency

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Introduction

In 1940, A. G. Kurosh [8] (and independently J. Levitzki [1]) formulated the question of whether a finitely generated algebraic associative algebra over a field is necessarily finite-dimensional. The answer is negative in general, but turns affirmative when we restrict to associative algebras which are algebraic of bounded degree (see [20], where the above problems and their solutions are extensively discussed, and precise references are included).

In 1971, Shirshov posed the Jordan version of the Kurosh problem for nil algebras [5, 1.156], which was solved affirmatively in the case of linear Jordan algebras in [18]. The proof was based on the local nilpotency of the McCrimmon radical of a linear Jordan algebra.

Recently, a proof of the local nilpotency of the McCrimmon radical of a Jordan system over an arbitrary ring of scalars has been given [4]. This paper is aimed at using this result to prove that Jordan systems which are nil of bounded degree, or

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more generally nil and PI (homotope-PI in the cases of pairs and triple systems) are locally nilpotent.

The paper is divided into three sections, apart from a preliminary one. In the first section, we use the local nilpotency of the McCrimmon radical to prove that a quadratic Jordan nil algebra of bounded degree is locally nilpotent. It should be remarked the fundamental role played by the so-called linear absorbers in the quadratic setting (see [13, 19]). In the next section, these results are extended to nil algebras satisfying a polynomial identity, so obtaining that the nil radical and the McCrimmon radical coincide for PI Jordan algebras. Finally, in the third section, the use of local algebras allows us to obtain analogues of the results of the two previous sections for Jordan pairs and triple systems.

0. Preliminaries

0.1 We will deal with Jordan systems (algebras, pairs and triple systems), and associative and Lie algebras over an arbitrary ring of scalars Φ. In particular, we will NOT assume $1/2 \in \Phi$.

The reader is referred to [7, 10, 14, 15] for basic results, notation, and terminology, though we will stress some notions.

— When dealing with an associative algebra, the (associative) products will be denoted by juxtaposition.

— Given a Jordan algebra J, its products will be denoted by x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y = V_x y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

— For a Jordan pair $V = (V^+, V^-)$, we have products $Q_x y \in V^\sigma$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$.

— A Jordan triple system J is given by its products $P_x y$, for any $x, y \in J$, with linearizations denoted by $P_{x,z} y = \{x, y, z\} = L_{x,y} z$.

0.2 We recall the following identities valid for arbitrary Jordan algebras which will be needed in the sequel:

(i) $U_{x^n} = U_x^n$, for any positive integer n,

(ii) $U_{U_x y} = U_x U_y U_x$,

(iii) $U_{x,y} = V_y V_x - V_{y,x}$,

(iv) $U_x U_y z = U_{x \circ y} z - \{\{x, y, z\}, x, y\} + z \circ U_y x^2 - U_y U_x z$,

(v) $U_x \{y, z, x\} = \{x, y, z\} \circ x^2 - \{z, y, x^3\}$,

(vi) $U_x y^2 = (x \circ y)^2 - x \circ U_y x - U_y x^2$,

(vii) $x \circ U_y z = \{x \circ y, z, y\} - U_y (x \circ z)$.

Indeed, (ii, iii, vii) are [7, QJ2, QJ14, QJ20'], respectively, and (i, iv, v, vi) follow from Macdonald's Theorem [9].

0.3 Given a Jordan algebra J, the *multiplication algebra* $\mathcal{M}(J)$ of J is the subalgebra of $\mathrm{End}_\Phi(J)$ generated by the *multiplication operators* U_x, V_x, for $x \in J$. By linearization, $\mathcal{M}(J)$ contains all the operators $U_{x,y}$, for $x, y \in J$, hence also the operators $V_{x,y}$ by (0.2)(iii).

0.4 (i) A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [10, 1.13, 1.14].

(ii) An associative system R gives rise to a Jordan system $R^{(+)}$ by *symmetrization*: over the same Φ-module (the same pair of Φ-modules, in the pair case), we define $x^2 = xx$, $U_x y = xyx$, for any $x, y \in R$ in the case of algebras, $P_x y = xyx$ in the case of triple systems, and $Q_{x^\sigma} y^{-\sigma} = x^\sigma y^{-\sigma} x^\sigma$, $\sigma = \pm$ in the pair case, where juxtaposition denotes the associative product in R.

0.5 A Jordan system J is called *special* if it is a subsystem of $R^{(+)}$, for some associative system R. Otherwise J is said to be *exceptional*.

0.6 *Local algebras* of Jordan systems are introduced in [16] generalizing the corresponding notion for associative systems:

— Given an associative triple system R, the *homotope* $R^{(a)}$ of R at $a \in R$ is the associative algebra over the same Φ-module as R with product $x \cdot_a y = xay$, for any $x, y \in R$. The subset $\mathrm{Ker} a = \mathrm{Ker}_R a = \{x \in R \mid axa = 0\}$ is an ideal of $R^{(a)}$ and the quotient $R_a = R^{(a)}/\mathrm{Ker} a$ is called *the local algebra of R at a*.

— Given an associative pair $R = (R^+, R^-)$, the *homotope* $R^{\sigma(a)}$ of R at $a \in R^{-\sigma}$ ($\sigma = \pm$) is the associative algebra over the same Φ-module as R^σ with product $x \cdot_a y = xay$, for any $x, y \in R^\sigma$. The subset $\mathrm{Ker} a = \mathrm{Ker}_R a = \{x \in R^\sigma \mid axa = 0\}$ is an ideal of $R^{\sigma(a)}$ and the quotient $R^\sigma_a = R^{\sigma(a)}/\mathrm{Ker} a$ is called *the local algebra of R at a*.

— Given a Jordan triple system J, the *homotope* $J^{(a)}$ of J at $a \in J$ is the Jordan algebra over the same Φ-module as J with products $x^{(2,a)} = x^2 = P_x a$, $U^{(a)}_x y = U_x y = P_x P_a y$, for any $x, y \in J$. The subset $\mathrm{Ker} a = \mathrm{Ker}_J a = \{x \in J \mid P_a x = P_a P_x a = 0\}$ is an ideal of $J^{(a)}$ and the quotient $J_a = J^{(a)}/\mathrm{Ker} a$ is called *the local algebra of J at a*. When J is nondegenerate, $\mathrm{Ker} a = \{x \in J \mid P_a x = 0\}$.

— Given a Jordan pair V, the *homotope* $V^{\sigma(a)}$ of V at $a \in V^{-\sigma}$ ($\sigma = \pm$) is the Jordan algebra over the same Φ-module as V^σ with products $x^{(2,a)} = x^2 = Q_x a$, $U^{(a)}_x y = U_x y = Q_x Q_a y$, for any $x, y \in J$. The subset $\mathrm{Ker} a = \mathrm{Ker}_V a = \{x \in V^\sigma \mid Q_a x = Q_a Q_x a = 0\}$ is an ideal of $V^{\sigma(a)}$ and the quotient $V^\sigma_a = V^{\sigma(a)}/\mathrm{Ker} a$ is called *the local algebra of V at a*. When V is nondegenerate, $\mathrm{Ker} a = \{x \in$

$V^{-\sigma} \mid Q_a x = 0\}$.

For an associative or Jordan algebra, local algebras are given by the above definitions applied to its underlying triple system.

0.7 An *absolute zero divisor* of a Jordan algebra (resp., triple system) J is an element $x \in J$ such that $U_x J = 0$ (resp., $P_x J = 0$). An absolute zero divisor in a Jordan pair (V^+, V^-) is any element $x \in V^\sigma$ such that $Q_x V^{-\sigma} = 0$. A Jordan system is said to be *nondegenerate* if it does not have nonzero absolute zero divisors.

0.8 We recall that the *McCrimmon radical* (also called *small radical* in [10, 4.5]) $\mathrm{Mc}(J)$ of a Jordan system J is the smallest ideal of J which produces a nondegenerate quotient. It can be obtained by a transfinite induction process as follows [10, 4.7]: Let $M_1(J)$ be the span of absolute zero divisors of J, which is an ideal of J by [10, 4.6]. Once we have the ideals $M_\alpha(J)$ for all ordinals $\alpha < \beta$, we define $M_\beta(J)$ by

(i) $M_\beta(J)/M_{\beta-1}(J) = M_1(J/M_{\beta-1}(J))$ when β is not a limit ordinal,

(ii) $M_\beta(J) = \cup_{\alpha<\beta} M_\alpha(J)$ when β is a limit ordinal.

Then $\mathrm{Mc}(J) = \lim_\alpha M_\alpha(J)$, so that for any Jordan system J, $\mathrm{Mc}(J) = M_\alpha(J)$ for some ordinal α (such that $M_1(J/M_\alpha(J)) = 0$, i.e., $J/M_\alpha(J)$ is nondegenerate).

1. Jordan Nil Algebras of Bounded Degree

We will follow the strategy of Section 2 of [18], introducing the necessary changes to allow arbitrary rings of scalars, i.e., extending the results to arbitrary quadratic Jordan algebras.

The following result can be found in [18] and its proof is valid for arbitrary quadratic Jordan algebras.

1.1 LEMMA [18, Lemma 15]. *Let J be a Jordan algebra, J_1 be subalgebra of J, and I be an ideal of J_1, and let M be the ideal of J_1 such that $M/I = \mathrm{Mc}(J_1/I)$. If $w, w^* \in \mathcal{M}(J)$ satisfy $w^*(J) \subseteq J_1$, $w(I) \subseteq \mathrm{Mc}(J)$, and $U_{w(a)} = wU_a w^*$, for any $a \in J_1$, then $w(M) \subseteq \mathrm{Mc}(J)$.* ■

1.2 Recall the Jacobson Counterexample [6, ex. 3, p. 12] of a Jordan algebra J over a ring of scalars Φ of characteristic two, such that there exists an element $a \in J$ with $a^2 = 0$, but $a^3 \neq 0$. Of course, this cannot happen in the linear case, and we will see in the next result that it is also impossible in a nondegenerate atmosphere.

1.3 LEMMA. *Given a Jordan algebra J, and an element $x \in J$, if $x^n \in \mathrm{Mc}(J)$ then $x^m \in \mathrm{Mc}(J)$ for any $m \geq n$. In particular, if J is nondegenerate and $x^n = 0$, then $x^m = 0$ for any $m \geq n$.*

PROOF: Notice that

$$U_{x^m} J =_{(0.2)(\mathrm{i})}= U_x^m J = U_x^n U_x^{m-n} J \subseteq U_x^n J =_{(0.2)(\mathrm{i})} U_{x^n} J \subseteq \mathrm{Mc}(J)$$

implies $x^m + \mathrm{Mc}(J)$ is an absolute zero divisor of $J/\mathrm{Mc}(J)$, hence $x^m + \mathrm{Mc}(J) = 0$, i.e., $x^m \in \mathrm{Mc}(J)$. ■

In the next lemma we obtain the main result of this section in the particular case of degree two. Though this is trivial for linear Jordan algebras, it requires a nontrivial proof if we are not assuming the existence of $1/2$ in the ring of scalars.

1.4 LEMMA. *If J is a Jordan algebra such that $a^2 = 0$ for any $a \in J$, then $J = \mathrm{Mc}(J)$.*

PROOF: By factoring out $\mathrm{Mc}(J)$, we can assume that $\mathrm{Mc}(J) = 0$, i.e., J is nondegenerate, and we will show in that case that $J = \mathrm{Mc}(J) = 0$.

By linearization,

$$a \circ b = 0, \tag{1}$$

for any $a, b \in J$. Using (1.3),

$$a^3 = 0, \tag{2}$$

for any $a \in J$. Now, given any $a, b, c \in J$, putting $x = b, y = a, z = c$ in (0.2)(iv) yields

$$\begin{aligned} U_bU_ac &= U_{b\circ a}c - \{\{b,a,c\},b,a\} + c \circ U_ab^2 - U_aU_bc \\ &= -\{\{b,a,c\},b,a\} - U_aU_bc, \end{aligned} \tag{3}$$

using (1). Let us write $t := \{b, a, c\}$. We have,

$$\begin{aligned} U_{U_ab}c &=_{(0.2)(\mathrm{ii})} U_aU_bU_ac =_{(3)} -U_a\{t,b,a\} - U_aU_aU_bc \\ &=_{(0.2)(\mathrm{i})} -U_a\{t,b,a\} - U_{a^2}U_bc = -U_a\{t,b,a\} \\ &=_{(0.2)(\mathrm{v})} -\{a,t,b\} \circ a^2 + \{b,t,a^3\} = 0 \end{aligned}$$

by (2), hence $U_ab = 0$ by nondegeneracy of J. We have shown that J is a trivial algebra, hence $J = \mathrm{Mc}(J) = 0$. ■

1.5 Let J be a special Jordan algebra, and let A be an associative algebra such that J is a subalgebra of $A^{(+)}$. We will say that (J, A) satisfies the *weak polynomial identity* $f \in \mathrm{FAss}[X]$ if all evaluations of f in elements of J vanish: for any map $X \longrightarrow J \subseteq A$, the induced associative algebra homomorphism $\varphi : \mathrm{FAss}[X] \longrightarrow A$ satisfies $\varphi(f) = 0$.

1.6 By the usual linearization process [17, 6.1.13], if (J, A) satisfies a monic weak polynomial identity f of degree n, then it also satisfies a monic multilinear polynomial identity

$$\tilde{f} = x_1x_2\cdots x_n + \sum_{\sigma\in\Sigma_n\setminus\{Id\}} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)},$$

where Σ_n denotes the permutation group of $\{1, 2, \ldots, n\}$, and $\alpha_\sigma \in \Phi$, for any $\sigma \in \Sigma_n \setminus \{Id\}$.

1.7 Given an inner ideal K of a Jordan algebra J, the (*linear*) *absorber* $\operatorname{abs}(K)$ of K is given by

$$\operatorname{abs}(K) = \operatorname{abs}_J(K) = \{k \in K \mid k \circ J \subseteq K\},$$

which is an inner ideal of J and an ideal of K such that $K/\operatorname{abs}(K)$ is special [13, 2.1 and its proof; 19, Lemma 1(c)].

1.8 LEMMA. *If J is a Jordan algebra, K is an inner ideal of J, and $K_1 := \operatorname{abs}_J(K)$, then*

(i) $U_J U_{K_1} K \subseteq K$,

(ii) $U_J K_1^2 \subseteq K$,

(iii) $U_J U_{K_1} K_1 \subseteq K_1$.

PROOF: (i) For any $x \in J$, $a \in K_1$, and $b \in K$,

$$U_x U_a b =_{(0.2)(\mathrm{iv})} U_{x \circ a} b - \{\{x, a, b\}, x, a\} + b \circ U_a x^2 - U_a U_x b \in K$$

because, using the definition of K_1, and the fact that K is an inner ideal,

$$U_{x \circ a} b \in U_{J \circ K_1} K \subseteq U_K K \subseteq K,$$

$$b \circ U_a x^2 \in K \circ U_K J \subseteq K \circ K \subseteq K,$$

$$U_a U_x b \in U_K J \subseteq K,$$

and

$$\{x, a, b\} =_{(0.2)(\mathrm{iii})} b \circ (x \circ a) - \{b, x, a\} \in K \circ (J \circ K_1) + U_K J \subseteq K \circ K + U_K J \subseteq K,$$

hence

$$\{\{x, a, b\}, x, a\} \in U_K J \subseteq K.$$

.

(ii) For any $x \in J$, $a \in K_1$,

$$U_x a^2 =_{(0.2)(\mathrm{vi})} (x \circ a)^2 - x \circ U_a x - U_a x^2 \in K$$

because, using the definition of K_1, and the fact that K is an inner ideal,

$$(x \circ a)^2 \in (J \circ K_1)^2 \subseteq K^2 \subseteq K,$$

$$U_a x^2 \in U_K J \subseteq K,$$

$$x \circ U_a x =_{(0.2)(\mathrm{vii})} \{x \circ a, x, a\} - U_a(x \circ x) \in \{J \circ K_1, J, K\} + U_K J \subseteq U_K J \subseteq K.$$

(iii) Let $x \in J$, $a, b \in K_1$. By (i), $U_x U_a b \in K$, so we only have to check that $U_x U_a b \circ y \in K$, for any $y \in J$. Indeed,

$$U_x U_a b \circ y =_{(0.2)(\mathrm{vii})} \{y \circ x, U_a b, x\} - U_x(y \circ U_a b) \in K$$

because, using the definition of K_1, and the facts that K is an inner ideal of J and K_1 is an ideal of K,

$$\{y \circ x, U_a b, x\} \in U_J U_{K_1} K_1 \subseteq U_J U_{K_1} K \subseteq_{(\mathrm{i})} K,$$

$$\begin{aligned} U_x(y \circ U_a b) &=_{(0.2)(\mathrm{vii})} U_x\{a, b, y \circ a\} - U_x U_a (y \circ b) \\ &=_{(0.2)(\mathrm{iii})} U_x\Big(a \circ \big(b \circ (y \circ a)\big)\Big) - U_x\{b, y \circ a, a\} - U_x U_a(y \circ b) \\ &\in U_J\Big(K_1 \circ \big(K_1 \circ (J \circ K_1)\big)\Big) + U_J U_{K_1}(J \circ K_1) \\ &\subseteq U_J\big(K_1 \circ (K_1 \circ K)\big) + U_J U_{K_1} K \subseteq U_J K_1^2 + U_J U_{K_1} K \subseteq_{(\mathrm{i,ii})} K. \ \blacksquare \end{aligned}$$

1.9 THEOREM. *A Jordan nil algebra of bounded degree is McCrimmon radical, hence locally nilpotent.*

PROOF: Let J be a Jordan algebra such that $x^p = 0$, for any $x \in J$, for some positive integer p. We will show that $J = \mathrm{Mc}(J)$ by induction on p, so that J will be locally nilpotent by [4, 2.11]. As it is customary, $\widehat{J}$ will denote the *unital hull* of J (see [11]).

In the case $p = 1$, $J = 0 = \mathrm{Mc}(J)$. In the case $p = 2$, $J = \mathrm{Mc}(J)$ by (1.4). Assume that the result is true for all Jordan algebras satisfying the identity $x^m = 0$, for all $m < p$, and $p \geq 3$.

We will study first the case when J is special. Take any associative algebra such that J is a subalgebra of $A^{(+)}$. Notice that the unital hull $\widehat{J}$ is also special, indeed it is a subalgebra of $\widehat{A}^{(+)}$. Since (J, A) satisfies the monic weak polynomial identity $x^p = 0$, by (1.6) it also satisfies an identity given by a monic multilinear polynomial of degree p which always can be expressed as

$$\begin{aligned} f(x_1, \ldots, x_p) &= x_1 x_2 \cdots x_p + \sum_{i \geq 2} x_i f_i(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_p) \\ &= \sum_{i=1}^{p} x_i f_i(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_p), \end{aligned} \tag{1}$$

where the f_i's are multilinear polynomials of degree $p - 1$, in which the variable x_i is missing, and

$$f_1(x_2, \ldots, x_p) = x_2 \cdots x_p. \tag{2}$$

Take arbitrary elements $b \in J$, $a, c \in \widehat{J}$, and let $K = \Phi b + U_b\widehat{J} = \Phi b + b\widehat{J}b$, which is an inner ideal of J. For any $k_2, \ldots, k_p \in K$, since $bac + cab = \{b, a, c\} \in J$, we have by (1) and (2) that

$$\begin{aligned} 0 &= b^{p-1}f(bac + cab, k_2, \ldots, k_p) \\ &= b^{p-1}(bac + cab)k_2 \cdots k_p + b^{p-1}\sum_{i\geq 2} k_i f_i(\ldots) \\ &= b^{p-1}(bac + cab)k_2 \cdots k_p = b^{p-1}cabk_2 \cdots k_p \end{aligned} \tag{3}$$

since $b^{p-1}k_i \in b^{p-1}K \subseteq b^{p-1}b\widehat{A} = b^p\widehat{A} = 0$. Let B be the subalgebra of A generated by K. We have shown in (3) that

$$k_2 \cdots k_p = f_1(k_2, \ldots, k_p) \in P := \{x \in B \mid b^{p-1}\widehat{J}\widehat{J}bx = 0\}. \tag{4}$$

We claim that

(5) P is an ideal of B.

Indeed, we just need to show that $KP + PK \subseteq P$, but $PK \subseteq P$ is obvious, so we just have to prove that, $b^{p-1}\widehat{J}\widehat{J}bKy = 0$ for any $y \in P$, which, in view of the definition of K, reduces to $b^{p-1}\widehat{J}\widehat{J}bby = 0$ and $b^{p-1}\widehat{J}\widehat{J}bb\widehat{J}by = 0$:

$$\begin{aligned} b^{p-1}\widehat{J}\widehat{J}bby &\subseteq b^{p-1}\{\widehat{J}\widehat{J}b\}by + b^{p-1}b\widehat{J}\widehat{J}by \\ &\subseteq b^{p-1}Jby + b^p\widehat{J}\widehat{J}by \subseteq_{[b^p=0]} b^{p-1}\widehat{J}\widehat{J}by = 0, \end{aligned}$$

$$\begin{aligned} b^{p-1}\widehat{J}\widehat{J}bb\widehat{J}by &\subseteq b^{p-1}\{\widehat{J}\widehat{J}b^2\}\widehat{J}by + b^{p-1}b^2\widehat{J}\widehat{J}\widehat{J}by \\ &\subseteq b^{p-1}J\widehat{J}by + b^pb\widehat{J}\widehat{J}\widehat{J}by \subseteq_{[b^p=0]} b^{p-1}\widehat{J}\widehat{J}by = 0. \end{aligned}$$

Now, the Jordan algebra $(K + P)/P$ is a subalgebra of $(B/P)^{(+)}$, the pair $((K + P)/P, B/P)$ satisfies the weak polynomial identity $f_1 = 0$, and, in particular, $(K+P)/P$ satisfies $x^{p-1} = 0$. By the induction assumption, $(K+P)/P = \mathrm{Mc}((K+P)/P)$. Notice that $(K + P)/P \cong K/(K \cap P)$, hence

$$K/(K \cap P) = \mathrm{Mc}(K/(K \cap P)). \tag{6}$$

Let x an arbitrary element in J.

Notice that $J_1 = K$ is a subalgebra of J, $I := K \cap P$ is an ideal of K such that $M = K$ satisfies $M/I = \mathrm{Mc}(J_1/I)$. Moreover, $w = U_{b^{p-1}}U_xU_b$, $w^* = U_bU_xU_{b^{p-1}}$ satisfy

$$w^*(J) = U_bU_xU_{b^{p-1}}J \subseteq U_bJ \subseteq K = J_1,$$

$$w(I) = b^{p-1}xbIbxb^{p-1} \subseteq b^{p-1}xbPbxb^{p-1} = 0 \subseteq \mathrm{Mc}(J)$$

since $b^{p-1}xbP \subseteq b^{p-1}JbP \subseteq b^{p-1}\widehat{J}\widehat{J}bP = 0$ by the definition of P, and

$$U_{w(k)} = wU_k w^*,$$

for any $k \in K$ by (0.2)(ii). Thus (1.1) applies to obtain $w(K) \subseteq \mathrm{Mc}(J)$, so that $w(U_bJ) \subseteq w(K) \subseteq \mathrm{Mc}(J)$, which means

$$U_{b^{p-1}}U_xU_b^2J \subseteq \mathrm{Mc}(J). \tag{7}$$

Since $p \geq 3$, $p-1 \geq 2$, and

$$\begin{aligned} U_{U_{b^{p-1}}x}J &=_{(0.2)(\mathrm{ii})} U_{b^{p-1}}U_xU_{b^{p-1}}J =_{(0.2)(\mathrm{i})} U_{b^{p-1}}U_xU_b^{p-1}J \\ &\subseteq U_{b^{p-1}}U_xU_b^2J \subseteq_{(7)} \mathrm{Mc}(J). \end{aligned}$$

This means that $U_{b^{p-1}}x+\mathrm{Mc}(J)$ is an absolute zero divisor of $J/\mathrm{Mc}(J)$, i.e, $U_{b^{p-1}}x+\mathrm{Mc}(J) = 0$, for any $x \in J$, i.e, $U_{b^{p-1}}J \subseteq \mathrm{Mc}(J)$, hence $b^{p-1} \in \mathrm{Mc}(J)$. We have shown that $J/\mathrm{Mc}(J)$ satisfies $x^{p-1} = 0$. By the induction assumption $J/\mathrm{Mc}(J) = \mathrm{Mc}(J/\mathrm{Mc}(J)) = 0$, i.e., $J = \mathrm{Mc}(J)$.

Let us take care now of the general case, when we do not assume that J is special. By factoring out $\mathrm{Mc}(J)$, we may assume that $\mathrm{Mc}(J) = 0$. We will suppose that J satisfies the identity $x^p = 0$, and will show that $J = 0$.

Let b, x be arbitrary elements of J, and take $K := \Phi b + U_b\widehat{J}$ which is an inner ideal of J, in particular, a subalgebra of J. Let $K_1 = \mathrm{abs}_J(K)$, and $I = U_{K_1}K_1$. Notice that I is an ideal of K by [12, Prop. 2]. Let $w = U_{b^{p-1}}U_xU_b$, $w^* = U_bU_xU_{b^{p-1}}$, so that $U_{w(a)} = wU_aw^*$, for any $a \in K$ (in fact, for any $a \in J$) by (0.2)(ii). Also

$$w^*(J) \subseteq U_bJ \subseteq K,$$

and, using $b \in K$ and the fact that I is an ideal of K,

$$w(I) = U_{b^{p-1}}U_xU_bI \subseteq U_{b^{p-1}}U_xI \subseteq U_{b^{p-1}}U_JU_{K_1}K_1 \subseteq_{(1.8)(\mathrm{iii})} U_{b^{p-1}}K_1 \subseteq U_{b^{p-1}}K = 0$$

by (0.2)(i) and the form of K since $b^p = 0$ ($U_{b^{p-1}}b = 0$ by (1.3), and $U_{b^{p-1}}U_b\widehat{J} \subseteq_{(0.2)(\mathrm{i})} U_{b^p}\widehat{J} = 0$). Moreover, K/K_1 is special (1.7), satisfying $x^p = 0$, hence $K/K_1 = \mathrm{Mc}(K/K_1)$. We have $(K/I)/(K_1/I) \cong K/K_1$ is McCrimmon radical, and K_1/I is also McCrimmon radical since it has zero cube. By the extension property, K/I is also McCrimmon radical, i.e,, $K/I = \mathrm{Mc}(K/I)$. Thus, we can apply (1.1) with $J_1 = M = K$, to obtain $w(K) \subseteq \mathrm{Mc}(J) = 0$. As a consequence, using $p \geq 3$,

$$\begin{aligned} U_{U_{b^{p-1}}x}J =_{(0.2)(\mathrm{ii})} & U_{b^{p-1}}U_xU_{b^{p-1}}J \subseteq_{(0.2)(\mathrm{i})} \\ & U_{b^{p-1}}U_xU_bU_bJ \subseteq U_{b^{p-1}}U_xU_bK = w(K) = 0, \end{aligned}$$

hence $U_{b^{p-1}}x = 0$ by nondegeneracy of J. We have shown $U_{b^{p-1}}J = 0$, hence $b^{p-1} = 0$, again by nondegeneracy. Since this happens for any $b \in J$, $J = \mathrm{Mc}(J) = 0$ by the induction assumption. ■

1.10 COROLLARY. *A finitely generated Jordan nil algebra of bounded degree is nilpotent.* ■

2. Jordan Nil Algebras Satisfying a Polynomial Identity

2.1 Let X be an arbitrary set. A Jordan polynomial $f \in \mathrm{FJ}[X]$ is said to be *essential* if it has a monic image $\sigma(f) \in \mathrm{FAss}[X]$ by the Jordan homomorphism $\sigma : \mathrm{FJ}[X] \longrightarrow \mathrm{FAss}[X]^{(+)}$ fixing the elements of X.

Following [15, 7.2], given an essential Jordan polynomial f, a Jordan algebra is said to *satisfy the polynomial identity* $f \equiv 0$, if the evaluations of f vanish in all scalar extensions of J, equivalently, the evaluations of all linearizations of f vanish in J. In this situation J is also said to be PI.

This is stronger than just saying $f(J) = 0$, i.e., all evaluations of f vanish in J. We will study the behaviour of nilpotent elements in such an algebra J.

2.2 Given a Jordan algebra J, the sum $\mathrm{Nil}(J)$ of all *nil ideals* (ideals which are nil as Jordan algebras) of J is the biggest nil ideal of J called the *nil radical* of J. By [4, 2.11], $\mathrm{Mc}(J)$ is a nil ideal of J, hence $\mathrm{Mc}(J) \subseteq \mathrm{Nil}(J)$.

The next result is the quadratic version of [18, Lemma 16].

2.3 LEMMA. *Let J be a special Jordan algebra that is a subalgebra of $A^{(+)}$ for an associative algebra A, and let us assume that (J, A) satisfies a weak monic polynomial identity of degree p. If an element $b \in J$ satisfies $b^m = 0$ with $m > 4^p$, then $b^{m-4} \in \mathrm{Mc}(J)$.*

PROOF: If $p = 1$, then $J = 0 = \mathrm{Mc}(J)$ and everything is obvious, so we will assume that $p \geq 2$.

As in the proof of (1.9), let $\widehat{J}$ denote the unital hull of J (see [11]).

By (1.6), (J, A) also satisfies an identity given by a monic multilinear polynomial of degree p which always can be expressed as

$$\begin{aligned} f(x_1, \ldots, x_p) &= x_1 x_2 \cdots x_p + \sum_{i \geq 2} x_i f_i(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_p) \\ &= \sum_{i=1}^{p} x_i f_i(x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_p), \end{aligned} \tag{1}$$

where the f_i's are multilinear polynomials of degree $p-1$ in which the variable x_i is missing, and

$$f_1(x_2, \ldots, x_p) = x_2 \cdots x_p. \tag{2}$$

Take arbitrary elements $b \in J$, $a, c \in \widehat{J}$, and let $K = \Phi b^4 + U_{b^4}\widehat{J} = \Phi b^4 + b^4\widehat{J}b^4$, which is an inner ideal of J. For any $k_2, \ldots, k_p \in K$, since $b^4ac + cab^4 = \{b^4, a, c\} \in J$, we have by (1) and (2) that

$$\begin{aligned} 0 &= b^{m-4}f(b^4ac + cab^4, k_2, \ldots, k_n) \\ &= b^{m-4}(b^4ac + cab^4)k_2 \cdots k_n + b^{m-4}\sum_{i\geq 2} k_i f_i(\ldots) \\ &= b^{m-4}(b^4ac + cab^4)k_2 \cdots k_p = b^{m-4}cab^4k_2 \cdots k_p \end{aligned} \tag{3}$$

since $b^{m-4}k_i \in b^{m-4}K \subseteq b^{m-4}b^4\widehat{A} = b^m\widehat{A} = 0$. Let B be the subalgebra of A generated by K. We have shown in (3) that

$$k_2 \cdots k_p = f_1(k_2, \ldots, k_p) \in P := \{x \in B \mid b^{m-4}\widehat{J}\widehat{J}b^4x = 0\}. \tag{4}$$

We claim that

(5) P is an ideal of B.

Indeed, we just need to show that $KP + PK \subseteq P$, but $PK \subseteq P$ is obvious, so we just have to prove that $b^{m-4}\widehat{J}\widehat{J}b^4Ky = 0$ for any $y \in P$, which, in view of the definition of K, reduces to $b^{m-4}\widehat{J}\widehat{J}b^4b^4y = 0$ and $b^{m-4}\widehat{J}\widehat{J}b^4b^4\widehat{J}b^4y = 0$:

$$\begin{aligned} b^{m-4}\widehat{J}\widehat{J}b^4b^4y &\subseteq b^{m-4}\{\widehat{J}\widehat{J}b^4\}b^4y + b^{m-4}b^4\widehat{J}\widehat{J}b^4y \\ &\subseteq b^{m-4}Jb^4y + b^m\widehat{J}\widehat{J}b^4y \subseteq_{[b^m=0]} b^{m-4}\widehat{J}\widehat{J}b^4y = 0, \end{aligned}$$

$$\begin{aligned} b^{m-4}\widehat{J}\widehat{J}b^4b^4\widehat{J}b^4y &\subseteq b^{m-4}\{\widehat{J}\widehat{J}b^8\}\widehat{J}b^4y + b^{m-4}b^8\widehat{J}\widehat{J}\widehat{J}b^4y \\ &\subseteq b^{m-4}J\widehat{J}b^4y + b^mb^4\widehat{J}\widehat{J}\widehat{J}b^4y \subseteq_{[b^m=0]} b^{m-4}\widehat{J}\widehat{J}b^4y = 0. \end{aligned}$$

Now, the Jordan algebra $(K + P)/P$ is a subalgebra of $(B/P)^{(+)}$, the pair $((K + P)/P, B/P)$ satisfies the weak polynomial identity $f_1 = 0$, and, in particular, $(K + P)/P$ satisfies $x^{p-1} = 0$. By (1.9), $(K + P)/P = \mathrm{Mc}((K + P)/P)$. Notice that $(K + P)/P \cong K/(K \cap P)$, hence

$$K/(K \cap P) = \mathrm{Mc}(K/(K \cap P)). \tag{6}$$

Let x be an arbitrary element in J.

Notice that $J_1 = K$ is a subalgebra of J, $I := K \cap P$ is an ideal of of K such that $M = K$ satisfies $M/I = \mathrm{Mc}(J_1/I)$. Moreover, $w = U_{b^{m-4}}U_xU_{b^4}$, $w^* = U_{b^4}U_xU_{b^{m-4}}$ satisfy

$$w^*(J) = U_{b^4}U_xU_{b^{m-4}}J \subseteq U_{b^4}J \subseteq K = J_1,$$

$$w(I) = b^{m-4}xb^4Ib^4xb^{m-4} \subseteq b^{m-4}xb^4Pb^4xb^{m-4} = 0 \subseteq \mathrm{Mc}(J)$$

since $b^{m-4}xb^4P \subseteq b^{m-4}\widehat{J}b^4P \subseteq b^{m-4}\widehat{J}\widehat{J}b^4P = 0$ by the definition of P, and

$$U_{w(k)} = wU_kw^*,$$

for any $k \in K$ by (0.2)(ii). Thus (1.1) applies to obtain $w(K) \subseteq \mathrm{Mc}(J)$, so that $w(U_{b^4}\widehat{J}) \subseteq w(K) \subseteq \mathrm{Mc}(J)$, which means

$$U_{b^{m-4}}U_xU_{b^4}^2\widehat{J} \subseteq \mathrm{Mc}(J). \tag{7}$$

Since $p \geq 2$, $m-4 > 4^p - 4 \geq 12$, and

$$\begin{aligned} U_{U_{b^{m-4}}x}J &=_{(0.2)(\mathrm{ii})} U_{b^{m-4}}U_xU_{b^{m-4}}J =_{(0.2)(\mathrm{i})} U_{b^{m-4}}U_xU_b^{m-4}J \\ &\subseteq U_{b^{m-4}}U_xU_b^8J =_{(0.2)(\mathrm{i})} U_{b^{m-4}}U_xU_{b^8}J =_{(0.2)(\mathrm{i})} U_{b^{m-4}}U_xU_{b^4}^2J \\ &\subseteq_{(7)} \mathrm{Mc}(J). \end{aligned}$$

This means that $U_{b^{m-4}}x+\mathrm{Mc}(J)$ is an absolute zero divisor of $J/\mathrm{Mc}(J)$, i.e, $U_{b^{m-4}}x+\mathrm{Mc}(J) = 0$, for any $x \in J$, i.e, $U_{b^{m-4}}J \subseteq \mathrm{Mc}(J)$, hence $b^{m-4} \in \mathrm{Mc}(J)$. ∎

2.4 Lemma. *Let J be a Jordan algebra. If $a \in J$ satisfies $U_aU_xa = 0$, for any $x \in J$, then $a \in \mathrm{Mc}(J)$.*

Proof: By factoring out $\mathrm{Mc}(J)$, we can assume that $\mathrm{Mc}(J) = 0$, and we will show in that case that if $a \in J$ satisfies $U_aU_xa = 0$, then $a = 0$. Otherwise the local algebra J_a is nonzero, and nondegenerate by [3, 0.2], hence $J_a \neq \mathrm{Mc}(J_a) = 0$: But this contradicts (1.4) since, for any $x + \mathrm{Ker}\, a \in J_a$, $(x + \mathrm{Ker}\, a)^2 = x^{(2,a)} + \mathrm{Ker}\, a = U_xa + \mathrm{Ker}\, a = 0$. ∎

2.5 Theorem. *If J is a Jordan algebra such that $f(J) = 0$ for some essential Jordan polynomial f, then $\mathrm{Nil}(J) = \mathrm{Mc}(J)$, hence it is locally nilpotent. In particular, this holds if J is PI.*

Proof: First, notice that local nilpotency of $\mathrm{Nil}(J)$ will be a consequence of the desired equality by [4, 2.11].

Let p be the degree of the associative image $\sigma(f)$ of f as in (2.1). By factoring out $\mathrm{Mc}(J)$ we can assume that $\mathrm{Mc}(J) = 0$, and we then have to prove that $\mathrm{Nil}(J) = 0$ (clearly $\mathrm{Nil}(J/\mathrm{Mc}(J)) = \mathrm{Nil}(J)/\mathrm{Mc}(J)$), so that we just take $J/\mathrm{Mc}(J)$ in the place of J).

If, for any element $b \in \mathrm{Nil}(J)$, $b^{4^p} = 0$, then $\mathrm{Nil}(J) = \mathrm{Mc}(\mathrm{Nil}(J))$ by (1.9). But the McCrimmon radical is hereditary for ideals [10, 4.13], hence $\mathrm{Mc}(\mathrm{Nil}(J)) = \mathrm{Mc}(J) \cap \mathrm{Nil}(J) = 0$, hence $\mathrm{Nil}(J) = 0$ as desired. Otherwise, there exists $b \in \mathrm{Nil}(J)$ such that $b^{4^p} \neq 0$. Since b is nilpotent anyway, there exists a positive integer m such that $m > 4^p$, $b^m = 0$, and $b^{m-1} \neq 0$.

Take $K := \Phi b + U_b\widehat{J}$ which is an inner ideal of J, in particular, a subalgebra of J. Let $K_1 = \mathrm{abs}_J(K)$, and $I = U_{K_1}K_1$. Notice that I is an ideal of K by [12, Prop.

2]. Take an arbitrary element $x \in J$, and let $w = U_{b^{m-1}} U_x U_b$, $w^* = U_b U_x U_{b^{m-1}}$, so that $U_{w(a)} = w U_a w^*$, for any $a \in K$ (in fact, for any $a \in J$) by (0.2)(ii). Also

$$w^*(J) \subseteq U_b J \subseteq K,$$

and, using $b \in K$ and the fact that I is an ideal of K,

$$\begin{aligned} w(I) = U_{b^{m-1}} U_x U_b I &\subseteq U_{b^{m-1}} U_x I \subseteq U_{b^{m-1}} U_J U_{K_1} K_1 \\ &\subseteq_{(1.8)(\mathrm{iii})} U_{b^{m-1}} K_1 \subseteq U_{b^{m-1}} K = 0 \end{aligned}$$

by (0.2)(i) and the form of K, since $b^m = 0$ ($U_{b^{m-1}} b = 0$ by (1.3), and also $U_{b^{m-1}} U_b \widehat{J} \subseteq_{(0.2)(\mathrm{i})} U_{b^m} \widehat{J} = 0$). Moreover, K/K_1 is special (1.7), so that, if A is an associative algebra such that K/K_1 is a subalgebra of $A^{(+)}$, then $(K/K_1, A)$ satisfies the weak identity given by $\sigma(f)$. The element $b + K_1 \in K/K_1$ satisfies $(b + K_1)^m = 0$, hence (2.3) applies to show $b^{m-4} + K_1 = (b + K_1)^{m-4} \in \mathrm{Mc}(K/K_1)$. As a consequence,

$$b^{m-3} + K_1 = (b + K_1)^{m-3} \in \mathrm{Mc}(K/K_1) \tag{1}$$

by (1.3). We claim

$$b^{m-3} + I \in \mathrm{Mc}(K/I). \tag{2}$$

Indeed, if $\mathrm{Mc}(K/K_1) = R/K_1$, where R is an ideal of K such that $K_1 \subseteq R$, and $\mathrm{Mc}(K/I) = M/I$, where M is an ideal of K such that $I \subseteq M$, we have that $b^{m-3} \in R$, hence we just need to prove $R \subseteq M$. Now

$$\big((K_1/I) + (M/I)\big)/(M/I) = \big((K_1 + M)/I\big)/(M/I)$$

is an ideal of $(K/I)/(M/I)$ which is nondegenerate, hence $\big((K_1+M)/I\big)/(M/I)$ is at the same time nodegenerate (using [10, 4.13]) and has zero cube because $I = U_{K_1} K_1$ and

$$\big((K_1/I) + (M/I)\big)/(M/I) \cong (K_1/I)/\big((K_1/I) \cap (M/I)\big).$$

Therefore $\big((K_1 + M)/I\big)/(M/I) = 0$, i.e., $K_1 + M = M$, i.e., $K_1 \subseteq M$. Thus $(K/K_1)/(M/K_1) \cong K/M \cong (K/I)/(M/I)$ is nondegenerate, and, as a consequence, $R/K_1 = \mathrm{Mc}(K/K_1) \subseteq M/K_1$, i.e, $R \subseteq M$.

By (1.1) with $J_1 = K$, we obtain $w(M) \subseteq \mathrm{Mc}(J) = 0$. In particular

$$U_{b^{m-1}} U_x b^{m-1} = U_{b^{m-1}} U_x U_b b^{m-3} = w(b^{m-3}) = 0.$$

Therefore, $b^{m-1} \in \mathrm{Mc}(J) = 0$ by (2.4), which is a contradiction. ∎

3. The Kurosh Problem for Jordan Nil Systems

3.1 Following [10, Sections 3 and 4], the nilpotency of elements in Jordan pairs and triple system is given in terms of their homotope algebras :

Let V be a Jordan pair. Given $x^+ \in V^+$, $x^- \in V^-$, we will say that the pair (x^+, x^-) is *nilpotent* if x^+ is nilpotent in the homotope algebra $V^{+(x^-)}$ of V at x^-, which is equivalent to x^- being nilpotent in the homotope algebra $V^{-(x^+)}$ by [10, 3.8]. We will say that the Jordan pair V is *nil* if (x^+, x^-) is nilpotent for any $x^+ \in V^+$, $x^- \in V^-$.

Let J be a Jordan triple system. Given $x, y \in J$, we will say that the pair (x, y) is *nilpotent* if it is nilpotent in the Jordan pair $V(J)$. We will say that the Jordan triple system J is *nil* if $V(J)$ is nil as a Jordan pair.

An ideal I of V (resp. of J) is said to be a *nil ideal* if it is nil as a Jordan pair (resp. as a Jordan triple system).

3.2 The nil radical $\mathrm{Nil}(J)$ of a Jordan pair or triple system J is the sum of all nil ideals of J. It turns out to be the biggest nil ideal of J by [10, 4.14]. Since $\mathrm{Mc}(J)$ is locally nilpotent by [4, 2.11], it is a nil ideal and, hence, $\mathrm{Mc}(J) \subseteq \mathrm{Nil}(J)$.

3.3 A Jordan pair V will be said to be nil *of bounded degree*, if there exists a positive integer n such that, $x^{+(n,x^-)} = 0$, for any $x^+ \in V^+$, $x^- \in V^-$, where $x^{+(n,x^-)}$ denotes de n-th power of x^+ in the homotope $V^{+(x^-)}$. Notice that, by [10, 3.8], $x^{-(n+1,x^+)} = 0$ too. A Jordan triple system J will be said to be nil *of bounded degree* if $V(J)$ is nil of bounded degree.

3.4 Given an essential Jordan polynomial $f \in \mathrm{FJ}[X]$, a Jordan system J is said to satisfy the homotope polynomial identity $f \equiv 0$, if all its homotope algebras satisfy the polynomial identity $f \equiv 0$, i.e, the linearizations of f vanish when evaluated in all the homotopes of J. In this situation J is said to be *homotope*-PI.

As for algebras, we will study the behaviour of nilpotent elements in a class of Jordan systems which includes those which are homotope-PI.

3.5 THEOREM. *Let V be a Jordan pair such that, for every local algebra V_y^σ, $y \in V^{-\sigma}$, $\sigma = \pm$, there exists an essential polynomial $f_y \in \mathrm{FJ}[X]$ such that $f_y(V_y^\sigma) = 0$. Then $\mathrm{Nil}(V) = \mathrm{Mc}(V)$, hence it is locally nilpotent. In particular, this holds if V is homotope-PI.*

PROOF: As in the proof of (2.5), it is enough to prove the equality. Moreover, we can assume that V is nondegenerate, so that we have to prove that $\mathrm{Nil}(V) = 0$.

Put $\mathrm{Nil}(V) = (N^+, N^-)$. Since $\mathrm{Nil}(V)$ is an ideal of V, $\mathrm{Nil}(V)$ is nondegenerate by [10, 4.13]. Hence, for any $y \in N^{-\sigma}$, $\sigma = \pm$,

(1) N_y^σ is nondegenerate

by [2, 3.1(i)]. It can be readily seen that the local algebras of $\mathrm{Nil}(V)$ are isomorphic to ideals of the local algebras of V, hence, for any $y \in N^\sigma$, $\sigma = \pm$,

(2) $f_y(N_y^\sigma) = 0$.

Since N_y^σ is nil, we can apply (2.5) to obtain $N_y^\sigma = \mathrm{Mc}(N_y^\sigma) = 0$ by (1). We have shown that all local algebras of $\mathrm{Nil}(V)$ are zero, i.e., $\mathrm{Nil}(V)$ is a trivial ideal of V. But $\mathrm{Nil}(V)$ was also nondegenerate, hence $\mathrm{Nil}(V) = 0$. ■

By using the functor $V(\ )$, we have an analogue of the previous theorem for Jordan triple systems.

3.6 COROLLARY. *Let J be a Jordan triple system such that, for every local algebra J_y, $y \in J$, there exists an essential polynomial $f_y \in \mathrm{FJ}[X]$ such that $f_y(J_y) = 0$. Then $\mathrm{Nil}(J) = \mathrm{Mc}(J)$, hence it is locally nilpotent. In particular, this holds if J is homotope-PI.* ■

Notice that a Jordan pair or triple system which is nil of bounded degree is, in particular, homotope-PI. Hence, we can apply [4, 2.11], (3.5) and (3.6) to obtain:

3.7 COROLLARY. *If a Jordan pair or triple system is nil of bounded degree, then it is McCrimmon radical, hence locally nilpotent.* ■

3.8 COROLLARY. *If a finitely generated Jordan pair or triple system is nil of bounded degree, then it is nilpotent.* ■

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