Speciality and Deformations of Algebras

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1. Introduction. The notion of speciality has come from the theory of Jordan algebras. A Jordan algebra is called *special* if it admits an isomorphic embedding into an associative algebra with respect to a symmetrized multiplication $a \circ b = 1/2(ab + ba)$. Not any Jordan algebra is special; moreover, the variety generated by all special algebras neither coincides with the class of all Jordan algebras, nor with the class of all special Jordan algebras. The algebras from this variety are called *i-special*. Both speciality and i-speciality can also be naturally defined for superalgebras.

The condition of speciality plays an important role in the theory of Jordan algebras and was investigated by a number of authors (see the references in [8, 9, 18]). In particular, several years ago the author in [13] proved the i-speciality of so called Jordan Poisson superalgebras. The proof was based on a construction of quantization deformations for corresponding Poisson superalgebras.

Here we consider the speciality problem in a more general framework that includes also the problem of embedding of Malcev algebras into skewsymmetrized alternative algebras, and that of a linear representability of Akivis algebras. Our main purpose is to show that the methods of the deformation theory could be applied to speciality problems.

The paper is organized as follows. In the next section we give the definitions and examples of speciality problems. The third section is devoted to the universal enveloping algebras and to their associated graded algebras. Here, in particular, we show that every Akivis algebra which is a free module over its coefficient ring is special, that is, has a faithful linear representation. In section 4 we remind the definition of a Poisson-Lie algebras and their deformations, and the Gutt–Drinfeld representation of a universal enveloping algebra $U(L)$ of a Lie algebra L as a quantization deformation of its associated graded Poisson Lie algebra $S(L)$.

Section 5 is devoted to the speciality problem for Malcev algebras. First we define a *Poisson Malcev algebra* as an associative commutative algebra with a Malcev bracket $\{,\}$ that satisfies the Leibniz identity. For any Malcev algebra M there exists a universal Poisson Malcev algebra $\tilde{S}(M)$ that contains M as a subalgebra with respect to the bracket $\{,\}$ and has a universal property for homomorphisms of M into Poisson Malcev algebras. Our main result states that if the Poisson Malcev algebra $\tilde{S}(M)$ admits an alternative quantization deformation, then the algebra M is special. As a corollary, we obtain a necessary condition for speciality of Malcev algebras.

2. Speciality problems: definitions and examples. Let $\underline{M}$ and $\underline{N}$ be two categories of (multi)linear algebras and assume that there exists a functor $F : \underline{M} \to \underline{N}$. Then we call an algebra $A \in \underline{N}$ to be *F-special* if there exists an algebra $B \in \underline{M}$ such that A is $\underline{N}$-isomorphic to a certain subalgebra of $F(B)$.

In fact, we will consider only the functors that conserve the underlying additive structure of A, that is, for any $A \in \underline{M}$ the algebra $F(A)$ carries the same additive structure (as a vector space or a module over a ring of scalars, possibly, with a certain topology) as A. In other words, F changes only multiplicative operations in A.

Two basic examples of such functors are the *symmetrization* and *skewsymmetrizatian* functors, that either symmetrize the binary multiplication ab in A to $a \circ b = 1/2(ab + ba)$ or skewsymmetrize it to $[a, b] = ab - ba$. We will denote these functors as $(.)^+$ and $(.)^-$.

Let us consider certain examples of speciality problems.

- **Lie algebras and associative algebras.**

It is well known that for any associative algebra A the algebra A^- is a Lie algebra, thus the functor $(.)^-$ maps the category $\underline{Ass}$ of associative algebras into the category $\underline{Lie}$ of Lie algebras. According to the celebrated Poincare—Birkhoff—Witt theorem, any Lie algebra which is a free module over its coefficient ring is $\underline{Ass}^-$-special. Therefore, in this case the speciality problem has a positive solution. Nevertheless, one may still consider this problem for certain subcategories of $\underline{Ass}$ and $\underline{Lie}$. For example, let $\underline{TM}$ be the category of topological algebras from $\underline{M}$, with continuous homomorphisms as morphisms. Then evidently $(.)^- : \underline{TAss} \to \underline{TLie}$, and, as far as the author knows, the corresponding problem of a *topological speciality* is open:

Problem 1. Is it true that any topological Lie algebra is topologically special?

In other words, is it true that any topological Lie algebra can be topologically embedded into a topological associative algebra?

- **Jordan algebras and associative algebras.**

A Jordan algebra is a commutative algebra that satisfies the identity

$$(x^2 y)x = x^2(yx).$$

We have already mentioned that the symmetrization functor $(.)^+$ maps the category $\underline{Ass}$ into the category $\underline{Jord}$ of Jordan algebras, and that not any Jordan algebra is $\underline{Ass}^+$-special. There are many results, open problems, and hypothesis on the structure and identities of special Jordan algebras, and on on speciality of certain Jordan algebras or of classes of algebras. We just refer the reader again to [8,9, 18] and to the references there.

We give here only two problems on a topological speciality of Jordan algebras. The first one is just an analogue of Problem 1:

Problem 2. Is it true that a topological Jordan algebra which is (algebraically) special, is also topologically special?

The recent results by A.Moreno-Galindo [11] give a certain evidence that this problem may have a negative answer.

The next problem is concerned with the categories $\underline{NormAss}$ and $\underline{NormJord}$ of normed associative and Jordan algebras (see, for example, [12]), with contractions as morphisms. It is well known that $(.)^+ : \underline{NormAss} \to \underline{NormJord}$, so the notion of a *normed speciality* naturally arises.

Problem 3. Let J be a normed Jordan algebra which is topologically special. Would it be *normed special*?

In other words, if a normed Jordan algebra admits a topological (Jordan) embedding into an associative topological algebra, would it admit an isometric (Jordan) embedding into an associative normed algebra?

Observe that the last problem could be formulated for Lie algebras as well.

- **Malcev algebras and alternative algebras.**

An algebra M is called a *Malcev algebra* if it satisfies the identities

$$\begin{aligned} x^2 &= 0, \\ J(xy, x, z) &= J(x, y, z)x, \end{aligned} \tag{1}$$

where $J(x, y, z) = [[x, y], z] + [[z, x], y] + [[y, z], x]$ is the *Jacobian* of the elements x, y, z.

An *alternative algebra* is an algebra that satisfies the identities

$$\begin{aligned} (xy)y &= x(yy), \\ (xx)y &= x(xy). \end{aligned}$$

It is evident that the class $\underline{Alt}$ of alternative algebras generalizes the class $\underline{Ass}$ of associative algebras, while the class $\underline{Malc}$ of Malcev algebras includes the Lie algebras. It was shown in [10] that for any $A \in \underline{Alt}$ the inclusion $A^- \in \underline{Malc}$ holds. Nevertheless, the following problem is still open.

Problem 4. Is it true that any Malcev algebra is $\underline{Alt}^-$-special?

This problem was written by E.N.Kuzmin in the Dniester Notebook [2] in the sixties. As A. T. Gainov testifies, A. I. Malcev already stated it in the fifties in the Ivanovo Pedagogical Institute, where A. T. Gainov was a student. We will return to this problem in the last section.

In fact, Malcev algebras first appeared in [10] as tangent algebras of so called analytic Moufang loops. These are the subject of our next example.

- **Analytic loops and Akivis algebras.**

A vector space A is called an *Akivis algebra* if it is endowed with two operations: an anticommutative bilinear operation $[x, y]$, (*a commutator*), and a trilinear operation $\mathcal{A}(x, y, z)$ (*an associator*), that are related by means of the identity

$$\begin{aligned}[][[x,y],z]+[[y,z],x]+[[z,x],y] &= \mathcal{A}(x,y,z)+\mathcal{A}(y,z,x)+\mathcal{A}(z,x,y)\\ &\quad -\mathcal{A}(y,x,z)-\mathcal{A}(x,z,y)-\mathcal{A}(z,y,x).\end{aligned} \tag{2}$$

These algebras were introduced in 1976 by M. A. Akivis [1], under the name *W-algebras*, as local algebras of three-webs (or of local analytic loops).

Let L be a local analytic loop with the multiplication $x \cdot y$, left division $y\backslash x$, and right division x/y (see, for example, [10, 7]). The tangent space of L at the unit 0 may be identified with L itself; and one may endow this space with the following two operations that represent the deviation from commutativity and from associativity of the multiplication $x \cdot y$ in the loop L :

$$\begin{aligned}[][x,y] &= \lim_{t\mapsto 0} t^{-2}\,((tx\cdot ty)\,/\,(ty\cdot tx))\\ (&= \lim_{t\mapsto 0} t^{-2}\,((ty\cdot tx)\setminus(tx\cdot ty))\\ &= \lim_{t\mapsto 0} t^{-2}\,(tx\cdot ty - ty\cdot tx)\ ),\\ \mathcal{A}(x,y,z) &= \lim_{t\mapsto 0} t^{-3}\,(((tx\cdot ty)\cdot tz)\,/\,(tx\cdot(ty\cdot tz)))\\ (&= \lim_{t\mapsto 0} t^{-3}\,((tx\cdot(ty\cdot tz))\setminus((tx\cdot ty)\cdot tz))\\ &= \lim_{t\mapsto 0} t^{-3}\,((tx\cdot ty)\cdot tz - tx\cdot(ty\cdot tz))\ ),\end{aligned}$$

where x, y, z are vectors from the tangent space, $t \in \mathbf{R}$. It was proved in [1] that with respect to these operations the tangent space of the loop L forms an Akivis algebra. We will denote this Akivis algebra as $\mathcal{A}(L)$.

If a loop L is associative, i.e., L is a Lie group, then the operation $\mathcal{A}(x, y, z)$ is trivial, and so the Akivis identity (2) converts to the well known Jacobi identity. Hence, in this case the algebra $\mathcal{A}(L)$ is a Lie algebra. If L satisfies the Moufang identity

$$(xy)(zx) = x(yz \cdot x),$$

then the function $\mathcal{A}(x, y, z)$ becomes skewsymmetric [1] and so by (2) this function can be represented in terms of the bilinear operation $[x, y]$:

$$\mathcal{A}(x, y, z) = 1/6J(x, y, z),$$

Moreover, in this case $\mathcal{A}(L)$ satisfies the Malcev identity (1), hence $\mathcal{A}(L)$ is a Malcev algebra (see [10]). In general case the operation $\mathcal{A}(x, y, z)$ is not expressed in terms of the commutator $[x, y]$.

Now, let B be a (not necessary associative) algebra with a bilinear multiplication $(x, y) \mapsto xy$. Consider in B the usual *commutator* $[x, y] = xy - yx$ and *associator* $\mathcal{A}(x, y, z) = (xy)z - x(yz)$ functions; then it is easily checked that these functions satisfy identity (2). Hence B is an Akivis algebra with respect to these operations, and we have a functor $Ak : B \mapsto Ak\,(B)$, where $Ak\,(B)$ denotes the corresponding Akivis algebra structure on B.

In [1] Akivis posed the following question (see also [4, Problem X.3.8], [7, Problem IX.6.12]):

Problem 5. Is it true that an arbitrary Akivis algebra can be isomorphically embedded into an Akivis algebra $Ak\,(B)$ for a suitable algebra B?

In other words, this problem asks whether every Akivis algebra is Ak-special for the functor Ak. If an Akivis algebra admits such a representation, it is called *linear*. K. H. Hofmann and K. Strambach in [7] also formulated some weaker versions of the Akivis problem; for example, *whether a free Akivis algebra is linear?* In the next section we show that the Akivis problem has a positive solution.

3. Universal enveloping algebras and associate graded algebras. Consider again the categories of (multi)linear algebras $\underline{M}$ and $\underline{N}$ with a functor $F : \underline{M} \to \underline{N}$. We will assume that the additive structures of algebras from $\underline{M}$ and $\underline{N}$ belong to the same category $\underline{Add}$ (of vector spaces or modules over a coefficient ring, possibly, with certain topology or norm conditions). Denote by Add the forgetful functor that assigns to an algebra A its additive structure $\langle A, + \rangle$. We will assume that the functor F satisfies the condition

$$Add\,(F(A)) = Add\,(A) \text{ for any } A \in \underline{M}. \tag{3}$$

Observe that this is the case for the functors $(.)^+,\ (.)^-,\ Ak\,(.)$.

For an algebra $A \in \underline{N}$ we define an *$\underline{M}$-representation* of A to be a morphism $\phi \in Hom_{\underline{N}}(A, F(B))$ for some $B \in \underline{M}$. The $\underline{M}$-representations of A form a category, if for any $\phi : A \to F(B)$ and $\psi : A \to F(C)$ we define

$$Hom\,(\phi, \psi) = \{\theta \in Hom_{\underline{M}}(B, C) |\, \psi = F(\theta) \circ \phi\}.$$

If this category has an inicial object $i : A \to F(U)$, we will denote this (unique) algebra $U \in \underline{M}$ by $U(A)$ and call it a *universal enveloping algebra* for $\underline{M}$-representations of A.

In other words, an algebra $U(A) \in \underline{M}$ is a universal enveloping algebra for $\underline{M}$-representations of A if there exists an $\underline{M}$-representation $i : A \to F(U(A))$ such that for any $\underline{M}$-representation $\phi : A \to F(B)$ there is a unique $\tilde{\phi} \in Hom_{\underline{M}}(U(A), B)$ such that $\phi = F(\tilde{\phi}) \circ i$. In this case we have an isomorphism of categories

$$Hom_{\underline{M}}(U(A), B) \cong Hom_{\underline{N}}(A, F(B)),$$

which shows that the mapping $A \mapsto U(A)$ is the left adjoint functor for the functor F. The universal mapping $i : A \to F(U(A))$ corresponds to the identical morphism of $U(A)$ under the above isomorphism (with $B = U(A)$); and A is F-special if and only if the mapping i is injective.

Now consider an additive object $X \in \underline{Add}$ and form the category with the objects $Hom_{\underline{Add}}(X, B)$, $B \in \underline{M}$, and the morphisms defined for $\phi : X \to B$, $\psi : X \to C$ by $Hom\,(\phi, \psi) = \{\theta \in Hom_{\underline{M}}(B, C) |\, \psi = Add(\theta) \circ \phi\}$. An initial object in this category, if it exists (or, more exactly, the corresponding algebra $B \in \underline{M}$ of this object), is called a *free* $\underline{M}$*-algebra over* X. We will denote this algebra by $\mathcal{F}_{\underline{M}}(X)$ or simply by $\mathcal{F}(X)$.

Proposition 1. *Let a functor* $F : \underline{M} \to \underline{N}$ *satisfies condition* (3)*. Assume that for any additive object* $X \in \underline{Add}$ *there exists a free* $\underline{M}$*-algebra* $\mathcal{F}_{\underline{M}}(X)$ *over* X*. Then for any algebra* $A \in \underline{N}$ *there exists a universal enveloping algebra* $U_{\underline{M}}(A)$ *for its* $\underline{M}$*-representations. Moreover, the algebra* A *is* $\underline{M}$*-special if and only if the corresponding universal mapping* $i : A \to FU(A)$ *is an* $\underline{N}$*-isomorphism of* A *to* $i(A)$*.*

Proof. Let the algebras in $\underline{N}$ have multiplicative operations $\varphi_1, \ldots, \varphi_k$. For any $A \in \underline{N}$ denote by $\mathcal{F}(A)$ the $\underline{M}$-free algebra over the additive object $Add\,A$, and by i the corresponding morphism $i \in Hom_{\underline{Add}}(A, \mathcal{F}(A))$. Consider in $\mathcal{F}(A)$ the ideal I generated by all elements of the form $i(\varphi_j(a, b, \ldots, c)) - \varphi_j(i(a), i(b), \ldots, i(c))$, where $a, b, \ldots, c \in A, j = 1, \ldots, k$, and the products $\varphi_j(i(a), i(b), \ldots, i(c))$ are calculated in the $\underline{N}$-algebra $F(\mathcal{F}(A))$. Then it is easily seen that the quotient algebra $\mathcal{F}(A)/I$ and the mapping $a \mapsto i(a) + I$ satisfy the requirements of the proposition. □

From now on we will consider only the case when a category $\underline{M}$ is just a variety of linear (binary) algebras over an associative commutative ring of scalars Φ, with the only one binary multiplication and without any additional topological structure. One can easily prove (see, for instance, [18]) that in this case the conditions of the proposition are satisfied and so universal enveloping algebras always exist.

Fix $A \in \underline{N}$ and consider the universal algebra $U = U_{\underline{M}}(A)$ with the universal mapping $i : A \to FU_{\underline{M}}(A)$. Observe that in this case the speciality of A is equivalent to the injectivity of i. Set $U_1 = i(A), U_2 = U_1 + i(A)^2, \ldots, U_k = U_{k-1} + i(A)^k, \ldots$, where the powers $i(A)^k$ are taken in the algebra $U(A)$ (not in $FU(A)$). Then $U_iU_j \subseteq U_{i+j}$, hence we have an ascending filtration

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \ldots .$$

Observe that U is generated by $U_1 = i(A)$; besides, $U = \cup_i U_i$, so the filtration is *exhaustive.*

Consider now the Z-graded algebra $gr\,U = \oplus_{i \in Z}(gr\,U)_i$ associated with the filtered algebra U. Its components are defined by the conditions: $(gr\,U)_n = 0$

for $n \leq 0$, $(gr\,U)_1 = U_1$, and $(gr\,U)_i = U_i/U_{i-1}$ for $i > 1$. If $\bar{a} = a + U_{i-1} \in (gr\,U)_i$, $\bar{b} = b + U_{j-1} \in (gr\,U)_j$, then $\bar{a} \cdot \bar{b} = ab + U_{i+j-1} \in (gr\,U)_{i+j}$.

Notice that $i(A) = U_1 = (gr\,U)_1$, hence the problem of the injectivity of i is reduced to the structure of the graded algebra $gr\,U(A)$. Usually a graded algebra associated with a filtered one is more easy to deal with, so we may turn our attention to the algebra $gr\,U(A)$.

In the case of Akivis algebras we can describe the structure of associated graded algebras $gr\,U(A)$ which immediately yields the solution to the speciality problem.

Let X be a Φ-module, and $S(X) = \sum_{i=0}^{\infty} S^i(X)$ be its symmetric algebra. We set $V^i(X) = S^i(X)$ if $1 \leq i \leq 3$, and for $i > 3$ we define by induction $V^i(X) = \oplus_{j=1}^{i-1} V^j(X) \otimes V^{i-j}(X)$. Consider the Φ-module direct sum $V(X) = \oplus_{i=1}^{\infty} V^i(X)$ and define a multiplication on it, by setting for $v_i \in V^i(X),\ v_j \in V^j(X)$

$$v_i \cdot v_j = \begin{cases} v_i v_j \in S^{i+j}(X),\ i+j \leq 3, \\ v_i \otimes v_j \in V^{i+j}(X),\ i+j > 3, \end{cases}$$

where the juxtaposition ab means the product of the elements a, b in the symmetric algebra $S(X)$.

Theorem 1 [14, 15]**.** *Let A be an Akivis algebra which is a free Φ-module. Then the graded algebra $gr\,U(A)$ is isomorphic to the algebra $V(A)$ defined above.*

Example 1. Let $A = \Phi \cdot a$ be a one-dimensional trivial Akivis algebra, with $[a, a] = \mathcal{A}(a, a, a) = 0$. Then $gr\,U(A) = U(A)$ is a free module over Φ with the base

$$\begin{aligned} &a,\ a^2,\ a^3, \\ &a^2a^2,\ a^3a,\ aa^3, \\ &a^2a^3,\ a^3a^2,\ (a^2a^2)a,\ (a^3a)a,\ (aa^3)a,\ a(a^2a^2),\ a(a^3a),\ a(aa^3), \\ &\ldots, \end{aligned}$$

where the basic elements are multiplied just as nonassociative words (with the subwords of length 3 being associative).

With the information on the structure of $gr\,U(A)$ at hand, we are able to describe the structure of the algebra $U(A)$ as well, provided that the Akivis algebra A is a free module over Φ.

So, let A be an Akivis algebra which is a free Φ-module with a base $\{e_i\}$. Consider again the Φ-module $V(A)$ and define on it a multiplication $*$. Let $a \in V^i(A),\ b \in V^j(A)$. We will distinguish two cases.

1. If $i + j > 3$, then we set $a * b = a \otimes b \in V^{i+j}(A)$.

2. For the case $i + j \leq 3$ we will use the base $\{e_i\}$ of A. Clearly, it is also a base of $V^1(A)$; while $V^2(A)$ and $V^3(A)$ are free Φ-modules with the bases $e_ie_j,\ i \leq j$ and $e_ie_je_k,\ i \leq j \leq k$, respectively, (remind that juxtaposition means the product in $S(A)$). Now, the product $a * b$ in the case $i + j \leq 3$ is

completely determined by the rules:

$$e_r * e_s = \begin{cases} e_r e_s, \ r \le s, \\ e_s e_r + [e_r, e_s], \ r > s, \end{cases}$$

$$(e_r e_s) * e_k = \begin{cases} e_r e_s e_k, \ r \le s \le k, \\ e_r e_k e_s + \mathcal{A}(e_r, e_s, e_k) - \mathcal{A}(e_r, e_k, e_s) + e_r * [e_s, e_k], \ r \le k < s, \\ e_k e_r e_s + e_r * [e_s, e_k] + [e_r, e_k] * e_s \\ +\mathcal{A}(e_r, e_s, e_k) - \mathcal{A}(e_r, e_k, e_s), \ k < r \le s, \end{cases}$$

$$e_r * (e_s e_k) = \begin{cases} e_r e_s e_k - \mathcal{A}(e_r, e_s, e_k), \ r \le s \le k, \\ e_s e_r e_k - \mathcal{A}(e_r, e_s, e_k) + [e_r, e_s] * e_k, \ s < r \le k, \\ e_s e_k e_r - \mathcal{A}(e_s, e_k, e_r) + \mathcal{A}(e_s, e_r, e_k) - \mathcal{A}(e_r, e_s, e_k) \\ -e_s * [e_k, e_r] - [e_s, e_r] * e_k, \ r > k \ge s. \end{cases}$$

Denote the algebra $\langle V(A), +, * \rangle$ by $\tilde{V}(A)$.

Theorem 2 [14, 15]**.** *Let A be an Akivis algebra which is a free Φ-module with a base $\{e_i\}$. Then the universal enveloping algebra $U(A)$ is isomorphic to the algebra $\tilde{V}(A)$, with the mapping $\varepsilon : A \to V^1(A)$, $\varepsilon(a) = a$, as a universal embedding mapping.*

Corollary 1. *Any Akivis algebra which is a free module over the ring of scalars Φ is linear.*

In conclusion of this section, we give an analogue of example 1 for the case of a nontrivial Akivis algebra.

Example 2. Let $A = \Phi \cdot a$ be an Akivis algebra with $[a, a] = 0$, $\mathcal{A}(a, a, a) = a$. Then $U(A)$ is a free module over Φ with the same base as in Example 1 and the multiplication $*$ defined by the rules

$$a * a = a^2,$$
$$a^2 * a = a^3,$$
$$a * a^2 = a^3 - a,$$

and the other products as in Example 1.

For instance, we have

$$a * (a * (a * a^2)) = a * (a * (a^3 - a)) = a * (aa^3 - a^2) = a(aa^3) - a^3 + a.$$

4. Deformations of Poisson Lie algebras. Before to consider universal alternative enveloping algebras for Malcev algebras and their associated graded algebras, we first remind certain facts concerning Lie algebras and their related associative algebras.

Let L be a Lie algebra which is a free Φ-module. Then it is well known and in fact is one of the equivalent formulations of the Poincare–Birkhoff–Witt theorem that $gr\, U(L)$ is isomorphic to the symmetric algebra $S(L)$.

The algebra $S(L)$ has a Lie algebra structure given by the *Poisson bracket* $\{,\}$ that is completely defined by the conditions

$$\begin{aligned}\{l_1, l_2\} &= [l_1, l_2],\ l_i \in L,\\ \{ab, c\} &= a\{b, c\} + \{a, c\}b \ (\textit{ Leibniz identity}).\end{aligned}$$

This structure was already discovered by S.Lie and then rediscovered by F. A. Berezin, A. A. Kirillov, and others (see [5]).

An associative commutative algebra that admits such a structure (a Lie bracket that satisfies the Leibniz identity) is called a *Poisson algebra*. We will call it a *Poisson Lie algebra*.

The algebra $S(L)$ admits the following characterization (cf. [17]).

Proposition 2. *Consider the category of Lie homomorphisms of the algebra L into Poisson algebras, with the morphisms defined as in the previous section. Then the canonical embedding $L \to S(L)$ is an initial object of this category.*

An other words, $S(L)$ plays a role of a universal enveloping algebra for Lie homomorphisms of L into Poisson algebras.

S.Gutt [6] and V.G.Drinfeld [3] observed that in the characteristic 0 case the universal enveloping algebra $U(L)$ is in fact a *quantization* of the algebra $S(L)$; that is, $U(L)$ can be obtained by a certain deformation of $S(L)$. In [13] this was proved for any Lie (super)algebra which is a free module over Φ.

Let us give a definition.

Definition 1. Let $A = \langle A, +, \cdot, \{,\}\rangle$ be a Poisson Lie algebra. An (algebraic) *quantization deformation* of A is an associative multiplication $*$ on the Φ-module of polynomials $A[t]$ such that

$$\begin{aligned} a * b &= ab\ (mod\ t),\\ a * b - b * a &= \{a, b\}t\ (mod\ t^2),\\ t * a = a * t &= at\end{aligned}$$

for any $a, b \in A$.

Substituting in this definition the polynomials by the formal power series, we get the notion of a *formal quantization deformation.*

Denote by L_t the Lie algebra $L[t]$ over $\Phi[t]$ with the multiplication $[l, l']_t = t[l, l']$. Then $L[t]$ is a free module over $\Phi[t]$, and the universal enveloping algebra $U(L_t)$ as a module over Φ is isomorphic to $U(L)[t]$. Furthermore, by the Poincare–Birkhoff–Witt theorem $S(L)$ and $U(L)$ are isomorphic as Φ-modules, hence the Φ-modules $U(L_t)$ and $S(L)[t]$ are isomorphic as well. So the multiplication $*$ in the algebra $U(L_t)$ can be considered as an associative multiplication in $S(L)[t]$.

Theorem 3 [3, 6, 13]**.** *The multiplication $*$ in the algebra $U(L_t)$ gives a quantization deformation of the Poisson Lie algebra $S(L)$.*

Thus we can construct the algebra $U(L)$ by a quantization of $S(L)$. In the next section we will try to apply this approach to Malcev algebras.

5. Poisson Malcev algebras and their alternative deformations. By analogue with the Lie algebra case, we will call an associative commutative algebra A a *Poisson Malcev algebra* if it admits an anticommutative Malcev bracket that satisfies the Leibniz identity.

Proposition 3 [16]**.** *For any Malcev algebra M the associated graded algebra $gr\,U(M)$ of the universal alternative enveloping algebra $U(M)$ is a Poisson Malcev algebra with respect to the bracket induced by the commutator in $U(M)$.*

In order to check whether a Malcev algebra M is special it would be enough to construct the Poisson Malcev algebra $gr\,U(M)$. In the Lie algebra case this algebra is isomorphic to the symmetric algebra $S(L)$. So, let us consider the symmetric algebra $S(M)$. As in the Lie algebra case, using the Leibniz identity one can extend the Malcev bracket $\{,\}$ given on M to a certain anticommutative bracket on $S(M)$. Although the extended bracket is no more a Malcev one, we can make it to be Malcev by factorizing $S(M)$ by a certain ideal.

Theorem 4 [16]**.** *Let M be a Malcev algebra and I be the ideal of the symmetric algebra $S(M)$ generated by the set $\{[a,b]J(a,b,c) \mid a,b,c \in M\}$. Then the quotient algebra $\tilde{S}(M) = S(M)/I$ is a Malcev Poisson algebra such that the embedding mapping $m \mapsto m + I$ of M into $\tilde{S}(M)$ is an initial object in the category of all (Malcev) homomorphisms of M into Poisson Malcev algebras.*

Now, as in the Lie algebra case, we can define alternative quantization deformations for Poisson Malcev algebras, substituting in definition 1 the associativity condition by alternativity. Our main result is the following

Theorem 5 [16]**.** *If the Poisson Malcev algebra $\tilde{S}(M)$ admits an algebraic alternative quantization deformation then M is special. Conversely, if M is special then the Poisson Malcev algebra $gr\,U(M)$ admits an alternative quantization deformation.*

By studying the obstructions to deformations of small orders for algebras $\tilde{S}(M)$, we have found the following necessary condition for speciality of Malcev algebras.

Proposition 4 [16]. *If a Malcev algebra M is special then it satisfies the following quasiidentity.*

$$\text{If} \qquad \sum_i J(a_i, b_i, c_i) \otimes [a_i, b_i] \otimes t_i = 0 \ \textit{in } S(M) \textit{ for some } a_i, b_i, c_i, t_i \in M,$$

$$\text{then} \qquad \sum_i ([[t_i, J(a_i, b_i, c_i)], [a_i, b_i]] + [[t_i, [a_i, b_i]], J(a_i, b_i, c_i)]) = 0 \ \textit{in } M.$$

The known examples of Malcev algebras satisfy this quasiidentity. As a possible algebra that might not satisfy it, we suggest the algebra

$$M = alg\langle x, y, z, u, v |\, J(x, y, z) = [u, v]\rangle.$$

One have to check whether the element

$$r = J([u, v], [x, y], J(u, v, z)) + 3[[J(u, v, z), [x, y]], [u, v]]$$

is nonzero in M. It seems that this task could be solved with a computer.

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