GENERALIZED KEPLER PROBLEMS I: WITHOUT MAGNETIC CHARGES

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ABSTRACT. Let V be a simple euclidean Jordan algebra of rank ρ and degree δ, T_Ω the tube domain naturally associated with V. Denote by $\mathrm{Co}(V)$ the universal cover of the identity component of the holomorphic automorphism group for T_Ω. It is known that the scalar-type unitary lowest weight representations π_ν of $\mathrm{Co}(V)$ are parametrized by the Wallach parameter $\nu \in \mathcal{W}(V) := \{k\frac{\delta}{2} \mid k = 0, 1, \ldots, (\rho-1)\} \cup ((\rho-1)\frac{\delta}{2}, \infty)$.

It is demonstrated here that, behind each non-trivial representation π_ν, there is precisely one quantum dynamic problem which shares the characteristic features of the quantum Kepler problem. For such a dynamic problem, the bound state spectra is $-\frac{1/2}{(I+\nu\frac{\rho}{2})^2}$, $I = 0, 1, \ldots$, and the configuration space is the cone consisting of semi-positive elements in V of rank $\rho(\nu)$, here, $\rho(\nu) = k$ if $\nu = k\frac{\delta}{2}$ and $= \rho$ if $\nu > (\rho-1)\frac{\delta}{2}$. As a by-product, we get two explicit realizations for π_ν, one is $L^2(\frac{1}{\mathrm{tr}\, x}\mathrm{vol})$, the other is $\mathscr{H}$ — a closed subspace of $L^2(\mathrm{vol})$. Here, vol is the volume form on the cone, $\mathrm{tr}\, x$ is the trace of x in the cone, and $\mathscr{H}$ is the Hilbert space of bound states for the dynamic problem. A few results in the literatures about these representations become more explicit and more refined.

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1. Introduction

The goal of this paper and its sequels is to develop a general mathematical theory for the Kepler problem of planetary motions. As with many endeavors of developing a general theory for a nice mathematical object, one usually starts with a reformulation of this object from a new perspective which captures the essence of such object — a crucial step which often requires introducing new concepts, and then work out all the necessary technical details.

For example, take continuous functions in calculus as the nice mathematical object, introduce the concept of open set and reformulate the notion of continuous functions in terms of open sets, then one arrive at the much broader notion of continuous maps between topological spaces. For a sophisticated example, take the Riemann integral for continuous functions over close intervals as the nice mathematical object, reformulate it as the Lebesgue integral — a step which requires introducing the concepts of measurable space and measure, then one arrives at the general integration theory. For still another sophisticated example, take the Euler number of closed manifolds as the nice mathematical object, reformulate it as the obstruction to the existence of nowhere vanishing vector fields — a step which requires introducing the concepts of vector bundles and obstructions, then one arrives at the theory of characteristic classes.

In our case, the Kepler problem of planetary motions is the nice mathematical object, and we reformulate it as a dynamic problem on the *open future light cone*

$$\{(t,\vec{x})\in\mathbb{R}^4 \mid t^2=|\vec{x}|^2, t>0\}$$

of the Minkowski space. To quickly convince readers that this is a much better formulation, one notes that the orbits (including the colliding ones) in this new formulation has a simpler and uniform description:

orbits are precisely the intersections of 2D planes with the open future light cone.

In this new reformulation, a pivotal role is played by the *euclidean Jordan algebra* structure, something that is more fundamental than the Lorentz structure. Once this formulation is discovered, it is just a matter of time to work out the details for the general theory. As a comparison with the integration theory, *Euclidean Jordan algebras* are like *measurable spaces*, *euclidean Jordan algebra structures* are like *σ-algebras*, *canonical cones* (cf. Subsection 4.1) are like *measures*, a *generalized Kepler problem* (cf. Section 5) on a canonical cone is like an *integration theory* on a measured space.

1.1. A general remark on euclidean Jordan algebras.

Euclidean Jordan algebras were first introduced by P. Jordan [1] to formalize the algebra of physics observables in quantum mechanics. With E. Wigner and J. von Neumann, Jordan [2] classified the finite dimensional euclidean Jordan algebras: *every finite dimensional euclidean Jordan algebra is a direct sum of simple ones, and the simple ones consist of four infinite series and one exceptional.*

Although euclidean Jordan algebras are abandoned by physicists quickly, they have been extensively studied at much more general settings by mathematicians since 1950's.

For an authoritative account of the history of Jordan algebras, readers may consult the recent book by K. McCrimmon [3]. However, for our specific purpose, the book by J. Faraut and A. Korányi [4] is sufficient — everything we really need either is already there or can be derived from there.

It is helpful for us to view euclidean Jordan algebras as the super-symmetric analogue of compact real Lie algebras. Just as compact real Lie algebras are the analogues of the infinitesimal gauge group for electromagnetism, euclidean Jordan algebras are the analogues of the infinitesimal space-time, but with a more refined euclidean Jordan algebra structure hidden behind. In our view, *it is this more refined hidden structure that is responsible for the existence of the inverse square law*. Therefore, we would not be surprised if someday euclidean Jordan algebras indeed play an indispensable role in the study of the fundamental physics.

1.2. **A quick review of euclidean Jordan algebras.** Throughout this paper and its sequels we always assume that V is a finite dimensional euclidean Jordan algebra. That means that V has both the inner product structure and the Jordan algebra (with an identity element e) structure on the underlying finite dimensional real vector space, such that, the two structures are compatible in the sense that the Jordan multiplication by $u \in V$, denoted by L_u, is always self-adjoint with respect to the inner product on V. We write the Jordan product of $u, v \in V$ as uv, so $uv = L_u v$, and the **Jordan triple product** of $u, v, w \in V$ as $\{uvw\}$. By definition, $\{uvw\} = S_{uv}w$, where

$$S_{uv} = [L_u, L_v] + L_{uv}, \tag{1.1}$$

i.e.,

$$\{uvw\} = u(vw) - v(uw) + (uv)w. \tag{1.2}$$

It is a fact that a simple euclidean Jordan algebra is uniquely determined by its rank ρ and degree δ.

V is a real Jordan algebra means that V is a real commutative algebra such that

$$[L_u, L_{u^2}] = 0, \quad u \in V.$$

We write the inner product of $u, v \in V$ as $\langle u \mid v\rangle$ and assume that the length of e is one: $||e|| = 1$. We further assume that V is simple, then the inner product is unique. A simple computation shows that

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}. \tag{1.3}$$

Therefore, these S_{uv} span a real Lie algebra — the **structure algebra** of V, denoted by $\mathfrak{str}(V)$ or simply $\mathfrak{str}$. It is a fact that the **derivation algebra** of V, denoted by $\mathfrak{der}(V)$ or simply $\mathfrak{der}$, is a Lie subalgebra of $\mathfrak{str}(V)$. Since $\mathfrak{der}$ is a generalization of $\mathfrak{so}(3)$, it is compact. Since $\mathfrak{str}$ is a generalization of $\mathfrak{so}(3,1) \oplus \mathbb{R}$, it is reductive.

The generalization of $\mathfrak{so}(4,2)$, denoted by $\mathfrak{co}(V)$ or simply $\mathfrak{co}$, was independently discovered by Tits, Kantor, and Kroecher [5]. Like $\mathfrak{so}(4,2)$, $\mathfrak{co}(V)$ — the **conformal Lie algebra** of V, is a simple real Lie algebra. In this paper and its sequels, the universal enveloping algebra of $\mathfrak{co}$ shall be referred to as the **TKK algebra** and the corresponding simply connected Lie group shall be referred to as the **conformal group** and denoted by $\mathrm{Co}(V)$ or Co.

As a vector space,

$$\mathfrak{co}(V) = V \oplus \mathfrak{str}(V) \oplus V^*.$$

If we rewrite $u \in V$ as X_u and $\langle v \mid \rangle \in V^*$ as Y_v, then the TKK algebra is determined by the **TKK commutation relations**:

$$\boxed{\begin{array}{c} [X_u, X_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, Y_v] = -2S_{uv}, \\ \\ [S_{uv}, X_z] = X_{\{uvz\}}, \quad [S_{uv}, Y_z] = -Y_{\{vuz\}}, \\ \\ [S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}. \end{array}} \tag{1.4}$$

Here, $u, v, z, w \in V$.

1.3. **Classical realization of TKK algebras.** The total cotangent space $T^*V = V \times V^*$ is a symplectic space. Let $\{e_\alpha\}$ be an orthonomal basis for V, with respect to which, a point $(x, \langle \pi \mid \rangle) \in T^*V$ can be represented by its coordinate (x^α, π^β). Then the basic Poisson bracket relations on T^*V are

$$\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}, \quad \{x^\alpha, x^\beta\} = 0, \quad \{\pi^\alpha, \pi^\beta\} = 0.$$

Introduce the moment functions

$$\mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi \rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\} \rangle, \quad \mathcal{Y}_v := \langle x \mid v \rangle \tag{1.5}$$

on T^*V. We shall show that, for any u, v, z and w in V, we have

$$\left\{ \begin{array}{c} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, \quad \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, \quad \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, \quad \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{array} \right. \tag{1.6}$$

The realization of O by $\mathcal{O}$ is referred to as the classical realization of the TKK algebra. The quantization of this classical realization leads to operator realizations of the TKK algebra.

1.4. **Operator realizations of TKK algebras.** The canonical quantization involves promoting classical physical variables $\mathcal{O}$ to differential operators $\acute{O}$ (or the duals $\grave{O}$) using recipe: $\pi^\alpha \to -i\frac{\partial}{\partial x^\alpha}$ (or $x^\alpha \to i\frac{\partial}{\partial \pi^\alpha}$). Here is a word of warning: in order to get anti-hermitian differential operators in the end, instead of using the quantized differential operators, we actually use the quantized differential operators multiplied by $-i$.

Due to the ambiguity with the operator ordering, the canonical quantization has an ambiguity, measured by a real parameter μ here. For simplicity, we write $\sum_\alpha e_\alpha \frac{\partial}{\partial x^\alpha}$ as $\partial\!\!\!/$ and $\sum_\alpha e_\alpha \frac{\partial}{\partial \pi^\alpha}$ as $\eth$. The quantization recipe, with the operator orderings taken into account, yields either differential operators on V:

$$\left\{ \begin{array}{rcl} \acute{S}_{uv}(\nu) & := & -\langle S_{uv}(x) \mid \partial\!\!\!/ \rangle - \frac{\nu}{2}\mathrm{tr}\,(uv), \\ \\ \acute{X}_u(\nu) & := & i\langle x \mid \{\partial\!\!\!/ u \partial\!\!\!/\} \rangle + i\nu\mathrm{tr}\,(u\partial\!\!\!/), \\ \\ \acute{Y}_v(\nu) & := & -i\langle x \mid v \rangle \end{array} \right. \tag{1.7}$$

or differential operators on V^*:

$$\left\{\begin{array}{lcl} \check{S}_{uv}(\nu) & := & \langle S_{vu}(\pi) \mid \eth\rangle - \frac{\nu^*}{2}\mathrm{tr}\,(uv), \\ \check{X}_u(\nu) & := & \langle \{\pi u \pi\} \mid \eth\rangle - \nu^* \mathrm{tr}\,(u\pi), \\ \check{Y}_v(\nu) & := & \langle v \mid \eth\rangle. \end{array}\right. \tag{1.8}$$

Here, $\nu^* = \nu - \frac{2n}{\rho}$ with $n = \dim V$.

Eqns (1.7) and (1.8) provide us two families of operator realizations for the TKK algebra. It is not hard to see that the two families are related by the Fourier transform and for the realizations to be unitary we must have $\nu \geq 0$. Note that the case $\nu^* = 0$ is well-known to physicists, cf. Ref. [6]; moreover, for $\nu > 1 + (\rho - 1)\delta$, the explicit formula in Eq. (1.7) has been obtained in Ref. [7] via an indirect route.

These operator realizations as given in Eq. (1.7) are not unitary with respect to the obvious L^2-inner product

$$(\psi_1, \psi_2) = \int_V \bar{\psi}_1 \psi_2 \, dm,$$

where dm is the Lebesgue measure. Therefore, the right hermitian inner products must be found in order to have unitary realizations. For us, the clues come out naturally in our [8, 9] study of the Kepler problem.

1.5. **Quantum realizations of TKK algebras.** In order to address the unitarity problem for the quantization given by the explicit formula in Eq. (1.7), we are led to introduce the notion of **canonical cones**. Let $k > 0$ be an integer no more than ρ. The canonical cone of rank k, denoted by $\mathcal{C}_k$, is an open Riemannian manifold. As a smooth manifold, it consists of all semi-positive elements in V of rank k. The Riemannian metric on $\mathcal{C}_k$ is defined in terms of Jordan multiplication, so it is invariant under the automorphism group of V.

Denote by $\mathcal{P}(V)$ the algebra of polynomial maps from V to $\mathbb{C}$, by $\mathcal{P}(\mathcal{C}_k)$ the algebra of functions on $\mathcal{C}_k$ coming from the restriction of elements in $\mathcal{P}(V)$, i.e.,

$$\mathcal{P}(\mathcal{C}_k) = \{p : \mathcal{C}_k \to \mathbb{C} \mid p \in \mathcal{P}(V)\}. \tag{1.9}$$

We use r to denote this function on V:

$$r(x) = \langle e \mid x\rangle. \tag{1.10}$$

We shall show that $e^{-r}\mathcal{P}(\mathcal{C}_k)$ is a common domain for the operators in Eq. (1.7) and is dense in $L^2(d\mu_\nu)$, here $d\mu_\nu$ is a measure on $\mathcal{C}_\rho$ (on $\mathcal{C}_k$) if $\nu > (\rho - 1)\frac{\delta}{2}$ (if $\nu = k\frac{\delta}{2}$, $k = 1$, $2, \ldots, \rho - 1$), and can be explicitly written down [1] in terms of the volume form vol on the canonical cone. It shall also be demonstrated that $L^2(d\mu_\nu)$ is a unitary lowest weight representation π_ν for Co with the lowest weigh equal to $\nu\lambda_0$, here λ_0 is the fundamental weight conjugate to the unique non-compact simple root of $\mathfrak{co}$ under a suitable choice of Cartan subalgebra. Note that, we have a unitary lowest weight representation for Co when either $\nu = k\frac{\delta}{2}$ with $k = 1, \ldots, \rho - 1$ or $\nu > (\rho - 1)\frac{\delta}{2}$, and these π_ν exhaust all nontrivial scale-type unitary lowest weight representations for Co, according to Ref. [11].

Even though we get the explicit formula for measure $d\mu_\nu$ via the Riemannian metric on canonical cones, as far as representations are concerned, the introduction of this Riemannian metric is probably not needed; however, it is essential if one wishes to go one step further, i.e., unravel the hidden dynamic models behind these unitary lowest weight representations.

[1] The explicit formula for $d\mu_\nu$ has already been given in Ref. [4] when $\nu > (\rho - 1)\frac{\delta}{2}$.

1.6. **Universal Kepler hamiltonian and universal Lenz vector.** The hidden dynamical models referred to in the proceeding paragraph are of the Kepler type in the sense that hidden conserved Lenz vector always exists. To see that, we need to introduce the **universal hamiltonian**

$$H := \frac{1}{2} Y_e^{-1} X_e + i Y_e^{-1} \tag{1.11}$$

and the **universal Lenz vector**

$$A_u := i Y_e^{-1} [L_u, Y_e^2 H], \quad u \in V. \tag{1.12}$$

Here, i is the imaginary unit, and Y_e^{-1} is the formal inverse of Y_e. Note that both H and A_u are elements of the complexified TKK algebra with Y_e inverted.

The manifest conserved quantity is $L_{u,v} := [L_u, L_v]$ — the analogue of angular momentum, and the hidden conserved quantity is A_u — the analogue of original Lenz vector; in fact, the following commutation relations have been verified in Ref. [9]:

$$\left\{ \begin{array}{lcl} [L_{u,v}, H] & = & 0\,, \\ [A_u, H] & = & 0\,, \\ [L_{u,v}, L_{z,w}] & = & L_{[L_u,L_v]z,w} + L_{z,[L_u,L_v]w}\,, \\ [L_{u,v}, A_z] & = & A_{[L_u,L_v]z}\,, \\ [A_u, A_v] & = & -2HL_{u,v}\,. \end{array} \right. \tag{1.13}$$

Now, under a unitary lowest weight representation π_ν in the proceeding subsection, both H and A_u become differential operators $\acute{H}(\nu)$ and $\acute{A}_u(\nu)$ on the canonical cone. However, $\acute{H}(\nu)$ is not quite right, because the term $-\acute{Y}_e^{-1}(\nu)\acute{X}_e(\nu)$ in $\acute{H}(\nu)$ is not the Laplace operator on the canonical cone, even up to an additive function.

1.7. **Generalized Kepler problems.** By comparing with the Laplace operator on the canonical cone, one realizes that $-\acute{Y}_e^{-1}(\nu)\acute{X}_e(\nu)$ is almost right: after conjugation by the multiplication with a positive function on the canonical cone, modulo an additive function, it becomes the Laplace operator on the canonical cone. After this conjugation, $\acute{O}$ becomes a new differential operator (shall be denoted by $\tilde{O}$), and $L^2(d\mu_\nu)$ becomes $L^2(\frac{1}{r}\mathrm{vol})$. Now $\tilde{H}(\nu)$ is the hamiltonian for the Kepler-type dynamic problem behind representation π_ν. It is in this sense we say that $L^2(\frac{1}{r}\mathrm{vol})$ is more natural than $L^2(d\mu_\nu)$.

The bound state spectrum for $\tilde{H}(\nu)$ is

$$-\frac{1/2}{(I + \nu\frac{\rho}{2})^2}, \quad I = 0, 1, 2, \ldots$$

and the hilbert space of bound states — a closed subspace of $L^2(\mathrm{vol})$, denoted by $\mathscr{H}(\nu)$, provides a new relization for representation π_ν.

The two natural realizations for π_ν, one by $L^2(\frac{1}{r}\mathrm{vol})$, one by $\mathscr{H}(\nu)$, implies that there are two kinds of orthogonalities for generalized Laguerre polynomials, generalizing the well-known fact that there are two kinds of orthogonalities for Laguerre polynomials, cf. Ref. [10].

1.8. **Outline of the paper.** In Section 2, we give a quick review of Jordan algebras and especially euclidean Jordan algebras. In Section 3, we first review TKK algebras, then we introduce the classical realization for TKK algebras. The operator realizations follows from formally quantizing the classical realization. Section 4 is the most technical section of this article, here, the unitarity of those operator realizations is addressed. In Section 5 we present the generalized Kepler problem behind each nontrivial unitary representation in Section 4, then we solve the bound state problem and prove that the Hilbert space of bound states provides another realization for the representation. Here, the proof is complete only after one proves an analogue of Theorem 2 in Ref. [10] for generalized Laguerre polynomials, something that can surely be done. Two appendixes are given in the end. In Appendix A, a technical theorem extending the existing one in Ref. [4] is proved. In Appendix B, for the convenience of readers, some basic notations for this paper are listed.

Ref. [4] is our most consulted book — really a bible for our purpose. However, we should warn the readers that there are some convention/notation differences: 1) our invariant inner product on V is the one such that the identity element e has unit length and is denoted by $\langle | \rangle$ rather than $(\,,\,)$, 2) the rank and degree of a Jordan algebra is denoted by ρ and δ rather than r and d because r is reserved for function $x \mapsto \langle e \mid x\rangle$ and d is reserved for the exterior derivative operator, 3) the Jordan multiplication by u is denoted by L_u rather than $L(u)$, 4) the Jordan triple product of u, v and w is denoted by $\{uvw\}$ rather than $\{u, v, w\}$, 5) we use S_{uv} rather than $u\square v$ for endomorphism $w \mapsto \{uvw\}$.

Note: Since the author is not an expert on most of the mathematical areas touched upon by the subject studied here, he apologizes in advance to those authors whose works are either not cited properly or not cited at all. It is his strong intention to rectify the problem as much as possible in the future versions of this paper.

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2. Jordan Algebras

Jordan algebra has become a big subject now; however, what is relevant for us here is very minimal. In this section, we shall review the bare essential of Jordan algebras, oriented towards our purpose. Apart from Lemma 2.1, everything else presented here can be found from the book by J. Faraut and A. Korányi [4].

2.1. **Basic definitions.**

Recall that an **algebra** V over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a $\mathbb{F}$-bilinear map $V \times V \to V$ which maps (u, v) to uv. This $\mathbb{F}$-bilinear map can be recast as a linear map $V \to \mathrm{End}_{\mathbb{F}}(V)$ which maps u to L_u: $v \mapsto uv$.

We say that algebra V is *commutative* if $uv = vu$ for any $u, v \in V$. As usual, we write u^2 for uu and u^{m+1} for uu^m inductively.

Definition 2.1. *A* **Jordan algebra** *over* $\mathbb{F}$ *is just a commutative algebra* V *over* $\mathbb{F}$ *such that*

$$[L_u, L_{u^2}] = 0 \tag{2.1}$$

for any $u \in V$.

As the first example, we note that $\mathbb{F}$ is a Jordan algebra over $\mathbb{F}$. Here is a *recipe* to produce Jordan algebras. Suppose that Φ is an associative algebra over field $\mathbb{F}$ with characteristic $\neq 2$, and $V \subset \Phi$ is a linear subspace of Φ, closed under square operation, i.e, $u \in V \Rightarrow u^2 \in V$. Then V is a Jordan algebra over $\mathbb{F}$ under the Jordan product

$$uv := \frac{(u+v)^2 - u^2 - v^2}{2}.$$

Applying this recipe, we have the following Jordan algebras over $\mathbb{R}$:

(1) The algebra $\Gamma(n)$. Here $\Phi = \mathrm{Cl}(\mathbb{R}^n)$—the Clifford algebra of $\mathbb{R}^n$ and $V = \mathbb{R} \oplus \mathbb{R}^n$.
(2) The algebra $\mathcal{H}_n(\mathbb{R})$. Here $\Phi = M_n(\mathbb{R})$—the algebra of real $n \times n$-matrices and $V \subset \Phi$ is the set of symmetric $n \times n$-matrices.
(3) The algebra $\mathcal{H}_n(\mathbb{C})$. Here $\Phi = M_n(\mathbb{C})$—the algebra of complex $n \times n$-matrices (considered as an algebra over $\mathbb{R}$) and $V \subset \Phi$ is the set of Hermitian $n \times n$-matrices.
(4) The algebra $\mathcal{H}_n(\mathbb{H})$. Here $\Phi = M_n(\mathbb{H})$—the algebra of quaternionic $n \times n$-matrices (considered as an algebra over $\mathbb{R}$) and $V \subset \Phi$ is the set of Hermitian $n \times n$-matrices.

The Jordan algebras over $\mathbb{R}$ listed above are *special* in the sense that they are derived from associated algebras via the above recipe. Let us use $\mathcal{H}_n(\mathbb{O})$ to denote the algebra for which the underlying real vector space is the set of Hermitian $n \times n$-matrices over $\mathbb{O}$ and the product is the symmetrization of the matrix product. One can show that $\mathcal{H}_n(\mathbb{O})$ is a Jordan algebra if and only if $n \leq 3$. However, only $\mathcal{H}_3(\mathbb{O})$ is new because $\mathcal{H}_1(\mathbb{O}) = \mathcal{H}_1(\mathbb{R})$, $\mathcal{H}_2(\mathbb{O}) \cong \Gamma(9)$. $\mathcal{H}_3(\mathbb{O})$ is called *exceptional* because it cannot be obtained via the above recipe, even if one allows the associative algebra Φ be infinite dimensional.

2.2. **Euclidean Jordan algebras.** Any Jordan algebra V comes with a canonical symmetric bilinear form

$$\tau(u, v) := \text{the trace of } L_{uv}. \tag{2.2}$$

It is a fact that L_u is self-adjoint with respect to τ.

We say that Jordan algebra V is *semi-simple* if the symmetric bilinear form τ is non-degenerate. We say that Jordan algebra V is *simple* if it is semi-simple and has no ideal other than $\{0\}$ and V itself.

By definition, an **euclidean Jordan algebra**[2] is a real Jordan algebra with an identity element e such that the symmetric bilinear form τ is positive definite. Therefore, an euclidean Jordan algebra is semi-simple and can be uniquely written as the direct sum of simple ideals — ideals which are simple as Jordan algebras.

Theorem 2.1 (Jordan, von Neumann and Wigner, Ref. [2])**.** *The complete list of simple euclidean Jordan algebras are*

(1) *The algebra* $\Gamma(n) = \mathbb{R} \oplus \mathbb{R}^n$ *(*$n \geq 2$*).*
(2) *The algebra* $\mathcal{H}_n(\mathbb{R})$ *(*$n \geq 3$ *or* $n = 1$*).*
(3) *The algebra* $\mathcal{H}_n(\mathbb{C})$ *(*$n \geq 3$*).*
(4) *The algebra* $\mathcal{H}_n(\mathbb{H})$ *(*$n \geq 3$*).*
(5) *The algebra* $\mathcal{H}_3(\mathbb{O})$*.*

Note that $\Gamma(1)$ is not simple and $\mathcal{H}_1(\mathbb{F})$ $(= \mathbb{R})$ is the only associative simple euclidean Jordan algebra. Note also that there are various isomorphisms: $\Gamma(2) \cong \mathcal{H}_2(\mathbb{R})$, $\Gamma(3) \cong \mathcal{H}_2(\mathbb{C})$, $\Gamma(5) \cong \mathcal{H}_2(\mathbb{H})$, $\Gamma(9) \cong \mathcal{H}_2(\mathbb{O})$.

Remark 2.1. *A simple euclidean Jordan algebra is uniquely specified by its rank* ρ *and degree* δ*:*

J	$\Gamma(n)$	$\mathcal{H}_n(\mathbb{R})$	$\mathcal{H}_n(\mathbb{C})$	$\mathcal{H}_n(\mathbb{H})$	$\mathcal{H}_3(\mathbb{O})$
ρ	2	n	n	n	3
δ	$n-1$	1	2	4	8

[2] Called *formally real Jordan algebra* in the old literatures.

Hence, for the simple euclidean Jordan algebras, there is one with rank-one, infinity many with rank two, four with rank three, and three with rank four or higher.

The notion of **trace** is valid for Jordan algebras. For the simple euclidean Jordan algebras, the trace can be easily described: For $\Gamma(n)$, we have

$$\mathrm{tr}\,(\lambda, \vec{u}) = 2\lambda,$$

and for all other types, it is the usual trace for matrices.

For the **inner product** on V, we take

$$\boxed{\langle u \mid v\rangle := \tfrac{1}{\rho}\mathrm{tr}\,(uv)} \tag{2.3}$$

so that the identity element e becomes a unit vector. One can check that L_u is self-adjoint with respect to this inner product: $\langle vu \mid w\rangle = \langle v \mid uw\rangle$, i.e., $L'_u = L_u$. For u, v in V, we introduce linear map $S_{uv} := [L_u, L_v] + L_{uv}$, and write $S_{uv}(w)$ as $\{uvw\}$. One can check that $S'_{uv} = S_{vu}$ and

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}.$$

In the remainder of this paper, we fix a simple euclidean Jordan algebra V, and use e, ρ, δ and n to denote its identity element, rank, degree and dimension respectively. We shall use $\{e_{ii}\}$ to denote a **Jordan frame** and V_{ij} to denote the resulting (i,j)-**Pierce component**. Choosing an orthogonal basis e_{ij}^μ for each V_{ij} ($1 \le i < j \le \rho$) with each basis vector has length $\frac{1}{\sqrt{\rho}}$, here $1 \le \mu \le \delta$. Then $\{e_{ii}, e_{ij}^\mu\}$ are an orthogonal basis for V, and each basis vector has length $\frac{1}{\sqrt{\rho}}$. Such a basis is referred to as a **Jordan basis** with respect to Jordan frame $\{e_{ii}\}$.

For any *orthonomal* basis $\{e_\alpha\}_{\alpha=1}^n$ of V, one can verify that

$$\begin{aligned} &\sum e_\alpha^2 = ne, \quad \sum L_{e_\alpha}^2(x) = \rho\left(1+\frac{\rho-2}{4}\delta\right)x + \frac{\rho\delta}{4}\mathrm{tr}\,x\; e,\\ &\sum \mid [L_{e_\alpha}, L_{e_\beta}]x\rangle\langle[L_{e_\alpha}, L_{e_\beta}]x \mid = \frac{1}{\rho}\left(1+\frac{\rho-2}{4}\delta\right)(L_{x^2} - {L_x}^2).\end{aligned} \tag{2.4}$$

Here is a convention we shall adopt: x is reserved for a point in the smooth space V, and u, v, z, w are reserved for vectors in vector space V. We shall also use V to denote the Euclidean space with underlying smooth space V and Riemannian metric ds_E^2:

$$\begin{array}{rcl} T_xV \times T_xV & \to & \mathbb{R} \\ ((x,u),(x,v)) & \mapsto & \langle u \mid v\rangle. \end{array} \tag{2.5}$$

Finally, we would like to remark that, if one takes Jordan algebras as analogues of Lie algebras, then euclidean Jordan algebras are the analogues of semi-simple compact Lie algebras.

2.3. **Structure algebras and TKK Algebras.** As always, we let V be a finite dimensional simple euclidean Jordan algebra. We use Ω to denote the **symmetric cone** of V and $\mathrm{Str}(V)$ to denote the **structure group** of V. By definition, Ω is the topological interior of

$$\{x^2 \mid x \in V\}$$

and

$$\mathrm{Str}(V) = \{g \in GL(V) \mid P(gx) = gP(x)g' \quad \forall x \in V\}.$$

Here $P(x) := 2L_x^2 - L_{x^2}$ and it is called the **quadratic representation** of x.

We write $V^{\mathbb{C}}$ for the complexification of V, denote by T_Ω the tube domain associated with V. By definition, $T_\Omega = V \oplus i\Omega$. We say that map $f\colon T_\Omega \to T_\Omega$ is a holomorphic automorphism of T_Ω if f is invertible and both f and f^{-1} are holomorphic. We use $\mathrm{Aut}(T_\Omega)$ to denote the group of holomorphic automorphisms of T_Ω.

It is a fact that both $\mathrm{Str}(V)$ and $\mathrm{Aut}(T_\Omega)$ are Lie groups. The Lie algebra of $\mathrm{Str}(V)$ is referred to as the **structure algebra** of V and is denoted by $\mathfrak{str}(V)$ or simply $\mathfrak{str}$, the Lie algebra of $\mathrm{Aut}(T_\Omega)$ is referred to as the **conformal algebra** of V and is denoted by $\mathfrak{co}(V)$ or simply $\mathfrak{co}$, and its universal enveloping algebra is called the **TKK algebra** of V. The simply connected Lie group with $\mathfrak{co}$ as its Lie algebra, denoted by $\mathrm{Co}(V)$ or simply Co, shall be referred to as the **conformal group** of V.

Both the structure algebra and the conformal algebra have a simple direct algebraic description, cf. Subsection 1.2. While the structure algebra is reductive, the conformal algebra is simple.

Since $\mathfrak{co}$ is a non-compact real simple Lie algebra, it admits a Cartan involution θ, unique up to conjugations by inner automorphisms. Indeed, one can choose θ such that

$$\theta(X_u) = Y_u, \quad \theta(Y_u) = X_u, \quad \theta(S_{uv}) = S_{uv}^* = -S_{vu}.$$

The resulting Cartan decomposition is $\mathfrak{co} = \mathfrak{u} \oplus \mathfrak{p}$ with

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{[L_u, L_v], X_w + Y_w \mid u, v, w \in V\}, \quad \mathfrak{p} = \mathrm{span}_{\mathbb{R}}\{L_u, X_v - Y_v \mid u, v \in V\}.$$

Note that $\mathfrak{u}$ is reductive with center spanned by $X_e + Y_e$ and its semi-simple part $\bar{\mathfrak{u}}$ is

$$\mathrm{span}_{\mathbb{R}}\{[L_u, L_v], X_w + Y_w \mid u, v, w \in (\mathbb{R}e)^\perp\}.$$

Sometime we need to emphasize the dependence on V, then we rewrite $\mathfrak{u}$ as $\mathfrak{u}(V)$. It is a fact that $\mathfrak{str}$ and $\mathfrak{u}$ are different real forms of the same complex reductive Lie algebra. In fact, one can identify their complexfications as follows:

$$[L_u, L_v] \leftrightarrow [L_u, L_v], \quad -\frac{i}{2}(X_w + Y_w) \leftrightarrow L_w. \tag{2.6}$$

Recall that e_{11} denotes the first element of a Jordan frame for V. The following lemma has been proved in Subsection 4.2 of Ref. [8].

Lemma 2.1. *There is a maximally compact θ-stable Cartan subalgebra for $\mathfrak{co}$, with respect to which, there is a simple root system consisting of imaginary roots α_0, α_1, …, α_r such that, for $i \ge 1$, α_i is compact with $H_{\alpha_i}, E_{\pm\alpha_i} \in \bar{\mathfrak{u}}^{\mathbb{C}}$, and α_0 is non-compact with*

$$H_{\alpha_0} = i(X_{e_{11}} + Y_{e_{11}}), \quad E_{\pm\alpha_0} = \frac{i}{2}(X_{e_{11}} - Y_{e_{11}}) \mp L_{e_{11}}.$$

2.4. **The decomposition of the action of $\mathrm{Str}(V)$ on $\mathcal{P}(V)$.** The structure group acts on V linearly, so it acts on $\mathcal{P}(V)$ — the set of complex-valued polynomial functions on V. The goal here is to describe the known decomposition of the action of $\mathrm{Str}(V)$ on $\mathcal{P}(V)$ into irreducible components.

With a Jordan frame $\{e_{ii}\}$ for V chosen, for $1 \le k \le \rho$, we let $e_k = e_{11} + \cdots + e_{kk}$. Denote by V_k the eigenspace of L_{e_k} with eigenvalue 1 and by P_k the orthogonal projection of V onto V_k. Then V_k is a simple euclidean Jordan algebra of rank k and there is a filtration of euclidean Jordan algebras:

$$V_1 \subset V_2 \cdots \subset V_\rho = V.$$

Let $\mathbf{m} \in \mathbb{Z}^\rho$. We write $\mathbf{m} = (m_1, \ldots, m_\rho)$ and say that $\mathbf{m} \geq \mathbf{0}$ if $m_1 \geq \ldots \geq m_\rho \geq 0$. Let

$$\Delta_{\mathbf{m}}(x) = \prod_{i=1}^{\rho} \Delta_i(x)^{m_i - m_{i+1}},$$

here $m_{\rho+1} = 0$, $\Delta_i(x)$ is the determinant of $P_i(x)$, considered as an element of V_i.

For $\mathbf{m} \geq 0$, we let $\mathcal{P}_{\mathbf{m}}(V)$ be the subspace of $\mathcal{P}(V)$ generated by the polynomials $g \cdot \Delta_{\mathbf{m}}$, $g \in \mathrm{Str}(V)$. The polynomials belonging to $\mathcal{P}_{\mathbf{m}}(V)$ are homogeneous of degree $|\mathbf{m}| = \sum m_i$, hence $\mathcal{P}_{\mathbf{m}}(V)$ is finite dimensional.

Theorem XI.2.4 of Ref. [4]**.** *The subspaces $\mathcal{P}_{\mathbf{m}}(V)$ are mutually inequivalent irreducible as representation spaces of $\mathrm{Str}(V)$, and $\mathcal{P}(V)$ is the direct sum*

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m} \geq \mathbf{0}} \mathcal{P}_{\mathbf{m}}(V).$$

Since $\mathcal{P}(V)$ consists of complex-valued polynomials on V, the representations in this theorem naturally extends to the complxification of $\mathrm{Str}(V)$. Using the identification in Eq. (2.6), these representations naturally becomes representations of $\mathfrak{u}$.

From here on, as representations of $\mathfrak{u}$, $\mathcal{P}(V)$ and $\mathcal{P}_{\mathbf{m}}(V)$ shall always be viewed in this sense. For later use, we use ξ_ν to denote the one-dimensional representation of $\mathfrak{u}$ such that $-\frac{i}{2}(X_e + Y_e)$ acts as the scalar multiplication by $-\nu\frac{\rho}{2}$.

3. Realizations Of TKK Algebras

The goal of this section is to realize the TKK algebras. The results and their presentations here are strongly influenced by the thinking/practice in physics. Although our perspective is different, we don't claim any originality here, because most (maybe all) of the materials presented here should be known to the experts in one area or another area.

We start with the classical realization on symplectic space T^*V, from which the operator realizations follow via the straightforward canonical quantization. Due to the operator ordering ambiguity, we get a family of operator realizations, parametrized by a real parameter ν. The case $\nu = 2n/\rho$ is well-known to physicists, cf. Ref. [6]. The case $\nu > 1 + (\rho - 1)\delta$ has been worked out by M. Aristidou, M. Davidson and G. Ólafsson [7] by an indirect method.

3.1. The classical realization of TKK algebras.

As is well-known, the total cotangent space T^*V is a natural symplectic space. By virtue of the euclidean metric ds_E^2 on V, one can identify T^*V with the total tangent space TV. Now the tangent bundle and cotangent bundle of V both have a natural trivialization, with respect to which, one can denote an element of T^*V by (x, p) and its corresponding element in TV by (x, π). We fix an orthonormal basis $\{e_\alpha\}$ for V so that we can write $x = x^\alpha e_\alpha$ and $\pi = \pi^\alpha e_\alpha$. Then the basic Poisson bracket relations on TV are $\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}$, $\{x^\alpha, x^\beta\} = 0$, and $\{\pi^\alpha, \pi^\beta\} = 0$.

Introduce the moment functions

$$\mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi \rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\} \rangle, \quad \mathcal{Y}_v := \langle x \mid v \rangle \tag{3.1}$$

on TV. The following theorem would be well-known to experts on Jordan algebra.

Theorem 1. *As polynomial functions on TV, $\mathcal{S}_{uv}$, $\mathcal{X}_u$ and $\mathcal{Y}_v$ satisfy the following Poisson bracket relations: for any u, v, z and w in V,*

$$\begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, \quad \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, \quad \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, \quad \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{cases} \tag{3.2}$$

Proof. It is clear that $\{\mathcal{Y}_u, \mathcal{Y}_v\} = 0$.

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{Y}_v\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid v\rangle\} \\ &= -2\langle x \mid \{vu\pi\}\rangle = -2\langle S_{uv}(x) \mid \pi\rangle \\ &= -2\mathcal{S}_{uv}. \end{aligned}$$

$$\begin{aligned} \{\mathcal{S}_{uv}, \mathcal{Y}_z\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle x \mid z\rangle\} \\ &= -\langle S_{uv}(x) \mid z\rangle = -\langle x \mid \{vuz\}\rangle \\ &= -\mathcal{Y}_{\{vuz\}}. \end{aligned}$$

$$\begin{aligned} \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle S_{zw}(x) \mid \pi\rangle\} \\ &= \langle S_{uv}S_{zw}(x) \mid z\rangle - \langle S_{zw}S_{uv}(x) \mid z\rangle \\ &= \langle [S_{uv}, S_{zw}](x) \mid z\rangle = \langle (S_{\{uvz\}w} - S_{z\{vuw\}})(x) \mid z\rangle \\ &= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{aligned}$$

$$\begin{aligned} \{\mathcal{S}_{uv}, \mathcal{X}_z\} &= \{\langle S_{uv}(x) \mid \pi\rangle, \langle x \mid \{\pi z \pi\}\rangle\} \\ &= -\langle S_{uv}(x) \mid \{\pi z \pi\}\rangle + 2\langle x \mid \{\pi z \{vu\pi\}\}\rangle \\ &= \langle x \mid 2S_{\pi z}S_{vu}(\pi) - S_{vu}S_{\pi z}(\pi)\}\rangle \\ &= \langle x \mid S_{\pi z}S_{vu}(\pi) - [S_{vu}, S_{\pi z}](\pi)\}\rangle \\ &= \langle x \mid S_{\pi z}(\{vu\pi\}) - S_{\{vu\pi\}z}(\pi) + S_{\pi\{uvz\}}(\pi)\}\rangle \\ &= \langle x \mid \{\pi\{uvz\}\pi\}\rangle \\ &= \mathcal{X}_{\{uvz\}}. \end{aligned}$$

Finally, since[3]

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{X}_e\} &= \{\langle x \mid \{\pi u \pi\}\rangle, \langle x \mid \pi^2\rangle\} \\ &= 2\langle x \mid \pi\{\pi u \pi\}\rangle - 2\langle x \mid \{\pi u \pi^2\}\rangle \\ &= 2\langle x \mid (2L_\pi^3 - 3L_\pi L_{\pi^2} + L_{\pi^3})u\rangle \\ &= 0 \end{aligned}$$

for any $u \in V$, we have

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{X}_v\} &= \{\mathcal{X}_u, \{\mathcal{L}_v, \mathcal{X}_e\}\} \\ &= \{\{\mathcal{X}_u, \mathcal{L}_v\}, \mathcal{X}_e\} + \{\mathcal{L}_v, \{\mathcal{X}_u, \mathcal{X}_e\}\} \\ &= -\{\mathcal{X}_{uv}, \mathcal{X}_e\} = 0. \end{aligned}$$

□

[3]Here, we use the fact that, for any Jordan algebra, identity $L_{u^3} = -2L_u^3 + 3L_uL_{u^2}$ holds for any $u \in V$. See line 12 on page 27 of Ref. [4].

3.2. **The operator realizations of TKK Algebras.** The canonical quantization involves promoting classical physical variables $\mathcal{O}$ to differential operators $\hat{O}$ (or the duals $\check{O}$) using recipe: $\pi_\alpha \to -i\frac{\partial}{\partial x^\alpha}$ (or $x_\alpha \to i\frac{\partial}{\partial \pi^\alpha}$). Here is a word of warning: in order to get anti-hermitian differential operators in the end, instead of using the quantized differential operators, we actually use the quantized differential operators multiplied by $-i$.

For simplicity we write $\sum_\alpha e_\alpha \frac{\partial}{\partial x^\alpha}$ as $\partial\!\!\!/$ and $\sum_\alpha e_\alpha \frac{\partial}{\partial \pi^\alpha}$ as $\eth$. We introduce differential operators on V:

$$\hat{S}_{uv} := -\langle S_{uv}(x) \mid \partial\!\!\!/\rangle, \quad \hat{X}_u := i\langle x \mid \{\partial\!\!\!/ u \partial\!\!\!/\}\rangle, \quad \hat{Y}_v := -i\langle x \mid v\rangle. \tag{3.3}$$

and differential operators on V^*:

$$\check{S}_{uv} := \langle S_{vu}(\pi) \mid \eth\rangle, \quad \check{X}_u := \langle \{\pi u \pi\} \mid \eth\rangle, \quad \check{Y}_v := \langle v \mid \eth\rangle. \tag{3.4}$$

It is easy to see that the TKK commutation relations (1.4) hold when all O there are replaced by either their hat version or their check version. Note that the check version is well-known to physicists, cf. Ref. [6]. However, the quantization has ambiguity because of the operator ordering problem. To get the general version of quantization, we let ν be a real parameter and introduce differential operators on V:

$$\acute{S}_{uv}(\nu) := \hat{S}_{uv} - \frac{\nu}{2}\mathrm{tr}\,(uv), \quad \acute{X}_u(\nu) := \hat{X}_u + i\nu\mathrm{tr}\,(u\partial\!\!\!/), \quad \acute{Y}_v(\nu) := \hat{Y}_v. \tag{3.5}$$

and differential operators on V^*:

$$\grave{S}_{uv}(\nu) := \check{S}_{uv} - \frac{\nu^*}{2}\mathrm{tr}\,(uv), \quad \grave{X}_u(\nu) := \check{X}_u - \nu^*\mathrm{tr}\,(u\pi), \quad \grave{Y}_v(\nu) := \check{Y}_v. \tag{3.6}$$

where $\nu^* = \nu - \frac{2n}{\rho}$.

Theorem 2. *The TKK commutation relations (1.4) still hold when all O there are replaced by either their acute version or their grave version.*

Proof. When $\nu = 0$, the proof is essentially the same as the proof of Theorem 1, so we skip it. For the general case, we shall verify the acute version and leave the grave version to the readers.

Verify that $[\acute{Y}_u(\nu), \acute{Y}_v(\nu)] = 0$:

$$[\acute{Y}_u(\nu), \acute{Y}_v(\nu)] = [\hat{Y}_u, \hat{Y}_v] = 0.$$

Verify that $[\acute{S}_{uv}(\nu), \acute{Y}_z(\nu)] = -\acute{Y}_{\{vuz\}}(\nu)$:

$$[\acute{S}_{uv}(\nu), \acute{Y}_z(\nu)] = [\hat{S}_{uv}, \hat{Y}_z] = -\hat{Y}_{\{vuz\}} = -\acute{Y}_{\{vuz\}}(\nu).$$

Verify that $[\acute{S}_{uv}, \acute{S}_{zw}] = \acute{S}_{\{uvz\}w} - \acute{S}_{z\{vuw\}}$: Since $[\hat{S}_{uv}, \hat{S}_{zw}] = \hat{S}_{\{uvz\}w} - \hat{S}_{z\{vuw\}}$, all we need to check is that

$$\mathrm{tr}\,(\{uvz\}w) = \mathrm{tr}\,(z\{vuw\}), \quad \text{i.e.,} \quad \langle S_{uv}(z) \mid w\rangle = \langle z \mid S_{vu}(w)\rangle$$

which is true because $S'_{uv} = S_{vu}$.

Verify that $[\acute{X}_u(\nu), \acute{Y}_v] = -2\acute{S}_{uv}$:

$$\begin{aligned} [\acute{X}_u(\nu), \acute{Y}_v(\nu)] &= [\hat{X}_u + i\nu\mathrm{tr}\,(u\partial\!\!\!/), \hat{Y}_v] \\ &= -2\hat{S}_{uv} + \nu\mathrm{tr}\,(uv) \\ &= -2\acute{S}_{uv}(\nu). \end{aligned}$$

Verify that $[\acute{S}_{uv}(\nu), \acute{X}_z(\nu)] = \acute{X}_{\{uvz\}}(\nu)$:

$$\begin{aligned} [\acute{S}_{uv}(\nu), \acute{X}_z(\nu)] &= [\hat{S}_{uv}(\nu), \hat{X}_z + i\nu\mathrm{tr}\,(z\partial\!\!\!/)] \\ &= \hat{X}_{\{uvz\}} + i\rho\nu[\hat{S}_{uv}, \langle z \mid \partial\!\!\!/\rangle] \end{aligned}$$

$$\begin{array}{rcl} & = & \hat X_{\{uvz\}}+i\rho\nu\langle S_{uv}(z)\mid \partial\!\!\!/\rangle\\ & = & \acute X_{\{uvz\}}(\nu).\end{array}$$

Verify that $[\acute X_u(\nu), \acute X_v(\nu)]=0$:

$$\begin{array}{rcl}[\acute X_u(\nu), \acute X_v(\nu)] & = & [\hat X_u+i\rho\nu\langle u\mid \partial\!\!\!/\rangle, \hat X_v++i\rho\nu\langle v\mid \partial\!\!\!/\rangle]\\ & = & i\rho\nu[\hat X_u, \langle v\mid \partial\!\!\!/\rangle] - < u\leftrightarrow v >\\ & = & -\rho\nu[\langle x\mid \{\partial\!\!\!/ u\partial\!\!\!/\}\rangle, \langle v\mid \partial\!\!\!/\rangle] - < u\leftrightarrow v >\\ & = & \rho\nu\langle v\mid \{\partial\!\!\!/ u\partial\!\!\!/\}\rangle - < u\leftrightarrow v >\\ & = & \rho\nu\left(2\langle v\partial\!\!\!/\mid u\partial\!\!\!/\}\rangle-\langle uv\mid \partial\!\!\!/^2\rangle\right) - < u\leftrightarrow v >\\ & = & 0.\end{array}$$

□

In the remainder of this paper, let us focus the attention on the operator realization on V: $O\to \acute O(\nu)$ for a fixed ν. Note that this operator realization provides a linear action of the TKK algebra on $C^\infty(V)$. We shall investigate the unitarity of this action in the next section. In order to do that, let us make some preparations here.

Let $\mathcal P(V)$ be the algebra of $\mathbb C$-valued polynomial functions on V, and $\mathcal P_I(V)$ be the vector subspace consisting of polynomials of degree at most I. Let

$$\acute D(V)=e^{-r}\mathcal P(V),\quad \acute D_I(V)=e^{-r}\mathcal P_I(V). \tag{3.7}$$

It is clear that the action of $\mathfrak{co}$ on $C^\infty(V)$, which maps O to $\acute O(\nu)$, leaves $\acute D(V)$ invariant. Let us denote by π_ν this action on $\acute D(V)$.

Recall that ν is a real parameter and $\mathfrak u$ is the maximal compact Lie subalgebra of $\mathfrak{co}$.

Theorem 3. *Let $H_e=i(X_e+Y_e)$, $\pi_\nu|_{\mathfrak u}$ be the restriction of π_ν from $\mathfrak{co}$ to $\mathfrak u$.*

i) $\pi_\nu|_{\mathfrak u}$ leaves $\acute D_I(V)$ invariant and commutes with the inclusion of $\acute D_{I-1}(V)$ into $\acute D_I(V)$, consequently it induces a linear action on $\acute D_I(V)/\acute D_{I-1}(V)$.

ii) As a representation of $\mathfrak u$, $\acute D_I(V)/\acute D_{I-1}(V)$ is isomorphic to

$$\mathcal P_I(V):=\bigoplus_{\mathbf m\geq 0, |\mathbf m|=I}^{m_{\rho(\nu)+1}=0}\xi_\nu\otimes\mathcal P_{\mathbf m}(V)$$

under the map sending $p\in\mathcal P_I(V)$ into $e^{-r}p+\acute D_{I-1}(V)\in \acute D_I(V)/\acute D_{I-1}(V)$.

iii) The induced linear map

$$\acute H_e:\quad \acute D_I(V)/\acute D_{I-1}(V)\to \acute D_I(V)/\acute D_{I-1}(V)$$

is the scalar multiplication by $(2I+\nu\rho)$.

iv) π_ν is unitarizable $\implies$ $\nu\geq 0$.

v) e^{-r} is a lowest weight state with weight $\nu\lambda_0$.

vi) π_ν is indecomposable.

Proof. i) Since

$$e^r\frac{-i}{2}(\acute X_u+\acute Y_u)e^{-r}=\frac12(\langle x\mid \{\partial\!\!\!/ u\partial\!\!\!/\}\rangle+\nu\mathrm{tr}\,(u\partial\!\!\!/))+\hat L_u-\frac{\nu}{2}\mathrm{tr}\,u, \tag{3.8}$$

$(\acute X_u+\acute Y_u)$ maps $\acute D_I$ into $\acute D_I$. It is also clear that $[\acute L_u, \acute L_v]$ maps $\acute D_I$ into $\acute D_I$. Therefore, in view of the fact that

$$\mathfrak u=\mathrm{span}_{\mathbb R}\{[L_u, L_v], X_w+Y_w\mid u,v,w\in V\},$$

the linear action of $\mathfrak u$ on $\acute D$ leaves $\acute D_I$ invariant. The rest is clear.

ii) That is clear from Eq. (3.8) and the last paragraph of Subsection 2.4.

iii) For any homogeneous degree I polynomial p, we have

$$\begin{aligned} i(\acute{X}_e+\acute{Y}_e)e^{-r}p &\equiv e^{-r}(-2\hat{L}_e+\nu\rho)p \mod \acute{D}_{I-1}\\ &\equiv (2I+\nu\rho)e^{-r}p \mod \acute{D}_{I-1}. \end{aligned}$$

iv) Let $E_\pm := \frac{i}{2}(X_e-Y_e)\mp S_{ee}$. Then

$$[H_e, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = -H_e.$$

Suppose that π_ν is unitarizable, and $(\,,)$ is the inner product on $\acute{D}$. Let $\boxed{\psi_0 = e^{-r}}$. Since $||\pi_\nu(E_+)\psi_0||^2 \geq 0$ and $\pi_\nu(E_-)\psi_0 = 0$, using relation $[E_+, E_-] = -H_e$, we arrive at $(\psi_0, \pi_\nu(H_e)\psi_0) \geq 0$, i.e., $\nu\rho||\psi_0||^2 \geq 0$. So $\nu \geq 0$.

v) Let us take the simple root system $\alpha_0, \ldots, \alpha_r$ specified in Lemma 2.1. Since $\acute{D}_0$ ($=\mathrm{span}_{\mathbb{C}}\{\psi_0\}$) is one dimensional and $\bar{\mathfrak{u}}$ is semi-simple, the action of $\bar{\mathfrak{u}}^{\mathbb{C}}$ on $\acute{D}_0$ must be trivial. Therefore, for $i \geq 1$, in view of the fact that $E_{\pm\alpha_i}, H_{\alpha_i} \in \bar{\mathfrak{u}}^{\mathbb{C}}$, we have

$$\pi_\nu(E_{-\alpha_i})\psi_0 = 0, \quad \pi_\nu(H_{\alpha_i})\psi_0 = 0. \tag{3.9}$$

On the other hand, since $E_{-\alpha_0} = \frac{i}{2}(X_{e_{11}} - Y_{e_{11}}) + L_{e_{11}}$ and $H_{\alpha_0} = i(X_{e_{11}} + Y_{e_{11}})$, by a computation, we have

$$\begin{aligned} e^r \acute{E}_{-\alpha_0} e^{-r} &= -\frac{1}{2}\left(\langle x \mid \{\partial\!\!\!/ e_{11}\partial\!\!\!/\}\rangle + \nu \mathrm{tr}\,(e_{11}\partial\!\!\!/)\right),\\ e^r \acute{H}_{\alpha_0} e^{-r} &= -\langle x \mid \{\partial\!\!\!/ e_{11}\partial\!\!\!/\}\rangle - \nu\mathrm{tr}\,(e_{11}\partial\!\!\!/) - 2\hat{L}_{e_{11}} + \nu, \end{aligned}$$

so it is easy to see that

$$\pi_\nu(E_{-\alpha_0})\psi_0 = 0, \quad \pi_\nu(H_{\alpha_0})\psi_0 = \nu\psi_0. \tag{3.10}$$

Therefore, in view of the fact that $\alpha_0(H_{\alpha_0}) = 2$, Eqns (3.9) and (3.10) imply that ψ_0 is a lowest weight state with weight $\nu\lambda_0$.

vi) In view of the fact that operator $\acute{Y}_v$ is the multiplication by $-i\langle v \mid x\rangle$, this is obvious:

$$e^{-r}\sum_{i_1,\ldots,i_n} \alpha_{i_1\cdots i_n} x_1^{i_1}\cdots x_n^{i_n} = \left(\sum_{i_1,\ldots,i_n} \alpha_{i_1\cdots i_n} (i\acute{Y}_{e_1})^{i_1}\cdots(i\acute{Y}_{e_n})^{i_n}\right)\psi_0.$$

□

We shall show in the next section that π_ν is irreducible when $\nu > (\rho-1)\frac{\delta}{2}$ and is not irreducible when $\nu = k\frac{\delta}{2}$, $k = 0, 1, \ldots, (\rho-1)$. The collection of theses values of ν, denoted by $\mathcal{W}(V)$, is called the **Wallach set** for V. So

$$\mathcal{W}(V) = \left\{k\frac{\delta}{2} \mid k = 0, 1, \ldots, \rho-1\right\} \cup \left((\rho-1)\frac{\delta}{2}, \infty\right).$$

It is known from Ref. [11] that the set of scalar-type unitary lowest weight representation of Co is isomorphic to $\mathcal{W}(V)$. We shall show in the next section that these representations are precisely the irreducible quotient of these π_ν, also denoted by π_ν.

As mentioned in the introduction, the operator realizations as given in Eq. (3.5) are not unitary with respect to the obvious L^2-inner product

$$(\psi_1, \psi_2) = \int_V \bar{\psi}_1\psi_2\, dm,$$

where dm is the Lebesgue measure. The right inner product must be found in order to have unitary realizations. For that, let us move on to the next section.

4. QUANTIZATIONS OF TKK ALGEBRAS

The goal of this section is to investigate the unitarity of representation π_ν obtained in the previous section. The question is to find a positive hermitian form $(,)_\nu$ on $\acute{D}(V)$ with respect to which operators $\acute{O}(\nu)$ are all anti-hermitian. More generally, $(,)_\nu$ can be semi-positive because then π_ν descends to a unitary representation by formally setting the "spurious states" (i.e., elements of $\acute{D}(V)$ with zero norm) as zero. It is a fact from Ref. [4] that such a $(,)_\nu$ does exist for ν in the Wallach set. Our purpose here is to present a new route towards this fact along with its refinements.

The case $\nu = \frac{\delta}{2}$ is already known form our work in Ref. [8]. Recall from Ref. [8], the J-Kepler problem for V is a dynamic problem on $\mathscr{P}$ — the submanifold of V consisting of semi-positive elements of rank one. By comparing Remark 8 and Proposition 8.1 from Ref. [8] with Theorem 2 here, we have

$$(\psi_1, \psi_2)_{\frac{\delta}{2}} = \int_{\mathscr{P}} \bar{\psi}_1\, \psi_2\, r^{-1-(\rho/2-1)d}\, \mathrm{vol}. \tag{4.1}$$

Here, vol is the volume form for the **Kepler metric**

$$ds_K^2 := \frac{2}{\rho} ds_E^2 - dr^2. \tag{4.2}$$

It is clear that, as a vector subspace of $\acute{D}(V)$, the space of "spurious states" consists of elements of $\acute{D}(V)$ which vanish on $\mathscr{P}$.

With this result in mind, it is not hard to imagine what the general picture should be: replacing $\mathscr{P}$ by the submanifold of V consisting of semi-positive elements of a fixed positive rank. Of course, some technical hurdles must be overcome. The initial hurdle is the generalization of the Kepler metric in Eq. (4.2). The second hurdle is the generalization of the extra factor $r^{-1-(\rho/2-1)d}$ in measure

$$d\mu_{\frac{\delta}{2}} := r^{-1-(\rho/2-1)d}\, \mathrm{vol}.$$

It turns out, the second hurdle simply disappears by itself as we walk along a natural path towards quantizations of TKK algebras. The clue for removing the first hurdle comes from the study of the universal Kepler problem in Ref. [9], as we shall sketch below.

We have noted in the past that the total tangent space of a *Riemannian* manifold is a symplectic manifold, and if N is a submanifold of M, then TN is a symplectic submanifold of TM. With this understood, we remarked in Ref. [9] that, by restricting the classical universal hamiltonian

$$\mathcal{H} = \frac{1}{2}\frac{\langle x \mid \pi^2\rangle}{r} - \frac{1}{r}$$

from TV to $T\mathscr{P}$, one obtains the classical hamiltonian for the J-Kepler problem. Since the first term in $\mathcal{H}$ should be identified with the kinetic energy, we must have the following new formula for the Kepler metric:

$$(\pi, \pi)_{ds_K^2} = \frac{\langle x \mid \pi^2\rangle}{r} = \frac{\langle \pi \mid L_x \mid \pi\rangle}{r}, \tag{4.3}$$

a fact which can be verified directly. Now it becomes clear how to generalize the Kepler metric.

4.1. **Canonical cones.** We say that an element $x \in V$ is *semi-positive* if $x = y^2$ for some $y \in V$. Let us denote by $\mathcal{Q}$ the space of semi-positive elements in V and recall that Str is the structure group of V. Then one can check that the action of Str on V leaves $\mathcal{Q}$ invariant, so we have a partition of $\mathcal{Q}$ into the disjoint union of Str-orbits:

$$\mathcal{Q} = \cup_{k=0}^{\rho} \mathcal{C}_k.$$

Here, homogeneous space $\mathcal{C}_k$ is the space of semi-positive elements of rank k. Note that, $\mathcal{C}_0 = \{0\}$, $\mathcal{C}_\rho$ is the symmetric cone Ω of V, and $\mathcal{Q}$ is the topological closure of Ω.

As a sub-manifold of the Euclidean space V, $\mathcal{C}_k$ has an induced Riemannian metric. When we say that $T\mathcal{C}_k$ is a symplectic manifold, it is this Riemannian metric that is used to identify $T\mathcal{C}_k$ with $T^*\mathcal{C}_k$. However, the Riemannian metric for the Kepler-type dynamics on $\mathcal{C}_k$ is a different one, which we shall describe below.

For any $u \in V$, since $L_u\colon V \to V$ is self-adjoint, we have an orthogonal decomposition $V = \mathrm{Im} L_u \oplus \ker L_u$, with respect to which, L_u decomposes as $L_u = \bar{L}_u \oplus 0$. Since $\bar{L}_u$ is invertible, we can introduce endomorphism

$$\frac{1}{L_u} \stackrel{\text{def}}{=} \bar{L}_u^{-1} \oplus 0 \tag{4.4}$$

on V. For any $x \in \mathcal{C}_k$, one can check that the tangent space $T_x\mathcal{C}_k$ is $\{x\} \times \mathrm{Im} L_x$, the normal space $N_x\mathcal{C}_k$ is $\{x\} \times \ker L_x$.

Definition 1 (Canonical Metric)**.** *The* **canonical metric** *on $\mathcal{C}_k$, denoted by ds_K^2, is defined as follows:*

$$\begin{array}{rcl} T_x\mathcal{C}_k \times T_x\mathcal{C}_k & \to & \mathbb{R} \\ ((x,u),(x,v)) & \mapsto & r\langle u \mid \dfrac{1}{L_x} \mid v\rangle = r\langle u \mid \bar{L}_x^{-1} \mid v\rangle. \end{array} \tag{4.5}$$

One can check that, on $\mathcal{C}_1$, the canonical metric is the Kepler metric introduced in Ref. [8]. On the symmetric cone Ω, the canonical metric is

$$\begin{array}{rcl} T_x\Omega \times T_x\Omega & \to & \mathbb{R} \\ ((x,u),(x,v)) & \mapsto & r\langle u \mid {L_x}^{-1} \mid v\rangle. \end{array} \tag{4.6}$$

Definition 2 (Canonical Cone)**.** *Let V be a simple Euclidean Jordan algebra of rank ρ, $\mathcal{C}_k$ be its submanifold consisting of the semi-positive elements of rank k, $1 \le k \le \rho$. The V's* **canonical cone** *of rank k is defined to be the Riemannian manifold $(\mathcal{C}_k, ds_K^2)$, where ds_K^2 is the canonical metric in Definition 1 .*

We remark that, since the action of structure group on V leaves $\mathcal{C}_k$ invariant, for any $u, v \in V$ and $x \in \mathcal{C}_k$, $\hat{S}_{uv}|_x \in T_x\mathcal{C}_k$; i.e., $\hat{S}_{uv}$ descends to a vector field on $\mathcal{C}_k$, which shall still be denoted by $\hat{S}_{uv}$. Since $L_u = S_{ue}$, we write $\hat{L}_u$ for $\hat{S}_{ue}$. In the remainder of this paper, we shall use Δ (vol resp.) denote the Laplace operator (the volume form resp.) on a Riemannian manifold, e.g., a canonical cone.

4.2. **Basic facts on canonical cones.** We use Tr to denote the trace for endomorphisms on V. For any x in the canonical cone, we use $P_x\colon V \to V$ to denote the orthogonal projection onto Im L_x. Throughout this subsection, we focus our attention on a fixed canonical cone $\mathcal{C}$ of rank k.

Let us start with a local analysis of the canonical metric around a point $x_0 \in \mathcal{C}$. Choose a Jordan frame $\{e_{ii}\}_{i=1}^{\rho}$ such that

$$x_0 = \sum_{i=1}^{k} \lambda_i e_{ii}$$

for some positive numbers $\lambda_1, \ldots, \lambda_k$. With this Jordan frame $\{e_{ii}\}_{i=1}^{\rho}$ fixed, we let $N := \bigoplus_{j\geq i\geq k} V_{ij}$ and T be the orthogonal complement of N in (euclidean vector space) V. Then $T_{x_0}\mathcal{C} = \{x_0\}\times T$.

For any $x\in V$, one can uniquely decompose $x = x_0 + t + y$ with $t\in T$ and $y\in N$. Now if we assume that x is always in $\mathcal{C}$, then

$$y = O(|t|^2) \quad \text{near } x_0.$$

Choose basis $\{e_i\}$ for T which is orthonormal with respect to the inner product on V. Write $t = t^i e_i$ and the canonical metric ds_K^2 as $h_{ij}dt^i\,dt^j$, then

$$h_{ij}(x) = \langle e \mid x\rangle\langle x_i \mid \bar{L}_x^{-1} \mid x_j\rangle,$$

where $x_i = \frac{\partial x}{\partial t^i}$. Let $g_{ij} := \langle x_i \mid x_j\rangle$, then

$$g_{ij}(x) = \delta_{ij} + O(|t|^2) \quad \text{near } x_0.$$

As usual, the inverse of $[g_{ij}]$ is denoted by $[g^{ij}]$ and the inverse of $[h_{ij}]$ is denoted by $[h^{ij}]$. It is easy to see that

$$h^{ij}(x) = \frac{g^{ik}(x)\langle x_k \mid L_x \mid x_l\rangle g^{lj}(x)}{\langle e \mid x\rangle}.$$

Consequently, when $x\in\mathcal{C}$ is near x_0, we have

$$\begin{aligned} h^{ij}(x) &= \frac{\langle x_i \mid L_x \mid x_j\rangle}{\langle e \mid x\rangle} + O(|t|^2) \\ &= \frac{\langle e_i \mid x_0 e_j\rangle}{\langle e \mid x_0\rangle}\left(1 - \frac{\langle e \mid t\rangle}{\langle e \mid x_0\rangle}\right) + \frac{\langle e_i \mid t e_j\rangle}{\langle e \mid x_0\rangle} + O(|t|^2). \end{aligned} \tag{4.7}$$

Proposition 4.1. *Fix a canonical cone. Let* vol *be its volume form, and* $\mathscr{L}_u$ *the Lie derivative with respect to vector field* $\hat{L}_u$. *Then*

$$\mathscr{L}_u(\frac{1}{r}\text{vol}) = -2\lambda_u\frac{1}{r}\text{vol}, \tag{4.8}$$

where

$$\boxed{2\lambda_u = -\tfrac{1}{2}Tr(\tfrac{1}{L_x}L_{ux}) + Tr(P_xL_u) + \tfrac{\langle u|x\rangle}{\langle e|x\rangle}\left(\tfrac{TrP_x}{2}-1\right).} \tag{4.9}$$

Consequently, λ_u *depends on* u *linearly and*

$$\hat{L}_u(\lambda_v) = \hat{L}_v(\lambda_u), \quad \hat{S}_{uv}(\lambda_{zw}) - \hat{S}_{zw}(\lambda_{uv}) = \lambda_{\{uvz\}w} - \lambda_{z\{vuw\}}. \tag{4.10}$$

Proof. We just need to show that

$$\mathscr{L}_u(\text{vol}) = -2\tilde{\lambda}_u\text{vol}$$

at a point x_0, where

$$-2\tilde{\lambda}_u = \frac{1}{2}\text{Tr}\left(-\frac{\langle u \mid x\rangle}{\langle e \mid x\rangle}P_x + \frac{1}{L_x}L_{xu}\right) - \text{Tr}(P_xL_u). \tag{4.11}$$

Recall that $ds_K^2 = h_{ij}\,dt^i\,dt^j$. Since $\text{vol} = \sqrt{h}\,dt^1\wedge dt^2\wedge\cdots$ where $h = \det[h_{ij}]$, we have

$$\mathscr{L}_u(\text{vol}) = \hat{L}_u(\ln\sqrt{h})\text{vol} + \sum_i \sqrt{h}\,dt^1\wedge\cdots\wedge\mathscr{L}_u(dt^i)\wedge\cdots.$$

Since $\mathscr{L}_u(dt^i) = d\mathscr{L}_u(\langle e_i \mid x\rangle) = -d(\langle xe_i \mid u\rangle)$, we have

$$\mathscr{L}_u(dt^i) = -\langle e_i^2 \mid u\rangle dt^i \quad \text{at } x_0,$$

consequently,

$$-2\tilde\lambda_u=\hat L_u(\ln\sqrt{h})-\langle\sum_i e_i^2\mid u\rangle\quad\text{at } x_0. \tag{4.12}$$

In view of Eq. (4.7), for x near x_0,

$$h=\det[\frac{\langle e_i\mid x_0e_j\rangle}{\langle e\mid x_0\rangle}]^{-1}\left(1+\mathrm{Tr}\left(\frac{\langle e\mid t\rangle}{\langle e\mid x_0\rangle}P_{x_0}-\frac{1}{L_{x_0}}L_t\right)\right)+O(|t|^2),$$

so

$$\hat L_u(\ln\sqrt{h})|_{x_0}=\frac{1}{2}\mathrm{Tr}\left(-\frac{\langle u\mid x_0\rangle}{\langle e\mid x_0\rangle}P_{x_0}+\frac{1}{L_{x_0}}L_{x_0u}\right). \tag{4.13}$$

Since

$$\frac{1}{\rho}\sum_i e_i^2=(1+\frac{\delta(\rho-k-1)}{2})\sum_{j=1}^k e_{jj}+\frac{\delta k}{2}e,$$

where k is the rank of the canonical cone, we have

$$\begin{aligned}\langle\sum_i e_i^2\mid u\rangle &= (1+\frac{\delta(\rho-k-1)}{2})\sum_{j=1}^k u_{jj}+\frac{\delta k}{2}\mathrm{tr}\,u\\ &= \mathrm{Tr}(P_{x_0}L_u).\end{aligned} \tag{4.14}$$

Plugging Eqs. (4.13) and (4.14) into Eq. (4.12), we arrive at Eq. (4.11).

Since $[\mathscr{L}_{\hat L_u},\mathscr{L}_{\hat L_v}]=\mathscr{L}_{[\hat L_u,\hat L_v]}$, in view of fact that the Kepler metric is invariant under the action of the $\mathrm{Aut}(J)$, we have $[\mathscr{L}_{\hat L_u},\mathscr{L}_{\hat L_v}](\frac{1}{r}\mathrm{vol})=0$. Then

$$-2\left(\hat L_u(\lambda_v)-\hat L_v(\lambda_u)\right)\cdot\frac{1}{r}\mathrm{vol}=0,$$

consequently $\hat L_u(\lambda_v)=\hat L_v(\lambda_u)$.

Since $[\mathscr{L}_{\hat S_{uv}},\mathscr{L}_{\hat S_{zw}}]=\mathscr{L}_{[\hat S_{uv},\hat S_{zw}]}=\mathscr{L}_{\hat S_{\{uvz\}w}-\hat S_{z\{vuw\}}}$, acting on $\frac{1}{r}\mathrm{vol}$, we have

$$\hat S_{uv}(\lambda_{zw})-\hat S_{zw}(\lambda_{uv})=\lambda_{\{uvz\}w}-\lambda_{z\{vuw\}}.$$

□

In the remainder of this paper, we let

$$\boxed{\tilde S_{uv}=\hat S_{uv}-\lambda_{uv},\quad \tilde L_u=\tilde S_{ue}.} \tag{4.15}$$

Proposition 4.2. *Fix a canonical cone and let Δ be its Laplace operator. Then*

$$[r\Delta,\langle u\mid x\rangle]=-2\tilde L_u,\quad u\in V. \tag{4.16}$$

Proof. Upon observing that $[r\Delta,\langle u\mid x\rangle]$ is a linear differential operator, it suffices to verify that

$$[[r\Delta,\langle u\mid x\rangle],\langle v\mid x\rangle](1)=[-2\tilde L_u,\langle v\mid x\rangle](1),\quad [r\Delta,\langle u\mid x\rangle](1)=-2\tilde L_u(1),$$

i.e.,

$$[[r\Delta,\langle u\mid x\rangle],\langle v\mid x\rangle](1)=2\langle uv\mid x\rangle,\quad r\Delta(\langle u\mid x\rangle)=2\lambda_u. \tag{4.17}$$

In view of the fact that $*\Delta f=d*df$, $[[*\Delta,\langle u\mid x\rangle],\langle v\mid x\rangle](1)$ is equal to

$$\begin{aligned}& d*d(\langle u\mid x\rangle\langle v\mid x\rangle)-\langle u\mid x\rangle d*d(\langle v\mid x\rangle)-\langle v\mid x\rangle d*d(\langle u\mid x\rangle)\\ =\;& d(\langle u\mid x\rangle)\wedge *d(\langle v\mid x\rangle)+d\langle v\mid x\rangle\wedge *d(\langle u\mid x\rangle)\\ =\;& 2\langle u\mid x_i\rangle h^{ij}\langle v\mid x_j\rangle\ \mathrm{vol},\end{aligned}$$

so

$$[[r\Delta, \langle u \mid x\rangle], \langle v \mid x\rangle](1)|_{x_0} = 2\langle u \mid e_i\rangle\langle e_i \mid L_{x_0}|e_j\rangle\langle v \mid e_j\rangle = 2\langle uv \mid x_0\rangle.$$

The first identity of Eq. (4.17) is verified.

In view of the fact that $*\Delta(f) = d\left(\sum_{i,j} h^{ij}\partial_i f \iota_{\partial_j}(\mathrm{vol})\right)$, we have

$$\begin{aligned}
*\, r\Delta(\langle u \mid x\rangle)|_{x_0} &= rd\left(\sum_{i,j} h^{ij}\langle u \mid x_i\rangle \iota_{\partial_j}(\mathrm{vol})\right)\Bigg|_{x_0} \\
&= -\, rd\left(\frac{1}{r}\iota_{\hat{L}_u}(\mathrm{vol})\right)\Bigg|_{x_0} \\
&= -\mathscr{L}_u(\mathrm{vol})|_{x_0} - \frac{\langle e \mid ux\rangle}{r}\mathrm{vol}|x_0 \\
&= -r\mathscr{L}_u(\frac{1}{r}\mathrm{vol})|_{x_0} \\
&\overset{Eq.(4.8)}{=} 2\lambda_u \mathrm{vol}|_{x_0}.
\end{aligned}$$

The second identity of Eq. (4.17) is verified. □

As we have demonstrated in Ref. [8], to check a commutation relation on the Kepler cone, it is easier to check the corresponding one on V. For this reason, we wish to lift $r\Delta$ to a second order differential operator on V with rational functions as its coefficients. In order to do that, we first need to lift λ_u to a rational function on V. Let $c_k(x)$ ($\tau_k(x)$ resp.) be the polynomial in $\mathrm{tr}\, x, \mathrm{tr}\, x^2, \ldots, \mathrm{tr}\, x^k$ such that, if $x = \sum_{i=1}^k \lambda_i e_{ii}$, then

$$c_k(x) = \prod_{i=1}^k \lambda_i \quad \left(\tau_k(x) = \prod_{1\leq i<j\leq k} (\lambda_i + \lambda_j) \quad \text{resp.}\right).$$

For example, $c_1(x) = \mathrm{tr}\, x$, $\tau_1(x) = 1$, $c_2(x) = \frac{1}{2}((\mathrm{tr}\, x)^2 - \mathrm{tr}\, x^2)$ and $\tau_2(x) = \mathrm{tr}\, x$. Let $D_k = k[1 + (\rho - \frac{k+1}{2})\delta]$ — the dimension of the canonical cone of rank k, and

$$\boxed{\varphi_k = \tau_k^\delta \cdot c_k^{\delta-1} \cdot r^{2-D_k}.} \tag{4.18}$$

For example, $D_1 = 1 + (\rho - 1)\delta$, and $\varphi_1 = r^{-\delta(\rho-2)}$ up to a multiplicative constant. Note that φ_k is a rational function on V and is positive on the canonical cone of rank k. From here one, we shall call φ_k the **phi-function** on the canonical cone of rank k.

With an orthonormal basis $\{e_\alpha\}$ for V chosen, we write x as $x^\alpha e_\alpha$, $\frac{\partial}{\partial x^\alpha}$ as ∂_α. Recall that $\not{\partial} = \sum_\alpha e_\alpha \partial_\alpha$.

Proposition 4.3. *Fix a canonical cone of rank k. Let φ be its phi-function, Δ its Laplace operator. Then*

i) λ_u can be lifted to a rational function on V:

$$\boxed{4\lambda_u = \hat{L}_u \ln\varphi + \delta k \mathrm{tr}\, u\ .} \tag{4.19}$$

ii) $r\Delta$ can be lifted to a second order differential operator on V with rational function coefficients:

$$\boxed{r\Delta = \langle x \mid \not{\partial}^2\rangle + 2\textstyle\sum_\alpha \lambda_{e_\alpha}\partial_\alpha\ .} \tag{4.20}$$

Proof. i) We just need to prove the identity at a point x_0 on the canonical cone. Choose a Jordan frame $\{e_{ii}\}$ such that $x_0 = \sum_{i=1}^k \lambda_i e_{ii}$ for some numbers $\lambda_1, \ldots, \lambda_k$. Then we extend $\{\sqrt{\rho} e_{ii}\}_{i=1}^\rho$ to an orthonomal basis $\{e_\alpha\}$ such that $e_i = \sqrt{\rho} e_{ii}$, $1 \le i \le k$, and $\{e_i\}_{i=1}^{D_k}$ is an basis of $\mathrm{Im} L_{x_0}$.

In view of that fact that $x^k x^l = x^{k+l}$ and $\mathrm{tr}\,(x_0^k e_j) = 0$ for $j > k$, by induction on m, we have

$$\hat{L}_u(\mathrm{tr}\, x^m)|_{x_0} = -\sum_{i=1}^k \langle u x_0 \mid e_i\rangle \partial_i (\mathrm{tr}\, x^m)|_{x_0},$$

consequently

$$\hat{L}_u \varphi|_{x_0} = -\sum_{i=1}^k \langle u x_0 \mid e_i\rangle \partial_i \varphi|_{x_0}. \tag{4.21}$$

The rest of the proof is just a straightforward computation based on identity (4.21), so we leave it to readers.

ii) Since both sides are differential operators without the zero-th order term, in view of identity (4.16), we just need to show that

$$[\langle x \mid \not{\partial}^2\rangle + 2\sum_\alpha \lambda_{e_\alpha} \partial_\alpha, \langle u \mid x\rangle] = -2\tilde{L}_u,$$

something that can be easily verified. □

For $\nu \in \mathcal{W}(V) \setminus \{0\}$, we introduce integer

$$\rho(\nu) = \begin{cases} k & \text{if } \nu = k\frac{\delta}{2}, \\ \\ \rho & \text{if } \nu > (\rho-1)\frac{\delta}{2}. \end{cases} \tag{4.22}$$

and rational function

$$\varphi(\nu) := \begin{cases} \varphi_k & \text{if } \nu = k\frac{\delta}{2}\ , \\ \\ \varphi_\rho\ \det(x)^{2\nu-\rho\delta} & \text{if } \nu > (\rho-1)\frac{\delta}{2} \end{cases} \tag{4.23}$$

on V. Note that $\varphi(\nu)$ is always positive on the canonical cone of rank $\rho(\nu)$.

For canonical cone $\mathcal{C}$, we let

$$\acute{D}(\mathcal{C}) := \{\psi:\ \mathcal{C} \to \mathbb{C} \mid \psi \in \acute{D}(V)\}, \quad \acute{D}_I(\mathcal{C}) := \{\psi:\ \mathcal{C} \to \mathbb{C} \mid \psi \in \acute{D}_I(V)\}.$$

Proposition 4.4. *Let $\nu \in \mathcal{W}(V) \setminus \{0\}$ and $\mathcal{C}$ be the canonical cone of rank $\rho(\nu)$. Then $\acute{D}(\mathcal{C})$ is dense in $L^2(\mathcal{C}, \frac{\sqrt{\varphi(\nu)}}{r}\mathrm{vol})$.*

Proof. Let us write $d\mu_\nu$ for $\frac{\sqrt{\varphi(\nu)}}{r}\mathrm{vol}$. Let $C_c(\mathcal{C})$ be the set of compactly-supported continuous complex-valued functions and

$$M = \int_{\mathcal{C}} e^{-2r}\, d\mu_\nu.$$

It is clear that $M > 0$. By applying Theorem A.1 in appendix A, one can easily check that $M < \infty$.

Suppose that $f \in L^2(\mathcal{C}, d\mu)$ and $\epsilon > 0$. By Theorem 3.14 in Ref. [12], there is $g \in C_c(\mathcal{C})$ such that

$$||f - g||_{L^2} < \frac{\epsilon}{2}. \tag{4.24}$$

Since $e^r g\in C_c(\mathcal C)$, by the Stone-Weierstrass Theorem in Ref. [13], there is a polynomial p such that

$$|e^r g - p| < \frac{\epsilon}{2\sqrt{M}} \quad \text{on } \mathcal{C}, \tag{4.25}$$

so

$$\begin{aligned} ||g-e^{-r}p||_{L^2} &= \left(\int_{\mathcal C} |g-e^{-r}p|^2\, d\mu_\nu\right)^{\frac12} \\ &< \frac{\epsilon}{2}\left(\frac{1}{M}\int_{\mathcal C} e^{-2r}\, d\mu_\nu\right)^{\frac12} \quad \text{using Eq. (4.25)} \\ &= \frac{\epsilon}{2}. \end{aligned} \tag{4.26}$$

Combining Eqs (4.24) and (4.26), we have

$$||f-e^{-r}p||_{L^2} \le ||f-g||_{L^2} + ||g-e^{-r}p||_{L^2} < \epsilon.$$

$\square$

Let

$$U(\nu) := \begin{cases} \frac{r}{4}\left(\Delta(\ln\varphi(\nu)) + \frac14 |d\ln\varphi(\nu)|^2\right) & \text{if } \nu \le (\rho-1)\frac{\delta}{2} \\ \frac{r}{4}\left(\Delta(\ln\varphi_\rho) + \frac14|d\ln\varphi_\rho|^2\right) & \\ +\frac{\rho}{4}\left((\nu-\frac{n}{\rho})^2 - (\frac{\delta}{2}-1)^2\right)\operatorname{tr} x^{-1} & \text{if } \nu > (\rho-1)\frac{\delta}{2}. \end{cases} \tag{4.27}$$

Here, d and $|\ |$ denote the exterior derivative operator and the point-wise norm for differential one-form on $\mathcal C$ respectively, x^{-1} denotes the Jordan inverse of $x\in\Omega$. Note that $U(\nu)$ can be lifted to a rational function on V. Recall that $\tilde L_u = \hat L_u - \lambda_u$.

Proposition 4.5. *Let $\nu\in\mathcal W(V)\setminus\{0\}$ and $\mathcal C$ be the canonical cone of rank $\rho(\nu)$.*

i) As differential operator on $\mathcal C$,

$$\tilde L_u = \sqrt[4]{\varphi_{\rho(\nu)}}\,\acute{L}_u(\nu)\frac{1}{\sqrt[4]{\varphi_{\rho(\nu)}}}. \tag{4.28}$$

ii) As differential operator on $\mathcal C$,

$$r\Delta = \sqrt[4]{\varphi(\nu)}(-i\acute{X}_e(\nu))\frac{1}{\sqrt[4]{\varphi(\nu)}} + U(\nu). \tag{4.29}$$

This proposition says that $\tilde L_u$ and $r\Delta$ are not as hard as they might look. To prove this proposition, with the help of Proposition 4.3, one just needs to do some straightforward and relative short computations, so we skip the proof.

4.3. **The unitary realizations of TKK algebras on canonical cones.** Let $\nu\in\mathcal W(V)\setminus\{0\}$ and $\mathcal C$ a canonical cone of rank $\rho(\nu)$. Recall that $\varphi(\nu)$, a rational function introduced in Eq. (4.23), is always positive on $\mathcal C$. Upon recalling the definitions of $\acute D(\mathcal C)$ and $\acute D_I(\mathcal C)$ in the paragraph preceding to Proposition 4.4, in view of Proposition 4.5, we introduce

$$\tilde D(\mathcal C) = \sqrt[4]{\varphi(\nu)}\,\acute D(\mathcal C), \quad \tilde D_I(\mathcal C) = \sqrt[4]{\varphi(\nu)}\,\acute D_I(\mathcal C)$$

and differential operators with common domain $\tilde D(\mathcal C)$:

$$\tilde S_{uv}(\nu) = \sqrt[4]{\varphi(\nu)}\acute S_{uv}(\nu)\frac{1}{\sqrt[4]{\varphi(\nu)}},$$

$$\begin{aligned}\tilde X_u(\nu) &= \sqrt[4]{\varphi(\nu)}\acute X_u(\nu)\frac{1}{\sqrt[4]{\varphi(\nu)}},\\ \tilde Y_v(\nu) &= \sqrt[4]{\varphi(\nu)}\acute Y_v(\nu)\frac{1}{\sqrt[4]{\varphi(\nu)}}.\end{aligned}$$

Note that these differential operators on $\mathcal{C}$ can be lifted to differential operators on V.

Proposition 4.6. *Let $\nu \in \mathcal{W}\setminus\{0\}$ and $\mathcal{C}$ a canonical cone of rank $\rho(\nu)$.*

i) The TKK commutation relations (1.4) hold under the replacement of O by $\tilde O(\nu)$.

ii) $\tilde D(\mathcal{C})$ is a dense subset of $L^2\left(\mathcal{C}, \frac{1}{r}\mathrm{vol}\right)$.

iii) $\tilde S_{uv}(\nu)$, $\tilde X_u(\nu)$ and $\tilde Y_v(\nu)$ are anti-hermitian operators on $\tilde D(\mathcal{C})$ with respect to hermitian inner product

$$(\psi_1,\psi_2)=\int_{\mathcal{C}}\bar\psi_1\,\psi_2\,\frac{1}{r}\mathrm{vol}.$$

iv) Let $\tilde{\mathscr{D}}_I(\mathcal{C})$ be the orthogonal complement of $\tilde D_{I-1}(\mathcal{C})$ in $\tilde D_I(\mathcal{C})$, then, under the unitary $\mathfrak{u}$-action, we have the following orthogonal decomposition

$$\tilde D(\mathcal{C})=\bigoplus_{I=0}^{\infty}\tilde{\mathscr{D}}_I(\mathcal{C}). \tag{4.30}$$

Moreover, the finite dimensional vector space $\tilde{\mathscr{D}}_I(\mathcal{C})$ is the eigenspace of $\tilde H_e := i(\tilde X_e+\tilde Y_e)$ with eigenvalue $(2I+\nu\rho)$.

v) Assume that $\mathbf{m}\in\mathbb{Z}^\rho$ with $\mathbf{m}\geq 0$ and $m_{\rho(\nu)+1}=0$. For and only for such $\mathbf{m}$, we let $\tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C})$ be the orthogonal projection of $\sqrt[4]{\varphi(\nu)}e^{-r}\mathcal{P}_{\mathbf{m}}(V)$ onto $\tilde D_{|\mathbf{m}|}(\mathcal{C})$. Then, as unitary representations of $\mathfrak{u}$, we have isomorphism $\tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C})\cong\xi_\nu\otimes\mathcal{P}_{\mathbf{m}}(V)$ and orthogonal decomposition into irreducibles:

$$\tilde{\mathscr{D}}_I(\mathcal{C})=\bigoplus_{\mathbf{m}\geq 0,|\mathbf{m}|=I}^{m_{\rho(\nu)+1}=0}\tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C}). \tag{4.31}$$

Proof. i) This quickly follows from Theorem 2.

ii) This quickly follows from Proposition 4.4.

iii) We start the proof with the following two observations: 1) multiplication by a real-valued function is hermitian, herece $\tilde Y_v$ is anti-hermitian; 2) $r\Delta$ is hermitian, hence $\tilde X_e$ is anti-hermitian in view of part ii) of Proposition 4.5. Combining these observations with the commutation relations in part i), the proof follows quickly.

iv) The orthogonal decomposition follows from the following two facts: 1) the $\mathfrak{u}$-action is unitary, a fact from part iii) above, 2) the $\mathfrak{u}$-action commutes with the inclusion of $D_{I-1}(\mathcal{C})$ into $D_I(\mathcal{C})$, a fact implied by part i) of Theorem 3. The remaining part follows from the fact that $\tilde H_e$ is hermitian and part iii) of Theorem 3.

v) This follows from part ii) of Theorem 3. □

Remark 4.1. *In view of Proposition 4.6, the semi-positive hermitian form $(,)_\nu$ mentioned in the beginning paragraph of this section is*

$$(\psi_1,\psi_2)_\nu=\int_{\mathcal{C}}\bar\psi_1\,\psi_2\,\frac{\sqrt{\varphi(\nu)}}{r}\mathrm{vol} \tag{4.32}$$

and the space of "spurious states" consists of functions in $\acute D(V)$ which vanish on $\mathcal{C}$. One can check that Eq. (4.32) is a generalization of Eq. (4.1) and $\acute D(\mathcal{C})$ is the quotient of $\acute D(V)$ by the the space of "spurious states". Therefore, the measure $d\mu_\nu$ mentioned in Subsection

1.5 is equal to $\frac{\sqrt{\varphi(\nu)}}{r}\mathrm{vol}$. *When* $\nu > (\rho-1)\frac{\delta}{2}$*, up to a multiplicative constant, this explicit formula for* $d\mu_\nu$ *agrees with the one retrieved from the bottom line of page 271 in Ref.* [4]. *Of course, our explicit formula works even if* ν *takes a discrete value* $k\frac{\delta}{2}$, $1 \le k < \rho$.

Denote by $\tilde{\mathrm{U}}(V)$ (or simply $\tilde{\mathrm{U}}$) the simply connected Lie group whose Lie algebra is $\mathfrak{u}(V)$. Recall that $\mathcal{W}(V)$ is the Wallach set of V, λ_0 is the fundamental weight conjugate to the unique non-compact simple root α_0 in the simple root system for $\mathfrak{co}$ in Lemma 2.1.

Theorem 4. *Let* V *be a simple euclidean Jordan algebra,* $\nu \in \mathcal{W}(V)\setminus\{0\}$*, and* $\mathcal{C}$ *be* V*'s canonical cone of rank* $\rho(\nu)$*. Under the action of* $\mathfrak{co}(V)$ *which maps* O *to* $\tilde{O}(\nu)$*,* $\tilde{D}(\mathcal{C})$ *becomes a unitary lowest weight* $(\mathfrak{co}(V), \tilde{\mathrm{U}}(V))$*-module with lowest weight* $\nu\lambda_0$ *and has the following multiplicity free* K*-type formula:*

$$\tilde{D}(\mathcal{C}) = \bigoplus_{\mathbf{m}\ge 0}^{m_{\rho(\nu)+1}=0} \tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C}). \tag{4.33}$$

Therefore, as a representation of $\tilde{\mathrm{U}}(V)$*,* $\tilde{D}(\mathcal{C}) \cong \bigoplus_{\mathbf{m}\ge 0}^{m_{\rho(\nu)+1}=0} \xi_\nu \otimes \mathcal{P}_{\mathbf{m}}(V)$.

Consequently, upon integration, $L^2\left(\mathcal{C}, \frac{1}{r}\mathrm{vol}\right)$ *becomes a unitary lowest weight representation* π_ν *for* $\mathrm{Co}(V)$ *with the same lowest weight.*

Proof. Parts i), iii) and iv) of Proposition 4.6 imply that $\tilde{D}(\mathcal{C})$ is a unitary $(\mathfrak{co}(V), \tilde{\mathrm{U}}(V))$-module. The K-type formula follows from parts iv) and v) of Proposition 4.6. Combining with parts v) and vi) of Theorem 3, we arrive at first part of this theorem. The second part follows from the first part, part ii) of Proposition 4.6, and a fundamental theorem of Harish-Chandra.

□

In view of the classification theorem in Ref. [11], the nontrivial scalar-type unitary lowest weight representations of $\mathrm{Co}(V)$ are exhausted by representations π_ν in the above theorem.

5. Generalized Quantum Kepler Problems Without Magnetic Charges

In Ref. [9], we introduce the universal hamiltonian for the Kepler problem in terms of the generators of TKK algebra, and remark that whenever we have a quantization for the TKK algebra, we have a super-integrable model of the Kepler-type. In view of the quantizations for the TKK algebra presented in the last section, we have some new super-integrable models of Kepler-type.

As before, V is a simple euclidean Jordan algebra of rank ρ and degree δ, $\mathcal{W}(V)$ is its Wallach set. For a canonical cone inside V of rank k, we use φ_k to denote the phi-function defined in Eq. (4.18) and Δ to denote its (non-positive) Laplace operator. For $\nu \in \mathcal{W}(V)\setminus\{0\}$, we let

$$V(\nu) := \begin{cases} \frac{1}{8}\left(\Delta(\ln\varphi_k) + \frac{1}{4}|d\ln\varphi_k|^2\right) & \text{if } \nu = k\frac{\delta}{2}, \\ \frac{1}{8}\left(\Delta(\ln\varphi_\rho) + \frac{1}{4}|d\ln\varphi_\rho|^2\right) & \\ +\frac{\rho}{8}\left((\nu-\frac{n}{\rho})^2 - (\frac{\delta}{2}-1)^2\right)\frac{\mathrm{tr}\, x^{-1}}{r} & \text{if } \nu > (\rho-1)\frac{\delta}{2}. \end{cases} \tag{5.1}$$

and call $V(\nu)$ the **quantum-correction potential** on the canonical cone of rank $\rho(\nu)$. Note that $V(\nu) = \frac{U(\nu)}{2r}$. Here is the definition of the generalized quantum Kepler problem attached to π_ν:

Definition 3 (Generalized Quantum Kepler Problems)**.** *Let V be a simple euclidean Jordan algebra and $\nu \in \mathcal{W}(V)\setminus\{0\}$. The ν-th generalized quantum Kepler problem of V is the quantum mechanical system for which the configuration space is the canonical cone of rank $\rho(\nu)$, and the hamiltonian $\tilde H(\nu)$ (or simply $\tilde H$) is*

$$-\frac{1}{2}\Delta + V(\nu) - \frac{1}{r}. \tag{5.2}$$

Here, Δ and $V(\nu)$ are the Laplace operator and the quantum-correction potential respectively.

One can verify that when $\nu = \frac{\delta}{2}$, generalized quantum Kepler problem is the J-Kepler problem in Ref. [8], and to get the original Kepler problem we need to take $V = \Gamma(3)$ and $\nu = 1$.

5.1. **Solution of the bound state problem.** Given a generalized quantum Kepler problem on a canonical cone $\mathcal{C}$, the bound state problem is primarily the following (energy) spectrum problem:

$$\left\{\begin{array}{rcl} \tilde H\psi & = & E\psi \\ \displaystyle\int_{\mathcal{C}} |\psi|^2\,\mathrm{vol} & < & \infty, \quad \psi \not\equiv 0. \end{array}\right. \tag{5.3}$$

It turns out that E has to take certain discrete values. For example, for the original Kepler problem, we have $E = -\frac{1}{2n^2}$, $n = 1, 2, \ldots$

We shall use $\mathscr{H}_I$ to denote the I-th **energy eigenspace**, $I = 0, 1, \ldots$ and $\mathscr{H}$ to denote the **Hilbert space of bound states** — the L^2-completion of $\bigoplus_{I=0}^{\infty} \mathscr{H}_I$.

Theorem 5. *Let V be a simple euclidean Jordan algebra and $\nu \in \mathcal{W}(V)\setminus\{0\}$. For the ν-th generalized quantum Kepler problem of V, the following statements are true:*

i) The bound state energy spectrum is

$$E_I = -\frac{1/2}{(I + \nu\frac{\rho}{2})^2}$$

where $I = 0, 1, 2, \ldots$

ii) As a representation of $\tilde{\mathrm{U}}(V)$, $\mathscr{H}_I \cong \bigoplus_{\mathbf{m}\geq 0, |\mathbf{m}|=I}^{m_{\rho(\nu)+1}=0} \xi_\nu \otimes \mathcal{P}_{\mathbf{m}}(V)$.

iii) $\mathscr{H}$ provides a realization for representation π_ν.

Proof. In view of part iv) of Proposition 4.6, we start with the eigenvalue problem for $-\frac{1}{2}\tilde H_e$:

$$-\frac{1}{2}\tilde H_e \tilde\psi = -n_I \tilde\psi, \tag{5.4}$$

where $n_I = I + \nu\frac{\rho}{2}$ and $\tilde\psi \not\equiv 0$ is square integrable with respect to measure $\frac{1}{r}$vol on the canonical cone of rank $\rho(\nu)$. The above equation can be recast as

$$-\frac{1}{2}\left(\Delta - \frac{U(\nu)}{r} + \frac{2n_I}{r}\right)\tilde\psi(x) = -\frac{1}{2}\tilde\psi(x).$$

Let $\psi(x) := \tilde\psi(\frac{x}{n_I})$, then the preceding equation becomes

$$\left(-\frac{1}{2}\Delta + V(\nu) - \frac{1}{r}\right)\psi(x) = -\frac{1/2}{n_I^2}\psi(x),$$

i.e.,

$$\tilde{H}\psi = -\frac{1/2}{n_I^2}\psi. \tag{5.5}$$

One can check that ψ is square integrable with respect to measure vol. Therefore, $\tilde{\psi}$ is an eigenfunction of $\tilde{H}_e$ with eigenvalue $2n_I \Rightarrow \psi$ is an eigenfunction of $\tilde{H}$ with eigenvalue $-\frac{1/2}{n_I^2}$. By turning the above arguments backward, with the help of an explicit form of the eigenfunctions for $\tilde{H}$, one can show that the converse of this statement is also true. Therefore, in view of parts iv) and v) of Proposition 4.6, we have

$$\mathscr{H}_I \cong \tilde{\mathscr{D}}_I(\mathcal{C}) \cong \bigoplus_{\mathbf{m}\geq 0, |\mathbf{m}|=I}^{m_{\rho(\nu)+1}=0} \xi_\nu \otimes \mathcal{P}_{\mathbf{m}}(V).$$

Introduce

$$\begin{array}{rccc} \tau: & \bigoplus_{I=0}^\infty \mathscr{H}_I & \longleftarrow & \tilde{D}(\mathcal{C}) = \bigoplus_{\mathbf{m}\geq 0}^{m_{\rho(\nu)+1}=0} \tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C}) \\ & c_{\mathbf{m}}\tilde{\psi}_{\mathbf{m}}(\frac{x}{n_{|\mathbf{m}|}}) & \longleftarrow\!\!\mid & \tilde{\psi}_{\mathbf{m}}(x) \in \tilde{\mathscr{D}}_{\mathbf{m}}(\mathcal{C}) \end{array} \tag{5.6}$$

Here $c_{\mathbf{m}}$ is a constant depending on $\mathbf{m}$. The value of $c_{\mathbf{m}}$ can be determined and τ can be shown to be an isometry, provided that an analogue of Theorem 2 in Ref. [10] for generalized Laguerre polynomials can be established, something that definitely can be done. Since $\tilde{D}(\mathcal{C})$ is a unitary highest weight Harish-Chandra module, and τ is an isometry, $\bigoplus_{I=0}^\infty \mathscr{H}_I$ becomes a unitary highest weight Harish-Chandra module. Since the L^2-completion of $\bigoplus_{I=0}^\infty \mathscr{H}_I$ is the Hilbert space of bound states, we arrive at part iii) of this theorem. □

We conclude this section with a remark. Generalized Kepler problems are natural generalizations of the J-Kepler problems, but with an important difference: the energy eigenspaces are no longer always irreducible representations of $\tilde{\mathrm{U}}(V)$, cf. part ii) of the theorem above.

Appendix A. Polar coordinates

The purpose of this section is to understand the polar coordinates on $\mathcal{C}_k$. The theorem obtained here is an extension of Theorem VI.2.3 in Ref. [4] from symmetric cones to canonical cones and the presentation follows that of Section 2 of Chapter VI in Ref. [4].

We fix a Jordan frame: $e_{11}, \ldots, e_{\rho\rho}$ and a Jordan basis $\{e_{ii}, e_{ij}^\mu\}$. We denote by V_{ij} the corresponding (i,j)-Peirce component of V. Let

$$R_k = \left\{\sum_{i=1}^k a_i e_{ii} \mid a_i \in \mathbb{R}\right\}, \quad R_k^+ = \left\{\sum_{i=1}^k a_i e_{ii} \mid a_1 > a_2 > \cdots > a_k > 0\right\}$$

Let K be the identity component of $\mathrm{Aut}(V)$ and M_k be the subgroup K fixing each point $a \in R_k$:

$$M_k = \{g \in K \mid \forall a \in R_k, ga = a\}$$

and m_k be its Lie algebra:

$$\mathfrak{m}_k = \{X \in \mathfrak{Der} \mid \forall a \in R_k, Xa = 0\}$$

For $i < j$, we define

$$\mathfrak{l}_{ij} = \{[L_{e_{ii}}, L_\xi] \mid \xi \in V_{ij}\}$$

Let

$$\mathfrak{l}_k = \sum_{1\le i\le k, i<j} \mathfrak{l}_{ij}$$

Proposition A.1. *Let $a = \sum_{i=1}^k a_i e_{ii} \in R_k^+$. For $X \in \mathfrak{Der}$, $(a, Xa) \in T_a\mathcal{C}_k$ is orthogonal to $T_aR_k^+$, and if $a_i \neq a_j$ for $i \neq j$, the map*

$$\begin{array}{rcl} \mathfrak{l}_k & \to & (T_aR_k^+)^\perp \\ X & \mapsto & (a, Xa) \end{array} \tag{A.1}$$

is an isomorphism.

Proof. For $X = [L_u, L_v]$, u and v in V, and for $a \in R_k^+$ and (x, b) in $T_aR_k^+$:

$$\begin{array}{rcl} ((a, Xa), (a, b)) & = & \langle a \mid e\rangle\langle Xa \mid \bar{L}_a^{-1}b\rangle \\ & = & \langle a \mid e\rangle\langle [L_a, L_{\bar{L}_a^{-1}b}]u \mid v\rangle \\ & = & 0 \end{array} \tag{A.2}$$

because both L_a and $L_{\bar{L}_a^{-1}b}$, being of diagonal form with respect to the Jordan basis, commute with each other.

Assume that $1 \le i \le k$ and $i < j \le \rho$. For X in $\mathfrak{l}_{ij}$:

$$X = [L_{e_{ii}}, L_\xi], \quad \xi \in V_{ij},$$

and for $a = \sum_{i=1}^k a_i e_{ii}$ in R_k^+ we have

$$Xa = \frac{1}{4}(a_j - a_i)\xi.$$

Here it is understood that $a_j = 0$ if $j > k$. Therefore, the range of the map $X \mapsto Xa$ contains the subspaces V_{ij} and the sum

$$\bigoplus_{1\le i\le k, i<j} V_{ij}.$$

Since $(T_aR_k^+)^\perp = \{a\} \times \bigoplus_{1\le i\le k, i<j} V_{ij}$, map (A.1) is onto, hence must be an isomorphism because the dimensions of the domain and the target are equal. □

Corollary A.1. *—i) As a vector space, we have*

$$\mathfrak{Der} = \mathfrak{m}_k \oplus \mathfrak{l}_k.$$

ii) Map

$$\begin{array}{rrcl} \phi: & K/M_k \times R_k^+ & \to & \mathcal{C}_k \\ & (gM_k, a) & \mapsto & ga \end{array}$$

has dense range and is a diffeomorphism onto its range.

Theorem A.1. *Write $d\mu_{k\frac{\delta}{2}}$ for $\frac{\sqrt{\varphi_k}}{r}\mathrm{vol}$. Under the identification map ϕ, we have*

$$d\mu_{k\frac{\delta}{2}} = C\,\mathrm{vol}_{K/M_k} \prod_{1\le i<j\le k} (a_i - a_j)^\delta \prod_{i=1}^k \left(a_i^{\frac{\delta}{2}(\rho-k+1)-1} da_i\right) \tag{A.3}$$

where vol_{K/M_k} is the K-invariant measure on K/M_k and C is a constant depending only on $\mathcal{C}_k$.

Proof. We start with a local parametrization of $\mathcal{C}_k$ around point $a \in R_k^+$:

$$x = \exp\left(\sum_{1\leq i\leq k, i<j\leq \rho}^{1\leq\alpha\leq\delta} x_{ij}^\alpha X_{ij}^\alpha\right) a.$$

Here, $\boxed{X_{ij}^\alpha = [L_{e_{ii}}, L_{e_{ij}^\alpha}]}$. Then

$$dx|_a = \sum_{i=1}^k e_{ii} da_i + \sum_{1\leq i\leq k, i<j\leq\rho}^{1\leq\alpha\leq\delta} \frac{1}{4}(a_j - a_i) e_{ij}^\alpha dx_{ij}^\alpha \mid_{x_{ij}^\alpha=0},$$

where it is understood that $a_j = 0$ if $j > k$. Therefore

$$\begin{aligned} ds_K^2|_a &= \langle a \mid e\rangle \langle dx \mid \bar{L}_a^{-1} dx\rangle \mid_a \\ &= \frac{\langle a\mid e\rangle}{\rho}\left(\sum_i \frac{1}{a_i} da_i^2 + \frac{1}{8}\sum_{1\leq i\leq k, i<j\leq\rho}^{1\leq\alpha\leq\delta} \frac{(a_j-a_i)^2}{a_i+a_j}(dx_{ij}^\alpha)^2 \mid_{x_{ij}^\alpha=0}\right), \end{aligned} \tag{A.4}$$

so, up to a multiplicative numerical constant, we have

$$\mathrm{vol}|_a = (r^{D_k/2} c_k^{\frac{1}{2}(\delta(\rho-k)-1)} \tau_k^{-\frac{\delta}{2}})|_a \prod_{1\leq i<j\leq k}(a_i - a_j)^\delta \bigwedge_{i=1}^k da_i \bigwedge \bigwedge_{1\leq i\leq k, 1\leq\alpha\leq\delta}^{i<j\leq\rho} dx_{ij}^\alpha|_{x_{ij}^\alpha=0}$$

and

$$d\mu_{k\frac{\delta}{2}}|_a = c_k(a)^{\frac{\delta}{2}(\rho-k+1)-1} \prod_{1\leq i<j\leq k}(a_i - a_j)^\delta \bigwedge_{i=1}^k da_i \bigwedge \bigwedge_{1\leq i\leq k, 1\leq\alpha\leq\delta}^{i<j\leq\rho} dx_{ij}^\alpha|_{x_{ij}^\alpha=0}.$$

On the other hand, since K is a simple Lie group, one can show that X_{ij}^α's are mutually orthogonal with respect to the negative-definite Cartan-Killing form[4], so the K-invariant volume form on K/M_k at eM_k is equal to $\bigwedge_{1\leq i\leq k, 1\leq\alpha\leq\delta}^{i<j\leq\rho} dx_{ij}^\alpha|_{x_{ij}^\alpha=0}$ modulo a multiplicative numerical constant. Since $d\mu_k$ is also K-invariant, up to a multiplicative numerical constant, we have

$$d\mu_{k\frac{\delta}{2}} = \mathrm{vol}_{K/M_k} \prod_{1\leq i<j\leq k}(a_i - a_j)^\delta \prod_{i=1}^k \left(a_i^{\frac{\delta}{2}(\rho-k+1)-1} da_i\right)$$

as a measure. □

As a side remark, we would like to mention the fact that integral

$$\int_\Omega e^{-2r}\det(x)^{\nu-\rho\frac{\delta}{2}}\, d\mu_{\rho\frac{\delta}{2}}$$

is finite if and only if $\nu > (\rho-1)\frac{\delta}{2}$.

[4] One just needs to show that the trace of $X_{ij}^\alpha X_{i'j'}^{\alpha'}$ is zero if $(i,j,\alpha) \neq (i',j',\alpha')$.

APPENDIX B. LIST OF NOTATIONS

The purpose here is to list some basic notations and terminologies for this paper and its sequels.

- V — a (finite dimensional) simple euclidean Jordan algebra;
- e, ρ, δ, and n — reserved for the identity element, rank, degree, and dimension of V;
- $\mathrm{tr}\, u$, $\det u$ — the trace, determinant of $u \in V$;
- $\langle u \mid v\rangle$ — the inner product of $u, v \in V$, and is chosen to be $\frac{1}{\rho}\mathrm{tr}\,(uv)$;
- x — reserved for a generic point in V when V is considered as a smooth space;
- r — reserved for function $\langle e \mid \rangle$ on smooth space V;
- $\{e_\alpha\}$ — an orthonormal basis for V;
- x^α — the coordinates of $x \in V$ with respect to basis $\{e_\alpha\}$;
- π — reserved for a generic point in V when V is considered as the tangent space of V;
- π^α — the coordinates of $\pi \in V$ with respect to basis $\{e_\alpha\}$;
- $\partial\!\!\!/$ — a shorthand notation for $\sum_\alpha e_\alpha \frac{\partial}{\partial x^\alpha}$;
- $\eth$ — a shorthand notation for $\sum_\alpha e_\alpha \frac{\partial}{\partial \pi^\alpha}$;
- d — the exterior derivative operator;
- vol — the volume form;
- uv — the Jordan product of $u, v \in V$;
- $\{uvw\}$— the Jordan triple product of $u, v, w \in V$;
- L_u — the multiplication by $u \in V$;
- S_{uv} — defined to be $[L_u, L_v] + L_{uv}$, so $S_{uv}w = \{uvw\}$;
- $\mathcal{W}(V)$ — the Wallach set of V;
- $\mathcal{P}(V)$ — the set of complex-valued polynomial functions on V;
- $\mathfrak{Der}(V)$, $\mathfrak{Der}$ — the derivation algebra of V;
- $\mathfrak{str}(V)$, $\mathfrak{str}$ — the structure algebra of V, it is generated by L_u, $u \in V$;
- $\mathfrak{co}(V)$, $\mathfrak{co}$ — the conformal algebra of V;
- $\mathfrak{u}(V)$, $\mathfrak{u}$ — the maximal compact Lie subalgebra of $\mathfrak{co}$;
- $\mathrm{Aut}(V)$ — the automorphism group of V;
- $\mathrm{Str}(V)$, Str — the structure group of V;
- $\mathrm{Co}(V)$, Co — the conformal group of V, and is defined to be the simply connected Lie group with $\mathfrak{co}$ as its Lie algebra;
- $\tilde{\mathrm{U}}(V)$, $\tilde{\mathrm{U}}$— the simply connected Lie group with $\mathfrak{u}$ as its Lie algebra;
- $\tilde{H}(\nu)$, $\tilde{H}$ — the hamiltonian of the generalized Kepler problem corresponding to Wallach parameter ν;
- $\mathscr{H}_I$ — the Ith energy eigenspace for $\tilde{H}$;
- $\mathscr{H}$ — the Hilbert space of bound states for $\tilde{H}$.

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