δ-DERIVATIONS OF SIMPLE FINITE-DIMENSIONAL JORDAN SUPERALGEBRAS

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We describe non-trivial δ-derivations of semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic not 2, and of simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0. For these classes of algebras and superalgebras, non-zero δ-derivations are shown to be missing for $\delta \neq 0, \frac{1}{2}, 1$, and we give a complete account of $\frac{1}{2}$-derivations.

INTRODUCTION

The notion of derivation for an algebra was generalized by many mathematicians along quite different lines. Thus, in [1], the reader can find the definitions of a derivation of a subalgebra into an algebra and of an (s_1, s_2)-derivation of one algebra into another, where s_1 and s_2 are some homomorphisms of the algebras. Back in the 1950s, Herstein explored Jordan derivations of prime associative rings of characteristic $p \neq 2$; see [2]. (Recall that a *Jordan derivation of an algebra A* is a linear mapping $j_d : A \to A$ satisfying the equality $j_d(xy + yx) = j_d(x)y + xj_d(y) + j_d(y)x + yj_d(x)$, for any $x, y \in A$.) He proved that the Jordan derivation of such a ring is properly a standard derivation. Later on, Hopkins in [3] dealt with antiderivations of Lie algebras (for definition of an antiderivation, see [1]). The antiderivation, on the other hand, is a special case of a δ-derivation — that is, a linear mapping μ of an algebra such that $\mu(xy) = \delta(\mu(x)y + x\mu(y))$, where δ is some fixed element of the ground field.

Subsequently, Filippov generalized Hopkin's results in [4] by treating prime Lie algebras over an associative commutative ring Φ with unity and $\frac{1}{2}$. It was proved that every prime Lie Φ-algebra, on which a non-degenerated symmetric invariant bilinear form is defined, has no non-zero δ-derivation if $\delta \neq -1, 0, \frac{1}{2}, 1$. In [4], also, $\frac{1}{2}$-derivations were described for an arbitrary prime Lie Φ-algebra A $\left(\frac{1}{6} \in \Phi\right)$ with a non-degenerate symmetric invariant bilinear form defined on the algebra. It was shown that the linear mapping $\phi : A \to A$ is a $\frac{1}{2}$-derivation iff $\phi \in \Gamma(A)$, where $\Gamma(A)$ is the centroid of A. This implies that if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ on which a non-degenerate symmetric invariant bilinear form is defined, then every $\frac{1}{2}$-derivation ϕ has the form $\phi(x) = \alpha x$, $\alpha \in \Phi$. At a later time, Filippov described δ-derivations for prime alternative and non-Lie Mal'tsev Φ-algebras with some restrictions on the operator ring Φ. In [5], for instance, it was stated that algebras in these classes have no non-zero δ-derivations if $\delta \neq 0, \frac{1}{2}, 1$.

In the present paper, we come up with an account of non-trivial δ-derivations for semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic not 2, and for simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0. For these classes of

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algebras and superalgebras, non-zero δ-derivations are shown to be missing for $\delta \neq 0, \frac{1}{2}, 1$, and we provide in a complete description of $\frac{1}{2}$-derivations.

The paper is divided into four parts. In Sec. 1, relevant definitions are given and known results cited. In Sec. 2, we deal with δ-Derivations of simple and semisimple finite-dimensional Jordan algebras. In Secs. 3 and 4, δ-derivations are described for simple finite-dimensional Jordan supercoalgebras over an algebraically closed field of characteristic 0. For some superalgebras, note, the condition on the characteristic may be weakened so as to be distinct from 2. A proof for the main theorem is based on the classification theorem for simple finite-dimensional superalgebras and on the results obtained in Secs. 3 and 4.

1. BASIC FACTS AND DEFINITIONS

Let F be a field of characteristic p, $p \neq 2$. An algebra A over F is *Jordan* if it satisfies the following identities:

$$xy = yx, \quad (x^2y)x = x^2(yx).$$

Jordan algebras arise naturally from the associative algebras. If in an associative algebra A we replace multiplication ab by symmetrized multiplication $a \circ b = \frac{1}{2}(ab + ba)$ then we will face a Jordan algebra. Denote this algebra by $A^{(+)}$. Below are essential examples of Jordan algebras.

(1) The *algebra $J(V, f)$ of bilinear form.* Let $f : V \times V \longrightarrow F$ be a symmetric bilinear form on a vector space V. On the direct sum $J = F \cdot 1 + V$ of vector spaces, we then define multiplication by setting $1 \cdot v = v \cdot 1 = v$ and $v_1 \cdot v_2 = f(v_1, v_2) \cdot 1$; under this multiplication, $J = J(V, f)$ is a Jordan algebra. If the form f is non-degenerate and $\dim V > 1$, then the algebra $J(V, f)$ is simple.

(2) The Jordan algebra $H(D_n, J)$. Here, $n \geqslant 3$, D is a composition algebra, which is associative for $n > 3$, $j : d \to \overline{d}$ is a canonical involution in D, and $J : X \to \overline{X}$ is a standard involution in D_n.

THEOREM 1.1 [6]. Every simple finite-dimensional Jordan algebra A over an algebraically closed field F of characteristic not 2 is isomorphic to one of the following algebras:

(1) $F \cdot 1$;

(2) $J(V, f)$;

(3) $H(D_n, J)$.

We recall the definition of a superalgebra. Let Γ be a Grassmann algebra over F, which is generated by elements $1, e_1, \ldots, e_n, \ldots$ and is defined by relations $e_i^2 = 0$, $e_ie_j = -e_je_i$. Products $1, e_{i_1}e_{i_2}\ldots e_{i_k}$, $i_1 < i_2 < \ldots < i_k$, form a basis for Γ over F. Denote by Γ_0 and Γ_1 the subspaces generated by products of even and odd lengths, respectively. Then Γ is represented as a direct sum of these subspaces, $\Gamma = \Gamma_0 + \Gamma_1$, with $\Gamma_i\Gamma_j \subseteq \Gamma_{i+j(\mathrm{mod}\,2)}$, $i, j = 0, 1$. In other words, Γ is a Z_2-graded algebra (or superalgebra) over F.

Now let $A = A_0 + A_1$ be any supersubalgebra over F. Consider a tensor product of F-algebras, $\Gamma \otimes A$. Its subalgebra

$$\Gamma(A) = \Gamma_0 \otimes A_0 + \Gamma_1 \otimes A_1$$

is called a *Grassmann envelope* for A.

Let Ω be some variety of algebras over F. A Z_2-graded algebra $A = A_0 + A_1$ is a Ω-*superalgebra* if its Grassmann envelope $\Gamma(A)$ is an algebra in Ω. In particular, $A = A_0 \oplus A_1$ is a *Jordan superalgebra* if its Grassmann envelope $\Gamma(A)$ is a Jordan algebra.

In [7], it was shown that every simple finite-dimensional associative superalgebra over an algebraically closed field F is isomorphic either to $A = M_{m,n}(F)$, which is the matrix algebra $M_{m+n}(F)$, or to $B = Q(n)$, which is a subalgebra of $M_{2n}(F)$. Gradings of superalgebras A and B are the following:

$$\begin{aligned} A_0 &= \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \middle| \, A \in M_m(F),\ D \in M_n(F) \right\}, \\ A_1 &= \left\{ \left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right) \middle| \, B \in M_{m,n}(F),\ C \in M_{n,m}(F) \right\}, \\ B_0 &= \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & A \end{array} \right) \middle| \, A \in M_n(F) \right\},\ B_1 = \left\{ \left(\begin{array}{cc} 0 & B \\ B & 0 \end{array} \right) \middle| \, B \in M_n(F) \right\}. \end{aligned}$$

Let $A = A_0 + A_1$ be an associative superalgebra. The vector space of A can be endowed with the structure of a Jordan supersubalgebra $A^{(+)}$, by defining new multiplication as follows: $a \circ b = \frac{1}{2}(ab + (-1)^{p(a)p(b)}ba)$. In this case $p(a) = i$ if $a \in A_i$.

Using the above construction, we arrive at superalgebras

$$M_{m,n}(F)^{(+)},\ m \geqslant 1,\ n \geqslant 1;$$

$$Q(n)^{(+)},\ n \geqslant 2.$$

Now, we define the superinvolution $j : A \to A$. A graded endomorphism $j : A \to A$ is called a *superinvolution* if $j(j(a)) = a$ and $j(ab) = (-1)^{p(a)p(b)}j(b)j(a)$. Let $H(A, j) = \{a \in A : j(a) = a\}$. Then $H(A, j) = H(A_0, j) + H(A_1, j)$ is a subsuperalgebra of $A^{(+)}$. Below are superalgebras which are obtained from $M_{n,m}(F)$ via a suitable superinvolution:

(1) the Jordan superalgebra $osp(n, m)$, consisting of matrices of the form $\left(\begin{array}{cc} A & B \\ C & D \end{array} \right)$, where $A^T = A \in M_n(F)$, $C = Q^{-1}B^T$, $D = Q^{-1}D^TQ \in M_{2m}(F)$, and $Q = \left(\begin{array}{cc} 0 & E_m \\ -E_m & 0 \end{array} \right)$;

(2) the Jordan superalgebra $P(n)$, consisting of matrices of the form $\left(\begin{array}{cc} A & B \\ C & D \end{array} \right)$, where $B^T = -B$, $C^T = C$, and $D = A^T$, with $A, B, C, D \in M_n(F)$.

THEOREM 1.2 [8, 9]. Every simple finite-dimensional non-trivial (i.e., with a non-zero odd part) Jordan superalgebra A over an algebraically closed field F of characteristic 0 is isomorphic to one of the following superalgebras:

$M_{m,n}(F)^{(+)}$; $Q(n)^{(+)}$; $osp(n, m)$; $P(n)$; $J(V, f)$; D_t, $t \neq 0$; K_3; K_{10}; $J(\Gamma_n)$, $n > 1$.

The superalgebras $J(V, f)$, D_t, K_3, K_{10}, and $J(\Gamma_n)$ will be defined below.

Let $\delta \in F$. A linear mapping ϕ of A is called a *δ-derivation* if

$$\phi(xy) = \delta(x\phi(y) + \phi(x)y) \tag{1}$$

for arbitrary elements $x, y \in A$.

The definition of a 1-derivation coincides with the conventional definition of a derivation. A 0-derivation is any endomorphism ϕ of A such that $\phi(A^2) = 0$. A *non-trivial* δ-derivation is a δ-derivation which is not a 1-derivation, nor a 0-derivation. Obviously, for any algebra, the multiplication operator by an element of the ground field F is a $\frac{1}{2}$-derivation. We are interested in the behavior of non-trivial δ-derivations of semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic not 2, and of simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0.

2. δ-DERIVATIONS FOR SEMISIMPLE FINITE-DIMENSIONAL JORDAN ALGEBRAS

In this section, we look at how non-trivial δ-derivations of simple finite-dimensional Jordan algebras behave over an algebraically closed field F of characteristic distinct from 2. As a consequence, we furnish a description of δ-derivations for semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic not 2.

THEOREM 2.1. Let ϕ be a non-trivial δ-derivation of a superalgebra A with unity e over a field F of characteristic not 2. Then $\delta = \frac{1}{2}$.

Proof. Let $\delta \neq \frac{1}{2}$. Then $\phi(e) = \phi(e \cdot e) = \delta(\phi(e) + \phi(e)) = 2\delta\phi(e)$, that is, $\phi(e) = 0$. Thus $\phi(x) = \phi(x \cdot e) = \delta(\phi(x) + x\phi(e)) = \delta\phi(x)$ for arbitrary $x \in A$. Contradiction. The theorem is proved.

LEMMA 2.2. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of a Jordan algebra A isomorphic to the ground field. Then $\phi(x) = \alpha x$, $\alpha \in F$.

Proof. Let e be unity in A. Then

$$\phi(x) = 2\phi(xe) - \phi(x) = x\phi(e), \tag{2}$$

that is, $\phi(x) = \alpha x$, $\alpha \in F$. The lemma is proved.

LEMMA 2.3. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of an algebra $J(V, f)$. Then $\phi(x) = \alpha x$ for $\alpha \in F$.

Proof. Let $\phi(e) = \alpha e + v$, where $\alpha \in F$ and $v \in V$. From (2), it follows that $\phi(x) = x\phi(e)$ for any $x \in J(V, f)$.

For $w \in V$, we then have

$$\begin{aligned}\alpha f(w,w)e + f(w,w)v = w^2(\alpha e + v) = \phi(w^2) &= \tfrac{1}{2}(w\phi(w) + \phi(w)w)\\ &= w\phi(w) = w(w(\alpha e + v)) = w(\alpha w + f(v,w)e)\\ &= \alpha f(w,w)e + f(w,v)w.\end{aligned}$$

As the result, $f(w,w)v = f(w,v)w$. Now, since w is arbitrary and $\dim(V) > 1$, we have $v = 0$. Thus $\phi(x) = \alpha x$ for any $x \in J(V, f)$. The lemma is proved.

LEMMA 2.4. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of an algebra $H(D_n, J)$, $n \geqslant 3$. Then $\phi(x) = \alpha x$ for $\alpha \in F$.

Proof. Relevant information on composition algebras can be found in [6]. Let $\phi(e) = \alpha e + v$, where $v = \sum\limits_{i,j=1} x_{i,j}e_{i,j}$, $x_{1,1} = 0$, $x_{i,j} = \overline{x_{j,i}}$, $\alpha \in F$, $x_{i,j} \in D$.

From (2), for $x \in H(D_n, J)$ arbitrary, we have

$$x^2 \circ (\alpha e + v) = \phi(x^2) = x \circ \phi(x) = x \circ (x \circ (\alpha e + v)),\ x^2 \circ v = x \circ (x \circ v). \tag{3}$$

If we put $x = e_{k,k}$ we obtain $\sum\limits_{j=1}^{n} x_{k,j}e_{k,j} + \sum\limits_{i=1}^{n} x_{i,k}e_{i,k} = 2e_{k,k}^2 \circ v = 2e_{k,k} \circ (e_{k,k} \circ v) = \frac{1}{2}(\sum\limits_{j=1}^{n} x_{k,j}e_{k,j} + x_{k,k}e_{k,k} + x_{k,k}e_{k,k} + \sum\limits_{i=1}^{n} x_{i,k}e_{i,k})$, whence $v = \sum\limits_{i=1}^{n} x_{i,i}e_{i,i}$.

For $x = e_{n,k} + e_{k,n}$ substituted in (3), we have $x_{n,n}e_{n,n} + x_{k,k}e_{k,k} = (e_{n,k} + e_{k,n})^2 \circ \sum\limits_{i=1}^{n} x_{i,i}e_{i,i} = (e_{n,k} + e_{k,n}) \circ ((e_{n,k} + e_{k,n}) \circ \sum\limits_{i=1}^{n} x_{i,i}e_{i,i}) = (e_{n,k} + e_{k,n}) \circ \frac{1}{2}(x_{n,n}e_{k,n} + x_{k,k}e_{k,n} + x_{k,k}e_{n,k} + x_{n,n}e_{n,k}) = \frac{1}{2}(x_{k,k}e_{k,k} + x_{k,k}e_{n,n} + x_{n,n}e_{k,k} + x_{n,n}e_{n,n})$, which yields $x_{n,n} = x_{n-1,n-1} = \ldots = x_{1,1} = 0$ and $v = 0$.

Consequently, $\phi(x) = \alpha x$ for any $x \in H(D_n, J)$. The lemma is proved.

THEOREM 2.5. Let ϕ be a non-trivial δ-derivation of a simple finite-dimensional Jordan algebra A over an algebraically closed field F of characteristic distinct from 2. Then $\delta = \frac{1}{2}$ and $\phi(x) = \alpha x$, $\alpha \in F$.

The **proof** follows from Theorems 1.1, 2.1 and Lemmas 2.2-2.4.

THEOREM 2.6. Let ϕ be a non-trivial δ-derivation of a semisimple finite-dimensional Jordan algebra $A = \bigoplus_{i=1}^{n} A_i$, where A_i are simple algebras, over an algebraically closed field of characteristic not 2. Then $\delta = \frac{1}{2}$, and for $x = \sum_{i=1}^{n} x_i$ where $x_i \in A_i$, we have $\phi(x) = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in F$.

Proof. Unity in A_k is denoted by e_k. If $x_i \in A_i$, then $\phi(x_i) = x_i^+ + x_i^-$, where $x_i^+ \in A_i$ and $x_i^- \notin A_i$. Put $e^i = \sum_{k=1}^{n} e_k - e_i$ and $\phi(e^i) = e^{i+} + e^{i-}$, where $e^{i+} \in A_i$ and $e^{i-} \notin A_i$. Then $0 = \phi(x_i \cdot e^i) = \delta(\phi(x_i) \cdot e^i + x_i \cdot \phi(e^i)) = \delta((x_i^+ + x_i^-)e^i + x_i(e^{i+} + e^{i-})) = \delta(x_i^- + x_i \cdot e^{i+})$, which yields $x_i^- = 0$. Consequently, the mapping ϕ is invariant on A_i. In virtue of Theorem 2.5, $\delta = \frac{1}{2}$ and $\phi(x_i) = \alpha_i x_i$ for some $\alpha_i \in F$ defined for A_i with $x_i \in A_i$ arbitrary. It is easy to verify that the mapping ϕ, given by the rule $\phi\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \alpha_i x_i$, $x_i \in A_i$, is a $\frac{1}{2}$-derivation. The theorem is proved.

3. δ-DERIVATIONS FOR SIMPLE FINITE-DIMENSIONAL JORDAN SUPERALGEBRAS WITH UNITY

In this section, all superalgebras but $J(\Gamma_n)$ are treated over a field of characteristic not 2. The superalgebra $J(\Gamma_n)$ is treated over a field of characteristic 0. Among the title superalgebras are $M_{m,n}(F)^{(+)}$, $Q(n)^{(+)}$, $osp(n,m)$, $P(n)$, $J(V,f)$, and $J(\Gamma_n)$. Theorem 2.1 implies that these superalgebras all lack in non-trivial δ-derivations, for $\delta \neq \frac{1}{2}$. Therefore, we need only consider the case of a $\frac{1}{2}$-derivation.

LEMMA 3.1. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of $M_{m,n}(F)^{(+)}$. Then $\phi(x) = \alpha x$ for some $\alpha \in F$.

Proof. It is easy to see that, for $1 \leqslant i,j \leqslant n+m$, elements $e_{i,j}$ form a basis for the superalgebra $M_{m,n}(F)^{(+)}$. Let $\phi(e_{i,j}) = \sum_{k,l=1}^{m+n} \alpha_{k,l}^{i,j} e_{k,l}$, where $\alpha_{k,l}^{i,j} \in F$, $i,j = 1, \ldots, n+m$.

If in (1) we put $x = y = e_{i,i}$ we arrive at

$$\sum_{k,l=1}^{m+n} \alpha_{k,l}^{i,i} e_{k,l} = \phi(e_{i,i}) = \phi(e_{i,i}^2) = \tfrac{1}{2}(e_{i,i} \circ \phi(e_{i,i}) + \phi(e_{i,i}) \circ e_{i,i}) = \tfrac{1}{2}\left(\sum_{l=1}^{n+m} \alpha_{i,l}^{i,i} e_{i,l} + \sum_{k=1}^{n+m} \alpha_{k,i}^{i,i} e_{k,i}\right),$$

whence $\phi(e_{i,i}) = \alpha_i e_{i,i}$, where $\alpha_i = \alpha_{i,i}^{i,i}$, $i = 1, \ldots, m+n$.

Substituting $x = e_{i,j}$ and $y = e_{i,i}$, $i \neq j$, in (1), we obtain

$$\sum_{k,l=1}^{m+n} \alpha_{k,l}^{i,j} e_{k,l} = \phi(e_{i,j}) = 2\phi(e_{i,j} \circ e_{i,i}) = \tfrac{1}{2}\left(\alpha_i e_{i,j} + \sum_{l=1}^{m+n} \alpha_{i,l}^{i,j} e_{i,l} + \sum_{k=1}^{m+n} \alpha_{k,i}^{i,j} e_{k,i}\right).$$

Analyzing the resulting equalities, we conclude that $\alpha_{i,j}^{i,j} = \alpha_i$. A similar argument for $e_{i,j}$ and $e_{j,j}$ yields $\alpha_{i,j}^{i,j} = \alpha_j$. Since ϕ is linear, $\phi(e) = \alpha e$. Using (2) gives $\phi(x) = \alpha x$, for any $x \in M_{n,m}(F)^{(+)}$. The lemma is proved.

LEMMA 3.2. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of $Q(n)^{(+)}$. Then $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Clearly, $\Delta_{i,j} = e_{i,j} + e_{n+i,n+j}$ and $\Delta^{i,j} = e_{n+i,j} + e_{i,n+j}$ form a basis for the superalgebra $Q(n)^{(+)}$.

On the basis elements, the following relations hold:

$$\Delta_{i,j} \circ \Delta_{k,l} = \tfrac{1}{2}(\delta_{j,k}\Delta_{i,l} + \delta_{l,i}\Delta_{k,j}), \quad \Delta_{i,j} \circ \Delta^{k,l} = \tfrac{1}{2}(\delta_{j,k}\Delta^{i,l} + \delta_{l,i}\Delta^{k,j}).$$

Let $\phi(\Delta_{i,j}) = \sum\limits_{k,l=1}^{n} \alpha_{k,l}^{i,j}\Delta_{k,l} + \sum\limits_{k,l=1}^{n} \alpha_{k,l}^{*i,j}\Delta^{k,l}$. Put $x = y = \Delta_{i,i}$ in (1). Then

$$\sum_{k,l=1}^{n} \alpha_{k,l}^{i,i}\Delta_{k,l} + \sum_{k,l=1}^{n} \alpha_{k,l}^{*i,i}\Delta^{k,l} = \phi(\Delta_{i,i}) = \phi(\Delta_{i,i}^2) = \tfrac{1}{2}(\Delta_{i,i} \circ \phi(\Delta_{i,i}) + \phi(\Delta_{i,i}) \circ \Delta_{i,i}) =$$
$$\frac{1}{2}\left(\sum_{l=1}^{n} \alpha_{i,l}^{i,i}\Delta_{i,l} + \sum_{k=1}^{n} \alpha_{k,i}^{i,i}\Delta_{k,i} + \sum_{k=1}^{n} \alpha_{k,i}^{*i,i}\Delta^{k,i} + \sum_{l=1}^{n} \alpha_{i,l}^{*i,i}\Delta^{i,l}\right).$$

Consequently, $\phi(\Delta_{i,i}) = \alpha_i\Delta_{i,i} + \alpha^i\Delta^{i,i}$, where $\alpha_i = \alpha_{i,i}^{i,i}$ and $\alpha^i = \alpha_{i,i}^{*i,i}$.

If we substitute $x = \Delta_{i,i}$ and $y = \Delta_{i,j}$, $i \neq j$, in (1) we obtain

$$\sum_{k,l=1}^{n} (\alpha_{k,l}^{i,j}\Delta_{k,l} + \alpha_{k,l}^{*i,j}\Delta^{k,l}) = \phi(\Delta_{i,i}) = 2\phi(\Delta_{i,i} \circ \Delta_{i,j}) =$$
$$\frac{1}{2}\left(\alpha_i\Delta_{i,j} + \alpha^i\Delta^{i,j} + \sum_{l=1}^{n} \alpha_{i,l}^{i,j}\Delta_{i,l} + \sum_{k=1}^{n} \alpha_{k,i}^{i,j}\Delta_{k,i} + \sum_{l=1}^{n} \alpha_{i,l}^{*i,j}\Delta^{i,l} + \sum_{k=1}^{n} \alpha_{k,i}^{*i,j}\Delta^{k,i}\right).$$

Hence $\alpha_{i,j}^{i,j} = \alpha_i$, $\alpha_{i,j}^{*i,j} = \alpha^i$.

A similar argument for $\Delta_{j,j}$ and $\Delta_{i,j}$ yields

$$\phi(\Delta_{i,j}) = \alpha_{j,j}^{i,j}\Delta_{j,j} + \alpha_j\Delta_{i,j} + \alpha_{j,j}^{*i,j}\Delta^{j,j} + \alpha^j\Delta^{i,j}.$$

These relations readily imply that $\alpha_i = \alpha_j = \alpha$ and $\alpha^i = \alpha^j = \beta$, that is, $\phi(\Delta_{i,i}) = \alpha\Delta_{i,i} + \beta\Delta^{i,i}$.

Clearly, $\phi(E) = \alpha E + \beta\Delta$, where E is unity in $Q(n)^{(+)}$, and $\Delta = \sum\limits_{i=1}^{n}(e_{i,n+i} + e_{n+i,i})$. Suppose that $\beta \neq 0$ and $\phi(x) = \alpha x + \beta\Delta \circ x$ is a $\frac{1}{2}$-derivation. A mapping $\psi : Q(n)^{(+)} \to Q(n)^{(+)}$, for which $\psi(x) = \Delta \circ x$, likewise is a $\frac{1}{2}$-derivation. Obviously, $\frac{1}{2}(\Delta^{i,i} - \Delta^{j,j}) = \psi(\Delta^{i,j} \circ \Delta^{j,i}) = \frac{1}{2}((\Delta^{i,j} \circ \Delta) \circ \Delta^{j,i} + \Delta^{i,j} \circ (\Delta^{j,i} \circ \Delta)) = 0$. On the other hand, $\Delta^{i,i} - \Delta^{j,j} \neq 0$. Consequently, $\beta = 0$, that is, $\phi(x) = \alpha x$. The lemma is proved.

LEMMA 3.3. Let ϕ be a non-trivial $\frac{1}{2}$-derivation of $osp(n, m)$. Then $\phi(x) = \alpha x$ for some $\alpha \in F$.

Proof. It is easy to see that $E = \sum\limits_{i=1}^{n} \Delta_i + \sum\limits_{j=1}^{m} \Delta^j$, where $\Delta^j = e_{n+j,n+j} + e_{n+m+j,n+m+j}$ and $\Delta_i = e_{i,i}$ is unity in the supersubalgebra $osp(n, m)$. Let

$$\phi(\Delta_i) = \sum_{k,l=1}^{n+2m} \alpha_{k,l}^{i} e_{k,l},\ i = 1, \ldots, n, \quad \phi(\Delta^j) = \sum_{k,l=1}^{n+2m} \beta_{k,l}^{j} e_{k,l},\ j = 1, \ldots, m.$$

If we put $x = y = \Delta_i$, $i = 1, \ldots, n$, in (1) we obtain $\sum\limits_{k,l=1}^{n+2m} \alpha_{k,l}^{i} e_{k,l} = \phi(\Delta_i) = \phi(\Delta_i^2) = \frac{1}{2}(\phi(\Delta_i) \circ \Delta_i + \Delta_i \circ \phi(\Delta_i)) = \frac{1}{2}\left(\sum\limits_{k=1}^{n+2m} \alpha_{k,i}^{i} e_{k,i} + \sum\limits_{l=1}^{n+2m} \alpha_{i,l}^{i} e_{i,l}\right)$, which yields $\phi(\Delta_i) = \alpha_i\Delta_i$, $i = 1, \ldots, n$.

Put $x = y = \Delta^i$, $i = 1, \ldots, m$, in (1). Then

$$\sum_{k,l=1}^{n+2m} \beta_{k,l}^{i} e_{k,l} = \phi(\Delta^i) = \phi((\Delta^i)^2) = \tfrac{1}{2}(\Delta^i \circ \phi(\Delta^i) + \phi(\Delta^i) \circ \Delta^i) =$$
$$\frac{1}{2}\left(\sum_{k=1}^{n+2m} \beta_{k,n+i}^{i} e_{k,n+i} + \sum_{k=1}^{n+2m} \beta_{k,n+m+i}^{i} e_{k,n+m+i} + \sum_{l=1}^{n+2m} \beta_{n+i,l}^{i} e_{n+i,l} + \sum_{l=1}^{n+2m} \beta_{n+m+i,l}^{i} e_{n+m+i,l}\right).$$

By the definition of $osp(n,m)$, we have $\beta^i_{n+i,n+m+i} = \beta^i_{m+n+i,n+i} = 0$ and $\beta^i_{n+i,n+i} = \beta^i_{n+m+i,n+m+i}$. Thus $\phi(\Delta^j) = \beta_j \Delta^j$, $j = 1,\dots,m$.

Let $(e_{i,j} + e_{j,i}) \in osp(n,m)$, $i,j = 1,\dots,n$, and $\phi(e_{i,j} + e_{j,i}) = \sum\limits_{k,l=1}^{2m+n} \gamma^{i,j}_{k,l} e_{k,l}$. If we put $x = e_{i,j} + e_{j,i}$ and $y = \Delta_i$ in (1) we arrive at

$$\sum_{k,l=1}^{2m+n} \gamma^{i,j}_{k,l} e_{k,l} = \phi(e_{i,j} + e_{j,i}) = 2\phi((e_{i,j} + e_{j,i}) \circ \Delta_i) = \frac{1}{2}\left(\sum_{k=1}^{2m+n} \gamma^{i,j}_{k,i} e_{k,i} + \sum_{l=1}^{2m+n} \gamma^{i,j}_{i,l} e_{i,l} + \alpha_i (e_{i,j} + e_{j,i})\right).$$

In view of the last relation, $\gamma^{i,j}_{j,i} = \gamma^{i,j}_{i,j} = \alpha_i$. Similar calculations for $e_{i,j} + e_{j,i}$ and Δ_j give $\gamma^{i,j}_{j,i} = \gamma^{i,j}_{i,j} = \alpha_j$. Ultimately, $\phi(\Delta_i) = \alpha \Delta_i$, $i = 1,\dots,n$.

Let $E_{ij} = (e_{n+i,n+j} + e_{n+m+j,n+m+i}) \in osp(n,m)$, $i,j = 1,\dots,m$, and $\phi(E_{ij}) = \sum\limits_{k,l=1}^{2m+n} \omega^{i,j}_{k,l} e_{k,l}$. Put $x = E_{ij}$ and $y = \Delta^i$ in (1); then

$$\sum_{k,l=1}^{2m+n} \omega^{i,j}_{k,l} e_{k,l} = \phi(E_{ij}) = 2\phi(E_{ij} \circ \Delta^i) = \frac{1}{2}\Bigg(\sum_{l=1}^{2m+n} \omega^{i,j}_{n+i,l} e_{n+i,l} + \sum_{k=1}^{2m+n} \omega^{i,j}_{k,n+i} e_{k,n+i} +$$
$$\sum_{l=1}^{2m+n} \omega^{i,j}_{n+m+i,l} e_{n+m+i,l} + \sum_{k=1}^{2m+n} \omega^{i,j}_{k,n+m+i} e_{k,n+m+i} + \beta_i E_{ij}\Bigg).$$

Consequently, $\omega^{i,j}_{n+i,n+j} = \omega^{i,j}_{n+m+j,n+m+i} = \beta_i$.

A similar argument for E_{ij} and Δ^j shows that $\omega^{i,j}_{n+i,n+j} = \omega^{i,j}_{n+m+j,n+m+i} = \beta_j$ with $1 \leqslant i,j \leqslant m$. Eventually we conclude that $\phi(\Delta^j) = \beta \Delta^j$, $j = 1,\dots,m$.

Let $E^{11} = e_{1,n+m+1} - e_{n+1,1} \in osp(n,m)$ and $\phi(E^{11}) = \sum\limits_{k,l=1}^{2m+n} \nu_{k,l} e_{k,l}$. If we put $x = E^{11}$ and $y = \Delta^1$ in (1) we have

$$\sum_{k,l=1}^{2m+n} \nu_{k,l} e_{k,l} = \phi(E^{11}) = 2\phi(E^{11} \circ \Delta^1) = \frac{1}{2}\Bigg(\sum_{k=1}^{2m+n} (\nu_{k,n+1} e_{k,n+1} + \nu_{k,n+m+1} e_{k,n+m+1}) +$$
$$\sum_{l=1}^{2m+n} (\nu_{n+1,l} e_{n+1,l} + \nu_{n+m+1,l} e_{n+m+1,l}) + \alpha E^{11}\Bigg),$$

whence $\nu_{1,m+n+1} = \nu_{n+1,1} = \alpha$. Further, for $x = E^{11}$ and $y = \Delta_1$ substituted in (1), we obtain

$$\sum_{k,l=1}^{2m+n} \nu_{k,l} e_{k,l} = \phi(E^{11}) = 2\phi((E^{11}) \circ \Delta_1) = \frac{1}{2}\left(\sum_{l=1}^{2m+n} \nu_{1,l} e_{1,l} + \sum_{k=1}^{2m+n} \nu_{k,1} e_{k,1} + \beta E^{11}\right)$$

and $\nu_{1,m+n+1} = \nu_{n+1,1} = \beta$. Thus $\alpha = \beta$ and $\phi(E) = \alpha E$. From (2), it follows that $\phi(y) = \alpha y$ for any element $y \in osp(n,m)$. The lemma is proved.

LEMMA 3.4. Let ϕ be a $\frac{1}{2}$-derivation of $P(n)$. Then $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let $\Delta_{i,j} = e_{i,j} + e_{n+j,n+i}$, $E = \sum\limits_{i=1}^{n} \Delta_{i,i}$ be unity in the superalgebra $P(n)$, and $\phi(\Delta_{i,j}) = \sum\limits_{k,l=1}^{2n} \alpha^{i,j}_{k,l} e_{k,l}$. If in (1) we put $x = y = \Delta_{i,i}$ we arrive at

$$\sum_{k,l=1}^{2n} \alpha^{i,i}_{k,l} e_{k,l} = \phi(\Delta_{i,i}) = \phi(\Delta^2_{i,i}) = \frac{1}{2}\left(\sum_{l=1}^{2n} \alpha^{i,i}_{n+i,l} e_{n+i,l} + \sum_{k=1}^{2n} \alpha^{i,i}_{k,n+i} e_{k,n+i} + \sum_{l=1}^{2n} \alpha^{i,i}_{i,l} e_{i,l} + \sum_{k=1}^{2n} \alpha^{i,i}_{k,i} e_{k,i}\right).$$

The definition of $P(n)$ implies $\alpha^{i,i}_{i,n+i} = 0$. Therefore, $\phi(\Delta_{i,i}) = \alpha^{i,i}_{i,i} e_{i,i} + \alpha^{i,i}_{n+i,n+i} e_{n+i,n+i} + \alpha^{i,i}_{n+i,i} e_{n+i,i}$.

Put $x = \Delta_{i,i}$ and $y = \Delta_{i,j}$ in (1). Then

$$\sum_{k,l=1}^{2n} \alpha_{k,l}^{i,j} e_{k,l} = \phi(\Delta_{i,j}) = 2\phi(\Delta_{i,i} \circ \Delta_{i,j})$$
$$= \frac{1}{2}\Bigg(\alpha_{i,i}^{i,i} e_{i,j} + \alpha_{n+i,n+i}^{i,i} e_{n+j,n+i} + \alpha_{n+i,i}^{i,i} e_{n+j,i} + \alpha_{n+i,i}^{i,i} e_{n+i,j}$$
$$+ \sum_{l=1}^{2n} \alpha_{i,l}^{i,j} e_{i,l} + \sum_{k=1}^{2n} \alpha_{k,i}^{i,j} e_{k,i} + \sum_{l=1}^{2n} \alpha_{n+i,l}^{i,j} e_{n+i,l} + \sum_{k=1}^{2n} \alpha_{k,n+i}^{i,j} e_{k,n+i} \Bigg).$$

Thus $\alpha_{i,i}^{i,i} = \alpha_{i,j}^{i,j}$, $\alpha_{n+i,n+i}^{i,i} = \alpha_{n+j,n+i}^{i,i}$, and $\alpha_{n+i,i}^{i,i} = \alpha_{n+j,i}^{i,j}$.

Arguing similarly for $\Delta_{j,j}$ and $\Delta_{i,j}$, we obtain $\alpha_{j,j}^{j,j} = \alpha_{i,j}^{i,j}$, $\alpha_{n+j,n+j}^{j,j} = \alpha_{n+j,n+i}^{i,i}$, and $\alpha_{n+j,j}^{j,j} = \alpha_{n+j,i}^{i,j}$. In view of the definition of $P(n)$ and the relations above, we have $\phi(\Delta_{i,i}) = \alpha\Delta_{i,i} + \beta e_{n+i,i}$. The fact that the mapping ϕ is linear implies $\phi(E) = \alpha E + \beta\Delta$, $\Delta = \sum_{i=1}^{n}(e_{n+i,i})$.

Suppose that $\beta \neq 0$ and $\phi(x) = \alpha x + \beta\Delta \circ x$ is a $\frac{1}{2}$-derivation. Then a mapping $\psi : P(n) \to P(n)$, where $\psi(x) = \Delta \circ x$, likewise is a $\frac{1}{2}$-derivation. We argue to show that this is not so. Let $b_{j,i} = e_{j,n+i} - e_{i,n+j}$. Then $\psi(\Delta_{i,j} \circ b_{j,i}) = \psi(0) = 0$; but $\frac{1}{2}(\psi(\Delta_{i,j}) \circ b_{j,i} + \Delta_{i,j} \circ \psi(b_{j,i})) = \frac{1}{2}((\Delta_{i,j} \circ \Delta) \circ b_{j,i} + \Delta_{i,j} \circ (b_{j,i} \circ \Delta)) = \frac{1}{4}((e_{n+j,i} + e_{n+i,j}) \circ (e_{j,n+i} - e_{i,n+j}) + (e_{j,i} - e_{i,j} - e_{n+j,n+i} + e_{n+i,n+j}) \circ (e_{i,j} + e_{n+j,n+i})) = \frac{1}{8}\Delta_{i,i} \neq 0$ on the other hand. Hence ψ is not a $\frac{1}{2}$-derivation. Therefore, $\beta = 0$ and $\phi(x) = \alpha x$. The lemma is proved.

We define the Jordan superalgebra $J(V, f)$. Let $V = V_0 + V_1$ be a Z_2-graded vector space on which a non-degenerate superform $f(.,.) : V \times V \to F$ is defined so that it is symmetric on V_0 and is skew-symmetric on V_1. Also $f(V_1, V_0) = f(V_0, V_1) = 0$. Consider a direct sum of vector spaces, $J = F \oplus V$. Let e be unity in the field F. Define, then, multiplication by the formula $(\alpha + v)(\beta + w) = (\alpha\beta + f(v, w))e + (\alpha w + \beta v)$. The given superalgebra has grading $J_0 = F + V_0$, $J_1 = V_1$. It is easy to see that e is unity in $J(V, f)$.

LEMMA 3.5. Let ϕ be a $\frac{1}{2}$-derivation of $J(V, f)$. Then $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let $\phi(e) = \alpha e + v_0 + v_1$, $v_i \in V_i$. Putting $x = z_i$, $y = e$, and $z_i \in V_i$ in (1), we obtain $\phi(z_i) = 2\phi(z_i e) - \phi(z_i) = \phi(z_i)e + z_i\phi(e) - \phi(z_i) = \alpha z_i + f(z_i, v_i)e$, whence $\phi(z_i) = \alpha z_i + f(z_i, v_i)e$.

If we put $x = z_0$ and $y = z_1$ in (1) we arrive at $0 = \phi(z_1 z_0) = \frac{1}{2}(\phi(z_1)z_0 + z_1\phi(z_0)) = f(z_1, v_1)z_0 + f(z_0, v_0)z_1$. By the definition of a superform f, we have $v_0 = 0$ and $v_1 = 0$, that is, $\phi(e) = \alpha e$. Using (2) yields $\phi(x) = \alpha x$, $\alpha \in F$, for any $x \in J(V, f)$. The lemma is proved.

Consider the Grassmann algebra Γ with (odd) anticommutative generators $e_1, e_2, \ldots, e_n, \ldots$. In order to define new multiplication, we use the operation

$$\frac{\partial}{\partial e_j}(e_{i_1} e_{i_2} \ldots e_{i_n}) = \begin{cases} (-1)^{k-1} e_{i_1} e_{i_2} \ldots e_{i_{k-1}} e_{i_{k+1}} \ldots e_{i_n} & \text{if } j = i_k, \\ 0 & \text{if } j \neq i_l,\ l = 1, \ldots, n. \end{cases}$$

For $f, g \in \Gamma_0 \bigcup \Gamma_1$, *Grassmann multiplication* is defined thus:

$$\{f, g\} = (-1)^{p(f)} \sum_{j=1}^{\infty} \frac{\partial f}{\partial e_j} \frac{\partial g}{\partial e_j}.$$

Let $\overline{\Gamma}$ be an isomorphic copy of Γ under the isomorphic mapping $x \to \overline{x}$. Consider a direct sum of vector spaces, $J(\Gamma) = \Gamma + \overline{\Gamma}$, and endow it with the structure of a Jordan superalgebra, setting $A_0 = \Gamma_0 + \overline{\Gamma_1}$ and $A_1 = \Gamma_1 + \overline{\Gamma_0}$, with multiplication $\bullet$. We obtain

$$a \bullet b = ab, \overline{a} \bullet b = (-1)^{p(b)}\overline{ab},\ a \bullet \overline{b} = \overline{ab},\ \overline{a} \bullet \overline{b} = (-1)^{p(b)}\{a, b\},$$

where $a, b \in \Gamma_0 \bigcup \Gamma_1$ and ab is the product in Γ. Let Γ_n be a subalgebra of Γ generated by elements $e_1, e_2, \dots, e_n$. By $J(\Gamma_n)$ we denote the subsuperalgebra $\Gamma_n + \overline{\Gamma_n}$ of $J(\Gamma)$. If $n \geqslant 2$ then $J(\Gamma_n)$ is a simple Jordan superalgebra.

LEMMA 3.6. Let ϕ be a $\frac{1}{2}$-derivation of $J(\Gamma_n)$. Then $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let $\phi(1) = \alpha\gamma + \beta\overline{\nu}$, where $\alpha, \beta \in F$, $\gamma \in \Gamma$, and $\overline{\nu} \in \overline{\Gamma}$. Put $y = 1$ in (1); then

$$\phi(x) = 2\phi(x \bullet 1) - \phi(x) = \phi(x) + x \bullet \phi(1) - \phi(x) = x \bullet \phi(1). \tag{4}$$

If in (1) we put $x = \overline{e_i}$, $y = \overline{e_i}$, $i = 1, \dots, n$, with (4) in mind, we arrive at

$$\phi(1) = \phi(\overline{e_i} \bullet \overline{e_i}) = \tfrac{1}{2}(\phi(\overline{e_i}) \bullet \overline{e_i} + \overline{e_i} \bullet \phi(\overline{e_i})) = \phi(\overline{e_i}) \bullet \overline{e_i} = \overline{e_i} \bullet (\overline{e_i} \bullet \phi(1)).$$

For any x of the form $e_{i_1} e_{i_2} \dots e_{i_k}$, obviously, we have

$$\overline{e_i} \bullet (\overline{e_i} \bullet x) = \begin{cases} x & \text{if } \frac{\partial x}{\partial e_i} = 0, \\ 0 & \text{otherwise;} \end{cases} \tag{5}$$

$$\overline{e_i} \bullet (\overline{e_i} \bullet \overline{x}) = \begin{cases} \overline{x} & \text{if } \frac{\partial x}{\partial e_i} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let $\gamma = \gamma^{i+} + e_i \gamma^{i-}$ and $\overline{\nu} = \overline{\nu^{i+}} + e_i \overline{\nu^{i-}}$, where $\gamma^{i-}, \gamma^{i+}, \nu^{i-}, \nu^{i+}$ do not contain e_i. Since i is arbitrary, in view of (5) and (6), we have $\gamma = 1$ and $\nu = e_1 \dots e_n$. Thus $\phi(1) = \alpha \cdot 1 + \beta \overline{e_1 \dots e_n}$. Relation (4) entails

$$\begin{aligned} \phi(e_1) &= e_1 \bullet \phi(1) = e_1 \bullet (\alpha \cdot 1 + \beta \overline{e_1 \dots e_n}) = \alpha e_1, \\ \phi(\overline{e_1}) &= \overline{e_1} \bullet \phi(1) = \overline{e_1} \bullet (\alpha \cdot 1 + \beta \overline{e_1 \dots e_n}) = \alpha \overline{e_1} + \beta e_2 \dots e_n. \end{aligned}$$

The relations above, combined with the condition in (1), imply $0 = \phi(e_1 \bullet \overline{e_1}) = \frac{1}{2}(e_1 \bullet \phi(\overline{e_1}) + \phi(e_1) \bullet \overline{e_1}) = \frac{\beta}{2} e_1 \dots e_n$; that is, $\phi(1) = \alpha \cdot 1$. From (2), we conclude that $\phi(x) = \alpha x$ for any element $x \in J(\Gamma_n)$. The lemma is proved.

4. δ-DERIVATIONS FOR JORDAN SUPERALGEBRAS K_3, D_t, K_{10}

In this section, we confine ourselves to non-trivial δ-derivations of simple finite-dimensional Jordan superalgebras K_3, K_{10}, and D_t over an algebraically closed field of characteristic p not equal to 2. For the superalgebra K_{10}, we require in addition that $p \neq 3$. In conclusion, we formulate a theorem on δ-derivations for simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0.

The *three-dimensional Kaplansky superalgebra* K_3 is defined thus:

$$(K_3)_0 = Fe, \ (K_3)_1 = Fz + Fw,$$

where $e^2 = e$, $ez = \frac{1}{2}z$, $ew = \frac{1}{2}w$, and $[z, w] = e$.

LEMMA 4.1. Let ϕ be a non-trivial δ-derivation of K_3. Then $\delta = \frac{1}{2}$ and $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let $\phi(e) = \alpha_e e + \beta_e z + \gamma_e w$, $\phi(z) = \alpha_1 e + \beta_1 z + \gamma_1 w$, and $\phi(w) = \alpha_2 e + \beta_2 z + \gamma_2 w$, where $\alpha_e, \alpha_1, \alpha_2, \beta_e, \beta_1, \beta_2, \gamma_e, \gamma_1, \gamma_2 \in F$. If we put $x = y = e$ in (1) we obtain

$$\alpha_e e + \beta_e z + \gamma_e w = \phi(e) = \phi(e^2) = \delta(e\phi(e) + \phi(e)e) = \delta(2\alpha_e e + \beta_e z + \gamma_e w).$$

Thus it suffices to consider the following two cases:

(1) $\delta = \frac{1}{2}$;

(2) $\delta \neq \frac{1}{2}$, $\phi(e) = 0$.

In the former case, $\phi(e) = \alpha e$, where $\alpha = \alpha_e$. Case (1), for $x = e$ and $y = z$, entails $\alpha_1 e + \beta_1 z + \gamma_1 w = \phi(z) = 2\phi(ez) = 2 \cdot \frac{1}{2}(e\phi(z) + \phi(e)z) = \alpha_1 e + \frac{1}{2}(\beta_1 z + \gamma_1 w + \alpha z)$, whence $\beta_1 = \frac{1}{2}(\beta_1 + \alpha)$ and $\gamma_1 = \frac{1}{2}\gamma_1$; that is, $\beta_1 = \alpha$ and $\gamma_1 = 0$. Similarly, substituting in (1) $x = e$ and $y = w$, we obtain $\gamma_2 = \alpha$ and $\beta_2 = 0$. For $x = z$ and $y = w$ in (1), we have $\alpha e = \phi(e) = \phi([z, w]) = \frac{1}{2}(z\phi(w) + \phi(z)w) = \frac{1}{2}(\frac{1}{2}\alpha_2 z + \alpha e + \frac{1}{2}\alpha_1 w + \alpha e)$, whence $\phi(e) = \alpha e$, $\phi(z) = \alpha z$, and $\phi(w) = \alpha w$, where $\alpha \in F$. Consequently, $\phi(x) = \alpha x$ for any $x \in K_3$.

We handle the second case. For $x = e$ and $y = z$ in (1), we have $\alpha_1 e + \beta_1 z + \gamma_1 w = \phi(z) = 2\phi(ez) = 2\delta(e\phi(z) + \phi(e)z) = \delta(2\alpha_1 e + \beta_1 z + \gamma_1 w)$, which yields $\phi(z) = 0$. Similarly, we arrive at $\phi(w) = 0$. The fact that ϕ is linear implies $\phi = 0$. The lemma is proved.

At the moment, we define a one-parameter family of four-dimensional superalgebras D_t. For $t \in F$ fixed, the given family is defined thus:

$$D_t = (D_t)_0 + (D_t)_1,$$

where $(D_t)_0 = Fe_1 + Fe_2$, $(D_t)_1 = Fx + Fy$, $e_i^2 = e_i$, $e_1 e_2 = 0$, $e_i x = \frac{1}{2}x$, $e_i y = \frac{1}{2}y$, $[x, y] = e_1 + te_2$, $i = 1, 2$.

LEMMA 4.2. Let ϕ be a non-trivial δ-derivation of D_t. Then $\delta = \frac{1}{2}$ and $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let

$$\begin{aligned} \phi(e_1) &= \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 z + \lambda_1 w, \ \phi(e_2) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 z + \lambda_2 w, \\ \phi(z) &= \alpha_z e_1 + \beta_z e_2 + \gamma_z z + \lambda_z w, \ \phi(w) = \alpha_w e_1 + \beta_w e_2 + \gamma_w z + \lambda_w w, \end{aligned}$$

with coefficients in F.

Putting $x = y = e_1$ and then $x = y = e_2$ in (1), we obtain $\alpha_1 e_1 + \beta_1 e_2 + \gamma_1 z + \lambda_1 w = \phi(e_1) = \phi(e_1^2) = 2\delta(e_1\phi(e_1)) = 2\delta\alpha_1 e_1 + \delta\gamma_1 z + \delta\lambda_1 w$ and $\alpha_2 e_1 + \beta_2 e_2 + \gamma_2 z + \lambda_2 w = 2\delta\beta_2 e_2 + \delta\gamma_2 z + \delta\lambda_2 w$, whence $\alpha_1 = 2\delta\alpha_1$, $\beta_1 = 0$, $\gamma_1 = \delta\gamma_1$, $\lambda_1 = \delta\lambda_1$, $\alpha_2 = 0$, $\beta_2 = 2\delta\beta_2$, $\gamma_2 = \delta\gamma_2$, $\lambda_2 = \delta\lambda_2$.

There are two cases to consider:

(1) $\delta = \frac{1}{2}$, $\beta_1 = \alpha_2 = \gamma_1 = \gamma_2 = \lambda_1 = \lambda_2 = 0$;

(2) $\delta \neq \frac{1}{2}$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \lambda_1 = \lambda_2 = 0$.

In the former case, $\phi(e_1) = \alpha_1 e_1$ and $\phi(e_2) = \beta_2 e_2$. Put $x = e_1$ and $y = z$ in condition (1); then $\alpha_z e_1 + \beta_z e_2 + \gamma_z z + \lambda_z w = \phi(z) = 2\phi(e_1 z) = 2 \cdot \frac{1}{2}(e_1\phi(z) + \phi(e_1)z) = \alpha_z e_1 + \frac{1}{2}(\gamma_z z + \lambda_z w + \alpha_1 z)$, which yields $\alpha_1 = \gamma_z$, $\beta_z = \lambda_z = 0$.

For $x = e_2$ and $y = z$ in (1), we have $\alpha_z e_1 + \gamma_z z = \phi(z) = 2\phi(e_2 z) = 2 \cdot \frac{1}{2}(e_2\phi(z) + \phi(e_2)z) = \frac{1}{2}(\gamma_z z + \beta_2 z)$, whence $\gamma_z + \beta_2 = 2\gamma_z$, $\alpha_z = 0$, $\alpha_1 = \beta_2$, and $\phi(z) = \alpha z$, where $\alpha = \alpha_1$. Similarly, we conclude that $\phi(w) = \alpha w$. The mapping ϕ is linear; so $\phi(x) = \alpha x$, $\alpha \in F$, for any $x \in D_t$.

We handle the second case. Put $x = e_1$ and $y = z$ in (1); then $\alpha_z e_1 + \beta_z e_2 + \lambda_z z + \gamma_z w = \phi(z) = 2\phi(e_1 z) = 2\delta(e_1\phi(z) + \phi(e_1)z) = \delta(2\alpha_z e_1 + \lambda_z z + \gamma_z w)$, which yields $\phi(z) = 0$. Arguing similarly for w, we arrive at $\alpha_w e_1 + \beta_w e_2 + \gamma_w z + \lambda_w w = \delta(2\alpha_w e_1 + \gamma_w z + \lambda_w w)$. Consequently, $\phi(w) = 0$. Ultimately, the linearity of ϕ implies $\phi = 0$. The lemma is proved.

The simple ten-dimensional *Kac superalgebra* K_{10} is defined thus:

$$K_{10} = A \oplus M, \ (K_{10})_0 = A, \ (K_{10})_1 = M, \quad \text{where } A = A_1 \oplus A_2,$$

$$A_1 = Fe_1 + Fuz + Fuw + Fvz + Fvw,$$

$$A_2 = Fe_2, M = Fz + Fw + Fu + Fv.$$

Multiplication is specified by the following conditions:

$$e_i^2 = e_i,\ e_1 \text{ is unity in } A_1,\ e_i m = \tfrac{1}{2}m \text{ for any } m \in M,$$
$$[u,z] = uz,\ [u,w] = uw,\ [v,z] = vz,\ [v,w] = vw,$$
$$[z,w] = e_1 - 3e_2,\ [u,z]w = -u,\ [v,z]w = -v,\ [u,z][v,w] = 2e_1;$$

all other non-zero products are obtained from the above either by applying one of the skew-symmetries $z \leftrightarrow w$ or $u \leftrightarrow v$ or by substituting $z \leftrightarrow u$ and $w \leftrightarrow v$ simultaneously.

LEMMA 4.3. Let ϕ be a non-trivial δ-derivation of K_{10}. Then $\delta = \frac{1}{2}$ and $\phi(x) = \alpha x$, where $\alpha \in F$.

Proof. Let

$$\phi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 z + \alpha_4 w + \alpha_5 u + \alpha_6 v + \alpha_7 uz + \alpha_8 uw + \alpha_9 vz + \alpha_{10} vw,$$
$$\phi(e_2) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 z + \beta_4 w + \beta_5 u + \beta_6 v + \beta_7 uz + \beta_8 uw + \beta_9 vz + \beta_{10} vw,$$
$$\phi(z) = \gamma_1^z e_1 + \gamma_2^z e_2 + \gamma_3^z z + \gamma_4^z w + \gamma_5^z u + \gamma_6^z v + \gamma_7^z uz + \gamma_8^z uw + \gamma_9^z vz + \gamma_{10}^z vw,$$
$$\phi(w) = \gamma_1^w e_1 + \gamma_2^w e_2 + \gamma_3^w z + \gamma_4^w w + \gamma_5^w u + \gamma_6^w v + \gamma_7^w uz + \gamma_8^w uw + \gamma_9^w vz + \gamma_{10}^w vw,$$
$$\phi(u) = \gamma_1^u e_1 + \gamma_2^u e_2 + \gamma_3^u z + \gamma_4^u w + \gamma_5^u u + \gamma_6^u v + \gamma_7^u uz + \gamma_8^u uw + \gamma_9^u vz + \gamma_{10}^u vw,$$
$$\phi(v) = \gamma_1^v e_1 + \gamma_2^v e_2 + \gamma_3^v z + \gamma_4^v w + \gamma_5^v u + \gamma_6^v v + \gamma_7^v uz + \gamma_8^v uw + \gamma_9^v vz + \gamma_{10}^v vw,$$

where all coefficients are in F.

For $x = y = e_1$ in (1), we have

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 z + \alpha_4 w + \alpha_5 u + \alpha_6 v + \alpha_7 uz + \alpha_8 uw + \alpha_9 vz + \alpha_{10} vw =$$
$$\phi(e_1) = \phi(e_1^2) = \delta(\phi(e_1)e_1 + e_1\phi(e_1)) =$$
$$2\delta(\alpha_1 e_1 + \tfrac{1}{2}\alpha_3 z + \tfrac{1}{2}\alpha_4 w + \tfrac{1}{2}\alpha_5 u + \tfrac{1}{2}\alpha_6 v + \alpha_7 uz + \alpha_8 uw + \alpha_9 vz + \alpha_{10} vw),$$

whence $\alpha_1 = 2\delta\alpha_1$, $\alpha_2 = 0$, $\alpha_3 = \delta\alpha_3$, $\alpha_4 = \delta\alpha_4$, $\alpha_5 = \delta\alpha_5$, $\alpha_6 = \delta\alpha_6$, $\alpha_7 = 2\delta\alpha_7$, $\alpha_8 = 2\delta\alpha_8$, $\alpha_9 = 2\delta\alpha_9$, $\alpha_{10} = 2\delta\alpha_{10}$.

Putting $x = y = e_2$ in (1), we obtain

$$\beta_1 e_1 + \beta_2 e_2 + \beta_3 z + \beta_4 w + \beta_5 u + \beta_6 v + \beta_7 uz + \beta_8 uw + \beta_9 vz + \beta_{10} vw =$$
$$\phi(e_2) = \phi(e_2^2) = \delta(\phi(e_2)e_2 + e_2\phi(e_2)) = 2\delta e_2\phi(e_2) =$$
$$2\delta(\beta_2 e_2 + \tfrac{1}{2}\beta_3 z + \tfrac{1}{2}\beta_4 w + \tfrac{1}{2}\beta_5 u + \tfrac{1}{2}\beta_6 v),$$

which yields $\beta_1 = 0$, $\beta_2 = 2\delta\beta_2$, $\beta_3 = \delta\beta_3$, $\beta_4 = \delta\beta_4$, $\beta_5 = \delta\beta_5$, $\beta_6 = \delta\beta_6$, $\beta_7 = \beta_8 = \beta_9 = \beta_{10} = 0$.

Consequently, it suffices to consider the following two cases:

(1) $\delta = \frac{1}{2}$;

(2) $\delta \neq \frac{1}{2}$, $\phi(e_1) = \phi(e_2) = 0$.

In the former case, $\phi(e_1) = \alpha_1 e_1 + \alpha_7 uz + \alpha_8 uw + \alpha_9 vz + \alpha_{10} vw$ and $\phi(e_2) = \alpha e_2$. Put $x = e_2$ and $y = z$ in (1); then

$$\gamma_1^z e_1 + \gamma_2^z e_2 + \gamma_3^z z + \gamma_4^z w + \gamma_5^z u + \gamma_6^z v + \gamma_7^z uz + \gamma_8^z uw + \gamma_9^z vz + \gamma_{10}^z vw =$$
$$\phi(z) = 2\phi(ze_2) = \phi(z)e_2 + z\phi(e_2) =$$
$$\gamma_2^z e_2 + \tfrac{1}{2}\gamma_3^z z + \tfrac{1}{2}\gamma_4^z w + \tfrac{1}{2}\gamma_5^z u + \tfrac{1}{2}\gamma_6^z v + \tfrac{1}{2}\alpha z,$$

and so $\phi(z) = \gamma_2^z e_2 + \alpha z$. If in (1) we put $x = e_1$ and $y = z$ we obtain $\gamma_2^z e_2 + \alpha z = \phi(z) = 2\phi(ze_1) = \phi(z)e_1 + z\phi(e_1) = (\gamma_2^z e_2 + \alpha z)e_1 + z(\alpha_1 e_1 + +\alpha_7 uz + \alpha_8 uw + \alpha_9 vz + \alpha_{10} vw)$, whence $\gamma_2^z = 0$ and $\alpha = \alpha_1$; that is, $\phi(z) = \alpha z$. Similarly, for w, u, and v, we have $\phi(u) = \alpha u$, $\phi(v) = \alpha v$, and $\phi(w) = \alpha w$. Hence

$\phi(uz) = \phi([u,z]) = \frac{1}{2}(\phi(u)z + u\phi(z)) = \frac{1}{2}(\alpha[u,z] + \alpha[u,z]) = \alpha uz$. Analogously, we obtain $\phi(uw) = \alpha uw$, $\phi(vz) = \alpha vz$, and $\phi(vw) = \alpha vw$.

Let $x = [u,z]$ and $y = [v,w]$ in (1); then

$$2\phi(e_1) = \phi([u,z][v,w]) = \tfrac{1}{2}(\phi([u,z])[v,w] + [u,z]\phi([v,w])) = \alpha[u,z][v,w] = 2\alpha e_1.$$

The fact that ϕ is linear implies $\phi(x) = \alpha x$, $\alpha \in F$, for $x \in K_{10}$ arbitrary.

We handle the second case. Put $x = z$ and $y = e_1$ in (1). Then

$$\gamma_1^z e_1 + \gamma_2^z e_2 + \gamma_3^z z + \gamma_4^z w + \gamma_5^z u + \gamma_6^z v + \gamma_7^z uz + \gamma_8^z uw + \gamma_9^z vz + \gamma_{10}^z vw = \phi(z) = 2\phi(ze_1) = 2\delta(\phi(z)e_1 + z\phi(e_1)) = 2\delta(\gamma_1^z e_1 + \tfrac{1}{2}\gamma_3^z z + \tfrac{1}{2}\gamma_4^z w + \tfrac{1}{2}\gamma_5^z u + \tfrac{1}{2}\gamma_6^z v + \gamma_7^z uz + \gamma_8^z uw + \gamma_9^z vz + \gamma_{10}^z vw),$$

which yields $\phi(z) = 0$. Similarly, we arrive at $\phi(w) = \phi(v) = \phi(u) = 0$. Since e_1, e_2, z, v, u, w generate K_{10}, we have $\phi = 0$. The lemma is proved.

THEOREM 4.4. Let A be a simple finite-dimensional Jordan superalgebra over an algebraically closed field of characteristic 0, and let ϕ be a non-trivial δ-derivation of A. Then $\delta = \frac{1}{2}$ and $\phi(x) = \alpha x$ for some $\alpha \in F$ and for any $x \in A$.

The **proof** follows from Theorems 1.2, 2.1 and Lemmas 3.1-3.6, 4.1-4.3.

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