CATEGORIES OF JORDAN STRUCTURES AND GRADED LIE ALGEBRAS

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Abstract. In the paper we describe the subcategory of the category of $\mathbb{Z}$-graded Lie algebras which is equivalent to the category of Jordan pairs via a functorial modification of the TKK construction. For instance, we prove that $L = L_{-1} \oplus L_0 \oplus L_1$ can be constructed from a Jordan pair if and only if $L_0 = [L_{-1}, L_1]$ and the second graded homology group $H_2^{\mathrm{gr}}(L)$ is trivial. Similar descriptions are obtained for Jordan triple systems and Jordan algebras. New functorial versions of the TKK construction are given for pairs and algebras.

1. Preliminaries.

Strong connections between Lie and Jordan algebras were discovered in the 1960s by Tits, Kantor, and Koecher. Independently and almost simultaneously they introduced three versions of a construction, known presently as TKK. Its importance was recognized immediately, as TKK allows a beautiful and fruitful interplay between the theory of Lie algebras and the theory of Jordan algebras [12]. At the same time the original TKK construction does not provide a functor for the categories at hand.

There are two ways to address this. The first is to restrict the considered objects or morphisms as was done by Koecher and Kantor and, more recently, by Kac in [10] and Bertram in [5]. The second is to modify the construction. These modifications have been available for some time even in more general settings (see [3] or [9]). One can also arrive at a functorial modification of TKK by considering the universal central extension of TKK as was done in [7] or [14]. It appears, however, that the properties of the functor have never been studied. These studies are the main goal of our paper.

The categories which naturally arise from the modified construction are the category of $\mathbb{Z}$-graded Lie algebras and related categories, on one hand, and the categories of Jordan algebras, triple systems, and pairs on the other. All three functors which appear from the modified TKK are full and faithful and are left adjoints of natural forgetful functors. The main result of our paper contains descriptions of the image of the functor from Jordan pairs to graded Lie algebras in terms of graded homology and cohomology groups.

Theorem A. *The category of Jordan pairs is equivalent to the category of* 3*-graded Lie algebras* $L = L_{-1} \oplus L_0 \oplus L_1$ *such that* $L_0 = [L_{-1}, L_1]$ *and* L *satisfies one of the equivalent conditions:*

(i) $H_2^{\mathrm{gr}}(L) = 0$
(ii) $H^2_{\mathrm{gr}}(L, M) = 0$ *for every module* M *with the trivial grading* $M = M_0$.

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As a corollary we have similar results for Jordan triple systems and Jordan algebras. The latter description has an interesting feature as it involves regular rather than graded homology and cohomology groups.

Theorem C. *The category of unital Jordan algebras is equivalent to the category of A_1-graded Lie algebras L satisfying one of the equivalent conditions:*

(i) $H_2(L) = 0$
(ii) $H^2(L, M) = 0$ *for every trivial module* M.

This paper is organized as follows. The categories of Jordan objects and corresponding categories of graded Lie algebras are described in Sections 2 and 3. We recall the original TKK construction and its versions for pairs and triples in Section 4. In Section 5, we introduce a modified version of TKK and prove its universal properties. The final section contains various descriptions of images of the functors.

The part of the present paper related to Jordan pairs is parallel to the content from [13]. It is a 3-graded version of the theory of central extensions of Lie algebras (e.g., see [2, Sect. 7.9]) while the latter is a $\mathbb{Z}_2$-graded version of that theory. Moreover, since the category of Jordan pairs is equivalent to the category of polarized Lie triple systems (see [5, Sect.III.3]), one can apply the results on Lie triple systems from [13] to invoke our results on Jordan pairs. We found, however, that the direct approach suitably modified works better than application of Lie triple systems results providing shorter arguments that are also self-contained. On the other hand the content related to triple systems and algebras has no analogs in [13].

All of our objects (algebras, vector spaces, and modules) will be defined over a unital commutative ring k that has no 2- or 3-torsion. In the text we often consider pairs of spaces/modules indexed by $\pm$. Throughout, σ stands for an element from $\{-, +\}$ with $-\sigma$ defined naturally.

2. Categories of Jordan objects

In this section we describe three categories of Jordan objects which arise naturally in the context of the TKK construction.

2.1. **Jordan Pairs.** A *(linear) Jordan pair* is a pair of k-modules $P = (P_-, P_+)$ with two trilinear maps $\{\ ,\ ,\ \}_\sigma : P_\sigma \times P_{-\sigma} \times P_\sigma \to P_\sigma$ such that for $a, c \in P_\sigma$ and $b, d \in P_{-\sigma}$, one has

$$\{a, b, c\}_\sigma = \{c, b, a\}_\sigma \quad \text{and} \quad [V_{a,b}, V_{c,d}] = V_{V_{a,b}c,d} - V_{c,V_{b,a}d} \tag{1}$$

for $V_{a,b}c := \{a, b, c\}_\sigma$. When it is clear which trilinear map applies, we often drop the subscript. For two Jordan pairs $P = (P_-, P_+)$ and $Q = (Q_-, Q_+)$, a pair of linear maps $\gamma = (\gamma_-, \gamma_+)$, $\gamma_\sigma : P_\sigma \to Q_\sigma$, is called a *Jordan pair homomorphism* provided $\gamma_\sigma(\{a, b, c\}) = \{\gamma_\sigma(a), \gamma_{-\sigma}(b), \gamma_\sigma(c)\}$ for $a, c \in P_\sigma$ and $b \in P_{-\sigma}$. We denote the category of all Jordan pairs and their homomorphisms by $\mathcal{JP}$.

Other Jordan objects, namely Jordan triple systems and Jordan algebras, can be viewed as Jordan pairs with additional structure.

2.2. **Jordan Triple Systems as Jordan Pairs.** A *Jordan triple system* is a k-module T together with a trilinear map $\{\ ,\ ,\ \} : T \times T \times T \to T$ satisfying (1) for $a, b, c, d \in T$. A homomorphism of two Jordan triple systems T and S is a linear map $\gamma : T \to S$ satisfying the equation $\gamma(\{a, b, c\}) = \{\gamma(a), \gamma(b), \gamma(c)\}$ for $a, b, c \in T$.

The category $\mathcal{JTS}$ of all Jordan triple systems with their homomorphisms is equivalent to the category $\mathcal{JP}^{\varepsilon}$ of Jordan pairs with involutions which is described below (see [11, Sect.1.13]).

For a Jordan pair $P = (P_-, P_+)$ with the trilinear products $\{\ ,\ ,\ \}_{\pm}$, the pair of spaces $P^{\mathrm{op}} = (P_+, P_-)$ with the products $\{\ ,\ ,\ \}^{\mathrm{op}}_{\pm} := \{\ ,\ ,\ \}_{\mp}$ is also a Jordan pair, termed the pair *opposite* to P. A Jordan pair homomorphism $\varepsilon : P \to P^{\mathrm{op}}$ is called an *involution of* P provided $\varepsilon_{-\sigma} \circ \varepsilon_{\sigma} = \mathrm{id}_{P_\sigma}$ for $\sigma = \pm$. The objects of the category $\mathcal{JP}^{\varepsilon}$ are pairs (P, ε_P) consisting of a Jordan pair P and its involution ε_P.

If (P, ε_P) and (Q, ε_Q) are two Jordan pairs with involutions, a homomorphism $\gamma : P \to Q$ is called *involutary* if it preserves the involutions, i.e., if $\varepsilon_Q \gamma = \gamma^{\mathrm{op}} \varepsilon_P$, where $\gamma^{\mathrm{op}} : P^{\mathrm{op}} \to Q^{\mathrm{op}}$ is simply $\gamma^{\mathrm{op}} = (\gamma_+, \gamma_-)$. The involutary homomorphisms constitute the collection of morphisms in $\mathcal{JP}^{\varepsilon}$.

Finally, an equivalence between the categories $\mathcal{JTS}$ and $\mathcal{JP}^{\varepsilon}$ is provided by the functor $\mathcal{P}_{\mathcal{T}}$ which sends a triple system T with the product $\{\ ,\ ,\ \}$ to the pair (T, T) with the products $\{\ ,\ ,\ \}_{\pm} := \{\ ,\ ,\ \}$ and with the *canonical involution* $\kappa_{\pm} = \mathrm{id}_T$. A Jordan triple system homomorphism $\gamma : T \to S$ is sent to the involutary Jordan pair homomorphism $\mathcal{P}_{\mathcal{T}}\gamma = (\gamma, \gamma) : (T, T) \to (S, S)$.

2.3. **Jordan Algebras as Jordan Pairs.** A *(linear) Jordan algebra* is a k-module J with a bilinear product ab satisfying the following identities

$$ab = ba \quad \text{and} \quad (a^2 b)a = a^2(ba). \tag{2}$$

We will work with the category of unital Jordan algebras, where the algebras as well as homomorphisms are unital. We denote this category by $\mathcal{JA}$. It is equivalent to the category $\mathcal{JP}^{\mathrm{inv}}$ of Jordan pairs with fixed invertible elements, defined in [11, Sect.1.10] as follows.

For a Jordan pair $P = (P_-, P_+)$ an element $b \in P_\sigma$ is called *invertible* if the map $P_{-\sigma} \to P_\sigma$ defined by $a \mapsto \{b, a, b\}_\sigma$ is invertible. An object of $\mathcal{JP}^{\mathrm{inv}}$ is a pair (P, b) where P is a Jordan pair and b is an invertible element from P_+. A morphism $\varphi : (P, b) \to (Q, d)$ is defined to be a Jordan pair homomorphism from P to Q such that $\varphi(b) = d$.

To describe an equivalence functor $\mathcal{P}_{\mathcal{J}A}$ from $\mathcal{JA}$ onto $\mathcal{JP}^{\mathrm{inv}}$ we note that any Jordan algebra J considered with a *triple product* $\{a, b, c\} := (ab)c + a(bc) - b(ca)$ is a Jordan triple system and hence $\mathcal{P}_{\mathcal{T}} J = (J, J)$ is a Jordan pair. Moreover, the identity element $1 \in J = (\mathcal{P}_{\mathcal{T}} J)_+$ is an invertible element of the pair. The functor $\mathcal{P}_{\mathcal{J}A}$ sends J to $((J, J), 1)$ and the Jordan algebra homomorphism γ to the pair homomorphism $\mathcal{P}_{\mathcal{J}A}\gamma = (\gamma, \gamma)$.

3. Categories of Lie Algebras

In this section we review the categories of Lie algebras related to the aforementioned Jordan structures.

3.1. **3-graded Lie Algebras and Jordan Pairs.** Let $L = \bigoplus_{i \in \mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded Lie algebra, that is L is a direct sum of indexed k-modules $\{L_i : i \in \mathbb{Z}\}$ such that $[L_i, L_j] \subseteq L_{i+j}$. If $L_i = 0$ for $|i| > 1$, we write $L = L_{-1} \oplus L_0 \oplus L_1$ and say that the algebra L is 3-*graded.* We will work with the category $\mathcal{LA}_{3\text{-gr}}$ of all 3-graded Lie algebras and their graded homomorphisms. For every such morphism $\alpha : L \to K$ we have $\alpha(L_i) \subseteq K_i$ for $i \in \{0, \pm 1\}$ and we write $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1$ where $\alpha_i = \alpha|_{L_i}$. It is well-known that for a 3-graded Lie algebra $L = L_{-1} \oplus L_0 \oplus L_1$ the pair (L_{-1}, L_1) is Jordan with respect to the operations $\{a,b,c\}_\sigma := [[a,b],c]$, with $a,c \in L_{\sigma 1}$ and $b \in L_{-\sigma 1}$. It is also clear that for any graded Lie homomorphism α the pair (α_{-1}, α_1) is a Jordan homomorphism. In other words, we have a forgetful functor $\mathcal{F}_\mathcal{P}$ from the category of 3-graded Lie algebras $\mathcal{LA}_{\text{3-gr}}$ to the category of Jordan pairs $\mathcal{JP}$.

3.2. **Involutions on 3-graded Lie Algebras and Jordan Triple Systems.** An *anti-graded involution* of a $\mathbb{Z}$-graded Lie algebra L is a homomorphism $\varepsilon : L \to L$ satisfying $\varepsilon^2 = \mathrm{id}_L$ and $\varepsilon(L_i) \subseteq L_{-i}$ for $i \in \mathbb{Z}$. For $\mathbb{Z}$-graded Lie algebras with fixed involutions, (L, ε_L) and (K, ε_K), we say that a homomorphism $\alpha : L \to K$ is *involutary* if it preserves the involutions; that is, if $\alpha\varepsilon_L = \varepsilon_K\alpha$. We let $\mathcal{LA}^{\varepsilon}_{\text{3-gr}}$ denote the category whose objects are 3-graded Lie algebras with fixed anti-graded involution and whose morphisms are graded involutary homomorphisms.

If ε is an anti-graded involution of a 3-graded algebra $L = L_{-1} \oplus L_0 \oplus L_1$, then the restriction $\widetilde{\varepsilon}$ of ε onto the pair (L_{-1}, L_1) is a Jordan involution. It is easy to see that in this case, the Jordan pair with involution $(\mathcal{F}_\mathcal{P} L, \widetilde{\varepsilon})$ is isomorphic to the pair $\mathcal{P}_\mathcal{T} L_1$ via the map $(a,b) \mapsto (\varepsilon(a), b)$ where L_1 is considered as Jordan triple system with the operation $\{a,b,c\} := [[a,\varepsilon(b)],c]$. This leads to a forgetful functor $\mathcal{F}_\mathcal{T}$ from $\mathcal{LA}^{\varepsilon}_{\text{3-gr}}$ to the category of Jordan triple systems $\mathcal{JTS}$. This forgetful functor takes involutary graded homomorphisms α to Jordan triple system homomorphisms $\mathcal{F}_\mathcal{T}\alpha = \alpha_1$.

3.3. **A_1-graded Lie Algebras and Jordan Algebras.** The third category of Lie algebras is part of the theory of Lie algebras graded by root systems initiated by Berman and Moody in [6]. The general definition of root-graded algebra over a field along with classification theorems can be found in [6], [9], and [4]. Here, we give its adaptation for the case when the root system is A_1 and k is a commutative ring.

Following [10], we say that an ordered triple $\mathfrak{s} = \langle h, e, f \rangle$ of elements of a Lie algebra is an $\mathfrak{sl}_2$-*triple* if one has the following commutator relations:

$$[e,f] = h, \quad [h,e] = 2e, \text{ and } [h,f] = -2f.$$

An A_1-*graded* Lie algebra is a pair $(L, \mathfrak{s})$ that consists of a Lie algebra L and an $\mathfrak{sl}_2$-triple $\mathfrak{s} \subseteq L$ such that $L = L_{-1} \oplus L_0 \oplus L_1$ where $L_i = \{x \in L : [h,x] = 2ix\}$ for $i \in \{0, \pm 1\}$ and $L_0 = [L_{-1}, L_1]$. All A_1-graded Lie algebras together with Lie algebra homomorphisms sending $\mathfrak{sl}_2$-triple to $\mathfrak{sl}_2$-triple element-wise form a category which we denote by $\mathcal{LA}_{\mathrm{A}_1-\mathrm{gr}}$.

To relate this category to the previous ones we note that for an A_1-graded algebra $(L, \mathfrak{s})$ the decomposition $L = L_{-1} \oplus L_0 \oplus L_1$ is a 3-grading and that $e \in L_1$ is an invertible element of the Jordan pair $\mathcal{F}_\mathcal{P} L$. Thus the pair $\mathcal{F}_\mathcal{P} L$ is obtained from the Jordan algebra L_1 considered with operation $ab := [[a,f],b]$ and $\frac{1}{2}e$ is the identity element of this algebra. Further, for any morphism α of A_1-graded algebras, the restriction $\alpha_1 = \alpha|_{L_1}$ is a homomorphism of unital Jordan algebras. Thus one has the third forgetful functor $\mathcal{F}_{\mathcal{JA}}$ from $\mathcal{LA}_{\mathrm{A}_1-\mathrm{gr}}$ to the category of unital Jordan algebras $\mathcal{JA}$.

4. Tits-Kantor-Koecher construction

In this section we recall the constructions of the structure Lie algebra $\mathfrak{instr}(P)$ and of $\mathrm{TKK}(P)$ for a Jordan pair P. We also describe additional structures arising on these Lie algebras when P is obtained from a Jordan triple system or a Jordan algebra. Throughout the paper $\mathfrak{gl}(V)$ stands for the Lie algebra of all linear transformations on a k-module space V.

4.1. **Inner Structure Algebra.** Let $P = (P_-, P_+)$ be a Jordan pair. For any pair $(a, b) \in P_- \times P_+$ we define $\nu(a, b) \in \mathfrak{gl}(P_-) \oplus \mathfrak{gl}(P_+)$ by $\nu(a, b) := (V_{a,b}, -V_{b,a})$. Identities (1) imply that

$$[\nu(a, b), \nu(c, d)] = \nu(\nu(a, b)c, d) + \nu(c, \nu(a, b)d) \tag{3}$$

for $a, c \in P_-$ and $b, d \in P_+$, so the span of the $\nu(a, b)$'s forms a subalgebra of the Lie algebra $\mathfrak{gl}(P_-) \oplus \mathfrak{gl}(P_+)$ called the *inner structure algebra* of P and denoted by $\mathfrak{instr}(P)$.

If $P = \mathcal{P}_{\mathcal{T}}T$ for a Jordan triple system T, we write $\mathfrak{instr}(T)$ to denote $\mathfrak{instr}(\mathcal{P}_{\mathcal{T}}T)$. In this case it is easy to see that the map $\nu(x, y) \mapsto -\nu(y, x)$ is an automorphism of the Lie algebra $\mathfrak{instr}(T)$ of period two. Hence there is a eigenspace decomposition

$$\mathfrak{instr}(T) = \mathfrak{instr}(T)_{-1} \oplus \mathfrak{instr}(T)_1. \tag{4}$$

Here $\mathfrak{instr}(T)_1 = \operatorname{span}\{\nu(a, b) - \nu(b, a) : a, b \in T\}$ is the subalgebra of *inner derivations of* T, denoted usually by $\mathfrak{inder}(T)$.

Furthermore, if $P = \mathcal{P}_{\mathcal{JA}}J$ for a unital Jordan algebra J, then the component $\mathfrak{instr}(J)_{-1} = \operatorname{span}\{\nu(a, a) : a \in J\}$ can be identified with the submodule R_J of operators of right multiplication because $\nu(a, a) = (R_{a^2}, -R_{a^2})$. So the decomposition above admits the form

$$\mathfrak{instr}(J) = R_J \oplus \mathfrak{inder}(J). \tag{5}$$

4.2. **TKK construction.** The following theorem introduces the celebrated Tits-Kantor-Koecher construction.

Theorem 4.1. *For any Jordan pair $P = (P_-, P_+)$, the k-module $P_- \oplus \mathfrak{instr}(P) \oplus P_+$ together with the bracket*

$$[a + X + b, c + Y + d] = (Xc - Ya) + ([X, Y] + \nu(a, d) - \nu(c, b)) + (Xd - Yb) \tag{6}$$

for $X, Y \in \mathfrak{instr}(P)$, $a, c \in P_-$, and $b, d \in P_+$, is a Lie algebra.

This Lie algebra is denoted by $\mathrm{TKK}(P)$ and is called the *Tits-Kantor-Koecher construction*, or *TKK* for short. It follows readily from (6) that the algebra $\mathrm{TKK}(P)$ is 3-graded and $\mathcal{F}_{\mathcal{P}}(\mathrm{TKK}(P)) = P$ for the forgetful functor $\mathcal{F}_{\mathcal{P}}$ defined in Section 3.1.

It was noted by Kantor and Koecher that the TKK construction has certain functorial properties. Specifically, if one allows only epimorphisms as morphisms in the category of Jordan pairs under consideration, then TKK constitutes a functor from this category to $\mathcal{LA}_{3\text{-gr}}$. In general however, to be able to extend a Jordan pair homomorphism $\gamma : P \to Q$ to a Lie algebra homomorphism $\widehat{\gamma} : \mathrm{TKK}(P) \to \mathrm{TKK}(Q)$, one needs the condition: $\sum_i V_{a_i,b_i} = 0$ implies $\sum_i V_{\gamma(a_i),\gamma(b_i)} = 0$.

4.3. **TKK construction for Jordan triple systems and algebras.** Here our goal is to show that whenever the pair P is obtained from a Jordan triple system or a Jordan algebra, the Lie algebra $\mathrm{TKK}(P)$ can be considered naturally as an object of $\mathcal{LA}^{\varepsilon}_{3\text{-gr}}$ or of $\mathcal{LA}_{\mathrm{A}_1-\mathrm{gr}}$, respectively. Assume that $P = \mathcal{P}_{\mathcal{T}}T$ for a Jordan triple system T. Then it is easy to see that the endomorphism $\bar{\kappa}$ on $\mathrm{TKK}(P)$ defined by

$$\bar{\kappa}(a_- + \nu(c, d) + b_+) = b_- - \nu(d, c) + a_+ \tag{7}$$

is an anti-graded involution. The restriction of $\bar{\kappa}$ onto (T, T) is the canonical involution κ of the pair $\mathcal{P}_{\mathcal{T}}T$ defined in Sect. 2.2, so we can conclude that $\mathcal{F}_{\mathcal{T}}\mathrm{TKK}(P) = T$.

Assume now that $P = \mathcal{P}_{\mathcal{JA}}J$ for a Jordan algebra J with the identity element 1. Then the triple $\bar{\mathfrak{s}} = \langle -\nu(1,2), 2_+, 1_- \rangle$ is an $\mathfrak{sl}_2$-triple of $\mathrm{TKK}(P)$, the 3-grading defined by this triple coincides with the standard grading, and $\mathcal{F}_{\mathcal{JA}}\mathrm{TKK}(P) = J$.

5. Universal Imbeddings of Jordan objects

In the previous section we saw that any Jordan object can be imbedded into a Lie algebra of a certain type. It is not difficult to see that there exists a universal imbedding, even in much more general settings (see [9, Sect.1.4], for example). In the next section we present a different construction of a universal object. The construction is obtained as a result of a general procedure that first appeared in [8] and was modified in [13]. We believe simplicity of the generating set to be another merit of our construction.

5.1. **Construction of universal imbedding.** Let $P = (P_-, P_+)$ be a Jordan pair and $\mathfrak{instr}(P)$ be its structure Lie algebra described in Sect. 4.1. Since $\mathfrak{instr}(P)$ is a subalgebra of $\mathfrak{gl}(P_-) \oplus \mathfrak{gl}(P_+)$, it acts canonically on P_- and P_+, and on the tensor product $P_- \otimes P_+$ via $X\big(\sum_i a_i \otimes b_i\big) = \sum_i Xa_i \otimes b_i + \sum_i a_i \otimes Xb_i$. Furthermore, it follows from (3) that the map $\lambda : P_- \otimes P_+ \to \mathfrak{instr}(P)$ defined by $\lambda(a \otimes b) = \nu(a, b)$ is a module homomorphism from $P_- \otimes P_+$ to the regular module $\mathfrak{instr}(P)$. Therefore we can apply the following

Lemma 5.1. ([13, Lemma 3.1]) *Let M be a module over a Lie algebra L and let $\lambda : M \to L$ be an L-module epimorphism. Then $A(M) = \mathrm{span}\{\lambda(m) \cdot m : m \in M\}$ is a submodule of M, the quotient module $Q = M/A(M)$ with the product $[p, q] = \mu(p) \cdot q$ is a Lie algebra, and the map $M/A(M) \to L$ induced by λ is a central extension.*

In our case the set $A(P_- \otimes P_+)$ is spanned by elements of the form $\nu(a, b)(a \otimes b)$ or equivalently by

$$\nu(a, b)(c \otimes d) + \nu(c, d)(a \otimes b) \tag{8}$$

where $a, c \in P_-$ and $b, d \in P_+$. We let $\langle P_-, P_+\rangle$ denote the quotient module $(P_- \otimes P_+)/A(P_- \otimes P_+)$ and let $\langle a, b\rangle$ denote the coset containing $a \otimes b$. Lemma 5.1 yields:

Corollary 5.2. *$\langle P_-, P_+\rangle$ is a Lie algebra relative to the product $[\langle a, b\rangle, \langle c, d\rangle] = \langle\{a, b, c\}, d\rangle - \langle c, \{b, a, d\}\rangle$. Moreover, the map $\mu : \langle P_-, P_+\rangle \to \mathfrak{instr}(P)$, defined by $\mu(\langle a, b\rangle) = \nu(a, b)$, is a central extension.*

In what follows, the $\mathfrak{instr}(P)$-modules P_- and P_+ will be viewed as $\langle P_-, P_+\rangle$-modules via μ.

Theorem 5.3. *For any Jordan pair $P = (P_-, P_+)$, the k-module*

$$\widehat{\mathrm{TKK}}(P) = P_- \oplus \langle P_-, P_+\rangle \oplus P_+$$

together with the bracket

$$[a + X + b, c + Y + d] = (Xc - Ya) + ([X, Y] + \langle a, d\rangle - \langle c, b\rangle) + (Xd - Yb) \tag{9}$$

for $X, Y \in \langle P_-, P_+\rangle$, $a, c \in P_-$, and $b, d \in P_+$, is a 3-graded Lie algebra and the map $\upsilon : \widehat{\mathrm{TKK}}(P) \to \mathrm{TKK}(P)$ defined by

$$\upsilon(a + X + b) = a + \mu(X) + b \tag{10}$$

is a graded central extension.

Moreover, in the special case when $P = \mathcal{P}_{\mathcal{T}}T$ for a Jordan triple system T, the map

$$\widehat{\kappa}(a_- + \langle c, d\rangle + b_+) = b_- - \langle d, c\rangle + a_+ \tag{11}$$

is an anti-graded involution of $\widehat{\mathrm{TKK}}(P)$ and υ is an involutory morphism from $(\widehat{\mathrm{TKK}}(P), \widehat{\kappa})$ to $(\mathrm{TKK}(P), \bar{\kappa})$.

In the special case when $P = \mathcal{P}_{\mathcal{JA}}J$ for a unital Jordan algebra J, the triple $\widehat{\mathfrak{s}} = \langle -\langle 1, 2\rangle, 2_+, 1_-\rangle$ is an $\mathfrak{sl}_2$-triple of $\widehat{\mathrm{TKK}}(P)$, the 3-grading defined by this triple coincides with the canonical grading, and υ is an A_1-graded morphism from $(\widehat{\mathrm{TKK}}(P), \widehat{\mathfrak{s}})$ to $(\mathrm{TKK}(P), \bar{\mathfrak{s}})$.

Proof. The fact that $\widehat{\mathrm{TKK}}(P) = P_- \oplus \langle P_-, P_+\rangle \oplus P_+$ is a 3-grading follows directly from (9). We write $|l| = i$ if $l \in \widehat{\mathrm{TKK}}(P)_i$.

It is easy to see from (9) that the bracket $[\,,\,]$ is anticommutative, and from (6) and (9) that υ is an epimorphism with $\mathrm{Ker}(\upsilon) \subseteq \widehat{\mathrm{TKK}}(P)_0 = \langle P_-, P_+\rangle$. It follows that the Jacobian $\mathfrak{j}(l, k, m)$, defined by $\mathfrak{j}(l, k, m) = [[l, k], m] + [[k, m], l] + [[m, l], k]$, is zero for $l, k, m \in \widehat{\mathrm{TKK}}(P)$ with $|l| + |k| + |m| \neq 0$. Assume that $|l| + |k| + |m| = 0$. Because $\mathfrak{j}(l, k, m)$ is an alternating function and because of the $+/-$ symmetry, it suffices to check that $\mathfrak{j}(l, k, m) = 0$ when $(|l|, |k|, |m|) = (0, -1, 1)$. Here is the calculation for this case:

$$\mathfrak{j}\big(X, a, b\big) = \langle Xa, b\rangle - [X, \langle a, b\rangle] + \langle a, Xb\rangle = 0.$$

Thus, $\widehat{\mathrm{TKK}}(P)$ is a 3-graded Lie algebra and υ is a graded epimorphism. It is easy to see that $\mathrm{Ker}(\upsilon) = \mathrm{Ker}(\mu) \subseteq \mathrm{Center}(\widehat{\mathrm{TKK}}(P))_0$.

Assume now that $P = \mathcal{P}_{\mathcal{T}}T$ for a Jordan triple system T. It follows from (9) that $\widehat{\kappa}$ is an anti-graded involution of $\widehat{\mathrm{TKK}}(P)$ and it is clear that $\upsilon : (\widehat{\mathrm{TKK}}(P), \widehat{\kappa}) \longrightarrow (\mathrm{TKK}(P), \bar{\kappa})$ is a morphism in $\mathcal{LA}^{\varepsilon}_{\text{3-gr}}$.

Finally, when $P = \mathcal{P}_{\mathcal{JA}}J$ for a unital Jordan algebra J, it is easy to check that $\widehat{\mathfrak{s}} = \langle\langle 2, 1\rangle, 2_+, 1_-\rangle$ is an $\mathfrak{sl}_2$-triple of $\widehat{\mathrm{TKK}}(P)$, the 3-grading defined by this triple coincides with the standard grading, and $\upsilon : (\widehat{\mathrm{TKK}}(P), \widehat{\mathfrak{s}}) \longrightarrow (\mathrm{TKK}(P), \bar{\mathfrak{s}})$ is a morphism in $\mathcal{LA}_{\mathrm{A_1-gr}}$.

$\square$

Our construction $\widehat{\mathrm{TKK}}(P)$ has an especially simple form for the case of Jordan algebras. To this end we need:

Lemma 5.4. *If J is a Jordan algebra with the identity element 1, then $\langle J, J\rangle = J \otimes J / A(J \otimes J)$ where $A(J \otimes J)$ is spanned by the elements of the form*

$$a^2 \otimes a - 1 \otimes a^3. \tag{12}$$

Proof. We need to prove that the set $A = A(J \otimes J)$ defined in Section 5.1 equals to the span of (12) denoted temporarily by B. In the proof we write $x \equiv_T y$ if $x - y$ is an element of a set T. We note that for the linear map $\omega : J \otimes J \longrightarrow J \otimes J$, defined by $\omega(a \otimes b) = b \otimes a$, one has $(\nu_{a,b}(c \otimes d))^{\omega} = -\nu_{b,a}(d \otimes c)$ for $a, b, c, d \in J$.

Therefore we have $A^\omega \subseteq A$ and hence $x \equiv_A y$ implies $x^\omega \equiv_A y^\omega$. First we show that $B \subseteq A$. Specializing to $b = c = d = 1$ in (8) and noting that $\nu_{1,1}$ acts as zero on $J \otimes J$, we have

$$a \otimes 1 \equiv_A 1 \otimes a. \tag{13}$$

Next we specialize to $b = c = a$ and $d = 1$ to obtain

$$a^3 \otimes 1 - a \otimes a^2 \equiv_A a \otimes a^2 - a^2 \otimes a. \tag{14}$$

Since the right-hand side is skew-symmetric relative to ω, (13) and (14) imply that

$$2(a^3 \otimes 1 - a \otimes a^2) \equiv_A (a^3 \otimes 1 - a \otimes a^2) - (a^3 \otimes 1 - a \otimes a^2)^\omega \equiv_A a^3 \otimes 1$$
$$-a \otimes a^2 - 1 \otimes a^3 + a^2 \otimes a \equiv_A -(a \otimes a^2 - a^2 \otimes a) \equiv_A -(a^3 \otimes 1 - a \otimes a^2)$$

so $a^3 \otimes 1 - a \otimes a^2 \equiv_A 0$ and $B \subseteq A$.

To show the reverse containment, we will use the linearized version of (12):

$$ab \otimes c + bc \otimes a + ac \otimes b \equiv_B 1 \otimes (ab)c + 1 \otimes (bc)a + 1 \otimes (ca)b. \tag{15}$$

Setting $b = c = 1$ in this equation results in

$$a \otimes 1 \equiv_B 1 \otimes a, \tag{16}$$

while setting $c = 1$ and then applying (16) gives us

$$a \otimes b + b \otimes a \equiv_B 2 \otimes ab. \tag{17}$$

It follows that for every element $X \in J \otimes J$

$$X \equiv_B X^\omega \text{ if and only if } X \equiv_B 1 \otimes \eta(X), \tag{18}$$

where $\eta : J \otimes J \to J$ is a linear map defined by $\eta(a \otimes b) = ab$. Our next goal is to prove that the generators (8) satisfy one of the equivalent conditions from (18). Let us denote the set of all such elements by S^B.

Specializing $c = ab$ in (15), we obtain $(ab)a \otimes b + b(ab) \otimes a \equiv_{S^B} 0$ and use (17) to write it as

$$(ab)a \otimes b - a \otimes b(ab) \equiv_{S^B} 0. \tag{19}$$

Replacing a with a^2 in (15) and setting $b = c$ yields $2a^2b \otimes b \equiv_{S^B} 0$. Adding this to $2b^2a \otimes a \equiv_{S^B} 0$ and using (17) gives us

$$a^2b \otimes b - a \otimes b^2a \equiv_{S^B} 0. \tag{20}$$

Now (19) and (20) together with (2.3) imply that a generator (8) of A is

$$\{a, b, a\} \otimes b - a \otimes \{b, a, b\} = 2(ab)a \otimes b - a^2b \otimes b - 2a \otimes b(ab) + a \otimes b^2a \equiv_{S^B} 0$$

so it is in S^B. However the result of application of η to this element is a special case of the linearized Jordan identity (see [15, Ident.(23), p.86]):

$$2((ab)a)b - (a^2b)b - 2a(b(ab)) + a(b^2a) = 0.$$

Consequently, (18) implies that any generator (8) of A is in B. $\square$

Corollary 5.5. *For a unital Jordan algebra J, set J_+ and J_- to be two copies of J and $\langle J, J\rangle = J \otimes J/\operatorname{span}\{a^2 \otimes a - 1 \otimes a^3\}$. Then the k-module*

$$\widehat{\mathrm{TKK}}(J) = J_- \oplus \langle J, J\rangle \oplus J_+$$

together with the bracket (9) is a 3-graded Lie algebra.

Remark 5.6. It is also possible to select symmetric/skew-symmetric tensors generating $A = A(J \otimes J)$. Let $S^2(J) = \mathrm{span}\{a \otimes a : a \in J\}$ and $J \wedge J = \mathrm{span}\{a \otimes b - b \otimes a : a, b \in J\}$ be the sets of symmetric and skew-symmetric tensors and let $A_{symm} = A(J \otimes J) \cap S^2(J)$ and $A_{skew} = A(J \otimes J) \cap (J \wedge J)$. In the proof we noted that $A^\omega \subseteq A$, so $A = A_{symm} \oplus A_{skew}$. Moreover one can prove that $A_{symm} = \mathrm{span}\{2a \otimes a + 1 \otimes a^2 + a^2 \otimes 1 : a \in J\}$ and $A_{skew} = \mathrm{span}\{a^2 \otimes a - a \otimes a^2 : a \in J\}$. So one has the following decomposition of the algebra $\langle J, J\rangle$

$$\langle J, J\rangle = S^2(J)/A_{symm} \oplus (J \wedge J)/A_{skew}$$

which covers the one in (5).

5.2. **Universal property of $\widehat{\mathrm{TKK}}(P)$.** In the theorem below we consider P as a subset of $\widehat{\mathrm{TKK}}(P)$.

Theorem 5.7. *Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded Lie algebra.*

(i) *For every Jordan pair homomorphism $\gamma : P \to \mathcal{F}_{\mathcal{P}}(L)$, there is a unique graded Lie algebra homomorphism $\widehat{\gamma} : \widehat{\mathrm{TKK}}(P) \to L$ extending γ, that is $\widehat{\gamma}_{\pm 1} = \gamma_\pm$. It is defined by the formula*

$$\widehat{\gamma}(a + \sum_i \langle c_i, d_i\rangle + b) = \gamma_-(a) + \sum_i [\gamma_-(c_i), \gamma_+(d_i)] + \gamma_+(b) \tag{21}$$

for $a, c_i \in P_-$ and $b, d_i \in P_+$.

(ii) *If L has an anti-graded involution ε, then for every Jordan triple system homomorphism $\gamma : T \to \mathcal{F}_{\mathcal{T}}(L, \varepsilon)$ there is a unique involutary Lie algebra homomorphism $\widehat{\gamma} : (\widehat{\mathrm{TKK}}(\mathcal{P}_{\mathcal{T}}T), \widehat{\kappa}) \to (L, \varepsilon)$ extending γ, where $\widehat{\kappa}$ is the canonical anti-graded involution on $\widehat{\mathrm{TKK}}(\mathcal{P}_{\mathcal{T}}T)$.*

(iii) *Furthermore, if $(L, \mathfrak{s})$ is an A_1-graded Lie algebra, then for every unital Jordan algebra homomorphism $\gamma : J \to \mathcal{F}_{\mathcal{JA}}(L, \mathfrak{s})$, there is a unique A_1-graded morphism $\widehat{\gamma} : (\widehat{\mathrm{TKK}}(\mathcal{P}_{\mathcal{JA}}J), \widehat{\mathfrak{s}}) \to (L, \mathfrak{s})$ extending γ.*

Proof. For a pair homomorphism $\gamma : P \longrightarrow \mathcal{F}_{\mathcal{P}}(L)$, we begin by constructing $\widehat{\gamma}_0 : \langle P_-, P_+\rangle \to L_0$. It is easy to see that the set $A(P_- \otimes P_+)$ is in the kernel of the linear map $\varphi : P_- \otimes P_+ \to L_0$ defined by $\varphi(a \otimes b) = [\gamma_-(a), \gamma_+(b)]$. Thus one can consider the map $\widehat{\gamma}_0$ induced by φ on $\langle P_-, P_+\rangle = (P_- \otimes P_+)/A(P_- \otimes P_+)$.

We let $\widehat{\gamma} := \gamma_- + \widehat{\gamma}_0 + \gamma_+ : P_- \oplus \langle P_-, P_+\rangle \oplus P_+ \to L_{-1} \oplus L_0 \oplus L_1$, which is clearly graded. It follows from the definition of $\widehat{\gamma}$ that $\widehat{\gamma}[a, b] = [\gamma(a), \gamma(b)] = [\widehat{\gamma}(a), \widehat{\gamma}(b)]$. Furthermore, for $a, c \in P_-$ and $b, d \in P_+$ one has $\widehat{\gamma}[\langle a, b\rangle, c] = \gamma\{a, b, c\} = [[\gamma(a), \gamma(b)], \gamma(c)] = [\widehat{\gamma}\langle a, b\rangle, \widehat{\gamma}(c)]$ and similarly $\widehat{\gamma}[\langle a, b\rangle, d] = [\widehat{\gamma}\langle a, b\rangle, \widehat{\gamma}(d)]$. Since the subspaces P_- and P_+ generate the algebra $\widehat{\mathrm{TKK}}(P)$, it follows that $\widehat{\gamma}$ is a Lie algebra homomorphism and a unique extension of γ.

To establish assertion (ii), we consider an anti-graded involution ε on L and a triple system homomorphism $\gamma : T \to \mathcal{F}_{\mathcal{T}}(L, \varepsilon)$. We lift γ to the Jordan pair homomorphism $(\varepsilon\gamma, \gamma) : (T, T) \to \mathcal{F}_{\mathcal{P}}(L)$. As previously, we define a linear map $\varphi : T \otimes T \to L_0$ by $\varphi(a \otimes b) := [\varepsilon\gamma(a), \gamma(b)]$, which gives us a well-defined map $\widehat{\gamma}_0$ on $\langle T, T\rangle$ and a unique Lie algebra homomorphism $\widehat{\gamma} := \varepsilon\gamma + \widehat{\gamma}_0 + \gamma$. Furthermore, $\widehat{\gamma} : (\widehat{\mathrm{TKK}}(\mathcal{P}_{\mathcal{T}}T), \widehat{\kappa}) \to (L, \varepsilon)$ is involutary.

We proceed similarly with assertion (iii). Assume that $(L, \mathfrak{s})$ is an object in $\mathcal{LA}_{\mathrm{A_1-gr}}$ for an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \langle h, e, f\rangle$ and that $\gamma : (J, 1) \to \mathcal{F}_{\mathcal{JA}}(L, \mathfrak{s})$ is a unital Jordan algebra homomorphism. We lift γ to a Jordan pair homomorphism $(\varepsilon\gamma, \gamma)$:

$(J, J) \to \mathcal{F}_{\mathcal{P}}(L)$ just as in the proof of assertion (ii), using the anti-graded involution $\varepsilon = (\varepsilon_-, \varepsilon_+)$ on L given by $(-\frac{1}{2}(\mathrm{ad}e)^2, -\frac{1}{2}(\mathrm{ad}f)^2)$. Then we define the unique Lie algebra homomorphism $\widehat{\gamma}$ as before. It is easy to see that this $\widehat{\gamma}$ takes the $\mathfrak{sl}_2$-triple $\langle\langle 2, 1\rangle, 2_+, 1_-\rangle$ in $\widehat{\mathrm{TKK}}(\mathcal{P}_{\mathcal{JA}}J)$ element-wise to the triple $\langle h, e, f\rangle$ in L. □

6. Categories of Lie algebras equivalent to $\mathcal{JP}$, $\mathcal{JTS}$, and $\mathcal{JA}$

6.1. **$\widehat{\mathrm{TKK}}$ as left adjoint.** Part (i) of Theorem 5.7 implies that the identity map $P \to \mathcal{F}_{\mathcal{P}}\widehat{\mathrm{TKK}}(P)$ is a universal arrow from P to $\mathcal{F}_{\mathcal{P}}$ (see [1] for definitions and properties). Hence, the construction $\widehat{\mathrm{TKK}}$ constitutes a left adjoint to the forgetful functor $\mathcal{F}_{\mathcal{P}} : \mathcal{LA}_{3\text{-gr}} \to \mathcal{JP}$ from Section 3.1. Similarly, it follows from (ii) and (iii) of the theorem that functors $\widehat{\mathrm{TKK}} : \mathcal{JTS} \to \mathcal{LA}^{\varepsilon}_{3\text{-gr}}$ and $\widehat{\mathrm{TKK}} : \mathcal{JA} \to \mathcal{LA}_{\mathrm{A}_1-\mathrm{gr}}$ are left adjoints to the forgetful functors $\mathcal{F}_{\mathcal{T}}$ and $\mathcal{F}_{\mathcal{JA}}$ described in Sections 3.2 and 3.3, respectively. We do not introduce additional notations for these functors since it will be clear from the context which functor is considered.

In addition, formula (21) implies that all three of these adjoints are full and faithful. It follows that $\widehat{\mathrm{TKK}}$ provides an equivalence between the categories of Jordan objects $\mathcal{JP}$, $\mathcal{JTS}$, $\mathcal{JA}$ on the one hand and full subcategories of $\mathcal{LA}_{3\text{-gr}}$, $\mathcal{LA}^{\varepsilon}_{3\text{-gr}}$, and $\mathcal{LA}_{\mathrm{A}_1-\mathrm{gr}}$, respectively, on the other hand. Our next goal is to give intrinsic descriptions of these subcategories.

6.2. **Images of $\widehat{\mathrm{TKK}}$.** The model for further development is the theory of central extensions of perfect Lie algebras (e.g. see [2, Sect. 7.9]) and its $\mathbb{Z}_2$-graded version described in [13]. Here, we provide only a brief version of these standard arguments.

To describe Lie algebras L isomorphic to $\widehat{\mathrm{TKK}}(P)$, we note first that definition (9) implies the equality $L_0 = [L_{-1}, L_1]$. So for a 3-graded Lie algebra $L = L_{-1} \oplus L_0 \oplus L_1$, we say that L is 0-*perfect* provided $L_0 = [L_{-1}, L_1]$. In particular the graded algebras $\mathrm{TKK}(P)$ and $\widehat{\mathrm{TKK}}(P)$ are 0-perfect. Furthermore, we noted in Theorem 5.3 that $\widehat{\mathrm{TKK}}(P)$ is a central extension of $\mathrm{TKK}(P)$. In fact this extension is universal in the class of central 0-extensions. Here we say that a graded homomorphism $\varphi : K_{-1} \oplus K_0 \oplus K_1 \to L_{-1} \oplus L_0 \oplus L_1$ is a 0-*extension* of L if φ is surjective and $\mathrm{Ker}(\varphi) \subseteq K_0$. A central 0-extension is said to be *universal* if every other central 0-extension of L factors uniquely through it. A graded algebra is said to be *centrally* 0-*closed* if every central 0-extension of the algebra splits in a unique way.

Theorem 6.1. *If a 3-graded Lie algebra L is 0-perfect, then $\upsilon : \widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}}L) \to L$ is a universal central 0-extension of L for the map $\upsilon = \widehat{(\mathrm{id}_{\mathcal{F}_{\mathcal{P}}L})}$ as in Theorem 5.7.*

Proof. If $\varphi : K \to L$ is a central 0-extension of L then the Jordan homomorphism $\mathcal{F}_{\mathcal{P}}\varphi : \mathcal{F}_{\mathcal{P}}K \to \mathcal{F}_{\mathcal{P}}L$ is invertible and the map $(\mathcal{F}_{\mathcal{P}}\varphi)^{-1} : \mathcal{F}_{\mathcal{P}}L \to \mathcal{F}_{\mathcal{P}}K$ can be extended uniquely to a Lie algebra homomorphism $\psi : \widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}}L) \to K$ by Theorem 5.7. Hence $(\mathcal{F}_{\mathcal{P}}\varphi) \circ (\mathcal{F}_{\mathcal{P}}\psi) = \mathrm{id}_{\mathcal{F}_{\mathcal{P}}L} = \mathcal{F}_{\mathcal{P}}\upsilon$ implies $\varphi \circ \psi = \upsilon$ since $\widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}}L)$ is 0-perfect. Since $L_{\pm 1}$ generates $\widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}}L)$, ψ is unique. □

Corollary 6.2. *A universal central* 0-*extension of a* 0-*perfect Lie algebra is* 0-*perfect.*

Corollary 6.3. *For every Jordan pair P the algebra $\widehat{\mathrm{TKK}}(P)$ is a universal central 0-extension of* $\mathrm{TKK}(P)$.

Theorem 6.4. *Assume that $\upsilon : U \to L$ is a central 0-extension of a 0-perfect Lie algebra L. This extension is universal if and only if U is 0-perfect and centrally* 0*-closed.*

Proof. Assume first that $\upsilon : U \to L$ is universal and note that U is 0-perfect by Corollary 6.2.

Let $\varphi : K \to U$ be a central 0-extension of U, and consider the subalgebra $K' = K_{-1} + [K_{-1}, K_1] + K_1$ of K together with the inclusion homomorphism $\iota : K' \to K$. It is straightforward to check that $\upsilon \circ \varphi \circ \iota$ is an epimorphism and $\mathrm{Ker}(\upsilon \circ \varphi \circ \iota) \subseteq K'_0$. Moreover, $\varphi(k) \in \mathrm{Ker}(\upsilon) \subseteq \mathrm{Center}(U)$ for every $k \in \mathrm{Ker}(\upsilon \circ \varphi \circ \iota)$ and therefore $[k, K_{\pm 1}] \in \mathrm{Ker}(\varphi) \cap K_{\pm 1} = 0$. Thus $\mathrm{Ker}(\upsilon \circ \varphi \circ \iota) \subseteq \mathrm{Center}(K')$ and $\upsilon \circ \varphi \circ \iota$ is a central 0-extension of L.

By universality of $\upsilon : U \to L$, there exists a map $\psi : U \to K'$ such that $(\upsilon \circ \varphi \circ \iota) \circ \psi = \upsilon$. It follows that $\varphi \circ \iota \circ \psi = \mathrm{id}_U$, so $\iota \circ \psi$ is the splitting map for φ. Since U is 0-perfect, the splitting map is unique.

Assume now that U is 0-perfect and centrally 0-closed, and consider a central 0-extension $\varphi : K \to L$.

To construct a map from U to K we consider the direct sum of algebras $K \oplus U$ with the 3-grading $(K \oplus U)_i = K_i \oplus U_i$ and the graded subalgebra $A = \{(k, u) : \varphi(k) = \upsilon(u)\}$. It is easy to see that the maps $\pi_K : A \to K$, $\pi_K(k, u) = k$ and $\pi_U : A \to U$, $\pi_U(k, u) = u$ are graded epimorphisms, and that $\varphi \circ \pi_K = \upsilon \circ \pi_U$.

We note that $\pi_U : A \to U$ is a central 0-extension of U because $\mathrm{Ker}(\pi_U) = \{(k, 0) : k \in \mathrm{Ker}(\varphi)\} \subseteq \mathrm{Center}(A) \cap A_0$. Thus there is a splitting morphism $\psi : U \to A$ with $\pi_U \circ \psi = \mathrm{id}_U$. It follows that $\varphi \circ (\pi_K \circ \psi) = (\upsilon \circ \pi_U) \circ \psi = \upsilon$. The map $\pi_K \circ \psi$ is unique with this property since U is 0-perfect. □

Corollary 6.5. *For a 3-graded Lie algebra L the map $\upsilon : \widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}} L) \to L$, defined in Theorem 6.1, is an isomorphism if and only if L is 0-perfect centrally 0-closed.*

In particular, the category of Jordan Pairs $\mathcal{JP}$ is equivalent to the category of 0-perfect centrally 0-closed 3-graded Lie algebras.

Proof. Recall that one has the full and faithful functor $\widehat{\mathrm{TKK}} : \mathcal{JP} \to \mathcal{LA}_{3\text{-gr}}$. If L is is 0-perfect and centrally 0-closed, then the map $\upsilon : \widehat{\mathrm{TKK}}(\mathcal{F}_{\mathcal{P}} L) \to L$ is an isomorphism. The converse follows immediately from Corollary 6.3 and Theorem 6.4. □

Corollary 6.6. *The category of Jordan triple systems $\mathcal{JTS}$ is equivalent to the category of 0-perfect centrally 0-closed 3-graded Lie algebras with involution.*

Proof. Let L be a 0-perfect centrally 0-closed Lie algebra with involution ε. Recall from Section 3.2 that the restriction $\widetilde{\varepsilon}$ of ε onto the pair (L_{-1}, L_1) is an involution of the Jordan pair $\mathcal{F}_{\mathcal{P}} L$, and there exists an involutory isomorphism φ from the Jordan pair with involution $(\mathcal{F}_{\mathcal{P}} L, \widetilde{\varepsilon})$ onto $(\mathcal{P}_{\mathcal{T}} L_1, \kappa)$ for the Jordan triple system $L_1 = \mathcal{F}_{\mathcal{T}}(L, \varepsilon)$ and the canonical involution κ on $\mathcal{P}_{\mathcal{T}} L_1$. Now one can verify readily that $\widehat{\mathrm{TKK}}\varphi$ is an isomorphism from (L, ε) onto $(\widehat{\mathrm{TKK}} L_1, \kappa)$. □

Corollary 6.7. *The category of unital Jordan algebras $\mathcal{JA}$ is equivalent to the category of centrally closed* A_1*-graded Lie algebras.*

Proof. Let L be a centrally closed Lie algebra with an A_1-grading. Then $L_1 = \mathcal{F}_{\mathcal{JA}} L$ is a unital Jordan algebra and the identity map $\mathrm{id} : L_1 \to \mathcal{F}_{\mathcal{JA}} L$ lifts to the A_1-graded morphism $\widehat{\mathrm{id}}$ from $\widehat{\mathrm{TKK}} L_1$ onto L. Since the kernel of $\widehat{\mathrm{id}}$ is an ideal contained in $(\widehat{\mathrm{TKK}} L_1)_0$ and hence belongs to the center of $\widehat{\mathrm{TKK}} L_1$, there is a splitting map for $\widehat{\mathrm{id}}$ which is surjective since $\widehat{\mathrm{TKK}} L_1$ is perfect. □

6.3. **Homological characterization of the extensions.** The condition on extensions in Theorem 6.4 has natural homological characterizations which we establish next using a graded version of the classical homology theory. Throughout this section we assume that k is a field of characteristic different from 2 and 3.

Recall that the homology groups $H_*(L) = H_*(L, k)$ of a Lie algebra L with coefficients in the one-dimensional trivial module k can be defined as the homology groups of the Chevalley-Eilenberg chain complex

$$... \longrightarrow \wedge^{n+1} L \xrightarrow{\delta_n} \wedge^n L \longrightarrow ...$$

where the boundary map δ_n is given by

$$\begin{aligned} &\delta_n(x_1 \wedge x_2 \wedge ... \wedge x_{n+1}) \\ (22)\qquad &= \sum_{1 \le i < j \le n+1} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge ... \wedge \widehat{x}_i \wedge ... \wedge \widehat{x}_j \wedge ... \wedge x_{n+1}, \end{aligned}$$

and $\widehat{x}_i$ indicates that x_i is omitted.

The cohomology groups $H^*(L, M)$ of L with coefficients in the any trivial L-module M are the cohomology groups of the dual cochain complex

$$... \longrightarrow \mathrm{Hom}_k(\wedge^n L, M) \xrightarrow{\delta^n} \mathrm{Hom}_k(\wedge^{n+1} L, M) \longrightarrow ...$$

with $\delta^n(f) = f \circ \delta_n$.

If L is a graded algebra and M is a graded module, one can introduce graded versions of the complexes above by setting

$$(\wedge^n L)_i = \sum_{i_1 + i_2 + ... + i_n = i} L_{i_1} \wedge L_{i_2} \wedge ... \wedge L_{i_n} \quad \text{and}$$

$$(\mathrm{Hom}_k(\wedge^n L, M))_i = \{f \in \mathrm{Hom}_k(\wedge^n L, M) : f(L_{i_1} \wedge L_{i_2} \wedge ... \wedge L_{i_n}) \subseteq M_{i_1 + i_2 + ... + i_n + i}\}$$

respectively and noting that the maps δ_* and δ^* are graded. In this case, the homology groups $H_*^{\mathrm{gr}}(L)$ of the graded Lie algebra L are the homology groups of the complex

$$... \longrightarrow (\wedge^{n+1} L)_0 \xrightarrow{\delta_n} (\wedge^n L)_0 \longrightarrow ...$$

and the cohomology groups $H^*_{\mathrm{gr}}(L, M)$ of the graded Lie algebra L are the cohomology groups of the complex

$$... \longrightarrow (\mathrm{Hom}_k(\wedge^n L, M))_0 \xrightarrow{\delta^n} (\mathrm{Hom}_k(\wedge^{n+1} L, M))_0 \longrightarrow ...$$

Theorem 6.8. *Assume that $L = L_{-1} \oplus L_0 \oplus L_1$ is a 0-perfect 3-graded Lie algebra over a field k. Then following are equivalent.*

(i) *Every central 0-extension of L splits.*
(ii) $H_2^{\mathrm{gr}}(L) = 0$.
(iii) $H^2_{\mathrm{gr}}(L, M) = 0$ *for every module M such that $M = M_0$.*

Proof. Assume that (i) holds. Then every central 0-extension of L splits in a unique way since L is 0-perfect, so by Corollary 6.5 the map $\upsilon = (\widehat{\mathrm{id}_{\mathcal{F}_{\mathcal{P}} L}})$ is an isomorphism.

Consider $\sum x_i \wedge y_i \in (\mathrm{Ker}(\delta_1))_0$. Since $L_0 = [L_{-1}, L_1]$ implies that $L_0 \wedge L_0 \subseteq L_{-1} \wedge L_1 + \mathrm{Im}(\delta_2)$, we can assume that $x_i \in L_{-1}$ and $y_i \in L_1$. Since $\sum [x_i, y_i] = 0$ and $[L_{-1}, L_1] \simeq \langle L_{-1}, L_1 \rangle$, in $\langle L_{-1}, L_1 \rangle$ one has $\sum \langle x_i, y_i \rangle = 0$, so $\sum x_i \otimes y_i \in A(L_{-1} \otimes L_1)$. It is only left to notice that identifying $L_{-1} \otimes L_1$ with $L_{-1} \wedge L_1$, we have $A(L_{-1} \otimes L_1) \subseteq \mathrm{Im}(\delta_2)$. Therefore $H_2^{\mathrm{gr}}(L) = 0$.

Assume now that $H_2^{\mathrm{gr}}(L) = 0$ and M is an L-module with the trivial grading $M = M_0$. Then $L_1 M = L_1 M_0 \subseteq M_1 = 0$ and hence $LM = ([L_1, L_1] + L_1)M = 0$. Thus M is a trivial L-module. If $\sigma : L \wedge L \to M$ is a 2-cocycle in $(\mathrm{Ker}(\delta^2))_0$, then it is easy to see that $\mathrm{Ker}(\delta_1) \subseteq \mathrm{Ker}(\sigma)$. So one can define $\tau_\sigma : L_0 \to M_0$ by setting $\tau_\sigma(x) = \sigma(\sum a_i \wedge b_i)$ for every $x = \sum [a_i, b_i] \in L_0 = [L_{-1}, L_1]$. Clearly, $\sigma = \tau_\sigma \circ \delta_1$, so σ is a co-boundary. Hence assertion (ii) implies (iii).

Assume finally that $H^2_{\mathrm{gr}}(L, M) = 0$ for every module M such that $M = M_0$. Let $\varphi : K \to L$ be a central 0-extension of L. Since $\mathrm{Ker}(\varphi) \subseteq K_0$ and L_0 is projective as k-module, there is a linear map $\eta : L \to K$ such that η preserves the grading and $\varphi \circ \eta = \mathrm{id}_L$.

We consider $\mathrm{Ker}(\varphi)$ as a trivial L-module with the grading $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi)_0$. It is known that the map $\sigma : L \wedge L \to \mathrm{Ker}(\varphi)$ defined by $\sigma(x \wedge y) = [\eta(x), \eta(y)] - \eta([x, y])$ is a 2-cocycle. Moreover, for every $x_i \in L_i$ and $y_j \in L_j$, we have $\sigma(x_i \wedge y_j) = [\eta(x_i), \eta(y_j)] - \eta([x_i, y_j]) \in M \cap K_{i+j} \subseteq M_{i+j}$. Hence $\sigma \in Z^2_{\mathrm{gr}}(L, \mathrm{Ker}(\varphi))$.

Since $H^2_{\mathrm{gr}}(L, \mathrm{Ker}(\varphi)) = 0$, there is a linear map $\tau : L \to \mathrm{Ker}(\varphi)$ such that $\sigma(x \wedge y) = \tau([x, y])$. Then the map $\psi = \eta + \tau$ is a Lie algebra homomorphism $\psi : L \to K$. Indeed,

$$\begin{aligned}\psi([x, y])) = \eta([x, y]) + \tau([x, y]) = [\eta(x), \eta(y)] &= [\eta(x) + \tau(x), \eta(y) + \tau(y)] \\ &= [\psi(x), \psi(y)].\end{aligned}$$

Besides, $\varphi \circ \psi = \mathrm{id}_L$ since $\varphi \circ \tau = 0$. Thus ψ is a splitting map for φ. $\square$

The characterizations above enable us to restate Corollaries 6.5 and 6.6 as follows:

Theorem A. *The category of Jordan pairs is equivalent to the category of* 3*-graded Lie algebras* $L = L_{-1} \oplus L_0 \oplus L_1$ *such that* L *is* 0*-perfect and satisfies one of the equivalent conditions:*

(i) $H_2^{\mathrm{gr}}(L) = 0$
(ii) $H^2_{\mathrm{gr}}(L, M) = 0$ *for every module* M *with the trivial grading* $M = M_0$.

Theorem B. *The category of Jordan triple systems is equivalent to the category of* 3*-graded Lie algebras* $L = L_{-1} \oplus L_0 \oplus L_1$ *with involution such that* L *is* 0*-perfect and satisfies one of the equivalent conditions:*

(i) $H_2^{\mathrm{gr}}(L) = 0$
(ii) $H^2_{\mathrm{gr}}(L, M) = 0$ *for every module* M *with the trivial grading* $M = M_0$.

Well-known facts on $H_2(L)$ and $H^2(L, M)$ and central extensions (see for example [2, Chapter 7]) together with Corollary 6.7 imply:

Theorem C. *The category of unital Jordan algebras is equivalent to the category of* A_1*-graded Lie algebras* L *satisfying one of the equivalent conditions:*

(i) $H_2(L) = 0$
(ii) $H^2(L, M) = 0$ *for every trivial module* M.

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