A JORDAN APPROACH TO FINITARY LIE ALGEBRAS

ANTONIO FERNÁNDEZ LÓPEZ

Dedicated to Professor Angel Rodríguez Palacios

Abstract. A Lie algebra L over a field $\mathbb{F}$ is said to be finitary if it is isomorphic to a subalgebra of the Lie algebra of finite rank linear transformations of a vector space over $\mathbb{F}$. A nonzero element $a \in L$ is said to be extremal if $\mathrm{ad}_a^2 L = \mathbb{F}a$

By using Baranov's classification, it is not difficult to verify that any simple finitary Lie algebra over an algebraically closed field of characteristic 0 is spanned by extremal elements. In this note, we provide a classification-free proof of this result by using Jordan theory instead of representation theory.

Introduction

Let L be a simple infinite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. According to Baranov's structure theorem [2, Corollary 1.2], L is finitary if and only if it is isomorphic to one of the following:

(1) $\mathfrak{f}sl_Y(X) := [\mathcal{F}_Y(X), \mathcal{F}_Y(X)]$,
(2) $\mathfrak{f}o(X) := \mathrm{Skew}(\mathcal{F}_X(X), \#)$,
(3) $\mathfrak{f}sp(X) := \mathrm{Skew}(\mathcal{F}_X(X), \#)$,

where in (1), X, Y are vector spaces over $\mathbb{F}$ dually paired by a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{F}$ and $\mathcal{F}_Y(X)$ denotes the associative algebra of finite rank linear transformations $a : X \to X$ having a (unique) adjoint $a^\# : Y \to Y$, $\langle ax, y \rangle = \langle x, a^\# y \rangle$ for all $x \in X, y \in Y$; and in (2) (resp. (3)), $Y = X$ and the form $\langle \cdot, \cdot \rangle$ is symmetric (resp. alternate).

As checked in [7, Proposition 6.4], each of the Lie algebras listed above contains an extremal element (in fact, these algebras are spanned by extremal elements). The converse of this result, i.e., every simple Lie algebra containing an extremal element over an algebraically closed field of characteristic 0 is finitary, also holds and was derived

---

Key words and phrases. Finitary Lie algebra, Jordan algebra, polynomial identity, algebraic adjoint representation, Jordan and extremal elements.

Supported by the MEC and Fondos FEDER, MTM2014-52470-P.

in [5] from a more general theorem, due to Zelmanov, on simple Lie algebras having a finite grading [15].

In this note, we prove the existence of extremal elements in any nondegenerate finitary Lie algebra L over an algebraically closed field $\mathbb{F}$ of characteristic 0 as follows:

• Any finitary Lie algebra has an algebraic adjoint representation. Hence L contains a nonzero Jordan element, i.e., an element a such that $\mathrm{ad}_a^3 = 0$.

• The Jordan algebra L_a attached to this Jordan element is algebraic of bounded degree and inherits nondegeneracy from L. Hence L_a has finite capacity.

• Any division idempotent of L_a yields an abelian minimal inner ideal B of L.

• Any $0 \neq b \in B$ is a Jordan element of L such that L_b is an algebraic division Jordan algebra over $\mathbb{F}$.

• Since $\mathbb{F}$ is algebraically closed, L_b is actually one-dimensional, equivalently, b is an extremal element of L.

1. Common features of Lie and Jordan algebras

1. Throughout this section $\mathbb{F}$ will stand for a field of characteristic different from 2. We will deal with Lie algebras L [10, 11], with $[x, y]$ denoting the Lie product and ad_x the inner derivation determined by x; Jordan algebras J [12, 13], with Jordan product $x \cdot y$, multiplication operators $m_x : y \mapsto x \cdot y$, quadratic operators $U_x = 2m_x^2 - m_{x^2}$ and triple product $V_{x,y}z = \{x, y, z\} = U_{x+z}y - U_x y - U_z y$; and associative algebras A, with product denoted by juxtaposition and left (resp. right) multiplication operators denoted by l_x (resp. r_x), $x \in L$. Algebras (Lie, Jordan or associative) will be understood over the field $\mathbb{F}$.

Any associative algebra A gives rise to a Lie algebra A^- with Lie product $[x, y] := xy - yx$, and a Jordan algebra A^+ with Jordan product $x \cdot y := 1/2(xy + yx)$. A Jordan algebra J is said to be *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A.

2. An element x of a Jordan algebra J is called an *absolute zero divisor* if $U_x = 0$. We say that J is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $B^2 = 0$ implies $B = 0$, and *prime* if $B \cdot C = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of J. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$ (for Lie algebras over a field of characteristic 2, the standard definition of absolute zero divisor requires $\mathrm{ad}_x^2 = 0 = \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x$ for every $y \in L$); L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[B, B] = 0$ implies $B = 0$, and *prime* if $[B, C] = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of L. Nondegeneracy for both Jordan and Lie algebras implies semiprimeness, but the converse does not hold. (Notice, however, that an associative algebra A is semiprime if and only if the Jordan algebra A^+ is nondegenerate.) A Jordan or Lie algebra is said to be *strongly prime* if it is prime and nondegenerate. *Simplicity,* for both Jordan and Lie algebras, means nonzero product and the absence of nonzero proper ideals.

3. Inner Ideals. An *inner ideal* of a Jordan algebra J is a vector subspace B of J such that $\{B, J, B\} \subseteq B$. Similarly, an *inner ideal* of a Lie algebra L is a vector subspace B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e. $[B, B] = 0$.

For any element $a \in J$, $U_a J$ is an inner ideal of J, as follows from the Fundamental Jordan Identity $U_{U_x y} = U_x U_y U_x$, $x, y \in J$. Hence a nonzero subspace B of a nondegenerate algebra J is a minimal inner ideal if and only if $B = U_b J$ for any nonzero $b \in J$. As will be seen later, only a special kind of elements in Lie algebras yield inner ideals in a similar way.

2. Algebraic Jordan algebras of bounded degree

The aim of this section is to prove that a nondegenerate Jordan algebra which is algebraic of bounded degree over a field of characteristic 0 is unital and has finite capacity. Although this result is essentially known, it is included here for the sake of completeness.

Throughout this section, and unless specified otherwise, $\mathbb{F}$ will denote a field of characteristic different from 2 and J a Jordan algebra over $\mathbb{F}$.

4. Jordan PI-algebras. A Jordan polynomial $p(x_1, \dots, x_n)$ of the free Jordan $\mathbb{F}$-algebra $J(X)$ is said to be an *s-identity* if it vanishes in all special Jordan algebras, but not in all Jordan algebras. A Jordan polynomial which is not an s-identity is called *admissible.* A Jordan algebra satisfying an admissible Jordan polynomial is called a *Jordan PI-algebra.*

5. Algebraic elements and degree. Let A now stand for an associative or Jordan $\mathbb{F}$-algebra. Denote by $\hat{A}$ the *unital hull* of A, i.e., $\hat{A} = A$ if A is unital and $\hat{A} = \mathbb{F}1 \oplus A$ otherwise. An element $a \in A$ is said to be *algebraic* if it is a root of a nonzero polynomial in $\mathbb{F}[\xi]$, equivalently, the subalgebra of A generated by a is finite dimensional (notice that this characterization makes sense for any non-associative algebra). In this case, $\deg(a) := \dim_{\mathbb{F}} \mathbb{F}[a]$ is the *degree* of a, where $\mathbb{F}[a]$ denotes the *unital* subalgebra of $\hat{A}$ generated by a. Then A is said to be *algebraic* if every $x \in A$ is algebraic, and *algebraic of bounded degree* if it is algebraic and there exists a positive integer n such that $\deg(x) \leq n$ for all $x \in A$.

6. I-algebras A Jordan algebra is said to be an *I-algebra* if every non-nil inner ideal of J contains a nonzero idempotent. By [13, I.8.1 (Algebraic I Proposition)], any algebraic Jordan algebra is an I-algebra.

Lemma 2.1. [1, 1.9] *Let J be algebraic of bounded degree, then J is a Jordan PI-algebra.*

Proof. Suppose that every element of J is algebraic of degree less than or equal to a fixed number n. Then J satisfies the admissible Jordan polynomial

$$p(x, y, z) := \mathcal{A}_{n+1}(V_{x^n,y}, \cdots, V_{x,y}, V_{1,y})z$$

for the alternating standard identity

$$\mathcal{A}_{n+1}(x_1, \ldots, x_n, x_{n+1}) := \sum_{\pi} sg(\pi) x_{\pi(1)} \cdots x_{\pi(n)} x_{\pi(n+1)},$$

which proves that J is PI. □

7. Semiprimitive Jordan algebras. A Jordan algebra is said to be *semiprimitive* if has no quasi-invertible ideals (see [13, III.1.3.1] for definition), i.e., its Jacobson radical vanishes. Any semiprimitive Jordan algebra is nondegenerate [13, III.1.6.1].

Lemma 2.2. *Let J be algebraic of bounded degree. If J is nondegenerate, then it is semiprimitive.*

Proof. By Lemma 2.1, J is PI, and hence, by Zelmanov PI-Radical Theorem [14, Theorem 4], J does not contain nonzero nil ideals. This and the fact that J is an I-algebra (6) imply that J is semiprimitive (otherwise the Jacobson radical of J would contain a nonzero idempotent, a contradiction). □

8. Division Jordan algebras and division idempotents. Let J be a Jordan algebra with 1. An element $x \in J$ is called *invertible* if there exists $y \in J$ such that $x \cdot y = 1$ and $x^2 \cdot y = x$. In this case U_x is invertible and the *inverse* of x, denoted by x^{-1} is uniquely determined: $x^{-1} = U_x^{-1}x$ [13, II.6.1.1-7].

A unital Jordan algebra in which every nonzero element is invertible is called a *division Jordan algebra.* If $J = A^+$ for an associative algebra A, then A^+ is a division Jordan algebra if and only if A is a division associative algebra [13, II.6.1.5].

A nonzero idempotent $e \in J$ is called a *division idempotent* if the unital Jordan algebra U_eJ is a division Jordan algebra, equivalently (if J is nondegenerate), U_eJ is a minimal inner ideal of J.

9. Capacity. A Jordan algebra J is said to have *capacity* n if J is unital and 1 can be written as a sum $1 = e_1 + \cdots + e_n$ of n orthogonal *division idempotents*, $U_{e_i}J$ is a division Jordan algebra. By Jacobson's capacity theorem [13, I.5.2], any nondegenerate Jordan algebra having finite capacity is a direct sum of ideals each of which is a simple Jordan algebra of finite capacity.

Theorem 2.3. [13, I.8.1 (I-Finite Capacity Theorem)] *Any semiprimitive I-algebra having no infinite family of nonzero orthogonal idempotents is unital and has finite capacity.*

Lemma 2.4. *Suppose that $\mathbb{F}$ is of characteristic 0 and that J is algebraic of bounded degree n. Then any family of nonzero orthogonal idempotents of J has a cardinal less than or equal to n.*

Proof. Given $m \geq 1$, let $\{e_1, \ldots, e_m\}$ be a sequence of nonzero orthogonal idempotents of J, let $\lambda_1, \ldots, \lambda_m$ be nonzero elements of $\mathbb{F}$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$, and set $a := \lambda_1 e_1 + \cdots + \lambda_m e_m$. Vandermonde determinant says to us that the vectors $a, a^2, \ldots, a^m$ are linearly independent. This proves that any sequence of nonzero orthogonal idempotents of J has a cardinal less than or equal to n. $\square$

Theorem 2.5. *Let J be a nondegenerate algebraic Jordan algebra of bounded degree over a field $\mathbb{F}$ of characteristic 0. Then J has finite capacity.*

Proof. We know by (6) that J is an I-algebra, and have proved in Lemmas 2.2 and 2.4 that J is semiprimitive with no infinite family of nonzero orthogonal idempotents. Hence J is unital and has finite capacity by Theorem 2.3. $\square$

Remark 2.6. E. Zelmanov shows in [14] that any strongly prime algebraic Jordan PI-algebra over a field of characteristic different from 2 is simple and has finite capacity. We encourage the reader to glance through the proof and observe the amazing reduction to the case of a special Jordan algebra [14, Lemma 20] to prove the idempotent finite condition [14, Lemma 24]. The reader is also referred to [6, Theorem 2.3] for a sketch of the proof of the whole theorem.

3. The Lie-Jordan connection

Throughout this section, and unless specified otherwise, $\mathbb{F}$ will denote a field of characteristic different from 2 and 3 and L a Lie algebra over $\mathbb{F}$.

10. Engel, Jordan and Extremal Elements. An element a of a Lie algebra L is called *Engel* if the inner derivation is ad_a nilpotent. Engel elements of index of nilpotency at most 3 are called *Jordan elements.* It is easy to verify that any element a of an associative algebra A such that $a^2 = 0$ is a Jordan element of the Lie algebra A^-.

Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [3, Lemma 1.8], any Jordan element a generates the *principal* abelian inner ideal $\mathrm{ad}_a^2 L$. As in the Jordan case, this result follows from an analogue of the Fundamental Jordan Identity:

$$\mathrm{ad}^2_{\mathrm{ad}^2_a x} = \mathrm{ad}^2_a \mathrm{ad}^2_x \mathrm{ad}^2_a$$

which holds for any Jordan element a and any x in L [3, Lemma 1.7(iii)]. This identity is a good justification for using the term Jordan element. Another reason for adopting this terminology will be given later.

A nonzero element $a \in L$ is said to be *extremal* if $\mathrm{ad}_a^2 L = \mathbb{F}a$, i.e., a generates a one-dimensional inner ideal. Notice that any extremal element is a Jordan element.

11. Jordan Algebra at a Jordan Element. Let a be a Jordan element of a Lie algebra L. It was proved in [8, Theorem 2.4] that the underlying vector space L with the new product defined by $x \cdot_a y := [[x, a], y]$ is a nonassociative algebra, denoted by $L^{(a)}$, such that

(i) $\mathrm{Ker}_L a := \{x \in L : [a, [a, x]] = 0\}$ is an ideal of $L^{(a)}$.
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ is a Jordan algebra, called *the Jordan algebra of* L *at* a.

We denote by $x \mapsto \bar{x}$ the natural epimorphism of $L^{(a)}$ onto L_a and by $U^{(a)}_{\bar{x}}$ the U-operator of $\bar{x}$ in L_a. As proved in [9], many properties of a Lie algebra can be transferred to its Jordan algebras. Moreover, the nature of the Jordan element in question is reflected in the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of the Jordan theory to Lie algebras. (See [9, Proposition 4.2].)

4. Finitary Lie algebras

Throughout this section, and unless specified otherwise, $\mathbb{F}$ will stand for an arbitrary field. Given a vector space X over $\mathbb{F}$, we denote by $\mathcal{L}(X)$ the associative $\mathbb{F}$-algebra of all linear transformations of X and by $\mathcal{F}(X)$ the ideal of $\mathcal{L}(X)$ consisting of all finite rank linear transformations.

12. For any $a \in \mathcal{F}(X)$, $\operatorname{rank}(a) := \dim(aX)$ denotes the *rank* of a. The following properties of the rank are immediate. Let $a, b \in \mathcal{F}(X)$ and $c \in \mathcal{L}(X)$, Then:

(i) $\operatorname{rank}(a+b) \leq \operatorname{rank}(a) + \operatorname{rank}(b)$,
(ii) $\max\{\operatorname{rank}(ac), \operatorname{rank}(ca)\} \leq \operatorname{rank}(a)$,
(iii) $\operatorname{rank}([a,c]) \leq 2\operatorname{rank}(a)$.

13. Following [2], a Lie algebra L is said to be *finitary* (over $\mathbb{F}$) if it is isomorphic to a subalgebra of $\mathcal{F}(X)^-$, the Lie algebra consisting of all finite rank operators on a vector space X over $\mathbb{F}$.

Lemma 4.1. *Let A be an associative $\mathbb{F}$-algebra and let a, b be algebraic elements of A such that $ab = ba$. Then for any $x \in \mathbb{F}[a,b]$, x is algebraic with* $\deg(x) \leq \deg(a)\deg(b)$.

Proof. Let $r = \deg(a)$ and $s = \deg(b)$. Then $\dim_{\mathbb{F}} \mathbb{F}[a,b] \leq rs$. □

Lemma 4.2. *Let $a \in \mathcal{F}(X)$ be a finite rank linear transformation with* $\operatorname{rank}(a) = r$. *Then:*

(i) *a is algebraic with* $\deg(a) \leq r^2 + 1$.
(ii) *l_a and r_a are algebraic with* $\max\{\deg(l_a), \deg(r_a)\} \leq r^2 + 1$.
(iii) ad_a *is algebraic with* $\deg(\operatorname{ad}_a) \leq (r^2+1)^2$.

Proof. (i) The restriction $\hat{a}$ of a to the r-dimensional subspace aX is algebraic with $\deg(\hat{a}) \leq r^2$, so a is algebraic with $\deg(a) \leq r^2 + 1$.

(ii) For any associative $\mathbb{F}$-algebra A, the map $x \mapsto l_x$ (resp. $x \mapsto r_x$) is a homomorphism (resp. anti-homomorphism) of A into $\mathcal{L}(A_{\mathbb{F}})$. Hence, by (i), both l_a and r_a are algebraic of degree less than or equal to $r^2 + 1$.

(iii) Since $[l_a, r_a] = 0$, $\deg(\operatorname{ad}_a) = \deg(l_a - r_a) \leq (r^2+1)^2$ by Lemma 4.1. □

Proposition 4.3. *Let $L \leq \mathcal{F}(X)^-$ be a finitary Lie algebra over a field $\mathbb{F}$ of characteristic different from 2 and 3 and let $a \in L$ be a Jordan element. Then L_a is algebraic of bounded degree.*

Proof. Let $\operatorname{rank}(a) = r$. It follows by induction that for each $c \in L$ and any integer $n \geq 1$,

$$\bar{c}^n = \overline{\mathrm{ad}_{[c,a]}^{n-1} c}.$$

Then, by Lemma 4.2(iii), $\bar{c}$ is algebraic with $\deg(\bar{c}) \leq (4r^2 + 1)^2 + 1$. This proves that the Jordan algebra L_a is algebraic of bounded degree. □

14. A Lie algebra L over a field $\mathbb{F}$ is said to be *algebraic* if for each x in L the inner derivation ad_x is algebraic over $\mathbb{F}$.

Theorem 4.4. *Let L be a nondegenerate finitary Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic* 0*. Then L contains extremal elements.*

Proof. By Lemma 4.2(iii), L is algebraic and hence, by [9, Corollary 2.3], L contains a nonzero Jordan element a. By [8, Proposition 2.15(i)], L_a is a nondegenerate Jordan algebra, which is algebraic of bounded degree by Proposition 4.3, so L_a has finite capacity by Theorem 2.5. Let $\bar{e}$ be a division idempotent of L_a, equivalently, $U_{\bar{e}}^{(a)} L_a$ is a minimal inner ideal of L_a. By [8, 2.14], $\mathrm{ad}^2_{\mathrm{ad}^2_a e} L$ is an abelian minimal inner ideal of L, and hence any nonzero element of $\mathrm{ad}^2_{\mathrm{ad}^2_a e} L$ is extremal by [9, Proposition 4.3] (since $\mathbb{F}$ is algebraically closed). This completes the proof. □

Corollary 4.5. *Let L be a simple finitary Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic* 0*. Then L is spanned by its extremal elements.*

Proof. Any simple Lie algebra over a field of characteristic 0 is nondegenerate, so L contains extremal elements by Theorem 4.4. Let $e \in L$ be an extremal element. By the Jacobson-Morozov Lemma (see [4] for the special case of an extremal element), L has a nontrivial finite grading and therefore it is generated by Engel elements. Thus any subspace of L which is invariant under automorphisms is an ideal. In particular, the linear span of extremal elements of L is a nonzero ideal and therefore equals L, since L is simple. □

References

[1] J. A. Anquela, T. Cortés, E. García, and M. Gómez Lozano, Polynomial identities and speciality of Martindale-like covers of Jordan algebras, *J. Pure Appl. Algebra* **202** (2005), 1-10.

[2] A. A. Baranov, Finitary simple Lie algebras, *J. Algebra* **219** (1999), 299-329.

[3] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61-81.

[4] A. M. Cohen, G. Ivanyos, and D. Roozemond, Simple Lie algebras having extremal elements, *Indag. Mathem. N.S.* **19** (2008), 177-188.

[5] C. Draper, A. Fernández López, E. García, and M. Gómez Lozano, The socle of a non-degenerate Lie algebra, *J. Algebra* **319** (2008), 2372-2394.

[6] A. Fernández López, Strongly prime algebraic Lie PI-algebras, *Jordan Theory Preprint Archives* (2015).

[7] A. Fernández López, E. García, and M. Gómez Lozano, The Jordan socle and finitary Lie algebras, *J. Algebra* **280** (2004), 635-654.

[8] A. Fernández López, E. García, and M. Gómez Lozano, The Jordan algebras of a Lie algebra, *J. Algebra* **308** (2007), 164-177.

[9] A. Fernández López and A. Yu. Golubkov, Lie algebras with an algebraic adjoint representation revisited, *Manuscripta Math.* **140** (2013), 363-376.

[10] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math. **9**, Springer-Verlag, Heidelberg-Berlin-New York, 1972.

[11] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962.

[12] N. Jacobson, *Structure and Representations of Jordan Algebras*, American Mathematical Society, 1968.

[13] K. McCrimmon, *A Taste on Jordan Algebras*, Springer, 2004.

[14] E. I. Zelmanov, Absolute zero-divisors and algebraic Jordan algebras, *Siberian Math. J.* **23** (1982), 100-116.

[15] E. I. Zelmanov, Lie algebras with a finite grading *Math. USSR Sb.* **52** (1985), 347-385.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
E-mail address: emalfer@uma.es