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Jordan triples and Riemannian symmetric spaces

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Abstract

We introduce a class of real Jordan triple systems, called *JH-triples*, and show, via the Tits–Kantor–Koecher construction of Lie algebras, that they correspond to a class of Riemannian symmetric spaces including the Hermitian symmetric spaces and the symmetric R-spaces.

MSC: 17C65; 53C35; 58B25; 17B70; 32M15

Keywords: Jordan triple system; Tits–Kantor–Koecher construction; Orthogonal involutive Lie algebra; Riemannian symmetric space

1. Introduction

It is well known that Jordan algebras and Jordan triple systems can be used to give an algebraic description of a large class of symmetric manifolds and thereby one can apply Jordan theory in the study of these manifolds. In finite dimensions, the symmetric R-spaces are known to be in one–one correspondence with the isomorphism classes of real Jordan triple systems equipped with a positive definite trace form, called *compact Jordan triples* [27]. In any dimension, the simply connected Hermitian symmetric spaces are in one–one correspondence with the complex Hilbert triples called JH*-*triples* [18] and hence these manifolds can be classified by JH*-triples. The one–one correspondence between bounded symmetric domains in complex Banach spaces and JB*-triples was established in [17]. Motivated by the fruitful connections between Jordan algebras, geometry and analysis (see, for example, [9,17,18,22,23,29,34,35]), and some recent developments in [5,6,8,14,19,20], we investigate a class of Riemannian symmetric spaces which

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correspond to real Jordan triple systems, including Hermitian symmetric spaces and symmetric R-spaces. We give a unified approach to these manifolds via the Tits–Kantor–Koecher construction of Lie algebras and show they correspond to a class of *real* Jordan triple systems called *JH-triples* which include the compact Jordan triples and JH*-triples described above. In fact, non-degenerate JH*-triples are exactly the JH-triples endowed with a compatible complex structure. Therefore one can also view the approach in this paper as a real extension of the complex theory in [18] as well as an infinite dimensional extension of the work in [27].

At the outset, we give an outline of the main ideas and results in various sections, and refer to [7,34] for undefined terminology and literature. We consider throughout Riemannian symmetric spaces of any dimension, namely, connected smooth manifolds modelled on real Hilbert spaces (cf. [21]), which are symmetric in that every point in the manifold is an isolated fixed point of an involutive isometry. The Lie algebras associated with these manifolds are involutive, and among involutive Lie algebras, the ones which are *orthogonal* correspond exactly to the symmetric spaces. Our task is to identify the class of Jordan triple systems which give rise to orthogonal involutive Lie algebras. This can be achieved in two stages. First, using the Tits–Kantor–Koecher construction, one obtains a class of *normed* involutive Lie algebras which correspond to normed Jordan triple systems with continuous triple product. Next, these involutive Lie algebras, called *quasi normed Tits–Kantor–Koecher Lie algebras*, contain two Lie subalgebras which correspond to a pair of mutually dual symmetric spaces if, and only if, they are orthogonal, in which case the involutive Lie algebra is said to *admit an orthogonal symmetric part*. Finally, one characterises Jordan triple systems which correspond to the quasi normed Tits–Kantor–Koecher Lie algebras with an orthogonal symmetric part. They are the JH-triples. Although not all symmetric spaces arise in this way, the class of symmetric spaces corresponding to JH-triples, which include Hermitian symmetric spaces and R-spaces, is sufficiently large to be of interest. It seems natural to call this class *Jordan symmetric spaces*.

We show in Section 2 that the category of normed Jordan triple systems with continuous left multiplication is equivalent to the category of *quasi normed* canonical Tits–Kantor–Koecher Lie algebras. We define the *symmetric part*, so-called because of the correspondence with symmetric spaces, of these Lie algebras and determine when the Lie product on the symmetric part is norm continuous. In Section 3, we show the correspondence between symmetric spaces and orthogonal involutive Lie algebras, where a Lie algebra $\mathfrak{g}$ with involution θ is called *orthogonal* if there is a positive definite quadratic form on $\mathfrak{p}$ which is invariant under the isotropy representation of $\mathfrak{k}$ on $\mathfrak{p}$, with $\mathfrak{k}$ and $\mathfrak{p}$ being the 1 and -1 eigenspace of θ respectively. A real non-degenerate Jordan triple system V with Jordan triple product $\{\cdot,\cdot,\cdot\}$ is called a *JH-triple* if it is a real Hilbert space with continuous left multiplication $(x,y) \in V^2 \mapsto \{x,y,\cdot\}: V \to V$ and the inner product satisfies $\langle\{x,y,z\},z\rangle = \langle z,\{y,x,z\}\rangle$. We give examples of JH-triples and prove that a Jordan triple system V is a JH-triple if, and only if, the corresponding Tits–Kantor–Koecher Lie algebra $\mathfrak{L}(V)$ admits an orthogonal symmetric part. This enables us to conclude the correspondence between Jordan symmetric spaces and JH-triples in Section 4.

2. Jordan triples and Tits–Kantor–Koecher Lie algebras

For later applications, we begin by showing the correspondence between Jordan triple systems and Tits–Kantor–Koecher Lie algebras which originate in the works of Tits [33], Kantor [15,16], Koecher [22] and Meyberg [26], while we adopt the terminology in [25,36]. In finite dimensions, this correspondence depends on a trace form which is not available in infinite dimension. We

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show a version of the correspondence, with norm and involution, to suit our purpose for arbitrary dimension.

By a *Jordan triple system*, we mean a real vector space V, equipped with a trilinear triple product $\{\cdot,\cdot,\cdot\}: V^3 \to V$ which is symmetric in the outer variables and satisfies the Jordan triple identity

$$\big\{a, b, \{x, y, z\}\big\} = \big\{\{a, b, x\}, y, z\big\} - \big\{x, \{b, a, y\}, z\big\} + \big\{x, y, \{a, b, z\}\big\}$$

for $a, b, x, y, z \in V$.

A *complex Jordan triple system* is one in which the vector space V is complex and the triple product is conjugate linear in the middle variable instead. By restricting to real scalars, a complex Jordan triple system is regarded as a real one.

Definition 2.1. A Jordan triple system V is called *non-degenerate* if for each $a \in V$, we have $a = 0$ whenever $Q_a = 0$, where $Q_a : x \in V \mapsto \{a, x, a\} \in V$ denotes the *quadratic operator* defined by a.

Given a Jordan triple system V and $x, y \in V$, we define a linear operator $x \square y : V \to V$, called a *box operator* or *left multiplication operator*, by

$$(x \square y)(v) = \{x, y, v\} \quad (v \in V).$$

Let $V \square V = \{x \square y\colon\ x, y \in V\}$ and let V_0 be the real linear vector space spanned by $V \square V$. Then V_0 is a real Lie algebra in the bracket product

$$[h, k] = hk - kh$$

due to the Jordan triple identity

$$[x \square y, u \square v] = \{x, y, u\} \square v - u \square \{v, x, y\}.$$

Lemma 2.2. *Let V be a non-degenerate Jordan triple system and let $a \in V$. If $x \square a = 0$ for all $x \in V$, then $a = 0$.*

Proof. We have

$$\begin{aligned} 0 &= \big\{x, a, \{y, a, y\}\big\} \\ &= \big\{\{x, a, y\}, a, y\big\} - \big\{y, \{a, x, a\}, y\big\} + \big\{y, a, \{x, a, y\}\big\} \\ &= -\big\{y, \{a, x, a\}, y\big\} \end{aligned}$$

for all $y \in V$ which implies $Q_{\{a,y,a\}} = 0$ for all $y \in V$. Hence $\{a, y, a\} = 0$ for all $y \in V$ and $a = 0$. □

Lemma 2.3. *Let V be a non-degenerate Jordan triple system and let $\sum_j a_j \square b_j = \sum_k u_k \square v_k$. Then we have $\sum_j b_j \square a_j = \sum_k v_k \square u_k$.*

Proof. We have

$$\begin{aligned}\Big[\sum_j a_j \,\square\, b_j, x \,\square\, y\Big] &= \Big(\sum_j (a_j \,\square\, b_j)x\Big) \,\square\, y - x \,\square\, \Big(\sum_j b_j \,\square\, a_j\Big)y \\ &= \Big[\sum_k u_k \,\square\, v_k, x \,\square\, y\Big] \\ &= \Big(\sum_k (u_k \,\square\, v_k)x\Big) \,\square\, y - x \,\square\, \Big(\sum_k v_k \,\square\, u_k\Big)y\end{aligned}$$

which gives $x \,\square\, (\sum_j b_j \,\square\, a_j)y = x \,\square\, (\sum_k v_k \,\square\, u_k)y$ for all $x, y \in V$. By Lemma 2.2, we conclude $\sum_j b_j \,\square\, a_j = \sum_k v_k \,\square\, u_k$. $\square$

In finite dimensions, non-degeneracy is sometimes defined in terms of the trace form. In fact, a finite dimensional Jordan triple system V is non-degenerate if it admits a non-degenerate trace form

$$\langle x, y\rangle = \mathrm{Tr}(x \,\square\, y) \quad (x, y \in V)$$

in which case, we have

$$\big\langle (x \,\square\, y)u, v\big\rangle = \big\langle u, (y \,\square\, x)v\big\rangle \quad (u, v \in V)$$

and hence $\mathrm{Tr}(x \,\square\, y) = \mathrm{Tr}(y \,\square\, x)$. Our definition of non-degeneracy applies to infinite dimensional Jordan triple systems.

We now consider Lie algebras. By an *involutive Lie algebra* $(\mathfrak{g}, \theta)$, we mean a real Lie algebra $\mathfrak{g}$ equipped with an involution θ, i.e. an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$. We will always denote the 1-eigenspace of θ by $\mathfrak{k}$, and $\mathfrak{p}$ the (-1)-eigenspace of θ so that $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Lie algebras with a finite grading have been classified by Zelmanov in [36] where the Tits–Kantor–Koecher construction plays an important part. Given a Lie algebra $\mathfrak{g}$ which admits a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

into subspaces, $\mathfrak{g}$ is said to be *graded* if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ with $\mathfrak{g}_\alpha = 0$ if $\alpha \neq 0, \pm 1$. We introduce the following definition for subsequent developments.

Definition 2.4. A graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a *Tits–Kantor–Koecher Lie algebra* or *TKK Lie algebra* if $\mathfrak{g}$ admits an involution θ satisfying

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}.$$

We call $\mathfrak{g}$ *canonical* if $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

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We define the *canonical* part of $\mathfrak{g}$ to be the Lie subalgebra

$$\mathfrak{g}^c = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}_1$$

which is also a TKK Lie algebra with the restriction of θ as its involution.

The *symmetric part* of $\mathfrak{g}$ is defined to be the following Lie subalgebra:

$$\mathfrak{g}_s = \{a \oplus h \oplus -\theta a\colon\ a \in \mathfrak{g}_{-1},\ \theta h = h \in \mathfrak{g}_0\}$$

where

$$[a \oplus h \oplus -\theta a, b \oplus k \oplus -\theta b] = \big([a,k] + [h,b]\big) \oplus [h,k] \oplus \big([-\theta a, k] - [h, \theta b]\big).$$

The restriction of θ to $\mathfrak{g}_s$ is an involution. The Lie subalgebra

$$\mathfrak{g}_s^* = \{a \oplus h \oplus \theta a\colon\ a \in \mathfrak{g}_{-1},\ \theta h = h \in \mathfrak{g}_0\}$$

is called the *dual symmetric part* of $\mathfrak{g}$, which is the 1-eigenspace of θ.

We define the *dual involution* θ^* on $\mathfrak{g}$ by $\theta^*(a \oplus h \oplus b) = -\theta b \oplus \theta h \oplus -\theta a$, which restricts to an involution on $\mathfrak{g}_s^*$.

Remark 2.5. With the dual involution, $(\mathfrak{g}, \theta^*)$ is also a TKK Lie algebra and $\mathfrak{g}_s^*$ now becomes the symmetric part of $(\mathfrak{g}, \theta^*)$.

To facilitate later application, we outline below the Tits–Kantor–Koecher construction of Lie algebras from Jordan triple systems.

Lemma 2.6. *Let V be a non-degenerate Jordan triple system. Then there is a canonical Tits–Kantor–Koecher Lie algebra $\mathfrak{L}(V)$ with grading*

$$\mathfrak{L}(V) = \mathfrak{L}(V)_{-1} \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_1$$

and an involution θ such that $\mathfrak{L}(V)_{-1} = V = \mathfrak{L}(V)_1$ and

$$\{x, y, z\} = \big[[x, \theta y], z\big]$$

for $x, y, z \in \mathfrak{L}(V)_{-1}$.

Proof. Form the algebraic direct sum

$$\mathfrak{L}(V) = V_{-1} \oplus V_0 \oplus V_1$$

where $V_{-1} = V_1 = V$ and V_0 is the linear span of $V \square V$. By Lemma 2.3, the mapping

$$x \square y \in V \square V \mapsto y \square x \in V \square V$$

is well defined and extends to an involution $^\natural : V_0 \to V_0$ satisfying

$$[x \square y, u \square v]^\natural = -[y \square x, v \square u].$$

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This enables us to define an involutive automorphism $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V)$ by

$$\theta(x \oplus h \oplus y) = y \oplus -h^\natural \oplus x \quad (x \oplus h \oplus y \in V_{-1} \oplus V_0 \oplus V_1)$$

where we also write (x, h, y) for $x \oplus h \oplus y$, x for $(x, 0, 0)$, $\bar{y}$ for $(0, 0, y)$, and h for $(0, h, 0)$ if there is no confusion. By identifying V_α naturally as subspaces of $\mathfrak{L}(V)$, we see immediately that $\theta(V_\alpha) = V_{-\alpha}$ for $\alpha = 0, \pm 1$.

One can show that $\mathfrak{L}(V)$ is a Lie algebra in the following product:

$$[x \oplus h \oplus y, u \oplus k \oplus v] = \big(h(u) - k(x), [h, k] + x \,\square\, v - u \,\square\, y, k^\natural(y) - h^\natural(v)\big)$$

and we have

$$\{x, y, z\} = \big[[x, \theta y], z\big] \quad (x, y, z \in V_{-1}).$$

Given $x \in V_{-1}$ and $\bar{y} \in V_1$, we have $[x, \bar{y}] = [(x, 0, 0), (0, 0, y)] = (0, x \,\square\, y, 0)$ which gives $[V_{-1}, V_1] = V_0$ and hence $\mathfrak{L}(V)$ is canonical. We also have $[V_{-1}, V_{-1}] = [V_1, V_1] = 0$. □

Remark 2.7. The involution θ in the Tits–Kantor–Koecher Lie algebra $\mathfrak{g} = \mathfrak{L}(V)$ is the unique involution satisfying

$$a \,\square\, b = [a, \theta b] = -\theta(b \,\square\, a) \quad (a, b \in V_{-1} = V)$$

and is called the *main involution* (cf. [22, p. 793]).

The *dual involution* $\theta^* : \mathfrak{L}(V) \to \mathfrak{L}(V)$ is given by

$$\theta^*(x \oplus h \oplus y) = -y \oplus -h^\natural \oplus -x$$

and we have $a \,\square\, b = -[a, \theta^* b]$.

Remark 2.8. The above construction translates the non-degeneracy of a Jordan triple system V into the following property of its TKK Lie algebra $\mathfrak{L}(V)$:

$$\big[[a, \theta y], a\big] = 0 \quad \text{for all } a, y \in \mathfrak{L}(V)_{-1} \quad \Rightarrow \quad a = 0$$

which is equivalent to the condition

$$(\operatorname{ad} a)^2 = 0 \quad \Rightarrow \quad a = 0 \quad \big(a \in \mathfrak{L}(V)_{-1}\big) \tag{2.1}$$

since $(\operatorname{ad} a)^2(x \oplus h \oplus y) = -Q_a(y)$ for $a \in \mathfrak{L}(V)_{-1}$ and $x \oplus h \oplus y \in \mathfrak{L}(V)$.

A TKK Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called *non-degenerate* if $(\operatorname{ad} a)^2 = 0 \Rightarrow a = 0$ for $a \in \mathfrak{g}_{-1}$. We now show the correspondence between non-degenerate Jordan triple systems and non-degenerate Tits–Kantor–Koecher Lie algebras. Given two Jordan triple systems V and V', a bijective linear map $\varphi : V \to V'$ is called a *triple isomorphism* if it preserves the triple product:

$$\varphi\{x, y, z\} = \{\varphi x, \varphi y, \varphi z\} \quad (x, y, z \in V).$$

Two TKK Lie algebras $(\mathfrak{g}, \theta)$ and $(\mathfrak{g}', \theta')$ are said to be *isomorphic* if there is a graded isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ which commutes with involutions:

$$\psi\theta = \theta'\psi.$$

Given a TKK Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with involution θ, we can identify $\mathfrak{g}_1$ with $\mathfrak{g}_{-1}$ by θ. In fact, every TKK Lie algebra $(\mathfrak{g}, \theta)$ is isomorphic to, and hence identified with, a TKK Lie algebra

$$\mathfrak{g}' = \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$$

in which $\mathfrak{g}'_{-1} = \mathfrak{g}'_1 = \mathfrak{g}_{-1}$ and $\mathfrak{g}'_0 = \mathfrak{g}_0$, with involution $\theta'(x \oplus h \oplus y) = y \oplus \theta h \oplus x$ and product $[\cdot,\cdot]'$ defined by

$$[x, y]' = [x, \theta y], \ [h, y]' = \big(0, 0, [\theta h, y]\big) \quad \big((x, h, y) \in \mathfrak{g}'_{-1} \times \mathfrak{g}'_0 \times \mathfrak{g}'_1\big)$$

but otherwise identical with the product $[\cdot,\cdot]$ of $\mathfrak{g}$. The graded isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ is given by $\psi(x \oplus h \oplus y) = x \oplus h \oplus \theta y$.

With the above identification, let $\mathcal{G}$ be the category of non-degenerate canonical TKK Lie algebras in which the morphisms are graded isomorphisms commuting with involutions. Let $\mathcal{V}$ be the category of non-degenerate Jordan triple systems in which the morphisms are triple isomorphisms.

Theorem 2.9. *For each V in the category $\mathcal{V}$ of non-degenerate Jordan triple systems, let $\mathfrak{L}(V) \in \mathcal{G}$ be the TKK Lie algebra constructed in Lemma* 2.6. *Then the functor $\mathfrak{L} : \mathcal{V} \to \mathcal{G}$ is an equivalence of the two categories $\mathcal{V}$ and $\mathcal{G}$.*

Proof. Given a triple isomorphism $\varphi : V \to V'$ between two non-degenerate Jordan triple systems, we have

$$\varphi a \,\square\, \varphi b = \varphi(a \,\square\, b)\varphi^{-1} \quad (a, b \in V).$$

Hence there is a graded isomorphism $\widetilde{\varphi} : (\mathfrak{L}(V), \theta) \to (\mathfrak{L}(V'), \theta')$ defined by

$$\widetilde{\varphi}(a \oplus h \oplus b) = \varphi a \oplus \varphi h \varphi^{-1} \oplus \varphi b$$

which satisfies

$$\widetilde{\varphi}\theta = \theta'\varphi.$$

Conversely, given a non-degenerate TKK Lie-algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with involution θ and $\mathfrak{g}_1 = \mathfrak{g}_{-1}$, we let $V = \mathfrak{g}_{-1}$. Then it follows from the Jacobi identity and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$ that V is a non-degenerate Jordan triple system with the Jordan triple product defined by

$$\{x, y, z\} = \big[[x, \theta y], z\big]$$

and we have $\mathfrak{L}(V) = \mathfrak{g}$ if $\mathfrak{g}$ is canonical.

If $\psi : (\mathfrak{L}(V), \theta) \to (\mathfrak{L}(V'), \theta')$ is a graded isomorphism satisfying $\psi\theta = \theta'\psi$, then the restriction $\psi|_V : V \to V'$ defines a triple isomorphism. $\square$

We note that TKK Lie algebras are reduced. We recall that an involutive Lie algebra $(\mathfrak{g}, \theta)$ with eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is *reduced* if $\mathfrak{k}$ does not contain any nonzero ideal of $\mathfrak{g}$, which is equivalent to the condition that the isotropy representation $\mathrm{ad}_{\mathfrak{k}} : X \in \mathfrak{k} \mapsto ad_{\mathfrak{g}} X|_{\mathfrak{p}} \in End(\mathfrak{p})$ is faithful (cf. [2, p. 21]).

Lemma 2.10. *The Tits–Kantor–Koecher Lie algebra $(\mathfrak{L}(V), \theta)$ of a non-degenerate Jordan triple system V is reduced.*

Proof. Let $\mathfrak{L}(V) = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into eigenspaces of θ, where

$$\mathfrak{k} = \{u \oplus h \oplus u\colon\ u \in V,\ \theta h = h\}.$$

Let $X = u \oplus h \oplus u \in \mathfrak{k}$ be such that $[X, Y] = 0$ for all $Y \in \mathfrak{p}$. For each $g \in V_0$ satisfying $\theta g = -g$, and for each $v \in V$, we have

$$[u \oplus h \oplus u, v \oplus g \oplus -v] = 0 = (hv - gu) \oplus \big([h, g] - u \square v - v \square u\big) \oplus (gu - hv)$$

which gives $hv = gu$ and in particular $hv = 0$ for all $v \in V$ if $g = 0$. Hence $h = 0$ and $u \square v + v \square u = 0$ for all $v \in V$. Choose $Y = (0, g, 0)$ with $g = v \square v$, then $(v \square v)(u) = 0$ and hence $(v \square u)(v) = -(u \square v)(v) = (v \square v)(u) = 0$ for all $v \in V$. This implies $v \square u = 0$ for all $v \in V$ since $\{x + v, u, x + v\} = 0$ for all $x, v \in V$. Therefore $u = 0$ by Lemma 2.2 which proves $X = 0$. $\square$

Remark 2.11. The above arguments also show that, in a non-degenerate Jordan triple system, $\{v, u, v\} = 0$ for all v implies $u = 0$.

We now consider topological structures for Jordan triple systems and their Tits–Kantor–Koecher Lie algebras. If a Jordan triple system V is equipped with a norm, we denote by $L(V)$ the normed space of linear continuous self-maps on V.

A Lie algebra $\mathfrak{g}$ is called a *normed Lie algebra* if $\mathfrak{g}$ is a normed linear space and the Lie product is continuous:

$$\big\|[X, Y]\big\| \leqslant C\|X\|\|Y\| \quad (X, Y \in \mathfrak{g})$$

for some $C > 0$. Further, $\mathfrak{g}$ is called a *Banach Lie algebra* if $\mathfrak{g}$ is a Banach space.

A non-degenerate Tits–Kantor–Koecher Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with involution θ is said to be *quasi normed* if $\mathfrak{g}$ is a normed linear space such that the maps $\theta : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ and $(a, b) \in \mathfrak{g}_\alpha \times \mathfrak{g}_\beta \mapsto [a, b] \in \mathfrak{g}_{\alpha+\beta}$ are continuous. In this case, the dual involution $\theta^*|_{\mathfrak{g}_{-1}} = -\theta|_{\mathfrak{g}_{-1}}$ is also continuous on $\mathfrak{g}_{-1}$.

Let V be a non-degenerate Jordan triple system and let

$$\mathfrak{L}(V) = V_{-1} \oplus V_0 \oplus V_1$$

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be the corresponding Tits–Kantor–Koecher Lie algebra with the main involution θ. We denote the symmetric part of $\mathfrak{L}(V)$ by

$$\mathfrak{g}(V)=\big\{(a,h,-a)\colon\ a\in V,\ \theta h=h\big\}.$$

The restriction of θ to $\mathfrak{g}(V)$ is an involution, also denoted by θ. The dual symmetric part of $\mathfrak{L}(V)$ will be denoted by $\mathfrak{g}^*(V)$.

Lemma 2.12. *Let V be a non-degenerate Jordan triple system and $\mathfrak{L}(V)=V_{-1}\oplus V_0\oplus V_1$ its Tits–Kantor–Koecher Lie algebra with symmetric part $\mathfrak{g}(V)$. The following conditions are equivalent.*

(i) *$\mathfrak{g}(V)$ can be normed to become a normed Lie algebra.*
(ii) *$\mathfrak{g}^*(V)$ can be normed to become a normed Lie algebra.*
(iii) *V can be normed to have continuous inner derivations, that is, one can define a norm on V such that the map $(a,b)\in V\times V\mapsto a\,\square\, b-b\,\square\, a\in L(V)$ is continuous:*

$$\|a\,\square\, b-b\,\square\, a\|\leqslant c\|a\|\|b\|\quad (a,b\in V)$$

for some $c>0$.

Proof. (i) $\Rightarrow$ (iii). Let $\mathfrak{g}(V)$ be a normed Lie algebra with norm $\|\cdot\|_{\mathfrak{g}(V)}$. We equip V with the norm

$$2\|a\|=\big\|(a,0,-a)\big\|_{\mathfrak{g}(V)}\quad (a\in V).$$

Then there is some constant $C>0$ such that

$$\|a\,\square\, b-b\,\square\, a\|_{\mathfrak{g}(V)}=\big\|\big[(a,0,-a),(b,0,-b)\big]\big\|_{\mathfrak{g}(V)}\leqslant 4C\|a\|\|b\|$$

for all $a,b\in V$. It follows that $a\,\square\, b-b\,\square\, a\in L(V)$ and $(a,b)\in V^2\mapsto a\,\square\, b-b\,\square\, a\in L(V)$ is continuous since, for $x\in V$, we have

$$\begin{aligned}2\big\|(a\,\square\, b-b\,\square\, a)(x)\big\|&=\big\|\big(\{a,b,x\}-\{b,a,x\},0,\{b,a,x\}-\{a,b,x\}\big)\big\|_{\mathfrak{g}(V)}\\&=\big\|\big[\big[(a,0,-a),(b,0,-b)\big],\ (x,0,-x)\big]\big\|_{\mathfrak{g}(V)}\\&\leqslant 8C^2\|a\|\|b\|\|x\|.\end{aligned}$$

(iii) $\Rightarrow$ (i). Let V be equipped with a norm $\|\cdot\|_V$ so that the inner derivations are continuous. Given $h=\sum_j a_j\,\square\, b_j\in V_0$ satisfying $\theta h=h$, we have

$$2h=\sum_j(a_j\,\square\, b_j-b_j\,\square\, a_j)\in L(V).$$

We equip the Lie algebra $\mathfrak{g}(V)$ with the norm

$$\big\|(a,h,-a)\big\|=2\|a\|_V+\|h\|_{L(V)}\quad \text{for } (a,h,-a)\in\mathfrak{g}(V).$$

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Then we have

$$\begin{aligned}\big\|[a \oplus h \oplus -a, u \oplus g \oplus -u]\big\| &= \big\|\big(hu - ga, [h,g] - a \square u + u \square a, ga - hu\big)\big\| \\ &\leqslant 2\|h\|\|u\| + 2\|g\|\|a\| + 2\|h\|\|g\| + c\|a\|\|u\| \\ &\leqslant (2+c)\|a \oplus h \oplus -a\|\|u \oplus g \oplus -u\|\end{aligned}$$

for some $c > 0$. Hence $\mathfrak{g}(V)$ is a normed Lie algebra with the above norm.

The equivalence of (ii) and (iii) is proved analogously since

$$[a \oplus h \oplus a, u \oplus g \oplus u] = \big(hu - ga, [h,g] + a \square u - u \square a, hu - ga\big). \qquad \square$$

The continuity of the triple product in a normed Jordan triple V requires its TKK Lie algebra $\mathfrak{L}(V)$ be quasi normed, as shown below.

Lemma 2.13. *Let V be a non-degenerate Jordan triple system and $\mathfrak{L}(V) = V_{-1} \oplus V_0 \oplus V_1$ its Tits–Kantor–Koecher Lie algebra. The following conditions are equivalent.*

(i) *$\mathfrak{L}(V)$ can be quasi normed.*
(ii) *V can be normed to have continuous left multiplication, that is, one can define a norm on V such that the map $(a,b) \in V \times V \mapsto a \square b \in L(V)$ is continuous:*

$$\|a \square b\| \leqslant c\|a\|\|b\| \quad (a, b \in V)$$

for some $c > 0$.

Proof. (i) $\Rightarrow$ (ii). Let $\mathfrak{L}(V)$ be quasi normed. Then $V = V_{-1}$ inherits the norm of $\mathfrak{L}(V)$. For $\alpha, \beta \in \{0, \pm 1\}$, there are positive constants $c_{\alpha,\beta}$ such that

$$\big\|[x,y]\big\|_{\mathfrak{L}(V)} \leqslant c_{\alpha,\beta}\|x\|_{\mathfrak{L}(V)}\|y\|_{\mathfrak{L}(V)} \quad \text{for } (x,y) \in \mathfrak{L}(V)_\alpha \times \mathfrak{L}(V)_\beta.$$

Given $a, b, x \in V$, we have

$$\begin{aligned}\big\|(a \square b)(x)\big\|_V &= \big\|\big[[a, \theta b], x\big]\big\|_{\mathfrak{L}(V)} \\ &\leqslant c_{0,-1}\big\|[a,\theta b]\big\|_{\mathfrak{L}(V)}\|x\|_V \\ &\leqslant c_{0,-1}c_{-1,1}\|a\|_V\|\theta b\|_V\|x\|_V \\ &\leqslant c_{0,-1}c_{-1,1}\|\theta\|\|a\|_V\|b\|_V\|x\|_V.\end{aligned}$$

Hence V satisfies condition (ii).

(ii) $\Rightarrow$ (i). We have $V_0 \subset L(V)$ and $\mathfrak{L}(V)$ can be equipped with a natural norm

$$\|x \oplus h \oplus y\| = \|x\|_V + \|h\|_{L(V)} + \|y\|_V \quad \big(x \oplus h \oplus y \in \mathfrak{L}(V)\big).$$

We have

$$\big\|\theta(x,0,0)\big\|_{\mathfrak{L}(V)} = \big\|(0,0,x)\big\|_{\mathfrak{L}(V)} = \|x\|_V = \|(x,0,0)\|_{\mathfrak{L}(V)}$$

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and hence $\theta : V_{-1} \to V_1$ is continuous. Given $a \in \mathfrak{L}(V)_\alpha$ and $b \in \mathfrak{L}(V)_\beta$ for $\alpha \neq \beta$, we have $[a, b] = \pm a \,\square\, b$ if $\alpha, \beta \neq 0$, and for $\alpha\beta = 0$, we have $[a, b] = \pm h(x)$ for some $h = \sum_k u_k \,\square\, v_k$ with $x = a$ or b. It follows from condition (ii) that the maps $(a, b) \in \mathfrak{L}(V)_\alpha \times \mathfrak{L}(V)_\beta \mapsto [a, b] \in \mathfrak{L}(V)_{\alpha+\beta}$ are continuous. $\square$

The symmetric part $\mathfrak{g}_s$ of a quasi normed TKK Lie algebra $\mathfrak{g}$ inherits the norm of $\mathfrak{g}$. The norm completion $\bar{\mathfrak{g}}_s$ is called the *complete symmetric part* of $\mathfrak{g}$. The Lie product of $\mathfrak{g}_s$ extends to $\bar{\mathfrak{g}}_s$ and the involution θ extends to $\bar{\theta}$ on $\bar{\mathfrak{g}}_s$. The *complete dual symmetric part* $\overline{\mathfrak{g}_s^*}$ is defined likewise.

We see from Theorem 2.9 and Lemma 2.13 that the category of quasi normed canonical TKK Lie algebras is equivalent to the category of normed Jordan triple systems with continuous left multiplication.

3. Symmetric spaces and orthogonal Lie algebras

In finite dimensions, Riemannian symmetric spaces correspond to orthogonal involutive Lie algebras. The first objective of this section is to extend this correspondence to infinite dimensional setting. This then leads to the second objective of characterising the class of Jordan triples whose TKK Lie algebras admit orthogonal symmetric parts.

In what follows, we shall not distinguish a quadratic form and its associated symmetric bilinear form, and use the same notation for both.

We adapt the notion of an *orthogonal involutive Lie algebra* in finite dimensions (cf. [13, p. 213], [1, p. 35] and [2, p. 21]) to our setting below.

Definition 3.1. Let $(\mathfrak{g}, \theta)$ be an involutive Lie algebra with the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into ± 1-eigenspaces of θ, with $\mathfrak{k}$ the 1-eigenspace. Then $\mathfrak{g}$ is called *orthogonal* if there is a quadratic form

$$q : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$$

which is positive definite, i.e. $q(X, X) > 0$ for $X \in \mathfrak{p}\backslash\{0\}$, and invariant under the isotropy representation $\mathrm{ad}_{\mathfrak{k}}$ of $\mathfrak{k}$ on $\mathfrak{p}$:

$$q\big(\mathrm{ad}\, Z(X), Y\big) + q\big(X, \mathrm{ad}\, Z(Y)\big) = 0$$

for $X, Y \in \mathfrak{p}$ and $Z \in \mathfrak{k}$, where the second condition above is equivalent to

$$q\big(\mathrm{ad}\, Z(X), X\big) = 0 \quad (Z \in \mathfrak{k}, X \in \mathfrak{p}).$$

We note that q is invariant under θ.

Unless otherwise stated, all manifolds are modelled on a real Hilbert space where, as usual, a complex Hilbert space is regarded as a real Hilbert space by restricting to real scalars and taking the real part of its inner product. Given a Hilbert space H, we denote by $L(H)$ the von Neumann algebra of bounded linear operators on H. The group $\mathcal{U}(H)$ of unitary operators on H is a topological group in the strong operator topology, and is *right amenable*, that is, there is a unital positive linear functional $\Psi : C_{\mathrm{ru}}(\mathcal{U}(H)) \to \mathbb{R}$ on the space $C_{\mathrm{ru}}(\mathcal{U}(H))$ of bounded right uniformly continuous functions on $\mathcal{U}(H)$ and Ψ is invariant under right translations [12], it is called a *right invariant mean* on $C_{\mathrm{ru}}(\mathcal{U}(H))$. If H is separable, then $\mathcal{U}(H)$ is a metric group and there is a positive linear functional φ in the predual $L(H)_*$ of $L(H)$, which is *faithful*, that is, $\varphi(TT^*)=0$ implies $T=0$ for all $T\in L(H)$ (cf. [32, p. 78]), where T^* denotes the adjoint of T.

We denote by e, unless otherwise stated, the identity of a group. By a *Banach Lie subgroup* of a Banach Lie group G, we mean a direct subgroup and a submanifold K of G, in which case K is closed and a Banach Lie group in the induced topology of G [34, p. 128], and further, the quotient space G/K of left cosets carries the structure of a Banach manifold and the quotient map $\pi: G\to G/K$ is a submersion [34, Theorem 8.19]. We call $M=G/K$ a *homogeneous space* and will assume that G is connected in the sequel. The natural action of G on G/K is denoted by

$$\tau_g: aK\in G/K\mapsto gaK\in G/K \quad (g\in G).$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K, which is a split Lie subalgebra of $\mathfrak{g}$. Let $p=\pi(e)=K$. Then the differential $(d\pi)_e:\mathfrak{g}\to T_pM$ has kernel $\ker d\pi_e=\mathfrak{k}$ and gives the canonical isomorphism $\mathfrak{g}/\mathfrak{k}\approx T_pM$.

Definition 3.2. Let G/K be a homogenous space. Following [13, p. 209], we call (G,K) a *symmetric pair* if there is an involutive automorphism $\sigma: G\to G$ such that $G^0_\sigma\subset K\subset G_\sigma$ where G_σ is the fixed point set of σ and G^0_σ the connected identity component.

Now let $M=G/K$ be a Riemannian symmetric space, then (G,K) is a symmetric pair in which the involution σ of G is induced by the symmetry s_p at $p=\pi(e)\in M$ and satisfies

$$\tau_g=s_p\tau_{\sigma(g)}s_p. \tag{3.1}$$

The tangent space T_pM at p is a Hilbert space in the Riemannian metric and we denote the unitary group of T_pM by $\mathcal{U}_p$ which is an amenable group in the strong operator topology of $L(T_pM)$. The Lie algebra $\mathfrak{g}$ of G has the canonical decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$$

into ±1-eigenspaces of the involution $\theta=(d\sigma)_e$. We have

$$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k} \quad\text{and}\quad [\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}.$$

If K is compact, there is an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product on $\mathfrak{k}$. In infinite dimension, there may still be a nonzero positive semidefinite quadratic form on $\mathfrak{k}$, invariant under $\mathrm{ad}_{\mathfrak{k}}$, as shown below.

Let $\psi=\mathrm{Ad}^{G/K}: K\to\mathcal{U}_p\subset GL(T_pM)$ be the isotropy representation of K on $\mathfrak{p}=T_pM$ given by the differential map

$$\mathrm{Ad}^{G/K}k=(d\tau_k)_p: T_pM\to T_pM$$

where $(d\tau_k)_p\in\mathcal{U}_p$ and ψ induces a Lie algebra homomorphism

$$\psi_*:=d\big(\mathrm{Ad}^{G/K}\big)_e:\mathfrak{k}\to L(T_pM).$$

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In the operator norm topology, $\mathcal{U}_p$ is a Banach Lie group and the adjoint representation $\mathrm{Ad}_{\mathcal{U}_p} : \mathcal{U}_p \to \mathrm{Aut}\, \mathfrak{u}_p$ of $\mathcal{U}_p$ on its Lie algebra $\mathfrak{u}_p = \{X \in L(T_pM):\ X + X^* = 0\}$ is given by

$$(\mathrm{Ad}_{\mathcal{U}_p} u)X = uXu^* \quad (u \in \mathcal{U}_p,\ X \in \mathfrak{u}_p).$$

Lemma 3.3. *Let $\aleph_0 \geqslant \dim T_pM > 1$. Then there is a nonzero positive semidefinite quadratic form q on the Lie algebra $\mathfrak{u}_p$ of $\mathcal{U}_p$, which is $\mathrm{Ad}_{\mathcal{U}_p}\mathcal{U}_p$-invariant:*

$$q\big((\mathrm{Ad}_{\mathcal{U}_p} u)X\big) = q(X) \quad (u \in \mathcal{U}_p,\ X \in \mathfrak{u}_p)$$

and q is continuous in the norm topology of $\mathfrak{u}_p$.

Proof. We make use of amenability of $\mathcal{U}_p$ as a topological group in the strong operator topology of $L(T_pM)$. Since T_pM is separable, there is a *faithful* normal real state φ on $L(T_pM)$.

For $X, Y \in \mathfrak{u}_p$, we define a bounded function $f_{X,Y} : \mathcal{U}_p \to \mathbb{R}$ by

$$f_{X,Y}(u) = \varphi(uXY^*u^*) \quad (u \in \mathcal{U}_p).$$

We have $f_{X,Y} = f_{Y,X}$ and $f_{X,X} > 0$ for $X \neq 0$ by faithfulness of φ. Moreover, $f_{X,Y}$ is *right uniformly continuous* on $\mathcal{U}_p$, in the strong operator topology, that is, given a net (u_α) converging strongly to the identity operator $I \in L(T_pM)$, we have

$$\sup\big\{\big|f_{X,Y}(vu_\alpha) - f_{X,Y}(v)\big|:\ v \in \mathcal{U}_p\big\} \to 0.$$

Indeed, since multiplication and the $*$-operation are continuous in $\mathcal{U}_p$ in the strong operator topology, we have, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \sup_{v\in\mathcal{U}_p} \big|f_{X,Y}(vu_\alpha) - f_{X,Y}(v)\big| &= \sup_{v\in\mathcal{U}_p} \big|\varphi\big(v\big(u_\alpha XY^*u_\alpha^* - XY^*\big)v^*\big)\big| \\ &\leqslant \varphi\big(\big(u_\alpha XY^*u_\alpha^* - XY^*\big)\big(u_\alpha XY^*u_\alpha^* - XY^*\big)^*\big)^{1/2} \to 0. \end{aligned}$$

By amenability of $\mathcal{U}_p$, there is a right invariant mean $\Psi : C_{\mathrm{ru}}(\mathcal{U}_p) \to \mathbb{R}$. We can define a quadratic form q on $\mathfrak{u}_p$ by

$$q(X, X) = \Psi(f_{X,X}) \quad (X \in \mathfrak{u}_p).$$

Then q is positive semidefinite and $\mathrm{Ad}_{\mathcal{U}_p}\mathcal{U}_p$-invariant since f_{uXu^*,uXu^*} is a right translation of $f_{X,X}$:

$$f_{uXu^*,uXu^*}(v) = \varphi(vuXX^*u^*v^*) = f_{X,X}(vu)$$

and right invariance of Ψ yields

$$q(uXu^*) = \Psi(f_{uXu^*,uXu^*}) = \Psi(f_{X,X}) = q(X).$$

The inequality $\sup_{v\in\mathcal{U}_p} |f_{X_n,Y_n}(v) - f_{X,Y}(v)| \leqslant \|X_nY_n^* - XY^*\|$ implies that q is norm continuous.

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Finally, one can find $X, Y \in \mathfrak{u}_p$ such that $XX^* + YY^* \geqslant I$. Indeed, if $\dim T_pM = 2, 4, 6, \ldots, \infty$, we can pick X and Y having block diagonal matrix representation of the form

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

If T_pM has odd dimension, we can pick

$$X = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & -1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

It follows that $f_{X,X}(u) + f_{Y,Y}(u) \geqslant 1$ for all $u \in \mathcal{U}_p$ and $q(X,X) + q(Y,Y) = \Psi(f_{X,X} + f_{Y,Y}) \geqslant 1$. Hence $q \neq 0$. $\square$

Remark 3.4. If $\dim T_pM < \infty$, then the quadratic form q in Lemma 3.3 is positive definite by compactness of $\mathcal{U}_p$ since, for $X \neq 0$, we have $f_{X,X} \geqslant c > 0$ on $\mathcal{U}_p$ which implies $\Psi(f_{X,X}) \geqslant \Psi(c1) = c > 0$. We note that Ψ need not be unique.

Denote by $\mathrm{Exp}: \mathfrak{u}_p \to \mathcal{U}_p$ the exponential map. We have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\psi} & \mathcal{U}_p \subset GL(T_pM) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \mathrm{Exp}} \\ \mathfrak{k} & \xrightarrow{\psi_*} & \mathfrak{u}_p \subset L(T_pM) \end{array}$$

where, for $h = \mathrm{Ad}^{G/K} k \in \mathcal{U}_p$ with $k \in K$ and for $Y \in \mathfrak{u}_p$, we have $(\mathrm{Ad}_{\mathcal{U}_p} h)Y = hYh^{-1}$ and hence

$$\mathrm{Exp}\bigl(\mathrm{Ad}^{G/K} k\bigr)Y\bigl(\mathrm{Ad}^{G/K} k^{-1}\bigr) = \bigl(\mathrm{Ad}^{G/K} k\bigr)\mathrm{Exp}Y\bigl(\mathrm{Ad}^{G/K} k^{-1}\bigr).$$

Let $X \in \mathfrak{k}$ and $t > 0$. Let $\mathrm{Ad}: K \to \mathrm{Aut}(\mathfrak{k})$ be the adjoint representation of K. From the commutative diagram above, we have

$$\begin{aligned} \mathrm{Exp}\, t\psi_*\bigl((\mathrm{Ad}\, k)X\bigr) &= \mathrm{Ad}^{G/K}\bigl(\exp(\mathrm{Ad}\, k)tX\bigr) \\ &= \mathrm{Ad}^{G/K}\bigl(k(\exp tX)k^{-1}\bigr) \\ &= \bigl(\mathrm{Ad}^{G/K} k\bigr)\bigl(\mathrm{Ad}^{G/K} \exp tX\bigr)\bigl(\mathrm{Ad}^{G/K} k^{-1}\bigr) \\ &= \bigl(\mathrm{Ad}^{G/K} k\bigr)(\mathrm{Exp}\, \psi_* tX)\bigl(\mathrm{Ad}^{G/K} k^{-1}\bigr) \\ &= \mathrm{Exp}\, t\bigl(\mathrm{Ad}^{G/K} k\bigr)(\psi_* X)\bigl(\mathrm{Ad}^{G/K} k^{-1}\bigr) \end{aligned}$$

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which gives

$$\psi_*\big((\operatorname{Ad} k)X\big) = \operatorname{Ad}^{G/K} k(\psi_* X) \operatorname{Ad}^{G/K} k^{-1} \quad (k \in K). \tag{3.2}$$

The adjoint representation $\operatorname{Ad}^G : G \to \operatorname{Aut}(\mathfrak{g})$ splits when restricted to K:

$$\operatorname{Ad}^G(k)(X+Y) = (\operatorname{Ad} k)X + \big(\operatorname{Ad}^{G/K} k\big)Y \quad (k \in K,\ X \in \mathfrak{k},\ Y \in \mathfrak{p}).$$

Let q be the quadratic form on $\mathfrak{u}_p$ in Lemma 3.3 and define a positive semidefinite quadratic form q_0 on $\mathfrak{k}$ by

$$q_0(X,X) = q(\psi_* X, \psi_* X) \quad (X \in \mathfrak{k}).$$

Then q_0 is invariant under $\operatorname{ad}_{\mathfrak{k}}$ since for $k \in K$, we have from (3.2) that

$$\begin{aligned} q_0\big((\operatorname{Ad} k)X\big) &= q\big(\psi_*(\operatorname{Ad} kX)\big) \\ &= q\big(\operatorname{Ad}^{G/K} k(\psi_* X) \operatorname{Ad}^{G/K} k^{-1}\big) = q(\psi_* X) \end{aligned}$$

and hence the infinitesimal version

$$q_0\big(\operatorname{ad} Z(X), Y\big) + q_0\big(X, \operatorname{ad} Z(Y)\big) = 0 \quad (X, Y, Z \in \mathfrak{k}).$$

Lemma 3.5. *Let $M = G/K$ be a Riemannian symmetric space. Then the Lie algebra $\mathfrak{g}$ of G is orthogonal with respect to an involution whose* 1*-eigenspace is the Lie algebra of K.*

Proof. Let $\sigma : G \to G$ be the involution satisfying (3.1) and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ with respect to the involution $\theta = (d\sigma)_e$. We first show that the 1-eigenspace

$$\mathfrak{k} = \big\{X \in \mathfrak{g}:\ (d\sigma)_e X = X\big\}$$

coincides with the Lie algebra $\{X \in \mathfrak{g}:\ \exp tX \in K,\ \forall t \in \mathbb{R}\}$ of K. Since (G,K) a symmetric pair, we have $G_\sigma^0 \subset K \subset G_\sigma$ where G_σ is the fixed point set of σ with connected identity component G_σ^0. Hence $\sigma(\exp tX) = \exp tX$ for all $t \in \mathbb{R}$ implies $(d\sigma)_e X = X$. Conversely, $X \in \mathfrak{k}$ implies $\sigma(\exp tX) = \exp tX$ for all $t \in \mathbb{R}$ since the two one-parameter subgroups have the same initial velocity. Hence $\exp tX \in G_\sigma^0$.

Next, let $\langle \cdot,\cdot \rangle_{\mathfrak{p}}$ be the $\operatorname{Ad}^{G/K} K$-invariant inner product on the tangent space $\mathfrak{p} = T_p M$ induced from the G-invariant Riemannian metric. Then $\langle \cdot,\cdot \rangle_{\mathfrak{p}}$ is $\operatorname{ad}_{\mathfrak{k}}$-invariant on $\mathfrak{p}$ and $\mathfrak{g}$ is orthogonal. $\square$

We now extend the characterisation of finite dimensional symmetric spaces by orthogonal Lie algebras [13] to infinite dimension. By [11, p. 73], if a Banach Lie algebra is the Lie algebra of a Banach Lie group, then it is the Lie algebra of a connected and simply connected Lie group. Hence, for our purpose in the sequel, we can confine our attention to simply connected Lie groups.

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Proposition 3.6. *Let $M = G/K$ be a homogeneous space where G is a simply connected Banach Lie group and K is connected. The following conditions are equivalent.*

(i) *M is a Riemannian symmetric space.*
(ii) *The Lie algebra $\mathfrak{g}$ of G is involutive and orthogonal, and the Lie algebra $\mathfrak{k}$ of K is the 1-eigenspace of the involution.*

In the above case, the involution of $\mathfrak{g}$ is given by $\theta = (d\sigma)_e$ where $\sigma : G \to G$ is the involution induced by the symmetry of M at $p = \pi(e)$.

Proof. (i) $\Rightarrow$ (ii). This follows from Lemma 3.5.

(ii) $\Rightarrow$ (i). We show that $M = G/K$, equipped with a G-invariant metric, is a symmetric space. The involution of $\mathfrak{g}$ induces a local involutive automorphism of G, which can then be extended to an involutive automorphism σ of G by simply connectedness (cf. [4, p. 49]). Since K is connected, we have $K \subset \{g \in G\colon \sigma g = g\}$ and (G, K) is a symmetric pair. We can define an involution s on $M = G/K$ by

$$s(gK) = \sigma(g)K \quad (gK \in G/K).$$

Evidently $p = K \in G/K$ is an isolated fixed point of s. We complete the proof by showing that G/K admits a G-invariant metric and s is an isometry for this metric.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the eigenspace decomposition with respect to the involution of $\mathfrak{g}$. Orthogonality provides a positive definite quadratic form q on $\mathfrak{p}$, invariant under $ad_{\mathfrak{k}}$. Under the identification $\mathfrak{p} \approx T_pG/K$, where $p = K$ is the identity of G/K, we have $(ds)_pX = -X$ for $X \in \mathfrak{p}$ and the quadratic form q is an $\mathrm{Ad}K$-invariant inner product on T_pG/K, which extends to a G-invariant metric $\langle \cdot,\cdot \rangle$ of G/K:

$$\langle X, Y\rangle_p = \big\langle d\tau_g(X), d\tau_g(Y)\big\rangle_{gK} \quad (g \in G \text{ and } X, Y \in T_pG/K)$$

where τ_g is the left multiplication by g.

Now we show that $s : G/K \to G/K$ is an isometry in this metric. We have

$$s\tau_{g^{-1}} = \tau_{\sigma(g^{-1})}s$$

since, for each $aK \in G/K$,

$$s\tau_{g^{-1}}(aK) = s\big(g^{-1}aK\big) = \sigma\big(g^{-1}\big)\sigma(a)K = \tau_{\sigma(g^{-1})}s(aK).$$

It follows from the G-invariance of the metric that

$$\begin{aligned}
\langle X, Y\rangle_{gK} &= \langle d\tau_{g^{-1}}X, d\tau_{g^{-1}}Y\rangle_K \\
&= \big\langle (ds)_p\, d\tau_{g^{-1}}X, (ds)_p\, d\tau_{g^{-1}}Y\big\rangle_K \\
&= \big\langle d\tau_{\sigma(g^{-1})}(ds)_{gK}X, d\tau_{\sigma(g^{-1})}(ds)_{gK}Y\big\rangle_{\sigma(g^{-1})s(gK)} \\
&= \big\langle (ds)_{gK}X, (ds)_{gK}Y\big\rangle_{s(gK)}. \qquad \square
\end{aligned}$$

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We now characterise Jordan triple systems which correspond to orthogonal involutive Lie algebras. We first consider the algebraic structures. Let V be a non-degenerate Jordan triple system and $\mathfrak{L}(V)$ its Tits–Kantor–Koecher Lie algebra with the main involution θ. Let

$$V_0^\theta = \{h \in V_0\colon\ \theta h = h\} = \{h \in V_0\colon\ h + h^\natural = 0\}$$

which is a Lie subalgebra of V_0 and the symmetric part $\mathfrak{g}(V)$ is contained in $V \oplus V_0^\theta \oplus V$. The dual symmetric part $\mathfrak{g}^*(V)$, with the dual involution $\theta^*(a, h, a) = (-a, h, -a)$, is also contained in $V \oplus V_0^\theta \oplus V$.

Lemma 3.7. *Let V be a non-degenerate Jordan triple system and let $\mathfrak{L}(V)$ be its Tits–Kantor–Koecher Lie algebra, with the main involution θ. The following conditions are equivalent.*

(i) *The symmetric part $(\mathfrak{g}(V), \theta)$ of $\mathfrak{L}(V)$ is orthogonal.*
(ii) *The dual symmetric part $(\mathfrak{g}(V), \theta^*)$ of $\mathfrak{L}(V)$ is orthogonal.*
(iii) *V admits an inner product $\langle\cdot,\cdot\rangle$ satisfying*

$$\big\langle (a \,\square\, b)x, x\big\rangle = \big\langle x, (b \,\square\, a)x\big\rangle \quad (a, b, x \in V).$$

Proof. (i) $\Rightarrow$ (iii). Let $(\mathfrak{g}(V), \theta)$ be orthogonal with eigenspace decomposition

$$\begin{aligned}\mathfrak{g}(V) &= \mathfrak{k} \oplus \mathfrak{p}\\ &= \big\{(0, h, 0)\colon\ \theta h = h\big\} \oplus \big\{(a, 0, -a)\colon\ a \in V\big\}.\end{aligned}$$

Let q be a positive definite quadratic form on $\mathfrak{p}$, invariant under $ad_{\mathfrak{k}}$. Then V is endowed with the following inner product:

$$\langle x, y\rangle := q\big((x, 0, -x), (y, 0, -y)\big).$$

Let $a, b \in V$. Then $\theta(a \,\square\, b - b \,\square\, a) = a \,\square\, b - b \,\square\, a$ and the $ad_{\mathfrak{k}}$-invariance of q gives

$$\begin{aligned}0 &= q\big(\big[(0, a \,\square\, b - b \,\square\, a, 0), (x, 0, -x)\big], (x, 0, -x)\big)\\ &= q\big(\big(\{abx\} - \{bax\}, 0, \{bax\} - \{abx\}\big), (x, 0, -x)\big)\\ &= \big\langle (a \,\square\, b)x, x\big\rangle - \big\langle (b \,\square\, a)x, x\big\rangle.\end{aligned}$$

(iii) $\Rightarrow$ (i). Let $\mathfrak{k} \subset \mathfrak{g}(V)$ be the 1-eigenspace of θ and $\mathfrak{p}$ the (-1)-eigenspace. We define a positive definite quadratic form $q_V : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ by the inner product $\langle\cdot,\cdot\rangle$ of V:

$$q_V\big((a, 0, -a), (u, 0, -u)\big) = \langle a, u\rangle \quad \big((a, 0, -a), (u, 0, -u) \in \mathfrak{p}\big).$$

Then q_V is $ad_{\mathfrak{k}}$-invariant. Indeed, let $Z = (0, h, 0) \in \mathfrak{k}$. Then we have

$$\begin{aligned}q_V\big(\big[Z, (u, 0, -u)\big], (u, 0, -u)\big) &= q_V\big((hu, 0, -hu), (u, 0, -u)\big)\\ &= \langle hu, u\rangle\end{aligned}$$

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which vanishes. Indeed, let $h = \sum_j \alpha_j x_j \square y_j$. Then $\theta h = h$ implies $\sum_j \alpha_j (x_j \square y_j + y_j \square x_j) = 0$ and hence

$$\begin{aligned} 2\langle hu, u\rangle &= \sum_j \bigl\langle 2\alpha_j (x_j \square y_j)u, u\bigr\rangle \\ &= \sum_j \bigl\langle \alpha_j (x_j \square y_j)u, u\bigr\rangle + \bigl\langle u, \alpha_j (x_j \square y_j)u\bigr\rangle \\ &= \sum_j \bigl\langle \alpha_j (x_j \square y_j + y_j \square x_j)u, u\bigr\rangle = 0. \end{aligned}$$

Therefore $\mathfrak{g}(V)$ is orthogonal.

The equivalence of (ii) and (iii) is proved similarly. □

We now consider topological structures. Our goal is to characterise Jordan triple systems V whose Tits–Kantor–Koecher Lie algebra $\mathfrak{L}(V)$ admits an *orthogonal normed* symmetric part.

Definition 3.8. A non-degenerate Jordan triple system V is called a *JH-triple* if V is a Hilbert space in which the inner product $\langle\cdot,\cdot\rangle$ satisfies

$$\bigl\langle (a \square b)x, x\bigr\rangle = \bigl\langle x, (b \square a)x\bigr\rangle \quad (a, b, x \in V)$$

and there is a constant $c > 0$ such that

$$\bigl\|(a \square b)(x)\bigr\| \leqslant c\|a\|\|b\|\|x\| \quad (a, b, x \in V)$$

where the latter condition is equivalent to continuity of the left multiplication $(a, b) \in V^2 \mapsto a \square b \in L(V)$, where $L(V)$ is the Banach space of continuous linear operators on V.

Let V be a JH-triple. Then V_0^θ is a subspace of $L(V)$ and by Lemma 2.13, the TKK Lie algebra $\mathfrak{L}(V)$ is quasi normed and the symmetric part $\mathfrak{g}(V)$ is a normed Lie algebra with norm

$$\bigl\|(u, h, -u)\bigr\| = 2\langle u, u\rangle^{1/2} + \|h\|.$$

Let $\overline{V_0^\theta}$ be the closure of V_0^θ in $L(V)$. Then

$$\bar{\mathfrak{g}}(V) = \bigl\{(u, h, -u)\colon\ u \in V,\ h \in \overline{V_0^\theta}\bigr\}$$

is the completion of $\mathfrak{g}(V)$. The Lie product of $\mathfrak{g}(V)$ extends naturally to $\bar{\mathfrak{g}}(V)$. With the extended involution

$$\bar\theta(u, h, -u) = (-u, h, u) \quad \bigl((u, h, -u) \in \bar{\mathfrak{g}}(V)\bigr),$$

$(\bar{\mathfrak{g}}(V), \bar\theta)$ is a real involutive Banach Lie algebra with involution $\bar\theta$ and eigenspace decomposition

$$\bar{\mathfrak{g}}(V) = \bigl\{(0, h, 0)\colon\ h \in \overline{V_0^\theta}\bigr\} \oplus \mathfrak{p} \tag{3.3}$$

where $\mathfrak{p} = \{(u, 0, -u)\colon\ u \in V\}$ is also the (-1)-eigenspace of θ in $\mathfrak{g}(V)$.

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Likewise $\overline{\mathfrak{g}^*}(V) = \{(u,h,u)\colon u \in V,\ h \in \overline{V_0^\theta}\}$ is the completion of the dual symmetric part $\mathfrak{g}^*(V)$, which is a real involutive Banach Lie algebra with the extended involution $\overline{\theta^*}(u,h,u) = (-u,h,-u)$ for $(u,h,u) \in \overline{\mathfrak{g}^*}(V)$.

Given $h \in V_0^\theta$, we have $\langle hu, u\rangle = 0$ for all $u \in V$, by the proof of (iii) $\Rightarrow$ (i) in Lemma 3.7. Hence h is skew-symmetric with respect to the inner product of V:

$$\langle hx, y\rangle + \langle x, hy\rangle = 0 \quad (x, y \in V),$$

in other words, $h^* = -h = h^\natural$ where h^* is the adjoint of h in $L(V)$.

Theorem 3.9. *Let V be a non-degenerate Jordan triple system and $(\mathfrak{L}(V), \theta)$ its Tits–Kantor–Koecher Lie algebra, with symmetric part $\mathfrak{g}(V)$ and (-1)-eigenspace $\mathfrak{p} \subset \mathfrak{g}(V)$ of θ. The following conditions are equivalent.*

(i) *V is a JH-triple.*
(ii) *$\mathfrak{L}(V)$ is quasi normed such that $\mathfrak{p}$, with the inherited norm, is a Hilbert space and, with the extended involution $\bar\theta$ from θ, the complete symmetric part $\bar{\mathfrak{g}}(V)$ is an involutive Banach Lie algebra, orthogonal with respect to the inner product of $\mathfrak{p}$.*

Proof. (i) $\Rightarrow$ (ii). Let $(V, \langle\cdot,\cdot\rangle)$ be a JH-triple. By the above remarks, $\mathfrak{L}(V)$ is quasi normed and the norm on $\mathfrak{p}$ is a Hilbert space norm:

$$\big\|(u, 0, -u)\big\| = 2\langle u, u\rangle^{1/2}.$$

The symmetric part $\mathfrak{g}(V)$ is a normed Lie algebra and its completion $\bar{\mathfrak{g}}(V)$, with the extended involution $\bar\theta$, is an involutive Banach Lie algebra. It remains to show orthogonality.

Let $\bar{\mathfrak{g}}(V) = \bar{\mathfrak{k}} \oplus \mathfrak{p}$ be the eigenspace decomposition with respect to $\bar\theta$, where $\bar{\mathfrak{k}} = \{(0,h,0)\colon h \in \overline{V_0^\theta}\}$. Let $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ be the inner product of $\mathfrak{p}$. For $h \in \overline{V_0^\theta}$ with $h = \lim_n h_n$ and $h_n \in V_0^\theta$, we have $\langle hu, u\rangle_{\mathfrak{p}} = 4\lim_n \langle h_n u, u\rangle = 0$ for all $u \in V$. Hence, as in the proof of Lemma 3.7, $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ is invariant under $ad_{\bar{\mathfrak{k}}}$, that is, $\bar{\mathfrak{g}}(V)$ is orthogonal.

(ii) $\Rightarrow$ (i). Let $(\bar{\mathfrak{g}}(V), \bar\theta)$ be orthogonal with respect to the complete inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ of $\mathfrak{p}$. Then $\mathfrak{g}(V) = \mathfrak{k} \oplus \mathfrak{p}$ is also orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$. We equip V with the complete inner product

$$\langle x, y\rangle = \big\langle (x, 0, -x), (y, 0, -y)\big\rangle_{\mathfrak{p}}$$

whose norm is equivalent to the inherited norm on V from $\mathfrak{L}(V)$. By Lemma 3.7, we have

$$\big\langle (a \square b)x, x\big\rangle = \big\langle x, (b \square a)x\big\rangle \quad (a, b, x \in V).$$

Finally, by Lemma 2.13, we have

$$\|a \square b\| \leqslant c\|a\|_V \|b\|_V \quad (a, b \in V)$$

for some $c > 0$. Hence $(V, \langle\cdot,\cdot\rangle)$ is a JH-triple. $\square$

Remark 3.10. Evidently, there is an equivalent version to condition (ii) above, in terms of the dual symmetric part $\mathfrak{g}^*(V)$. We omit the details to avoid repetition.

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Remark 3.11. If V is finite dimensional in Theorem 3.9(i), then one can define an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k} = \{(0, h, 0)\colon\ h \in V_0^\theta\}$ by the trace:

$$\big\langle(0, h, 0), (0, g, 0)\big\rangle_{\mathfrak{k}} = \operatorname{Tr}(hg^*) \quad \big(h, g \in V_0^\theta\big)$$

which is $ad_{\mathfrak{k}}$-invariant since

$$\big\langle\big[(0, h, 0), (0, g, 0)\big], (0, g, 0)\big\rangle_{\mathfrak{k}} = \operatorname{Tr}\big([h, g]g^*\big) = 0$$

where $[h, g]g^* = hgg^* - ghg^* = -hg^2 + ghg$ by skew-symmetry of g. It follows that the positive definite quadratic form $q : \mathfrak{g}(V) \times \mathfrak{g}(V) \to \mathbb{R}$ defined by

$$q\big((a, h, -a), (a, h, -a)\big) = \big\langle(0, h, 0), (0, h, 0)\big\rangle_{\mathfrak{k}} + \big\langle(a, 0, -a), (a, 0, -a)\big\rangle_{\mathfrak{p}}$$

is invariant under θ and $ad_{\mathfrak{k}}$.

By [1, p. 40], a finite dimensional reduced orthogonal involutive Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is semisimple if, and only if, the centralizer $\mathfrak{z}(\mathfrak{p}) = \{X \in \mathfrak{g}\colon\ [X, \mathfrak{p}] = 0\}$ is trivial.

Corollary 3.12. *Let V be a finite dimensional JH-triple. Then the TKK Lie algebra $\mathfrak{L}(V)$ is semisimple.*

Proof. By Lemma 2.10, $\mathfrak{L}(V)$ is reduced. Hence $\mathfrak{z}(\mathfrak{p}) \subset \mathfrak{p}$ by [1, Proposition 4.26]. We show that $\mathfrak{z}(\mathfrak{p}) = \{0\}$. Fix $X = a \oplus h \oplus -a \in \mathfrak{z}(\mathfrak{p})$ where $\theta h = -h$. For $u \oplus p \oplus -u \in \mathfrak{p}$, we have

$$\begin{aligned} 0 &= [a \oplus h \oplus -a, u \oplus p \oplus -u] \\ &= \big(hu - pa, [h, p] - a \,\square\, u + u \,\square\, a, hu - pa\big). \end{aligned}$$

Choose $p = 0$, then $hu = pa = 0$ for all $u \in V$ and hence $h = 0$ and $a \,\square\, u = u \,\square\, a$. If we choose $p = u \,\square\, u$, then $u \,\square\, u(a) = 0$ implies $u \,\square\, a(u) = a \,\square\, u(u) = 0$ for all $u \in V$. By Remark 2.11, we have $a = 0$. This proves $\mathfrak{z}(\mathfrak{p}) = \{0\}$. $\square$

We now give some examples of JH-triples.

Example 3.13. A finite dimensional real Jordan algebra V is called *formally real* if $a^2 + b^2 = 0$ implies $a = b = 0$ in V. Such a Jordan algebra admits a non-degenerate trace form [24, p. 118]. A finite dimensional real Jordan algebra is formally real if, and only if, it has an identity and is equipped with an inner product $\langle\cdot,\cdot\rangle$ which is associative, that is,

$$\langle ab, c\rangle = \langle b, ac\rangle$$

(cf. [9] and [3, p. 320]). It follows that a finite dimensional formally real Jordan algebra V is a JH-triple in the canonical Jordan triple product

$$\{a, b, c\} = (ab)c + a(bc) - b(ac).$$

We note that the finite dimensional formally real Jordan algebras are exactly the finite dimensional JB-algebras defined in [10].

The real Jordan algebras, called *JH-algebras*, introduced in [29] are also JH-triples. A familiar example is the *spin factor* which is an orthogonal direct sum $H \oplus \mathbb{R}$ of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and $\mathbb{R}$, with the Jordan product

$$(x \oplus \alpha)(y \oplus \beta) = (\beta x + \alpha y) \oplus \big(\langle x, y\rangle + \alpha\beta\big).$$

Example 3.14. A finite dimensional Jordan triple system V is called *positive* if the trace form $\mathrm{Tr}(x \,\square\, y)$ is positive definite. A positive Jordan triple V is a JH-triple with the inner product $\langle x, y\rangle = \mathrm{Tr}(x \,\square\, y)$. These Jordan triple systems are also called *compact* in [27] because they correspond to symmetric R-spaces which are compact. This will be seen later as a special case of the correspondence between JH-triples and symmetric spaces.

Example 3.15. A *JH*-triple*, as defined in [18], is a complex Jordan triple system V which is a complex Hilbert space in which the triple product is continuous and every operator $a \,\square\, a : V \to V$ is Hermitian. The latter condition is equivalent to the *associativity* of the inner product:

$$\big\langle (a \,\square\, b)x, y\big\rangle = \big\langle x, (b \,\square\, a)y\big\rangle \quad (a, b, x, y \in V).$$

JH*-triples can be regarded as real Jordan triple systems with *associative* real inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$, in which case the non-degenerate ones are JH-triples. However, the inner product of a JH-triple need not be associative in the above sense, even in finite dimensions. Let M_2 be the C*-algebra of 2×2 complex matrices, regarded as a real non-degenerate Jordan triple system, with the triple product

$$\{x, y, z\} = \frac{1}{2}(xyz + zyx).$$

Define a faithful positive functional $\varphi : M_2 \to \mathbb{C}$ by

$$\varphi \begin{pmatrix} s & t \\ u & v \end{pmatrix} = 2s + v.$$

Let V be a real subtriple generated by a single element $a = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$ in M_2. Then V is a real Hilbert space with the inner product

$$\langle x, y\rangle = \mathrm{Re}\,\varphi(xy^*) \quad (x, y \in V)$$

and is a JH-triple since $b \,\square\, c = c \,\square\, b$ for all $b, c \in V$, by power associativity. Nevertheless, we have $\langle (a \,\square\, a)a, a^3\rangle \neq \langle a, (a \,\square\, a)a^3\rangle$.

In fact, non-degenerate JH*-triples are exactly those JH-triples which admit compatible complex structures and they have been classified in [18,28].

Proposition 3.16. *Let V be a non-degenerate Jordan triple system. The following conditions are equivalent.*

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(i) *V is a JH*-triple.*
(ii) *V is a JH-triple with an inner product $\langle\cdot,\cdot\rangle$ and a complex structure $J : V \to V$ satisfying*

$$\langle Jx, Jy\rangle = \langle x, y\rangle \quad \text{and} \quad J(x \,\square\, y) = (x \,\square\, y)J$$

for $x, y \in V$.

Proof. We only need to show (ii) ⇒ (i). As usual, V is a complex Hilbert space with the complex inner product

$$\langle\langle x, y\rangle\rangle = \langle x, y\rangle - i\langle Jx, y\rangle.$$

Further, it is a complex Jordan triple system with the Jordan triple product

$$\{\{x, y, z\}\} = \{x, y, z\} + i\{x, Jy, z\}$$

which is conjugate linear in the middle variable. Substituting $x + y$ and $x + iy$ for z in the identity

$$\langle\langle (a \,\square\, b)z, z\rangle\rangle = \langle\langle z, (b \,\square\, a)z\rangle\rangle$$

yields $\langle\langle (a \,\square\, b)x, y\rangle\rangle = \langle\langle x, (b \,\square\, a)y\rangle\rangle$. Hence $(V, \{\{\cdot,\cdot,\cdot\}\}, \langle\langle\cdot,\cdot\rangle\rangle)$ is a JH*-triple. □

Remark 3.17. If we define a *general JH-triple* to be a Jordan triple system satisfying all the conditions for a JH-triple except non-degeneracy, then the above proof shows that the JH*-triples are precisely the general JH-triples with a compatible complex structure. In a quasi normed TKK Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, without non-degeneracy, $\mathfrak{g}_{-1}$ is a general JH-triple if the complete symmetric part $\bar{\mathfrak{g}}_s$ is orthogonal as in Theorem 3.9(ii).

Proposition 3.18. *Every finite dimensional real JB*-triple carries the structure of a JH-triple.*

Proof. We note that every non-degenerate subtriple of a JH-triple is also a JH-triple with the inherited inner product. The real JB*-triples are norm closed real subtriples of complex JB*-triples. Therefore we need only consider complex JB*-triples. Let H be a finite dimensional complex Hilbert space and $L(H)$ the complex JB*-triple of linear operators on H. Then $\|a \,\square\, b\| \leqslant \|a\|\|b\|$ for all $a, b \in L(H)$ and every subtriple of $L(H)$ is non-degenerate since $Q_a = 0$ implies $\{a, a, a\} = aa^*a = 0$. Define the canonical inner product $\langle\cdot,\cdot\rangle$ on $L(H)$ by the trace:

$$\langle x, y\rangle = \operatorname{Re}\operatorname{Tr}(xy^*) \quad \big(x, y \in L(H)\big).$$

Then $\langle\cdot,\cdot\rangle$ is associative:

$$\begin{aligned}\big\langle (a \,\square\, b)x, y\big\rangle &= \frac{1}{2}\operatorname{Re}\operatorname{Tr}(ab^*xy^* + xb^*ay^*)\\ &= \frac{1}{2}\operatorname{Re}\operatorname{Tr}(xy^*ab^* + xb^*ay^*)\\ &= \big\langle x, (b \,\square\, a)y\big\rangle \quad \big(a, b, x, y \in L(H)\big).\end{aligned}$$

It follows that $L(H)$ and all finite dimensional JW*-triples are JH-triples.

We now consider the exceptional ones. The JB*-triple $M_3(\mathcal{O})$ of 3×3 Hermitian matrices over the Cayley algebra $\mathcal{O}$ is a JB*-algebra which is the complexification of a JB-algebra and by Example 3.13, admits an associative inner product via complexification.

Finally the JB*-triple $M_{12}(\mathcal{O})$ of 1×2 matrices over $\mathcal{O}$ is a subtriple of $M_3(\mathcal{O})$ and is therefore a JH-triple. $\square$

We refer to [28] for a complete list of simple positive JH*-triples.

4. Jordan triples and symmetric spaces

We show in this final section the correspondence between JH-triples and Riemannian symmetric spaces, and conclude with some examples. We denote the centre of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ by

$$\mathfrak{z}(\mathfrak{g}) = \bigl\{X \in \mathfrak{g}\colon\ [X, \mathfrak{g}] = 0\bigr\}.$$

Although not all infinite dimensional Lie algebras are enlargeable, we have the following useful condition for enlargeability (cf. [11, p. 7] or [30]).

Lemma 4.1. *A Banach Lie algebra with trivial centre is the Lie algebra of a Banach Lie group.*

We first consider the centre of the complete symmetric part $\bar{\mathfrak{g}}(V)$ for a JH-triple V.

Lemma 4.2. *Let V be a JH-triple and $\mathfrak{g}(V)$ the symmetric part of its Tits–Kantor–Koecher Lie algebra $(\mathfrak{L}(V), \theta)$. The centre of $\bar{\mathfrak{g}}(V)$ is given by*

$$\mathfrak{z}\bigl(\bar{\mathfrak{g}}(V)\bigr) = \bigl\{(a, 0, -a)\colon\ a \,\square\, x = x \,\square\, a,\ \forall x \in V\bigr\}.$$

Proof. Let $X = a \oplus h_0 \oplus -a \in \bar{\mathfrak{g}}(V)$. If $X \in \mathfrak{z}(\bar{\mathfrak{g}}(V))$, then for $x \oplus g \oplus -x \in \mathfrak{g}(V)$, we have

$$\begin{aligned} 0 &= [a \oplus h_0 \oplus -a, x \oplus g \oplus -x] \\ &= \bigl(h_0 x - ga, [h_0, g] - a \,\square\, x + x \,\square\, a, ga - h_0 x\bigr). \end{aligned}$$

Choose $g = 0$, then $h_0 x = ga = 0$ for all $x \in V$ and hence $h_0 = 0$ and $a \,\square\, x = x \,\square\, a$. The arguments can be reversed since $a \,\square\, x = x \,\square\, a$ for all $x \in V$ implies $ga = 0$ for all $g \in V_0^{\theta}$. $\square$

The above lemma leads to the definition of the following closed subspace of a JH-triple V:

$$Z(V) = \{a \in V\colon\ a \,\square\, x = x \,\square\, a,\ \forall x \in V\}.$$

As V is a Hilbert space, we have the direct sum decomposition

$$V = Z(V) \oplus Z(V)^{\perp}.$$

Lemma 4.3. *Given a JH-triple V, the subspace $Z(V)$ is an associative subtriple of V.*

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Proof. We need only to show that $Z(V)$ is a subtriple of V for which, it suffices to show $a \in Z(V)$ implies $a^3 = \{a, a, a\} \in Z(V)$, by the polarization formulae

$$6\{b, a, b\} = (a + b)^3 + (a - b)^3 - 2a^3,$$
$$2\{a, x, b\} = \{a + b, x, a + b\} - \{a, x, a\} - \{b, x, b\}.$$

Let $a \in Z(V)$. We have

$$\begin{aligned}
\bigl\{a, x, \{a, a, y\}\bigr\} &= \bigl\{\{a, x, a\}, a, y\bigr\} - \bigl\{a, \{a, x, a\}, y\bigr\} + \bigl\{a, a, \{a, x, y\}\bigr\} \\
&= \bigl\{a, a, \{a, x, y\}\bigr\} \\
&= \bigl\{a^3, x, y\bigr\} - \bigl\{a, \{a, a, x\}, y\bigr\} + \bigl\{a, x, \{a, a, y\}\bigr\} \\
&= \bigl\{a^3, x, y\bigr\}
\end{aligned}$$

where $\{a, \{a, a, x\}, y\} = \{a, y, \{a, a, x\}\} = \{a, a, \{a, y, x\}\}$. Hence

$$\begin{aligned}
\bigl(a^3 \square x\bigr)(y) &= \bigl\{a, a, \{x, a, y\}\bigr\} \\
&= \bigl\{\{a, a, x\}, a, y\bigr\} - \bigl\{x, a^3, y\bigr\} + \bigl\{x, a, \{a, a, y\}\bigr\} \\
&= 2\bigl\{a^3, x, y\bigr\} - \bigl\{x, a^3, y\bigr\} \\
&= \bigl(x \square a^3\bigr)(y)
\end{aligned}$$

and $a^3 \in Z(V)$. $\square$

Lemma 4.4. *Let V be a JH-triple and let $\mathfrak{z}(\bar{\mathfrak{g}})$ be the centre of $\bar{\mathfrak{g}}(V)$. Then we have the decomposition $\bar{\mathfrak{g}}(V) = \mathfrak{z}(\bar{\mathfrak{g}}) \oplus \mathfrak{z}(\bar{\mathfrak{g}})^o$ where*

$$\mathfrak{z}(\bar{\mathfrak{g}})^o = \bigl\{(x, h, -x)\colon\ x \in Z(V)^{\perp} \text{ and } h \in \overline{V_0^{\theta}}\bigr\}$$

is a Lie subalgebra of $\bar{\mathfrak{g}}(V)$ with trivial centre.

Proof. The direct sum decomposition follows from Lemma 4.2 and the decomposition $V = Z(V) \oplus Z(V)^{\perp}$. We note that $\overline{V_0^{\theta}} \subset L(V)$ and by the remarks before Theorem 3.9, each $g \in \overline{V_0^{\theta}}$ is skew-symmetric with respect to the inner product of V. Given $x \in Z(V)^{\perp}$ and $g \in \overline{V_0^{\theta}}$, we have

$$\langle gx, a\rangle = -\langle x, ga\rangle = 0$$

for all $a \in Z(V)$, as in the proof of Lemma 4.2. Hence $gx \in Z(V)^{\perp}$.

Let $X = (x, h, -x)$, $Y = (y, g, -y) \in \mathfrak{z}(\bar{\mathfrak{g}})^o$. We have

$$[X, Y] = \bigl(hy - gx, [h, g] - x \square y + y \square x, gx - hy\bigr)$$

where $hy - gx \in Z(V)^{\perp}$ implies $[X, Y] \in \mathfrak{z}(\bar{\mathfrak{g}})^o$.

Further, if $[X,Y]=0$, then choosing $g=0$, we have $hy=0$ for all $y \in Z(V)^{\perp}$ and hence $h=0$ since $ha=0$ for all $a \in Z(V)$. It follows that $x \,\square\, y = y \,\square\, x$ for all $y \in Z(V)^{\perp}$. But $x \,\square\, a = a \,\square\, x$ for all $a \in Z(V)$. Hence $x \in Z(V) \cap Z(V)^{\perp}$ and $x=0$. This proves the triviality of the centre of $\mathfrak{z}(\bar{\mathfrak{g}})^{o}$. $\square$

Given a JH-triple V with TKK Lie algebra $(\mathfrak{L}(V),\theta)$, we note from (3.3) that the (-1)-eigenspace of θ in $\mathfrak{g}(V)$ coincides with the (-1)-eigenspace of $\bar{\theta}$ in the complete symmetric part $\bar{\mathfrak{g}}(V)$. This motivates the following definition.

Definition 4.5. A Riemannian symmetric space G/K is said to be *associated to a TKK Lie algebra* if the involutive Lie algebra $(\mathfrak{g},\sigma)$ of G is the complete symmetric part $(\bar{\mathfrak{h}}_s,\bar{\theta})$ of a quasi normed Tits–Kantor–Koecher Lie algebra $(\mathfrak{h},\theta)$ and the (-1)-eigenspace of θ in $\mathfrak{h}_s$ coincides with that of $\bar{\theta}$ in $\bar{\mathfrak{h}}_s$, in which case, G/K is called a *Jordan symmetric space*.

Remark 4.6. A symmetric space G/K is a Jordan symmetric space if the Lie algebra $(\mathfrak{g},\sigma)$ of G is the complete *dual* symmetric part $(\overline{\mathfrak{h}_s^*},\overline{\theta^*})$ of a quasi normed TKK Lie algebra $(\mathfrak{h},\theta)$ and the (-1)-eigenspace of θ^* in $\mathfrak{h}_s^*$ coincides with that of $\overline{\theta^*}$ in $\overline{\mathfrak{h}_s^*}$. Indeed, $\overline{\mathfrak{h}_s^*}$ is the complete symmetric part of $(\mathfrak{h},\theta^*)$ by Remark 2.5.

Theorem 4.7. *We have the following correspondence between JH-triples and Jordan symmetric spaces.*

(i) *Let V be a JH-triple. Then there is a Riemannian symmetric space G/K such that the Lie algebra of G is the complete symmetric part $\bar{\mathfrak{g}}(V)$ of the TKK Lie algebra $\mathfrak{L}(V)$ of V.*
(ii) *Let G/K be a Jordan symmetric space. Then there is a JH-triple V such that $\mathfrak{L}(V)$ is the canonical part of the TKK Lie algebra associated to G/K. Further, if G/K is associated to a canonical TKK Lie algebra, then the complete symmetric part $\bar{\mathfrak{g}}(V)$ of $\mathfrak{L}(V)$ is the Lie algebra of G.*

Proof. (i) Let V be a JH-triple and let $\mathfrak{L}(V)$ be its TKK Lie algebra with involution θ. Then the complete symmetric part $\bar{\mathfrak{g}}(V)$ is a Banach Lie algebra and θ extends to an involution $\bar{\theta}$ on $\bar{\mathfrak{g}}(V)$. By Lemma 4.4, we have $\bar{\mathfrak{g}}(V)=\mathfrak{z}(\bar{\mathfrak{g}})\oplus\mathfrak{z}(\bar{\mathfrak{g}})^{o}$ where the centre $\mathfrak{z}(\bar{\mathfrak{g}})$ is the Lie algebra of itself as a Lie group, and the Lie algebra $\mathfrak{z}(\bar{\mathfrak{g}})^{o}$ has trivial centre and is therefore the Lie algebra of a Banach Lie group. It follows that $\bar{\mathfrak{g}}(V)$ is the Lie algebra of a simply connected Banach Lie group G, and the 1-eigenspace $\mathfrak{k}=\{(0,h,0)\colon h\in\overline{V_0^{\theta}}\}\subset\bar{\mathfrak{g}}(V)$ of $\bar{\theta}$ is the Lie algebra of a connected Banach Lie subgroup K of G. By Theorem 3.9, $\bar{\mathfrak{g}}(V)$ is an orthogonal involutive Lie algebra and by Proposition 3.6, G/K is a Riemannian symmetric space.

(ii) Conversely, let G/K be a symmetric space such that the Lie algebra $(\mathfrak{g},\sigma)$ of G is the complete symmetric part $(\bar{\mathfrak{h}}_s,\bar{\theta})$ of a quasi normed Tits–Kantor–Koecher Lie algebra $(\mathfrak{h},\theta)$ and the (-1)-eigenspace of θ in $\mathfrak{h}_s$ coincides with that of $\bar{\theta}$ in $\bar{\mathfrak{h}}_s$. Then, by Theorem 2.9 and the construction in Lemma 2.6, there is a normed Jordan triple V such that

$$\mathfrak{h}=\mathfrak{h}_{-1}\oplus\mathfrak{h}_0\oplus\mathfrak{h}_1\supset\mathfrak{L}(V)=V\oplus V_0\oplus V$$

with $\mathfrak{h}_{\pm1}=V$ and $\bar{\mathfrak{h}}_s$ contains the complete symmetric part $\bar{\mathfrak{g}}(V)$ of the TKK Lie algebra $\mathfrak{L}(V)$ which is the canonical part $\mathfrak{h}^c$ of $\mathfrak{h}$.

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By Proposition 3.6, $(\bar{\mathfrak{h}}_s, \bar{\theta})$ is orthogonal with respect to a complete inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ on the (-1)-eigenspace of $\bar{\theta}$, which equals the (-1)-eigenspace $\mathfrak{p} = \{(a, 0, -a) : a \in \mathfrak{h}_{-1}\}$ of θ in $\mathfrak{h}_s$. Since $\mathfrak{p}$ is also the (-1)-eigenspace of θ in $\mathfrak{g}(V)$, it follows that $(\bar{\mathfrak{g}}(V), \bar{\theta})$ is also orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$. Hence V is a JH-triple by Theorem 3.9.

Finally, if $\mathfrak{h}$ is canonical, then $\mathfrak{h} = \mathfrak{L}(V)$ and $(\bar{\mathfrak{h}}_s, \bar{\theta}) = (\bar{\mathfrak{g}}(V), \bar{\theta})$. □

In the correspondence between Jordan symmetric spaces and JH-triples, a pair of mutually dual symmetric spaces corresponds to the symmetric part and the dual symmetric part of a TKK Lie algebra $\mathfrak{L}(V)$. Let $(\mathfrak{g}, \sigma)$ be the involutive Lie algebra of a Jordan symmetric space G/K and let V be a JH-triple such that $\mathfrak{g} = \bar{\mathfrak{g}}(V)$ and $\sigma = \bar{\theta}$, where $\mathfrak{g}(V)$ is the symmetric part of the TKK Lie algebra $(\mathfrak{L}(V), \theta)$ of V. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the eigenspace decomposition. The dual of G/K has the Lie algebra $(\mathfrak{g}', \sigma')$ where

$$g' = \mathfrak{k} \oplus i\mathfrak{p} \quad \text{and} \quad \sigma'(k + ip) = k - ip.$$

Given $\mathfrak{g} = \bar{\mathfrak{g}}(V) = \{(0, h, 0)\colon \theta h = h \in \overline{V_0^\theta}\} \oplus \{(a, 0, -a)\colon a \in V\}$, we have

$$\mathfrak{g}' = \bigl\{(0, h, 0) : \theta h = h \in \overline{V_0^\theta}\bigr\} \oplus \bigl\{(ia, 0, -ia)\colon a \in V\bigr\}.$$

The following Lie algebra isomorphism identifies $\mathfrak{g}'$ with the dual symmetric part $\overline{\mathfrak{g}*}(V)$:

$$\psi : (ia, h, -ia) \in \mathfrak{g}' \mapsto (a, h, a) \in \overline{\mathfrak{g}*}(V)$$

where $\psi\sigma' = \theta^*\psi$ and θ^* is the dual involution of $\mathfrak{L}(V)$.

We conclude with the following examples showing the correspondence between JH-triples, Hermitian symmetric spaces and R-spaces. Other examples of infinite dimensional symmetric spaces have been studied in [11].

Example 4.8. The correspondence between Hermitian symmetric spaces and JH*-triples have been shown by Kaup [18]. This correspondence can be seen from the perspective of TKK Lie algebras.

Let M be a Hermitian symmetric space and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Lie algebra and decomposition with respect to the involution $\theta = \operatorname{Ad} s_{x_0}$ induced by the symmetry s_{x_0} at a base point $x_0 \in M$. Let

$$J : \mathfrak{p} \to \mathfrak{p}$$

be the complex structure of $\mathfrak{p}$ satisfying $\theta J = J\theta$. Let $\mathfrak{p}_c$ be the complexification of $\mathfrak{p}$ and extend J to a complex linear map on $\mathfrak{p}_c$, also denoted by J. Let

$$\mathfrak{p}_+ = \{X \in \mathfrak{p}_c\colon JX = iX\}, \qquad \mathfrak{p}_- = \{X \in \mathfrak{p}_c\colon JX = -iX\}$$

be the $\pm i$-eigenspaces of J so that

$$\mathfrak{p}_c = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

and hence the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ has a decomposition

$$\mathfrak{g}_c = \mathfrak{p}_+ \oplus \mathfrak{k}_c \oplus \mathfrak{p}_-$$

where

$$X \in \mathfrak{p} \mapsto X - iJX \in \mathfrak{p}_+$$

is a complex linear isomorphism. The real Lie algebra $\mathfrak{g}_c$ has an involution given by

$$\sigma X = \theta \overline{X}$$

where the natural extension of θ to $\mathfrak{g}_c$ is still denoted by θ. We have $\sigma(\mathfrak{p}_+) = \mathfrak{p}_-$ and

$$[\mathfrak{k}_c, \mathfrak{p}_{\pm 1}] \subset \mathfrak{p}_{\pm 1}, \qquad [\mathfrak{p}_\pm, \mathfrak{p}_\pm] = 0$$

since $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{p}$. We also have $[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}_c$. Hence $(\mathfrak{g}_c, \sigma)$, as a real Lie algebra, is a Tits–Kantor–Koecher Lie algebra and the space $\mathfrak{p}_+ \approx \mathfrak{p}$ is a Jordan triple system with Jordan triple product

$$\{X, Y, Z\} = \big[[X, \sigma Y], Z\big].$$

Moreover, $\mathfrak{g}$ is orthogonal by Lemma 3.5, and identifies with the complete symmetric part of $\mathfrak{g}_c$ by the map

$$(Y, X) \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \mapsto (X - iJX, Y, X + iJX) \in \mathfrak{g}_c$$

where $X + iJX = -\sigma(X - iJX)$. Hence $\mathfrak{p}$ is a general JH-triple and also a JH*-triple by Remark 3.17.

If $\mathfrak{g}$ is finite dimensional, reduced and semisimple, then it is the symmetric part of a canonical TKK Lie algebra since in this case, we have $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ by [2, p. 23].

Conversely, let V be a non-degenerate JH*-triple and let

$$\mathfrak{L}(V) = V_{-1} \oplus V_0 \oplus V_1$$

be its Tits–Kantor–Koecher Lie algebra with the main involution θ. Define

$$\sigma X = (-1)^\alpha X \quad (X \in V_\alpha)$$

and $\overline{X} = \sigma\theta X$ for $X \in \mathfrak{L}(V)$. Let $\mathfrak{g} = \{X \in \mathfrak{L}(V)\colon \overline{X} = X\}$ be the real form of $X \mapsto \overline{X}$. Then σ is an involution on $\mathfrak{g}$. Since $X \in \mathfrak{g}$ if, and only if, $X = \sigma\theta X$, in which case X is of the form $(a, h, -a)$ with $a \in V$ and $\theta h = h$, it follows that $\sigma|_{\mathfrak{g}} = \theta|_{\mathfrak{g}}$ and $(\mathfrak{g}, \sigma)$ is the symmetric part of $\mathfrak{L}(V)$ with orthogonal completion $\bar{\mathfrak{g}} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} = \{(a, 0, -a)\colon a \in V\}$ inherits the Hermitian structure from V and $\bar{\mathfrak{g}}$ gives rise to a Hermitian symmetric space.

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Example 4.9. Loos [27] has shown the one–one correspondence between symmetric R-spaces and finite dimensional positive Jordan triple systems.

The construction in [27], same as in [31], is essentially equivalent to the TKK construction in Lemma 2.6 apart from a *sign* difference. Indeed, given a finite dimensional *positive* Jordan triple system $(V, \{\cdot,\cdot,\cdot\})$, which is a JH-triple, the Lie algebra $\mathfrak{L}$ constructed in [27] is the TKK Lie algebra $(\mathfrak{L}(V'), \theta)$, where $(V', \{\cdot,\cdot,\cdot\}')$ is the Jordan triple $(V, -2\{\cdot,\cdot,\cdot\})$.

The positive definiteness of the trace form

$$(x, y) \in V \times V \mapsto \operatorname{Tr}(x \square y) \in \mathbb{R}$$

implies that θ is a Cartan involution since the Killing form B of $\mathfrak{L}(V')$ satisfies

$$\begin{aligned} B(X, \theta X) &= -B_{\mathfrak{L}(V')_0}(h, h^\natural) - 2\operatorname{Tr}(hh^\natural) + 2\operatorname{Tr}(a \square' a) + 2\operatorname{Tr}(b \square' b) \\ &= -B_{\mathfrak{L}(V')_0}(h, h^\natural) - 2\operatorname{Tr}(hh^\natural) - 4\operatorname{Tr}(a \square a) - 4\operatorname{Tr}(b \square b) \end{aligned}$$

which is negative definite for $X = (a, h, b) \in \mathfrak{L}(V')$, where $B_{\mathfrak{L}(V')_0}$ is the Killing form of $\mathfrak{L}(V')_0 = V'_0$ and $a \square' b(\cdot) = \{a, b, \cdot\}' = -2a \square b$ (cf. [31, p. 38]). The Lie algebra of the corresponding symmetric R-space is the *dual* symmetric part $\mathfrak{g}^*(V')$ of $\mathfrak{L}(V')$, with dual involution $\theta^*(x, h, y) = (-y, -h^\natural, -x)$, where

$$\begin{aligned} \mathfrak{g}^*(V') &= \big\{(a, h, a)\colon\ a \in V',\ \theta^* h = h\big\} \\ &= \mathfrak{k}' \oplus \mathfrak{p}' \\ &= \big\{(0, h, 0)\colon\ \theta^* h = h\big\} \oplus \big\{(a, 0, a)\colon\ a \in V'\big\}. \end{aligned}$$

The restriction of the Killing form B on $\mathfrak{p}'$ is negative definite:

$$B\big((a, 0, a), (a, 0, a)\big) = 4\operatorname{Tr}(a \square' a) = -8\operatorname{Tr}(a \square a) < 0.$$

We note that $(\mathfrak{L}(V'), \theta)$ is orthogonal with respect to the positive definite quadratic form $-B(X, \theta X)$ and $\mathfrak{g}^*(V')$ is the 1-eigenspace of θ on $\mathfrak{L}(V')$, which is the Lie algebra of a compact subgroup of the Lie group of $\mathfrak{L}(V')$.

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