MATHEMATICAL LECTURES

BOUNDED SYMMETRIC DOMAINS AND JORDAN PAIRS

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Introduction

These notes are based on a series of lectures given at the University of California at Irvine in the spring of 1977. My main aim was to show how the theory of Jordan algebras and, more generally, Jordan triple systems and Jordan pairs, may be applied to study the geometry of bounded symmetric domains.

The Jordan triple system associated with a bounded symmetric domain can be described as follows. Let $\mathcal{D}$ be a bounded symmetric domain in $V = \mathbb{C}^n$, circled with respect to the origin. (This is no essential restriction since every bounded symmetric domain is biholomorphically equivalent to a circled domain which is unique up to a linear transformation.) Let $k(z,\bar{w})$ be the Bergman kernel function and $ds^2 = \sum h_{ij} dz_i d\bar{z}_j$ the Bergman metric of $\mathcal{D}$. After a linear coordinate change we may assume that $h_{ij}(0) = \delta_{ij}$. Define structure constants $C_{ijk\ell}$ by

$$C_{ijk\ell} = \left.\frac{\partial^4 \log k(z,\bar{z})}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_\ell}\right|_{z=0},$$

and for $u,v,w \in V$ define $\{u\bar{v}w\} \in V$ by

$$\{u\bar{v}w\} = \sum_{i,j,k,\ell} C_{ijk\ell} u_i \bar{v}_j w_k \vec{e}_\ell$$

where $\vec{e}_1,\cdots,\vec{e}_n$ is the standard basis of $\mathbb{C}^n$ and $u = \sum u_i\vec{e}_i$, etc. Clearly the triple product $\{u\bar{v}w\}$ is $\mathbb{C}$-linear and symmetric in u and w and $\mathbb{C}$-antilinear in v (indicated by $\bar{v}$). It turns out that it satisfies the algebraic identity

(1) $\{x\bar{y}\{u\bar{v}w\}\} - \{u\bar{v}\{x\bar{y}w\}\} = \{\{x\bar{y}u\}\bar{v}w\} - \{u\overline{\{y\bar{x}v\}}w\}$

and the positivity condition

(2) $\{u\bar{u}u\} = \lambda u \ (\lambda \in \mathbb{C}) \Rightarrow \lambda > 0$,

for all $0 \neq u \in V$.

A complex vector space V with a composition $\{u\bar{v}w\}$ as above, satisfying (1) and (2) is called a <u>positive hermitian Jordan triple system</u> (= PHJTS for short). We have associated with each circled bounded symmetric domain a PHJTS. What makes this interesting is the result that this establishes a bijection between circled bounded symmetric domains and PHJTS's.

As a typical example, the reader should keep in mind the case where $V = M_{p,q}(\mathbb{C})$, $p \times q$ matrices, and $\mathcal{D}$ is the set of all $z \in V$ for which $1_p - zz^*$ is positive definite (where $z^* = {}^t\bar{z}$ and 1_p is the $p \times p$ unit matrix). Then the Jordan triple product is given by

$$\{u\bar{v}w\} = uv^*w + wv^*u .$$

Let us take another look at the definition of a hermitian Jordan triple system. On closer inspection, this turns out to be a composite object. Namely, let V^- be the complex conjugate vector space of V; that is, $V^- = V$ as abelian groups, but scalar multiplication $\alpha\cdot v$ $(\alpha \in \mathbb{C}, v \in V^-)$ in V^- is related to scalar multiplication in V by $\alpha\cdot v = \bar{\alpha}v$. Then the maps $V \to V^-$ and $V^- \to V$ which are the identity on the underlying abelian groups are $\mathbb{C}$-antilinear isomorphisms which we shall denote by $u \to \bar{u}$. Also, the triple product $\{u\bar{v}w\}$ defines $\mathbb{C}$-trilinear maps $V \times V^- \times V \to V$, $(u,a,v) \mapsto \{uav\}$, and $V^- \times V \times V^- \to V^-$, $(a,u,b) \mapsto \{aub\} = \overline{\{\bar{a}u\bar{b}\}}$, symmetric in the outer variables, which satisfy

(J) $\{ua\{vbw\}\} - \{vb\{uaw\}\} = \{\{uav\}bw\} - \{v\{aub\}w\}$

for all $u,v,w \in V$, $a,b \in V^-$ (resp. $u,v,w \in V^-, a,b \in V$). This leads to the following definition: A pair (V^+,V^-) of vector spaces (where now V^- is not necessarily the complex conjugate of V^+) together with trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ $(\sigma = \pm)$, symmetric in the outer variables and satisfying (J), is called a <u>Jordan pair</u>. The point of introducing this concept is that it makes sense over arbitrary base fields or even rings (although (J) has to be supplemented by further identities in characteristic 2 or 3). Every Jordan triple system determines a Jordan pair (V,V^-), called the underlying Jordan pair, but different Jordan triple systems may have the same underlying Jordan pair. Remarkably enough, several objects associated with a bounded symmetric domain depend only on the Jordan pair of the corresponding PHJTS; for instance, the compact dual X of $\mathcal{D}$ (considered as an algebraic variety) or the complexification of the automorphism group of $\mathcal{D}$.

Abstracting now from the properties of the map $V \to V^-$, $v \to \bar{v}$, we define: A hermitian involution of a Jordan pair (V^+,V^-) is a $\mathbb{C}$-antilinear isomorphism $\tau : V^+ \to V^-$ such that $\tau\{uav\} = \{\tau u, \tau^{-1}a, \tau v\}$ for all $u,v \in V^+$, $a \in V^-$. Then $V = V^+$ becomes a hermitian Jordan triple system with $\{u\bar{v}w\} = \{u,\tau v,w\}$, and we say τ is positive if this Jordan triple system satisfies (2). Thus

PHJTS = Jordan pair + positive hermitian involution.

It turns out that every complex semisimple Jordan pair admits positive hermitian involutions, and any two of them are conjugate by an inner automorphism. Thus from the classification of complex semisimple Jordan pairs one obtains yet another classification of bounded symmetric domains.

Here is a short description of the contents. After recalling some basic facts on bounded domains in §1, the PHJTS (resp. Jordan pair with involution) of a circled bounded symmetric domain is introduced in §2. For technical reasons, we define this in terms of the Lie algebra of $\mathrm{Aut}(\mathcal{D})$ rather than by the fourth derivatives of the kernel function. We also express the kernel function, the Bergman metric, and the curvatune tensor in terms of the Jordan structure. In §3, we develop the algebraic machinery required to deal with Jordan triple systems.

Surprisingly little is needed, and the proofs use only basic linear algebra. Of fundamental importance is the concept of tripotent. We define odd powers in V inductively by $x^{(1)} = x$, $x^{(2n+1)} = \frac{1}{2}\{x\bar{x}x^{(2n-1)}\}$. An element e is called a tripotent if $e^{(3)} = e$. Thus tripotents are the analogues of idempotents of an algebra. Every $x \in V$ has a "spectral decomposition"

$$x = \lambda_1 e_1 + \cdots + \lambda_n e_n$$

where the e_i are orthogonal tripotents and the "eigenvalues" λ_i are positive. Moreover, the "spectral norm" $|x| = \max \lambda_i$ is a norm on V . A tripotent e gives rise to the Peirce decomposition

$$V = V_2 \oplus V_1 \oplus V_0 \tag{3}$$

of V which later leads naturally to unbounded realizations of $\mathcal{D}$ as a Siegel domain. In §4, we prove the correspondence between circled bounded symmetric domains and PHJTS's. The domain associated with a PHJTS V is simply the open unit ball of the spectral norm; in particular, it is convex. It may also be described as the set of topologically nilpotent elements of V (relative to the powers defined above). We also give a description of $\mathcal{D}$ by finitely many polynomial inequalities, and the classification based on that of semisimple Jordan pairs.

In §§5,6 the previous results are applied to study the set M of tripotents of V and the boundary of $\mathcal{D}$.

We show that M is fibered over a compact hermitian symmetric space and parametrizes the boundary components of $\mathcal{D}$. The boundary of $\mathcal{D}$ is not smooth but composed of finitely many submanifolds (a "curvilinear polyhedron"). We determine the normal cones of the convex body $\bar{\mathcal{D}}$; the cones occurring here are the selfdual cones associated with formally real Jordan algebras.

In §7, we show how to imbed the vector space $V = V^+$ of a complex semisimple Jordan pair (V^+, V^-) into a projective algebraic variety X as a dense open subset, essentially by adding the singularities of the quasi-inverse at infinity. X is the underlying algebraic variety of the compact dual of $\mathcal{D}$. Next (§8) we show that the connected automorphism group G of X is a semisimple algebraic group acting transitively on X, and by "fractional quadratic transformations" on V (generalizing the fractional linear transformations of $\mathbb{C}$). The Lie algebra of G consists of polynomial vector fields of degree ≤ 2 (when restricted to V). Also, we give a description of G by generators and relations. The complex group G has a real form $\underline{G}_0$ whose connected set of real points is the identity component of the automorphism group of $\mathcal{D}$. In §9, we show that every tripotent e of V defines a one-dimensional $\mathbb{R}$-split torus $\underline{T}_e$ of $\underline{G}_0$ and that the parabolic subgroup of $\underline{G}_0$ defined by $\underline{T}_e$ is essentially the normalizer of the boundary component corresponding to E.

In §10 we study partial Cayley transformations and Siegel domain realizations. In the Peirce decomposition (3), V_2 is a complex Jordan algebra with unit element e, and moreover, $V_2 = A \oplus iA$ where A is formally real. There is a hermitian positive definite form F on V_2 with values in V_2 given by $F(u,\bar{v}) = \{u\bar{v}e\}$ and an "action" of V_0 on V_1 by $\varphi(z)\cdot\bar{v} = \{e\bar{v}z\}$. These data define a Siegel domain of type three, namely, the set $\mathcal{S}_e$ of all $x_2 \oplus x_1 \oplus x_0 \in V_2 \oplus V_1 \oplus V_0$ for which $x_0 \in \mathcal{D}$ and $\mathrm{Re}(x_2 - \frac{1}{2}F(x_1, (\mathrm{Id} + \varphi(x_0))^{-1}\bar{x}_1)) > 0$, and $\mathcal{S}_e$ is isomorphic with $\mathcal{D}$ under the partial Cayley transformation defined by e.

In §11, we consider real bounded symmetric domains. They are defined as domains of the form $T \cap \mathcal{D}$ where $T \subset V$ is a real form of the complex vector space V with the property that complex conjugation with respect to T leaves $\mathcal{D}$ invariant. This leads immediately to real positive Jordan triple systems resp. real Jordan pairs with positive involution, and hence to an easy classification of real bounded symmetric domains. Most of the results on complex domains remain valid; in fact, some of the concepts introduced for complex domains appear more natural in the real setting.

With a few exceptions, the results of these notes are not new. The existing proofs, though, use Lie theoretic methods instead of Jordan theory. I have made no effort to give credit in each case, nor is the bibliography in any sense complete. The Jordan theoretic approach to bounded symmetric domains was pioneered by M. Koecher; in fact, much of this work arose out of the attempt to understand [K5]. It is a pleasure to acknowledge the mathematical debt I owe him.

I wish to thank R. A. Kunze and expecially H. L. Resnikoff without whose support and encouragement these notes would probably not have been written. I am grateful to the University of California at Irvine for their hospitality, and to Lynn Addington for her expert typing of the manuscript.

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§0. Notations

0.1. Let V be a finite-dimensional complex vector space, and ${}_{R}V$ the underlying real vector space. The complex conjugate of V, denoted by V^-, has the same underlying real vector space as V but scalar multiplication in V^- is twisted by complex conjugation: if αv (resp. $\alpha \cdot v$) denotes scalar multiplication of $\alpha \in \mathbb{C}$ and $v \in V$ in V (resp. V^-) then

$$\alpha \cdot v = \bar{\alpha} v .$$

A more convenient way to describe this is to introduce the maps $v \mapsto \bar{v}$ from V to V^- and V^- to V given by the identity on ${}_{R}V = {}_{R}V^-$. Then $v \mapsto \bar{v}$ is complex antilinear and $\bar{\bar{v}} = v$. If $f : V \to W$ is a linear map of complex vector spaces we denote by $\bar{f} : V^- \to W^-$ the complex-linear map given by

$$\bar{f}(\bar{v}) = \overline{f(v)} .$$

0.2. A sesquilinear form on V may and will be considered as a $\mathbb{C}$-bilinear form $\langle\, ,\, \rangle : V \times V^- \to \mathbb{C}$, written $\langle u, \bar{v} \rangle$ for $u, v \in V$. It is hermitian if $\overline{\langle u, \bar{v} \rangle} = \langle v, \bar{u} \rangle$ and a (positive definite) scalar product if $\langle u, \bar{u} \rangle > 0$ for $u \neq 0$. The transpose of an endomorphism f of V is then defined by $\langle f(u),\bar{v}\rangle = \langle u,{}^t f(\bar{v})\rangle$, and the adjoint of f is

$$f^* = {}^t\bar{f} .$$

0.3. Let $\mathcal{D} \subset V$ be open, $f : \mathcal{D} \to W$ a differentiable map, and $df(z) : {}_R V \to {}_R W$ the differential of f at $z \in \mathcal{D}$. Then $df(z)$ decomposes

$$df(z) = \partial f(z) + \bar{\partial} f(z)$$

where $\partial f(z)$ is $\mathbb{C}$-linear and $\bar{\partial} f(z)$ is $\mathbb{C}$-antilinear ; i.e., $\bar{\partial} f(z) : V^- \to W$ is $\mathbb{C}$-linear . For directional derivatives, we use the notations

$$d_v f(z) = df(z)\cdot v , \ \partial_v f(z) = \partial f(z)\cdot v , \ \bar{\partial}_{\bar{v}} f(z) = \bar{\partial} f(z)\cdot \bar{v} .$$

In terms of a $\mathbb{C}$-basis of V,

$$\partial_v f(z) = \Sigma \frac{\partial f}{\partial z_j} v_j , \ \bar{\partial}_{\bar{v}} f(z) = \Sigma \frac{\partial f}{\partial \bar{z}_j} \bar{v}_j .$$

Higher derivatives like

$$\partial_u \partial_v f(z) , \ \partial_u \bar{\partial}_{\bar{v}} f(z) , \cdots$$

are defined in an obvious way. In particular, if f is real valued then

$$h_z(u,\bar{v}) = \partial_u \bar{\partial}_{\bar{v}} f(z)$$

is a hermitian sesquilinear form on V, for every $z \in \mathcal{D}$.

0.4. For $\mathcal{D}$ as above, let $\mathcal{D}^-$ denote the domain $\mathcal{D}$ considered as a subset of V^-, with the complex structure

induced from V^-. That is, the holomorphic functions on $\mathcal{D}^-$ are the functions

$$\bar{f}(\bar{w}) = \overline{f(w)} \quad (\bar{w} \in D^-)$$

where f is holomorphic on $\mathcal{D}$. Then the map $z \mapsto \bar{z}$ from $\mathcal{D}$ to $\mathcal{D}^-$ is an antiholomorphic isomorphism.

0.5. A vector field on $\mathcal{D}$ will be identified with a map $\xi : \mathcal{D} \to V$. The Lie bracket of two vector fields ξ and η is then given by

$$[\xi,\eta](z) = d\xi(z)\cdot\eta(z) - d\eta(z)\cdot\xi(z) .$$

Note that this differs from the usual convention by sign.

0.6. The Lie algebra of a Lie group G is denoted by $\mathfrak{g}$ or $\mathrm{Lie}(G)$. For $h \in G$ we denote by $\mathrm{Int}(h)$ the inner automorphism $g \to hgh^{-1}$ of G and by $\mathrm{Ad}(h) = \mathrm{Lie}(\mathrm{Int}(h))$ the differential at the unit element; i.e., the adjoint representation of G on $\mathfrak{g}$. The adjoint representation of $\mathfrak{g}$ on itself is denoted by $\mathrm{ad}x\cdot y = [x,y]$.

0.7. Real algebraic groups and varieties are denoted by underlined symbols $\underline{G}$, $\underline{X}$, $\underline{\mathrm{Aut}}(V,V^-)$ etc. The group of real points of $\underline{G}$ is $\underline{G}(R)$, its topological identity component is often denoted by $G = \underline{G}(R)^0$.

0.8. We follow the notations and conventious of [L5] regarding Jordan algebras, pairs and triple systems. The reader should be alerted to the fact that there are two definitions of the Jordan triple product $\{xyz\}$ in the literature, differing by a factor 2. If (say) A is an associative algebra then we define $\{xyz\} = xyz + zyx$ whereas one often finds $\frac{1}{2}(xyz + zyx)$ as the Jordan triple product. In our definition, the Peirce spaces are indexed by the integers 0, 1, 2 (instead of $0, \frac{1}{2}, 1$) - as they should be, since they really are weight spaces of the multiplicative group.

0.9. The notation a.b.c refers to formula (c) of section a.b . The notation JP ij refers to the list of identities of the Appendix.

§1. Bounded symmetric domains

1.1. We begin by reviewing some basic facts on the Bergmann kernel function. Let $\mathcal{D}$ be a domain in a finite-dimensional complex vector space V, let μ be a Haar measure on the additive group of V, and let $H(\mathcal{D})$ be the set of square-integrable (with respect to μ) holomorphic functions on $\mathcal{D}$. Then $H(\mathcal{D})$ is closed in $L^2(\mathcal{D})$ and hence is a separable Hilbert space. Since evaluation at a point $w \in \mathcal{D}$ is a continuous functional on $H(\mathcal{D})$ there exists $k_w \in H(\mathcal{D})$ such that

$$(1) \qquad f(w) = \int_{\mathcal{D}} f(z)\overline{k_w(z)}\,d\mu(z)$$

for all $f \in H(\mathcal{D})$. Choosing an orthonormal basis $\varphi_0, \varphi_1, \cdots$ of $H(\mathcal{D})$ one sees that

$$(2) \qquad k_w(z) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}$$

The Bergmann kernel function $k(z,\overline{w}) \underset{\text{def}}{=} k_w(z)$ is holomorphic on $\mathcal{D} \times \mathcal{D}^-$ and clearly satisfies $\overline{k(z,\overline{w})} = k(w,\overline{z})$, $k(z,\overline{z}) \geq 0$. If g is an isomorphism of $\mathcal{D}$ onto a domain $\mathcal{D}'$ in V with kernel function k' then

(3) $$k(z,\overline{w}) = k'(g(z),\overline{g(w)})J_g(z)\overline{J_g(w)}$$

where $J_g(z) = \det(dg(z))$ is the complex Jacobian of g .

1.2. Suppose $\mathcal{D}$ is bounded. Then $k(z,\overline{z}) > 0$, and the formula

(1) $$h_z(u,\overline{v}) = \partial_u \partial_{\overline{v}} \log k(z,\overline{z})$$

defines a Kaehler metric on $\mathcal{D}$, the <u>Bergmann metric</u> (cf. Helgason [H]). From 1.1.3 it follows that h is invariant under the group $\mathrm{Aut}(\mathcal{D})$ of biholomorphic automorphisms of $\mathcal{D}$. It is a well-known result of Myers and Steenrod that the group of isometries of a Riemannian manifold is a Lie group in the compact-open topology. Since $\mathrm{Aut}(\mathcal{D})$ is a closed subgroup of the group of isometries of the Bergmann metric, we have the result, due to H. Cartan, that $\mathrm{Aut}(\mathcal{D})$ is a Lie transformation group of $\mathcal{D}$ with the compact-open topology.

1.3. A bounded domain $\mathcal{D}$ of V is called <u>symmetric</u> if, for every $z \in \mathcal{D}$, there exists an automorphism s_z of period two of $\mathcal{D}$, having z as isolated fixed point. Since s_z leaves the Bergmann metric invariant, it follows easily that it is the geodesic symmetry around z , and thus $\mathcal{D}$ is a hermitian symmetric space in the sense of E. Cartan. The Bergmann metric is then complete since any geodesic may be extended indefinitely by repeated

geodesic symmetries. Moreover, $\mathcal{D}$ is <u>homogeneous</u> ($\mathrm{Aut}(\mathcal{D})$ acts transitively on $\mathcal{D}$) as one sees by joining two given points by a geodesic and reflecting in the midpoint.

1.4. A domain $\mathcal{D}$ of V is called <u>circled</u> (with respect to 0) if $0 \in \mathcal{D}$, and $z \cdot e^{it} \in \mathcal{D}$ for all $z \in \mathcal{D}$, $t \in \mathbb{R}$.

By 1.1.3 the Bergmann kernel function of a bounded circled domain satisfies $k(z,\overline{w}) = k(ze^{it},\overline{we^{it}})$. For $w = 0$ it follows that $k(z,0)$ is constant, and applying 1.1.1 for $f = 1$ we get

(1) $$k(z,0) = k(0,0) = \mu(\mathcal{D})^{-1} .$$

1.5. Let f be a holomorphic map of a circled domain $\mathcal{D}$ into a finite-dimensional vector space W . Then the expansion

$$f = \sum_{n=0}^{\infty} f_n$$

of f into homogeneous polynomials around the origin converges uniformly on compact subsets of $\mathcal{D}$, and

$$f_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(z \cdot e^{it})\,dt .$$

For a proof see H. Cartan [C1]. Now suppose that $\mathcal{D}$ is bounded and that f is an isomorphism of $\mathcal{D}$ onto a bounded circled domain $\mathcal{D}'$ in W , carrying the origin into the origin. Then f is linear; more precisely, it is induced by a linear isomorphism of the ambient vector spaces. For the proof, due to H. Cartan, let g be the inverse of f , and define an automorphism h of $\mathcal{D}$ by $h(z) = e^{-it}g(e^{it}f(z))$, for any $t \in R$. Then h leaves the origin fixed and its differential at the origin is the identity. By H. Cartan's uniqueness theorem (or, using the Bergmann metric, since it is an isometry which is the identity on one tangent space) it follows that h is the identity. This means $f(e^{it}z) = e^{it}f(z)$ and f is linear.

1.6. THEOREM. *Every bounded symmetric domain in* V *is isomorphic to a bounded symmetric and circled domain which is unique up to a linear isomorphism of* V .

This theorem was first proved by E. Cartan [C2] by a case-by-case verification, then by Harish-Chandra (see Helgason [H]) using Lie group methods. A relatively elementary proof which even remains valid in infinite dimensions is due to J-P. Vigué [V]. The uniqueness part is of course immediate from 1.5.

§2. The Jordan pair associated with a circled bounded symmetric domain

2.1. In this section, $\mathcal{D}$ denotes a circled bounded symmetric domain in a finite-dimensional vector space V . The automorphism group of $\mathcal{D}$ is denoted by $\mathrm{Aut}(\mathcal{D})$, its connected component of the identity by G_o , the isotropy group of 0 in G_o by K . The Lie algebras of G_o and K are $\mathfrak{g}_o$ and $\mathfrak{k}$ respectively, considered as Lie algebras of holomorphic vector fields on $\mathcal{D}$. The Bergmann kernel function is $k(z,\overline{w})$, the Bergmann metric $h_z(u,\overline{v})$, and V is equipped with the hermitian scalar product

$$h_0(u,\overline{v}) = \partial_u \overline{\partial}_{\overline{v}} \log k(z,\overline{z})|_{z=0}$$

The adjoint of $f \in \mathrm{End}(V)$ with respect to h_0 is f^* . By 1.2, K is a compact subgroup of $GL(V)$ leaving this scalar product invariant. By 1.3 and standard facts on Lie transformation groups, the natural map $G_o/K \to \mathcal{D}$ is an isomorphism of real manifolds. The symmetry around the origin of $\mathcal{D}$ is given by $s(z) = -z$. Since K contains the one-dimensional circle group of all transformations $e^{it}\mathrm{Id}$, we have $s = e^{i\pi}\cdot\mathrm{Id} \in K$.

2.2. PROPOSITION. (a) K *is the centralizer of* s *in* G_o .

(b) *The centre of* G_o *is trivial*.

(c) $\mathfrak{g}_o = \mathfrak{k} \oplus \mathfrak{p}$ *where* $\mathfrak{p}$ *is the* (-1) *-eigenspace of* $\mathrm{Ad}\, s$ *on* $\mathfrak{g}_o$. *The spaces* $\mathfrak{k}$ *and* $\mathfrak{p}$ *satisfy*

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \; , \; [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \; ; \; [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \; , \; \mathrm{Ad}K\cdot\mathfrak{p} \subset \mathfrak{p} \; .$$

(d) *The evaluation map* $\xi \to \xi(0)$ *is an isomorphism* $\mathfrak{g} \to {}_{R}V$ *of real vector spaces*.

Proof. If $gs = sg$ then $g(0) = gs(0) = sg(0) = -g(0)$ implies $g(0) = 0$ and hence $g \in K$. The converse is trivial. The centre of G_o is contained in K by (a). But K contains no non-trivial subgroup of G_o since G_o acts effectively on $\mathcal{D}$. This proves (b). By (a), the $(+1)$-eigenspace of $\mathrm{Ad}\, s$ on $\mathfrak{g}_o$ is $\mathfrak{k}$. This proves the decomposition $\mathfrak{g}_o = \mathfrak{k} \oplus \mathfrak{p}$, and since $\mathrm{Ad}\, s$ is an automorphism of period two of $\mathfrak{g}_o$ we have (c). Finally, (d) follows from the manifold isomorphism $G_o/K \cong \mathcal{D}$.

2.3. LEMMA. *For every* $v \in V$ *let* ξ_v *be the unique vector field in* $\mathfrak{g}$ *which takes the value* v *at the origin. Then* $v - \xi_v(z)$ *is a homogeneous quadratic polynomial in* z *and is* $\mathbb{C}$*-antilinear in* v .

Proof. Let $\eta \in \mathfrak{k}$ be the vector field tangent to the circle group $\{e^{it}\cdot \mathrm{Id}\}$; i.e., $\eta(z) = \frac{d}{dt}\big|_{t=0} e^{it}\cdot z = iz$. Then $[\eta,\xi_v] \in \mathfrak{p}$, and since $[\eta,\xi_v](0) = d\eta(0)\cdot\xi_v(0) - d\xi_v(0)\cdot\eta(0) = iv$, we have

(1) $$[\eta,\xi_v] = \xi_{iv} \; ,$$

by 2.2. It follows that

(2) $$[\eta[\eta,\xi_v]] = \xi_{-v} = -\xi_v \; .$$

On the other hand, let $\xi_v(z) = \sum_{n=0}^{\infty} f_n(z)$ be the expansion of ξ_v into homogeneous polynomials. Then by Euler's differential equation,

$$[\eta,\xi_v](z) = i\xi_v(z) - \Sigma\, df_n(z)\cdot iz$$

$$= i\Sigma(1-n)f_n(z) \; ,$$

and hence by (2) ,

$$-[\eta[\eta,\xi_v]] = \Sigma(1-n)^2 f_n = \Sigma f_n \; .$$

It follows that $((1-n)^2 - 1)f_n = n\cdot(n-2)f_n = 0$ and hence $f_n = 0$ for $n \neq 0,2$. Thus $\xi_v(z) = f_0 + f_{2,v}(z) = v + f_{2,v}(z)$ where we have written $f_{2,v}$ instead of f_2 to indicate the dependence on v . Since ξ_v and hence $f_{2,v}$ is R-linear in v it remains to show that $f_{2,iv} = -if_{2,v}$. By (1) and (3), $\xi_{iv} = iv + f_{2,iv} = [\eta,\xi_v] = i\Sigma(1-n)f_n = iv - if_{2,v}$.

2.4. *Example*. Let V be the vector space of complex $p \times q$-matrices, and let

$$\mathcal{D} = \{z \in V \mid 1 - {}^{t}\bar{z}z \text{ positive definite}\} \; ,$$

where 1 denotes the unit matrix of the appropriate size. Then $\mathcal{D}$ is bounded and circled. Let $U(p,q)$ denote the unitary group of the hermitian form

$z_1\overline{w}_1 + \cdots + z_p\overline{w}_p - z_{p+1}\overline{w}_{p+1} - \cdots - z_{p+q}\overline{w}_{p+q}$ on $\mathbb{C}^{p+q}$

and write the elements of $U(p,q)$ in block form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a is $p \times p$, b is $p \times q$, etc. Then $U(p,q)$ acts transitively on $\mathcal{D}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)(cz + d)^{-1}$$

The Lie algebra of $U(p,q)$ decomposes into $\mathfrak{k}' \oplus \mathfrak{p}'$ where $\mathfrak{k}'$ consists of all matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with a, b skew-hermitian of size $p \times p$, $q \times q$ respectively, and $\mathfrak{p}'$ consists of all matrices $\begin{pmatrix} 0 & v \\ {}^t\overline{v} & 0 \end{pmatrix}$ where $v \in V$. The vector fields induced on $\mathcal{D}$ are given by

$$z \to az - zb \ , \ z \to v - z\,{}^t\overline{v}z$$

respectively. It follows that the latter vector fields are the ξ_v's, and hence

$$v - \xi_v(z) = z\,{}^t\overline{v}z \ .$$

2.5. Returning to the general situation, we shall write

$$v - \xi_v(z) = Q(z)\overline{v}$$

so that $Q(z): V^- \to V$ is a complex linear map and

$Q: V \to \mathrm{Hom}(V^-, V)$ is a homogeneous quadratic polynomial. Hence

$$Q(x,z) = Q(x + z) - Q(x) - Q(z) : V^- \to V$$

is bilinear and symmetric in x and z. For $x,y,z \in V$ we define

$$\{x\overline{y}z\} = D(x,\overline{y})\cdot z = Q(x,z)\cdot\overline{y} \ .$$

Thus $\{x\overline{y}z\}$ is complex bilinear and symmetric in x and z and complex antilinear in y, and $D(x,\overline{y})$ is the endomorphism $z \to \{x\overline{y}z\}$ of V. In the example above,

$$\{x\overline{y}z\} = x\,{}^t\overline{y}z + z\,{}^t\overline{y}x \ .$$

Finally, we define $\overline{Q}: V^- \to \mathrm{Hom}(V, V^-)$, $\overline{D}: V^- \times V \to \mathrm{End}(V^-)$ and $\{\overline{x}y\overline{z}\} \in V^-$ for $x,y,z \in V$ by

$$\overline{Q(z)}\cdot v = \overline{Q}(\overline{z})\cdot v = \overline{Q(z)\cdot\overline{v}}$$

$$\{\overline{x}y\overline{z}\} = \overline{D}(\overline{x},y)\cdot\overline{z} = \overline{\{x\overline{y}z\}} \ .$$

2.6. LEMMA. <u>The following formulae hold</u>.

(1) $[\xi_u,\xi_v] = D(u,\overline{v}) - D(v,\overline{u})$,

(2) $[[\xi_u,\xi_v],\xi_w] = \xi_{\{u\overline{v}w\}} - \{v\overline{u}w\}$,

(3) $\{u\overline{v}\{x\overline{y}z\}\} - \{x\overline{y}\{u\overline{v}z\}\} = \{\{u\overline{v}x\}\overline{y}z\} - \{x\overline{\{v\overline{u}y\}}z\}$,

(3)' $[D(u,\overline{v}), D(x,\overline{y})] = D(\{u\overline{v}x\},\overline{y}) - D(x,\overline{\{v\overline{u}y\}})$,

(4) $h_0(\{u\overline{v}w\},\overline{z}) = h_0(w,\overline{\{v\overline{u}z\}})$,

(4)' $D(u,\overline{v})^* = D(\overline{v},u)$.

Proof. Since $\xi_v(z) = v - Q(z)\bar{v}$ and $Q(z)$ is quadratic in z, it follows that

(5) $$d\xi_v(z)\cdot w = -Q(z,w)\cdot\bar{v} = -D(z,\bar{v})\cdot w .$$

Hence $[\xi_u,\xi_v](z) = -D(z,\bar{u})\cdot(v-Q(z)\bar{v}) + D(z,\bar{v})\cdot(u-Q(z)\bar{u})$ $= D(u,\bar{v})z - D(v,\bar{u})z +$ terms of degree 3 in z. By 2.2, $[\xi_u,\xi_v]$ belongs to $\mathfrak{k}$ and is therefore linear in z. This proves (1). By 2.2, the left hand side of (2) is in $\mathfrak{p}$ and is therefore equal to ξ_a where a is the value of the vector field at 0. Now by (1) and (5),

$$[[\xi_u,\xi_v],\ \xi_w](z) = (D(u,\bar{v}) - D(v,\bar{u}))\cdot(w - Q(z)\bar{w})$$
$$+ D(z,\bar{w})\cdot(D(u,\bar{v}) - D(v,\bar{u}))\cdot z ,$$

and at $z = 0$ this takes the value $D(u,\bar{v})w - D(v,\bar{u})\cdot w$ $\{u\bar{v}w\} - \{v\bar{u}w\}$, proving (2). Hence we also have

$$[[\xi_u,\xi_v],\ \xi_w](z) = \{u\bar{v}w\} - \{v\bar{u}w\}$$
$$- Q(z)(\overline{\{u\bar{v}w\}} - \overline{\{v\bar{u}w\}}) .$$

Comparison of the terms which are $\mathbb{C}$-linear in u yields

$$-D(u,\bar{v})Q(z)\bar{w} + D(z,\bar{w})D(u,\bar{v})z = Q(z)\cdot\overline{\{v\bar{u}w\}} .$$

Linearize with respect to z (i.e., replace z by $z + x$ and retain the terms linear in x) to get

$$-\{u\bar{v}\{x\bar{w}z\}\} + \{x\bar{w}\{u\bar{v}z\}\} + \{z\bar{w}\{u\bar{v}x\}\}$$
$$= \{x\overline{\{v\bar{u}w\}}z\} .$$

This is (3) (after a change of notation), and (3)' is just a different way of writing it. Finally, since $D(u,\bar{v}) - D(v,\bar{u})$ is in $\mathfrak{k}$ it is skew-hermitian (cf. 2.1):

$$D(u,\bar{v})^* - D(v,\bar{u})^* = -D(u,\bar{v}) + D(v,\bar{u}) .$$

Comparing the terms $\mathbb{C}$-linear in u we get (4)', and (4) is equivalent to it.

2.7. LEMMA. If $0 \neq v \in V$ and $Q(v)\bar{v} = \lambda v$ for some $\lambda \in \mathbb{C}$ then λ is positive.

Proof. $Q(v)\bar{v} = \frac{1}{2}D(v,\bar{v})v = \lambda v$ says that 2λ is an eigenvalue of $D(v,\bar{v})$. By 2.6, this transformation is self-adjoint and hence λ is real. The integral curve of ξ_v through 0 is given by

$$x(t) = \frac{1}{\sqrt{\lambda}}\tanh(\sqrt{\lambda}\,t)v \quad (=tv \ \text{ if } \ \lambda = 0) .$$

Indeed, $\dot{x} = \cosh^{-2}(\sqrt{\lambda}\,t)\ v$ and

$$\xi_v(x) = v - Q(x)\bar{v} = v - \frac{1}{\lambda}\tanh^2(\sqrt{\lambda}\,t)Q(v)\bar{v}$$
$$= (1 - \tanh^2(\sqrt{\lambda}\,t))v .$$

If $\lambda \leq 0$ then $x(t) = \frac{1}{\sqrt{-\lambda}}\tan(\sqrt{-\lambda}\,t)v$ which is unbounded, contradicting boundedness of $\mathcal{D}$.

2.8. DEFINITION. Let V^+ and V^- be complex vector spaces, and let $Q_+ : V^+ \to \mathrm{Hom}(V^-, V^+)$ and $Q_- : V^- \to \mathrm{Hom}(V^+, V^-)$

be quadratic maps. Define trilinear maps $\{\}: V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ and bilinear maps $D_\sigma: V^\sigma \times V^{-\sigma} \to \mathrm{End}(V^\sigma)$ by

$$\{xyz\} = D_\sigma(x,y)\cdot z = Q_\sigma(x+z)y - Q_\sigma(x)y - Q_\sigma(z)y ,$$

for $x,z \in V^\sigma$, $y \in V^{-\sigma}$, $(\sigma = \pm)$. The pair (V^+,V^-) together with the quadratic maps (Q_+,Q_-) is called a <u>Jordan pair</u> if the identity

$$\text{(J)} \quad \{uv\{xyz\}\} - \{xy\{uvz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\}$$

holds for all $u,x,z \in V^\sigma$, $v,y \in V^{-\sigma}$, $\sigma = \pm$. This definition may be extended to an arbitrary base field (or even base ring) but then the identity (J) turns out to be too weak to develop a satisfactory theory, and has to be replaced by more complicated identities (cf. [L5,1.2]). As long as 2 and 3 are invertible in the base ring, however, (J) is sufficient (cf. [L5,2.2]). From 2.5 and 2.6 it is now clear that a circled bounded symmetric domain $\mathcal{D}$ in V gives rise to a Jordan pair $(V,\bar V)$ with quadratic maps $(Q,\bar Q)$.

Abstracting from the properties of the complex antilinear map $v \to \bar v$ from V to $\bar V$, we define: a <u>hermitian involution</u> of a complex Jordan pair is an invertible complex antilinear map $\tau: V^+ \to V^-$ such that $\tau(Q_+(x)\tau y) = Q_-(\tau x)$, for all $x,y \in V^+$. We say τ is <u>positive</u> if $Q_+(x)\tau x = \lambda x$ for $\lambda \in \mathbb{C}$ implies $\lambda > 0$ for all non-zero $x \in V^+$. Then clearly the map $v \to \bar v$ of the Jordan pair $(V,\bar V)$ associated with $\mathcal{D}$ is a hermitian involution and by 2.7 it

is positive. We will prove in §4 that this establishes a one-to-one correspondence between circled bounded symmetric domains and Jordan pairs with positive hermitian involution.

2.9. An equivalent way of describing a Jordan pair with hermitian involution is to introduce the concept of <u>hermitian Jordan triple system</u>, this being defined as a complex vector space V with a map $\langle\,\rangle: V \times V \times V \to V$, $(x,y,z) \to \langle xyz\rangle$ which is $\mathbb{C}$-linear and symmetric in x and z and $\mathbb{C}$-antilinear in y , and satisfies (J) (with $\{\}$ replaced by $\langle\,\rangle$). Indeed, given a hermitian Jordan triple system, we obtain a Jordan pair with hermitian involution by letting $V^+ = V$, V^- the complex conjugate vector space of V , τ the canonical map $v \to \bar v$, and defining $Q_+(x)y = \frac{1}{2}\langle x\bar y z\rangle$, $Q_-(y)x = \overline{Q_+(\bar y)\bar x}$. In these notes, we have preferred to phrase things in terms of Jordan pairs with involution because this makes it easier to quote [L5] and also because Jordan pairs make sense over arbitrary base fields, thus allowing at least part of the theory to be generalized.

Returning now to the Jordan pair $(V,\bar V)$ associated with $\mathcal{D}$, introduce the endomorphisms

$$B(x,\bar y) = \mathrm{Id} - D(x,\bar y) + Q(x)\overline{Q(y)}$$

of V , for $x,y \in V$ (cf. [L5,2.11]). Their significance is shown by the following.

2.10. THEOREM. (a) *The Bergmann kernel function of* $\mathcal{D}$ *is*

$$k(x,\bar{y}) = \mu(\mathcal{D})^{-1}\cdot \det B(x,\bar{y})^{-1} .$$

(b) *The Bergmann metric at* 0 *is*

$$h_0(u,\bar{v}) = \operatorname{trace} D(u,\bar{v}) , \tag{2}$$

and at an arbitrary point $z \in \mathcal{D}$:

$$h_z(u,\bar{v}) = h_0(B(z,\bar{z})^{-1}u,\bar{v}) . \tag{3}$$

(c) *The triple product* $\{u\bar{v}w\}$ *is given by the fourth logarithmic derivative of* $k(z,\bar{z})$ *at* 0 :

$$h_0(\{u\bar{v}w\},\bar{y}) = \partial_u \bar{\partial}_{\bar{v}} \partial_w \bar{\partial}_{\bar{y}} \log k(z,\bar{z})\big|_{z=0} \tag{4}$$

(d) *The curvature tensor of the Bergmann metric at* 0 *is*

$$R_0(u,v)\cdot w = -\{u\bar{v}w\} + \{v\bar{u}w\} . \tag{5}$$

The proof rests on

2.11. LEMMA. (a) K *acts by automorphisms of the Jordan structure*:

$$g(Q(x)\bar{y}) = Q(gx)\overline{gy} , \text{ for } x,y \in V,\ g \in K . \tag{1}$$

(b) *For* $x,y \in \mathcal{D}$, $g \in G_o$ *we have*

$$B(gx,\overline{gy}) = dg(x)\cdot B(x,y)\cdot dg(y)^* . \tag{2}$$

Proof. (a) By 2.2 (c), Ad g leaves $\mathfrak{p}$ invariant. Now $(\operatorname{Ad} g\cdot\xi_v)(0) = g\cdot\xi_v(g^{-1}(0)) = gv$ and hence $\operatorname{Ad} g\cdot\xi_v = \xi_{gv}$. This proves (a) in view of 2.5.

(b) Using the chain rule one sees that it suffices to prove this for a set of generators of G_o. From 2.2 and standard facts on Lie groups it follows that G_o is generated by K and $\exp(\mathfrak{p})$. For $g \in K$ we have $g^* = g^{-1}$ and (1) implies by linearization that $gD(x,\bar{y})g^{-1} = D(gx,\overline{gy})$, proving (2) (remember that K consists of unitary transformations). Now let $\xi = \xi_v \in \mathfrak{p}$, and let φ_t be the one-parameter group of holomorphic automorphisms of $\mathcal{D}$ generated by ξ. Let $x_0,y_0 \in \mathcal{D}$ and $x = x(t) = \varphi_t(x_0)$, $y = y(t) = \varphi_t(y_0)$ the integral curves of ξ through x_0, y_0. We will show that $X(t) = d\varphi_t(x_0)B(x_0,\bar{y}_0)d\varphi_t(y_0)^*$ and $Y(t) = B(x(t),\overline{y(t)})$ satisfy the same differential equation with the same initial condition $X(0) = Y(0) = B(x_0,\bar{y}_0)$ and are therefore identical. We have $\dot{x} = \xi(x)$, and $d\dot{\varphi}_t(x_0) = d\xi(x(t))\cdot d\varphi_t(x_0)$. Hence $X(t)$ satisfies the differential equation

$$\dot{X} = d\xi(x)\cdot X + X\cdot d\xi(y)^* .$$

Now $Y = \frac{d}{dt}(\mathrm{Id} - D(x,\bar{y}) + Q(x)\overline{Q(y)}) = - D(x,\bar{y}) - D(x,\bar{\dot{y}}) + Q(x,\dot{x})\overline{Q(y)} + Q(x)\overline{Q(y,\dot{y})} = - D(v-Q(x)\bar{v},\bar{y}) - D(x,\bar{v}-\overline{Q(y)}v) + Q(x,v-Q(x)\bar{v})\overline{Q(y)} + Q(x)\overline{Q(y,v-Q(y)v)}$. On the other hand, $d\xi(x) = - D(x,\bar{v})$ (cf. 2.6.5) and hence $d\xi(x)\cdot Y + Y\cdot d\xi(y)^* = - D(x,\bar{v})B(x,\bar{y}) - B(x,\bar{y})D(y,\bar{v})^*$. By expanding, using $D(y,\bar{v})^* = D(v,\bar{y})$ and the identities JP4, JP9, JP13, we see $\dot{Y} = d\xi(x)\cdot Y + Y\cdot d\xi(y)^*$. This proves the Lemma.

2.12. Proof of 2.10.

(a) By taking determinants in 2.11.1 it follows that $\det B(x,\bar y)^{-1}$ transforms like $k(x,\bar y)$ under G_o. Since $k(x,0) = \mu(\mathcal{D})^{-1}$ and $\det B(x,0) = 1$, homogeneity of $\mathcal{D}$ under G_o implies 2.10.1 for $x = y$. Since both sides are holomorphic functions on the $\mathcal{D} \times \mathcal{D}^-$ agreeing on the diagonal whose complexification is $\mathcal{D} \times \mathcal{D}^-$, they agree on $\mathcal{D} \times \mathcal{D}^-$.

(b) By definition, $h_0(u,\bar v) = \partial_u \bar\partial_{\bar v} \log k(z,\bar z)|_{z=0}$, and by (a), we may replace k by $\det B^{-1}$. Since $\det B(z,\bar w)$ is holomorphic in z and antiholomorphic in w, $h_0(u,\bar v)$ is the coefficient of $\epsilon\bar\delta$ in the power series expansion of $\log \det B(\epsilon u,\overline{\delta v})^{-1}$ where ϵ and δ are complex parameters. Computing modulo ϵ^2 and δ^2, we have

$$\begin{aligned}\log \det B(\epsilon u,\overline{\delta v})^{-1} &\equiv \log \det (Id - \epsilon\bar\delta D(u,\bar v))^{-1} \\ &\equiv \log \det (Id + \epsilon\bar\delta D(u,\bar v)) \\ &= \log (1 + \epsilon\bar\delta \ \operatorname{trace} D(u,\bar v)) \\ &= \epsilon\bar\delta \ \operatorname{trace} D(u,\bar v) .\end{aligned}$$

Writing now $h_z(u,\bar v) = h_0(A(z)u,\bar v)$ where $A(z) \in End(V)$ is positive definite and self-adjoint, we have $A(0) = Id$ and the G_o-invariance of h means that $dg(z)^*.A(g(z))\cdot dg(z) = A(z)$, for $z \in \mathcal{D}$, $g \in G_o$. Now $A(z) = B(z,\bar z)^{-1}$ by homogeneity of $\mathcal{D}$ and the Lemma.

(c) By (b), $\partial_w \bar\partial_{\bar y} \log k(z,\bar z) = h_0(B(z,\bar z)^{-1}w,\bar y)$. As before, we have to compute the coefficient of $\epsilon\bar\delta$ in $h_0(B(\epsilon u,\overline{\delta v})^{-1}w,\bar y)$:

$$\begin{aligned}h_0(B(\epsilon u,\overline{\delta v})^{-1}w,\bar y) &\equiv h_0((Id - \epsilon\bar\delta D(u,v))^{-1}w,\bar y) \\ &= h_0((Id + \epsilon\bar\delta D(u,\bar v))w,\bar y) \\ &= h_0(w,\bar y) + \epsilon\bar\delta h_0(\{u\bar v w\},\bar y)\end{aligned}$$

(d) The curvature tensor of a symmetric space at a point 0 is given by $R_0(u,v)\cdot w = -[[\tilde u,\tilde v],\tilde w](0)$ where $\tilde v$ is the unique "infinitesimal transvection" taking the value v at 0 (cf. [L2, vol. 1]). In our case, these transvections are just the vector fields ξ_v, and hence (d) follows from 2.6.2.

§3. Tripotents and Peirce decomposition

3.1. In this section, (V^+,V^-) denotes a finite dimensional Jordan pair over $\mathbb{C}$. To simplify notation, we shall write $V = V^+$ and often drop subscripts $\pm$ on the quadratic operators $Q(x)y = Q_x y$, and their linearizations $D(x,y)z = Q(x,z)y = \{xyz\}$. The definitions and notations of [L5] will be adopted; for convenience, we list some of them. A homomorphism $(f,f_-)\colon (V,V^-) \to (W,W^-)$ of Jordan pairs is a pair of $\mathbb{C}$-linear maps $f\colon V \to W$, $f_-\colon V^- \to W^-$ satisfying $fQ(x)y = Q(f(x))f_-(y)$ and $f_-Q_-(y)x = Q_-(f_-(y))f(x)$. Automorphisms, subpairs, and direct products are defined in the obvious way. The automorphism group of (V,V^-) is denoted by $\mathrm{Aut}(V,V^-)$; it is an algebraic subgroup of $GL(V) \times GL(V^-)$. The Lie algebra of $\mathrm{Aut}(V,V^-)$ is the derivation algebra $\mathrm{Der}(V,V^-)$, consisting of all $(\Delta,\Delta_-) \in \mathrm{End}(V) \times \mathrm{End}(V^-)$ which satisfy $\Delta Q(x)y = Q(x,\Delta x)y + Q(x)\Delta_- y$ and $\Delta_- Q_-(y)x = Q_-(y,\Delta_- y)x + Q_-(y)\Delta x$. By linearization, this is equivalent with $\Delta\{xyz\} = \{\Delta x,y,z\} + \{x,\Delta_- y,z\} + \{x,y,\Delta z\}$ and the analogous formula for Δ_-. Define endomorphisms of V and V^- by $B(x,y) = \mathrm{Id} - D(x,y) + Q(x)Q_-(y)$, $B_-(y,x) = \mathrm{Id} - D_-(y,x) + Q_-(y)Q(x)$. Then if $B(x,y)$ is invertible so is $B_-(y,x)$, and

$$\beta(x,y) = (B(x,y)\ ,\ B_-(y,x)^{-1})$$

is an automorphism of (V,V^-), called an <u>inner automorphism</u>. The subgroup of $\mathrm{Aut}(V,V^-)$ generated by the $\beta(x,y)$ is denoted $\mathrm{Inn}(V,V^-)$; it is a normal connected algebraic subgroup. The Lie algebra of $\mathrm{Inn}(V,V^-)$ is spanned by the <u>inner derivations</u>

$$\delta(x,y) = (D(x,y),\ -D_-(y,x)) \quad (x \in V,\ y \in V^-)\ .$$

For more details see [L5,L6].

3.2. Let now $\tau: x \to \bar{x}$ be a hermitian involution of (V,V^-), as defined in 2.8. Thus V^- may be identified with the complex conjugate vector space of V, and for every $f \in \mathrm{End}(V)$ we have $\bar{f} \in \mathrm{End}(V^-)$ given by $\bar{f}(\bar{x}) = \overline{f(x)}$. Then τ induces a complex conjugation (i.e., a Galois action of the Galois group of $\mathbb{C}/\mathbb{R}$) on $\mathrm{Aut}(V,V^-)$ by $(f,f_-) \to (\bar{f}_-,\bar{f})$, whose fixed point set may be identified (by projection onto the first factor) with the group $\mathrm{Aut}(V)$ of all $f \in GL(V)$ satisfying

$$fQ(x)\bar{y} = Q(f(x))\overline{f(y)}$$

for all $x,y \in V$. In other words, τ defines an $\mathbb{R}$-structure on the complex algebraic group $\mathrm{Aut}(V,V^-)$ whose group of real points is $\mathrm{Aut}(V)$. Similar remarks apply to $\mathrm{Inn}(V,V^-)$, and we denote by $\mathrm{Inn}(V)$ the group of real points of $\mathrm{Inn}(V,V^-)$, considered as a subgroup of

$\mathrm{Aut}(V)$. The Lie algebra $\mathrm{Der}(V)$ of $\mathrm{Aut}(V)$ is the set of all $\Delta \in \mathrm{End}(V)$ with

$$\Delta Q(x)\bar{y} = Q(x,\Delta x)\bar{y} + Q(x)\overline{\Delta y}$$

for all $x,y \in V$. In particular, it follows from the Jordan identity that

(1) $\quad iD(x,\bar{x}) \in \mathrm{Der}(V)$ and $D(x,\bar{y}) - D(y,\bar{x}) \in \mathrm{Der}(V)$

for all $x,y \in V$.

3.3. The <u>odd powers</u> of an element $x \in V$ are defined by

$$x^{(1)} = x,\ x^{(3)} = Q(x)\bar{x}, \cdots, x^{(2n+1)} = Q(x)\ \overline{x^{(2n-1)}}\ .$$

We say x is <u>nilpotent</u> if $x^{(2n+1)} = 0$ for n sufficiently large. One shows by induction (cf. [M1]) that

(1) $\qquad Q(x^{(2n+1)}) = (Q(x)Q(\bar{x}))^n Q(x)$,

(2) $\qquad (x^{(m)})^{(n)} = x^{(mn)}$,

(3) $\qquad \{x^{(m)}, \overline{x^{(n)}}, x^{(p)}\} = 2x^{(m+n+p)}$.

Note that these powers depend on the involution τ, not only on the underlying Jordan pair. In particular, τ positive means that $x^{(3)} = \lambda x$ implies $\lambda > 0$ for $x \neq 0$, and hence there are no nilpotent elements different from zero (the converse is false).

3.4. PROPOSITION. *Let the involution* τ *be positive (more generally, assume* V *contains no non-zero nilpotent elements). Then* $\mathrm{Aut}(V)$ *is compact, and there exists an* $\mathrm{Aut}(V)$*-invariant positive definite hermitian scalar product on* V *such that the adjoint of* $D(x,\bar{y})$ *is*

$$D(x,\bar{y})^* = D(y,\bar{x}) .$$

Proof (cf. [P3]). Let $m(T,x,y) = \sum_{i=0}^{s}(-1)^i m_i(x,y)T^{h-i}$ be the generic minimum polynomial of the Jordan pair (V,V^-) (cf. [L5 §16]). The $m_i(x,y)$ are polynomial functions on $V \times V^-$, homogeneous of bidegree (i,i), invariant under $\mathrm{Aut}(V,V^-)$, and the roots of $m(T,x,y)$ and the minimum polynomial $\mu_{x,y}(T)$ coincide [P2, Prop. 1]. By [L5,16.5] it is clear that $\mu_{x,\bar{x}}(T)$ is a power of T and hence $m_i(x,\bar{x}) = 0$, $i = 1,\cdots,s$, if and only if x is nilpotent. Since there are no nonzero nilpotent elements by assumption, the $\mathbb{R}$-homogeneous polynomials $f_i(x) = m_i(x,\bar{x})$ on ${}_{\mathbb{R}}V$ have no non-trivial common zero, and they are invariant under $\mathrm{Aut}(V)$. By [P3, Lemma 7] the subgroup of $GL({}_{\mathbb{R}}V)$ leaving the f_i invariant is compact, and hence so is $\mathrm{Aut}(V)$. Choose now a positive definite hermitian scalar product on V which is $\mathrm{Aut}(V)$-invariant. Then $\mathrm{Der}(V)$ consists of skew-adjoint transformations, and 3.2.1 implies that $D(x,\bar{x})$ is self-adjoint. Now the assertion follows by linearizing and comparing terms $\mathbb{C}$-linear in x.

3.5. *Remark*. The condition $D(x,\bar{y})^* = D(y,\bar{x})$ is equivalent to the "*associativity*"

$$\langle\{z\bar{y}x\},\bar{w}\rangle = \langle z,\{\bar{y}x\bar{w}\}\rangle .$$

The proof shows that any $\mathrm{Aut}(V)$-invariant hermitian scalar product (not necessarily positive definite) is associative in this sense. Conversely, any associative scalar product is invariant under $\mathrm{Inn}(V)$ but not necessarily under $\mathrm{Aut}(V)$.

Suppose V admits a non-degenerate $\mathrm{Aut}(V)$-invariant hermitian scalar product $\langle,\rangle$. Then $\langle,\rangle$ defines a non-degenerate bilinear form $V \times V^- \to \mathbb{C}$ which is invariant under $\mathrm{Aut}(V,V^-)$; i.e., $\langle fx,f_-y\rangle = \langle x,y\rangle$ for all $(x,y) \in V \times V^-$, $(f,f_-) \in \mathrm{Aut}(V,V^-)$. Indeed, this is true for the real subgroup $\{(f,\bar{f}) \mid f \in \mathrm{Aut}(V)\}$ whose complexification is $\mathrm{Aut}(V,V^-)$, and hence follows by analytic continuation. As a consequence,

$$f_- = {}^t f^{-1}$$

(the transpose being defined as in 0.2); in particular, the projection onto the first factor $\mathrm{Aut}(V,V^-) \to GL(V)$, is injective.

3.6. A (linear) *Jordan algebra* is a vector space A with a commutative bilinear multiplication xy satisfying

$$x^2(xy) = x(x^2y)$$

where $x^2 = xx$. (Over fields of characteristic 2 or 3 or over a base ring, this definition has to be modified.) It is often convenient to also use the circle product

$$x \circ y = (x+y)^2 - x^2 - y^2 = 2xy .$$

We define linear maps $P(x): A \to A$ which are quadratic in x by

$$P(x)y = 2x(xy) - x^2y = \tfrac{1}{2}(x\circ(x\circ y) - x^2\circ y) .$$

Our standard reference for Jordan algebras is [B-K]. Every Jordan algebra gives rise to a Jordan pair by setting $V^+ = V^- = A$ and $Q_+(x)y = Q(x)y = P(x)y$. This is immediate from well-known identities for Jordan algebras. On the other hand, let (V,V^-) be a Jordan pair, and let $a \in V^-$. Then the vector space V becomes a Jordan algebra, denoted by $V^{(a)}$, with

$$xy = \tfrac{1}{2}\{xay\},\ x^2 = Q(x)a .$$

Circle product and P-operators are given by

$$x\circ y = \{xay\},\ P(x)y = Q(x)Q_-(a)y .$$

For the proof see [L5,1.9].

3.7. Let A be a real Jordan algebra, and let $z^* = x - iy$ denote the complex conjugate of $z = x + iy \in A_{\mathbb{C}} = A \oplus iA$, the complexification of A . Then the Jordan pair $(V,V^-) = (A_{\mathbb{C}},A_{\mathbb{C}})$ carries a hermitian involution given by

$\tau(z) = z^*$; in other words, we have

$$Q(z)\overline{w} = P(z)w^*$$

for $z,w \in V = A_{\mathbb{C}}$. This follows easily from the fact that $z \to z^*$ is an antilinear automorphism of period 2 of the Jordan algebra $A_{\mathbb{C}}$. The Jordan pair with involution obtained in this way is called the <u>hermitification</u> of A .

We say A is <u>formally real</u> if $x^2 + y^2 = 0$ implies $x = y = 0$. In this case, the involution defined above is positive. Indeed, suppose $Q(z)\overline{z} = P(z)z^* = \lambda z$ for $z \in A_{\mathbb{C}}$. Let σ be the trace form of $A_{\mathbb{C}}$, given by $\sigma(x) =$ trace $(y \to xy)$. Then σ is associative and positive definite on A in the sense that $\sigma((uv)w) = \sigma(u(vw))$ and $\sigma(x^2) > 0$ for $0 \neq x \in A$ (cf. [B-K]). It follows that $\sigma((P(z)z^*)z^*) = \sigma((zz^*)^2) = \lambda\sigma(zz^*)$ and $zz^* = x^2 + y^2 \in A$ for $z = x + iy$. Since A is formally real we get $\lambda > 0$ for $z \neq 0$.

3.8. An <u>idempotent</u> in a Jordan pair (V,V^-) is a pair $(a,b) \in V \times V^-$ such that $Q(a)b = a$ and $Q(b)a = b$. Note that this means in particular that $a^2 = a$ in the Jordan algebra $V^{(b)}$. An idempotent defines a one-dimensional torus in $\mathrm{Inn}(V,V^-)$ by $t \to \beta(a,(1-t)b)$ $(t \in \mathbb{C}^*)$, cf. [L5,§5]. The set I of idempotents is clearly a subvariety of $V \times V^-$ and it can be shown to be smooth. Now let τ be a hermitian involution of (V,V^-) . If (a,b) is an idempotent then so is $(\overline{b},\overline{a})$, and thus τ

defines an R-structure on I. The set of real points is the set of all idempotents of the form $(a,\bar{a})$ and thus may be identified (by projection onto the first factor) with the set M of tripotents of V, where $e \in V$ is called a tripotent if $e^{(3)} = e$. Clearly $\mathrm{Aut}(V,\bar{V})$ acts on I, the action is compatible with the Galois action defined by τ, and $\mathrm{Aut}(V)$ acts on M.

3.9. LEMMA. For tripotents c and e of V the following conditions are equivalent.

(i) $D(c,\bar{e}) = 0$;

(ii) $D(e,\bar{c}) = 0$;

(iii) $\{c\bar{c}e\} = 0$;

(iv) $\{e\bar{e}c\} = 0$.

If they hold then $D(c,\bar{c})$ and $D(e,\bar{e})$ commute and $e + c$ is a tripotent.

Proof. Clearly (i) implies (iv) and (ii) implies (iii). Assume that (iv) holds. Then by JP4, $2Q(e)\bar{c} = \{e\bar{c}e\} = \{e\bar{c}e^{(3)}\} = D(e,\bar{c})Q(e)\bar{e} = Q(e)\{\bar{e}e\bar{c}\} = 0$. Hence by the Jordan identity, $[D(e,\bar{e}),D(e,\bar{c})] = D(\{e\bar{e}e\},\bar{c}) - D(e,\{\bar{e}e\bar{c}\}) = 2D(e,\bar{c}) = -[D(e,\bar{c}),D(e,\bar{e})] = -D(\{e\bar{c}e\},\bar{e}) + D(e,\{\bar{c}e\bar{e}\}) = 0$ which proves (ii). By interchanging the roles of c and e we see that all four conditions are equivalent. Finally, $[D(e,\bar{e}), D(c,\bar{c})] = D(\{e\bar{e}c\},\bar{c}) - D(c,\{\bar{e}e\bar{c}\}) = 0$ and $(e + c)^{(3)} = e^{(3)} + c^{(3)} + Q(e)\bar{c} + Q(c)\bar{e} + \{e\bar{e}c\} + \{c\bar{c}e\} = e + c$.

3.10. Two tripotents c and e are called orthogonal if they satisfy the conditions of 3.9. It is easy to show that this is equivalent with the orthogonality of the idempotents $(c,\bar{c})$ and $(e,\bar{e})$ of $(V,\bar{V})$ in the sense of [L5,5.12].

A real subspace $W \subset V$ is called flat if $\{W\bar{W}W\} \subset W$ and $\{x\bar{y}z\} = \{y\bar{x}z\}$ for all $x,y,z \in W$. Flat subspaces are totally real in the sense that $W \cap iW = 0$, provided V contains no nilpotent elements different from zero. Indeed, if x and ix are in W then $2ix^{(3)} = \{ix,\bar{x},x\} = \{x,\overline{ix},x\} = -2ix^{(3)}$. (The term "flat" is suggested by the relation with the curvature tensor; cf. 2.10). For example, the R-linear span of the powers $x,x^{(3)},x^{(5)},\ldots$ of $x \in V$ is flat by 3.3.3.

3.11. THEOREM. Let $(V,\bar{V})$ be a Jordan pair with a positive hermitian involution, and let $W \subset V$ be flat. Then

$$W = R\cdot e_1 \oplus \cdots \oplus R\cdot e_n$$

where the e_i are pairwise orthogonal non-zero tripotents, uniquely determined up to sign and order. Conversely, any subspace of this form is flat.

Proof. Let $S \subset \mathrm{End}_R(W)$ be the R-linear span of all $D(x,\bar{x})\,|\,W$ where $x \in W$. Note that $D(x,\bar{y})\,|\,W$ belong to S for all $x,y \in W$ since $2D(x,\bar{y})z = D(x+y,\bar{x}+\bar{y})z - D(x,\bar{x})z - D(y,\bar{y})z$ by flatness of W. Choose a scalar product as in 3.4. By taking the real part of it and restricting to W, we obtain a Euclidean scalar product on W such that S consists of the selfadjoint transformations. Furthermore, S is commutative since $[D(x,\bar{x}), D(y,\bar{y})] = \{\{x\bar{x}y\},\bar{y}z\} - \{x,\overline{\{x\bar{x}y\}},z\} = 0$ for $x,y,z \in W$ by flatness of W. By a standard result of linear algebra, S may be simultaneously diagonalized. Hence there exists a basis $e_1,\cdots,e_n$ of W such that $fe_i \in Re_i$ for all $f \in S$. In particular, $e_i^{(3)} = \frac{1}{2}D(e_i,\bar{e}_i)e_i = \lambda e_i$ and $\lambda > 0$ by positivity. Replacing e_i by $\lambda^{-\frac{1}{2}}e_i$ we may assume that e_i is a tripotent. For $i \neq j$ we have $\{e_i\bar{e}_ie_j\} = D(e_i,\bar{e}_i)e_j = D(e_j,\bar{e}_i)e_i \in R\cdot e_i \cap R\cdot e_j = 0$ and hence the e_i are mutually orthogonal. If $c = \Sigma\lambda_ie_i$ is a tripotent in W then $c = c^{(3)} = \Sigma\lambda_i^3e_i$ implies $\lambda_i = 0, \pm 1$. From this it follows easily that the e_i are unique up to sign and order. The last statement is immediate from 3.9.

3.12. COROLLARY. <u>Every</u> $x \in V$ <u>can be written uniquely</u>

(1) $$x = \lambda_1e_1 + \cdots + \lambda_ne_n$$

<u>where the</u> e_i <u>are pairwise orthogonal non-zero tripotents which are real linear combinations of powers of</u> x, <u>and the</u> λ_i <u>satisfy</u>

(2) $$0 < \lambda_1 < \cdots < \lambda_n .$$

<u>Proof</u>. Apply 3.11 to the real subspace W spanned by the powers of x. Then $x = \lambda_1e_1 + \cdots + \lambda_ne_n$ with real λ_i, and after permutation and sign change we may assume that $0 \le \lambda_1 \le \cdots \le \lambda_n$. By orthogonality of the e_i, the powers of x are

$$x^{(2k+1)} = \lambda^{2k+1}e_1 + \cdots + \lambda_n^{2k+1}e_n .$$

Since the powers of x span W this implies (2). Now suppose that

$$x = \mu_1c_1 + \cdots + \mu_mc_m$$

with non-zero orthogonal tripotents c_j and $\mu_1 < \cdots < \mu_m$. Then $x^{(2k+1)} = \mu_1^{2k+1}c_1 + \cdots + \mu_m^{2k+1}c_m$, and a Vandermonde argument shows $c_j \in W$. Hence the c_j from a basis of W, and by 3.11, the c_j agree with the e_i up to sign and order. But now the inequalities on the coefficients imply $c_i = e_i$ and $\lambda_i = \mu_i$.

We call (1) the <u>spectral decomposition</u> and the λ_i the <u>eigenvalues</u> of x.

3.13. THEOREM. (Peirce decomposition) <u>Let</u> (V,V^-) <u>be a Jordan pair with hermitian involution</u> τ, <u>and let</u> e <u>be a tripotent of</u> V.

(a) V <u>decomposes</u>

(1) $$V = V_2 \oplus V_1 \oplus V_0$$

<u>where</u> $V_\alpha = V_\alpha(e)$ <u>is the</u> α-<u>eigenspace of</u> $D(e,\bar{e})$. <u>The</u>

V_α are orthogonal with respect to any associative scalar product and satisfy the multiplication rules

(2) $\{V_\alpha,\overline{V}_\beta,V_\gamma\}\subset V_{\alpha-\beta+\gamma}$, $\{V_2,\overline{V}_0,V\}=\{V_0\overline{V}_2V\}=0$.

In particular, $(V_\alpha,\overline{V}_\alpha)$ is a τ-invariant subpair.

(b) V_2 is a complex Jordan algebra with multiplication $xy=\frac{1}{2}\{x\bar{e}y\}$ and unit element e . The map $z\to z^*=Q(e)\bar{z}$ is a complex antilinear automorphism of period 2 of the Jordan algebra V_2 .

(c) The fixed point set $A=A(e)$ of the map $z\to z^*$ is a real Jordan algebra, and $V_2=A\oplus iA$ is the hermitification of A (cf. 3.7). If τ is positive then A is formally real.

Proof. (a) Associated with any idempotent (a,b) of a Jordan pair (V,V^-) we have the Peirce decomposition $V=V_2\oplus V_1\oplus V_0$, $V^-=V_2^-\oplus V_1^-\oplus V_0^-$, where the V_α (resp. V_α^-) are the weight spaces of the $\mathbb{C}^*$-action on V (resp. V^-) given by $t\to B(a,(1-t)b)$ (resp. $t\to B_-(b,(1-t)a)$) . Also, V_α is contained in the α-eigenspace of $D(a,b)$, and since the characteristic of the base field is not 2 we have equality (see [L5,5.4]). If $a=\bar{b}=e$, it follows easily that $V_\alpha^-=\overline{V}_\alpha$. Thus (a) follows from [L5,5.4] and 3.5.

(b) For $z,w\in V_2$ we have $zw=\frac{1}{2}\{z\bar{e}w\}\in V_2$ by (2) and $ez=\frac{1}{2}\{e\bar{e}z\}=z$. By 3.6, V_2 is a Jordan algebra with unit element e . Moreover, by JP12,

$$2z=\{e\bar{e}z\}=\{e^{(3)}\bar{e}z\}=D(z,\bar{e})Q(e)\bar{e}$$

$$=-Q(e)\{\bar{e}z\bar{e}\}+\{\{z\bar{e}e\}\bar{e}e\}=-2Q(e)\overline{Q(e)z}+$$

$$+2\{e\bar{e}z\}=-2(z^*)^*+4z\ .$$

Hence $z\to z^*$ is antilinear and of period 2. To show that it preserves the Jordan product it suffices to verify that $(z^*)^2=(z^2)^*$. Now $(z^2)^*=Q(e)\overline{Q(z)\bar{e}}$ $Q(e)\overline{Q}(\bar{z})Q(e)\bar{e}=Q(Q(e)\bar{z})\cdot\bar{e}=Q(z^*)\bar{e}=(z^*)^2$, by JP3.

(c) Clearly $V_2=A\oplus iA$, and $P(z)w^*=Q(z)\overline{Q}(\bar{e})w^*$ $=Q(z)\overline{Q}(\bar{e})Q(e)\bar{w}=Q(z)\bar{w}$ since $w=w^{**}=Q(e)\overline{Q}(\bar{e})w$ and hence $\bar{w}=\overline{Q}(\bar{e})Q(e)\bar{w}$. Therefore $(V_2,\overline{V}_2)$ is the hermitification of A . Suppose τ is positive, and choose a positive definite hermitian scalar product $\langle,\rangle$ on V as in 3.4. Then for $x,y\in A$ we have

$$\langle x^2+y^2,\bar{e}\rangle=\tfrac{1}{2}\langle\{x\bar{e}x\},\bar{e}\rangle+\tfrac{1}{2}\langle\{y\bar{e}y\},\bar{e}\rangle$$

$$=\tfrac{1}{2}\langle x,\{\bar{e}x\bar{e}\}\rangle+\tfrac{1}{2}\langle y,\{\bar{e}y\bar{e}\}\rangle=$$

$$=\langle x,\overline{x^*}\rangle+\langle y,\overline{y^*}\rangle=\langle x,\bar{x}\rangle+\langle y,\bar{y}\rangle\ .$$

This shows that A is formally real.

For orthogonal systems of tripotents, we have

3.14. THEOREM. Let $e_1,\cdots,e_n$ be orthogonal tripotents of V . Then

(1) $$V=\sum_{0\le i\le j\le n}V_{ij}$$

(direct sum of subspaces) where

$$(2)\quad \begin{cases} V_{ii} = V_2(e_i)\ , \ i = 1,\cdots,n\ ; \\ V_{ij} = V_{ji} = V_1(e_i) \cap V_1(e_j)\ , \ 1 \le i < j \le n\ ; \\ V_{i0} = V_{0i} = V_1(e_i) \cap \bigcap_{j\neq i} V_0(e_j)\ , \ i = 1,\cdots,n\ ; \\ V_{00} = V_0(e_1) \cap \cdots \cap V_0(e_n)\ . \end{cases}$$

The spaces V_{ij} *are orthogonal with respect to any associative scalar product. If* $I \subset \{1,\cdots,n\}$ *and* $e_I = \sum_{i\in I} e_i$ *then the Peirce spaces of* e_I *are given by*

$$(3)\quad V_2(e_I) = \sum_{i,j\in I} V_{ij},\quad V_1(e_I) = \sum_{\substack{i\in I\\ j\notin I}} V_{ij},\quad V_0(e_I) = \sum_{i,j\notin I} V_{ij}\ .$$

We have the multiplication rule

$$(4)\qquad \{V_{ij}, \overline{V}_{jk}, V_{k\ell}\} \subset V_{i\ell}$$

and all other types of products are zero.

Proof. See [L5,5.14].

3.15. COROLLARY. *Let* $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ *where* $\lambda_i \in \mathbb{C}$ *, and set* $\lambda_0 = 0$ *. Let* $y_{ij} \in V_{ij}$ *. Then*

$$(1)\qquad D(x,\overline{x})y_{ij} = (|\lambda_i|^2 + |\lambda_j|^2)y_{ij}\ ,$$

$$(2)\qquad Q(x)\overline{y}_{ij} = \lambda_i\lambda_j y^*_{ij}\ ,$$

$$(3)\qquad Q(x)\overline{Q(x)}y_{ij} = |\lambda_i\lambda_j|^2 y_{ij}\ ,$$

$$(4)\qquad B(x,\overline{x})y_{ij} = (1 - |\lambda_i|^2)(1 - |\lambda_j|^2)y_{ij}\ .$$

Here $y^*_{ij} = Q(e)\overline{y_{ij}}$ *and* $e = e_1 + \cdots + e_n$. This follows by straightforward computation from 3.13 and 3.14. The details are left to the reader (distinguish the 4 cases $i = j > 0$, $1 \le i < j \le n$, $i = 0 < j$, $i = j = 0$).

3.16. COROLLARY. *A hermitian involution* τ *is positive if and only if* trace $D(x,\overline{x}) > 0$ *for all* $0 \neq x \in V$. *In this case,* $\langle x,\overline{y}\rangle = \operatorname{trace} D(x,\overline{y})$ *is an associative* Aut(V)*-invariant hermitian scalar product.*

Proof. Suppose τ is positive, and let $0 \neq x \in V$. Then 3.12 and 3.15 imply $x = \sum \lambda_i e_i$ with positive λ_i , and trace $D(x,\overline{x}) \ge 2\sum\lambda_i^2$, since $e_i \in V_{ii}$. Conversely, assume $x^{(3)} = \lambda x$, $\lambda \in \mathbb{C}$. If $\lambda = 0$ then one shows, using JP13 and JP4 that $D(x,\overline{x})^3 = 0$. Hence trace $D(x,\overline{x}) = 0$ which implies $x = 0$. If $x \neq 0$ then $\lambda \neq 0$ and $(\lambda^{-1}x,\overline{x})$ is an idempotent of the Jordan pair (V,V^-) . Hence the eigenvalues of $D(\lambda^{-1}x,\overline{x})$ are $0,1,2$. This shows that the eigenvalues of $D(x,\overline{x})$ are $0,\lambda,2\lambda$, and thus trace $D(x,\overline{x})$ is a positive integer multiple of λ . Therefore $\lambda > 0$. For the last assertion, note that $D(fx,\overline{fy}) = fD(x,\overline{y})f^{-1}$ for $f \in \operatorname{Aut}(V)$ which shows that trace $D(x,\overline{y})$ is Aut(V)-invariant and hence associative (cf. 3.5).

3.17. THEOREM. *Let* (V,V^-) *be a Jordan pair with a positive hermitian involution. For every* $x \in V$ *let* $|x|$ *denote the* largest eigenvalue of x (cf. 3.12). Then $|.|$ is an $\mathrm{Aut}(V)$-invariant norm on V, called the spectral norm, with the following properties.

(a) $|x^{(3)}| = |x|^3$ and $|Q(x)\bar{y}| \le |x|^2|y|$.

(b) $|x|^2 = \|Q(x)\| = \frac{1}{2}\|D(x,\bar{x})\|$, where the operator norm $\|f\|$ for $f \in \mathrm{End}({}_R V)$ is computed relative to some associative scalar product $<,>$ on V by $\|f\| = \sup\{\|fx\| \mid \|x\| = 1\}$, and $\|x\| = <x,\bar{x}>^{\frac{1}{2}}$.

(c) If (U,U^-) is a τ-invariant subpair of V then the norm of an element $x \in U$ is the same whether computed in U or in V.

(d) If (V,V^-) is a direct product of Jordan pairs (V_1,V_1^-) and (V_2,V_2^-) then the norm of $x = (x_1,x_2) \in V_1 \times V_2$ is $\max(|x_1|, |x_2|)$.

Proof. From 3.12 it is clear that $|.|$ is $\mathrm{Aut}(V)$-invariant and satisfies all properties of a norm except the triangle inequality. The latter is equivalent to the convexity of the closed unit ball $B = \{x \in V \mid |x| \le 1\}$. Pick an $\mathrm{Aut}(V)$-invariant inner product $<x,\bar{y}>$ on V (cf. 3.4). Let $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ be the spectral decomposition of x so that $|x| = \lambda_n$. By 3.15.1, $|x| \le 1$ if and only if $2\mathrm{Id} - D(x,\bar{x})$ is positive semidefinite with respect to the scalar product $<,>$; i.e., $f_y(x) = <D(x,\bar{x})y,\bar{y}> \le 2$ for all $y \in V$ such that $<y,\bar{y}> = 1$. By 3.15.1, $D(x,\bar{x})$ is positive semidefinite,

and hence f_y is a positive semidefinite quadratic form on ${}_R V$. It follows from the Cauchy-Schwarz inequality that the sets $\{x \in V \mid f_y(x) \le 2\}$ are convex, and hence so is B, being their intersection. This shows that 1.1 is a norm on V. From 3.15 it is clear that we have (b). The first formula of (a) follows from $x^{(3)} = \lambda_1^3 e_1 + \cdots + \lambda_n^3 e_n$. For the second, use (b) and JP3 and the inequality $\|fg\| \le \|f\| \cdot \|g\|$ for the operator norm:

$$|Q(x)\bar{y}|^2 = \|Q(Q(x)\bar{y})\| = \|Q(x)\overline{Q(y)}Q(x)\|$$
$$\le \|Q(x)\|^2 \cdot \|Q(y)\| = |x|^4 \cdot |y|^2 .$$

(Note here that the linear map $Q(x) = f\colon V^- \to V$ may be considered as an R-linear map from ${}_R V$ into itself and therefore $\|f\|$ makes sense. Also, the conjugate map $\bar{f}\colon V \to V^-$ (cf. 0.1) has the same norm as f). Finally, (c) and (d) follow immediately from the definitions.

3.18. As an application, we have a kind of functional calculus for Jordan pairs with positive involution. Let $f(t)$ be an odd complex-valued function of the real variable t, defined for $|t| < \rho$. For every $x \in V$ with $|x| < \rho$ define $f(x) \in V$ by

$$f(x) = f(\lambda_1)e_1 + \cdots + f(\lambda_n)e_n$$

where $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ is the spectral decomposition. If $g(t)$ and $h(t)$ are also odd functions of t then one checks (under obvious assumptions on the domains and ranges) that

(1) $(f+g)(x) = f(x) + g(x)$, $2(f\bar{g}h)(x) = \{f(x)\overline{g(x)}h(x)\}$,

(2) $(g\circ f)(x) = g(f(x))$.

Also, $|f(t)| < \alpha$ for $|t| < \rho$ implies $|f(x)| < \alpha$ for $|x| < \rho$ $(x \in V)$. If $f(t)$ is real analytic and the power series expansion $f(t) = \sum_{n=0}^{\infty} a_n t^{2n+1}$ of f around 0 converges for $|t| < \rho_1$ then $\sum_{n=0}^{\infty} a_n x^{(2n+1)}$ converges to $f(x)$ for $|x| < \rho_1$; in particular, $f(x)$ is real analytic for $|x| < \rho_1$. This follows easily from $|x^{(2n+1)}| = |x|^{2n+1}$. Moreover, we have

3.19. PROPOSITION. *If* $f(t)$ *is real analytic for* $|t| < \rho$ *then the function* $x \to f(x)$ *is real analytic on the domain* $\{x \in V \mid |x| < \rho\}$.

Proof. By considering real and imaginary part, we may assume that f is real valued. Then f can be extended uniquely to a holomorphic function in a neighborhood of the open real interval $(-\rho,\rho)$ in $\mathbb{C}$. For $0 < \rho' < \rho$ we can find a closed rectangle $R \subset \mathbb{C}$, parallel to the real axis, containing the closed interval $[-\rho',\rho']$ in its interior, and such that f is holomorphic on a neighborhood of R . Since f is odd, Cauchy's integral formula shows that

$$f(t) = \frac{1}{2\pi i}\int_{\partial R} f(\zeta)\frac{t}{\zeta^2 - t^2}\,d\zeta$$

for all $t \in [-\rho',\rho']$. If $x = \sum \lambda_j e_j \in V$ and $|x| \le \rho'$ then

$$f(x) = \frac{1}{2\pi i}\int_{\partial R} \sum_j f(\zeta)\frac{\lambda_j}{\zeta^2 - \lambda_j^2}\,e_j\,d\zeta .$$

Let $\tilde{V}$ be the complexification of ${}_RV$ and let $\tilde{B} : \tilde{V} \to \mathrm{End}(\tilde{V})$ be the complex extension of the real polynomial map $x \to B(x,\bar{x})$ from ${}_RV$ into $\mathrm{End}({}_RV)$. Similarly extend the real polynomial map $x \to x^{(3)}$ from ${}_RV$ into itself to a complex polynomial map $z \to z^{[3]}$ of $\tilde{V}$. By a simple computation using orthogonality of the e_j , we have

$$\sum_j \frac{\lambda_j}{\zeta^2 - \lambda_j^2}\,e_j = \zeta^{-1}\tilde{B}(\zeta^{-1}x)^{-1}\cdot(\zeta^{-1}x - (\zeta^{-1}x)^{[3]}) .$$

Hence we can extend $x \to f(x)$ to a holomorphic function $\tilde{f}$ on a neighborhood of $\{x \in V \mid |x| \le \rho'\}$ in $\tilde{V}$ by

$$\tilde{f}(z) = \frac{1}{2\pi i}\int_{\partial R} f(\zeta)\tilde{B}(\zeta^{-1}z)^{-1}(\zeta^2 z - z^{[3]})\cdot\frac{d\zeta}{\zeta^4} .$$

This completes the proof.

§4. Correspondence between bounded symmetric domains and Jordan pairs; classification

4.1. THEOREM. *Let $\mathcal{D}$ be a circled bounded symmetric domain. Then $\mathcal{D}$ is the open unit ball of the spectral norm of the associated Jordan pair with involution. Conversely, given a Jordan pair with positive hermitian involution, the open unit ball of the spectral norm is a circled bounded symmetric domain whose associated Jordan pair is the given one.*

We first establish some Lemmas. Recall that a Riemannian manifold is called locally symmetric if the local geodesic symmetry s_p around every point p is a local isometry. If all local symmetries extend (necessarily uniquely) to global isometries, we speak of a Riemannian symmetric space. The following Lemma goes back to E. Cartan.

4.2. LEMMA. *Let ξ be a Killing vector field on a locally symmetric Riemannian manifold, and assume that $ds_p \cdot \xi \cdot s_p = -\xi$ for some point p. Then the integral curve of ξ through p is a geodesic.*

Proof. Let φ_t be the local one-parameter group generated by ξ, and let $x(t) = \varphi_t(p)$ be the integral curve of ξ through p. Since ξ is Killing, φ_t consists of local isometries and therefore $s_{x(t)} = s_{\varphi_t(p)} = \varphi_t \circ s_p \circ \varphi_t^{-1}$. The assumption on ξ implies that $s_p(x(t)) = x(-t)$. It follows that

$$s_{x(t/2)}(x(t')) = \varphi_{t/2} \cdot s_p(x(t'-\tfrac{1}{2}t)) = \varphi_{t/2} \cdot x(\tfrac{1}{2}t-t')$$

$$= x(t-t') \ .$$

By differentiating with respect to t' at $t' = 0$ we see that $x(t)$ satisfies the differential equation

$$x = - ds_{x(t/2)} \cdot v$$

where $v = x(0) = \xi(p)$. On the other hand, let $y(t)$ be the geodesic through p with initial vector v . Then $\dot{y}(t/2)$ is obtained from v by parallel transport along $y(t)$, and since $s_{y(t/2)}$ is an isometry, $ds_{y(t/2)} \cdot v$ is obtained from $ds_{y(t/2)} \cdot \dot{y}(t/2) = - \dot{y}(t/2)$ by parallel transport along $y(t)$. This shows that

$$\dot{y} = - ds_{y(t/2)} \cdot v \ ,$$

and hence $x(t)$ and $y(t)$ satisfy the same differential equation with the same initial condition $x(0) = y(0) = p$. The Lemma follows.

4.3. LEMMA. Let (V, V^-) be a Jordan pair with positive hermitian involution, and for $v \in V$ define the vector field ξ_v on V by

$$\xi_v(z) = v - Q(z) \cdot \bar{v} \ .$$

Then the integral curve of ξ_v through 0 is defined for all t and is given by

$$x(t) = \tanh(tv)$$

(where $\tanh: V \to V$ is defined as in 3.18).

Proof. Let $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$ be the spectral decomposition. Then we have $x(0) = 0$ and $\dot{x} = \sum \lambda_i \cosh^{-2}(\lambda_i t) e_i$. By orthogonality of the e_i ,

$$\xi(x(t)) = v - Q(x(t))\bar{v} = \sum \lambda_i (1 - \tanh^2(\lambda_i t)) e_i = \dot{x} \ .$$

4.4. LEMMA. With the notations and assumptions of 4.3, let $\mathcal{D}$ be the open unit ball of the spectral norm, and pick an Aut(V)-invariant scalar product $\langle \, , \rangle$ on V (cf. 3.4). Then

$$(1) \qquad h_z(u, \bar{v}) = \langle B(z, \bar{z})^{-1} \cdot u, \bar{v} \rangle$$

defines a hermitian metric on $\mathcal{D}$ with respect to which the vector fields ξ_v are Killing vector fields. The geodesic symmetry around 0 is $s_0 = - \mathrm{Id}$; it is a global isometry.

Proof. From 3.15 and the definition of the spectral norm it is clear that $B(z, \bar{z})$ is positive and selfadjoint relative to $\langle \, , \rangle$. Thus (1) defines a hermitian metric on $\mathcal{D}$. An easy computation shows that a vector field ξ

is Killing if and only if

(2) $$d_{\xi(z)}B(z,\bar{z}) = d\xi(z)\cdot B(z,\bar{z}) + B(z,\bar{z})\cdot d\xi(z)^* ,$$

where $*$ denotes the adjoint relative to $\langle\, ,\rangle$. For $\xi = \xi_v$ we have $d\xi(z) = -D(z,\bar{v})$ and hence $d\xi(z)^* = -D(v,\bar{z})$. The left hand side of (2) is $-D(\xi(z),\bar{z}) - D(z,\overline{\xi(z)}) + Q(z,\xi(z))\bar{Q}(\bar{z}) + Q(z)\bar{Q}(\bar{z},\overline{\xi(z)})$. Now (2) follows by a straight forward computation, by expanding both sides and using the identities JP4, JP9, and JP13. From $B(z,\bar{z}) = B(-z,-\bar{z})$ it follows that $z \to -z$ is an isometry and since its only fixed point is 0 , it is the geodesic symmetry.

4.5. <u>Proof of</u> 4.1. Suppose $\mathcal{D}$ is a circled bounded symmetric domain in V , and let ξ_v be as in 2.3. Then ξ_v satisfies the hypotheses of 4.2 (with $p = 0$). Now $\xi_v(z) = v - Q(z)\bar{v}$ by definition of the Jordan structure associated with $\mathcal{D}$ and by 4.3, the geodesic through 0 with initial vector v is given by $\mathrm{Exp}_0(tv) = \tanh(tv)$. Since the Bergmann metric is complete, $\mathcal{D} = \mathrm{Exp}_0(V)$. The function tanh is a real analytic diffeomorphism of $\mathbb{R}$ onto the open interval $(-1,1)$. It follows from 3.18 and 3.19 that $x \to \tanh(x)$ is a real analytic diffeomorphism of V onto $\{x \in V \mid |x| < 1\}$, and hence $\mathcal{D}$ is the open unit ball of the spectral norm.

Conversely, let now $\mathcal{D}$ be the open unit ball of the spectral norm of a Jordan pair with positive involution.

Clearly $\mathcal{D}$ is bounded and circled. From 4.3 and the properties of tanh, every point of $\mathcal{D}$ lies on an integral curve of some vector field ξ_v and these integral curves are defined for all $t \in \mathbb{R}$. Since the ξ_v are Killing by 4.4, it follows that every point of $\mathcal{D}$ has a neighborhood which is isometric to a neighborhood of 0 . Again by 4.4, $\mathcal{D}$ is locally symmetric. By 4.2, the geodesics through 0 are precisely the integral curves through 0 of the ξ_v and are therefore defined for all $t \in \mathbb{R}$. This shows that $\mathcal{D}$ is complete as a Riemannian manifold (cf. [H1, p.58, Remark]). Since Killing vector fields on a complete Riemannian manifold are complete, the holomorphic vector fields ξ_v generate global one-parameter groups of holomorphic automorphisms of $\mathcal{D}$, and therefore $\mathcal{D}$ is homogeneous. Being symmetric around 0 , it is symmetric around every point by homogeneity. Finally, it is clear that the vector fields ξ_v are precisely the vector fields used in §2 to define the Jordan structure associated with $\mathcal{D}$. Hence the Jordan pair associated with $\mathcal{D}$ is the given one, and our proof is complete.

4.6. COROLLARY. <u>A circled bounded symmetric domain is convex</u>.

4.7. COROLLARY. <u>The domain associated with an involution-invariant subpair</u> (W,W^-) <u>of</u> (V,V^-) <u>is</u> $\mathcal{D}\cap W$ <u>where</u> $\mathcal{D}$ <u>is the domain associated with</u> (V,V^-) . <u>The domain associated with a direct product of Jordan pairs is the direct</u>

product of the domains associated with the factors.
This follows immediately from properties of the spectral norm (3.17(d)).

4.8. COROLLARY. *The exponential map* $\mathrm{Exp}_0 : V \to \mathcal{D}$ *of the Bergmann metric at the origin is a real analytic diffeomorphism given by* $\mathrm{Exp}_0(v) = \tanh(v)$. *Its inverse is* $x \to \operatorname{artanh}(x)$. *The non-zero tripotents of* V *are precisely the limit points of geodesic rays emanating from the origin.*

Proof. The first assertion was proved in 4.5. The second follows from 3.18.2. Finally, if $v = \Sigma\lambda_j e_j$ is the spectral decomposition then $\lim_{t\to\infty} \mathrm{Exp}(tv) = \lim_{t\to\infty}\Sigma \tanh(\lambda_j t)e_j = \Sigma e_j$ is a tripotent and obviously every tripotent can be obtained in this way.

4.9. COROLLARY. *With the notations of 2.1 and 3.2, we have:*
(a) $\mathrm{Aut}(V)$ *is the isotropy group of* 0 *in* $\mathrm{Aut}(\mathcal{D})$ *and* K *is the identity component of* $\mathrm{Aut}(V)$. *In particular,* $\mathfrak{k} = \mathrm{Der}(V)$.
(b) G_0 *is real analytically diffeomorphic with* ${}_R V \times K$ *under the map* $(v,k) \to \exp(\xi_v)\cdot k$.

Proof. (a) Since $\mathrm{Aut}(V)$ preserves the spectral norm, it is contained in $\mathrm{Aut}(\mathcal{D})$. Conversely, the same argument as in the proof of 2.11(a) shows that an automorphism of $\mathcal{D}$ fixing the origin (which is linear by 1.5) belongs to $\mathrm{Aut}(V)$. Now $K = \mathrm{Aut}(V) \cap G_0$ contains $\mathrm{Aut}(V)^0$, and since

$\mathcal{D} \cong G_0/K$ is simply connected, K is connected and we have equality.
(b) Clearly this map is real analytic. Its real analytic inverse is $g \to (v, \exp(\xi_v)^{-1}g)$ where $v = \operatorname{artanh}(g(0))$.

4.10. We now wish to classify circled bounded symmetric domains up to isomorphism. Suppose $f : \mathcal{D} \to \mathcal{D}'$ is an isomorphism between two such domains. After composing f with a suitable automorphism of $\mathcal{D}'$ we may assume that $f(0) = 0$. Then f is linear by 1.5. Also, it will preserve the vector fields ξ_v ; i.e., $f\circ\xi_v = \xi'_{f(v)}\circ f$ for all $v \in V$. This shows that $f(Q(x)\bar{y}) = Q'(f(x))\overline{f(y)}$, and hence $(f,\bar{f}) : (V,V^-) \to (V',V'^-)$ is an isomorphism of the associated Jordan pairs which commutes with the involutions in the sense that $\tau'\cdot f = \bar{f}\cdot\tau$. Conversely, such an isomorphism will map $\mathcal{D}$ isomorphically onto $\mathcal{D}'$, since it preserves spectral norms. Thus we are reduced to classifying Jordan pairs with positive involution. The first observation is that such a Jordan pair is necessarily *semisimple*. Semisimplicity is defined by the vanishing of the radical. There are various kinds of radicals for Jordan pairs (cf. [L5,§4]) but they all agree in the finite-dimensional case. The one most easily described in our situation is the nilradical; that is, the set of all $x \in V$ which are nilpotent in every Jordan algebra $V^{(a)}$, $a \in V^-$ (cf. 3.6). If τ is an involution then one checks easily that the n-th power of x in $V^{(\tau x)}$ is $x^{(2n+1)}$. If τ

is positive there are no nilpotent elements different from zero and hence V is semisimple.

4.11. A Jordan pair (V,V^-) is called <u>simple</u> if the Q-operators are non-trivial, and if it contains no proper ideals. Here an ideal of (V,V^-) is a pair (I,I^-) of subspaces such that $\{IV^-V\} + \{VI^-V\} \subset I$ and $\{I^-VV^-\} + \{V^-IV^-\} \subset I^-$. By [L5,10.14] a finite-dimensional semisimple Jordan pair is the direct sum of simple ideals which are unique up to order. As a consequence, every ideal of a semisimple Jordan pair has a unique complement.

The corresponding concept for domains is that of irreducibility. A circled bounded symmetric domain $\mathcal{D}$ is called <u>irreducible</u> if it is not isomorphic to a direct product $\mathcal{D}' \times \mathcal{D}''$ of lower-dimensional circled bounded symmetric domains. If $\mathcal{D}$ is not irreducible then clearly the associated Jordan pair (V,V^-) is the direct product of the Jordan pairs associated with $\mathcal{D}'$ and $\mathcal{D}''$ and hence is not simple. Conversely, let (I,I^-) be an ideal of (V,V^-) and (J,J^-) the complementary ideal. If we can show that they are stable under the involution τ (i.e. $\tau(I) = I^-$ and $\tau(J) = J^-$) then by 4.7, $\mathcal{D}$ is not irreducible. Assume $x \in I$ and $\bar{x} = \tau x \in J^-$. Then $2x^{(3)} = \{x,\bar{x},x\} \in I \cap J = 0$, and by positivity, $x = 0$. Similarly, $J \cap \tau^{-1}(I^-) = 0$. Since the ideal $(\tau^{-1}(I^-), \tau(I))$ is the sum of its intersections with (I,I^-) and (J,J^-) we have $\tau(I) = I^-$. Therefore, we see that $\mathcal{D}$

is irreducible if and only if (V,V^-) is simple, and we now have to classify simple Jordan pairs with positive involution. This amounts to just classifying simple Jordan pairs by the following

4.12. THEOREM. <u>Every semisimple complex Jordan pair</u> (V,V^-) <u>admits a positive hermitian involution. If</u> τ_1 <u>and</u> τ_2 <u>are two such involutions then there exists an automorphism</u> $(f,f_-) \in \mathrm{Aut}(V,V^-)^0$ <u>such that</u> $\tau_1 \cdot f = f_- \cdot \tau_2$.

<u>Proof</u>. By the preceding remarks, it suffices to prove this for (V,V^-) simple. The existence of a positive involution will be proved below (4.14) by a case-by-case verification. (A proof avoiding the classification is also possible by choosing a suitable Cartan involution of the Koecher-Tits algebra of (V,V^-)). Now let τ_1, τ_2 be positive hermitian involutions, let $H = \mathrm{Aut}(V,V^-)^0$, and let $K_i\,(i = 1,2)$ be the group of real points of the $\mathbb{R}$-structure defined by τ_i on H (cf. 3.2); i.e.,

$$K_i = \{(f,f_-) \in H \mid f_- = \tau_i \cdot f \cdot \tau_i^{-1}\} .$$

By 3.4, K_i is compact, and is therefore a maximal compact subgroup of H. By conjugacy of maximal compact subgroups, there exists $(g,g_-) \in H$ such that $K_1 = (g,g_-)\cdot K_2 \cdot (g^{-1}, g_-^{-1})$ (componentwise operations). Let $\tau = g_- \cdot \tau_2 \cdot g^{-1}$. Then one checks that τ is a positive involution, and the maximal compact subgroup of H defined by τ is also K_1. After replacing τ_2 by τ we may therefore assume that τ_1 and τ_2 define the same maximal compact subgroup K of H. Since a Cartan involution is uniquely determined by the maximal compact subgroup it defines, we have $\tau_1 \cdot f \cdot \tau_1^{-1} = \tau_2 \cdot f \cdot \tau_2^{-1}$ and $\tau_1^{-1} \cdot f_- \cdot \tau_1 = \tau_2^{-1} \cdot f_- \cdot \tau_2$ for all $(f,f_-) \in H$; in other words, $(h,h_-) = (\tau_1^{-1} \cdot \tau_2, \tau_1 \cdot \tau_2^{-1})$ belongs to the centralizer of H in $GL(V) \times GL(V^-)$. By the Lemma below, $(h,h_-) = (\alpha \cdot Id, \alpha^{-1} \cdot Id)$ with $\alpha \in \mathbb{C}$, or $\tau_2 = \alpha\tau_1$. Choose a non-zero tripotent e relative to τ_1. Then $Q(e) \cdot \tau_2 e = Q(e)\alpha\tau_1 e = \bar{\alpha}Q(e) \cdot \tau_1 e = \bar{\alpha}e$, and since τ_2 is a positive involution, $\alpha = \bar{\alpha} > 0$. Now $(f,f_-) = (\alpha^{\frac{1}{2}} \cdot Id, \alpha^{-\frac{1}{2}} \cdot Id) \in H$ and $\tau_1 \cdot f = f_- \cdot \tau_2$.

4.13. LEMMA. Let (V,V^-) be a simple Jordan pair over an algebraically closed field k. Then the centralizer of $Inn(V,V^-)$ in $GL(V) \times GL(V^-)$ consists of all $(\alpha \cdot Id, \alpha^{-1} \cdot Id)$, $\alpha \in k$.

Proof. Suppose (h,h_-) centralizes $Inn(V,V^-)$. Pick a frame $E = ((e_1,e_1^-),\cdots,(e_r,e_r^-))$ of (V,V^-) (cf. [L5, 10.12]). Since the Peirce spaces of E are weight spaces of the torus of $Inn(V,V^-)$ defined by E ([L5,§5]) and $V_{ii} = k \cdot e_i$, $V_{ii}^- = k \cdot e_i^-$, we have $he_i = \alpha_i e_i$ and $h_- e_i^- = \alpha_i^{-1} e_i^-$. By conjugacy of frames ([L5,17.1]) we can permute the (e_i,e_i^-) by inner automorphisms. Hence $\alpha_i = \alpha_j = \alpha$ for all i,j. Thus (h,h_-) is of the form $(\alpha Id, \alpha^{-1} Id)$ when restricted to the Cartan subpair $(\Sigma V_{ii}, \Sigma V_{ii}^-)$, and by [L5,15.15] everywhere.

4.14. We now go through the list of simple complex Jordan pairs ([L5,17.4]), exhibiting in each case a positive hermitian involution.

Type $I_{p,q}$. $V = V^- = M_{p,q}(\mathbb{C})$, complex $p \times q$-matrices.

Type II_n. $V = V^- = A_n(\mathbb{C})$, alternating (= skew-symmetric) complex $n \times n$-matrices

Type III_n. $V = V^- = S_n(\mathbb{C})$, symmetric complex $n \times n$-matrices.

In these 3 cases, the Jordan pair structure is given by $Q(x)y = x \cdot {}^t y \cdot x$ (matrix product). A positive involution is $\tau(x) = \bar{x}$ (complex conjugate matrix). Indeed, suppose $Q(x) \cdot \tau(x) = x \cdot {}^t\bar{x} \cdot x = \lambda x$ where $\lambda \in \mathbb{C}$. Then $(x \cdot {}^t\bar{x})^2 = \lambda(x \cdot {}^t\bar{x})$, and since $x \cdot {}^t\bar{x}$ is positive semidefinite hermitian, it follows (for instance by taking traces) that $\lambda > 0$ for $x \neq 0$.

Type IV_n. $V = V^- = \mathbb{C}^n$, with $Q(x)y = q(x,y)x - q(x) \cdot y$ where q is a non-degenerate quadratic form on $\mathbb{C}^n$ and $q(x,y) = q(x+y) - q(x) - q(y)$. After a change of basis, we may assume that $q(x) = \langle x,x \rangle = \Sigma x_i^2$, and then $q(x,y) = 2\langle x,y \rangle$. A positive involution is given by $\tau(x) = \bar{x}$ (complex conjugation in each variable). Indeed, assume that $x \neq 0$ and $Q(x)\bar{x} = 2\langle x,\bar{x} \rangle x - \langle x,x \rangle \bar{x} = \lambda x$, $\lambda \in \mathbb{C}$. Then if $\langle x,x \rangle = 0$, we have $\lambda = 2\langle x,\bar{x} \rangle > 0$ and if $\langle x,x \rangle \neq 0$, it follows by taking the scalar product with x that $2\langle x,\bar{x} \rangle\langle x,x \rangle - \langle x,x \rangle\langle x,\bar{x} \rangle = \langle x,\bar{x} \rangle\langle x,x \rangle = \lambda\langle x,x \rangle$, hence $\lambda = \langle x,\bar{x} \rangle > 0$.

<u>Type</u> V. $V = V^- = M_{1,2}(\mathbb{O}_{\mathbb{C}})$, 1×2 -matrices with entries from the complex 8-dimensional Cayley algebra $\mathbb{O}_{\mathbb{C}}$, with $Q(x)y = x\cdot({}^t\tilde{y}\cdot x)$. There $\tilde{}$ denotes the ($\mathbb{C}$-linear) canonical involution of $\mathbb{O}_{\mathbb{C}}$.

<u>Type</u> VI. $V = V^- = H_3(\mathbb{O}_{\mathbb{C}})$, 3×3 -matrices with entries from $\mathbb{O}_{\mathbb{C}}$, which are hermitian with respect to the canonical involution; i.e., they satisfy ${}^t\tilde{x} = x$. The Jordan structure is the one induced from the Jordan algebra $H_3(\mathbb{O}_{\mathbb{C}})$; thus $Q(x)y = \frac{1}{2}(x\circ(x\circ y) - x^2\circ y)$ where $x\circ y = x\cdot y + y\cdot x$. Note that type V may be imbedded into type VI by

$$(x_1,x_2) \longrightarrow \begin{pmatrix} 0 & x_1 & x_2 \\ \tilde{x}_1 & 0 & 0 \\ \tilde{x}_2 & 0 & 0 \end{pmatrix} .$$

A positive involution for these two cases is given by $\tau(x) = \bar{x}$, complex conjugation relative to the real Cayley division algebra $\mathbb{O}$. Since the above imbedding commutes with τ , it suffices to prove this for type VI. Here the fixed point set of τ is the real Jordan algebra $A = H_3(\mathbb{O})$ which is known to be formally real (cf. [B-K]); and hence $H_3(\mathbb{O}_{\mathbb{C}})$ is the hermitification of A in the sense of 3.7. Therefore τ is positive.

4.15. We now describe the domains associated with (V,V^-) and τ in more detail. This will be done in terms of the generic minimun polynomial of (V,V^-) for which we refer to [L5,§16]. Suffice it to say here that the generic minimum polynomial

$$m(T,x,y) = \sum_{i=0}^{r} (-1)^i m_i(x,y) T^{r-i}$$

of a semisimple Jordan pair is a monic polynomial in the indeterminate T with coefficients $m_i(x,y)$ which are polynomial functions on $V \times V^-$, homogeneous of bidegree (i,i) . The degree r is the rank of (V,V^-) . If (V,V^-) is simple then the polynomial function $\det B(x,y)$ on $V \times V^-$ is a power of a unique irreducible polynomial function $N(x,y)$, normalized such that $N(0,0) = 1$, called the <u>generic norm</u> of (V,V^-) . It is related to the generic minimum polynomial by $N(x,y) = m(1,x,y)$, and hence we can recover $m(T,x,y)$ from $N(x,y)$ by $m(T,x,y) = T^r\cdot N(T^{-1}x,y)$.

4.16. PROPOSITION. <u>Let (V,V^-) be a Jordan pair with positive involution $\tau : x \to \bar{x}$. Define polynomial functions $f_j (j = 1,\cdots,r)$ on $V \times V^-$ by</u>

$$f_j(x,y) = \sum_{i=0}^{j} (-1)^i \binom{r-i}{r-j} m_i(x,y)$$

<u>where the m_i are the coefficients of the generic minimum polynomial, and r is the rank of (V,V^-) . Then $|x|^2$ is the largest root of $m(T,x,\bar{x})$, and the domain associated with (V,V^-) and τ is</u>

$$\mathcal{D} = \{x \in V \mid f_j(x,\bar{x}) > 0 \ , \ j = 1,\cdots,r\} .$$

Proof. Let $x \in V$. By decomposing the tripotents occurring in the spectral decomposition of x into primitive ones, we may assume that $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$ where $(e_1, \cdots, e_r)$ is a maximal orthogonal system of primitive tripotents (cf. §5). Then $((e_1,\bar{e}_1),\cdots,(e_r,\bar{e}_r))$ is a frame of (V,V^-) and $(C,C^-) = (\Sigma V_{ii}, \Sigma V_{ii}^-)$ is a Cartan subpair ([L5,15.9]). The restriction of $m(T,x,y)$ to (C,C^-) is given by

$$m(T,x,y) = \prod_{i=1}^{r} (T - \lambda_i\mu_i)$$

where $x = \Sigma\lambda_i e_i$, $y = \Sigma\mu_i\bar{e}_i$ (cf. [L5,16.15,16.16]). In particular, the real polynomial

$$f(T) = m(T,x,\bar{x}) = \prod_{i=1}^{r} (T - |\lambda_i|^2)$$

has roots $|\lambda_1|^2,\cdots,|\lambda_r|^2$, and $|x| = \max\{|\lambda_1|,\cdots,|\lambda_r|\} < 1$ if and only if all roots of $f(T)$ are less than one. Let $g(T) = f(T+1)$. Then all roots of f are less than one if and only if all roots of g are negative. But a real polynomial all of whose roots are real will have only negative roots if and only if all its coefficients are positive. Since $f(T) = g(T-1)$, the coefficients of g are the coefficients of the expansion

$$f(T) = \sum_{j=0}^{r} a_j (T-1)^{r-j}$$

of $f(T)$ in powers of $T-1$:

$$a_j = \frac{1}{(r-j)!}\left[\frac{\partial^{r-j} f}{\partial T^{r-j}}\right]_{T\to 1} =$$

$$= \frac{1}{(r-j)!}\left[\frac{\partial^{r-j}}{\partial T^{r-j}} \sum_{i=0}^{r} (-1)^i m_i(x,\bar{x}) T^{r-i}\right]_{T\to 1} =$$

$$= \sum_{i=0}^{j} (-1)^i \binom{r-i}{r-j} m_i(x,\bar{x}) = f_j(x,\bar{x}) .$$

This completes the proof.

4.17. Let us now work out the domains associated with the Jordan pairs listed in 4.14.

Type $I_{p,q}$. The transpose $x \to {}^t x$ is an isomorphism: $I_{p,q} \cong I_{q,p}$, so we assume $p \leq q$. Then the rank is p, and the generic minimum polynomial is

$$m(T,x,y) = \det(T\cdot 1 - x\cdot {}^t y) .$$

Hence $|x|^2$ is the largest eigenvalue of $x\cdot{}^t\bar{x}$, and $\mathcal{D}$ consists of all $x \in M_{p,q}(\mathbb{C})$ for which $1 - x\cdot{}^t\bar{x}$ is positive definite.

Type II_n. The generic minimum polynomial is the square root of $\det(T\cdot 1 - x\cdot{}^t y)$:

$$\det(T\cdot 1 - x\cdot{}^t y) = T^{n-2r}\cdot m(T,x,y)^2$$

where $r = \left[\frac{n}{2}\right]$ is the rank. Again $|x|^2$ is the largest eigenvalue of $x\cdot{}^t\bar{x} = -x\cdot\bar{x}$, and $\mathcal{D}$ is the set of all $x \in A_n(\mathbb{C})$ such that $1 + x\bar{x}$ is positive definite.

<u>Type</u> III_n . The generic minimum polynomial is

$$m(T,x,y) = \det(T\cdot 1 - x\cdot {}^t y) = \det(T\cdot 1 - x\cdot y) ,$$

$|x|^2$ is the largest eigenvalue of $x\cdot {}^t\bar{x} = x\cdot\bar{x}$, and $\mathcal{D}$ is the set of all $x \in S_n(\mathbb{C})$ such that $1 - x\cdot\bar{x}$ is positive definite.

<u>Type</u> IV_n . The generic minimum polynomial is

$$m(T,x,y) = T^2 - q(x,y)\cdot T + q(x)q(y)$$

$$= T^2 - 2\langle x,y\rangle\cdot T + \langle x,x\rangle\langle y,y\rangle ,$$

with the conventions introduced in 4.14. By 4.17, $f_1(x,y) = 2 - q(x,y) = 2\cdot(1 - \langle x,y\rangle)$ and $f_2(x,y) = N(x,y) = 1 - 2\langle x,y\rangle + \langle x,x\rangle\langle y,y\rangle$. Hence $\mathcal{D}$ is the set of all $x \in \mathbb{C}^n$ for which

$$\langle x,\bar{x}\rangle < 1 \quad\text{and}\quad 1 - 2\langle x,\bar{x}\rangle + |\langle x,x\rangle|^2 > 0 .$$

<u>Type</u> V . The generic minimum polynomial is

$$m(T,x,y) = T^2 - m_1(x,y)\cdot T + m_2(x,y)$$

where

$$m_1(x,y) = t(x_1\tilde{y}_1 + x_2\tilde{y}_2) ,$$

$$m_2(x,y) = n(x_1)n(y_1) + n(x_2)n(y_2) + t((\tilde{x}_1x_2)(\tilde{y}_2y_1)) ,$$

for $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $M_{1,2}(\mathbb{O}_{\mathbb{C}})$. Here $n(a) = a\tilde{a}$ and $t(a) = a + \tilde{a}$ are norm and trace of $\mathbb{O}_{\mathbb{C}}$, and $\sim$ is the canonical involution (Note: the formula for the generic norm given in [L5,17.9] is wrong; the correct one is obtained by imbedding V into VI and restricting the generic norm of VI). For $a \in \mathbb{O}_{\mathbb{C}}$ let $a^* = \bar{\tilde{a}} = \tilde{\bar{a}}$ where $-$ is conjugation with respect to $\mathbb{O}$. Then by 4.16, $\mathcal{D}$ consists of all $(a,b) \in M_{1,2}(\mathbb{O}_{\mathbb{C}})$ which satisfy

$$2 - t(aa^* + bb^*) > 0$$

and $1 - t(aa^* + bb^*) + |n(a)|^2 + |n(b)|^2 + t((\tilde{a}b)(b^*\bar{a})) > 0$.

<u>Type</u> VI . The generic minimum polynomial is

$$m(T,x,y) = T^3 - t(x,y)\cdot T^2 + t(x^\#,y^\#)\cdot T - n(x)n(y) ,$$

where n is the generic norm of the Jordan algebra $H_3(\mathbb{O}_{\mathbb{C}})$, $t(x,y) = -\partial_x\partial_y \log n(z)\,|_{z=e}$ (e the unit element), and $x^\#$ is the "adjoint" defined by $t(x^\#,y) = \partial_y n(x)$ (cf. [L5,17.10] and [Mc]). Denote conjugation with respect to $H_3(\mathbb{O})$ by $^-$. Then by 4.16, $\mathcal{D}$ is the set of all $x \in H_3(\mathbb{O}_{\mathbb{C}})$ for which

$$3 - t(x,\bar{x}) > 0,\ 3 - 2t(x,\bar{x}) + t(x^\#,\bar{x}^\#) > 0 ,$$

$$1 - t(x,\bar{x}) + t(x^\#,\bar{x}^\#) - |n(x)|^2 > 0 .$$

4.18. Finally, we remark that the only isomorphisms among the types listed are the following ([L5, 17.11]).

(1) $\quad I_{1,1} \cong II_1 \cong III_1 \cong IV_1$,

(2) $\quad IV_2 \cong IV_1 \times IV_1$,

(3) $I_{1,3} \cong II_3$,

(4) $III_2 \cong IV_3$,

(5) $I_{2.2} \cong IV_4$,

(6) $II_4 \cong IV_6$,

(7) $I_{p,q} \cong I_{q,p}$.

(The isomorphism (5) was overlooked in [L5]).

§5. The manifold of tripotents

5.1. In this section, (V,V^-) denotes a finite-dimensional complex Jordan pair, and $\tau: x \to \overline{x}$ a positive hermitian involution. Let $K = \mathrm{Aut}(V)^0$ and let $\langle\, , \rangle$ be a K-invariant hermitian scalar product on V. We denote by M the set of tripotents of V, and define an ordering on M by

$$c < e \Leftrightarrow e - c \in M \text{ and } c \perp e - c .$$

In other words, $c < e$ means that $e = c + c'$ where c' is a non-zero tripotent orthogonal to c. One checks easily that this is indeed an ordering. A tripotent e is <u>maximal</u> with respect to this ordering if and only if the Peirce space $V_0(e) = 0$. This follows from the fact that the tripotents orthogonal to e are precisely the tripotents contained in $V_0(e)$ (cf. 3.9, 3.13). A tripotent is called <u>primitive</u> if it is minimal among non-zero tripotents, i.e., if it cannot be written as a sum of orthogonal tripotents in a non-trivial way. Since an orthogonal set of non-zero orthogonal tripotents is linearly independent, it follows by finite-dimensionality that every tripotent can be written as a sum of primitive orthogonal tripotents. Finally, we define a <u>frame</u> to be a maximal orthogonal system of primitive tripotents.

5.2. PROPOSITION. (a) *A tripotent* e *is primitive if and only if* $A(e) = \mathbb{R}.e$ (*where* $A(e) \oplus iA(e) = V_2(e)$, *cf*. 3.13).

(b) *Let* $(e_1,\dots,e_r)$ *be an orthogonal system of tripotents. The following conditions are equivalent.*

(i) $(e_1,\dots,e_r)$ *is a frame*;

(ii) *the* e_i *are primitive and* $e_1 + \dots + e_r$ *is maximal*.

(iii) $\mathbb{R}.e_1 + \dots + \mathbb{R}.e_r$ *is a maximal flat subspace of* V .

Proof. (a) Let e be primitive. If c is an idempotent of $A(e)$ then $c^2 = c$ implies $c^{(3)} = c^3 = c$, and hence c and $e - c$ are orthogonal tripotents whose sum is e . It follows that $c = 0$ or $c = e$. Thus the only non-zero idempotent of $A(e)$ is e . By standard facts on formally real Jordan algebras ([B-K, Chapter XI]), $A(e) = \mathbb{R}.e$. Conversely, if $e = c + d$ is not primitive we have $Q_e\bar{c} = Q_c\bar{c} = c$ and $Q_e\bar{d} = Q_d\bar{d} = d$. Hence $c,d \in A(e)$ and thus $A(e)$ is not one dimensional.

(b) This follows easily from 3.11.

5.3. THEOREM. (a) *Any two maximal flat subspaces of* V *are conjugate by an element of* K .

(b) *Any two maximal tripotents are conjugate by an element of* K .

Proof. (a) The proof uses an idea due to Hunt. Let W be a maximal flat subspace, and write $W = \mathbb{R}.e_1 + \dots + \mathbb{R}.e_r$ as in 3.11. Choose $x \in W$ such that the powers of x span

W . For instance, let $x = \lambda_1 e_1 + \dots + \lambda_r e_r$ with $0 < \lambda_1 < \dots < \lambda_r$ (use a Vandermonde type argument). Similarly, choose x' for the maximal flat subspace W' . The function

$$\varphi(g) = \langle g(x'),\bar{x}\rangle$$

on the compact group K attains its maximum, say for $g = k$. We claim that $k(x') = y \in W$. This will imply $k(W') \subset W$ and hence $k(W') = W$ by maximality of W' . Now by the choice of k , we have, for any derivation $\Delta \in \mathfrak{k} = \mathrm{Lie}(K)$, that $0 = \frac{d}{dt}\Big|_{t=0} \langle \exp(t\Delta).y,\bar{x}\rangle = \langle \Delta.y,\bar{x}\rangle$. In particular, for $\Delta = D(u,\bar{v}) - D(v,\bar{u})$ (cf. 3.2) we get $\langle\{u\bar{v}y\},\bar{x}\rangle = \langle\{v\bar{u}y\},\bar{x}\rangle$ which implies $\langle u,\overline{\{v\bar{y}x\}}\rangle = \langle v,\overline{\{u\bar{y}x\}}\rangle = \langle\{u\bar{y}x\},\bar{v}\rangle = \langle u,\{\bar{y}x\bar{v}\}\rangle = \langle u,\overline{\{y\bar{x}v\}}\rangle$. Since this holds for all u it follows that $\{x\bar{y}v\} = \{y\bar{x}v\}$ for all v . Let $V = \Sigma V_{ij}$ be the Peirce decomposition of V with respect to $(e_1,\dots,e_r)$. Since $(e_1,\dots,e_r)$ is a frame, we have $V_{00} = 0$ and $V_{ii} = \mathbb{C}.e_i$, by 5.2. Let $y = \Sigma y_{ij}$ be the corresponding decomposition of y . Then by 3.15,

$$\{x\bar{y}x\} = 2\sum_{1\le i\le j\le r} \lambda_i\lambda_j y^*_{ij} = \{x\bar{x}y\} = \sum_{i\le j}(\lambda_i^2+\lambda_j^2)y_{ij} .$$

This implies $y_{ij} = 0$ for $0 \le i < j \le r$ and $y^*_{ii} = y_{ii} \in A(e_i) = \mathbb{R}.e_i$. Hence $y = \Sigma y_{ii} \in W$.

(b) Let e and e' be maximal tripotents. By 5.2, $e = e_1 + \dots + e_r$ and $e' = e'_1 + \dots + e'_r$ where $(e_1,\dots,e_r)$ and $(e'_1,\dots,e'_r)$ are frames, spanning maximal flat subspaces W and W' . By (a) and 3.11, there exists $g \in K$ such that $g(e_j') = \pm e_{\pi(j)}$ where π is a permutation of $1,\ldots,r$. Hence $g(e') = \epsilon_1 e_1 + \ldots + \epsilon_r e_r$ where $\epsilon_j = \pm 1$. Now $iD(e_j,\bar{e}_j) \in \mathfrak{k} = Der(V)$ by 3.2, and hence $\exp(\frac{\pi}{2}\, iD(e_j,\bar{e}_j)) = g_j \in K$ with the property that $g_j(e_j) = -e_j$ and $g_j(e_k) = e_k$ for $k \neq j$, by orthogonality of the e_j . Thus we obtain an automorphism in K carrying e' into e by following g with a suitable product of the g_j .

5.4. We define the <u>rank</u> of V to be the common dimension of the maximal flat subsystems of V , and the <u>rank of a tripotent</u> e to be the rank of $V_2(e)$. If $e = e_1 + \ldots + e_s$ is a decomposition of e as a sum of orthogonal primitive tripotents then s is the rank of e . Indeed, $e_j \in V_2(e)$ and e is a maximal tripotent of $V_2(e)$. Hence $(e_1,\ldots,e_s)$ is a frame of $V_2(e)$ which means s is the rank of $V_2(e)$. In particular, the primitive tripotents are those of rank one, and the maximal ones those of rank equal to rank(V) .

If $(e_1,\ldots e_r)$ is a frame of V then $((e_1,\bar{e}_r),\ldots,(e_r,\bar{e}_r))$ is a frame of the Jordan pair (V,V^-) (cf [L5, 10.12]). Hence the rank of V defined here agrees with the rank of the Jordan pair (V,V^-) as defined in [L5, 15.18]. We will show later that rank(V) is also the real rank of the group G_0 .

5.5. PROPOSITION. <u>Let</u> A <u>be a formally real Jordan algebra</u>. <u>Then the set of maximal tripotents of the hermification</u> $V = A_{\mathbb{C}}$ (cf. 3.7) <u>is the "unit circle"</u>

$$C = \{z \in A_{\mathbb{C}} \mid z^* = z^{-1}\} = \exp(iA) ,$$

where exp denotes the exponential function in the Jordan algebra $A_{\mathbb{C}}$.

<u>Proof</u>. Clearly the unit element e of A is a maximal tripotent of $A_{\mathbb{C}}$. Since K is transitive on the set of maximal tripotents by 5.3, and K is contained in the structure group $Str(A_{\mathbb{C}})$ of the complex Jordan algebra $A_{\mathbb{C}}$, every maximal tripotent is invertible. (In fact, K is the set of all $g \in Str(A_{\mathbb{C}})^0$ which satisfy $(gz)^* = g^{\#-1}(z^*)$, where $\#$ denotes the canonical involution of $Str(A_{\mathbb{C}})$, and is therefore a compact real form of $Str(A_{\mathbb{C}})^0$) . Now $z \in A_{\mathbb{C}}$ is a maximal tripotent if and only if $z = z^{(3)} = Q(z)\bar{z} = P(z)z^*$; i.e., $z^* = P(z)^{-1}z = z^{-1}$, by standard properties of the inverse in a Jordan algebra.

Now let $a \in A$. Then $(ia)^* = -ia$, and hence $[\exp(ia)]^* = \exp(-ia) = [\exp(ia)]^{-1}$, by elementary facts about the exponential function in a Jordan algebra. Conversely, let $z^* = z^{-1}$, and let B be the subalgebra of $A_{\mathbb{C}}$ generated by z and e . Then $z^* \in B$ and hence B is invariant under $*$. It follows that $B = B_0 \oplus iB_0$ where $B_0 = B \cap A$, and B_0 is an associative formally real subalgebra of A . Hence $B = \mathbb{R}.c_1 + \ldots + \mathbb{R}.c_s$ where the c_j are orthogonal idempotents of A whose sum is e. Thus $z = \alpha_1 c_1 + \dots + \alpha_s c_s$ with $\overline{\alpha}_j = \alpha_j^{-1}$, or $\alpha_j = \exp(i\lambda_j)$, $\lambda_j \in \mathbb{R}$. Thus $z = \exp i(\lambda_1 c_1 + \dots + \lambda_s c_s) \in \exp(iA)$. (This proof is due to U. Hirzebruch [Hi]).

5.6. THEOREM. (a) The set M of tripotents of V is a compact submanifold of V, and K acts transitively on every connected component of M. The tangent space of M at e (identified with a subspace of V) is

$$T_e(M) = iA(e) \oplus V_1(e) .$$

(b) Define an equivalence relation R on M by

$$d \sim e \Leftrightarrow d \text{ and } e \text{ have the same Peirce spaces.}$$

Then the set $S = M/R$ of equivalence classes has a unique manifold structure such that the canonical map $p: M \to S$ is a K-equivariant fibration. The fibre through e (i.e., the equivalence classes of e) is the unit circle $C(e)$ of $A(e)_{\mathbb{C}}$. Moreover, S is a compact (not connected) hermitian symmetric space with symmetry around $p(e)$ induced by the "Peirce reflection"

$$s_e = B(e, 2\overline{e}) = \exp \pi i D(e,\overline{e}) (= (-1)^{\alpha}.\mathrm{Id} \text{ on } V_\alpha(e)) .$$

Proof. (a) M is compact since $x^{(3)} = x$ implies $|x|^3 = |x|$ hence $|x| \leq 1$. Let N be the orbit of $e \in M$ under K. Then N is a compact submanifold of M, and we claim that the tangent space of N at e is

$$(1) \qquad T_e(N) = iA(e) \oplus V_1(e) .$$

Indeed, by differentiating the equation $x^{(3)} = x$ at $x = e$ in direction v we get

$$(2) \qquad v = \{e\overline{e}v\} + Q_e \overline{v} ,$$

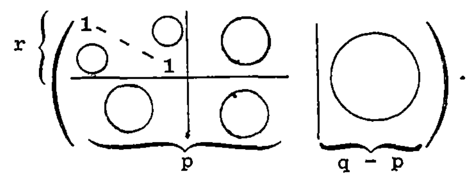
and if $v = v_2 + v_1 + v_0$ is the Peirce decomposition with respect to e then (2) implies $v = 2v_2 + v_1 + v_2^*$ or $v_2^* = -v_2$, $v_0 = 0$. This proves the inclusion $\subset$ in (1). Conversely let $v_2 + v_1 \in iA(e) \oplus V_1(e)$ and let $\Delta = D(u,\overline{e}) - D(e,\overline{u}) \in \mathfrak{k}$ where $u = \frac{1}{4}v_2 + v_1$. Then $\exp(t\Delta).e$ is a curve through e in N whose tangent vector at $t = 0$ is $\Delta.e = \{u\overline{e}e\} - \{e\overline{u}e\} = \frac{1}{2}v_2 + v_1 - \frac{1}{2}v_2^* = v$. This establishes (1).

To complete the proof of (a) it suffices to show that N is a neighborhood of e in M (where M has the topology induced from V). Assume this is not the case. Then there exists a sequence $z_n \in M - N$ converging to e. Every z in a neighborhood of e in V can be written uniquely in the form $z = x + y$ where $x \in N$ and y belongs to the normal space $T_e(N)^\perp$ of N at e. Writing $z_n = x_n + y_n$ in this way, we have $x_n \in N$ converging to e and $y_n \in T_e(N)^\perp$ converging to zero. After passing to a subsequence, we may assume that the sequence $y_n/|y_n|$ converges to a unit vector $u \in T_e(N)^\perp$. By expanding the equation $z_n^{(3)} = z_n$ we get $y_n = y_n^{(3)} + Q(x_n)\overline{y}_n + Q(y_n)\overline{x}_n + \{x_n\overline{x}_n y_n\} + \{x_n\overline{y}_n y_n\}$. If we divide by $|y_n|$ and let $n \to \infty$ it follows that $u = \{e\overline{e}u\} + Q_e\overline{u}$ which implies, in view of (2), that $u \in T_e(N)$. Contradiction.

(b) We show first that $d \sim e$ if and only if $d \in C(e)$. Let $d \sim e$. Then $d \in V_2(d) = V_2(e)$, and the map $z \to Q(d)\bar{z}$ $(z \in V_2(e))$ is invertible since it is conjugation relative to $A(d)$. Now $Q(d)\bar{z} = P(d)z^*$ where P denotes the quadratic operators of $A(e)_{\mathbb{C}}$ and $*$ conjugation relative to $A(e)$. It follows that d is invertible in the Jordan algebra $A(e)_{\mathbb{C}}$, and $d = d^{(3)} = P(d)d^*$ implies $d^{-1} = d^*$; i.e., $d \in C(e)$. Now let $d \in C(e)$. For $z \in V_2(e)$ we have $\{d\bar{d}z\} = P(d,z).d^* = P(d,z).d^{-1}$ $= 2d(d^{-1}z) + 2d^{-1}(dz) - 2(dd^{-1})z = 2z$ (where ab denotes the product in the Jordan algebra $A(e)_{\mathbb{C}}$), and for $z \in V_0(e)$ we have $\{d\bar{d}z\} = 0$ by the Peirce rules since $d \in V_2(e)$. Thus $V_\alpha(e) \subset V_\alpha(d)$ for $\alpha = 0,2$. Since $C(e)$ is connected by 5.5 and dim $V_\alpha(e)$ depends continuously on d, we have equality. Now $V_1(d) = V_1(e)$ since it is the orthogonal complement of $V_2(e) \oplus V_0(e)$ with respect to $\langle\, ,\rangle$, and therefore we have shown that $d \sim e$. As a consequence, $d \sim e$ if and only if $C(d) = C(e)$, and thus $C(e)$ is the equivalence class of e. Clearly $k.C(e) = C(k.e)$ for $k \in K$ since K consists of automorphisms of V. Let $M(e)$ be the connected component of M containing e, K' the isotropy group of e in K, and K'' the normalizer of $C(e)$ in K. Then $M(e) = K/K'$, and the canonical map $p\colon M(e) \to S$ is equivalent with the map $K/K' \to K/K''$ which is a K-equivariant fibration. The tangent space of $C(e)$ at e is $iA(e)$. This follows from (a) applied to $V_2(e)$ in place of V. Hence the canonical map $p\colon M \to S$ induces

a vector space isomorphism $V_1(e) \cong T_{p(e)}(S)$, which depends only on $\pi(e)$. By transferring the complex structure structure and the scalar product from $V_1(e)$ to $T_{p(e)}(S)$ we see that S has an almost complex structure and hermitian metric which are K-invariant. The Peirce reflection $s_e \in K$ defines a diffeomorphism of S having $p(e)$ as isolated fixed point since $s_e \mid V_1(e) = -\mathrm{Id}$. It follows that S is a *hermitian symmetric space.*

5.7. Example. Let $V = M_{p,q}(\mathbb{C})$ with $p \le q$, and $Q_z\bar{w} = zw^*z$, $w^* = {}^t\bar{w}$. Let

$e =$ 

Then the connected component M' of M containing e consists of all matrices $d \in V$ of rank r which satisfy $dd^*d = d$. The Peirce spaces of e are

$$\left(\begin{array}{c|c} V_2 & V_1 \\ \hline V_1 & V_0 \end{array}\right);$$

i.e., $V_2(e) \cong M_r(\mathbb{C})$, $A(e) \cong H_r(\mathbb{C})$ (hermitian $r \times r$ matrices), $V_1(e) \cong M_{r,q-r}(\mathbb{C}) \times M_{p-r,r}(\mathbb{C})$, $V_0(e)$ $\cong M_{p-r,q-r}(\mathbb{C})$. The unit circle $C(e)$ is isomorphic with the group of unitary $r \times r$ matrices. Every $d \in M'$ may be considered as a linear map $d\colon \mathbb{C}^q \to \mathbb{C}^p$ of rank r.

The quotient $S' = M'/\sim$ may be identified with $\mathrm{Grass}_{q-r}(\mathbb{C}^q) \times \mathrm{Grass}_r(\mathbb{C}^p)$, with p being given by $p(d) = (\mathrm{Ker}(d), \mathrm{Im}(d))$. The group K consists of all transformations $z \to uzv$ with u,v unitary matrices of size $p \times p$ and $q \times q$ respectively.

5.8. Recall that $M = I(\mathbb{R})$, the set of real points of the variety of idempotents of the Jordan pair (V,V^-), with the $\mathbb{R}$-structure given by the Galois action $(a,b) \to (\bar{b},\bar{a})$ (cf. 3.8). Similarly, let $F \subset (V \times V^-)^r$ (where $r = \mathrm{rank}(V,V^-)$) be the variety of frames of (V,V^-). Then τ defines an $\mathbb{R}$-structure on F and $F(\mathbb{R})$ may be identified with the set of frames of tripotents of V. By [L5, 17.1] we have: if (V,V^-) is simple, then any two frames of (V,V^-) are conjugate under an inner automorphism; in other words, the group $H = \mathrm{Aut}(V,V^-)^0$ is transitive on F. The analogous statement for frames of tripotents is

5.9. THEOREM. <u>If</u> (V,V^-) <u>is simple then any two frames of tripotents of</u> V <u>are conjugate under</u> K.

Since $K = H(\mathbb{R})$, the set of real points of the $\mathbb{R}$-structure defined by τ (cf. 3.2), and the action of H on F is compatible with the Galois actions (i.e., it is defined over $\mathbb{R}$), this will follow from

5.10. LEMMA. <u>Let</u> G <u>be a complex affine algebraic group acting transitively on a complex variety</u> X. <u>Suppose</u> G, X, <u>and the action are all defined over</u> $\mathbb{R}$, <u>and that</u> $G(\mathbb{R})$ <u>is compact</u>. <u>Then</u> $G(\mathbb{R})$ <u>acts transitively on</u> $X(\mathbb{R})$.

For later applications, we prove the following more general "real version" of 5.10. (To see that 5.10 is a special case, imbed G into $GL_m(\mathbb{C})$ in such a way that $G(\mathbb{R})$ is the intersection of G with the unitary group $U(n)$, then restrict scalars to $\mathbb{R}$. The maps σ and θ are just the Galois actions).

5.11. LEMMA. <u>Let</u> $G \subset GL_n(\mathbb{R})$ <u>be an open subgroup of the set of real points of a real algebraic group, and suppose</u> G <u>is self-adjoint; i.e.,</u> $g \in G$ <u>implies</u> $\theta(g) = {}^t g^{-1} \in G$. <u>Assume</u> G <u>acts transitively on a real manifold</u> X. <u>Let</u> σ <u>be a diffeomorphism of period</u> 2 <u>of</u> X <u>compatible with</u> θ $(\sigma(gx) = \theta(g)\sigma(x))$. <u>Then the fixed point set</u> G^θ <u>of</u> θ <u>in</u> G <u>acts transitively on the fixed point set</u> X^σ <u>of</u> σ <u>in</u> X.

<u>Proof</u>. Let $x_0 \in X^\sigma$ (if $X^\sigma = \emptyset$ there is nothing to prove) and let H be the isotropy group of x_0 in G. Then H is selfadjoint and hence the restriction of the trace form $(A,B) = \mathrm{trace}(AB)$ $(A,B \in \mathfrak{gl}_n(\mathbb{R}))$ to $\mathfrak{h} = \mathrm{Lie}(H)$ is non-degenerate. Thus $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m} = \mathfrak{h}^\perp$ is both θ - and $\mathrm{Ad}\, H$ -invariant. Let $K = G^\theta$ with Lie algebra $\mathfrak{k}$ and let $\mathfrak{p},\mathfrak{s},\mathfrak{t}$ be the (-1)-eigenspace of θ on $\mathfrak{g},\mathfrak{h},\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{t}$ and $\mathfrak{s}$ is a Lie triple system. By [H, p. 218, Th.1.4], G decomposes topologically (in fact, diffeomorphically)

$G = K.\exp(\mathfrak{t}).\exp(\mathfrak{s})$ (the proof in [H] is for the case G semisimple but applies equally well to the present case where G is reductive). Let $L = H \cap K$ and consider the vector bundle $E = K \times_L \mathfrak{t}$ associated with the principal bundle $K \to K/L$, with typical fibre $\mathfrak{t}$, where L acts on $\mathfrak{t}$ via Ad. Then one shows that the map $E \to G/H = X$ induced from the map $(k,x) \to k.\exp(x).H$ $(k \in K,\ x \in \mathfrak{t})$ is a K-equivariant diffeomorphism, and that the action of σ on X corresponds to the map $v \to -v$ (fibrewise) on E. Hence $X^{\sigma} \cong$ zero-section of $E \cong K/L$.

As a consequence of 5.9, we have

5.12. COROLLARY. *Let* (V,V^-) *be simple. Then two tripotents of* V *are conjugate under* K *if and only if they have the same rank.*

Thus if $r = \mathrm{rank}(V)$ then M has $r+1$ connected components: $M = M_0 \cup M_1 \cup \ldots \cup M_r$, where M_i is the set of tripotents of rank i.

§6. The boundary of $\mathcal{D}$

6.1. We keep the notation introduced in the previous section, and denote by $\mathcal{D}$ the circled bounded symmetric domain associated with (V,V^-) and τ. By a *holomorphic arc* we mean the image of a holomorphic map from the open unit disc of $\mathbb{C}$ into V. A *segment* is a subset of V of the form $\{u + tv \mid 0 < t < 1\}$ where $u,v \in V$ $(v = 0$ is possible). Let $\mathcal{X}$ be a subset of V. The *holomorphic arc components* of $\mathcal{X}$ are the equivalence classes of $\mathcal{X}$ under the equivalence relation

$$x \sim y \Leftrightarrow x \text{ and } y \text{ can be connected by a chain of holomorphic arcs in } \mathcal{X}.$$

Analogously, one defines *affine arc components* by replacing holomorphic arcs with segments. The holomorphic (resp. affine) *boundary components* of $\mathcal{D}$ are by definition the holomorphic (resp. affine) arc components of the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ in V.

Recall that a real hyperplane $\mathfrak{H}$ supports $\mathcal{X}$ if $\mathfrak{H}$ meets $\overline{\mathcal{X}}$ (the closure of $\mathcal{X}$), and $\mathcal{X}$ is contained in one of the half-spaces defined by $\mathfrak{H}$.

6.2. LEMMA. For a non-zero tripotent e of V let $\mathfrak{H}$ be the real hyperplane

$$\mathfrak{H} = \{x \in V \mid \mathrm{Re}\langle x,\bar{e}\rangle = \langle e,\bar{e}\rangle\}$$

and $\mathfrak{K}$ the complex hyperplane

$$\mathfrak{K} = \{x \in V \mid \langle x,\bar{e}\rangle = \langle e,\bar{e}\rangle\} .$$

Then $\mathfrak{H}$ supports $\mathcal{D}$ and

(1) $$\mathfrak{H} \cap \bar{\mathcal{D}} = \mathfrak{K} \cap \bar{\mathcal{D}} = e + (\bar{\mathcal{D}} \cap V_0(e)) .$$

Moreover,

(2) $$|\langle x,\bar{e}\rangle| < \langle e,\bar{e}\rangle$$

for all $x \in \mathcal{D}$.

Proof. Clearly $\mathfrak{K} \cap \bar{\mathcal{D}} \subset \mathfrak{H} \cap \bar{\mathcal{D}}$. Let $e + x \in \mathfrak{H} \cap \bar{\mathcal{D}}$ (hence $\mathrm{Re}\langle x,\bar{e}\rangle = 0$) and let $x = x_2 + x_1 + x_0$ be the Peirce decomposition of x with respect to e. Then $|e + x| \le 1$ (cf. 4.1), and by 3.17(b), $D(e+x,\overline{e+x}) \le 2\mathrm{Id}$. It follows that $2\langle e,\bar{e}\rangle \ge \langle D(e+x,\overline{e+x})e,\bar{e}\rangle = \langle\{e\bar{e}e\} + \{e\bar{x}e\} + \{e\bar{e}x\} + \{x\bar{x}e\},\bar{e}\rangle = 2\langle e^{(3)},\bar{e}\rangle + 2\langle e^{(3)},\bar{x}\rangle + 2\langle x,\overline{e^{(3)}}\rangle + \langle x,\overline{\{x\bar{e}e\}}\rangle$
$= 2\langle e,\bar{e}\rangle + 4\mathrm{Re}\langle x,\bar{e}\rangle + \langle x,2\bar{x}_2+\bar{x}_1\rangle = 2\langle e,\bar{e}\rangle + 2\langle x_2,\bar{x}_2\rangle + \langle x_1,\bar{x}_2\rangle$, using the orthogonality of the Peirce spaces $V_i(e)$ with respect to $\langle\,,\rangle$. Since $\langle\,,\rangle$ is positive definite, $x_2 = x_1 = 0$; i.e., $x = x_0 \in V_0(e)$. By the composition rules for the Peirce spaces (3.13), $V_2(e)$ and $V_0(e)$ annihilate each other, and hence $V_2(e) \oplus V_0(e) \cong V_2(e) \times V_0(e)$. By 3.17(c),(d), $|e + x| = \max(|e|,|x|) \le 1$

if and only if $|x| \le 1$ (since $|e| = 1$) which means $x \in \bar{\mathcal{D}} \cap V_0(e)$. Thus $\mathfrak{H} \cap \bar{\mathcal{D}} \subset e + \bar{\mathcal{D}} \cap V_0(e)$, and $e + \bar{\mathcal{D}} \cap V_0(e) \subset \mathfrak{K} \cap \bar{\mathcal{D}}$ is clear since $V_0(e)$ is perpendicular to e with respect to $\langle\,,\rangle$. This proves (1). To see that $\mathfrak{H}$ supports $\mathcal{D}$ note that $\mathfrak{H} \cap \bar{\mathcal{D}}$ is contained in the boundary of $\mathcal{D}$ (since $|e+x| = 1$ for $x \in \bar{\mathcal{D}} \cap V_0(e)$), and use convexity of $\mathcal{D}$ (4.6). Consider now the complex linear form $\alpha(x) = \langle x,\bar{e}\rangle/\langle e,\bar{e}\rangle$ on V. Then $\alpha(\mathcal{D}) \subset \mathbb{C}$ is a circled domain and $1 \notin \alpha(\mathcal{D})$ by (1). Hence $\alpha(\mathcal{D})$ is contained in the open unit disc. This proves (2).

6.3. THEOREM. (a) Holomorphic and affine boundary components of $\mathcal{D}$ coincide. They are precisely the sets

$$\mathcal{F}_e = e + \mathcal{D}_e$$

where e is a non-zero tripotent of V, and $\mathcal{D}_e = \mathcal{D} \cap V_0(e)$ in the bounded symmetric domain associated with $V_0(e)$. The map $e \to \mathcal{F}_e$ is a bijection between the set of non-zero tripotents of V and the set of boundary components of $\mathcal{D}$.

(b) An element $x \in V$ belongs to $\mathcal{F}_e$ if and only if

$$e = \lim_{n\to\infty} x^{(2n+1)} .$$

(c) The boundary components of $\mathcal{F}_e$ are precisely the $\mathcal{F}_d$ with $d > e$. In particular, a boundary component of a boundary component of $\mathcal{D}$ is itself a boundary component of $\mathcal{D}$.

Proof. We first show that $\partial\mathcal{D}$ is the disjoint union of the sets $\mathcal{F}_e$. Let $x = \lambda_1 e_1 + \ldots + \lambda_n e_n$ be the spectral decomposition of an element $x \in \partial\mathcal{D}$. Then $|x| = \lambda_n = 1$, and hence $x = e_n + y$ where $y = \lambda_1 e_1 + \ldots + \lambda_{n-1} e_{n-1} \in V_0(e_n)$ with $|y| = \lambda_{n-1} < 1$; i.e., $x \in \mathcal{F}_{e_n}$. Now assume $x = e + y = d + z \in \mathcal{F}_e \cap \mathcal{F}_d$. Then by the Peirce rules (3.13),

$$x^{(2n+1)} = e + y^{(2n+1)} = d + z^{(2n+1)} ,$$

and $|y^{(2n+1)}| = |y|^{2n+1} \to 0$ as $n \to \infty$ since $|y| < 1$. In the same way $z^{(2n+1)} \to 0$ as $n \to \infty$, and therefore $e = d$. This also shows that the map $e \to \mathcal{F}_e$ is bijective. Next we show that the boundary of $\mathcal{F}_e$ (in the affine subspace $e + V_0(e)$) is the union of all $\mathcal{F}_d$, $d > e$. Indeed, since everything we proved applies to $\mathcal{D}_e$ in place of $\mathcal{D}$, the boundary of $\mathcal{D}_e$ in $V_0(e)$ is the union of all $\mathcal{F}_c$, c a non-zero tripotent of $V_0(e)$. But $\mathcal{F}_c = c + \mathcal{D}_{e+c}$, since the Peirce zero-space of c in $V_0(e)$ is $V_0(e+c)$. Hence $\partial\mathcal{F}_e = e + \partial\mathcal{D}_e$ is the union of all $e + c + \mathcal{D}_{e+c} = \mathcal{F}_{e+c}$, and the $e + c$ with $0 \neq c \perp e$ are precisely the tripotents $d > e$. Since each $\mathcal{F}_e$ is a star-shaped circled domain with respect to e it follows that every point of $\mathcal{F}_e$ can be connected to e within $\mathcal{F}_e$ by a holomorphic arc. Thus $\mathcal{F}_e$ is contained in the holomorphic arc component of e. For the converse, we have to show: every holomorphic arc $\gamma: \Delta \to \partial\mathcal{D}$ ($\Delta \subset \mathbb{C}$ the open unit disc) which meets $\mathcal{F}_e$ is entirely contained in $\mathcal{F}_e$. If

not then $\gamma(\Delta)$ would meet the boundary of $\mathcal{F}_e$; say $\gamma(\Delta)$ meets $\mathcal{F}_d$, $d > e$. Let α be the linear form $\alpha(z) = \langle z,\bar{d}\rangle/\langle d,\bar{d}\rangle$ on V. By 6.2, $\alpha\circ\gamma: \Delta \to \mathbb{C}$ is bounded in absolute value by 1, and the maximum is attained since $\gamma(\Delta)$ meets $\mathcal{F}_d \subset \{x \in V | \alpha(z) = 1\}$. By the maximum principle, $\alpha\circ\gamma$ is constant equal to one, which means, in view of 6.2, that $\gamma(\Delta) \subset d + (\bar{\mathcal{D}} \cap V_0(d)) = \bar{\mathcal{F}}_d \subset \partial\mathcal{F}_e$, a contradiction. This completes the proof that the $\mathcal{F}_e$ are precisely the holomorphic boundary components of $\mathcal{D}$. The proof that they are also the affine boundary components is similar, replacing holomorphic arcs by segments and the complex linear form α by the real linear form $\mathrm{Re}(\alpha)$.

6.4. LEMMA. Let f be a holomorphic function on $\mathcal{D}$ which extends continuously to $\bar{\mathcal{D}}$. Then the restriction of f to every boundary component $\mathcal{F}_e$ is holomorphic.

Proof. If $e + z \in \mathcal{F}_e$ ($z \in \mathcal{D}_e$) then $(1 - \frac{1}{n})e + z \in \mathcal{D}$, for $n = 1,2,3,\ldots$. Define $f_n(e+z) = f((1 - \frac{1}{n})e + z)$. Since f is continuous on $\bar{\mathcal{D}}$ and $\bar{\mathcal{D}}$ is compact, the restriction of f to $\mathcal{F}_e$ is the uniform limit of the holomorphic functions f_n and is therefore holomorphic.

6.5. THEOREM. The Bergmann-Shilov boundary Ш of $\mathcal{D}$ coincides with each of the following sets:

(i) The set of maximal tripotents of V;

(ii) the set of extremal points of $\bar{\mathcal{D}}$;

(iii) *the set of points of maximal Euclidean distance from the origin in* $\bar{\mathcal{D}}$.

In particular, $\mathcal{S}$ *is a compact connected submanifold of* V *on which* K *acts transitively.*

Proof. Recall that $\mathcal{S}$ is the minimal closed subset of $\bar{\mathcal{D}}$ where every continuous function on $\bar{\mathcal{D}}$ which is holomorphic on $\mathcal{D}$ attains maximum absolute value. Let e be a maximal tripotent, and consider the function $f(x) = \frac{1}{2}(1 + \langle x,\bar{e}\rangle / \langle e,\bar{e}\rangle)$ on V . By 6.2, $|f(x)| \leq 1$ for all $x \in \bar{\mathcal{D}}$, and $|f(x)| = 1$ if and only if $f(x) = 1$ if and only if $x \in e + \bar{\mathcal{D}} \cap V_0(e) = \{e\}$ since $V_0(e) = 0$. Hence $\mathcal{S}$ contains all maximal tripotents. Conversely, let f be holomorphic on $\mathcal{D}$ and continuous on $\bar{\mathcal{D}}$. By the maximum principle and 6.4 and 6.3, f takes its maximum at a point of $\partial\mathcal{D}$ which is not an interior point of any boundary component, and such a point is a maximal tripotent. Hence $\mathcal{S}$ is the set of maximal tripotents and also the set of extremal points, since an extremal point is just a one-point affine boundary component. By 5.3, $\mathcal{S}$ is an orbit of K , in particular, all $e \in \mathcal{S}$ have a common Euclidean distance $\delta = \langle e,\bar{e}\rangle^{\frac{1}{2}}$ from 0 . Every $x \in \bar{\mathcal{D}}$ is contained in some maximal flat subspace and thus can be written $x = \sum \lambda_i e_i$ where $0 \leq \lambda_i \leq 1$ and $(e_1,\ldots,e_r)$ is a frame, hence $e = e_1 + \ldots + e_r$ a maximal tripotent (5.1, 5.2). It follows that $\langle x,\bar{x}\rangle = \sum \lambda_i^2 \langle e_i,\bar{e}_i\rangle \leq \sum \langle e_i,\bar{e}_i\rangle = \langle e,\bar{e}\rangle = \delta^2$.

6.6. Let M be the manifold of tripotents of V and let

$$E = \{(e,v) \in V \times V \mid e \in M \text{ and } v \in V_0(e)\} .$$

Then E is a submanifold of $V \times V$; in fact, it is a vector bundle over M with projection $\pi: E \to M$ given by $\pi(e,v) = e$. The spectral norm defines a norm on each fibre of E , and we set

$$B = \{(e,z) \in E \mid |z| < 1\} ,$$

the open unit disc bundle of E . Clearly B is an open submanifold of E .

6.7. LEMMA. *For* $(e,z) \in B$, *the* R-*linear endomorphism* $v \to \{e\bar{v}z\}$ *of* $V_1(e)$ *has all eigenvalues less than* 1 *in absolute value. In particular,* $v = \{e\bar{v}z\}$ *for* $v \in V_1(e)$ *implies* $v = 0$.

Proof. Note first that $v \to \{e\bar{v}z\}$ is self-adjoint relative to the real scalar product $(u,v) = \mathrm{Re}\langle u,\bar{v}\rangle$ on $V_1(e)$. This follows from $\langle\{e\bar{v}z\},\bar{u}\rangle = \langle e,\{\bar{v}z\bar{u}\}\rangle = \langle e,\{\bar{u}z\bar{v}\}\rangle = \langle\{e\bar{u}z\},\bar{v}\rangle$. Hence all eigenvalues are real. Furthermore, $\{e\bar{v}z\} = Q(e+z).\bar{v}$ by the Peirce rules. Suppose now that $Q(e+z)\bar{v} = \lambda v,\ v \neq 0$. Then by 3.3, $Q((e+z)^{(2n+1)}).\bar{v} = \lambda^{2n+1}v$. But $(e+z)^{(2n+1)} = e + z^{(2n+1)}$ by the Peirce rules; and $z^{(2n+1)} \to 0$ as $n \to \infty$ since $|z| < 1$. It follows that $\lim_{n\to\infty} \lambda^{2n+1}v = Q(e)\bar{v} = 0$ by the Peirce rules, which shows $|\lambda| < 1$.

6.8. PROPOSITION. The map $f: B \to \bar{\mathcal{B}}$ given by $f(e,z) = e + z$ is a bijective immersion. The restriction of f to each connected component of B is an imbedding.

Proof. In view of 6.3 it is clear that f is bijective. We compute the tangent space of E at a point (e,z). Since E is defined by the equations $e^{(3)} = e$, $\{e\bar{e}z\} = 0$, a tangent vector $(u,v) \in T_{(e,z)}(E) \subset V \times V$ satisfies $u \in T_e(M) = iA(e) \oplus V_1(e)$ (by 5.6) and $\{e\bar{u}z\} + \{u\bar{e}z\} + \{e\bar{e}v\} = 0$. This implies $\{e\bar{e}v\} = 2v_2 + v_1 = -\{e\bar{u}z\} \in V_1(e)$ since $\{u\bar{e}z\} = 0$ by the Peirce rules. Hence $v_2 = 0$ and $v_1 = -\{e\bar{u}z\}$. Comparing dimensions, we see that these conditions are also sufficient for (u,v) to be in $T_{(e,z)}(E)$. Now assume $df(e,z).(u,v) = u + v = 0$. Then $u_2 + v_2 = u_2 = 0$, $u_1 + v_1 = u_1 - \{e\bar{u}z\} = u_1 - \{e\bar{u}_1 z\} = 0$, and $u_0 + v_0 = v_0 = 0$. By Lemma 6.7, we also have $u_1 = 0$ and hence f is an immersion. To show that the restriction of f to a connected component B' of B is an imbedding it suffices to show that $f: B' \to f(B')$ is a proper map. We have $B' = \pi^{-1}(M')$ where M' is a connected component of M. Let $C \subset f(B')$ be compact. We claim that there exists a $\rho < 1$ such that $C \subset f(B'_\rho)$ where $B'_\rho = \{(e,z) \in B \mid |z| \leq \rho\}$. If this were not the case, we could find a convergent sequence $e_n + z_n \in C$ such that $\lim_{n\to\infty} |z_n| = 1$. Then $\lim_{n\to\infty}(e_n + z_n) = e + z \in \partial\mathcal{J}_e \cap f(B')$ and $e \in M'$. By 6.3, the boundary of $\mathcal{J}_e$ is the union of all $\mathcal{J}_d$, $d > e$. Since $\operatorname{rank}(d) > \operatorname{rank}(e)$ and $\operatorname{rank}(e)$ is constant on M' we have $d \notin M'$ and therefore $\mathcal{J}_d \cap f(B') = \phi$ which implies $\partial\mathcal{J} \cap f(B') = \phi$, a contradiction. Now $f^{-1}f(B'_\rho) = B'_\rho$ is clearly compact in B' and hence $f^{-1}(C)$, which is closed in B'_ρ, is compact.

6.9. Remark. By the Proposition, the boundary of $\mathcal{B}$ is the disjoint union of the finitely many imbedded submanifolds

$$X_i = \bigcup_{e \in M_i} \mathcal{J}_e$$

where M_i ranges over the connected components of the set of non-zero tripotents. If V is simple of rank r there are precisely r of them (cf. 5.12). Each X_i is fibered over M_i by its boundary components. Thus we may describe $\partial\mathcal{B}$ as a "convex curvilinear polyhedron" whose "faces" are the X_i. We shall see later that the X_i are precisely the orbits of G_o on $\partial\mathcal{B}$.

6.10. We recall some facts on cones defined by formally real Jordan algebras (cf. [B-K, Chapter XI]). Let A be a formally real Jordan algebra with unit element e, and let Y be the connected component of e of the set of invertible elements of A. Then Y is an open convex cone in A and $\bar{Y}$ is the set of squares of A. Let (x,y) be an associative scalar product on A (i.e., $(xy,z) = (x,yz)$). Then $\bar{Y}$ is self-dual in the sense that $x \in A$ belongs to $\bar{Y}$ if and only if $(x,y) \geq 0$ for all $y \in \bar{Y}$. Every $x \in A$ has a unique spectral decomposition $x = \lambda_1 c_1 + \ldots + \lambda_n c_n$ where $c_1, \ldots, c_n$ are orthogonal idempotents of A whose sum is

e . Then $x \in Y$ if and only if all $\lambda_i \geq 0$ and $x \in Y$ if and only if all $\lambda_i > 0$. For $x,y \in A$ define

$$x < y \Leftrightarrow y - x \in Y \ ; \quad x \leq y \Leftrightarrow y - x \in \overline{Y} \ .$$

These order relations are compatible with the vector space structure of A . Note, however, that $x \leq y$ and $x \neq y$ does not imply $x < y$.

6.11. LEMMA. *Let* e *be a tripotent of* V *and let* $Y(e)$ *be the cone of the Jordan algebra* $A(e) \subset V_2(e)$. *Then*

$$\mathcal{D} \cap A(e) = \{x \in A(e) \mid -e < x < e\} \ ,$$

$$\overline{\mathcal{D}} \cap A(e) = \{x \in A(e) \mid -e \leq x \leq e\} \ .$$

Proof. Let $x = \sum \lambda_i c_i$ be the spectral decomposition of x in A . Then the c_i are orthogonal tripotents of V and hence $|x| = \max|\lambda_i|$. Since the eigenvalues of $e \pm x$ are $1 \pm \lambda_i$ we have $|x| < 1$ if and only if $e \pm x > 0$ and $|x| \leq 1$ if and only if $e \pm x \geq 0$.

6.12. THEOREM. *Let* b *be a boundary point of* $\mathcal{D}$, *belonging to the boundary component* $\mathcal{T}_e$.

(a) *The normal cone of the convex body* $\overline{\mathcal{D}}$ (*i.e., the closed cone generated by all outward normal vectors of supporting hyperplanes through* b) is $\overline{Y(e)}$.

(b) *There exists a real analytic diffeomorphism* φ *of a neighborhood* N *of* b *in* V *onto a neighborhood* N' *of*

0 *in* V *such that* $\varphi(N \cap \overline{\mathcal{D}}) = N' \cap (\overline{Y(e)} \oplus T)$ (*where* $T = iA(e) \oplus V_1(e) \oplus V_0(e)$)

Proof. (a) Since $\mathcal{T}_e$ is the relative interior of the intersection of $\overline{\mathcal{D}}$ with a supporting hyperplane (6.2), the normal cones at b and at e are the same. Thus we may assume that $b = e$. Let M' be the connected component of M containing e and let $X' = \bigcup_{e \in M'} \mathcal{T}_e$. By 6.8, 6.9, X' is an imbedded submanifold of V and the tangent space of X' at e is T . Let $e + \mathfrak{H}$ be a supporting hyperplane of $\overline{\mathcal{D}}$ at e with outward normal vector n . Setting $(x,y) = \mathrm{Re}\langle x,\overline{y}\rangle$, we have therefore

(1) $$\mathfrak{H} = \{h \in V \mid (h,n) = 0\}$$

and $(e,n) > 0$. Then $\mathfrak{H} \supset T$ and hence $n \in T^{\perp} = A(e)$. Since $e + \mathfrak{H}$ supports $\overline{\mathcal{D}}$ we have $(x,n) \leq (e,n)$ for all $x \in \overline{\mathcal{D}}$. In particular, let $x \in \overline{\mathcal{D}} \cap A(e)$ and set $y = e - x$. By 6.11, $(y,n) \geq 0$ for all $y \in A(e)$ with $0 \leq y \leq 2e$. Every element of $\overline{Y(e)}$ is a positive multiple of an element y with $0 \leq y \leq 2e$ and hence $(y,n) \geq 0$ holds for all $y \in \overline{Y(e)}$. By self-duality, $n \in \overline{Y}$. Conversely, let $0 \neq n \in \overline{Y(e)}$ and define $\mathfrak{H}$ by (1). Since $\overline{Y(e)}$ is the set of squares of $A(e)$ we have $n = y^2$, $y \in A(e)$. Hence $(e,n) = (e,y^2) = (ey,y) = (y,y) > 0$. Any $x \in V$ can be written uniquely in the form $x = \lambda e + h$ where $h \in \mathfrak{H}$ and $\lambda = (x,n)/(e,n)$. We have to show that $x \in \overline{\mathcal{D}}$ implies $\lambda \leq 1$. Now $|x| \leq 1$ is equivalent with $D(x,\overline{x}) \leq 2\mathrm{Id}$ by 3.17, and hence $2(y,y) \geq (\{x\overline{x}y\},y)$

$= \lambda^2(\{e\bar{e}y\},y) + \lambda(\{e\bar{h}y\},y) + \lambda(\{h\bar{e}y\},y) + (\{h\bar{h}y\},y)$
$= 2(\lambda^2 y,y) + 2\lambda(h,\{e\bar{y}y\}) + (\{h\bar{h}y\},y) \geq 2\lambda^2(y,y)$ since
$(\{e\bar{h}y\},y) = (\{h\bar{e}y\},y) = (h,\{e\bar{y}y\}) = 2(h,y^2) = 2(h,n) = 0$
and $(\{h\bar{h}y\},y) = (D(h,\bar{h})y,y) \geq 0$ by 3.15. This completes the proof.

(b) Let $b = e + z_0$ where $z_0 \in \mathcal{B} \cap V_0(e)$. Consider the map $f: V \to V$ defined by $f(x+u+z) = \exp(D(u,\bar{e}) - D(e,\bar{u})).(e-x+z-z_0)$ where $x \in A(e)$, $u \in iA(e) + V_1(e)$, $z \in V_0(e)$. Then $\exp(D(u,\bar{e}) - D(e,\bar{u})) \in K$ and therefore $f(x+u+z) \in \bar{\mathcal{B}}$ if and only if $f(x+0+z) = (e-x) + (z+z_0) \in \bar{\mathcal{B}}$. Since $e - x \in V_2(e)$ and $z + z_0 \in V_0(e)$, this means $|e - x + z + z_0| = \max(|e - x|, |z + z_0|) \leq 1$; i.e., $e - x \in \bar{\mathcal{B}} \cap A(e)$ and $z + z_0 \in \bar{\mathcal{B}} \cap V_0(e)$. By 6.11, $e - x \in \bar{\mathcal{B}} \cap A(e)$ if and only if $0 \leq x \leq 2e$. Clearly $f(0) = b$, and a simple computation, using 6.7, shows that $df(0)$ is invertible. Now (b) follows from the implicit function theorem.

6.13. COROLLARY. *The boundary of $\mathcal{B}$ is smooth at a point b if and only if $b \in \mathcal{T}_e$ where e is a primitive tripotent. In particular, the following conditions are equivalent.*

(i) $\partial\mathcal{B}$ *is smooth,*

(ii) (V,V^-) *has rank one;*

(iii) $\mathcal{B}$ *is the open unit ball of a finite-dimensional Hilbert space.*

Indeed, $\partial\mathcal{B}$ is smooth at b if and only if the normal cone of b is just a half line.

§7. The compactification of V

7.1. Let (V,V^-) be a semisimple finite-dimensional Jordan pair over $\mathbb{C}$. In this section we show how to imbed V in a natural way into a compact manifold X (actually, a projective variety) as a dense open subset. As an example, consider the Jordan pair of $p \times q$-matrices: $V = V^- = M_{p,q}(\mathbb{C})$, with $Q_x y = x.{}^t y.x$. Then V imbeds into the Grassmann manifold $X = \mathrm{Grass}_p(\mathbb{C}^{p+q})$ of p-dimensional subspaces of $\mathbb{C}^{p+q}$ by associating with every $x \in V$ the subspace of $\mathbb{C}^{p+q}$ spanned by the row vectors of the $p \times (p+q)$ matrix $(1|x)$ (where 1 denotes the $p \times p$ unit matrix). X itself may be identified with the quotient of the set of matrices of rank p in $M_{p,p+q}(\mathbb{C})$ by the action of $GL_p(\mathbb{C})$ given by left multiplication. Unfortunately, this construction does not generalize immediately to an arbitrary Jordan pair, but the following one does. It is an exercise in linear algebra to show that, if $x,y \in M_{p,q}(\mathbb{C})$ then the $p \times (p+q)$ matrix $(1-x.{}^t y|x)$ has rank p, and that every such matrix is of the form $g.(1-x.{}^t y|x)$ with $g \in GL_p(\mathbb{C})$. Thus we have a surjective map $V \times V^- \to X$. When do (x,y) and (x',y') in $V \times V^-$ determine the same point in X? If and only if there exists an invertible $p \times p$ matrix g such that

$g.(1-x.{}^t y|x) = (1-x'.{}^t y'|x')$. A simple computation shows that this is equivalent with $g.(1-x.{}^t(y-y')) = 1$ and $x' = g.x$; in other words, $1 - x.{}^t(y-y')$ is invertible and $x' = (1-x.{}^t(y-y'))^{-1}.x$. It turns out that these conditions may be phrased in terms of the quasi-inverse in the Jordan algebra $V^{(y-y')}$ (cf. 3.6), and we therefore consider first inverses and quasi-inverses in Jordan algebras.

7.2. Let A be a Jordan algebra over $\mathbb{C}$ with unit element 1 . An element $a \in A$ is called invertible if there exists $b \in A$ such that $P(a)b = a$ and $P(a)b^2 = 1$ (where $P(x) = 2L(x)^2 - L(x^2)$ are the quadratic operators and $L(x)y = xy$ the left multiplication of A). This is the case if and only if $P(a)$ is invertible, and the inverse of a is by definition $b = a^{-1} = P(a)^{-1}a$. Note that this implies $ab = 1$ but this condition is not sufficient for a being invertible, as one sees, e.g., for A the matrix algebra with $ab = \frac{1}{2}(a.b+b.a)$, $a.b$ being the matrix product. In fact, the definition of invertibility is chosen so as to coincide with invertibility in the associative sense if A is a Jordan subalgebra of an associative algebra.

We say $x \in A$ is quasi-invertible if $1 - x$ is invertible. Writing $(1-x)^{-1} = 1 + z$, the above conditions become

(1) $$z - 2xz + P(x)z = x - x^2 ,$$

(2) $$z^2 - 2xz^2 + P(x)z^2 = x^2 .$$

These conditions make sense even if A has no unit element, and they hold if and only if $Id - 2L(x) + P(x)$ is invertible. By definition, the quasi-inverse of x is $z = (Id-2L(x)+P(x))^{-1}.(x-x^2)$.

Now let (V,V^-) be a Jordan pair, and let $(x,y) \in V \times V^-$. We say (x,y) is quasi-invertible if x is quasi-invertible in the Jordan algebra $V^{(y)}$ (cf. 3.6). Then $x^2 = Q_x y$, $2xz = \{xyz\} = D(x,y)z$, $P(x)z = Q_x Q_y z$, hence $Id - 2L(x) + P(x) = B(x,y)$ and (1) and (2) become

(3) $$B(x,y)z = x - Q_x y ,$$

(4) $$B(x,y)Q_z y = Q_x y .$$

The quasi-inverse of (x,y) is

(5) $$z = x^y = B(x,y)^{-1}(x-Q_x y) .$$

For example, if (V,V^-) is the Jordan pair of $p \times q$-matrices then (x,y) is quasi-invertible if and only if $1 - x.{}^t y$ is invertible, and the quasi-inverse is

(6) $$x^y = (1-x.{}^t y)^{-1}.x .$$

7.3. We now list the main properties of the quasi-inverse in a Jordan pair. For proofs, see [L5, §3].

(i) For all $t \in \mathbb{C}$, (tx,y) is quasi-invertible if and only if (x,ty) is quasi-invertible, and then

(1) $$(tx)^y = t.(x^{ty}) .$$

(ii) (x,y) is quasi-invertible if and only if (y,x) is quasi-invertible (in the "opposite" Jordan pair (V^-,V)), and then

(2) $$x^y = x + Q_x.y^x .$$

(iii) Let (x,y) be quasi-invertible and $z \in V^-$. Then $(x,y+z)$ is quasi-invertible if and only if (x^y,z) is quasi-invertible, and then

(3) $$x^{y+z} = (x^y)^z .$$

(iv) Let (x,y) be quasi-invertible and $z \in V$. Then $(x+z,y)$ is quasi-invertible if and only if (z,y^x) is quasi-invertible, and then

(4) $$(x+z)^y = x^y + B(x,y)^{-1}.z^{(y^x)} .$$

(v) The B-operators satisfy the identities

(5) $$B(x,y)B(x^y,z) = B(x,y+z) ,$$

(6) $$B(z,y^x)B(x,y) = B(x+z,y) .$$

(vi) If $(f,f_-): (V,V^-) \to (W,W^-)$ is a homomorphism of Jordan pairs then (x,y) quasi-invertible implies that (fx,f_-y) is quasi-invertible, and

(7) $$f(x^y) = (fx)^{f_-y} .$$

Suppose that (V,V^-) is a semisimple Jordan pair over $\mathbb{C}$. Let τ be a positive involution and $|x|$ the spectral norm. For $y \in V^-$ define $|y|=|\tau^{-1}y|$. Also denote by $x^{(n,y)}$ the n-th power of x in the Jordan algebra $V^{(y)}$. Then we have:

(vii) If $|x||y| < 1$ then (x,y) is quasi-invertible and x^y is given by the geometric series

(8) $$x^y = \sum_{n=1}^{\infty} x^{(n,y)} .$$

Indeed one shows easily by induction that $x^{(n+2,y)} = Q_xQ_yx^{(n,y)}$ and hence, using 3.17(a), that

$$|x^{(n,y)}| \leq |x|^n|y|^{n-1} .$$

Therefore the series converges, and computing in the associative subalgebra of $V^{(y)}$ generated by x one checks that (8) holds.

7.4. Let (V,V^-) be a finite-dimensional Jordan pair over $\mathbb{C}$. Clearly the map $(x,y) \to x^y$ is a rational map from $V \times V^-$ into V. A polynomial function $\delta: V \times V^- \to \mathbb{C}$, normalized such that $\delta(0,0) = 1$, is called a <u>denominator</u> of the quasi-inverse if

(i) $\delta(x,y) \neq 0$ if and only if (x,y) is quasi-invertible,

(ii) $\delta(x,y).x^y$ is a polynomial function.

Thus $x^y = \frac{\nu(x,y)}{\delta(x,y)}$ where $\nu\colon V \times V^- \to V$ is a polynomial function, called the numerator of x^y (with respect to δ). For example, we can take $\delta(x,y) = \det B(x,y)$ and then $\nu(x,y) = B(x,y)^{\#}(x-Q_x y)$ where $\#$ denotes the adjoint matrix. There is a unique minimal denomiator obtained by cancelling all common factors of δ and ν, and this is just the generic norm $N(x,y)$ of (V,V^-) (cf. [L5, 16.9]). If (V,V^-) is simple then $\det B(x,y)$ is a power of $N(x,y)$ and $N(x,y)$ is an irreducible polynomial function ([L5, 17.3]). Note that $\delta(x,0) = \delta(0,y) = 1$ since $(x,0)$ and $(0,y)$ are quasi-invertible, for all $x \in V$, $y \in V^-$.

7.5. LEMMA. Let δ,ν be a denominator and numerator for the quasi-inverse. Then

(1) $$\delta(tx,y) = \delta(x,ty),$$

(2) $$\delta(x,y)\delta(x^y,z) = \delta(x,y+z),$$

(3) $$\nu(tx,y) = t.\nu(x,ty),$$

(4) $$\delta(x,y)\nu(x^y,z) = \nu(x,y+z),$$

for all $x \in V$, $y,z \in V^-$, $t \in \mathbb{C}$.

Proof. By property (i) of a denominator, $\delta(x,y)$ and $\det B(x,y)$ have the same irreducible factors. Since $\det B(x,y)$ obviously satisfies (1) and by 7.3.5 satisfies (2), it suffices to show that so does each of its irreducible

factors. Thus let $f_1(x,y),\ldots,f_n(x,y)$ be the different irreducible factors of $\det B(x,y)$, normalized by $f_i(0,0) = 1$. For each fixed $t \in \mathbb{C}$, $f_1(tx,y)$ is an irreducible polynomial in (x,y), and $\prod f_i(tx,y)^{m_i} = \prod f_i(x,ty)^{m_i}$. Hence there exists an $i = i(t)$ such that $f_1(tx,y) = f_i(x,ty)$, for all (x,y). It follows that there exists one index, say j, such that $f_1(tx,y) = f_j(x,ty)$ for all (x,y) and for an infinite number of t's. Since an infinite subset of $\mathbb{C}$ is Zariski-dense, this equation holds for all t, and for $t = 1$ we see that $j = 1$. This proves (1). Next, let F be the field of rational functions on $V \times V^-$. Then $x^y \in V \otimes_{\mathbb{C}} F$, and we may consider $f_i(x,y+z)$ and $f_i(x^y,z)$ (as functions of z) as polynomial functions on $V^- \otimes_{\mathbb{C}} F$. By Gauss' Lemma, they are irreducible. Since $\prod f_i(x,y+z)^{m_i} = g.\prod f_i(x^y,z)^{m_i}$ (with $g = \prod f_i(x,y) \in F$), there exists an index j such that $f_1(x,y+z) = h.f_j(x^y,z)$ with $h = h(x,y) \in F$. For $z = 0$ we get $h(x,y) = f_1(x,y)$ since $\det B(x,0) \equiv 1$ and hence $f_j(x,0) \equiv 1$. Now for $y = 0$ it follows that $j = 1$. This proves (2). Now (3) and (4) follow immediately from (1) and (2) and 7.3.1 and 7.3.3.

7.6. Let (V,V^-) be a finite-dimensional Jordan pair over $\mathbb{C}$. Motivated by 7.1 and 7.2, we define an equivalence relation on $V \times V^-$ by $(x,y) \sim (x',y') \Leftrightarrow (x,y-y')$ quasi-invertible and $x' = x^{y-y'}$. Using 7.3.3 one checks easily that this is indeed an equivalence relation. We denote the equivalence class of (x,y) by $(x:y)$ and the set of equivalence classes by X. It is easily seen that the map $x \to (x:0)$ is injective, and we shall thus identify V with a subset of X. Note that X depends functorially on (V,V^-): if $(f,f_-):(V,V^-) \to (V',V'^-)$ is a homomorphism of Jordan pairs then we have a map $X \to X'$ by $(x:y) \to (fx:f_-y)$.

7.7. PROPOSITION. <u>For every</u> $a \in V^-$ <u>let</u> $U_a = \{(x:a) | x \in V\} \subset X$. <u>Then the map</u> $\varphi_a: U_a \to V$, $(x:a) \to x$, <u>is bijective, and the</u> U_a <u>form a covering of</u> X. <u>There exists a unique structure of a smooth algebraic variety on</u> X <u>such that each</u> U_a <u>is an open affine subvariety, isomorphic with</u> V <u>under</u> φ_a. <u>In particular,</u> $V = U_0$ <u>is open and dense in</u> X. <u>Every finite subset of</u> X <u>is contained in one of the</u> U_a.

<u>Proof</u>. If $(x:a) = (x':a)$ then by definition of the equivalence relation, $x' = x^{a-a} = x^0 = x$. This shows that φ_a is a bijection. We show that $\varphi_a(U_a \cap U_b)$ is Zariski open and dense in V and that the transition functions $\varphi_{ba} = \varphi_b \circ \varphi_a^{-1}: \varphi_a(U_a \cap U_b) \to \varphi_b(U_a \cap U_b)$ are morphisms. Indeed, $\varphi_a(U_a \cap U_b) = \{x \in V | (x,a-b)$ quasi-invertible$\}$, and $\varphi_{ba}(x) = x^{a-b}$. Denoting by δ and ν a denominator and numerator for the quasi-inverse, we have $\varphi_a(U_a \cap U_b) = \{x \in V | \delta(x,a-b) \neq 0\}$ which is Zariski open and dense (since $\delta(0,a-b) = 1$) and $x^{a-b} = \nu(x,a-b)/\delta(x,a-b)$

is a morphism. Next we show that finitely many of the U_a cover X. For every $a \in V^-$ let $f_a(x) = \delta(x,-a)$, a polynomial function on V. Then the f_a are of bounded degree and hence span a finite-dimensional vector space. Let $f_{a_1},\dots,f_{a_n}$ be a basis of this vector space. We claim that the U_{a_i}, $i = 1,\dots,n$, cover X. For every $a \in V^-$ there exist $\lambda_i \in \mathbb{C}$ such that $\delta(x,-a) = \sum \lambda_i \delta(x,-a_i)$, for all $x \in V$. This implies $\delta(x,b-a) = \sum \lambda_i \delta(x,b-a_i)$ for all $x \in V$, $b \in V^-$. Indeed, since this is a polynomial relation, it suffices to check it on the open dense subset of all quasi-invertible pairs $(x,b) \in V \times V^-$. Then we have by 7.5.2, $\delta(x,b-a) = \delta(x,b)\delta(x^b,-a) = \sum \delta(x,b)\lambda_i\delta(x^b,-a_i) = \sum \lambda_i \delta(x,b-a_i)$. Now let $(x:a) \in X$. Then $1 = \delta(x,0) = \delta(x,a-a) = \sum \lambda_i \delta(x,a-a_i)$ and hence $\delta(x,a-a_j) \neq 0$ for some j. This means that $(x,a-a_j)$ is quasi-invertible and therefore $(x:a) = (x^{a-a_j}:a_j) \in U_{a_j}$. So far, we have shown that X is a prevariety, and it remains to check that X is separated. By [Mu, p. 71, Prop. 5] it suffices to show that any two points of X are contained in an open affine subset. More generally, we show that any finite subset of X is contained in some U_a. Let $(x_i:a_i) \in X$, $i = 1,\dots,n$. Then the polynomial functions $f_i(a) = \delta(x_i,a_i-a)$ on V^- are not zero since $f_i(a_i) = 1$. Hence there exists $a \in V^-$ such that $f_i(a) \neq 0$ for all i which means (x_i,a_i-a) quasi-invertible and therefore $(x_i:a_i) \in U_a$.

7.8. Let $\mathcal{O}$ be the sheaf of germs of regular functions on X and $\mathcal{O}^*$ the subsheaf of invertible functions. Also let δ be a denominator for the quasi-inverse. We define a 1-cocycle (f_{ab}) with values in $\mathcal{O}^*$ on X (relative to the covering $(U_a)_{a \in V^-)}$ by

$$f_{ab}(p) = \delta(x,a-b) ,$$

for $p = (x:a) \in U_a \cap U_b$. Indeed, if $p = (x:a) = (y:b) \in U_a \cap U_b \cap U_c$ we have $y = x^{a-b}$ and hence by 7.5,

$$f_{ab}(p)f_{bc}(p) = \delta(x,a-b)\delta(y,b-c) = \delta(x,a-b)\delta(x^{a-b},b-c)$$

$$= \delta(x,a-b+b-c) = \delta(x,a-c) = f_{ac}(p) .$$

Let $\mathcal{L}(\delta)$ be the line bundle on X defined by (f_{ab}). We remark that if $\delta(x,y) = \det B(x,y)$ then $\mathcal{L}(\delta)$ is the dual of the canonical bundle $\mathcal{K}$ of X. Indeed, the transition functions for the tangent bundle of X are given by $\theta_{ab}(p) = d\varphi_{ab}(\varphi_b(p))$ where $\varphi_{ab} = \varphi_a \circ \varphi_b^{-1}$. Now $\varphi_{ba}(x) = x^{a-b}$ (cf. the proof of 7.7) and we have to compute the derivative of x^y with respect to x. By 7.3.4, we have

$$(x+\varepsilon z)^y = x^y + B(x,y)^{-1}(\varepsilon z)^{(y^x)} .$$

Now $z^{\varepsilon v} \equiv z(\mathrm{mod}\ \varepsilon)$ since $z^0 = z$, and by 7.3.1, $(\varepsilon z)^v = \varepsilon.(z^{\varepsilon v}) \equiv \varepsilon z(\mathrm{mod}\ \varepsilon^2)$. Hence

$$(x+\varepsilon z)^y \equiv x^y + \varepsilon B(x,y)^{-1}.z(\mathrm{mod}\ \varepsilon^2) ,$$

and the derivative of x^y with respect to x is $B(x,y)^{-1}$. It follows that $d\varphi_{ba}(x) = B(x,a-b)^{-1}$ and therefore

$d\varphi_{ab}(\varphi_b(p)) = [d\varphi_{ba}(\varphi_a(p))]^{-1} = B(x,a-b)$, where $p = (x:a)$. Hence the transition functions for the canonical bundle are $\det \theta_{ab}(p)^{-1} = \det B(x,a-b)^{-1}$. If (V,V^-) is simple then by 7.4, $\det B(x,y) = N(x,y)^g$ is a power of the generic norm, and hence $\mathcal{L}(N)^g = \mathcal{K}^{-1}$. It can be shown that in this case $\mathcal{L}(N)$ generates the Picard group of X (see [L7]).

7.9. Recall the connection between line bundles and maps from X into projective spaces: if $\mathcal{L}$ is a line bundle on X and $s^{(0)},\dots,s^{(n)}$ are sections of $\mathcal{L}$ without common zeroes then we have a morphism $X \to P^n(\mathbb{C})$ given locally by $p \to [s_a^{(0)}(p),\dots,s_a^{(n)}(p)]$ where the brackets indicate the point in $P^n(\mathbb{C})$ defined by a vector in $\mathbb{C}^{n+1}-\{0\}$. Here $s_a^{(i)}$ denotes the function on U_a defined by $s^{(i)}$, assuming that the covering (U_a) trivializes $\mathcal{L}$. The bundle $\mathcal{L}$ is called very ample if there exist sections of $\mathcal{L}$ which define an imbedding of X into some $P^n(\mathbb{C})$. It will be more convenient for us to describe this situation as follows: Let E be a finite dimensional vector space, and consider the vector bundle $E \otimes \mathcal{L}$ on X (which is isomorphic with the sum of $\dim E$ copies of $\mathcal{L}$). Then a nowhere vanishing section s of $E \otimes \mathcal{L}$ defines a morphism $\sigma: X \to P(E)$ (the projective space of E) by $p \to [s_a(p)]$ locally.

7.10. THEOREM. The line bundles $\mathcal{L}(\delta)$ are very ample. If (V,V^-) is semisimple then the imbedding defined by

$\mathcal{L}(\delta)$ is closed and hence X is a projective variety.

Proof. Let ν be the numerator of the quasi-inverse belonging to δ. From (1) and (3) of Lemma 7.5 it follows that the expansions of δ and ν into bihomogeneous components are of the form

(1) $$\delta(x,y) = \sum_{i=0}^{m} (-1)^i \delta_i(x,y) ,$$

(2) $$\nu(x,y) = \sum_{i=0}^{n} (-1)^i \nu_i(x,y) ,$$

where $\delta_i(x,y)$ is homogeneous of bidegree (i,i) and $\nu_i(x,y)$ is homogeneous of bidegree $(i+1,i)$ in (x,y). Moreover, $\delta_0 = 1$ and $\nu_0(x,y) = x$ since $x^0 = x$. Let E be the finite-dimensional vector space of all polynomial functions (φ,f) on V^- with values in $\mathbb{C} \times V$ (i.e., $\varphi: V^- \to \mathbb{C}$ is a scalar polynomial and $f: V^- \to V$ a vector valued polynomial) such that φ is of degree $\leq m$ and f of degree $\leq n$. Define a section s of $E \otimes \mathcal{L}(\delta)$ locally on U_a by

$$s_a(x:a)(y) = (\delta(x,a-y),\ \nu(x,a-y)) ,$$

for all $y \in V^-$. We check that $s_a = f_{ab}s_b$ and hence the s_a define indeed a section: by 7.5, we have for $p = (x:a) = (x^{a-b}:b) \in U_a \cap U_b$ and all $y \in V^-$ that

$$\begin{aligned} f_{ab}(p)s_b(p)(y) &= \delta(x,a-b)(\delta(x^{a-b},b-y),\ \nu(x^{a-b},b-y)) \\ &= (\delta(x,a-b+b-y),\ \nu(x,a-b+b-y)) \\ &= s_a(p)(y) . \end{aligned}$$

The section s vanishes nowhere since $s_a(x:a)(a) = (\delta(x,0),\ \nu(x,0)) = (1,x)$. Hence s defines a morphism $\sigma: X \to P(E)$.

Every $a \in V^-$ defines a linear form on E by $(\varphi,f) \to \varphi(a)$. Thus

$$P(E)_a = \{[\varphi,f] \in P(E) \mid \varphi(a) \neq 0\}$$

is the complement of a hyperplane of $P(E)$, and we have a morphism $\eta_a: P(E)_a \to U_a$ by sending $[\varphi,f]$ into $\left(\frac{f(a)}{\varphi(a)} : a\right)$. Clearly σ maps U_a into $P(E)_a$, and $\eta_a \circ \sigma$ is the identity on U_a. Since by 7.7 any two points of X are contained in some U_a it follows that σ is injective.

For $(\varphi,f) \in E$ consider the conditions

(3) $$\delta_i(f(z),y) = \varphi(z)^{i-1}\varphi(z+y)_i ,\quad i = 1,\dots,m$$

(4) $$\nu_i(f(z),y) = \varphi(z)^i f(z+y)_i ,\quad i = 1,\dots,n ,$$

for all $y,z \in V^-$.

Here $\varphi(z+y)_i$ and $f(z+y)_i$ denotes the component homogeneous of degree i in y of the polynomial functions $\varphi(z+y)$ and $f(z+y)$ on $V^- \times V^-$. Then (3) and (4) is an (infinite) set of homogeneous polynomial equations for (φ,f) (indexed by i and $(y,z) \in V^- \times V^-$), and hence defines a closed subvariety Z of $P(E)$. Also let

$$W = \bigcup_{a \in V^-} P(E)_a ,$$

an open subvariety of $P(E)$. We claim that σ maps X isomorphically onto $Z \cap W$.

By what we proved above, σ maps X into W . To show that $\sigma(X) \subset Z$ let $[\varphi,f] \in \sigma(X)$; say, $[\varphi,f] = \sigma(x:a)$ which means (after multiplication by a non-zero scalar) $\varphi(y) = \delta(x,a-y)$ and $f(y) = \nu(x,a-y)$ for all $y \in V^-$. Then for all $z \in V^-$ for which $(x,a-z)$ is quasi-invertible (these z form a dense open subset), we have by 7.5.

$$\varphi(z+y) = \delta(x,a-z-y) = \delta(x,a-z)\delta(x^{a-z},y)$$

$$= \sum \varphi(z)(-1)^i\delta_i\left(\frac{f(z)}{\varphi(z)}, -y\right) = \sum \varphi(z)^{1-i}\delta_i(f(z),y) .$$

Comparing homogeneous components of degree i in y on both sides and multiplying by $\varphi(z)^{n-1}$ we get (3) for all z in a dense open subset and hence everywhere. Similarly,

$$f(z+y) = \nu(x,a-z-y) = \delta(x,a-z)\nu(x^{a-z},y)$$

$$= \sum \varphi(z)(-1)^i\nu_i\left(\frac{f(z)}{\varphi(z)}, -y\right) = \sum \varphi(z)^{-i}\nu_i(f(z),y)$$

implies (4).

Next we show that $\sigma\circ\eta_a$ is the identity on $Z \cap P(E)_a$. Let $[\varphi,f] \in Z \cap P(E)_a$. We may assume that $\varphi(a) = 1$. Then $\eta_a([\varphi,f]) = (f(a):a) \in U_a$. From (3) and (4) we get for $z = a$ by summing up: $\sum (-1)^i\delta_i(f(a),-y) = \sum_i \varphi(a+y)_i = \varphi(a+y) = \delta(f(a),-y)$ and $\sum (-1)^i\nu_i(f(a),-y) = \sum f(a+y)_i = f(a+y) = \nu(f(a),-y)$. Replacing y by $y-a$ we have

$$\varphi(y) = \delta(f(a),a-y) , \quad f(y) = \nu(f(a),a-y)$$

which says that $[\varphi,f] = \sigma(f(a):a)$. Thus σ induces isomorphisms between U_a and $Z \cap P(E)_a$. Since σ is injective it is now clear that $X \cong Z \cap W$ under σ .

Finally, we show that $Z \subset W$ if (V,V^-) is semisimple. Suppose to the contrary that $[\varphi,f] \in Z$ but $\varphi = 0$. Then by (3), $\delta_i(f(z),y) = 0$ for all y,z and $i=1,\dots,m$ which implies $\delta(f(z),y) = 1$. Hence $(f(z),y)$ is quasi-invertible for all $y \in V^-$; i.e., $f(z)$ is strictly quasi-invertible and hence belongs to Rad V (cf. [L5, §4]). By semisimplicity, $f(z) = 0$ for all z , which means $f = 0$, a contradiction.

7.11. As an example, consider the Jordan pair of $p \times q$-matrices. Here the generic norm is given by $N(x,y) = \det(1-x.{}^ty)$, and $\det B(x,y) = N(x,y)^{p+q}$. The projective imbedding of $X = \mathrm{Grass}_p(\mathbb{C}^{p+q})$ given by the line bundle $\mathcal{L}(N)$ is precisely the Plücker imbedding. The proof is left as an exercise.

§8. The automorphism group of X

8.1. In this section, (V,V^-) is a semisimple complex Jordan pair, and X is the projective algebraic variety constructed in the previous section. We denote by $\mathrm{Aut}(X)$ the group of automorphisms of X (as an algebraic variety). Since V is open and dense in X, every automorphism of X induces a birational transformation of V, and thus we may consider $\mathrm{Aut}(X)$ as a subgroup of the group of birational transformations of V.

8.2. PROPOSITION. *There exists a unique structure of an affine algebraic group on* $\mathrm{Aut}(X)$ *with the following property: if* A *is an algebraic group and* $A \times X \to X$ *a morphic action of* A *on* X *then the natural map* $A \to \mathrm{Aut}(X)$ *is a homomorphism of algebraic groups.*

Proof. Let $\delta(x,y) = \det B(x,y)$ so that $\mathcal{L}(\delta) = \mathcal{K}^{-1}$ is the inverse of the canonical bundle (cf. 7.8). Then $\mathrm{Aut}(X)$ acts naturally on $\mathcal{K}^{-1}$ and hence on the space of sections $\Gamma(X,\mathcal{K}^{-1})$. Since X is projective, this space is finite-dimensional, and picking a basis $s_0,\ldots,s_n$ of $\Gamma(X,\mathcal{K}^{-1})$, we obtain by 7.10 an imbedding $\sigma\colon X \to P^n(\mathbb{C})$, and a monomorphism $\varphi\colon \mathrm{Aut}(X) \to PGL_{n+1}(\mathbb{C}) = \mathrm{Aut}(P^n(\mathbb{C}))$ which are compatible. The image of φ is the normalizer of $\sigma(X)$ in $PGL_{n+1}(\mathbb{C})$. Since $\sigma(X)$ is a closed subvariety of $P^n(\mathbb{C})$, this is a Zariski closed subgroup of the affine algebraic group $PGL_{n+1}(\mathbb{C})$ and hence itself an affine algebraic group. By transport of structure, so is $\mathrm{Aut}(X)$. An action of A on X induces an action of A on $\Gamma(X,\mathcal{K}^{-1})$ and hence a homomorphism (of algebraic groups) $A \to \varphi(\mathrm{Aut}\ X)$. This completes the proof.

8.3. Since X is a smooth projective variety, we may consider it also as a compact complex manifold X_{an} . By Chow's Lemma, every holomorphic automorphism of X is algebraic, and hence $\mathrm{Aut}(X) = \mathrm{Aut}(X_{an})$ (as abstract groups). It is known that $\mathrm{Aut}(X_{an})$ is a complex Lie transformation group of X_{an} in the compact-open topology. On the other hand, the algebraic group $\mathrm{Aut}(X)$ may be regarded as a complex Lie group $\mathrm{Aut}(X)_{an}$ and from the definition of the algebraic group structure on $\mathrm{Aut}(X)$ it follows that $\mathrm{Aut}(X_{an}) = \mathrm{Aut}(X)_{an}$. The Lie algebra of $\mathrm{Aut}(X_{an})$ and therefore of $\mathrm{Aut}(X)$ is the set of all holomorphic vector fields on X (which, by GAGA , is the same as the set of algebraic vector fields).

We denote by G the connected of the identity of $\mathrm{Aut}(X)$ in the Zariski topology, this is also the connected component in the compact-open topology. The Lie algebra of G is $\mathfrak{g}$. By restriction to V , we may consider $\mathfrak{g}$ as a Lie algebra of vector fields on V .

8.4. PROPOSITION. (a) *There is a monomorphism* $\tilde{t}: V^- \to G$ *of the additive group of* V^- *into* G *given by*

$$\tilde{t}_v(x:a) = (x:a+v) , \quad (v \in V^-, (x:a) \in X) .$$

If we identify the Lie algebra of V^- *with* V^- *then the vector field* $\tilde{v} = \mathrm{Lie}(\tilde{t})(v)$ *corresponding to* $v \in V^-$ *is*

$$\tilde{v}(x) = Q_x v \quad (x \in V) .$$

(b) *There is a monomorphism* $\mathrm{Aut}(V,V^-) \to \mathrm{Aut}(X)$ *given by* $(h,h_-)\cdot(x:a) = (hx:h_-a)$, $((h,h_-) \in \mathrm{Aut}(V,V^-))$, $(x:a) \in X)$.

(c) *There is a monomorphism* $t: V \to G$ *of the additive group of* V *into* G *extending the action of* V *on itself by translations*:

$$t_u(x) = u + x ,$$

for $u \in V$, $x = (x:0) \in V \subset X$.

Proof. (a) One checks immediately that $\tilde{t}_v$ is well defined, and from 7.7 it is clear that the map $V^- \times X \to X$, $(v,p) \to \tilde{t}_v(p)$, is a morphism. By 8.2, we have a homomorphism $\tilde{t}: V^- \to \mathrm{Aut}(X)$ of algebraic groups, and $\tilde{t}(V^-) \subset G$ since V^- is connected. Assume $\tilde{t}_v = \mathrm{Id}$. Then for all $x \in V$ we have $(x:0) = (x:v)$, i.e., $x = x^v$. In particular, v is strictly quasi-invertible, and hence $v \in \mathrm{Rad}\ V^- = 0$. The birational map of V induced by $\tilde{t}_v$ is given by $\tilde{t}_v(x) = x^v$, for (x,v) quasi-invertible, since $\tilde{t}_v(x) = (x:v) = (x^v:0)$. The vector field $\tilde{v}$ on X induced by the one-parameter group

$\tilde{t}_{\epsilon v}$ is given by $\tilde{t}_{\epsilon v}(x) \equiv x + \epsilon\tilde{v}(x) \pmod{\epsilon^2}$. By 7.3.2 and the computation in 7.8, $\tilde{t}_{\epsilon v}(x) = x^{\epsilon v} = x + Q_x(\epsilon v)^x$ $\equiv x + \epsilon Q_x v \pmod{\epsilon^2}$.

(b) By 7.6, we have a homomorphism $\mathrm{Aut}(V,V^-) \to \mathrm{Aut}(X)$ which is clearly a homomorphism of algebraic groups. Since (V,V^-) admits a non-degenerate $\mathrm{Aut}(V,V^-)$-invariant bilinear form $\langle\, ,\rangle : V \times V^- \to \mathbb{C}$ (for instance, $\langle x,y\rangle = \mathrm{trace}\, D(x,y)$) , we have $h_- = {}^t h^{-1}$ (cf. 3.5), and therefore the homomorphism is injective.

(c) Let U^+ be the set of all $g \in G$ such that the derivative of g (considered as a birational transformation of V) is the identity. This is clearly an algebraic subgroup of G , and $g \in U^+$ if and only if g restricted to V is a translation t_u , for some $u \in V$. Thus U^+ may be identified with an algebraic subgroup of V . The Lie algebra $\mathfrak{u}^+$ of U^+ consists of all vector fields in $\mathfrak{g}$ which, when restricted to V , are constant. Hence to show that $U^+ \cong V$ it suffices to show that $\mathfrak{u}^+ \cong V$; i.e., every constant vector field on V extends to X . For $u \in V$ define a vector field ξ_a on U_a (cf. 7.7) by $\xi_a(x{:}a) = B(x,a)u$ (since $U_a \cong V$ we regard a vector field on U_a as a map $U_a \to V$) . Then the ξ_a define a global vector field ξ on X since for $p = (x{:}a) = (y{:}b) \in U_a \cap U_b$ we have $y = x^{a-b}$ and therefore $\theta_{ab}(p)\xi_b(p)$ $= B(x,a-b)B(x^{a-b},b)\cdot u = B(x,a-b+b)\cdot u = B(x,a)u = \xi_a(p)$, by 7.3.5 and 7.8. Now $\xi_0(x{:}0) = B(x,0)\cdot u = u$ is the constant vector field u , and hence ξ extends u . This completes the proof.

8.5. COROLLARY. G <u>acts transitively on</u> X .

Indeed, $(x{:}a) = \tilde{t}_a(x{:}0) = \tilde{t}_a t_x(0{:}0)$.

8.6. We denote by U^- the subgroup $\{\tilde{t}_v \mid v \in V^-\}$ of G , and identify $\mathrm{Aut}(V,V^-)$ with a subgroup of $\mathrm{Aut}(X)$. Let $H = \mathrm{Aut}(V,V^-)^0 \subset G$, and $U^+ = \{t_u \mid u \in V\} \subset G$. The Lie algebras of U^-,H,U^+ are denoted by $\mathfrak{u}^-,\mathfrak{h},\mathfrak{u}^+$. Thus

$$\mathfrak{u}^- = \{\tilde{v} \mid v \in V^-\}$$

consists of vector fields which, when restricted to V , are homogeneous polynomial functions of degree 2,

$$\mathfrak{h} = \{\Delta \mid (\Delta,\Delta_-) \in \mathrm{Der}(V,V^-)\}$$

is isomorphic with the derivation algebra of (V,V^-) (cf. 3.1) and consists of linear vector fields, and $\mathfrak{u}^+ \cong V$ consists of all constant vector fields. The subgroups U^+,U^- are unipotent, being isomorphic (as algebraic groups) with the vector groups V,V^- . The exponential map $\exp\colon \mathfrak{u}^\pm \to U^\pm$ is given by

$$\exp(u) = t_u , \quad \exp(\tilde{v}) = \tilde{t}_v$$

for $u \in V = \mathfrak{u}^+$ and $v \in V^-$. Note that H normalizes U^+ and U^- ; more precisely, we have

(1) $$h.t_u.h^{-1} = t_{hu} , \quad h.\tilde{t}_v.h^{-1} = \tilde{t}_{h_- v} ,$$

for $(h,h_-) \in \mathrm{Aut}(V,V^-)$, $u \in V$, $v \in V^-$. Indeed, for $x \in V$ we have $h.t_u.h^{-1}(x) = h(u+h^{-1}(x)) = hu + x$, and if (x,v) is quasi-invertible, $h.\tilde{t}_v.h^{-1}(x) = h((h^{-1}x)^v) = x^{h_-v}$, by 7.3.7. By a density argument, we have (1). On the Lie algebra level, (1) implies

(2) $$[\Delta,u] = \Delta u, \quad [\Delta,\tilde{v}] = \widetilde{\Delta_- v},$$

for $(\Delta,\Delta_-) \in \mathrm{Der}(V,V^-)$, $u \in V$, $v \in V^-$. Also, we have

(3) $$[u,\tilde{v}] = -D(u,v),$$

and thus $[\mathfrak{u}^+,\mathfrak{u}^-] \subset \mathfrak{h}$. Indeed, for $x \in V$, $[u,\tilde{v}](x) = du(x).\tilde{v}(x) - d\tilde{v}(x).u(x) = -d\tilde{v}(x).u = -\{xvu\} = -D(u,v).x$.

Finally, let $S \cong \mathbb{C}^*$ be the one-dimensional torus in the centre of H consisting of all transformations $(x:a) \to (sx:s^{-1}a)$, $s \in \mathbb{C}^*$, and let $\zeta \in \mathfrak{h}$ be the vector field tangent to S, given on V by $\zeta(x) = x$. Clearly, $\mathrm{Ad}\,s.\xi = s^{\pm 1}.\xi$ and $[\zeta,\xi] = \pm\xi$ for $\xi \in \mathfrak{u}^{\pm}$.

8.7. THEOREM. *G is a semisimple group, generated by* U^+ *and* U^-. *The Lie algebra of* G *is* $\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{h} \oplus \mathfrak{u}^-$.

Proof. Let R be the radical of G; i.e., the largest connected solvable normal subgroup of G. Then R has a fixed point on the complete variety X. Since R is normal in G and G is transitive on X by 8.5, R acts trivially on X and therefore $R = \{1\}$. For $\xi \in \mathfrak{g}$, let $\xi = \sum_{n\geq 0} \xi_n$ be the expansion of ξ into homogeneous polynomials. Then for all $s \in S$, $\mathrm{Ad}\,s.\xi = \sum_{n\geq 0} s^{1-n}.\xi_n \in \mathfrak{g}$, and therefore $\xi_n \in \mathfrak{g}$. Denoting by $\mathfrak{g}_n$ the set of

all $\xi \in \mathfrak{g}$ which are homogeneous of degree $n+1$, we have $\mathfrak{g} = \sum_{n\geq -1} \mathfrak{g}_n$ and $[\mathfrak{g}_n,\mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$; i.e., $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra. This implies that $\mathfrak{g}_m$ is orthogonal to $\mathfrak{g}_n$ with respect to the Killing form of $\mathfrak{g}$ except when $m + n = 0$. Since $\mathfrak{g}$ is semisimple, the Killing form is non-degenerate, and thus $\mathfrak{g}_n$ is isomorphic with the dual of $\mathfrak{g}_{-n}$. Hence $\mathfrak{g}_n = 0$ for $n > 1$. Clearly $\mathfrak{u}^+ = \mathfrak{g}_{-1}$, $\mathfrak{h} \subset \mathfrak{g}_0$, and $\mathfrak{u}^- \subset \mathfrak{g}_1$. Since $\dim \mathfrak{u}^- = \dim V^- = \dim V = \dim \mathfrak{u}^+ = \dim \mathfrak{g}_1$, we have $\mathfrak{u}^- = \mathfrak{g}_1$. From the composition rules for the $\mathfrak{g}_n$ it follows that $\mathfrak{g}' = \mathfrak{u}^+ \oplus [\mathfrak{u}^+,\mathfrak{u}^-] \oplus \mathfrak{u}^-$ is an ideal of $\mathfrak{g}$. By semisimplicity, $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ (direct sum of ideals) where $\mathfrak{g}''$ is the orthogonal complement of $\mathfrak{g}'$ with respect to the Killing form. Since the orthogonal complement of $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ is $\mathfrak{g}_0$ it follows that $\mathfrak{g}'' \subset \mathfrak{g}_0$. Since the vector fields in $\mathfrak{g}_0$ vanish at 0, the normal subgroup of G generated by $\mathfrak{g}''$ fixes the origin and is therefore trivial (by the same argument as used for R). Hence $\mathfrak{g} = \mathfrak{u}^+ \oplus [\mathfrak{u}^+,\mathfrak{u}^-] \oplus \mathfrak{u}^-$, and $\mathfrak{h} = [\mathfrak{u}^+,\mathfrak{u}^-]$. It follows that G is generated by U^+ and U^-.

8.8. COROLLARY. (a) H *is the centralizer of* S *in* G; *it is a reductive group*.

(b) *The centre of* G *is trivial*.

Proof. By 8.6 and 8.7, the centralizer of S in $\mathfrak{g}$ is $\mathfrak{h}$. Since the centralizer of a torus in a connected reductive group is connected and reductive, we have (a). Let g belong to the centre of G. Then $g \in H$, and by 8.6,

$gt_ug^{-1} = t_{gu} = t_u$ implies $gu = u$ for all $u \in V$. Since we may identify H with a subgroup of $GL(V)$, $g = \mathrm{Id}$.

8.9. COROLLARY. *Every derivation of the Jordan pair* (V,V^-) *is inner. The group* $\mathrm{Aut}(V,V^-)$ *is reductive and its identity component is* $\mathrm{Inn}(V,V^-)$.

Proof. By 8.6.3, the relation $[\mathfrak{u}^+,\mathfrak{u}^-] = \mathfrak{h}$ implies that every derivation of (V,V^-) is inner (cf. 3.1). Now the assertion follows from 8.8.

8.10. PROPOSITION. (a) *The isotropy group of* 0 *in* G *is* $H\cdot U^-$, *isomorphic with the semidirect product* $H \ltimes U^-$.

(b) *Let* $\Omega = \{g \in G \mid g(0) \in V\}$. *Then* $\Omega = U^+\cdot H\cdot U^-$, *isomorphic with* $U^+ \times H \times U^-$ *under multiplication*.

(c) $G = U^-\cdot\Omega = \Omega\cdot U^+$.

Proof. (a) The isotropy group P of 0 in G is parabolic and therefore connected, since $G/P \cong X$ is a complete variety. The Lie algebra of P is the set of all vector fields in $\mathfrak{g}$ vanishing at 0. By 8.6, this is $\mathfrak{h} \oplus \mathfrak{u}^-$. Hence $P = H\cdot U^-$. We show that $H \cap U^- = \{1\}$. By the computation in 7.8, the derivative of $\tilde{t}_v$ at $x \in V$ is $d\tilde{t}_v(x) = B(x,v)^{-1}$. In particular, $d\tilde{t}_v(0) = B(0,v)^{-1} = \mathrm{Id}$. Thus if $h = \tilde{t}_v \in H \cap U^-$ then, since h induces a linear map of V, $h = dh(0) = \mathrm{Id}$.

(b) Ω is the inverse image of $V \subset X$ under the orbit map

$g \to g(0)$ and therefore Zariski-open. Since U^+ acts simply transitively on V, we have $\Omega = U^+\cdot P \cong U^+ \times P \cong U^+ \times H \times U^-$.

(c) Since G is connected, Ω and $\Omega^{-1} = U^-\cdot H\cdot U^+$ are open and dense in G. Hence $G = \Omega^{-1}\cdot\Omega = U^-HU^+U^+HU^- = U^-U^+HU^- = U^-\Omega$, using that H normalizes U^+. Similarly, $G = \Omega\cdot\Omega^{-1} = \Omega\cdot U^+$.

Next, we give a description of G by generators and relations.

8.11. THEOREM. (a) *For* $(u,v) \in V \times V^-$, *we have* $\tilde{t}_v\cdot t_u \in \Omega$ *if and only if* (u,v) *is quasi-invertible, and then*

$$(1) \qquad \tilde{t}_v\cdot t_u = t_{u^v}\cdot B(u,v)^{-1}\cdot\tilde{t}_{v^u}.$$

(b) *Let* Γ *be a group, and let* $f_0\colon H \to \Gamma$, $f_\pm\colon U^\pm \to \Gamma$ *be homomorphisms. Then* f_0, f_+, f_- *extend to a (unique) homomorphism* $f\colon G \to \Gamma$ *if and only if*

$$(2) \qquad f_\pm(h\cdot b\cdot h^{-1}) = f_0(h)\cdot f_\pm(b)\cdot f_0(h)^{-1},$$

$$(3) \qquad f_-(\tilde{t}_v)\cdot f_+(t_u) = f_+(t_{u^v})\cdot f_0(B(u,v)^{-1})\cdot f_-(\tilde{t}_{v^u}),$$

for all $h \in H$, $b \in U^\pm$, $(u,v) \in V \times V^-$ *quasi-invertible*.

Proof. (a) We have $\tilde{t}_v\cdot t_u(0) = \tilde{t}_v(u:0) = (u:v) \in V$ if and only if $(u:v) = (x:0)$, which means (u,v) quasi-invertible (and $x = u^v$), by 7.6. Now (1) needs to be checked only for the dense open subset of all $z \in V$ for which $(u+z,v)$ is quasi-invertible, and then by 7.3.4, we have

$$\tilde{t}_v \cdot t_u(z) = (u+z)^v = u^v + B(u,v)^{-1} z^{(v^u)} = t_{u^v} \cdot B(u,v)^{-1} \cdot \tilde{t}_{v^u}(z) \ .$$

(b) Clearly (2) and (3) are necessary for f to exist. Conversely, let $g \in G$, and write

(4) $$g = \tilde{t}_v \cdot t_u \cdot h \cdot \tilde{t}_w$$

which is possible by 8.10. If f exists then $f(g) = f_-(\tilde{t}_v) f_+(t_u) f_0(h) f_-(\tilde{t}_w)$. We check that this is well defined. If

$$\tilde{t}_v \cdot t_u \cdot h \cdot \tilde{t}_w = \tilde{t}_{v'} \cdot t_{u'} \cdot h' \cdot \tilde{t}_{w'} \ ,$$

then by 8.6.1,

$$\tilde{t}_{v-v'} \cdot t_u = t_{u'} \cdot h' h^{-1} \cdot \tilde{t}_{h_- \cdot (w'-w)} \ ,$$

and by (1) we have $u' = u^{v-v'}$, $h'h^{-1} = B(u, v-v')^{-1}$, $(v-v')^u = h_- \cdot (w'-w)$. Hence by (2) and (3),

$$f_-(\tilde{t}_{v-v'}) f_+(t_u) = f_+(t_{u'}) \cdot f_0(h'h^{-1}) \cdot f_-(\tilde{t}_{h_- \cdot (w'-w)})$$

$$= f_+(t_{u'}) \cdot f_0(h') \cdot f_-(\tilde{t}_{w'-w}) \cdot f_0(h)^{-1}$$

which implies $f_-(\tilde{t}_v) \cdot f_+(t_u) \cdot f_0(h) \cdot f_-(\tilde{t}_w) = f_-(\tilde{t}_{v'}) \cdot f_+(t_{u'}) \cdot f_0(h') \cdot f_-(\tilde{t}_{w'})$. Now we show that $f \colon G \to \Gamma$ is a homomorphism. If $b \in U^-$ and $g \in G$ then

(5) $$f(bg) = f(b)f(g), \quad f(gb) = f(g)f(b) \ .$$

Indeed, writing g as in (4) this is immediate since $f|U^- = f_-$ is a homomorphism. If $h \in H$ then

(6) $$f(hg) = f(h)f(g), \quad f(gh) = f(g)f(h) \ .$$

This follows similarly, using (2). Finally, we show that

(7) $$f(abc) = f(a)f(b)f(c)$$

for $a,c \in U^+$, $b \in U^-$. Write $a = t_u$, $b = \tilde{t}_v$, $c = t_w$, and pick an element $y \in V^-$ such that $(-u,y)$ and $(w,y+v)$ are quasi-invertible. This is possible since the Zariski-open subsets $\{y \in V^- | \det B(-u,y) \neq 0\}$ and $\{y \in V^- | \det B(w,y+v) \neq 0\}$ of V^- not empty and therefore have non-empty intersection. Setting $d = \tilde{t}_y$ it follows by (1) that $da^{-1} \in \Omega$ and $dbc \in \Omega$. Let $ad^{-1} = zhx$, $dbc = x'h'z'$ where $x,x' \in U^+$, $h,h' \in H$, $z,z' \in U^-$. Then we have by (5) and (6),

$$f(abc) = f(ad^{-1}.dbc) = f(zhx.x'h'z')$$

$$= f(z)f(h)f(x)f(x')f(h')f(z') = f(zhx)f(x'h'z')$$

$$= f(ad^{-1})f(dbc) = f(a)f(d)^{-1}f(d)f(bc)$$

$$= f(a)f(b)f(c) \ .$$

Now it follows easily from (5) - (7) that f is a homomorphism.

8.12. Example. Let (V,V^-) be the Jordan pair of $p \times q$ matrices, thus $X = \mathrm{Grass}_p(\mathbb{C}^n)$ where $n = p + q$. Then $G = PGL_n(\mathbb{C})$ and the natural action of $GL_n(\mathbb{C})$ on X induces a surjective homomorphism $\kappa \colon GL_n(\mathbb{C}) \to G$ whose kernel consists of all multiples of the identity. If $x \in V = M_{p,q}(\mathbb{C})$ and

(1) $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathbb{C}) ,$$

divided into 4 blocks of sizes $p \times p$, $p \times q$, $q \times p$, $q \times q$, then the action of $\kappa(g)$ on V as a birational transformation is given by

(2) $$\kappa(g)(x) = (ax+b)(cx+d)^{-1} .$$

For $u \in V$, $v \in V^- = M_{p,q}(\mathbb{C})$ we have

(3) $$t_u = \kappa\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} , \quad \tilde{t}_v = \kappa\begin{pmatrix} 1 & 0 \\ -{}^t v & 1 \end{pmatrix} ,$$

and $\kappa^{-1}(\Omega)$ is the set of all g of the form (1) for which d is invertible.

In general, G will not consist of fractional linear transformations as in (2) but of fractional quadratic transformations in the following sense.

8.13. PROPOSITION. *For every* $g \in G$ *there exist uniquely determined polynomial functions* $\nu_g : V \to V$, $\Delta_g : V \to \operatorname{End} V$ *of degree* ≤ 2 *such that* $g(x) \in V$ (*for* $x \in V$) *if and only if* $\Delta_g(x)$ *is invertible, and then*

(1) $$g(x) = \Delta_g(x)^{-1} . \nu_g(x) ,$$

(2) $$dg(x) = \Delta_g(x)^{-1} .$$

The maps $g \to \nu_g$ *and* $g \to \Delta_g$ *are morphisms of algebraic varieties. The "denominators"* Δ_g *satisfy*

(3) $$\Delta_{g \circ h}(x) = \Delta_h(x) \circ \Delta_g(h(x)) ,$$

for all $g,h \in G$ *and all* $x \in V$ *such that* $h(x) \in V$.

Proof. Let $\xi \in \mathfrak{g}$, considered as a polynomial vector field on V . Then the adjoint representation of $g \in G$ is given by

$$(\operatorname{Ad} g^{-1} . \xi)(x) = dg(x)^{-1} . \xi(g(x)) .$$

In particular, for $\xi = u \in \mathfrak{u}^+$ (constant vector field) we have $(\operatorname{Ad} g^{-1} . u)(x) = dg(x)^{-1} . u$, and since $\operatorname{Ad} g^{-1} . u \in \mathfrak{g}$ and hence is a polynomial of degree ≤ 2 in x we see that $\Delta_g(x) = dg(x)^{-1}$ is a polynomial of degree ≤ 2 with values in $\operatorname{End} V$. Now let $\zeta \in \mathfrak{g}$ be the vector field $\zeta(x) = x$ (cf. 8.6). Then $(\operatorname{Ad} g^{-1} . \zeta)(x) = dg(x)^{-1} . g(x) = \nu_g(x)$ is a polynomial of degree ≤ 2 , and therefore we have (1) and (2). Since the adjoint representation of G on $\mathfrak{g}$ is a morphism so are the maps $g \to \nu_g$ and $g \to \Delta_g$. Finally (3) follows from the chain rule.

8.14. COROLLARY. *Let* $\chi : G \to \mathbb{C}$ *be the polynomial function* $\chi(g) = \det \Delta_g(0)$. *Then* $\Omega = \{g \in G \mid \chi(g) \neq 0\}$.

This is immediate from 8.13 and 8.10.

8.15. For elements of the subgroup U^+, H, U^- the numerators and denominators are easily determined. Indeed,

(1) $$\nu_{t_u}(x) = x + u , \quad \Delta_{t_u}(x) = \operatorname{Id} \quad (u \in V) ,$$

(2) $$\nu_h(x) = x , \quad \Delta_h(x) = h^{-1} \quad (h \in H) ,$$

(3) $\quad \nu_{\tilde{t}_v}(x) = x - Q_x v\,, \quad \Delta_{\tilde{t}_v}(x) = B(x,v) \quad (v \in V^-)\,.$

The first two formulas are obvious (remember that H may be considered as a subgroup of $GL(V)$) . For (3), use $\tilde{t}_v(x) = x^v = B(x,v)^{-1}.(x - Q_x v)$ and $d\tilde{t}_v(x) = B(x,v)^{-1}$ (cf. 7.2.5 and 7.8). From these formulas, 8.13.3, and 8.10 it follows that $\Delta_g(x)$ belongs to the submonoid of $\operatorname{End} V$ generated by all $B(u,v)$, $u \in V$, $v \in V^-$. Since $H \cong \operatorname{Inn}(V,V^-)$ is generated by all $B(u,v)$, (u,v) quasi-invertible, we have $\Delta_g(x) \in H$ whenever it is invertible.

§9. G as a real algebraic group

9.1. Let (V,V^-) be a semisimple Jordan pair over $\mathbb{C}$, X , G , $U^{\pm}$, H , etc. as in §§7,8. Recall that

$$\exp(u) = t_u \quad \text{(translation by } u)$$
$$\exp(\tilde{v}) = \tilde{t}_v \quad \text{(quasi-inverse with respect to } v)$$

for $u \in V = \mathfrak{u}^+$ (identified with the constant vector fields on V) , and that $\tilde{v} \in \mathfrak{u}^-$ is the vector field on X whose restriction to V is the quadratic vector field

$$\tilde{v}(x) = Q(x)v \;,$$

for $v \in V^-$. Also let $\tau : V \to V^-$, $u \to \bar{u}$, be a positive hermitian involution of (V,V^-) , $\mathcal{D} \subset V$ the associated bounded symmetric domain, G_o the connected component of $\operatorname{Aut}(D)$, K the isotropy group of 0 in G_o , $\mathfrak{g}_o = \mathfrak{k} \oplus \mathfrak{p}$ the Lie algebra of G_o (cf. §§2,4). We fix a positive definite K-invariant hermitian scalar product $\langle , \rangle$ on V , and denote the adjoint with respect to $\langle , \rangle$ by $*$. Then $H (\cong \operatorname{Aut}(V,V^-)^0)$ is invariant under $*$ and $K = \{h \in H \mid h^* = h^{-1}\}$ is a compact real form of the complex algebraic group H (cf. 3.2, 3.5, 4.9).

9.2. PROPOSITION. <u>There exists a unique complex conjugation (antiholomorphic automorphism of period 2)</u> σ <u>on</u> G <u>such that</u>

(1) $$\sigma(h) = h^{*-1} \quad (h \in H) ,$$

(2) $$\sigma(\exp u) = \exp(-\tilde{u}) \quad (u \in \mathfrak{u}^+ = V) ,$$

(3) $$\sigma(\exp \tilde{v}) = \exp(-\bar{v}) \quad (v \in V^-) .$$

<u>The fixed point set of</u> σ <u>on</u> $\mathfrak{g}$ <u>is</u> $\mathfrak{g}_o$.

<u>Proof</u>. Define $f_0 : H \to G$ by $h \to h^{*-1}$, $f_+ : U^+ \to G$ by $\exp(u) \to \exp(-\tilde{u})$, and $f_- : U^- \to G$ by $\exp(\tilde{v}) \to \exp(-\bar{v})$. Then one checks that (2) and (3) of 8.11 hold, and hence we have a homomorphism $\sigma : G \to G$ satisfying (1) - (3). Clearly σ is of period 2 and antiholomorphic. Note that σ leaves H invariant and interchanges U^+ and U^- . The fixed point set on $\mathfrak{h} = \mathrm{Lie}(H)$ is $\mathfrak{k} = \mathrm{Lie}(K)$ and on $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ it is $\mathfrak{p} = \{u - \tilde{u} | u \in \mathfrak{u}^+ = V\}$ since $u - \tilde{u}$ is just the vector field ξ_u (cf. 2.3, 2.5).

9.3. The automorphism σ defines a Galois action of the Galois group of $\mathbb{C}/\mathbb{R}$ on G , and hence a real algebraic group structure, denoted by $\underline{G}_o$. If A is an $\mathbb{R}$-algebra we denote by $\underline{G}_o(A)$ the group of A-valued points of $\underline{G}_o$. (According to his persuasion, the reader may think of $\underline{G}_o$ as of G with the $\mathbb{R}$-structure defined by σ , or as the functor $A \to \underline{G}_o(A)$ on the category of $\mathbb{R}$-algebras) . In particular, $\underline{G}_o(\mathbb{R})$ is just the fixed point set of σ in $G = \underline{G}_o(\mathbb{C})$. The Lie algebra of the real algebraic group $\underline{G}_o$ is $\mathfrak{g}_o$. This is also the Lie algebra of the real Lie group $\underline{G}_o(\mathbb{R})$.

9.4. PROPOSITION. <u>Every automorphism of</u> $\mathcal{D}$ <u>extends uniquely to an automorphism of</u> X . <u>Considering thus</u> G_o <u>as a subgroup of</u> G , <u>we have</u>

$$G_o = \underline{G}_o(\mathbb{R})^0 = \{g \in G | g(\mathcal{D}) = \mathcal{D}\} .$$

<u>Proof</u>. By 4.9, $\mathrm{Aut}(\mathcal{D}) = \exp(\mathfrak{p}) \cdot \mathrm{Aut}(V)$ and $G_o = \exp(\mathfrak{p}) \cdot K$. Since $\mathrm{Aut}(V) \subset \mathrm{Aut}(V,V^-) \subset \mathrm{Aut}(X)$ (cf. 8.4(b)) and $\mathfrak{p} \subset \mathfrak{g}_o \subset \mathfrak{g}$, we have the first assertion. Clearly $G_o = \underline{G}_o(\mathbb{R})^0 \subset \{g \in G | g(\mathcal{D}) = \mathcal{D}\}$. Conversely, let $g(\mathcal{D}) = \mathcal{D}$ for $g \in G$. Composing g with an element of G_o we may assume $g(0) = 0$. Then $g \in \mathrm{Aut}(V)$ by 4.9, and we have to show that $\mathrm{Aut}(V) \cap G = K$. But $\mathrm{Aut}(V) \cap G$ is contained in the centralizer of S in G which is H (8.8), and $\mathrm{Aut}(V) \cap H$ is the fixed point set of σ in H which is K .

9.5. Let s be the symmetry around the origin ($s(x) = -x$) . Then $s \in K$, and $\mathrm{Int}(s)$ is an automorphism of period 2 of G and of G_o whose fixed point set in G_o is K ; in fact, it is a Cartan involution of G_o . One checks easily that $\mathrm{Int}(s)$ commutes with σ . Hence $\theta = \sigma \circ \mathrm{Int}(s)$ is also a complex conjugation of G . The real algebraic group structure on G defined by θ is denoted by $\underline{G}_c$, its Lie algebra by $\mathfrak{g}_c$. Explicitly, θ is given by

$$\theta(h) = h^{*-1} = \sigma(h) , \quad (h \in H) ,$$
$$\theta(\exp u) = \exp(\tilde{u}) = \sigma(\exp(-u)) \quad (u \in V) .$$

The fixed point set of θ on $\mathfrak{h}$ is $\mathfrak{k}$, and on $\mathfrak{u}^+ \oplus \mathfrak{u}^-$

it is $\{u+\tilde{u} \mid u \in V\} = i\mathfrak{p}$. Thus the fixed point set of θ on $\mathfrak{g}$, i.e., the Lie algebra of $\underline{G}_c$, is $\mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{p}$.

9.6. Example. Let (V,V^-) be the Jordan pair of rectangular matrices (cf. 8.12). Working in the covering group $SL_n(\mathbb{C})$ of $G = PGL_n(\mathbb{C})$, θ is the automorphism $g \to {}^t\bar{g}^{-1}$ of $SL_n(\mathbb{C})$, and $\sigma = \mathrm{Int}(s)\circ\theta$ where

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In particular, if $n = 2$ (thus $V = \mathbb{C}$) we have the explicit formulae

(1) $$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} ,$$

(2) $$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. The fixed point set of σ in $SL_2(\mathbb{C})$ is $SU(1,1)$ and that of θ is $SU(2)$. Note, however, that the fixed point set of σ in $G = PGL_2(\mathbb{C})$, i.e., the group $\underline{G}_o(\mathbb{R})$, has two connected components.

9.7. LEMMA. Let e be a tripotent. Then there exists a unique homomorphism $f : SL_2(\mathbb{C}) \to G$ such that

(1) $$f\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = t_{\alpha e} = \exp(\alpha e) \quad (\alpha \in \mathbb{C}) ,$$

(2) $$f\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} = \tilde{t}_{\alpha\bar{e}} = \exp(\alpha\tilde{\bar{e}}) \quad (\alpha \in \mathbb{C}) ,$$

(3) $$f\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = B(e,(1-\mu)\bar{e}) \quad (\mu \in \mathbb{C}^*) .$$

The homomorphism f commutes with the automorphisms σ and θ of $SL_2(\mathbb{C})$ and G .

Proof. Define a homomorphism on the Lie algebra level by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to e \in \mathfrak{u}^+ \quad \text{(constant vector field)},$$

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \to \tilde{\bar{e}} \in \mathfrak{u}^- \quad \text{(quadratic vector field)},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to D(e,\bar{e}) \in \mathfrak{h} \quad \text{(linear vector field)}.$$

One checks that this is indeed a Lie algebra homomorphism. Since $SL_2(\mathbb{C})$ is simply connected, it induces a group homomorphism f which clearly satisfies (1) and (2). To prove (3), note that, by the rules for the Peirce decomposition (3.13), $D(e,\bar{e})x = nx$ and $B(e,(1-\mu)\bar{e})x = \mu^n x$ for $x \in V_n(e)$, $n = 0,1,2$. Finally, the compatibility of f with σ and θ is easily verified. We remark that f is injective if and only if the Peirce space $V_1(e) \neq 0$, and has kernel $\pm I$ otherwise.

We can now give explicit formulae for the one-parameter groups generated by the vector fields $v - \tilde{\bar{v}} = \xi_v \in \mathfrak{p}$ and $v + \tilde{\bar{v}} \in i\mathfrak{p} (v \in V)$. Recall (3.18, 3.19) that any odd real analytic function $\varphi(t)$ gives rise to a map $x \to \varphi(x)$ $(x \in V)$. In particular, this is so for the functions tanh and tan.

9.8. PROPOSITION. <u>For</u> $v \in V$ <u>let</u> $u = \tanh(v)$ <u>and</u> $w = \tan(v)$, <u>the latter provided</u> $|v| < \pi/2$. <u>Then we have</u>

(1) $$\exp(v-\tilde{v}) = \exp(\xi_v) = t_u \circ B(u,\bar{u})^{\frac{1}{2}} \circ \tilde{t}_{-\bar{u}} ,$$

(2) $$\exp(v+\tilde{v}) = t_w \circ B(w,-\bar{w})^{\frac{1}{2}} \circ \tilde{t}_{\bar{w}} .$$

<u>Proof</u>. Let $\lambda \in \mathbb{R}$. An elementary matrix computation shows that

$$\exp\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \tanh\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\cosh\lambda)^{-1} & 0 \\ 0 & \cosh\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh\lambda & 1 \end{pmatrix} .$$

Applying the homomorphism f of 9.7, we get

$$\exp(\lambda(e-\tilde{e})) = t_{\alpha e} \circ B(e,(1-(\cosh\lambda)^{-1})\bar{e}) \circ \tilde{t}_{-\alpha\bar{e}}$$

where $\alpha = \tanh\lambda$. Now $\tanh^2\lambda = 1 - (\cosh\lambda)^{-2}$ and hence $B(\alpha e,\alpha\bar{e}) = B(e,\alpha^2\bar{e}) = B(e,(1-(\cosh\lambda)^{-1})\bar{e})^2$. Since $\tanh(\lambda e) = \tanh(\lambda)\cdot e$, this proves (1) in case $v = \lambda e$. For the general case, let $v = \lambda_1 e_1 + \cdots + \lambda_v e_v$ be the spectral decomposition. By orthogonality, the homomorphisms $f_i : SL_2(\mathbb{C}) \to G$ defined by the e_i commute and thus we are reduced to the previous case. Note that there is a well-defined positive definite square root of $B(u,\bar{u})$ since $u \in \mathcal{D}$ (cf. 4.8) and hence $B(u,\bar{u})$ is positive definite (4.4 or 3.15). The proof of (2) follows the same lines. Note that $\tan$ is a diffeomorphism of $\frac{\pi}{2}\mathcal{D}$ onto V , and $B(w,-\bar{w})$ is positive definite for all $w \in V$ by 3.15.

9.9. PROPOSITION. θ <u>is a Cartan involution of</u> G . <u>Its fixed point set</u> $G_c = \underline{G}_c(\mathbb{R}) = \exp(i\mathfrak{p})\cdot K$ <u>is a compact connected subgroup of</u> G , <u>acting transitively on</u> X . <u>The isotropy group of</u> 0 <u>in</u> G_c <u>is</u> K .

<u>Proof</u>. Let $g \in G_c$ with $g(0) = 0$. By 8.10, $g = h\cdot\tilde{t}_v$. Since θ interchanges U^+ and U^- , we have $v = 0$ and $g = h$ with $h = h^{*-1} \in K$. Hence the isotropy group of 0 in G_c is K . Next, we show that G_c acts transitively on X . Let W be a maximal flat subspace of V and Y its closure in X . Then $K\cdot W = V$ (5.3) and hence $K\cdot Y = X$. Now $W \cong \mathbb{R}^n$ and Y is isomorphic with the product of n copies of the real projective line. By (2) of 9.8, V belongs to the orbit of 0 under G_c . Thus it suffices to show that the "point at infinity" $(e:\bar{e}) \in X - V$ of the line $\mathbb{R}\cdot e$ belongs to the orbit of 0 under G_c , for any tripotent e . Now $(e:\bar{e})$ is the image of 0 under $g = \tilde{t}_{\bar{e}} \circ t_e \circ \tilde{t}_{\bar{e}}$ and $g = \theta(g)$ since

$$g = f\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where f is the homomorphism $SL_2(\mathbb{C}) \to G$ defined by e . Therefore G_c acts transitively on X . Since X and K are compact and connected so is G_c . By standard properties of compact Lie groups, $G_c = \exp(i\mathfrak{p})\cdot K$.

9.10. <u>Metric boundary components</u>. As a further application of 9.8, we show that the boundary components of $\mathcal{D}$ may be characterized in terms of the Bergman metric on $\mathcal{D}$.

For $x,y \in \mathcal{D}$ let $d(x,y)$ denote the Riemannian distance of x and y. Call two points a,b of the boundary of $\mathcal{D}$ equivalent $(a \sim b)$ if there exist sequences $x_n, y_n \in \mathcal{D}$ with $\lim x_n = a$, $\lim y_n = b$, and $d(x_n,y_n)$ bounded. It is easily verified that this is an equivalence relation. The equivalence classes are called the <u>metric boundary components</u> of $\mathcal{D}$.

9.11. THEOREM. <u>The metric boundary components agree with the holomorphic resp. affine boundary components of</u> $\mathcal{D}$ (cf. 6.1).

<u>Proof</u>. Let $\mathcal{J}_e$ be a holomorphic boundary component, e a tripotent, and let $e + x \in \mathcal{J}_e$, $x \in V_0(e)$, $|x| < 1$ (cf. 6.3). We show that $e + x \sim e$. Let $g_n = \exp(n(e-\tilde{e})) = \exp(n\xi_e) \in G_o$, and set $x_n = g_n(x) \in \mathcal{D}$, $y_n = g_n(0) = \tanh(ne) = \alpha_n e$ where $\alpha_n = \tanh(n)$. Since $\alpha_n \to 1$ as $n \to \infty$ we have $\lim y_n = e$ and

$$\begin{aligned}\lim x_n &= \lim(\alpha_n e + B(\alpha_n e, \alpha_n \bar{e})^{\frac{1}{2}} \cdot x^{-\alpha_n \bar{e}}) \\ &= e + B(e,\bar{e})^{\frac{1}{2}} \cdot x^{-\bar{e}} = e + x ,\end{aligned}$$

since $x^{-\alpha_n \bar{e}} = x$ by the Peirce rules, and $B(e,\bar{e})$ is the projection onto $V_0(e)$. Also, $d(x_n,y_n) = d(x,0)$ is bounded.

Now let b be a boundary point of $\mathcal{D}$ and $b \sim e$. We show that $b \in \mathcal{J}_e$. Let $\lim x_n = b$, $\lim y_n = e$, and $d(x_n,y_n)$ bounded. We can write $y_n = \tanh(v_n)$ with unique $v_n \in V$. Let $g_n = \exp(v_n - \tilde{v}_n) \in G_o$. Then

$y_n = g_n(0)$. Also let $x_n = g_n(z_n)$ with $z_n \in \mathcal{D}$, and write $z_n = \tanh(w_n)$, $w_n \in V$. Then $d(x_n,y_n) = d(z_n,0)$. Since the exponential map $\tanh = \mathrm{Exp} : V \to \mathcal{D}$ is a diffeomorphism (4.8), $d(z_n,0) = \|w_n\|$ (Euclidean length in the Bergman metric at 0). After passing to a subsequence, we may therefore assume that the w_n converge to $w \in V$, and hence the z_n to $z = \tanh(w) \in \mathcal{D}$. Now

$$\begin{aligned} b &= \lim x_n = \lim g_n(z_n) \\ &= \lim(y_n + B(y_n,\bar{y}_n)^{\frac{1}{2}} \cdot z_n^{-\bar{y}_n}) \\ &= e + B(e,\bar{e})^{\frac{1}{2}}(z^{-\bar{e}}) = e + x \end{aligned}$$

with $x \in V_0(e)$ since $B(e,\bar{e})$ is the projection onto $V_0(e)$. Moreover, $|b| = |e+x| = \max(|e|,|x|) = 1$ shows $|x| \leq 1$. Thus b belongs to the closure of $\mathcal{J}_e$. If b were not in $\mathcal{J}_e$ then $b \in \mathcal{J}_d$ where $d > e$ (cf. 6.3). Hence $d \sim b \sim e$ and therefore e belongs to the closure of $\mathcal{J}_d$ which implies $e \geq d$, a contradiction. This completes the proof.

9.12. Let E be a tripotent and $f : SL_2(\mathbb{C}) \to G$ the associated homomorphism (9.7). In $SL_2(\mathbb{C})$, considered as a real algebraic group via σ (cf. 9.6.1) (so its group of real points is $SU(1,1)$, <u>not</u> $SL_2(\mathbb{R})$) we have the one-dimensional $\mathbb{R}$-split torus $\underline{T}$ consisting of all matrices

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a^2 - b^2 = 1 .$$

Denote by $\underline{T}_e \subset \underline{G}_o$ the one-dimensional R-split torus which is the image of $\underline{T}$ under f. The Lie algebra of $\underline{T}$ has the canonical generator $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and hence the Lie algebra of $\underline{T}_e$ is spanned by $\mathrm{Lie}(f)\begin{pmatrix}0&1\\1&0\end{pmatrix} = e-\tilde{e} = \xi_e \in \mathfrak{p}$. Let $\mathfrak{g}_o = \sum_{n\in Z} \mathfrak{g}^n(e)$ be the decomposition of $\mathfrak{g}_o$ into the weight spaces with respect to $\underline{T}_e$; i.e., $\mathfrak{g}^n(e)$ is the n-eigenspace of $\mathrm{ad}\,\xi_e$ on $\mathfrak{g}_o$. By standard facts on semisimple algebraic groups, there is a unique parabolic subgroup $\underline{N}_e$ of $\underline{G}_o$ whose Lie algebra is $\sum_{n\geq o} \mathfrak{g}^n(e)$.

9.13. LEMMA. *Let* $V = V_2 \oplus V_1 \oplus V_0$ *be the Peirce decomposition with respect to* e, *and* $* : V_2 \to V_2$ *as in* 3.13. *Then*

$$D(a,\bar{e}) = D(e,\overline{a^*}) \quad \text{for} \quad a \in V_2 .$$

Proof. We have to show $\{a\bar{e}x\} = \{e\overline{a^*}x\}$ for all $x \in V$. For $x \in V_0$ this is true since both sides vanish by the Peirce rules. For $x \in V_1$ we have, by JP9, $\{a\bar{e}x\} = \{e\bar{e}\{a\bar{e}x\}\} = \{a, Q(\bar{e})x, e\} + D(e, Q(\bar{e})a)x = \{e\overline{a^*}x\}$ since $Q(\bar{e})x \in V_3 = 0$ and $\overline{a^*} = Q(\bar{e})a$ for $a \in V_2$. For $x \in V_2$ we have $\{a\bar{e}x\} = a\circ x$ and $\{e\overline{a^*}x\} = P(e,x)a^{**} = P(e,x)a = a\circ x$.

9.14. LEMMA. *Let* $\mathfrak{k}^e = \{\Delta \in \mathfrak{k} \mid \Delta(e) = 0\}$. *Then the spaces* $\mathfrak{g}^n(e)$ *are given by*

(1) $\mathfrak{g}^0(e) = \mathfrak{k}^e \oplus \{\xi_v \mid v \in A \oplus V_0\}$,

(2) $\mathfrak{g}^{\pm1}(e) = \{\xi_v \mp (D(v,\bar{e}) - D(e,\bar{v})) \mid v \in V_1\}$,

(3) $\mathfrak{g}^{\pm2}(e) = \{\xi_v \mp D(v,\bar{e}) \mid v \in iA\}$,

(4) $\mathfrak{g}^n(e) = 0$ for $|n| > 2$.

Here $V_2 = A \oplus iA$ *as in* 3.13.

Proof. $\mathfrak{g}_o$ consists of all $\Delta + \xi_v$, $\Delta \in \mathfrak{k}$, $v \in V$. We have $\Delta(e) = b + u$ where $b \in iA$, $u \in V_1$, since $\Delta(e)$ is tangent to the manifold of tripotents at e (cf. 5.6). Let $\Delta' = \Delta - \frac{1}{2} D(b,\bar{e}) - (D(u,\bar{e}) - D(e,\bar{u}))$. Note that $2D(b,\bar{e}) = D(b,\bar{e}) - D(e,\bar{b}) \in \mathfrak{k}$ by 9.13 and 3.2.1. Then $\Delta'(e) = 0$ so $\Delta' \in \mathfrak{k}^e$. Let $v = c + d + v_1 + v_0$ be the components of v in the Peirce spaces ($c \in A, d \in iA, v_j \in V_j$). Then

$$\begin{aligned}\Delta + \xi_v = {} & \Delta' + \xi_c + \xi_{v_0} + \xi_{w_1} - (D(w_1,\bar{e}) - D(e,\bar{w}_1)) \\ & + \xi_{w_{-1}} + (D(w_{-1},\bar{e}) - D(e,\bar{w}_{-1})) \\ & + \xi_{w_2} - D(w_2,\bar{e}) + \xi_{w_{-2}} + D(w_{-2},\bar{e})\end{aligned}$$

where $w_{\pm1} = \frac{1}{2}(v_1 \mp u) \in V_1$ and $w_{\pm2} = \frac{1}{2}d \mp \frac{1}{4}b \in iA$. Hence $\mathfrak{g}_o$ is the sum of the subspaces indicated on the right hand sides of (1) - (3). A straightforward computation, using the fact that $[\xi_u, \xi_v] = D(u,\bar{v}) - D(v,\bar{u})$, $[\Delta, \xi_u] = \xi_{\Delta u}$, and the Peirce rules, shows that these subspaces are the $0, \pm1, \pm2$-eigenspaces of $\mathrm{ad}\,\xi_e$.

9.15. THEOREM. *The normalizer of the boundary component* $\mathcal{F}_e$ *in* G_o *is* $N(\mathcal{F}_e) = \underline{N}_e(\mathbb{R}) \cap G_o$ *(hence an open subgroup of the group of real points* $\underline{N}_e(\mathbb{R})$*). Moreover,*

$$N(\mathcal{J}_e) = K^e \cdot \exp(\mathfrak{g}^0(e) \cap \mathfrak{p}) \cdot \exp(\mathfrak{g}^1(e)) \cdot \exp(\mathfrak{g}^2(e))$$

where $K^e = \{k \in K | k(e) = e\}$.

Proof. Let $N = N(\mathcal{J}_e)$ and let $\mathfrak{n}$ be its Lie algebra. (Since G_o acts on X and stabilizes $\mathcal{D}$ it stabilizes the boundary of $\mathcal{D}$ and clearly permutes the boundary components). We first show that $\mathfrak{n} = \mathrm{Lie}(\underline{N}_e) = \mathfrak{g}^0(e) \oplus \mathfrak{g}^1(e) \oplus \mathfrak{g}^2(e)$. Clearly $\mathfrak{n}$ consists of all vector fields $\xi \in \mathfrak{g}_o$ which are tangent to $\mathcal{J}_e$; i.e., satisfy $\xi(e+z) \in V_0$ for all $z \in V_0 \cap \mathcal{D}$. For $\Delta \in \mathfrak{k}^e$ we have $\Delta(e) = 0$ and Δ leaves the Peirce spaces invariant. Hence $\Delta \in \mathfrak{n}$. If $a \in A$ then $\xi_a(e+z) = a - Q(e+z)\bar{a} = a - Q(e)\bar{a} = a - a^* = a - a = 0$. If $v \in V_0$ then $\xi_v(e+z) = v - Q(e+z)\bar{v} = v - Q(z)\bar{v} \in V_0$. This shows $\mathfrak{g}^0(e) \subset \mathfrak{n}$. For $v \in V_1$ and $\varepsilon = \pm 1$ we have, by the Peirce rules,

$$\xi_v(e+z) - \varepsilon(D(v,\bar{e}) - D(e,\bar{v}))(e+z)$$
$$= v - Q(e+z)\bar{v} - \varepsilon(\{v,\bar{e},e+z\} - \{e,\bar{v},e+z\})$$
$$= (1-\varepsilon)(v - \{e\bar{v}z\}) \ .$$

Note that $0 \neq v - \{e\bar{v}z\} \in V_1$ for $v \neq 0$ by 6.7. For $b \in iA$ we have

$$\xi_b(e+z) - \varepsilon D(b,\bar{e})(e+z) = b - Q(e+z)\bar{b} - \varepsilon\{b\bar{e}e\}$$
$$= b - Q(e)\bar{b} - 2\varepsilon b = 2(1-\varepsilon)b \ ,$$

since $Q(e)\bar{b} = b^* = -b$. From these relations it follows that $\mathfrak{n} = \mathrm{Lie}(\underline{N}_e)$. Since a parabolic subgroup equals the normalizer of its Lie algebra, we have

$$\underline{N}_e(\mathbb{R}) \cap G_o = \mathrm{Norm}_{G_o}(\mathfrak{n}) \supset N \ .$$

Let $\underline{U}$ be the unipotent radical of $\underline{N}_e$ and $\underline{L}$ the centralizer of $\underline{T}_e$ in $\underline{N}_e$, a Levi subgroup. Then $\underline{N}_e(\mathbb{R}) \cap G_o = (\underline{L}(\mathbb{R}) \cap G_o) \cdot \underline{U}(\mathbb{R})$ since $\underline{U}(\mathbb{R})$ is connected. Also $\underline{U}(\mathbb{R}) = \exp(\mathfrak{g}^1(e)) \cdot \exp(\mathfrak{g}^2(e))$. The group $\underline{L}(\mathbb{R}) \cap G_o$ is stable under θ and hence has the Cartan decomposition $\underline{L}(\mathbb{R}) \cap G_o = (\underline{L}(\mathbb{R}) \cap K) \cdot \exp(\mathfrak{g}^0(e) \cap \mathfrak{p})$. Thus it remains to show that $\underline{L}(\mathbb{R}) \cap K \subset N$. But clearly $\underline{L}(\mathbb{R}) \cap K = K^e \subset N$. This completes the proof.

9.16. COROLLARY. <u>The normalizer (in G_o) of a boundary component of $\mathcal{D}$ is transitive on $\mathcal{D}$</u> .

Proof. If we evaluate all the vector fields in $\mathfrak{n}$ at 0 we get all of V . Hence the orbit of 0 under N is open. It is also closed in $\mathcal{D}$ since G_o acts properly on $\mathcal{D}$.

9.17. COROLLARY. <u>The orbit of a tripotent e under G_o equals the orbit of $\mathcal{J}_e$ under K . The orbits of G_o on the boundary of $\mathcal{D}$ are just the submanifolds X_i described in</u> 6.9.

Proof. By the proof of 9.15, the vector fields $\xi_v (v \in V_0)$ satisfy $\xi_v(e+z) = \xi_v(z)$ for $e + z \in \mathcal{J}_e$ and hence $\exp(\xi_v)(e) = e + \exp(\xi_v)(0) = e + \tanh v$. It follows that the orbit of e under N is $\mathcal{J}_e$. By 9.16, $G_o = N \cdot K = K \cdot N$. Hence $G_o \cdot e = K \cdot N \cdot e = K \cdot \mathcal{J}_e$.

9.18. (The following results are not needed for §10). Let $\underline{S} \subset \underline{G}_o$ be a maximal R-split torus of $\underline{G}_o$. Then

$$\mathfrak{g}_o = \mathfrak{g}^{\underline{S}} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}^{\alpha}$$

where Φ, the *real root system* of $\underline{G}_o$, is the set of weights of $\underline{S}$ in $\mathfrak{g}_o$ (with respect to the adjoint representation), $\mathfrak{g}^{\alpha}$ is the weight space for the weight α, and $\mathfrak{g}^{\underline{S}}$ is the centralizer of $\underline{S}$ in $\mathfrak{g}_o$. By associating with $\alpha \in \Phi$ its differential we may identify Φ with a subset of the dual of $\mathfrak{s} = \mathrm{Lie}(\underline{S})$. Then Φ consists of all linear forms α on $\mathfrak{s}$ for which $\mathfrak{g}^{\alpha} = \{x \in \mathfrak{g}_o \mid [h,x] = \alpha(h)x$ for all $h \in \mathfrak{s}\}$ is not zero.

Now let $\mathfrak{g}_o = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}_o$ and assume $\mathfrak{s} \subset \mathfrak{p}$ (such tori always exist). Then

$$\mathfrak{p} = \mathfrak{s} \oplus \Sigma\, \mathfrak{p}^{\alpha}$$

where

$$\mathfrak{p}^{\alpha} = \{x - \theta(x) \mid x \in \mathfrak{g}^{\alpha}\}$$

$$= \{y \in \mathfrak{p} \mid [[y,h],h] = \alpha(h)^2 y\,,\ \text{for all } h \in \mathfrak{s}\}\,.$$

Since $\theta(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha}$ we have $\mathfrak{p}^{\alpha} = \mathfrak{p}^{-\alpha}$. This shows that we can compute Φ once we know the Lie triple system $\mathfrak{p}$ with triple product $[xyz] = [[x,y],z]$. (All this holds for any reductive real algebraic group in place of $\underline{G}_o$).

Note now that *the real vector space* ${}_{R}V$ *is a real Lie triple system with*

$$[uvw] = \{u\bar{v}w\} - \{v\bar{u}w\}$$

and that ${}_{R}V \cong \mathfrak{p}$ *under the map* $v \to \xi_v$. This is an immediate consequence of 2.6.2. Now we can compute the real root system of $\underline{G}_o$.

9.19. PROPOSITION. *Let* $(e_1,\cdots,e_r)$ *be a frame of tripotents of* V, *and* $\underline{T}_{e_i}$ *the one-dimensional* R-*split torus of* $\underline{G}_o$ *associated with* e_i (cf. 9.12). *Then*

(a) $\underline{S} = \underline{T}_{e1}\cdots\underline{T}_{e_r}$ *is a maximal* R-*split torus of* $\underline{G}_o$ *(and hence the real rank of* $\underline{G}_o$ *equals the rank of the Jordan pair* (V,V^-)).

(b) *Define linear forms* ω_i *on* $\mathfrak{s} = \mathrm{Lie}(\underline{S})$ *by* $\omega_j(\xi_{e_i}) = \delta_{ij}$. *Let* $V = \Sigma V_{ij}$ *be the Peirce decomposition of* V *with respect to* $(e_1,\cdots,e_r)$. *Also let* $V_{ij} = A_{ij} \oplus B_{ij}$ $(1 \le i \le j \le r)$ *be the decomposition into real and imaginary part relative to the real form* $A(e)$ *of* $V_2(e)$ *(where* $e = e_1 + \cdots + e_r$ *and hence* $V_2(e) = \sum_{i,j>0} V_{ij}$; cf. 3.14). *Then* $\Phi \subset \{\pm\omega_i, \pm 2\omega_i, \pm\omega_i \pm \omega_j\}$ *and*

$$\mathfrak{p}^{\omega_i} \cong V_{i0}\,,\ \mathfrak{p}^{2\omega_i} \cong B_{ii}\,,\ \mathfrak{p}^{\omega_i-\omega_j} \cong A_{ij}\,,\ \mathfrak{p}^{\omega_i+\omega_j} \cong B_{ij}$$

under the isomorphism $\mathfrak{p} \cong {}_{R}V$ *of* 9.18.

(c) *If* (V,V^-) *is simple then either* $\Phi = \{\pm 2\omega_i, \pm\omega_i \pm \omega_j\}$ *is of type* C_r *or* $\Phi = \{\pm\omega_i, \pm 2\omega_i, \pm\omega_i \pm \omega_j\}$ *is of type* BC_r.

Remark. The first case in (c) occurs if and only if (V,V^-) contains invertible elements, or equivalently, the domain $\mathcal{D}$

is equivalent to a tube domain (cf. 10.9). The multiplicities of the roots (i.e., the dimensions of the spaces $\mathfrak{g}^{\alpha} \cong \mathfrak{p}^{\alpha}$) can be looked up, for each simple type, in [L5, §17]. Note that $2\omega_i$ always has multiplicity 1, and that $\omega_i + \omega_j$ and $\omega_i - \omega_j$ have the same multiplicity, since $B_{ii} = \sqrt{-1}\,\mathrm{R}e_i$ and $B_{ij} = \sqrt{-1}\,A_{ij}$.

Proof. Clearly $\mathfrak{s} \cong W = \sum \mathrm{R}.e_i$ under the isomorphism of 9.18. Let $x = \lambda_1 e_1 + \cdots + \lambda_r e_r \in W$ and $y_{ij} \in V_{ij}$. Then we have by 3.15,

$$[y_{ij}xx] = \{y_{ij}\bar{x}x\} - \{x\bar{y}_{ij}x\} = (\lambda_i^2 + \lambda_j^2)y_{ij} - 2\lambda_i\lambda_j y_{ij}^* .$$

This implies

$$[a_{ij}xx] = (\lambda_i - \lambda_j)^2 a_{ij}, \quad \text{for } a_{ij} \in A_{ij},$$

$$[b_{ij}xx] = (\lambda_i + \lambda_j)^2 b_{ij}, \quad \text{for } b_{ij} \in B_{ij},$$

$$[y_{i0}xx] = \lambda_i^2 y_{i0}, \quad \text{for } y_{i0} \in V_{i0}.$$

From these formulae, it follows in particular that $\mathfrak{s} \cong W$ is a maximal abelian subspace of $\mathfrak{p}$ which implies that $\underline{S}$ is a maximal R-split torus of $\underline{G}_o$. Also, $\omega_i(\xi_x) = \lambda_i$ and hence we have (a) and (b). For (c), note that for a simple Jordan pair, the spaces V_{ij} $(1 \leq i < j \leq r)$ have the same dimension, and so do the spaces V_{i0}. Moreover, $V_{ij} \neq 0$ for $1 \leq i < j \leq r$ (cf. [L5, 17.2]).

9.20. Recall the relation between parabolic subgroups of $\underline{G}_o$ and subsets of Φ: If $\Sigma \subset \Phi$ is a simple root system and $\Psi \subset \Sigma$ a subset then $\underline{N}(\Psi)$ is the parabolic subgroup whose Lie algebra is $\mathfrak{g}^{\underline{S}} \oplus \sum \mathfrak{g}^{\alpha}$ where the sum is over all $\alpha \in \Phi$ which, when written as linear combinations of simple roots, involve the elements of $\Sigma - \Psi$ with non-negative coefficients. This establishes a one-to-one correspondence between subsets of Σ and conjugacy classes of parabolic subgroups of $\underline{G}_o$ (cf. [B-T]).

9.21. PROPOSITION. *Let* (V, V^-) *be simple. Then the map* $e \to \underline{N}_e$ *is a bijection from the set of non-zero tripotents of* V *onto the set of proper maximal parabolic subgroups of* $\underline{G}_o$. *If* $(e_1, \cdots, e_r)$ *is a frame and* $\Sigma = \{\alpha_1, \cdots, \alpha_r\}$ *is the simple root system with* $\alpha_1 = \omega_1 - \omega_2, \cdots, \alpha_{r-1} = \omega_{r-1} - \omega_r$, $\alpha_r = 2\omega_r$ *in case* $\Phi = C_r$ *and* $\alpha_r = \omega_r$ *in case* $\Phi = BC_r$ *then* $\underline{N}_{e_1 + \cdots + e_k}$ *is the parabolic subgroup associated with the subset* $\Sigma - \{\alpha_k\}$ *of* Σ.

Proof. Let e be a tripotent and write $e = e_1 + \cdots + e_k$ where the e_i are primitive orthogonal tripotents. Complete $(e_1, \cdots, e_k)$ to a frame $(e_1, \cdots, e_r)$. Then we have $\alpha_i(\xi_e) = 0$ for $1 \leq i \leq k - 1$ or $k + 1 \leq i \leq r$, and $\alpha_k(\xi_e) > 0$ ($= 1$ or $= 2$). In view of the definition of $\underline{N}_e$, this shows that $\underline{N}_e = \underline{N}(\Sigma - \{\alpha_k\})$; in particular, every maximal parabolic subgroup of $\underline{G}_o$ is conjugate to one of the $\underline{N}_e$. By [B-T, 14.2], the conjugacy class of $\underline{N}_e$ is isomorphic with $(\underline{G}_o/\underline{N}_e)(\mathrm{R}) \cong \underline{G}_o(\mathrm{R})/\underline{N}_e(\mathrm{R}) \cong G_o/N(\mathcal{T}_e)$

(recall $G_o = \underline{G}_o(R)^0$ and $N(\mathcal{J}_e) = \underline{N}_e(R) \cap G_o)$. By 9.15, $K \cap N(\mathcal{J}_e) = K^e$, and hence by 9.16, $G_o/N(\mathcal{J}_e) = K.N(\mathcal{J}_e)/N(\mathcal{J}_e) \cong K/K^e \cong$ the component containing e of the manifold of tripotents. This completes the proof.

9.22. More generally, one can show easily that there is a bijection between the set of all parabolic subgroups of $\underline{G}_o$ and the set of <u>flags of tripotents</u>, where a flag of tripotents is a k-tuple $(f_1,\cdots,f_k)$ of tripotents such that $0 < f_1 < \cdots < f_k$. The bijection is given by $(f_1,\cdots,f_k) \to \underline{N}_{f_1} \cap \cdots \cap \underline{N}_{f_k}$.

§10. Cayley transformations and Siegel domain realizations

10.0. We keep the notations of §9. Throughout this section, e is a tripotent and $V = V_2 \oplus V_1 \oplus V_0$ the Peirce decomposition with respect to e . The vector space V is a Jordan algebra with product $xy = \frac{1}{2}\, x \circ y = \frac{1}{2}\{x\bar{e}y\}$ and quadratic operators $P(x)y = Q(x)Q(\bar{e})y$ (cf. 3.6). We denote by J the subalgebra whose underlying vector space is V_2 . Thus J has unit element e , and $J = A \oplus iA$ where $A = \{z \in V_2 = J \mid z^* = Q(e)\bar{z} = z\}$ is formally real, and $*$ is complex conjugation with respect to A in J (cf. 3.13). The positive cone of A is denoted by Y , and $a > 0$ (resp. $a \geq 0$) stands for $a \in Y$ (resp. $a \in \bar{Y}$) (cf. 6.10).

10.1. <u>The partial Cayley transformation defined by</u> e <u>is</u>

$$\gamma_e = \exp\frac{\pi}{4}(e + \tilde{e}) \ .$$

Thus $\gamma_e \in G_c$. By 9.8 and since $\tan(\frac{\pi}{4}e) = \tan(\frac{\pi}{4})e = e$, we have the explicit formula

(1) $$\gamma_e = t_e \cdot B(e, -\bar{e})^{\frac{1}{2}} \cdot \tilde{t}_{\bar{e}} \ .$$

One checks easily that

(2) $$B(e, -\bar{e})^{\frac{1}{2}} \circ x = (\sqrt{2})^n x \quad \text{for } x \in V_n \ , \ n = 0,1,2 \ .$$

Also,

(3) $$\gamma_e = f\left(\exp\frac{\pi}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) = f\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right)$$

where f is as in 9.7. In $SL_2(\mathbb{C})$ we have the relations

$$\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right)^2 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

By applying f, this yields

(4) $$\gamma_e^2 = t_e \circ \tilde{t}_{\bar{e}} \circ t_e = \tilde{t}_{\bar{e}} \circ t_e \circ \tilde{t}_{\bar{e}},$$

(5) $$\gamma_e^4 = B(e, 2\bar{e}),$$

(the Peirce reflection; cf. 5.6),

(6) $$\gamma_e^8 = \mathrm{Id}.$$

Also, we see that $\gamma_e^4 = \mathrm{Id}$ if and only if $V_1 = 0$. Note further that $\gamma_0 = \mathrm{Id}$,

(7) $$\gamma_e^{-1} = \gamma_{-e},$$

(8) $$k\gamma_e k^{-1} = \gamma_{ke} \quad \text{for } k \in K,$$

(9) $$\gamma_c\gamma_d = \gamma_{c+d}$$

for orthogonal tripotents c and d.

We call $j_e = \gamma_e^2 = f\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ the <u>partial inverse</u> defined by e. This is motivated by the explicit formula derived below. First we prove

10.2. LEMMA. <u>For $a \in J$ let $R_a \in \mathrm{End}(V_1)$ be defined by $R_a(x) = a\circ x = \{a\bar{e}x\}$. Then the map $a \to R_a$ is a homomorphism of unital Jordan algebras (V_1 is a "special J-module"); i.e.,</u>

(1) $$(a\circ b)\circ x = a\circ(b\circ x) + b\circ(a\circ x), \quad e\circ x = x,$$

<u>for</u> $a,b \in J$, $x \in V_1$. <u>Moreover,</u>

(2) $$a\circ(b^*\circ x) = \{a\bar{b}x\},$$

(3) $$R_a^* = R_{a^*},$$

(4) R_a <u>positive definite for</u> $a > 0$.

<u>Here R_a^* denotes the adjoint with respect to the scalar product</u> $\langle,\rangle$.

<u>Proof.</u> By JP 13,

$$a\circ(a\circ x) = D(a,\bar{e})^2x = D(Q(a)\bar{e},\bar{e})x + 2\,Q(a)Q(\bar{e})x = a^2\circ x$$

since $a^2 = Q(a)\bar{e}$ and $Q(\bar{e})x \in V_3 = 0$. Linearization yields (1) (clearly $e\circ x = \{e\bar{e}x\} = x$ by definition of V_1). For (2) we use JP 9 : $a\circ(b^*\circ x) = D(a,\bar{e})D(b^*,\bar{e})x = \{b^*,Q(\bar{e})x,a\} + D(a,Q(\bar{e})b^*)x = \{a\bar{b}x\}$. Next, $\langle a\circ x,\bar{y}\rangle = \langle\{a\bar{e}x\},\bar{y}\rangle = \langle x,\{\bar{e}a\bar{y}\}\rangle = \langle x,\overline{\{e\bar{a}y\}}\rangle = \langle x,\overline{a^*\circ y}\rangle$ (by (2)) which proves (3). Finally, $a > 0$ means $a = b^2$, $b = b^* \in A$ invertible. Hence $R_a = R_b^2$ is the square of an invertible self-adjoint endomorphism and therefore positive definite.

10.3. PROPOSITION. <u>Let</u> $x = x_2 \oplus x_1 \oplus x_0$ <u>be the components of</u> $x \in V$ <u>in the Peirce spaces</u>. <u>Then</u> $\gamma_e(x) \in V$ <u>if and only if</u> $e - x_2$ <u>is invertible in</u> J , <u>and then</u>

$$(1)\quad \gamma_e(x) = \left(\frac{e+x_2}{e-x_2}\right) \oplus \sqrt{2}\cdot(e-x_2)^{-1}\circ x_1 \oplus (x_0 + P(x_1)(e-x_2)^{-1}) \ .$$

<u>Also</u>, $j_e(x) \in V$ <u>if and only if</u> x_2 <u>is invertible in</u> J , <u>and then</u>

$$(2)\qquad j_e(x) = (-x_2^{-1}) \oplus (-x_2^{-1}\circ x_1) \oplus (x_0 - P(x_1)x_2^{-1}) \ .$$

<u>Here</u> a^{-1} <u>denotes the inverse in</u> J , <u>and</u> $\frac{e+x_2}{e-x_2} = (e-x_2)^{-1}\cdot(e+x_2)$ (which can be computed in the commutative associative subalgebra generated by e and x_2) .

<u>Proof</u>. By 10.1.1, $\gamma_e(x)$, which a priori belongs to X , lies in V if and only if $(x,\bar{e})$ is quasi-invertible. By 7.3.4,

$$\tilde{t}_{\bar{e}}(x_2 + x_1 + x_0) = x_2^{\bar{e}} + B(x_2,\bar{e})^{-1}\cdot(x_1 + x_0)^{w}$$

where $w = \tilde{t}_{x_2}(\bar{e}) \in V^-$. By the Peirce rules, the pair $(x_1 + x_0, w)$ is nilpotent in the sense of [L5,3.8], and moreover,

$$(x_1 + x_0)^{(2,w)} = Q(x_1 + x_0)w = Q(x_1)w \in V_0 \ ,$$

$$(x_1 + x_0)^{(n,w)} = 0 \quad \text{for} \quad n > 2 \ .$$

Further, by [L5, 3.13], $\tilde{t}_{\bar{x}_2}(e) = (e - x_2^*)^{-1}$ and therefore $Q(e)w = ((e - x_2^*)^{-1})^* = (e - x_2)^{-1}$ which implies $Q(x_1)w = Q(x_1)Q(\bar{e})Q(e)w = P(x_1)(e - x_2)^{-1}$. Next, $B(x_2,\bar{e})$ leaves the Peirce spaces invariant, acts like the identity on V_0 , and for $y_1 \in V_1$ satisfies $B(x_2,\bar{e})y_1 = y_1 - \{x_2,\bar{e},y_1\} + Q(x_2)Q(\bar{e})y_1 = y_1 - x_2\circ y_1 = (e - x_2)\circ y_1$, since $Q(\bar{e})y_1 = 0$. Hence we get

$$(3)\quad \tilde{t}_{\bar{e}}(x) = x_2^{\bar{e}} \oplus (e - x_2)^{-1}\circ x_1 \oplus (x_0 + P(x_1)(e - x_2)^{-1}) \ .$$

Now by (1) and (2) of 10.1,

$$\gamma_e(x) = (e + 2x_2^{\bar{e}}) \oplus \sqrt{2}\cdot(e - x_2)^{-1}\circ x_1 \oplus (x_0 + P(x_1)(e - x_2)^{-1})$$

and this proves (1) since $e + 2x_2^{\bar{e}} = e + 2\frac{x_2}{e-x_2} = \frac{e+x_2}{e-x_2}$. Next, by (3) and 10.1.4 (replace x_2 by $e + x_2$!),

$$j_e(x) = e + [(e + x_2) + x_1 + x_0]^{\bar{e}} =$$

$$- x_2^{-1} \oplus (-x_2^{-1}\circ x_1) \oplus (x_0 - P(x_1)x_2^{-1})$$

since $e + (e + x_2)^{\bar{e}} = e + \frac{e+x_2}{e-(e+x_2)} = - x_2^{-1}$.

10.4. <u>Siegel domains</u>. Define $F : V_1 \times V_1^- \to J$ by

$$F(u,\bar{v}) = \{u\bar{v}e\} \quad (u,v \in V_1) \ .$$

Then F is <u>hermitian</u> and <u>positive definite</u> in the sense that

$$(1)\qquad F(u,\bar{v})^* = F(v,\bar{u}) \ ,$$

(2) $F(u,\bar{u}) \geq 0$, and $F(u,\bar{u}) = 0$ only for $u = 0$.

Indeed, by JP 12,

$$\{v\bar{u}e\} = \{v,\bar{u},Q(e)\bar{e}\} = - Q(e)\{\bar{u}v\bar{e}\} + \{\{v\bar{u}e\}\bar{e}e\} =$$
$$- \{u\bar{v}e\}^* + 2\{v\bar{u}e\} .$$

For (2), let $a \in \bar{Y}$. Then $<\{u\bar{u}e\} , \bar{a}> = <\{u\bar{e}a\} , \bar{u}> = <R_a(u),\bar{u}> \geq 0$ since R_a is positive semidefinite by 10.2. Now Y is a self-dual cone and hence we have $F(u,\bar{u}) \geq 0$. If $F(u,\bar{u}) = 0$ then $0 = <\{u\bar{u}e\},\bar{e}> = <u,\{\bar{u}e\bar{e}\}> = <u,\bar{u}>$ implies $u = 0$.

For $z \in V_0$ define the $\mathbb{C}$-antilinear endomorphism $\varphi(z)$ of V_1 by

$$\varphi(z)\bar{v} = \{e\bar{v}z\} .$$

Then $\varphi(z)$ is self-adjoint with respect to F in the sense that

(3) $$F(\varphi(z)\bar{u},\bar{v}) = F(\varphi(z)\bar{v},\bar{u}) ,$$

and satisfies

(4) $$0 \neq F(v,\bar{v}) - F(\varphi(z)\bar{v} , \overline{\varphi(z)\bar{v}}) \in \bar{Y}$$

for $0 \neq v \in V_1$ and $z \in \mathcal{D}_e = \mathcal{D} \cap V_0$. Indeed, by JP 13,

$$F(\varphi(z)\bar{u},\bar{v}) = \{\{e\bar{u}z\}\bar{v}e\} = D(e,\bar{v})D(e,\bar{u})z =$$
$$D(Q(e)\bar{v},\bar{u})z + Q(e)\{\bar{v}z\bar{u}\} = Q(e)\{\bar{v}z\bar{u}\}$$

(since $Q(e)\bar{v} \in V_3 = 0$) which is symmetric in u and v . Now let $z \in \mathcal{D}_e$, and set $f = \varphi(z)$. Then

$F(f\bar{v},\overline{f\bar{v}}) = F(f^2v,\bar{v}) = F(v,\overline{f^2v})$ and f^2 is also self-adjoint with respect to the hermitian scalar product $<,>$ on V_1 . The eigenspaces of f^2 are orthogonal both with respect to F and to $<,>$. Let $f^2v = \sum \lambda_j v_j$ where the λ_j are distinct eigenvalues of f^2 . Then $F(v,\bar{v}) - F(f\bar{v},\overline{f\bar{v}}) = \sum (1-\lambda_j)F(v_j,\bar{v}_j) \geq 0$ since by 6.7, $0 \leq \lambda_j < 1$, and $F(v,\bar{v}) - F(f\bar{v},\overline{f\bar{v}}) = 0$ implies $F(v_j,\bar{v}_j) = 0$ (since Y is a convex cone containing no straight line) and hence $v = 0$.

Now the data $(A,Y,F,\mathcal{D}_e,\varphi)$ define a Siegel domain of type three in $V = V_2 \oplus V_1 \oplus V_0$ with base $\mathcal{D}_e$ which is

(5) $$\mathcal{S}_e = \{x_2 \oplus x_1 \oplus x_0 \in V \mid |x_0| < 1 , \operatorname{Re}(x_2 - \tfrac{1}{2} F_{x_0}(x_1,\bar{x}_1) > 0\} .$$

Here we set

(6) $$F_z(u,\bar{v}) = F(u,(Id+\varphi(z))^{-1}\bar{v})$$

for $z \in \mathcal{D}_e$, $u,v \in V_1$, and Re denotes the real part with respect to the real form A of J (cf. [W1]; actually, (5) is a generalized right half plane rather than upper half plane as in [W1], see also 10.10). We call $\mathcal{S}_e$ the <u>Siegel domain defined by the tripotent</u> e . We will show that $\mathcal{S}_e = \gamma_e(\mathcal{D})$. To do so, we shall use the fact the normalizer N of the boundary component $\mathcal{F}_e$ in G_o is transitive on $\mathcal{D}$ (9.16) and hence $N' = \gamma_e N \gamma_e^{-1}$ is transitive on $\gamma_e(\mathcal{D})$. It turns out that N' consists essentially of affine transformations of the vector space V , except for a subgroup corresponding to the automorphism group of $\mathcal{D}_e$ which is trivial if e is maximal and hence $\mathcal{D}_e$ is a point. Our first goal is to describe the Lie algebra of N'.

10.5. LEMMA. Let $v \in V = \mathfrak{u}^+$ (constant vector field), $v = v_2 + v_1 + v_0$ the Peirce components, and $v_2 = a + b$ where $a \in A$, $b \in iA$. Then

(1) $$\mathrm{Ad}\,\gamma_e \cdot v_0 = v_0 ,$$

(2) $$\mathrm{Ad}\,\gamma_e \cdot v_1 = \frac{1}{\sqrt{2}}(v_1 + D(v_1,\bar{e})) ,$$

(3) $$\mathrm{Ad}\,\gamma_e \cdot a = \tfrac{1}{2}(a + \tilde{a} + D(a,\bar{e})) ,$$

(4) $$\mathrm{Ad}\,\gamma_e \cdot b = \tfrac{1}{2}(b - \tilde{b} + D(b,\bar{e})) .$$

Proof. For any $g \in G$ (considered as a birational transformation of V) and $\xi \in \mathfrak{g}$ (considered as a vector field on V) we have

$$(\mathrm{Ad}\,g^{-1} \cdot \xi)(x) = dg(x)^{-1} \cdot \xi(g(x)) .$$

Since $\gamma_e^{-1} = \gamma_{-e}$ this implies

$$(\mathrm{Ad}\,\gamma_e \cdot \xi)(x) = d\gamma_{-e}(x)^{-1} \cdot \xi(\gamma_{-e}(x)) .$$

Now $d\gamma_{-e}(x) = B(e,-\bar{e})^{\frac{1}{2}} B(x,-\bar{e})^{-1}$ by 10.1.1, since $d\tilde{t}_y(x) = B(x,y)^{-1}$ (see the computation in 7.8). Hence

$$\begin{aligned}(\mathrm{Ad}\,\gamma_e \cdot v)(x) &= B(x,-\bar{e}) \cdot B(e,-\bar{e})^{-\frac{1}{2}} \cdot v \\ &= B(x,-\bar{e})(\tfrac{1}{2}v_2 + \tfrac{1}{\sqrt{2}} v_1 + v_0) \\ &= \tfrac{1}{2}(v_2 + \{x\bar{e}v_2\} + Q(x)Q(\bar{e})v_2) + \tfrac{1}{\sqrt{2}}(v_1 + \{x\bar{e}v_1\}) + v_0\end{aligned}$$

and this implies (1) - (4) since $\{x\bar{e}v_j\} = D(v_j,\bar{e})x$ and

$$Q(x)Q(\bar{e})v_2 = Q(x)\bar{v}_2^* = Q(x)(\bar{a} - \bar{b}) = \tilde{a}(x) - \tilde{b}(x) .$$

10.6. PROPOSITION. Let $\mathfrak{n} = \mathfrak{g}^0(e) \oplus \mathfrak{g}^1(e) \oplus \mathfrak{g}^2(e)$ be the Lie algebra of $N = N(\mathcal{J}_e)$ (cf. 9.14, 9.15). The following formulae describe the Lie algebra $\mathfrak{n}' = \mathrm{Ad}\,\gamma_e(\mathfrak{n})$.

(1) $$\mathrm{Ad}\,\gamma_e \cdot \Delta = \Delta , \text{ for } \Delta \in \mathfrak{k}^e ,$$

(2) $$\mathrm{Ad}\,\gamma_e \cdot \xi_v = \xi_v , \text{ for } v \in V_0 ,$$

(3) $$\mathrm{Ad}\,\gamma_e \cdot \xi_a = D(a,\bar{e}) , \text{ for } a \in A ,$$

(4) $$\mathrm{Ad}\,\gamma_e \cdot (\xi_b - (D(v,\bar{e}) - D(e,\bar{v})) = \sqrt{2} \cdot (v + D(e,\bar{v})) ,$$
for $v \in V_1$,

(5) $$\mathrm{Ad}\,\gamma_e \cdot (\xi_v - D(b,\bar{e})) = 2b , \text{ for } b \in iA .$$

Proof. (1) follows from 10.1.8. (2) follows from 10.5.1 and $\mathrm{Ad}\,\gamma_e \cdot \tilde{v} = \tilde{v}$, obtained by applying the automorphism θ to it and recalling that γ_e is fixed under θ since it belongs to G_c. Apply θ to 10.5.3 and obtain

$$\mathrm{Ad}\,\gamma_e \cdot \tilde{a} = \tfrac{1}{2}(\tilde{a} + a - D(a,\bar{e}))$$

since $\theta(D(a,\bar{e})) = -D(a,\bar{e})^* = -D(e,\bar{a}) = -D(a,\bar{e})$ by 9.13. If we subtract this from 10.5.3 we get (3). Next, replace e by $-e$ in 10.5.2:

$$\mathrm{Ad}\,\gamma_{-e} \cdot v = \frac{1}{\sqrt{2}}(v - D(v,\bar{e})) .$$

Applying $\mathrm{Ad}\,\gamma_e$ to this equation it follows, since $\gamma_e^{-1} = \gamma_{-e}$, that

$$v = \frac{1}{\sqrt{2}} \mathrm{Ad}\,\gamma_e (v - D(v,\bar{e})) ,$$

and combining this with 10.5.2, one gets

(6) $$\operatorname{Ad}\gamma_e \cdot D(v,\bar e) = \frac{1}{\sqrt{2}}(D(v,\bar e) - v) \ .$$

Now apply θ to 10.5.2 and to (6) and add all four equations (with appropriate signs) to get (4). Finally, replace e by $-e$ in 10.5.4:

$$\operatorname{Ad}\gamma_{-e}\cdot b = \tfrac{1}{2}(b - \tilde b - D(b,\bar e)) \ ,$$

and apply $\operatorname{Ad}\gamma_e$ to obtain

$$b = \tfrac{1}{2}\operatorname{Ad}\gamma_e\cdot(b - \tilde b - D(b,\bar e))$$

which is (5).

10.7. LEMMA. Every element of $N' = \gamma_e N \gamma_e^{-1}$ can be written uniquely

(1) $$g = t_b \cdot t_{v+\frac{1}{2}F(v,\bar v)} \cdot \exp(D(e,\bar v)) \cdot B(e-y,\bar e) \cdot \exp(\xi_w) \cdot k$$

where $b \in iA$, $v \in V_1$, $y \in Y$, $w \in V_0$, $k \in K^e$, and this establishes a diffeomorphism $N' \approx iA \times V_1 \times Y \times V_0 \times K^e$. Explicitly, $B(e-y,\bar e)$ and $\exp D(e,\bar v)$ are given by

(2) $$B(e-y,\bar e)x = P(y)x_2 \oplus y\circ x_1 \oplus x_0 \ ,$$

(3) $$\exp(D(e,\bar v))x = $$
$$= (x_2 + F(x_1,\bar v) + \tfrac{1}{2}F(\varphi(x_0)\bar v,\bar v)) \oplus (x_1 + \varphi(x_0)\bar v) \oplus x_0 \ ,$$

for $x = x_2 \oplus x_1 \oplus x_0 \in V_2 \oplus V_1 \oplus V_0$.

Proof. In view of 9.15 and 10.6, this essentially amounts to computing $\exp D(a,\bar e)$ for $a \in A$ and $\exp(v + D(e,\bar v))$

for $v \in V_1$. For $x_2 \in J = A \oplus iA$ we have $D(a,\bar e)x_2 = 2L_a(x_2)$ where L_a is left multiplication by a in the Jordan algebra J. By [B-K, Satz 2.2, p. 317], $\exp 2L_a = P(\exp a)$ where $\exp a = \sum a^n/n!$ is the exponential map of J. Also, $\exp : A \to Y$ is a diffeomorphism ([B-K, p. 333]). For $x_1 \in V_1$ we have $\exp(D(a,\bar e))\cdot x_1 = (\exp R_a)\cdot x_1 = R_{\exp a}\cdot x_1$, by 10.2. For $x_0 \in V_0$ clearly $D(a,\bar e)x_0 = 0$. On the other hand, one checks easily that (2) holds. Hence $B(e - \exp a,\bar e) = \exp D(a,\bar e)$.

For $u \in V$ (considered as a constant vector field) and $f \in \operatorname{End} V$ (considered as a linear vector field) we have $[f,u] = f(u)$. Hence the Campbell-Hausdorff formula is

$$\exp(u)\exp(f) = \exp(u + f - \tfrac{1}{2}f(u) + \tfrac{1}{12}f^2(u) + \cdots) \ .$$

In particular, let $u = v + \frac{1}{2}F(v,\bar v)$ and $f = D(e,\bar v)$. By the Peirce rules,

(4) $$D(e,\bar v)\cdot V_j \subset V_{j+1}$$

and hence

$$\begin{aligned}
&\exp(v + \tfrac{1}{2}F(v,\bar v))\exp(D(e,\bar v)) \\
&= \exp(v + \tfrac{1}{2}\{v\bar v e\} + D(e,\bar v) + \tfrac{1}{2}[v + \tfrac{1}{2}\{v\bar v e\}, D(e,\bar v)]) \\
&= \exp(v + \tfrac{1}{2}\{v\bar v e\} + D(e,\bar v) - \tfrac{1}{2}D(e,\bar v)v) = \exp(v + D(e,\bar v)) \ .
\end{aligned}$$

By (4), $D(e,\bar v)^3 = 0$ and therefore

$$\exp D(e,\bar v) = \operatorname{Id} + D(e,\bar v) + \tfrac{1}{2}D(e,\bar v)^2 \ .$$

From this, (3) follows easily by a straightforward computation.

10.8. THEOREM. *The image of* $\mathcal{D}$ *under the Cayley transformation* γ_e *is the Siegel domain of type three*

$$\mathcal{S}_e = \{x_2 \oplus x_1 \oplus x_0 \in V_2 \oplus V_1 \oplus V_0 \mid |x_0| < 1, \mathrm{Re}(x_2 - \tfrac{1}{2} F_{x_0}(x_1, \bar{x}_1)) > 0\} .$$

Proof. We have $\mathcal{D} = N \cdot 0$ (9.16) and $\gamma_e(0) = e$. Hence $\gamma_e(\mathcal{D}) = N' \cdot e$. For $w \in V_0$ we have $\exp(\xi_w) \cdot e = e + \tanh w = e + z$, $z \in \mathcal{D}_e$. Thus by 10.7, the typical element of $\gamma_e(\mathcal{D})$ is

$$\begin{aligned} & t_b \cdot t_{v+\frac{1}{2}F(v,\bar{v})} \cdot \exp(D(e,\bar{v})) \cdot B(e-y,\bar{e}) \cdot (e+z) \\ &= (y^2 + \tfrac{1}{2} F(v + \varphi(z)\bar{v}, \bar{v}) + b) \oplus (v + \varphi(z)\bar{v}) \oplus z \\ &= x_2 \oplus x_1 \oplus x_0 , \end{aligned}$$

where $y \in Y$, $v \in V_1$, $b \in iA$, $z \in \mathcal{D}_e$. Therefore

$$\mathrm{Re}(x_2 - \tfrac{1}{2} F_{x_0}(x_1, \bar{x}_1)) = \mathrm{Re}(y^2 + b) = y^2 > 0 ,$$

which proves $\gamma_e(\mathcal{D}) \subset \mathcal{S}_e$. Conversely, let

$$x = x_2 \oplus x_1 \oplus x_0 \in \mathcal{S}_e ,$$

and let y be the unique positive square root of

$$\mathrm{Re}(x_2 - \tfrac{1}{2} F_{x_0}(x_1, \bar{x}_1))$$

(cf. [B-K, Kap. XI, §6]), let

$$b = i\,\mathrm{Im}(x_2 - \tfrac{1}{2} F_{x_0}(x_1, \bar{x}_1)) \in iA , \quad v = (\mathrm{Id} + \varphi(x_0))^{-1} x_1 \in V_1 ,$$

and $w = \mathrm{ar\,tanh}(x_0)$. Then $x = g(e) \in \gamma_e(\mathcal{D})$ where $g \in N'$ is as in 10.7.1 (with $k = \mathrm{Id}$) .

10.9. COROLLARY. *Let* e *be a maximal tripotent (hence* $V_0 = 0$) . *Then* $\gamma_e(\mathcal{D})$ *is the Siegel domain of type two*

$$\mathcal{S} = \{x_2 \oplus x_1 \in V_2 \oplus V_1 \mid x_2 + x_2^* - F(x_1, \bar{x}_1) > 0\} .$$

(By 5.3, $\mathcal{S}$ is independent of the choice of e , up to a transformation of K) . *In the special case where* $V_1 = 0$ *as well (hence* $V = V_2 = A \oplus iA$), $\gamma_e(\mathcal{D})$ *is the tube domain*

$$\mathcal{S} = Y \oplus iA$$

over the cone $Y \subset A$.

This is immediate from 10.8. Note that $V = V_2$ with respect to a maximal tripotent if and only if the Jordan pair (V, V^-) contains invertible elements in the sense of [L5, 1.10]. For the simple Jordan pairs listed in 4.14, this happens precisely for the types

$$I_{p,p} , \ II_{2n} , \ III_n , \ IV_n , \ VI .$$

10.10. *Remark*. The reader will have noticed that it is generalized *right* half planes rather than the more familiar upper half planes which occur naturally as Cayley transforms of bounded symmetric domains. This is confirmed by the fact that right half planes generalize to the case of real Jordan pairs and are, in fact, the Cayley transforms of real bounded symmetric domains, whereas upper half planes don't make sense in the real case. In the complex case, the upper half plane picture is obtained by replacing e by ie *but keeping the decomposition* $V_2 = A \oplus iA$ *with respect to* e. (Note that the Peirce spaces of e and ie are the same but $A(ie) = iA(e)$). One checks easily that

(1) $\gamma_{ie}(\mathcal{D}) = \{x_2 \oplus x_1 \oplus x_0 \mid |x_0| < 1, \operatorname{Im}(x_2) - \tfrac{1}{2}\operatorname{Re} F_{ix_0}(x_1,\bar{x}_1) > 0\}$

where Re and Im are understood with respect to the real form $A = A(e)$ of $V_2(e) = V_2(ie)$, and $F_z(u,\bar{v})$ is defined as in 10.4.6. If e is maximal we get

(2) $$\gamma_{ie}(\mathcal{D}) = \{x_2 \oplus x_1 \mid \operatorname{Im}(x_2) - \tfrac{1}{2}F(x_1,\bar{x}_1) > 0\}$$

and in case $V = V_2$, the upper half plane $A \oplus iY$.

We finally prove a number of formulae (see also [D], [S1], [S2], [T3]).

10.11. PROPOSITION. (a) *For* $a \in V_2$, $u,v \in V_1$ *we have*

(1) $$a\circ F(u,\bar{v}) = F(a\circ u,\bar{v}) + F(u,\overline{a^*\circ v}),$$

(2) $$P(a)F(u,\bar{v}) = F(a\circ u,\overline{a^*\circ v}).$$

(b) *The actions of* V_2 *and* V_0 *on* V_1 *given by* $a \to R_a$ *and* $z \to \varphi(z)$ *commute in the sense that*

(3) $$R_a\varphi(z) = \varphi(z)R_{a^*},$$

for $a \in V_2$, $z \in V_0$.

(c) *If* e *is maximal* $(V_0 = 0)$ *then*

(4) $$Q(u)\bar{v} = F(u,\bar{v})\circ u,$$

(5) $$F(u,\bar{v})\circ(a\circ u) = F(u,\overline{a^*\circ v})\circ u,$$

for $u,v \in V_1$. $a \in V_2$.

Proof. (1): By the Jordan identity and 10.2.2,

$$a\circ F(u,\bar{v}) = \{a\bar{e}\{u\bar{v}e\}\} = \{\{a\bar{e}u\}\bar{v}e\} - \{u\{\bar{e}a\bar{v}\}e\} + \{u\bar{v}\{a\bar{e}e\}\}$$
$$= F(a\circ u,\bar{v}) - F(u,\overline{a^*\circ v}) + 2\{u\bar{v}a\}.$$

Now by JP10, $\{u\bar{v}a\} = \{u,\bar{v},Q(e)\overline{a^*}\} = D(u,\bar{v})Q(e)\overline{a^*} = -\{Q(e)\bar{v},\overline{a^*},u\} + \{u,\overline{a^*\circ v},e\} = F(u,\overline{a^*\circ v})$ since $Q(e)\bar{v} \in V_3 = 0$. (Note that $u \in V$ can be arbitrary in this computation).

(2): By exponentiating (1) and observing $\exp 2L_x = P(\exp x)$ (cf. the proof of 10.7) we get (2) for a of the form $\exp x$, $x \in V_2$. Since these elements are open in V_2 and (2) is a polynomial identity, the assertion follows.

(3): $R_a\varphi(z)(\bar{v}) = a\circ\{e\bar{v}z\} = \{a\bar{e}\{e\bar{v}z\}\}$

$= \{\{a\bar{e}e\}\bar{v}z\} - \{e\{\bar{e}a\bar{v}\}z\} + \{e\bar{v}\{a\bar{e}z\}\}$

$= 2\{z\bar{v}a\} - \{e,\overline{a^*\circ v},z\} = \{e,\overline{a^*\circ v},z\}$

$= \varphi(z)\overline{R_{a^*}v}$ by 10.2.2, since $\{a\bar{e}z\} = 0$ by the Peirce rules, and $\{z\bar{v}a\} = \{z,\overline{a^*\circ v},e\}$ by the remark in the proof of (1).

(4): By JP11, $Q(u)\bar{v} = Q(u)\{\bar{e}e\bar{v}\}$

$= D(u,e)Q(u,e)\bar{v} - \{Q(u)\bar{e},\bar{v},e\}$

$= \{u\bar{e}\{u\bar{v}e\}\} = F(u,\bar{v})\circ u$ since $Q(u)\bar{e} \in V_0 = 0$.

(5): By JP12, $F(u,\overline{a^{*}\circ v})\circ u = Q(u)(\overline{a^{*}\circ v})$

$= Q(u)D(\bar{v},e)\overline{a^{*}} = \{u\overline{a^{*}}\{e\bar{v}u\}\} - \{e,\bar{v},Q(u)\overline{a^{*}}\}$

$= \{F(u,\bar{v}),\overline{a^{*}},u\} = F(u,\bar{v})\circ(a\circ u)$, using 10.2.2 and $Q(u)\overline{a^{*}} \in V_0 = 0$.

Remark. If $V_0 = 0$ then (V_1,V_1^{-}) is an alternative pair in the sense of [L5] with composition $\langle xyz\rangle = \{\{xye\}\bar{e}z\}$, and (4) and (5) of 10.11 correspond to 8.2.2 and 8.3.3 of [L5].

10.12. PROPOSITION. The geodesic symmetry around e of $\mathscr{S}_e$ is given by

(1) $x_2 \oplus x_1 \oplus x_0 \to x_2^{-1} \oplus (-x_2^{-1}\circ x_1) \oplus (P(x_1)x_2^{-1} - x_0)$.

The geodesic symmetry of $\mathscr{S}_{ie}$ (in the form 10.10.1) around ie is given by

(2) $x_2 \oplus x_1 \oplus x_0 \to -x_2^{-1} \oplus (-ix_2^{-1}\circ x_1) \oplus (P(x_1)x_2^{-1} - x_0)$.

(Here x_2^{-1} is the inverse in the Jordan algebra $J = V_2$ with unit element e) .

Proof. Since $\gamma_e(0) = e$, the geodesic symmetry around e is $s = \gamma_e\cdot(-Id)\cdot\gamma_e^{-1}$. Now $\gamma_e\cdot(-Id) = -Id\cdot\gamma_{-e} = -Id\cdot\gamma_e^{-1}$ and hence $s = \gamma_e^2\cdot(-Id) = j_e\cdot(-Id)$, and (1) follows from 10.3.2. Formula (2) follows similarly, by observing that the inverse in the Jordan algebra $J' = V_2$ with unit element ie (an isotope of J) is given by the negative inverse in J .

§11. Real bounded symmetric domains

11.1. PROPOSITION. Let (V,V^{-}) be a complex Jordan pair with positive involution τ and let $\mathscr{D} \subset V$ be the associated bounded symmetric domain. Let $T \subset V$ be a real form of the complex vector space V , and let η be complex conjugation with respect to T . Then $\eta(\mathscr{D}) = \mathscr{D}$ if and only if T is a sub-triple system; i.e., $\{T\bar{T}T\} \subset T$.

Proof. Assume $\mathscr{D}$ is invariant under η . Let $k(z,\bar{w})$ be the Bergman kernel function of $\mathscr{D}$. Then

$$k(\eta(z),\overline{\eta(w)}) = k(w,\bar{z}) .$$

This implies that η is an isometry of the Bergman metric and hence the fixed point set of η , which is $\mathscr{D} \cap T$, is a totally geodesic submanifold of $\mathscr{D}$. By 4.8, $\mathscr{D} \cap T = \tanh(T)$. In particular, if $v \in T$ then $\tanh(tv) = tv + (t^3/3)v^{(3)} + \cdots \in T$ for all $t \in \mathbb{R}$ which implies $v^{(3)} \in T$. By linearization, we have $\{u\bar{u}v\} + Q(u)\bar{v} \in T$ for all $u,v \in T$. By standard facts on symmetric spaces, T , which may be identified with the tangent space of $T \cap \mathscr{D}$ at 0 , is a sub-Lie triple system of the Lie triple system V . Hence $\{u\bar{v}w\} - \{v\bar{u}w\} \in T$ for all $u,v,w \in T$ (cf. 9.18). Therefore, $3Q(u)\bar{v} = Q(u)\bar{v} + \{u\bar{v}u\} = (Q(u)\bar{v} + \{u\bar{u}v\}) + (\{u\bar{v}u\} - \{v\bar{u}u\}) \in T$, and T is a sub-triple system.

Conversely, let T be a sub-triple system. Then a computation shows that $\eta(\{x\bar{y}z\}) = \{\eta(x),\overline{\eta(y)},\eta(z)\}$ and hence $\eta(z)^{(n)} = \eta(z^{(n)})$. Since $z \in \mathcal{D}$ if and only if $z^{(n)} \to 0$ as $n \to \infty$ it follows that $\eta(\mathcal{D}) = \mathcal{D}$.

11.2. Suppose that $T \subset V$ satisfies the conditions of 11.1. Clearly the pair (T,T^-) (where $T^- = \bar{T} \subset V^-$) is a real Jordan pair whose complexification is (V,V^-) . Also, the restriction $\tau_0 : T \to T^-$ is an involution of (T,T^-) which is positive in the sense that $Q(v)\tau_0(v) = \lambda v$ $(\lambda \in R)$ implies $\lambda > 0$, and the hermitian involution $\tau : V \to V^-$ is obtained by $\mathbb{C}$-antilinear extension from τ_0 . Conversely, let (T,T^-) be a real Jordan pair with positive involution τ_0 . We shall see below that this is equivalent with the trace form trace $(z \to \{x,\tau_0 y,z\})$ being positive definite. Let $(V,V^-) = (T_{\mathbb{C}},T^-_{\mathbb{C}})$ be the complexification and extend τ_0 to a $\mathbb{C}$-antilinear involution τ of (V,V^-) . Then by 3.16, τ is a positive hermitian involution of (V,V^-) .

We shall call <u>real bounded symmetric domain</u> a domain $\mathcal{D} \cap T$ as in 11.1. Thus real bounded symmetric domains are in one-to-one correspondence with real Jordan pairs with positive involution (or, in analogy to 2.9, real Jordan triple systems with positive definite trace form). It would be nice to have an intrinsic characterization of such domains in analogy to the complex case.

11.3. The theory developed in §3 can be carried over almost word for word to the case of a real Jordan pair with positive involution. In particular, 3.4, 3.9-3.12, 3.14-3.19 all remain valid (with some obvious changes in formulation). The details are left to the reader. Let us point out some of the differences to the complex case. Changing slightly the notation used in 11.1 and 11.2, let (V,V^-) be a real Jordan pair. The automorphism group $\mathrm{Aut}(V,V^-)$ is the set of real points of a real algebraic group $\underline{\mathrm{Aut}}(V,V^-)$, and a positive involution τ of (V,V^-) induces a Cartan involution of $\mathrm{Aut}(V,V^-)$ whose fixed point set, denoted $\mathrm{Aut}(V)$, is the automorphism group of the Jordan triple system on V defined by τ . The idempotents of a real Jordan pair are the real points of a real algebraic variety, and τ defines an involution of that manifold whose fixed points may be identified with the tripotents of V . The important Peirce decomposition 3.13 differs from the complex case as follows. If e is a tripotent of a real Jordan pair with involution then $V = V_2 \oplus V_1 \oplus V_0$ as before where V_α is the α-eigenspace of $D(e,\bar{e})$. But the vector space V_2 is now a <u>real semisimple Jordan algebra</u> J with unit element e and product $xy = \frac{1}{2}\{x\bar{e}y\}$. The map $z \to z^* = Q(e)\bar{z}$ is an automorphism of period 2 of J . Denoting by A and B the $(+1)$ - and (-1) -eigenspace of $*$, respectively,

$$J = A \oplus B$$

is a <u>Cartan decomposition</u> of J in the sense that A is formally real, and the trace form of J is negative definite on B. (We say that $*$ is a Cartan involution of J).
The Jordan triple structure of V_2 is given in terms of J and $*$ by

(1) $$Q(x)\overline{y} = P(x)y^*$$

(where $P(x)y = 2x(xy) - x^2y$). Indeed, one proves A formally real and formula (1) as in 3.13. From (1) it follows that trace $D(x,\overline{y}) = 2$ trace $L(xy^*)$ where $L(x)y = xy$. Since the former is positive definite, we have trace $L(x^2) < 0$ for $x \in B$.

Conversely, let $J = A \oplus B$ be the Cartan decomposition of a real semisimple Jordan algebra J with Cartan involution $*$. Then the real Jordan pair (J,J) has a positive involution τ given by $\tau x = x^*$, and thus J is a positive Jordan triple system with (1). The proof is similar to 3.7. Observe that, in contrast to the complex case where $B = iA$, B and A are not very closely tied together; for instance B may be zero, or $A = \mathbb{R}\cdot e$ and B of arbitrary dimension.

As in the complex case, the real bounded symmetric domain associated with (V,V^-) and τ is simply the open unit ball of the spectral norm, or may be described in terms of the generic minimum polynomial as in 4.16.

11.4. The <u>classification</u> of real bounded symmetric domains is obtained as follows. As in 4.11, one is reduced to the case of a <u>simple</u> real Jordan pair (V,V^-). If the complexification $(V_{\mathbb{C}},V_{\mathbb{C}}^-)$ is not simple then (V,V^-) is a complex Jordan pair, considered as a real Jordan pair by restriction of scalars. This case was treated in 4.14. We may therefore assume $(V_{\mathbb{C}},V_{\mathbb{C}}^-)$ simple; i.e., (V,V^-) <u>absolutely simple</u>. Then 4.12 and 4.13 continue to hold, and we only have to classify absolutely simple real Jordan pairs. From the results of [L5, §12] together with well-known facts on real associative division algebras and their involutions, real Cayley algebras, and real Jordan division algebras (which are all constructed from positive definite quadratic forms) one obtains without much difficulty the following list of absolutely simple real Jordan pairs. Here the symbols $I^{\mathbb{R}}_{p,q}$, $II^{\mathbb{H}}_{2n}$ etc. have been chosen in such a way that the complexifications are obtained by simply erasing the superscripts $\mathbb{R}$, $\mathbb{H}$, etc.

<u>Type</u> $I^{\mathbb{R}}_{p,q}$. $V = V^- = M_{p,q}(\mathbb{R})$, real $p \times q$ - matrices.

<u>Type</u> $I^{\mathbb{H}}_{2p,2q}$. $V = V^- = M_{p,q}(\mathbb{H})$, $p \times q$ - matrices with quaternion entries.

<u>Type</u> $I^{\mathbb{C}}_{n,n}$. $V = V^- = H_n(\mathbb{C})$, $n \times n$ complex hermitian matrices.

<u>Type</u> $II^{\mathbb{R}}_{n}$. $V = V^- = A_n(\mathbb{R})$, real alternating $n \times n$ - matrices

<u>Type</u> $II^{\mathbb{H}}_{2n}$. $V = V^- = H_n(\mathbb{H})$, quaternionic $n \times n$ - matrices, hermitian with respect to the standard involution of $\mathbb{H}$.

<u>Type</u> $III_n^{\mathbb{R}}$. $V = V^- = S_n(\mathbb{R})$, real symmetric $n \times n$ - matrices.

<u>Type</u> $III_{2n}^{\mathbb{H}}$. $V = V^- = SH_n(\mathbb{H})$, skew-hermitian quaternionic $n \times n$ - matrices.

In these cases, the Jordan pair structure is given by $Q(x)y = x \cdot {}^t\bar{y} \cdot x$, where ${}^t\bar{y}$ is the transpose - conjugate of y (with respect to the standard involution of the coefficient algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}$) . A positive involution is given by the identity; in other words, the vector spaces V listed are positive Jordan triple systems with $x \cdot {}^t\bar{y} \cdot x$. Note that $III_{2n}^{\mathbb{H}}$ is isomorphic with the Jordan pair of the Jordan algebra of $n \times n$ quaternionic matrices, hermitian relative to an involution of $\mathbb{H}$ with 3-dimensional fixed point set.

<u>Type</u> $IV_n^{\mathbb{R},p}$. $V = V^- = \mathbb{R}^n$, with $Q(x)y = q(x,y)x - q(x)y$ where q is the quadratic form $-x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2$ of index p , and $q(x,y) = q(x+y) - q(x) - q(y)$. A positive involution is given by $\tau x = (-x_1, \cdots, -x_p, x_{p+1}, \cdots, x_n)$. Note here that $IV_2^{\mathbb{R},0}$ is not absolutely simple and $IV_2^{\mathbb{R},1}$ is not simple.

<u>Type</u> $V^{\mathbb{O}}$. $V = V^- = M_{1,2}(\mathbb{O})$.

<u>Type</u> $V^{\mathbb{O}_o}$. $V = V^- = M_{1,2}(\mathbb{O}_o)$.

Here $\mathbb{O}$ is the real Cayley division algebra, and $\mathbb{O}_o$ the real split Cayley algebra. The Jordan pair structure is $Q(x)y = x \cdot ({}^t\bar{y} \cdot x)$ where the bar refers to the canonical involution.

<u>Type</u> $VI^{\mathbb{O}}$. $V = V^- = H_3(\mathbb{O})$.

<u>Type</u> $VI^{\mathbb{O}_o}$. $V = V^- = H_3(\mathbb{O}_o)$, 3×3 - matrices with entries from $\mathbb{O}$ resp. $\mathbb{O}_o$, hermitian with respect to the standard involution. The Jordan pair structure is induced from the Jordan algebra $H_3(\mathbb{O})$ resp. $H_3(\mathbb{O}_o)$.

As in 4.14, the type $V^{\mathbb{O}}$ imbeds into $VI^{\mathbb{O}}$ and $V^{\mathbb{O}_o}$ into $VI^{\mathbb{O}_o}$. A positive involution is given by the identity in case $V^{\mathbb{O}}$ and $VI^{\mathbb{O}}$, and by reflection in a quaternion subalgebra $\mathbb{H}$ of $\mathbb{O}_o$ in the other two cases.

11.5. Among the real Jordan pairs listed above, precisely the following isomorphism occur.

(1) $I_{1,1}^{\mathbb{R}} \cong II_2^{\mathbb{R}} \cong III_1^{\mathbb{R}} \cong I_{1,1}^{\mathbb{C}} \cong II_2^{\mathbb{H}} \cong IV_1^{\mathbb{R},0} \cong (\mathbb{R},\mathbb{R})$;

(2) $IV_2^{\mathbb{R},1} \cong IV_1^{\mathbb{R},0} \times IV_1^{\mathbb{R},0}$; $IV_2^{\mathbb{R},0} \cong (\mathbb{C},\mathbb{C})$;

(3) $I_{1,3}^{\mathbb{R}} \cong II_3^{\mathbb{R}}$;

(4) $III_2^{\mathbb{H}} \cong IV_3^{\mathbb{R},0}$; $III_2^{\mathbb{R}} \cong IV_3^{\mathbb{R},1}$;

(5) $I_{2,2}^{\mathbb{H}} \cong IV_4^{\mathbb{R},0}$; $I_{2,2}^{\mathbb{C}} \cong IV_4^{\mathbb{R},1}$; $I_{2,2}^{\mathbb{R}} \cong IV_4^{\mathbb{R},2}$;

(6) $II_4^{\mathbb{R}} \cong IV_6^{\mathbb{R},3}$; $II_4^{\mathbb{H}} \cong IV_6^{\mathbb{R},1}$;

(7) $I^{\mathbb{R}}_{p,q} \cong I^{\mathbb{R}}_{q,p}$; $I^{\mathbb{H}}_{2p,2q} \cong I^{\mathbb{H}}_{2q,2p}$;

(8) $IV^{\mathbb{R},p}_{n} \cong IV^{\mathbb{R},n-p}_{n}$.

11.6. Let (V,V^-) be a real Jordan pair with positive involution τ . Let $K = \mathrm{Aut}(V)^0$, a compact connected group, and $\langle\ ,\ \rangle$ a positive definite $\mathrm{Aut}(V)$-invariant scalar product on V . We may consider $\mathrm{Aut}(V,V^-)$ as a subgroup of $GL(V)$ and then $\mathrm{Aut}(V)$ is the intersection of $\mathrm{Aut}(V,V^-)$ with the orthogonal group of $\langle\ ,\ \rangle$ (cf. 11.3). With the definitions of 5.1, Proposition 5.2 remains valid. Note, however, that $A = \mathbb{R}.e$ does not tell much about $V_2(e) = A \oplus B$ (cf. 11.3) except that it is a real Jordan division algebra; B may be zero or more-than-onedimensional. Part (a) of 5.3 continues to hold, and we define the *real rank* of V to be the dimension of a maximal flat subspace. This is the same as the capacity of (V,V^-) as defined in [P3], but not the same as the rank of (V,V^-) as defined in [L5, 15.18] (rank and real rank agree if (V,V^-) is reduced; see below). Part (b) of 5.3 is no longer true in the real case, and the situation is more complicated.

11.7. LEMMA. *Let* (V,V^-) *be a simple real Jordan pair,* e *a maximal tripotent,* $V_2(e) = J = A \oplus B$ *as in* 11.3. *Then* J *is a simple real Jordan algebra, and* A *is either simple or the sum of two simple ideals. In the second case, there exists* $c \in A$ *such that* (i) $c^2 = e$, (ii)

the Cartan involution $*$ *of* J *is given by* $z^* = P(c)z$, (iii) *the Peirce spaces of of* c *and* e *in* V *agree,* (iv) *the Jordan algebra* $J' = V_2(c)$ *(with unit* c *and quadratic representation* $P'(x)y = Q(x)Q(\bar{c})y$) *is formally real.*

Proof. The Jordan pair $(V_2(e), V_2^-(e))$ is simple and hence so is J ([L5, 10.14, 1.6]). Now (i) and (ii) follow from [He 1, Satz 2.3, Kor.]. We have $c^* = P(c)c = c^3 = c = c^{-1}$. Now 10.2, which continues to hold in the real case, implies $D(e,\bar{e}) = D(c,\bar{c})$ and hence e and c have the same Peirce spaces. Moreover, $P(x)y^* = Q(x)Q(\bar{e})Q(c)Q(\bar{e})y = Q(x)Q(\overline{Q_e\bar{c}})y = Q(x)Q(\overline{c^*})y = P'(x)y$, and since $*$ is a Cartan involution, it follows that J' is formally real.

11.8. PROPOSITION. *Let* (V,V^-) *be a simple real Jordan pair with positive involution. If* $(e_1,\cdots,e_r)$ *and* $(e_1',\cdots,e_r')$ *are frames of tripotents then there exists* $f \in K$ *such that* $f(e_i') = \pm e_i$, $i = 1,\cdots,r$.

Proof. In view of 5.3 (a) it suffices to show that, for one particular frame $(e_1,\cdots,e_r)$ and any permutation σ of $\{1,\cdots,r\}$ there exists $f \in K$ with $f(e_i) = e_{\sigma(i)}$. Let $e = e_1 + \cdots + e_r$ and $V_2(e) = A \oplus B$ as before. By 11.7, we may assume that A is simple. (Otherwise replace e by c and decompose $c = c_1 + \cdots + c_r$ into a frame). Now $(e_1,\cdots,e_r)$ is an orthogonal system of primitive idempotents of A and therefore there exist $a \in A_{ij}$ for

$i \neq j$ (where the A_{ij} are the Peirce spaces of A relative to $e_1, \cdots, e_r$) with $a^2 = e_i + e_j$. Let $\Delta = D(a, \bar{e}_i) - D(e_i, \bar{a}) \in \mathfrak{k} = \mathrm{Lie}(K)$. Then one checks that $\Delta(e_i) = a = -\Delta(e_j)$, $\Delta(a) = 2(e_j - e_i)$, and $\Delta(e_k) = 0$ for $i \neq k \neq j$. It follows that $f = \exp(\frac{\pi}{2}\Delta) \in K$ interchanges e_i and e_j and leaves the other e_k fixed. The Proposition follows.

Remark. From [P3, Th. 6] together with 5.11 one gets the somewhat weaker result that $f \in \mathrm{Inn}(V, V^-) \cap \mathrm{Aut}(V)$, an open subgroup of $\mathrm{Aut}(V)$ which is in general not connected (hence $\neq K$).

11.9. Let (V, V^-) be simple real, e a primitive tripotent. Then the isomorphism class of the Jordan division algebra $V_2(e)$ is independent of the choice of e (and in fact determined by $\dim V_2(e)$). We say (V, V^-) is reduced if $V_2(e) = \mathbb{R}.e$. Note that $V_2(e)$ has (absolute) rank 2 unless it is one-dimensional, being the Jordan algebra of a positive definite quadratic form. This implies that (V, V^-) is reduced if and only if the real rank agrees with the rank of the complexification, and the latter is twice the former in the non-reduced case. In the list 11.4, the non-reduced types are $I^{\mathbb{H}}_{2p,2q}$, $III^{\mathbb{H}}_{2n}$, $IV^{\mathbb{R},0}_n$, $V^{\mathbb{O}}$, and in these cases, the dimension of $V_2(e)$ (e primitive) is 4, 3, n, 8 respectively.

11.10. Let J be a real semisimple Jordan algebra, $J = A \oplus B$ a Cartan decomposition with Cartan involution $*$, and $(V, V^-) = (J, J)$ the associated Jordan pair with positive involution $\tau = *$. Then the set of maximal tripotents is the "unit circle"

$$C = \{x \in J \mid x^* = x^{-1}\}$$

(see [He2] for an extensive study). Indeed, the condition for a tripotent is $x = x^{(3)} = P(x)x^*$. Thus we have to show that a maximal tripotent is invertible in J. Clearly this is the case for the unit element e of J. If c is a maximal tripotent then 5.3 (a) implies $f(V_2(c)) = V_2(e) = J$ for some $f \in K$, and hence $J = V_2(c)$ which implies c invertible. In contrast to the complex case, C is in general not connected (see below for more information). The tangent space of C at e is B, and $\exp(B) \subset C$; in fact, $\exp(B) = \{c^2 \mid c \in C\}$ (cf. [He2, Satz 1.1]). The Cartan involution $*$ of J induces a Cartan involution θ of the structure group $\mathrm{Str}(J)$ by $\theta(g)(x) = [g^{\#-1}(x^*)]^*$ where $\#$ is the canonical involution of $\mathrm{Str}(J)$, and K is the fixed point set of θ in $\mathrm{Str}(J)^0$.

11.11. PROPOSITION. Let (V, V^-) be a simple real Jordan pair with positive involution.

(i) If $(V, V^-) = (A, A)$ is the Jordan pair associated with a simple formally real Jordan algebra of degree (= rank) r then the set C of maximal tripotents of V is $\{a \in A \mid a = a^{-1}\}$ and has $r + 1$ connected components. These are the cases $III^{\mathbb{R}}_n$, $I^{\mathbb{C}}_{n,n}$, $II^{\mathbb{H}}_{2n}$, $IV^{\mathbb{R},1}_n$, $VI^{\mathbb{O}}$.

(ii) If $(V,V^-) = (J,J)$, J a simple reduced real Jordan algebra with Cartan involution $* = \tau$, and J is not isotopic with a formally real algebra, then the set C of maximal tripotents has 2 connected components. These are the cases $I^R_{n,n}$, II^R_{2n}, $IV^{R,p}_n$ with $2 \le p \le [\frac{n}{2}]$, $VI^{\mathbb{O}}_o$.

(iii) If (V,V^-) is not reduced, or contains no invertible elements, then K acts transitively on frames, and the set of maximal tripotents is connected. (The rest of the cases, including all complex Jordan pairs).

Proof. (i) C is isomorphic with the set of idempotents of A via $a \to \frac{1}{2}(a+e)$. Now the assertion follows from well-known facts on formally real Jordan algebras (see also [He2, Satz 5.3 (c)]).

(ii) See [He2, Satz 5.3 (B_1), (B_2)] .

(iii) Let $(e_1,\cdots,e_r)$ be a frame of tripotents. Let $e = e_1+\cdots+e_r$, $J = V_2(e) = A \oplus B$. By 11.8, it suffices to show that there exists $f \in K$ with $f(e_i) = -e_i$, $f(e_j) = e_j$, for all $i \neq j$. If V is not reduced then $V_{ii} = \mathbb{R}.e_i \oplus B_i$ is a Cartan decomposition with $B_i = V_{ii} \cap B \neq 0$. Choose $x \in B_i$ such that $x^2 = -e_i$. Then $\Delta = D(x,\bar{e}) - D(e,\bar{x}) \in \mathfrak{k} = \mathrm{Lie}(K)$, and by 9.13, we have $D(x,\bar{e}) = D(e,\bar{x}^*) = -D(e,\bar{x})$ hence $\Delta = 2D(x,\bar{e})$. Now $\Delta(e_i) = 2\{x\bar{e}e_i\} = 4x$, $\Delta(x) = 2\{x\bar{e}x\} = 4x^2 = -4e_i$, and $\Delta(e_j) = 0$ for $j \neq i$. Hence $f = \exp(\frac{\pi}{4}\Delta)$ has the required properties.

If V contains no invertible elements then $V_{i0} \neq 0$, $i = 1,\cdots,r$. Choose $x \in V_{i0}$ such that $\{x\bar{x}e_i\} = e_i$. This is possible since $\{x\bar{x}e_i\} \in A \cap V_{ii} = \mathbb{R}.e_i$ by the Peirce rules, and $\{x\bar{x}e_i\}$ is a positive multiple of e_i, by 10.4.2. Then $\Delta = D(x,\bar{e}) - D(e,\bar{x}) \in \mathfrak{k}$, and $\Delta(e_i) = x$, $\Delta(x) = -e_i$, $\Delta(e_j) = 0$ for $j \neq i$. Hence we may set $f = \exp(\pi\Delta)$.

11.12. Let M be the set of tripotents of a real Jordan pair with positive involution. Then Theorem 5.6 holds if we replace $iA(e)$ by $B(e)$, the (-1)-eigenspace of $*$ in $V_2(e)$. Also, $S = M/R$ is now only a compact Riemannian symmetric space. Note that the Peirce reflection $B(e,2\bar{e})$ is in general not in K, nor are the "unit circles" $C(e)$ connected. The proof that $d \sim e$ if and only if $d \in C(e)$ has to be modified; cf. the proof of 11.7. The fibration $p : M \to S$ is obtained as follows. Let $e \in M$ and consider the map $f : V_1(e) \times C(e) \to M$ given by $f(u,c) = k(u).c$ where $k(u) = \exp(D(u,\bar{e}) - D(e,\bar{u})) \in K$. Then $f(0,c) = c$, $f(u,C(e)) = k(u).C(e) = C(k(u).e)$. For the differential of f at $(0,c)$ one obtains by a simple computation

$$df(0,c)\cdot(v,w) = w + \{v\bar{e}c\} - \{e\bar{v}c\} = w + D(c,\bar{e})v$$

where $v \in V_1(e)$ and $w \in T_e(C(e)) = B(c)$. Thus $df(0,c)\cdot(v,w) = 0$ implies $w = D(c,\bar{e})v = 0$, and since c is invertible in $V_2(e)$, we get $v = 0$ by 10.2. Hence

f is an immersion at all points of $\{0\} \times C(e)$, and by comparing dimensions, we see that it is a local diffeomorphism. By compactness of $C(e)$, there exists an open neighborhood U of 0 in $V_1(e)$ such that $f: U \times C(e) \to M$ is an open imbedding. Now it follows easily that S is a compact manifold and $p: M \to S$ a fibration.

11.13. Let $\mathcal{D}$ be a real bounded symmetric domain. All of the results of §6 on the boundary structure of D which make sense for real domains remain valid. There are now no holomorphic boundary components but we do have affine (and metric) boundary components, and they are described by 6.3. The sets (i)-(iii) of 6.5 still coincide but are no longer connected. It would be interesting to characterize the set of maximal tripotents as the Shilov boundary of a suitable function space.

11.14. Let (V,V^-) be a real semisimple Jordan pair and $(\tilde{V},\tilde{V}^-) = (V,V^-) \otimes \mathbb{C}$ its complexification. Let $X = V \times V^-/\sim$ be as in 7.6. Then X is a compact real manifold; moreover, $X = \underline{X}(\mathbb{R})$ is the set of real points of a real projective algebraic variety $\underline{X}$ whose complexification is the variety $\tilde{X}$ defined by $(\tilde{V},\tilde{V}^-)$. Thus one may think of $\underline{X}$ also as of $\tilde{X}$ together with the Galois action induced by complex conjugation relative to (V,V^-) . The manifolds X obtained in this way are precisely the symmetric R-spaces studied in [T1].

Let $G = \underline{\mathrm{Aut}}(\underline{X})^0$, a real semisimple connected algebraic group with trivial centre (whose complexification is $\tilde{G} = \mathrm{Aut}(\tilde{X})^0$) . Let $G = \underline{G}(\mathbb{R})^0$, $H = G \cap \underline{H}(\mathbb{R})$ where $\underline{H} = \underline{\mathrm{Aut}}(V,V^-)^0$, and $U^{\pm}$ as in 8.6. Then the results of §8 all remain valid. In 8.9, we have $\underline{\mathrm{Aut}}(V,V^-)^0 = \underline{\mathrm{Inn}}(V,V^-)$ as <u>algebraic</u> groups (!) . In general, $\mathrm{Inn}(V,V^-)$, the subgroup of $GL(V) \times GL(V^-)$ generated by all $(B(x,y), (B(y,x)^{-1})$, (x,y) quasi-invertible, is neither topologically connected nor equal to $\underline{\mathrm{Inn}}(V,V^-)(\mathbb{R})$. For an extension of these results to arbitrary base fields see [L6, L7].

11.15. Now let τ be a positive involution of the real Jordan pair (V,V^-) . Then τ defines commuting automorphisms σ and θ of period 2 of the real algebraic group $\underline{G}$ as in 9.2 and 9.5. Let $\underline{G}_o$ and $\underline{G}_c$ be the fixed point sets of σ and θ in $\underline{G}$ and let

$$G_o = \underline{G}_o(\mathbb{R})^0 , \quad G_c = \underline{G}_c(\mathbb{R})^0$$

be the topological identity components of the sets of real points. Then $\underline{G}_o$ and $\underline{G}_c$ are real reductive (in general <u>not semisimple</u>) algebraic groups with Lie algebras

$$\mathfrak{g}_o = \mathfrak{k} \oplus \mathfrak{p} , \quad \mathfrak{g}_c = \mathfrak{k} \oplus \mathfrak{m}$$

where $\mathfrak{k} = \mathrm{Der}(V) = \mathrm{Lie}(K)$, $\mathfrak{p} = \{u - \tilde{\tilde{u}} \mid u \in V\}$, $\mathfrak{m} = \{u + \tilde{\tilde{u}} \mid u \in V\}$. Thus $u^+ \oplus u^- = \mathfrak{p} \oplus \mathfrak{m}$. The complexifications of $\mathfrak{g}_o$ and $\mathfrak{g}_c$ are isomorphic (but have not much to do with $\mathfrak{g} = \mathrm{Lie}(\underline{G})$, in contrast to the complex case). We note that

(1) $\mathfrak{k} = [\mathfrak{p},\mathfrak{p}] = [\mathfrak{m},\mathfrak{m}]$.

Indeed, the second equality is immediate from the definitions. By 8.7, $\mathfrak{h} = [\mathfrak{u}^+ \oplus \mathfrak{u}^-, \mathfrak{u}^+ \oplus \mathfrak{u}^-] = [\mathfrak{m} \oplus \mathfrak{p}, \mathfrak{m} \oplus \mathfrak{p}] = [\mathfrak{p},\mathfrak{p}] \oplus [\mathfrak{m},\mathfrak{p}]$, and this is the decomposition of $\mathfrak{h}$ into the (± 1)-eigenspaces of θ. On the other hand, the fixed point set of θ in $\mathfrak{h}$ is $\mathfrak{k}$.

As in 9.7, a tripotent e of V defines a homomorphism $f : SL_2(R) \to G$, and we have 9.8 and 9.9, with identical proofs (replace $i\mathfrak{p}$ by $\mathfrak{m}$). It follows that $\theta \mid G_o$ is a Cartan involution and hence

(2) $G_o = \exp(\mathfrak{p}) \cdot K$

is a Cartan decomposition.

11.16. Let $\mathcal{D} \subset V$ be the real bounded symmetric domain defined by τ. The restriction of the Bergman metric of the complexification $\mathcal{D}_{\mathbb{C}}$ of $\mathcal{D}$ to $\mathcal{D}$ is a Riemannian metric on $\mathcal{D}$, given by $ds^2 = \langle B(x,\bar{x})^{-1}dx, d\bar{x}\rangle$ (cf. 2.10). Since $\mathcal{D}$ is totally geodesic it is itself a Riemannian symmetric space, and its exponential map at 0 is given by 4.8. We claim that G_o <u>is</u> (by restriction to $\mathcal{D}$ isomorphic with) <u>the group of displacements</u> of the symmetric space $\mathcal{D}$; i.e., the group generated by all $s_x s_y, x,y \in \mathcal{D}$, where s_x denotes the geodesic symmetry around the point x (cf. [L2]). Indeed, K acts on $\mathcal{D}$ by isometries. By

4.4, the vector fields in $\mathfrak{p}$ restrict to Killing vector fields on $\mathcal{D}$. Hence 11.15.2 shows that G_o is contained in the group of isometries of $\mathcal{D}$. Now it follows from 11.15.1 and general facts on symmetric spaces that G_o is the group of displacements of $\mathcal{D}$.

Finally, note that metric boundary components of $\mathcal{D}$ can be defined as in the complex case (9.10), and the proof of 9.11 shows that they agree with the affine boundary components.

11.17. Let e be a tripotent. Then the homomorphism $f : \underline{SL}_2 \to \underline{G}$ induced by e maps the $\mathbb{R}$-split torus $\underline{T} = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1\}$ of $\underline{SL}_2$ into $\underline{G}_o$. Denoting the image of $\underline{T}$ by $\underline{T}_e$, 9.12-9.17 all remain valid if we replace iA by B (where $V_2(e) = A \oplus B$). Note that G_o acts on $\bar{\mathcal{D}}$ and permutes the boundary components. This follows from their characterization as metric boundary components since G_o acts by isometries on $\mathcal{D}$.

The real root system Φ of G_o may be computed as in 9.18, 9.19. In particular, 9.19 (a), (b) hold in the real case as well (where now $B_{ij} = B \cap V_{ij}$). Part (c), however, now takes the following form.

11.18. PROPOSITION. <u>Let</u> (V,V^-) <u>be an absolutely simple real Jordan pair with positive involution</u> τ, <u>and let</u> $(e_1,\cdots e_r)$ <u>be a frame of tripotents with the property that for</u> $e = e_1 + \cdots + e_r$ <u>we have</u> $V_2(e) = A \oplus B$ <u>with</u> A <u>simple</u> (cf. 11.7). <u>Then there are the following possibilities</u>.

(A) $\Phi = \{\pm(\omega_i - \omega_j)\}$ *is of type* A_{r-1} *for the types* $III^{\mathbb{R}}_r$, $I^{\mathbb{C}}_{r,r}$, $II^{\mathbb{H}}_{2r}$, $IV^{\mathbb{R},1}_n(r=2)$, *and* $VI^{\mathbb{O}}(r=3)$.

(B) $\Phi = \{\pm\,\omega_i,\ \pm\,\omega_i \pm \omega_j\}$ *is of type* B_r *for the cases* $I^{\mathbb{R}}_{r,q}$ *with* $r < q$, $II^{\mathbb{R}}_{2r+1}$, $V^{\mathbb{O}_o}(r=2)$.

(C) $\Phi = \{\pm\, 2\omega_i,\ \pm\,\omega_i \pm \omega_j\}$ *is of type* C_r *for the cases* $I^{\mathbb{H}}_{2r,2r}$, $III^{\mathbb{H}}_{2r}$, $IV^{\mathbb{R},0}_n(r=1)$.

(D) $\Phi = \{\pm\,\omega_i \pm \omega_j\}$ *is of type* D_r *for the cases* $I^{\mathbb{R}}_{r,r}$, $II^{\mathbb{R}}_{2r}$, $IV^{\mathbb{R},p}_n$ $(2 \le p \le [\frac{n}{2}],\ r = 2)$, *and* $VI^{\mathbb{O}_o}(r=3)$.

(BC) $\Phi = \{\pm\,\omega_i,\ \pm\, 2\omega_i,\ \pm\,\omega_i \pm \omega_j\}$ *is of type* BC_r *in the cases* $I^{\mathbb{H}}_{2r,2q}(r<q)$, $V^{\mathbb{O}}$ $(r=1)$.

Observe that (A) and (D) correspond to the cases (i) and (ii) of 11.11, respectively, and (BC), (B), and (C) to case (iii). In fact, the number of connected components of the set of maximal tripotents is equal to the index of the Weyl group of Φ in the group of all signed permutations of $\omega_1,\cdots,\omega_r$ (resp. $e_1,\cdots,e_r$) .

11.19. In contrast to the complex case, the real algebraic group $\underline{G}_o$ (resp. the Riemannian symmetric space $\mathcal{D}$) does not determine the Jordan pair uniquely. For example, the domains associated with the non-isomorphic Jordan pairs $V^{\mathbb{O}_o}$ and $I^{\mathbb{H}}_{4,4}$ are isomorphic as symmetric spaces. The same happens for $II^{\mathbb{H}}_8$ and $VI^{\mathbb{O}_o} \times (\mathbb{R},\mathbb{R})$. This is all the more surprising as $V^{\mathbb{O}_o}$ is an exceptional Jordan pair (cf. [L-Mc] and $I^{\mathbb{H}}_{4,4}$ is not.

11.20. Partial Cayley transformations and Siegel domain realizations work the same way as in the complex case, provided we replace iA by B ($J = A \oplus B = V_2(e)$) throughout, and define the "real part" Re(x) for $x \in V_2$ by $\frac{1}{2}(x + x^*)$. The endomorphisms $\varphi(z)$ of V_1 (cf. 10.4) are now of course $\mathbb{R}$-linear . Also, there is no way of converting right half planes into upper half planes as in the complex case.

Appendix:
List of identities for Jordan pairs

(For proofs see [L5])

Let (V^+,V^-) be a Jordan pair, with quadratic maps $Q\colon V^{\pm} \to \mathrm{Hom}(V^{\mp},V^{\pm})$. Thus for $x \in V^{\pm}$, $y \in V^{\mp}$, we have $Q(x)y = Q_x y \in V^{\pm}$ and $Q_x y$ is quadratic in x and linear in y. We set

$$\{xyz\} = D(x,y)z = Q(x,z)y = Q(x+z)y - Q_x y - Q_z y\ ,$$

$$B(x,y) = \mathrm{Id} - D(x,y) + Q_x Q_y$$

$$x^y = B(x,y)^{-1}(x - Q_x y) \quad \text{(the quasi-inverse)}\ .$$

Then the following identities hold (the first three are the defining identities of a Jordan pair)

JP1 $\quad D(x,y)Q_x = Q_x D(y,x)$,

JP2 $\quad D(Q_x y,y) = D(x,Q_y x)$,

JP3 $\quad Q(Q_x y) = Q_x Q_y Q_x$,

JP4 $\quad D(x,y)Q_x = Q(x,Q_x y)$

JP5 $\quad Q(x,z)D(y,x) + Q_x D(y,z) = Q(x,\{xyz\}) + Q(z,Q_x y)$
$\quad\quad = D(x,y)Q(x,z) + D(z,y)Q_x$,

JP6 $\quad D(x,\{yxz\}) + Q_x Q(y,z) = D(x,z)D(x,y) + D(Q_x y,z)$
$\quad\quad = D(x,y)D(x,z) + D(Q_x z,y)$,

JP7 $D(\{xyz\},y) = D(z,Q_y x) + D(x,Q_y z)$,

JP8 $D(x,\{yxz\}) = D(Q_x y,z) + D(Q_x z,y)$,

JP9 $D(x,y)D(z,y) = Q(x,z)Q_y + D(x,Q_y z)$,

JP10 $Q(x,z)D(y,x) = Q(Q_x y,z) + D(z,y)Q_x$,

JP11 $D(x,y)Q(x,z) = Q(Q_x y,z) + Q_x D(y,z)$,

JP12 $D(x,y)Q_z + Q_z D(y,x) = Q(z,\{xyz\})$,

JP13 $D(x,y)D(x,z) = D(Q_x y,z) + Q_x Q(y,z)$,

JP14 $\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\}$,

JP15 $[D(x,y),D(u,v)] = D(\{xyu\},v) - D(u,\{yxv\})$,

JP16 $\{\{xyu\}vz\} - \{u\{yxv\}z\} = \{x\{vuy\}z\} - \{\{uvx\}yz\}$.

JP17 $D(Q_x y,z)Q_x = Q_x D(y,Q_x z)$,

JP18 $D(Q_x y,z)D(x,y) = Q_x Q_y D(x,z) + D(x,Q_y Q_x z)$,

JP19 $Q(Q_x y,\{xyz\}) = Q_x Q_y Q(x,z) + Q(x,z)Q_y Q_x$,

JP20 $Q(\{xyz\}) + Q(Q_x y,Q_z y) = Q_x Q_y Q_z + Q_z Q_y Q_x + Q(x,z)Q_y Q(x,z)$,

JP21 $Q(\{xyz\}) + Q(Q_x Q_y z,z) = Q_x Q_y Q_z + Q_z Q_y Q_x + D(x,y)Q_z D(y,x)$,

JP22 $Q(Q_x Q_y z,\{xyz\}) = Q_x Q_y Q_z D(y,x) + D(x,y)Q_z Q_y Q_x$,

JP23 $B(x,y)Q_x = Q_x B(y,x) = Q(x - Q_x y)$,

JP24 $B(Q_x y,y) = B(x,Q_y x) = B(x,y)B(x,-y)$,

JP25 $B(x,y)^2 = B(2x - Q_x y,y) = B(x,2y - Q_y x)$,

JP26 $Q(B(x,y)z) = B(x,y)Q_z B(y,x)$,

JP27 $Q(B(x,y)z,\ x - Q_x y) = B(x,y)(Q(x,z) - D(z,y)Q_x)$

$= (Q(x,z) - Q_x D(y,z))B(y,x)$,

JP28 $B(x,y)Q(x^y) = Q(x^y)B(y,x) = Q_x$,

JP29 $B(x,y)Q(x^y,z) + Q_x D(y,z) = Q(x^y,z)B(y,x) + D(z,y)Q_x = Q(x,z)$,

JP30 $B(x,y)D(x^y,z) = D(x,z) - Q_x Q(y,z)$,

JP31 $D(z,x^y)B(y,x) = D(z,x) - Q(y,z)Q_x$,

JP32 $D(x^y,y - Q_y x) = D(x - Q_x y,y^x) = D(x,y)$,

JP33 $B(x,y)B(x^y,z) = B(x,y+z)$,

JP34 $B(z,x^y)B(y,x) = B(y+z,x)$,

JP35 $B(x,y)^{-1} = B(x^y,-y) = B(-x,y^x)$.

The quasi-inverse satisfies

$$x^y = x + Q_x(y^x)\ , \qquad \text{(symmetry)}$$

$$\left.\begin{aligned} Q_y\left(x^{Q_y z}\right) &= (Q_y x)^z\ , \\ B(u,v)\left(x^{B(v,u)y}\right) &= (B(u,v)x)^y\ , \end{aligned}\right\} \text{(shifting)}$$

$$\left.\begin{aligned} x^{y+z} &= (x^y)^z\ , \\ (x+z)^y &= x^y + B(x,y)^{-1}\left(z^{(y^x)}\right)\ . \end{aligned}\right\} \text{(addition formulae)}$$

The following identity of degree 11 (an analogue of Glennie's identity) holds in all special Jordan pairs but not, e.g., in the Jordan pairs of type V or VI (cf. [L-Mc]).

(G) $$\{Q_x z,w,Q(x,y)Q_z Q_y w\} - Q_x Q_z Q(x,y)Q_w Q_y z$$

$$= \{Q_y z,w,Q(y,x)Q_z Q_x w\} - Q_y Q_z Q(y,x)Q_w Q_x z\ .$$

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