INNER IDEALS OF THE LIE ALGEBRA OF THE SKEW ELEMENTS OF CENTRALLY CLOSED PRIME ALGEBRAS WITH A RING INVOLUTION

JOSE BROX[1] AND ANTONIO FERNÁNDEZ LÓPEZ[2]

Dedicated to Professor W.S. Martindale, 3rd

ABSTRACT. In this note we extend the Lie inner ideal structure of simple Artinian rings with involution, initiated by Benkart and completed by Benkart and Fernández López, to centrally closed prime algebras with a ring involution over a field of characteristic not 2 or 3. New Lie inner ideals (which we call special) occur when making this extension. We also give a a purely algebraic description of the so-called Clifford inner ideals, which had been described only in geometric terms. Our main tool is a theorem by Martindale and Miers on nilpotent inner derivations of the skew-symmetric elements of prime rings with involution.

1. INTRODUCTION

Inner ideals of Lie algebras are the analogues of one-sided ideals in associative rings and algebras. They are Φ-submodules B of a Lie algebra L (over a ring of scalars Φ) such that $[[B, L], B] \subseteq B$. An abelian inner ideal of L is an inner ideal B which is also an abelian subalgebra, i.e., such that $[B, B] = 0$. Since their introduction over 30 years ago ([9],[4]), abelian inner ideals have proved to be a useful tool for classifying both finite-dimensional and infinite-dimensional simple Lie algebras. Premet ([19],[20]) has shown that every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic not 2 or 3 must contain one-dimensional inner ideals. Moreover, it follows from ([4],[21]) (see also [6]) that when the field is algebraically closed of characteristic $p > 5$, the classical Lie algebras (modular versions of the complex finite-dimensional simple Lie algebras) can be characterized as the finite-dimensional simple Lie algebras satisfying the following two equivalent conditions:

Date: June 20, 2012.

2000 *Mathematics Subject Classification.* Primary 17B60; Secondary 16W10, 17C50.

[1] Supported by the Spanish MEC through the FPU grant AP2009-4848.

[2] Supported by the Spanish MEC and Fondos FEDER, MTM2010-19482.

(i) They are generated by one-dimensional inner ideals.

(ii) They are nondegenerate, that is, they have no nonzero absolute zero divisors (where by an absolute zero divisor or sandwich element we mean an element x such that $[x, [x, L]] = 0$).

Further evidence of the usefulness of inner ideals comes from [15], where it is shown that an abelian inner ideal B of finite length in a nondegenerate Lie algebra L over a ring of scalars Φ such that 2 and 3 are invertible gives rise to a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ with $B = L_n$. Zelmanov in [24] described the simple Lie algebras over fields of characteristic 0 or $p > 4n + 1$ with such gradings in terms of finite $\mathbb{Z}$-gradings of simple associative rings with involution. A description of these associative rings and their gradings was later provided by Smirnov in [22],[23]. As a result, any nondegenerate simple Lie algebra with a nonzero abelian inner ideal of finite length comes either from a simple associative ring with a finite $\mathbb{Z}$-grading by taking the Lie commutator, from the skew-symmetric elements of such a simple associative ring with involution or from the Tits-Kantor-Koecher construction of a Jordan algebra of Clifford type, or is of exceptional type E_6, E_7, E_8, F_4 or G_2.

We also note that there is a strong connection between inner ideals of Lie algebras and inner ideals of Jordan systems (algebras and pairs, [16]) which has been developed in a series of articles during the last five years ([11]-[15],[7],[8]). In particular, results from [15] enable us to adopt a Jordan approach based on the notion of a subquotient of an abelian inner ideal to obtain the desired Lie theoretic theorems.

Let A be a non-necessarily unital associative algebra with involution $*$ over an arbitrary ring of scalars Φ, and let $K := \mathrm{Skew}(A, *)$ be the Lie algebra of the skew-symmetric elements of A. Denote by T the Jordan triple system defined on the Φ-module K by the quadratic operator $P_a b = aba$. Any abelian inner ideal of K (respectively, any inner ideal of T) will be called a Lie inner ideal (respectively, Jordan inner ideal) of K.

If V is a submodule of K such that $VV = 0$, then V is a Lie inner ideal if and only if it is a Jordan inner ideal. In this case V will be called a Jordan-Lie inner ideal.

It is clear that if V is a Jordan-Lie inner ideal of K and Ω is a Φ-submodule of $\mathrm{Skew}(Z(A), *)$ then $V + \Omega$ is a Lie inner ideal of K. Lie inner ideals of K of the form $V + \Omega$ with V and Ω as above will be called standard.

Let A be a unital (associative) ring whose centre is a field, with an involution $*$ of the second kind, so that K is a Lie algebra over the field $\mathrm{Sym}(Z(A), *)$. Let V be a Jordan-Lie inner ideal of K and let V_0 be a hyperplane of V such that $[[V, K], V] \subseteq V_0$. If $f : V \to \mathrm{Skew}(Z(A), *)$ is a $\mathrm{Sym}(Z(A), *)$-linear map with $\mathrm{Ker}(f) = V_0$, then the set

$\mathrm{Inn}(V, V_0, f) := \{v + f(v) : v \in V\}$ is a Lie inner ideal of K which is not standard. This kind of Lie inner ideal will be called special. We have the following result: K contains a special inner ideal if and only if there exists a skew-symmetric element a in A such that $a^2 = 0$ and a is not von Neumann regular.

Suppose now that A is a prime associative algebra with nonzero socle and with an involution $*$ of transpose type. Then K can be regarded as a subalgebra of the orthogonal algebra $\mathfrak{o}(X)$ containing the finitary orthogonal algebra $\mathfrak{fo}(X)$ [1], where X is a vector space (possibly infinite-dimensional) with a nonsingular symmetric bilinear form $\langle ., . \rangle$. Given $x, z \in X$, we set x^*z to denote the linear map defined by $x^*z(y) = \langle y, x\rangle z$ for all $y \in X$. It is easy to check that $\mathfrak{fo}(X)$ is the linear span of the linear maps of the form $x^*z - z^*x$.

Given a hyperbolic plane H of X and a nonzero isotropic vector x of H, we have that the set $[x, H^\perp] := \{x^*z - z^*x : z \in H^\perp\}$ is a Lie inner ideal of $\mathfrak{o}(X)$ contained in $\mathfrak{fo}(X)$. Since the subquotient of such an inner ideal $[x, H^\perp]$ is a Clifford Jordan pair, an inner ideal of the form $[x, H^\perp]$ will be called a Clifford inner ideal. They can also be described in algebraic terms as $\kappa(eA(1-e)) := \{a - a^* : a \in eA(1-e)\}$, where $e \in A$ is a rank-one isotropic idempotent.

Our last example of Lie inner ideal is a very singular one. Let A be the algebra $\mathrm{M}_2(\mathbb{F})$ of 2 by 2 matrices over a field $\mathbb{F}$ of characteristic not 2 and let $*$ be the transpose involution. Then $K = \mathrm{Skew}(\mathrm{M}_2(\mathbb{F}), *)$ is itself a Lie inner ideal.

A prime associative algebra A over a field $\mathbb{F}$ is centrally closed if its extended centroid is $\mathbb{F}$ itself. Then any ring involution $*$ of A induces an involution, also denoted by $*$, in $\mathbb{F}$. We say that $*$ is of the first kind if it is the identity on $\mathbb{F}$; otherwise $*$ is of the second kind. Note that K is then a Lie algebra over the field $\mathrm{Sym}(\mathbb{F}, *)$.

The main result of this paper proves that the examples of Lie inner ideals listed above essentially exhaust all possibilities in the case of a centrally closed prime associative algebra with a ring involution.

Theorem (Main Theorem)**.** *Let A be a centrally closed prime associative algebra with a ring involution over a field $\mathbb{F}$ of characteristic not 2 or 3, let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, and suppose that $\overline{\mathbb{F}} \otimes_{\mathbb{F}} A$ is not the full matrix algebra $M_2(\overline{\mathbb{F}})$ with the transpose involution. If B is a Lie inner ideal of* $\mathrm{Skew}(A, *)$*, then either*

(i) $B = V$ *is a Jordan-Lie inner ideal,*
(ii) $B = V \oplus \mathrm{Skew}(Z(A), *)$ *is a standard inner ideal,*
(iii) $B = \mathrm{Inn}(V, V_0, f)$ *is special, or*

(iv) $B = \kappa(eA(1-e))$ *is Clifford.*

Moreover, in cases (ii) and (iii) A is unital and $*$ is of the second kind, while in case (iv) A has nonzero socle and $*$ is the transpose involution.

As will be shown in a forthcoming paper, this theorem provides a new approach to the inner ideal structure of the skew-symmetric elements of an associative algebra initiated by Benkart [3] and completed by in Benkart and Fernández López in [5].

2. Lie inner ideals and Jordan inner ideals

Throughout this section, and unless otherwise specified, we will be dealing with non-necessarily unital associative algebras A, with product xy; Lie algebras L, with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x; and Jordan triple systems T (see [16] for a definition), with quadratic Jordan operator P_x and triple product $\{x, y, z\}$; all of them over a ring of scalars Φ. We denote by $Z(A)$ the centre of A. Note that any ring is an associative algebra over $\mathbb{Z}$.

2.1. Any associative algebra A with involution $*$ gives rise to:

(i) A Lie algebra K defined on the Φ-submodule $\mathrm{Skew}(A, *)$ by the Lie bracket $[x, y] := xy - yx$.

(ii) A Jordan triple system T defined on $\mathrm{Skew}(A, *)$ by the quadratic Jordan operator $P_x y := xyx$, with triple product $\{x, y, z\} = xyz + zyx$.

2.2. Given a Jordan triple system T, an *inner ideal* of T is any Φ-submodule V of T such that $\{V, T, V\} \subseteq V$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., such that $[B, B] = 0$.

2.3. An abelian inner ideal of the Lie algebra K will be called a *Lie inner ideal.* Similarly, an inner ideal of the Jordan triple system T will be called a *Jordan inner ideal.*

2.4. Let $V = (V^+, V^-)$ be a Jordan pair (see [16]). An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x V^{-\sigma} = 0$. A Jordan pair V is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, and L is said to be *nondegenerate* if it has no nonzero absolute zero divisors.

2.5. Let $B \subseteq V^+$ be an inner ideal of a Jordan pair V. Following [17], the *kernel* of B is the set $\mathrm{Ker}_V B := \{y \in V^- \mid Q_B y = 0\}$. Then $(0, \mathrm{Ker}_V B)$ is an ideal of the Jordan pair (B, V^-), and the quotient $\mathrm{Sub}_V B := (B, V^-)/(0, \mathrm{Ker}_V B) = (B, V^-/\mathrm{Ker}_V B)$ is a Jordan pair called the *subquotient* of B. The kernel and the corresponding subquotient of an inner ideal $B \subseteq V^-$ are defined in a similar way.

The analogue of this result holds for the abelian inner ideals of a Lie algebra if we replace the Jordan triple product $\{x, y, z\}$ by the left double commutator $[[x, y], z]$ as we describe next.

2.6. Let M be an abelian inner ideal of a Lie algebra L.

(i) The *kernel* of M is the set $\mathrm{Ker}_L M := \{y \in L \mid [M, [M, y]] = 0\}$.

(ii) The pair of Φ-modules $\mathrm{Sub}_L M := (M, L/\mathrm{Ker}_L M)$ with triple products given by

$$\begin{aligned} \{m, \overline{a}, n\} &:= [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L, \\ \{\overline{a}, m, \overline{b}\} &:= \overline{[[a, m], b]} \quad \text{for every } m \in M \text{ and } a, b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\mathrm{Ker}_L M$, is a Jordan pair called the *subquotient of* M [15, Lemma 3.2].

(iii) A Φ-submodule B of M is an inner ideal of L if and only if it is an inner ideal of $\mathrm{Sub}_L M$ [15, Proposition 3.5 (i)].

2.7. (See [15, Proposition 3.3].) Let $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ be a $(2n+1)$-grading. Then L_n and L_{-n} are abelian inner ideals. Moreover, if L is nondegenerate, then $\mathrm{Sub}_L L_n$ is isomorphic to the Jordan pair (L_n, L_{-n}); the triple products are given by the left double commutator.

Definition 2.8. Let B and C be abelian inner ideals of L. We will say that B and C are *Jordan isomorphic* ($B \cong C$) if their subquotients $\mathrm{Sub}_L B$ and $\mathrm{Sub}_L C$ are isomorphic as Jordan pairs.

3. Standard inner ideals

Throughout this section A will denote an associative algebra with an involution $*$ over a ring of scalars Φ containing $\frac{1}{2}$, and $K = \mathrm{Skew}(A, *)$ will be the Lie algebra of the skew-symmetric elements of A.

3.1. If V is a Φ-submodule of $\mathrm{Skew}(A, *)$ such that $VV = 0$, then V is a Jordan inner ideal of K if and only if it is a Lie inner ideal: for $u, v \in V$ and $x \in K$, $VV = 0$ implies

$[[u,x],v] = uxv + vxu = \{u,x,v\}$. In this case, V will be called a *Jordan-Lie inner ideal.*

Note that any Lie inner ideal B of K such that $b^2 = 0$ for all $b \in B$ is Jordan-Lie: for any $b, c \in B$, $0 = (b+c)^2 = 2bc$, so $BB = 0$ and B is Jordan-Lie.

Definition 3.2. A Lie inner ideal B of K will be said to be *standard* if $B = V + \Omega$, where V is a Jordan-Lie inner ideal of K and Ω is a Φ-submodule of $\mathrm{Skew}(Z(A), *)$.

3.3. Given a Lie inner ideal B of K, we denote by V_B the subset of all zero square elements of the commutative set $B + \mathrm{Skew}(Z(A), *)$.

Lemma 3.4. *Let A be semiprime and let B be a Lie inner ideal of K.*

(i) *If $B \subseteq V_B + \mathrm{Skew}(Z(A), *)$, then V_B is a Jordan-Lie inner ideal and $\{V_B, K, V_B\} \subseteq B$.*

(ii) *If in addition $V_B \subseteq B$, then $B = V_B \oplus (B \cap \mathrm{Skew}(Z(A), *))$ is standard. In particular, this is so if $\mathrm{Skew}(Z(A), *) \subseteq B$, or if every $v \in V_B$ is von Neumann regular.*

Proof. (i) Since $B + \mathrm{Skew}(Z(A), *)$ is closed under the sum, we have

$$V_B + V_B \subseteq B + \mathrm{Skew}(Z(A), *) \subseteq V_B + \mathrm{Skew}(Z(A), *),$$

which implies $V_B + V_B \subseteq V_B$ because $V_B \subseteq B + Z(A)$ is commutative and $Z(A)$ contains no nonzero nilpotent elements, since A is semiprime. Then, for any $u, v \in V_B$, we have $0 = (u+v)^2 = 2uv$, and hence $\{u,x,v\} = uxv + vxu = [[u,x],v] \in B$ with $\{u,x,v\}^2 = 0$. This proves that V_B has the required properties.

(ii) Suppose in addition that $V_B \subseteq B$. Then the Modular Law applied to the inclusion $B \subseteq V_B \oplus \mathrm{Skew}(Z(A), *)$ yields $B = V_B \oplus (\mathrm{Skew}(Z(A), *) \cap B)$. Note finally that if $\mathrm{Skew}(Z(A), *) \subseteq B$ then $V_B \subseteq B + \mathrm{Skew}(Z(A), *) \subseteq B$, and that if every $v \in V_B$ is von Neumann regular, then $V_B = \{V_B, K, V_B\} \subseteq B$ by (i). This completes the proof. □

Theorem 3.5. *Let A be semiprime and let B be a Lie inner ideal of K. Then B is standard if and only if the following condition holds:*

$$V_B \subseteq B \subseteq V_B + \mathrm{Skew}(Z(A), *). \tag{ST}$$

Proof. By Lemma 3.4, condition (ST) is sufficient for B to be standard. Suppose then that $B = V \oplus \Omega$ is standard. Clearly, $V \subseteq V_B$, and $V_B \subseteq B + \mathrm{Skew}(Z(A), *) \subseteq V \oplus \mathrm{Skew}(Z(A), *)$ implies $V_B = V$, which proves that B satisfies (ST). □

4. Special inner ideals

Recall that an involution $*$ of a unital associative ring A is of *the first kind* if $*$ is the identity map on $Z(A)$; otherwise $*$ is said to be of *the second kind.* As above, we denote by K the Lie algebra $\mathrm{Skew}(A,*)$ over the ring of scalars $\mathrm{Sym}(Z(A),*)$.

4.1. We show a way of constructing Lie inner ideals of K which are not standard. For this purpose we need:

(i) A unital ring A whose centre is a field of characteristic not 2, with an involution $*$ of the second kind,

(ii) a Jordan-Lie inner ideal V of K containing a hyperplane V_0 such that $[[V,K],V]\subseteq V_0$, and

(iii) a $\mathrm{Sym}(Z(A),*)$-linear map $f: V\to \mathrm{Skew}(Z(A),*)$ such that $\mathrm{Ker}(f)=V_0$.

Theorem 4.2. *Let A, $*$, K, V, V_0 and f be as above. Then the set* $\mathrm{Inn}(V,V_0,f):=\{v+f(v): v\in V\}$ *is a Lie inner ideal of K which is not standard.*

Proof. Set $B:=\mathrm{Inn}(V,V_0,f)$.

(1) B is a Lie inner ideal of K. Indeed,

$$[[B,K],B]=[[V,K],V]\subseteq V_0\subseteq B$$

and

$$[B,B]=[V,V]=0.$$

(2) $B\cap Z(A)=0$, since $v+f(v)=z\in Z(A)$ implies $v=z-f(v)\in Z(A)$ and hence $z=f(v)=0$ because A is semiprime and v is nilpotent.

(3) $V_B=V$: From the very definition of B, $V\subseteq B+\mathrm{Skew}(Z(A),*)$, and since $VV=0$ we have $V\subseteq V_B$. Conversely, let $x:=b+z\in V_B$ with $b=v+f(v)$. If $v=0$ then $b=0$ and hence $x^2=0$ implies $z=0$, so we may assume $v\neq 0$. Then

$$0=x^2=(b+z)^2=(v+(f(v)+z))^2=2(f(v)+z)v+(f(v)+z)^2$$

implies $f(v)+z=0$. Thus $x=v\in V$, which proves that $V_B\subseteq V$.

(4) Let $u\in V$ be such that $f(u)\neq 0$. Since $V\cap B=\mathrm{Ker}(f)$, we have that u does not belong to B. Thus $V_B=V$ is not contained in B, so B is not standard by Theorem 3.5. □

Lie inner ideals of the form $\mathrm{Inn}(V,V_0,f)$ will be called *special.* Note that a Jordan-Lie inner ideal V giving rise to an special inner ideal necessarily contains an element which is not von Neumann regular. Conversely:

Example 4.3. *Let A be a unital ring with an involution $*$ of the second kind whose centre is a field of characteristic not 2. Then any element $x \in K$ which is of zero square but not von Neumann regular yields the special inner ideal* $\mathrm{Inn}(V, V_0, f)$*, where* $V = \mathrm{Sym}(Z(A), *)x \oplus xKx$*,* $V_0 = xKx$ *and* $f : V \to \mathrm{Skew}(Z(A), *)$ *is given by* $f(x) = 1$ *and* $f(xKx) = 0$.

5. Clifford inner ideals

Throughout this section $\mathbb{F}$ will denote a field of characteristic not 2.

5.1. Let X be a vector space over $\mathbb{F}$ endowed with a nonsingular symmetric bilinear form $\langle \cdot , \cdot \rangle$. Denote by $\mathcal{L}_X(X)$ the associative algebra of the linear maps $a : X \to X$ having a (unique) adjoint $a^* : X \to X$, that is, such that $\langle ax, y\rangle = \langle x, a^*y\rangle$ for all $x, y \in X$. Then:

- (i) $\mathcal{L}_X(X)$ is a primitive algebra with involution $*$ (the adjoint).
- (ii) The socle of $\mathcal{L}_X(X)$ is the ideal $\mathcal{F}_X(X)$ of all $a \in \mathcal{L}_X(X)$ having finite rank (cf. [2, Theorems 4.3.7 and 4.6.8]).
- (iii) $\mathrm{Skew}(\mathcal{L}_X(X), *)$ is the *orthogonal algebra* $\mathfrak{o}(X)$ and $\mathrm{Skew}(\mathcal{F}_X(X), *)$ is the *finitary orthogonal algebra* $\mathfrak{fo}(X)$ [1].
- (iv) If $b \in \mathfrak{o}(X)$, then $\langle bx, x\rangle = 0$ for every $x \in X$.

5.2. Given $x, y \in X$, write y^*x to denote the linear map on X defined by $y^*x(x') = \langle x', y\rangle x$ for all $x' \in X$. We have:

- (i) $(y^*x)^* = x^*y$ and therefore $y^*x \in \mathcal{F}_X(X)$. In fact, $\mathcal{F}_X(X)$ is the additive span of these rank-one linear maps.
- (ii) $a(y^*x) = y^*ax$ and $(y^*x)b = (b^*y)^*x$ for all $x, y \in X$, any linear map a on X and any $b \in \mathcal{L}_X(X)$.
- (iii) $(y^*x)(z^*w) = \langle w, y\rangle z^*x$ for all $x, y, z, w \in X$.
- (iv) The linear map defined by $[x, y] := x^*y - y^*x$, $x, y \in X$, belongs to $\mathfrak{fo}(X)$. If V and W are subspaces of X, we write $[V, W]$ to denote the additive span of all the linear maps $[v, w]$, $v \in V$, $w \in W$. With this convention, $\mathfrak{fo}(X) = [X, X]$.

5.3. A *hyperbolic pair* is a pair of isotropic vectors (x, y) of X such that $\langle x, y\rangle = 1$, i.e, such that $H = \mathbb{F}x \oplus \mathbb{F}y$ is a hyperbolic plane.

- (i) If H is a hyperbolic plane then $X = H \oplus H^\perp$, where $H^\perp = \{z \in X : \langle z, H\rangle = 0\}$.
- (ii) Any nonzero isotropic vector $x \in X$ is part of a hyperbolic pair.

5.4. An idempotent $e \in \mathfrak{L}_X(X)$ is *isotropic* if $ee^* = 0 = e^*e$. It is easy to see that e is a rank-one isotropic idempotent if and only if $e = x^*y$ where (x, y) is a hyperbolic pair.

Proposition 5.5. *Let X be a vector space over $\mathbb{F}$, $\dim_{\mathbb{F}} X > 2$, endowed with a non-singular symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, let H be a hyperbolic plane of X and let $x \in H$ be a nonzero isotropic vector. Then:*

(i) $[x, H^\perp]$ *is a Lie inner ideal of* $\mathfrak{o}(X)$ *contained in* $\mathfrak{fo}(X)$.

(ii) *For any* $z \in H^\perp$, $[x, z]^3 = 0$ *and* $[x, z]^2 = -\langle z, z\rangle x^*x$, *with* $[x, z]^2 = 0$ *if and only if z is isotropic. Hence there exists $b \in [x, H^\perp]$ such that $b^3 = 0$ and b^2 is a rank-one linear map.*

(iii) $[x, H^\perp] = \mathrm{ad}^2_{[x,z]}(\mathfrak{fo}(X))$ *for any nonisotropic vector* $z \in H^\perp$.

(iv) $[x, H^\perp]$ *is minimal if and only if* $H^\perp$ *has no nonzero isotropic vectors.*

(v) $[x, H^\perp]$ *is a maximal commutative subset, and therefore a maximal Lie inner ideal, of* $\mathfrak{o}(X)$.

Proof. (i)-(iv) are proved in [12, Lemma 3.7].

(v) Let $a \in \mathfrak{o}(X)$ be such that

$$a(x^*z - z^*x) = (x^*z - z^*x)a,\ \ z \in H^\perp. \tag{1}$$

We will show that for any isotropic vector $y \in H$ such that $\langle x, y\rangle = 1$ we have that $ay \in H^\perp$ and $a = [x, ay]$.

Since $a^* = -a$, equation (1) can be written as

$$x^*az - z^*ax = (az)^*x - (ax)^*z,\ \ z \in H^\perp, \tag{2}$$

which evaluated in y yields $az = \langle y, az\rangle x - \langle y, ax\rangle z,\ z \in H^\perp$. Taking $z_1 \in H^\perp$ such that $\langle z_1, z_1\rangle \neq 0$, which is possible because $\dim_{\mathbb{F}} X > 2$, and since $\langle az_1, z_1\rangle = 0$ because a is skew-symmetric, we have that $\langle y, ax\rangle = 0$. Thus

$$az = \langle y, az\rangle x,\ \ z \in H^\perp. \tag{3}$$

Taking again $z_1 \in H^\perp$ such that $\langle z_1, z_1\rangle \neq 0$, it follows from (2) and (3) that $ax = \langle z_1, ax\rangle z_1 = -\langle az_1, x\rangle z_1 = -\langle y, az_1\rangle\langle x, x\rangle z_1 = 0$. Thus

$$ax = 0. \tag{4}$$

It follows from (4) that $\langle ay, x\rangle = -\langle y, ax\rangle = 0$, and since $\langle ay, y\rangle = 0$, we have that $ay \in H^\perp$. Using the decomposition $X = \mathbb{F}x \oplus \mathbb{F}y \oplus H^\perp$ we prove that $a = [x, ay] \in [x, H^\perp]$. □

5.6. By [11, 5.11] (see also [11, 5.7] for the definition of a Clifford pair), the finitary orthogonal algebra $\mathfrak{fo}(X)$ can be realized as the Tits-Kantor-Koecher algebra of the Clifford pair $(H^\perp, H^\perp)$. Moreover, we can verify (using formula (12) of [12]) that the pair of maps $(z \mapsto [x,z],\ v \mapsto -[y,v])$ defines an isomorphism of the Clifford Jordan pair $(H^\perp, H^\perp)$ onto the Jordan pair $([x,H^\perp],[y,H^\perp])$, which is isomorphic to $\mathrm{Sub}_L H^\perp$ by **2.7**. For this reason, Lie inner ideals of the form $[x,H^\perp]$ will be called *Clifford inner ideals.*

We describe Clifford inner ideals in algebraic terms. To this end we introduce the following notation, which makes sense for any subset S of a ring A with involution: $\kappa(S) := \{a + a^* : a \in S\}$.

Proposition 5.7. *A Lie inner ideal B of $\mathfrak{o}(X)$ is Clifford if and only if $B = \kappa(eA(1-e))$, where $A = \mathfrak{L}_X(X)$ and $e \in A$ is a rank-one isotropic idempotent.*

Proof. As previously noted, cf. (5.4), $e \in A$ is a rank-one isotropic idempotent if and only if $e = x^*y$, where (x,y) is a hyperbolic pair. Let $H = \mathbb{F}x \oplus \mathbb{F}y$ be the associated hyperbolic plane and set $f := e + e^*$. Then $Ae = A(x^*y) = x^*Ay = x^*X$, and since $1-f$ is the orthogonal projection on the subspace $H^\perp$, we have:

$$(1-f)Ae = (1-f)x^*X = x^*(1-f)X = x^*H^\perp.$$

Now consider $b \in \mathfrak{o}(X)$ and note that $e^*be = (y^*x)b(x^*y) = (y^*x)(x^*by) = x^*\langle by, y\rangle x = 0$ by (5.1)(iv). Then $\kappa((1-f)ae) = \kappa((1-e)ae - e^*ae) = \kappa((1-e)ae) - e^*\kappa(a)e = \kappa(ea(1-e))$ for every $a \in A$. Hence $[x,H^\perp] = \kappa(x^*H^\perp) = \kappa((1-f)Ae) = \kappa(eA(1-e))$. □

Theorem 5.8. *Let X be a vector space over a field $\mathbb{F}$ of characteristic not 2, $\dim_{\mathbb{F}} X > 2$, which is endowed with a nonsingular symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let L be a subalgebra of the Lie algebra $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$ and let B be an abelian inner ideal of L. Then B is a Clifford inner ideal of L if and only if there exists $b \in B$ such that $b^3 = 0$ and b^2 is a rank-one linear map.*

Proof. By (5.2), $b^2 = \alpha x^* x$, where both $\alpha \in \mathbb{F}$ and $x \in X$ are nonzero. Moreover, since $b^3 = 0$, x is isotropic. Let $y \in X$ be such that (x,y) is a hyperbolic pair (cf. 5.3(ii)). Then the vectors $y, by, b^2y = \alpha x$ are linearly independent and satisfy the following metric relations:

$$\langle y,y\rangle = 0,\ \langle y,by\rangle = 0,\ \langle y,b^2y\rangle = \alpha,\ \langle by,by\rangle = -\alpha,\ \langle by,b^2y\rangle = 0.$$

Hence the vector subspace $V := \mathbb{F}y \oplus \mathbb{F}by \oplus \mathbb{F}b^2y$ is the orthogonal sum $V = H \oplus \mathbb{F}by$, where $H = \mathbb{F}y \oplus \mathbb{F}b^2y = \mathbb{F}y \oplus \mathbb{F}x$ is a hyperbolic plane and the vector by is anisotropic, so V is nonsingular and $X = V \oplus V^\perp$. Since V is nonsingular and invariant under b, and b is skew-symmetric, $bV^\perp \subseteq V^\perp$. Set $b = b_1 \oplus b_0$, where $b_1 v = bv$, $b_1 w = 0$ and $b_0 v = 0$, $b_0 w = w$ for all $v \in V$, $w \in V^\perp$. Set $c = [x, by]$. We claim that $c = b_1$. Indeed:

(i) $cy = (x^*by - (by)^*x)y = by$,
(ii) $c(by) = (x^*by - (by)^*x)by = -\langle by, by\rangle x = \alpha x = b(by)$,
(iii) $c(b^2y) = (x^*by - (by)^*x)b^2y = 0 = b(b^2y)$, and
(iv) $cw = (x^*by - (by)^*x)w = 0$ for all $w \in V^\perp$.

As previously noted, cf. (5.6), $([x, H^\perp], [y, H^\perp])$ is a Clifford Jordan pair, and hence von Neumann regular. Then there exists $w \in H^\perp$ such that $\mathrm{ad}_c^2 d = c$, where $d = [y, w]$. As above we can see that $dV \subseteq V$ and $dV^\perp = 0$. Hence $\mathrm{ad}_b^2 d = \mathrm{ad}_c^2 d = c$, so $[x, by] = c \in B$. Since $[x, H^\perp]$ is a maximal abelian inner ideal by Theorem 5.8(v), we conclude that $B = [x, H^\perp]$ is Clifford. □

6. CENTRALLY CLOSED PRIME ALGEBRAS WITH A RING INVOLUTION

6.1. Let A be a prime associative algebra over a field $\mathbb{F}$. Following [2] denote by C the *extended centroid* of A. Then C is a field extension of $\mathbb{F}$ and the *central closure* CA of A is a prime associative algebra over C. We say that A is *centrally closed* if $\mathbb{F}$ is itself the extended centroid of A.

Any $*$-subalgebra A of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, where X is a vector space over a field $\mathbb{F}$ endowed with a nonsingular symmetric bilinear form (see 5.1), is a centrally closed primitive algebra with involution over $\mathbb{F}$.

Suppose now that A has a ring involution $*$. Then $*$ induces an involution on C and therefore it can be extended to an involution of the central closure. We say that $*$ is of the *first kind* if it is the identity on C, otherwise $*$ is of the *second kind*.

The following proposition is a corollary of a theorem due to Martindale and Miers, [18, Main Theorem], for prime rings with involution. Although the general result requires the characteristic to be zero, in our particular case only characteristic different from 2 and 3 is needed.

Proposition 6.2. *Let A be a centrally closed prime associative algebra with involution $*$ over a field $\mathbb{F}$ of characteristic not 2 or 3. Let K be the Lie algebra of its skew-symmetric elements, and let $b \in K$ be such that $\mathrm{ad}_b^3 K = 0$. Suppose that $\overline{\mathbb{F}} \otimes_{\mathbb{F}} A$ is not the algebra $M_2(\overline{\mathbb{F}})$ with the transpose involution, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. Then $b^2 = 0$, unless A is a $*$-subalgebra of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, where X is* *a vector space with a nonsingular symmetric bilinear form over* $\mathbb{F}$ *and* $*$ *is the adjoint involution,* $b^3 = 0$ *and* b^2 *is a rank-one symmetric linear map.*

Theorem 6.3 (Main Theorem)**.** *Let* A *be a centrally closed prime associative algebra with a ring involution over a field* $\mathbb{F}$ *of characteristic not* 2 *or* 3*, let* $\overline{\mathbb{F}}$ *be the algebraic closure of* $\mathbb{F}$*, and suppose that* $\overline{\mathbb{F}} \otimes_{\mathbb{F}} A$ *is not the full matrix algebra* $M_2(\overline{\mathbb{F}})$ *with the transpose involution. If* B *is a Lie inner ideal of* $\mathrm{Skew}(A, *)$*, then either*

(i) $B = V$ *is a Jordan-Lie inner ideal,*
(ii) $B = V \oplus \mathrm{Skew}(Z(A), *)$ *is a standard inner ideal,*
(iii) $B = \mathrm{Inn}(V, V_0, f)$ *is special, or*
(iv) $B = \kappa(eA(1 - e))$ *is Clifford.*

Moreover, in cases (ii) and (iii) A *is unital and* $*$ *is of the second kind, while in case (iv)* A *has nonzero socle and* $*$ *is the transpose involution.*

Proof. Let A be a centrally closed prime associative algebra with a ring involution $*$ over a field $\mathbb{F}$ of characteristic not $2, 3$ such that $\overline{\mathbb{F}} \otimes_{\mathbb{F}} A$ is not $M_2(\overline{\mathbb{F}})$ with the transpose involution, with $\overline{\mathbb{F}}$ being the algebraic closure of $\mathbb{F}$, and let B be an abelian inner ideal of $K := \mathrm{Skew}(A, *)$. Suppose first that $*$ is of the second kind and let ξ be a nonzero skew-symmetric element of $\mathbb{F}$. Then $A = K \oplus \xi K$. Set $C := B \oplus \xi B$. It is straightforward to see that C is an abelian inner ideal of the Lie algebra A^-. By [10, Theorem 5.5], either (i) $C = U$, where U is an inner ideal of A^- with $UU = 0$; (ii) $C = U \oplus \mathbb{F}1$, where U is as in (i); or (iii) $C = \{u + f(u)1 : u \in U\}$, where U is as in (i) and $f : U \to \mathbb{F}$ is a nonzero linear form. If $C = U$ as in (i), then $B = \mathrm{Skew}(U, *)$ is a Jordan-Lie inner ideal of K (cf. 3.1). Suppose then that A is unital and C is as in (ii) or (iii). In both cases U is $*$-invariant: $U^* \subseteq C^* = C$ implies $[U^*, U] = 0$ and hence $u^* = v + \alpha 1$, $u, v \in U$, $\alpha \in \mathbb{F}$, implies $u^* = v \in U$ since $UU = 0$ and u^* is nilpotent. If (ii), then $B = \mathrm{Skew}(U, *) \oplus \mathrm{Skew}(\mathbb{F}, *)1$, with $\mathrm{Skew}(U, *)$ being a Jordan-Lie inner ideal of K; if (iii), then $B = \{v + f(v)1 : v \in V\}$, where $V = \mathrm{Skew}(U, *)$ is a Jordan-Lie inner ideal of K and $f : V \to \mathrm{Skew}(\mathbb{F}, *)$ is a nonzero linear map.

Suppose now that the involution $*$ is of the first kind. If $b^2 = 0$ for every $b \in B$, then B is a Jordan-Lie inner ideal. Thus we may assume that $b^2 \neq 0$ for some $b \in B$. Then we have by Proposition 6.2 that A is a $*$-subalgebra of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, where X is a vector space with a nonsingular symmetric bilinear form over $\mathbb{F}$ and $*$ is the adjoint involution, $b^3 = 0$ and b^2 is a rank-one symmetric linear map. By Theorem 5.8, B is a Clifford inner ideal. □

Note that the Lie algebra $\mathrm{Skew}(M_2(\mathbb{F}), *)$, where $*$ is the transpose involution, is an abelian inner ideal in itself which does not lie in any of the four cases of the theorem above. Thus the exception in the statement is not superfluous.

References

[1] A.A. Baranov, Finitary simple Lie algebras, *J. Algebra* **219** (1999), 299-329.

[2] K.I. Beidar, W.S. Martindale and A.V. Mikhalev, *Rings with Generalized Identities.* Pure and Applied Mathematics; Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.

[3] G. Benkart, The Lie inner ideal structure of associative rings, *J. Algebra* **43** (1976), 561-584.

[4] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61-81.

[5] G. Benkart and A. Fernández López, The Lie inner ideal structure of associative rings revisited, *Comm. Algebra* **37** (2009), 3833-3850.

[6] A. Cohen, G. Ivanyos, and D. Roozemond, Simple Lie algebras having extremal elements, *Indag. Math. (N. S.)* **19** (2008), 177-188.

[7] C. Draper, A. Fernández López, E. García and M. Gómez Lozano, The socle of a nondegenerate Lie algebra, *J. Algebra* **319** (2008), 2372-2394.

[8] C. Draper, A. Fernández López, E. García and M. Gómez Lozano, The inner ideals of the classical Lie algebras, related gradings and Jordan pairs, *J. Lie Theory* (to appear).

[9] J.R. Faulkner, On the geometry of inner ideals, *J. Algebra* **26** (1973), 1-9.

[10] A. Fernández López, Lie inner ideals are nearly Jordan inner ideals, *Proc. Amer. Math. Soc.* (to appear).

[11] A. Fernández López, E. García and M. Gómez Lozano, The Jordan socle and finitary Lie algebras, *J. Algebra* **280** (2004), 635-654.

[12] A. Fernández López, E. García and M. Gómez Lozano, Inner ideals of finitary simple Lie algebras, *J. Lie Theory* **16** (2006), 97-114.

[13] A. Fernández López, E. García, and M. Gómez Lozano, The Jordan algebras of a Lie algebra, *J. Algebra* **308** (2007), 164-177.

[14] A. Fernández López, E. García and M. Gómez Lozano, An Artinian theory for Lie algebras, *J. Algebra* **319** (2008), 938-951.

[15] A. Fernández López, E. García, M. Gómez Lozano and E. Neher, A construction of gradings of Lie algebras, *Int. Math. Res. Not.* **16** (2007), doi:10.1093/imrn/rmn051.

[16] O. Loos, *Jordan pairs.* Lecture Notes in Math., vol. 460, Springer-Verlag, New York, 1975.

[17] O. Loos and E. Neher, Complementation of inner ideals in Jordan pairs, *J. Algebra* **166** (1994), 255-295.

[18] W.S. Martindale III and C. Robert Miers, Nilpotent inner derivations of the skew elements of prime rings with involution, *Can. J. Math.* **43** (1991), 1045-1954.

[19] A.A. Premet, Lie algebras without strong degeneration, *Mat. Sb.* **129** (1986), 140-153, translated in Math. USSR Sbornik **57** (1987), 151-164.

[20] A.A. Premet, Inner ideals in modular Lie algebras, *Vestsi Akad. Navuk BSSR Ser. Fõz.-Mat. Navuk* **5** (1986), 1115.

[21] A.A. Premet and H. Strade, Simple Lie algebras of small characteristic: I. Sandwich elements, *J. Algebra* **189** (1997), 419-480.

[22] O.N. Smirnov, Simple associative algebras with finite Z-grading, *J. Algebra* **196** (1997), 171-184.

[23] O.N. Smirnov, Finite Z-gradings of Lie algebras and symplectic involutions, *J. Algebra* **218** (1999), 246-275.

[24] E. Zelmanov, Lie algebras with finite grading, *Math. USSR Sbornik* **52** (1985), 347-385.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071, Málaga, Spain

E-mail address: brox@agt.cie.uma.es[1]

E-mail address: emalfer@uma.es[2]