THE LIE INNER IDEAL STRUCTURE OF ASSOCIATIVE RINGS REVISITED

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ABSTRACT. The aim of this note is to complete the description of the Lie inner ideal structure of simple Artinian rings with involution and of simple rings with involution and minimal one-sided ideals. Inner ideals are classified by adopting a Jordan approach based on the notion of a subquotient of an abelian inner ideal.

1. Introduction

Inner ideals of Lie algebras are the analogues of one-sided ideals in associative rings and algebras. They are subspaces B of a Lie algebra L such that $[B,[B,L]] \subseteq B$. Since their introduction over 30 years ago ([F], [B2]), they have proven to be a useful tool for classifying both finite-dimensional and infinite-dimensional simple Lie algebras.

One-dimensional inner ideals of a Lie algebra L are spanned by an extremal element, that is, an element x with the property that $[x,[x,y]]$ is a multiple of x for all $y \in L$. Premet ([P1],[P2]) has shown that every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic not 2 or 3 must have nonzero extremal elements. Moreover, it follows from ([B2], [PS]) (see also [CIR]) that when the field is algebraically closed of characteristic $p > 5$, the classical Lie algebras (modular versions of the complex finite-dimensional simple Lie algebras) can be characterized as the finite-dimensional simple Lie algebras satisfying the following two equivalent conditions:

(i) they are generated by extremal elements;

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(ii) they are nondegenerate, (that is, they have no nonzero absolute zero divisors, where by an absolute zero divisor (or sandwich element) we mean an element x such that $[x,[x,L]]=0$).

Thus, these special types of inner ideals play an essential role in the theory of simple modular Lie algebras and in on-going efforts (see [St]) to streamline their classification and to extend it to small characteristics.

Further evidence of the usefulness of inner ideals comes from [FGGN], where it is shown that an abelian inner ideal B of finite length in an arbitrary nondegenerate Lie algebra L over a commutative ring Φ such that 2 and 3 are invertible in Φ gives rise to a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ with $B = L_n$. Zelmanov [Z] described the simple Lie algebras over fields of characteristic 0 or $p > 4n+1$ with such gradings in terms of finite $\mathbb{Z}$-gradings of simple associative rings with involution. A description of these associative rings and their gradings was later provided by Smirnov in [S1],[S2]. As a result, any nondegenerate simple Lie algebra with a nonzero abelian inner ideal of finite length comes from a simple associative ring with a finite $\mathbb{Z}$-grading by taking the Lie commutator, from the skew-symmetric elements of such a simple associative ring with involution, or from the Tits-Kantor-Koecher construction of a Jordan algebra of a symmetric bilinear form, or it is of exceptional type E_6, E_7, E_8, F_4, G_2.

Let R be a simple associative ring of characteristic not 2 or 3 with an involution $*$ and with minimal one-sided ideals, and let Z denote the center of R. Such a simple ring R can be realized as the ring $\mathcal{F}(X)$ of all finite rank linear operators $a : X \to X$ on a left vector space X over a division ring with involution $(\Delta, -)$, where the involution $*$ on $\mathcal{F}(X)$ is given by the adjoint $a^* : X \to X$ $((ax, y) = (x, a^*y)$, for all $x, y \in X)$ with respect to a nondegenerate Hermitian or skew-Hermitian form $(\,,)$, $((x,y) = \epsilon\overline{(y,x)}, \epsilon = \pm 1)$ on X. The ring $R = \mathcal{F}(X)$ is Artinian if and only if X is finite-dimensional over Δ, in which case R is the complete ring $\mathrm{End}_\Delta X$ of linear transformations of the vector space X. In this paper we consider Lie algebras of the form

$$L = [K,K]/Z \cap [K,K],$$

where $K = \mathrm{Skew}(R, *)$, the set of skew-symmetric elements of R with respect to $*$. When R has dimension greater than 16 over its center, L is a nondegenerate central simple Lie algebra over the symmetric elements of the center of Δ (relative to the involution $-$). The description of the inner ideals of such Lie algebras L was begun in [B1, Thm. 5.5]. However, there is a case missing from that theorem and its proof; namely, when the inner ideal B is such that $b^2 = 0$ for all $b \in B$ and B cannot be written in the form eKe^* for any idempotent $e \in R$ with $e^*e = 0$. (See **4.1** below for further discussion.) It is the goal of this paper to finish the classification of the inner ideals of the Lie algebras $L = [K, K]/Z \cap [K, K]$. Theorem 6.1 gives the complete classification result - the previously omitted case is (ii.2) in the statement of that theorem.

The finitary orthogonal Lie algebras $L = \mathfrak{fo}(X, (\,,\,))$ are Lie algebras of the form $[K, K]/Z \cap [K, K]$, where $K = \mathrm{Skew}(R, *)$, $R = \mathfrak{F}(X)$, X is a vector space over a field $\mathbb{F}$ of arbitrary (possibly infinite) dimension greater than 4, and $*$ is the adjoint involution of a symmetric bilinear form on X. Their inner ideals were described in [FGG2, Prop. 3.6 (iv)]. However, point spaces (abelian inner ideals all of whose nonzero elements x satisfy $[x, [x, L]] = \mathbb{F}x$) are missing from the statement of that proposition because of the omission above. A further consequence of this paper is that the description of the inner ideals in finitary orthogonal Lie algebras is now complete. (See Remarks 6.7 to follow.)

There is a strong connection between inner ideals of Lie algebras and inner ideals of Jordan pairs (see [L1]) which has been developed in a series of articles during the last five years ([FGG1]-[FGG3], [FGGN], [DFGG1], [DFGG2]). In particular, results from [FGGN] enable us to adopt a Jordan approach based on the notion of a subquotient of an abelian inner ideal to obtain the desired Lie theoretic theorems.

2. Lie algebras and Jordan pairs

2.1. Throughout this paper, and unless specified otherwise, we will be dealing with Lie algebras L, with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, and with Jordan pairs $V = (V^+, V^-)$ [L1] with Jordan triple products $\{x, y, z\}$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$,

$\sigma = \pm$ and quadratic operators $Q_x(y) = \frac{1}{2}\{x, y, x\}$ over a ring of scalars Φ containing $\frac{1}{6}$.

2.2. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$, and V is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Similarly, $x \in L$ is an *absolute zero divisor* if $\mathrm{ad}_x^2 = 0$, and L is *nondegenerate* if it has no nonzero absolute zero divisors.

2.3. (INNER IDEALS) An *inner ideal* of V is a Φ-submodule B of V^σ such that $\{B, V^{-\sigma}, B\} \subseteq B$. Similarly, an *inner ideal* of L is a Φ-submodule B of L such that $[B, [B, L]] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$.

(i) The *socle* of a nondegenerate Jordan pair V is

$$\mathrm{Soc}\, V = (\mathrm{Soc}\, V^+, \mathrm{Soc}\, V^-),$$

where $\mathrm{Soc}\, V^\sigma$ is the sum of all minimal inner ideals of V contained in V^σ [L2]. The *socle* of a nondegenerate Lie algebra L, $\mathrm{Soc}\, L$, is defined as the sum of all minimal inner ideals of L [DFGG1].

(ii) By [L2, Thm. 2] (for Jordan pairs) and [DFGG1, Thm. 2.5] (for Lie algebras), the socle of a nondegenerate Jordan pair or Lie algebra is the direct sum of its simple ideals. Moreover, each simple component of $\mathrm{Soc}\, L$ is either inner simple or contains an abelian minimal inner ideal.

(iii) A Lie algebra L or Jordan pair V is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals.

2.4. (JORDAN SUBQUOTIENTS) Let $B \subseteq V^+$ be an inner ideal of V. Following [LN], the *kernel* of B is the set $\mathrm{Ker}_V\, B = \{y \in V^- \mid Q_B y = 0\}$. Then $(0, \mathrm{Ker}_V\, B)$ is an ideal of the Jordan pair (B, V^-), and the quotient $\mathrm{Sub}_V\, B = (B, V^-)/(0, \mathrm{Ker}_V\, B) = (B, V^-/\,\mathrm{Ker}_V\, B)$ is called the *subquotient* of B. The kernel and the corresponding subquotient of an inner ideal $B \subseteq V^-$ are defined in a similar way.

The analogues of all these results hold for abelian inner ideals of a Lie algebra, if we replace the Jordan triple product $\{x, y, z\}$ by the left double commutator $[[x, y], z]$ as we describe next.

2.5. (LIE SUBQUOTIENTS) Let M be an abelian inner ideal of L.

(i) The *kernel* of M is the set $\operatorname{Ker}_L M := \{y \in L \mid [M, [M, y]] = 0\}$.

(ii) The pair of Φ-modules $\operatorname{Sub}_L M := (M, L/\operatorname{Ker}_L M)$ with the triple products given by

$$\begin{aligned} \{m, \overline{a}, n\} &:= [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L \\ \{\overline{a}, m, \overline{b}\} &:= \overline{[[a, m], b]} \quad \text{for every } m \in M \text{ and } a, b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\operatorname{Ker}_L M$, is a Jordan pair called the *subquotient of* M [FGGN, Lem. 3.2].

(iii) A Φ-submodule B of M is an inner ideal of L if and only if it is an inner ideal of $\operatorname{Sub}_L M$ [FGGN, 3.5 (i)].

2.6. (GRADINGS) (See [FGGN, Prop. 3.3].) Let $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ be a $(2n+1)$-grading. Then L_n and L_{-n} are abelian inner ideals. Moreover, if L is nondegenerate, then $\operatorname{Sub}_L L_n$ is isomorphic to the Jordan pair (L_n, L_{-n}); the triple products are given by the left double commutator.

Definition 2.7. Let B and B' be abelian inner ideals of Lie algebras L and L' respectively. Then B and B' are said to be *isomorphic* if $\operatorname{Sub}_L B \cong \operatorname{Sub}_{L'} B'$ as Jordan pairs. The same definition makes sense for inner ideals of Jordan pairs.

Note that if $\varphi : L \to L'$ is an isomorphism of Lie algebras and $\varphi(B) = B'$, then B and B' are isomorphic in the above sense, but the converse is not true. For instance, the Lie algebra of type E_7 contains two abelian inner ideals of dimension 5 which are not conjugate under any isomorphism, but whose respective subquotients are isomorphic [DFGG2]. A similar phenomenon happens for the Jordan algebra of Albert type [M, p. 457].

Lemma 2.8. *Let B and C be abelian inner ideals of a Lie algebra L with $C \subset B$, and put $S := \operatorname{Sub}_L B$. Then the Jordan pairs $\operatorname{Sub}_L C$ and $\operatorname{Sub}_S C$ are isomorphic.*

Proof. For $\overline{x} \in L/\operatorname{Ker}_L B$, $\overline{x} \in \operatorname{Ker}_S C$ if and only if $x \in \operatorname{Ker}_L C$. Hence the identity mapping on L induces a linear isomorphism φ of

$L/\operatorname{Ker}_L C$ onto $S^-/\operatorname{Ker}_S C$. Then $(\mathrm{Id}_C, \varphi) : \operatorname{Sub}_L C \to \operatorname{Sub}_S C$ is the required isomorphism. $\square$

This lemma together with **2.5** (iii) reduces the classification, up to isomorphism, of the abelian inner ideals of a Lie algebra L to that of the inner ideals of the subquotients of the maximal abelian inner ideals of L.

3. Isotropic idempotents

3.1. In this paper, our primary focus is on Lie algebras of the form

$$L = [K, K]/Z \cap [K, K],$$

where $K = \operatorname{Skew}(R, *)$ is the set of skew-symmetric elements of a simple associative ring R of characteristic $\neq 2, 3$ with involution $*$, center Z, and minimal one-sided ideals (in particular, we consider the case when R is an Artinian ring).

3.2. Recall that a simple ring R with involution $*$ and minimal one-sided ideals can be realized as the ring $\mathcal{F}(X)$ of all finite rank linear operators $a : X \to X$ on a left vector space X over a division ring with involution $(\Delta, -)$, where the involution $*$ is given by the adjoint $a^* : X \to X$ ($(ax, y) = (x, a^*y)$, for all $x, y \in X$) with respect to a nondegenerate Hermitian or skew-Hermitian form $(,)$, ($(x, y) = \epsilon\overline{(y, x)}, \epsilon = \pm 1$) [BMM].

Note that $R = \mathcal{F}(X)$ is Artinian if and only if X is finite-dimensional over Δ. In this case, R is the complete ring $\operatorname{End}_\Delta X$ of linear transformations of the vector space X.

3.3. Let X be a left vector space over $(\Delta, -)$ endowed with a nondegenerate Hermitian or skew-Hermitian form $(,)$. Given $x, y \in X$, write y^*x to denote the linear operator on X defined by $y^*x(x') = (x', y)x$ for all $x' \in X$.

(i) $(y^*x)^* = \epsilon x^*y$ and therefore $y^*x \in \mathcal{F}(X)$. In fact, any $a \in \mathcal{F}(X)$ can be written as $a = \sum_{j=1}^n y_j^* x_j$, where both the y_j and the x_j are linearly independent.

(ii) $(y^*x)(z^*w) = z^*(w, y)x$ for all $x, y, z, w \in X$.

(iii) The operator defined by $[x,y] := x^*y - \epsilon y^*x$, $x, y \in X$, belongs to $K := \mathrm{Skew}(\mathcal{F}(X), *)$, and it will be called a *skew-trace*. If V and W are subspaces of X, by $[V, W]$ we will mean the additive span of the skew-traces $[v, w]$, $v \in V$, $w \in W$. With this convention, $K = [X, X]$.

3.4. Recall that if there exists $0 \neq \xi \in \mathrm{Skew}(\Delta, -)$, the involution $-$ on Δ can be replaced by $\widetilde{\ }$, defined as $\widetilde{\alpha} = \xi^{-1}\overline{\alpha}\xi$ for all $\alpha \in \Delta$, and the Hermitian form (respectively the skew-Hermitian form) $(\,,)$ over $(\Delta, -)$ can be replaced by $(\,,)^\xi$, where $(x,y)^\xi := (x,y)\xi$ is a skew-Hermitian form (respectively Hermitian form) over $(\Delta, \widetilde{\ })$, without changing the adjoint involution [K, 1.13 (a)]. So, when working with Lie algebras of skew-symmetric operators, we can consider two types of inner products: symmetric, that is, Δ is a field with the identity map as the involution (this is the case when $\mathrm{Skew}(\Delta, -) = 0$ and $(\,,)$ is Hermitian), and skew-Hermitian (in the rest of situations), after possibly changing the involution in Δ.

Assume that Δ is a field $\mathbb{F}$ with the identity map as the involution and X is an $\mathbb{F}$-vector space of dimension (possibly infinite) greater than 4. If $(\,,)$ is symmetric (respectively, skew-symmetric), then $\mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ is a nondegenerate central simple Lie algebra over $\mathbb{F}$, called the *finitary orthogonal algebra* $\mathfrak{fo}(X, (\,,))$ (respectively, the *finitary symplectic algebra* $\mathfrak{fsp}(X, (\,,))$ (see [B] for more details).

3.5. Let $e = \sum_{j=1}^n y_j^* x_j$, where both the y_j and the x_j are linearly independent. Then e is an idempotent if, and only if, the subsets $\{x_i\}$, $\{y_j\}$ are *dual*, i.e., $(x_i, y_j) = \delta_{ij}$ for all x_i, y_j.

3.6. For an idempotent $e = \sum_{j=1}^n y_j^* x_j$ as above, the following conditions are equivalent: (i) $e^*e = 0$, (ii) eX is a totally isotropic subspace. An idempotent e satisfying these equivalent conditions is called *isotropic*. If both e and e^* are isotropic, we say that e is a $*$-*orthogonal idempotent*.

Lemma 3.7. *Let $L = [K,K]/Z \cap [K,K]$, where Z is the center of $R = \mathcal{F}(X)$, and let $e = \sum_{j=1}^{n} y_j^* x_j$ be an isotropic idempotent of R. Then*

(i) *$eKe^* = [eX, eX]$ is an abelian inner ideal of L, and*

(ii) *there exists a $*$-orthogonal idempotent $f \in R$ such that $eKe^* = fKf^*$. In fact, $f = \sum_{j=1}^{n} z_j^* x_j$, where the pairs $\{x_j, z_j\}$ span pairwise hyperbolic planes, so* $\operatorname{rank} e = \operatorname{rank} f$. *Moreover, if R is Artinian, then* $\operatorname{rank} f \leq r$, *where r is the Witt index of $(,)$.*

Proof. (i) This follows from [B1, Thm. 5.5] and [FGG2, (11)].

(ii) Since e is isotropic, $e = \sum_{j=1}^{n} y_j^* x_j$, where the x_j are linearly independent and they span a totally isotropic subspace. Using [K, 1-13(h)], which also works in the skew-Hermitian case, we can construct a sequence $z_1, ..., z_n$ of vectors of X such that $(x_i, z_j) = \delta_{ij}$ and $(z_i, z_j) = 0$ for all i, j. Then $f = \sum_{j=1}^{n} z_j^* x_j$ is the required $*$-orthogonal idempotent. □

4. Maximal abelian inner ideals

Let R be a simple Artinian ring with involution, realized as in the previous section as the complete ring $R = \operatorname{End}_\Delta X$ of linear transformations on a finite-dimensional vector space X with a symmetric or skew-Hermitian form $(,)$. Assume further that the division ring Δ has characteristic $\neq 2, 3$ and that $\dim_Z R > 16$. (Note that the center of R is given by $Z = Z(\Delta)\,\mathrm{Id}_X$, where $Z(\Delta)$ is the center of Δ.) Then $L = [K,K]/Z \cap [K,K]$ is a nondegenerate central simple Lie algebra over the set $\mathrm{Sym}(Z(\Delta), -)$ of symmetric elements of $Z(\Delta)$ with respect to the involution $-$ (c.f. [DFGG1, Lem. 4.9]).

4.1. Let B be a proper (equivalently abelian, by [B1, Thm. 4.2]) inner ideal of the Lie algebra $L = [K,K]/Z \cap [K,K]$. Theorem 5.5 of [B1] states that one of two possibilities holds:

(i) $B = eKe^*$ for some isotropic idempotent e (in this case $a^2 = 0$ for each $a \in B$), or

(ii) Δ is a field, say $\mathbb{F}$, with the identity map as the involution, and there is a basis $\{x_1, ..., x_n\}$ of X such that B is the $\mathbb{F}$-span of the matrix units $e_{1j} - e_{j2}$, $j \geq 3$ with respect to this basis (in this case $a^2 \neq 0$ for some $a \in B$); equivalently, $B = [x, H^\perp]$, where H is a hyperbolic plane of the inner product space X and x is a nonzero isotropic vector of H (c.f. [FGG2, Prop. 3.8]).

However, there do exist proper inner ideals B in L such that $a^2 = 0$ for each $a \in B$, but which cannot be written in the form eKe^* for any isotropic idempotent. This case was omitted from the statement and proof of Theorem 5.5 of [B1]. The reason for this omission occurs in lines 6 and 7 of [B1, p. 583], where it is asserted that $V[K, K]V \subseteq V$. The correct statement is $\{V, [K, K], V\} \subseteq V$, and this leads to additional cases which must be considered.

Example 4.2. Let $L = \mathfrak{o}(8, \mathbb{F})$ be a split simple Lie algebra of type D_4 over a field $\mathbb{F}$ of characteristic not 2. Thus, in the ring $R = \mathrm{End}_{\mathbb{F}}(\mathbb{F}^8)$ of linear transformations on the vector space $\mathbb{F}^8$, L is the subspace of transformations which are skew-symmetric relative to a nondegenerate symmetric bilinear form on $\mathbb{F}^8$, so $L = K = \mathrm{Skew}(R, *)$ in this case. Viewing L as 8×8 matrices over $\mathbb{F}$ as in [J, p. 141]), let B be the 2-dimensional subspace of L spanned by the matrices $e_{16} - e_{25}$, $e_{17} - e_{35}$, where the e_{ij} are matrix units. Then B is an inner ideal of L such that $a^2 = 0$ for every $a \in B$, but B is not of the form eKe^* for any isotropic idempotent e. Indeed, by Lemma 3.7 (i), $eKe^* = [eX, eX]$ (where $X = \mathbb{F}^8$) cannot have dimension 2 for any isotropic idempotent e.

Nevertheless, we can enlarge B to get an inner ideal of $L = \mathfrak{o}(8, \mathbb{F})$ of type eKe^*. Let $C := B + \mathbb{F}(e_{27} - e_{36})$. Then $C = eKe^*$ for $e = e_{11} + e_{22} + e_{33}$, ($e^* = e_{55} + e_{66} + e_{77}$).

The proof of [B1, Thm. 5.5] actually proves the following result.

Proposition 4.3. *Let B be a proper (equivalently, abelian) inner ideal of $L = [K, K]/Z \cap [K, K]$.*

(i) *If $a^2 \neq 0$ for some $a \in B$, then $(,)$ is symmetric and $B = [x, H^\perp]$, where H is a hyperbolic plane of X and x is a nonzero isotropic vector of H. Moreover, B is a maximal abelian inner ideal* (see the proof of [FGG2, Prop. 3.8]).

(ii) *If $a^2 = 0$ for each $a \in B$, then $B \subset eKe^* = [eX, eX]$ for some $*$-orthogonal idempotent e. Moreover, eKe^* is a maximal abelian inner ideal if and only if* rank e *is equal to the Witt index of* $(,)$ (see Lemma 3.7 (ii)).

The information provided by this proposition is actually sufficient to determine all the inner ideals of L, by the relationship between abelian inner ideals and Jordan pairs, via the notion of subquotient.

Proposition 4.4. *Let M be an maximal abelian inner ideal of the Lie algebra $L = [K, K]/Z \cap [K, K]$.*

(i) *If $M = [x, H^\perp]$, where H is a hyperbolic plane of the inner product space X and x is a nonzero isotropic vector of H, then* $\mathrm{Sub}_L\, M$ *is isomorphic to the Clifford pair $(H^\perp, H^\perp)$, defined by the restriction of* $(,)$ *to $H^\perp$.*

(ii) *Suppose that $M = eKe^*$, where e is a $*$-orthogonal idempotent of R. If* $(,)$ *is skew-Hermitian (respectively, symmetric), then* $\mathrm{Sub}_L\, M$ *is isomorphic to the Jordan pair of Hermitian matrices $(H_r(\Delta, -), H_r(\Delta, -))$ (respectively,* $\mathrm{Sub}_L\, M$ *is isomorphic to the Jordan pair of alternating matrices $(A_r(\mathbb{F}), A_r(\mathbb{F}))$, where r is the Witt index in both cases.*

Proof. (i) In this case, L is the finitary orthogonal algebra $\mathfrak{fo}(X, (,))$, so it can be realized as the Tits-Kantor-Koecher algebra of the Clifford pair $(H^\perp, H^\perp)$ by [FGG1, 5.11] (see also [FGG1, 5.7] for the definition of a Clifford pair). Moreover, extending x to a hyperbolic basis $\{x, y\}$ of H, we can verify (using formula (12) of [FGG2]) that the pair of mappings $(z \mapsto [x, z],\ v \mapsto [y, v])$ defines an isomorphism of the Clifford Jordan pair $(H^\perp, H^\perp)$ onto the Jordan pair $([x, H^\perp], [y, H^\perp])$, which is isomorphic to $\mathrm{Sub}_L\, M$ by **2.6**.

(ii) The $*$-orthogonal idempotent e induces a 5-grading $R = R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2$ such that $R_2 = eRe^*$ and $R_{-2} = e^*Re$, and a 5-grading in L with $L_2 = eKe^*$ and $L_{-2} = e^*Ke$, so by **2.6**, $\mathrm{Sub}_L\, eKe^* \cong (eKe^*, e^*Ke)$. By Lemma 3.7 (iii), $e = \sum_{i=1}^r y_i^* x_i$, where the pairs $\{x_i, y_i\}$ span pairwise orthogonal hyperbolic planes H_i, and r is the Witt index of $(,)$. Then $v := \sum_{i=1}^r y_i^* y_i \in e^*Re$ is an invertible element of the associative pair (eRe^*, e^*Re) with inverse $u = \epsilon \sum_{i=1}^{r} x_i^* x_i \in eRe^*$, ($\epsilon = \pm$ according to whether $(,)$ is symmetric or skew-Hermitian). Hence (eRe^*, e^*Re) is isomorphic to the associative pair (S, S), where S is the unital ring with unit element u defined on the abelian group $(eRe^*, +)$ by the product $a \cdot b := avb$, for all $a, b \in eRe^*$. Moreover, since $v^* = \epsilon v$, the mapping $\star := \epsilon *$ defines an involution on the ring S such that $eKe^* = \mathrm{Skew}(eRe^*, *) = \mathrm{Sym}(S, \star)$ if $(,)$ is skew-Hermitian, and $eKe^* = \mathrm{Skew}(eRe^*, *) = \mathrm{Skew}(S, \star)$ if $(,)$ is symmetric. Then we have the Jordan pair isomorphisms $(eKe^*, e^*Ke) \cong (\mathrm{Sym}(S, \star), \mathrm{Sym}(S, \star)) \cong H_r(\Delta, -), H_r(\Delta, -))$ if $(,)$ is skew-Hermitian, and $(eKe^*, e^*Ke) \cong (\mathrm{Skew}(S, \star), \mathrm{Skew}(S, \star)) \cong (A_r(\Delta), A_r(\Delta))$ with Δ a field, $\mathbb{F}$, if $(,)$ is symmetric. $\square$

5. Point spaces

5.1. POINT SPACES OF JORDAN PAIRS

Assume here that our ring of scalars Φ is a field $\mathbb{F}$.

Let V be a Jordan pair. A subspace $P \subseteq V^\sigma$, $\sigma = \pm$, is called a *point space* if $Q_x V^{-\sigma} = \mathbb{F}x$ for any nonzero $x \in P$. Note that any subspace of P is also a point space, and $\mathbb{F}x$ is a minimal inner ideal of V for any nonzero $x \in P$.

5.2. Let V^+ be a left module and V^- be a right module over an $\mathbb{F}$-algebra R, and let $\langle , \rangle : V^+ \times V^- \to R$ be an R-bilinear form. Then $V = (V^+, V^-)$ is a Jordan pair over $\mathbb{F}$ with $Q_x y = \langle x, y\rangle x$ and $Q_y x = y\langle x, y\rangle$, for $x \in V^+$, $y \in V^-$.

Example 5.3. Let $V = (V^+, V^-)$, the Jordan pair defined by a bilinear form $\langle , \rangle$ over an $\mathbb{F}$-algebra R. Then $V^\pm$ are point spaces if and only $\langle , \rangle$ is nondegenerate and $R = \mathbb{F}$ (in particular, the subspace $M_{1\times r}(\mathbb{F})$ is a point space of the Jordan pair $(M_{1\times r}(\mathbb{F}), M_{r\times 1}(\mathbb{F}))$ of row matrices and column matrices). As the next theorem indicates, every point space has this form when V^+ is finite-dimensional.

Theorem 5.4. *Let $V = (V^+, V^-)$ be a nondegenerate Jordan pair. If V^+ is a point space, then V is the Jordan pair defined by a nondegenerate bilinear form. Moreover, if V^+ has finite dimension, say r, then V is the Jordan pair $(M_{1\times r}(\mathbb{F}), M_{r\times 1}(\mathbb{F}))$.*

Proof. For any nonzero $x \in V^+$, $Q_x V^- = \mathbb{F}x$ is a minimal inner ideal of V, so $\operatorname{Soc} V^+ = V^+$. Let (e^+, e^-) be a nonzero idempotent of V and let $V = V_2 \oplus V_1 \oplus V_0$ be the corresponding Peirce decomposition. Then V_0 is a nondegenerate Jordan pair with V_0^+ being a point space. We claim that $V_0^+ = 0$, and hence $V_0^- = 0$ too, by the nondegeneracy of V_0. If $V_0^+ \neq 0$, then V_0 would contain a nonzero idempotent $f = (f^+, f^-)$, and hence $Q_{e^+}V^- \cap Q_{f^+}V^- = 0$ by [L1, 5.4], which leads to the contradiction:

$$\mathbb{F}(e^+ + f^+) = Q_{e^+ + f^+}V^- = Q_{e^+}V^- \oplus Q_{f^+}V^- = \mathbb{F}e^+ \oplus \mathbb{F}f^+.$$

Therefore, $V = V_2 \oplus V_1$ is a nondegenerate Jordan pair coinciding with its socle. Further, V has capacity one (and hence it is simple) and two additional properties: (i) V has no invertible elements unless $V = (\mathbb{F}, \mathbb{F})$, and (ii) the coordinate system of V is the field $\mathbb{F}$ itself. By the classification of Jordan pairs of finite capacity [L1, 12.12], V is the Jordan pair defined by a nondegenerate bilinear form $\langle , \rangle : V^+ \times V^- \to \mathbb{F}$ (see [LN, 5.11] for a related result). Finally, if V^+ has finite dimension, say r, then V^- is canonically isomorphic to the dual of V^+. Via the canonical isomorphism, we can identify the Jordan pair V with $(M_{1\times r}(\mathbb{F}), M_{r\times 1}(\mathbb{F}))$. □

5.5. POINT SPACES IN LIE ALGEBRAS

For the remainder of this section, L will denote a Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2, 3$.

Definition 5.6. A subspace P of L will be called a *point space* if $[P, P] = 0$ and if $\operatorname{ad}_x^2 L = \mathbb{F}x$ for every nonzero element $x \in P$.

5.7. Note that the following hold:

(i) If P is a point space, then P is an abelian inner ideal of L, P is a point space of the Jordan pair $\operatorname{Sub}_L P$, and any subspace of P is also a point space.

(ii) Let $[x, H^\perp]$ be as in Proposition 4.4 (i). An $\mathbb{F}$-subspace P of $[x, H^\perp]$ is a point space if, and only if, $P = [x, S]$, where S is a totally isotropic subspace of $H^\perp$.

(iii) The abelian inner ideals B and C given in Example 4.2 are point spaces.

(iv) In [DFGG2], it is shown that the classical Lie algebras of types A_n, B_{n+1}, and D_{n+1} contain point spaces of dimension n, hence, by (i), point spaces P_i of dimension $i = 1, \ldots, n$. In fact, $\bigoplus_{j=1}^{n} \mathbb{F}e_{j,n+1}$ is a point space of $\mathrm{A}_n = \mathfrak{sl}_{n+1}(\mathbb{F})$. On the contrary, any nonzero point space of a classical Lie algebras of type C_n is one-dimensional.

Corollary 5.8. *Assume that L is nondegenerate, and let P be a point space of L. Then $\mathrm{Sub}_L\, P$ is the Jordan pair defined by a nondegenerate bilinear form over $\mathbb{F}$. In particular, if P has finite dimension, say r, then $\mathrm{Sub}_L\, P \cong (M_{1\times r}(\mathbb{F}), M_{r\times 1}(\mathbb{F}))$.*

Proof. This follows from Theorem 5.4, since P is a point space of the nondegenerate Jordan pair $\mathrm{Sub}_L\, P = (P, L/\operatorname{Ker} P)$ by [FGGN, 3.5 (iii)]. □

5.9. POINT SPACES OF FINITARY ORTHOGONAL ALGEBRAS

In this subsection, we describe the point spaces of a finitary orthogonal algebra $\mathfrak{fo}(X, (\,,\,))$ over an $\mathbb{F}$-vector space X of (possibly infinite) dimension greater than 4 (see **3.4** for definitions). This is by no means a restriction since, as will be seen later, in a Lie algebra $L = [K, K]/Z \cap [K, K]$ coming from a simple ring R with involution $*$ and minimal one-sided ideals (as in **3.1**), point spaces of dimension greater than 1 occur only when $*$ is the adjoint involution of a symmetric bilinear form, i.e., when L is a finitary orthogonal algebra.

The next technical result will used in what follows.

Lemma 5.10. *Any nonzero $b \in \mathfrak{fo}(X, (\,,\,))$ has an expression of the form $b = \sum_{k=1}^{n} [x_{2k-1}, x_{2k}]$, where $\{x_1, ..., x_{2n}\}$ is linearly independent.*

Proof. This is straightforward to show. □

Lemma 5.11. *An $\mathbb{F}$-subspace P of $L = \mathfrak{fo}(X, (\,,\,))$ is a point space if, and only if, all of its elements have rank ≤ 2 and square 0; equivalently, for any nonzero element $a \in P$, $a = [x_1, x_2]$, where $\mathbb{F}x_1 + \mathbb{F}x_2$ is a totally isotropic two-dimensional subspace.*

Proof. Let P be a point space of $\mathfrak{fo}(X, (\,,\,))$. For any $a \in P$, $a^2 = 0$ (otherwise, by [FGG2, Prop. 3.8], P would be of the form $[x, H^{\perp}]$, which is not a point space by **5.7** (ii)). By Lemma 5.10, the rank of every element of L is even. Suppose that P contains an element a whose rank is ≥ 4, i.e., $a = \sum_{k=1}^{n}[x_{2k-1}, x_{2k}] \in P$ where $n \geq 2$ and the x_i are linearly independent. Taking $\{y_i\}$ dual to $\{x_i\}$, we obtain by **3.3** (ii) that $b = [y_2, y_1]$ is an element of L satisfying $1/2[[a, b], a] = aba = [x_1, x_2] \notin \mathbb{F}a$, which is a contradiction. Conversely, if every nonzero element a of P has rank 2 and $a^2 = 0$, then P is a point space. Indeed, let $a = [x_1, x_2] \in P$ and $c \in L$. Then

$$\begin{aligned}
\frac{1}{2}[[a, c], a] &= aca = [x_1, x_2]c[x_1, x_2] = (x_1^* x_2 - x_2^* x_1)c(x_1^* x_2 - x_2^* x_1) \\
&= (x_1^* x_2 - x_2^* x_1)(x_1^* c x_2 - x_2^* c x_1) \\
&= x_1^*(cx_2, x_1)x_2 - x_2^*(cx_1, x_1)x_2 \\
&\qquad -x_1^*(cx_2, x_2)x_1 + x_2^*(cx_1, x_2)x_1 = (cx_2, x_1)a \in \mathbb{F}a,
\end{aligned}$$

since $(cx, x) = 0$ for any $x \in X$ because c is skew-symmetric and $(,)$ is symmetric. Therefore, we only need to show that P is abelian. First note that if $\{x_1, x_2\} \subset X$ is linearly independent,

$$[x_1, x_2]^2 = x_1^*(x_2, x_1)x_2 - x_2^*(x_1, x_1)x_2 - x_1^*(x_2, x_2)x_1 + x_2^*(x_1, x_2)x_1 = 0$$

if, and only if, $(x_1, x_1) = (x_1, x_2) = (x_2, x_2) = 0$. Now let $a = [x_1, x_2]$ and $b = [x_3, x_4]$ be two nonzero elements of P. Since $a+b$ has rank ≤ 2, $\{x_1, x_2, x_3, x_4\}$ is linearly dependent, say $x_4 = \alpha x_1 + \beta x_2 + \gamma x_3$. Then $b = [x_3, \alpha x_1 + \beta x_2]$ where x_1, x_2 and x_3 satisfy the following orthogonal relations:

$$(x_i, x_i) = 0, \;\text{ for }\; i = 1, 2, 3,\; (x_1, x_2) = 0, \;\text{ and }\; (x_3, \alpha x_1 + \beta x_2) = 0.$$

Now we have

$$\begin{aligned}
[b, a] &= ba - ab = b[x_1, x_2] - [x_1, x_2]b \\
&= b(x_1^* x_2 - x_2^* x_1) - (x_1^* x_2 - x_2^* x_1)b \\
&= x_1^*(bx_2) - x_2^*(bx_1) + (bx_1)^* x_2 - (bx_2)^* x_1 \\
&= [x_1, bx_2] + [bx_1, x_2].
\end{aligned}$$

But

$$\begin{aligned} b &= [x_3, \alpha x_1 + \beta x_2] = \alpha[x_3, x_1] + \beta[x_3, x_2] \\ &= \alpha(x_3^* x_1 - x_1^* x_3) + \beta(x_3^* x_2 - x_2^* x_3) \end{aligned}$$

implies by the orthogonal relations,

$$bx_1 = \alpha(x_1, x_3)x_1 + \beta(x_1, x_3)x_2 \text{ and } bx_2 = \alpha(x_2, x_3)x_1 + \beta(x_2, x_3)x_2.$$

Hence,

$$[b, a] = [x_1, bx_2] + [bx_1, x_2] = \beta(x_2, x_3)[x_1, x_2] + \alpha(x_1, x_3)[x_1, x_2] = 0$$

by the orthogonal relations again. □

5.12. THE TYPE OF A POINT SPACE IN $\mathfrak{fo}(X, (,)))$

A point space P of $\mathfrak{fo}(X, (,))$ is said to be of *type 1* if there exists a nonzero vector, u, in the image of any nonzero $a \in P$. Point spaces which are not of type 1 are said to be of *type 2*. Let S be a totally isotropic subspace of X of dimension greater than 1. If u is a nonzero vector in S, then $P = [u, S]$ is a point space of $\mathfrak{fo}(X, (,))$ by Lemma 5.11, and P is of type 1 since u is in the image of any nonzero element of P. As we see next, every point space of type 1 in $\mathfrak{fo}(X, (,))$ has this form.

Proposition 5.13. *Every point space P of type 1 of the finitary orthogonal algebra $\mathfrak{fo}(X, (,))$ is of the form $[u, S]$, where S is a totally isotropic subspace of X of dimension > 1 and u is a nonzero vector of S. Moreover, S is uniquely determined by P, and if* $\dim S > 2$, *$[u, S] = [v, S]$ implies $v = \alpha u$ for some $\alpha \in \mathbb{F}$.*

Proof. Let u be a nonzero vector which lies in the image of every nonzero element of P. Set $S := \{x \in X \mid [u, x] \in P\}$. Clearly, S is a subspace of X and $[u, S] \subseteq P$. The reverse inclusion also holds: let $a = [x_1, x_2] \in P$, where $\{x_1, x_2\}$ linearly independent. Since $u \in \mathrm{Im}(a)$, $u = \alpha_1 x_1 + \alpha_2 x_2$, and hence $[u, x_2] = [\alpha_1 x_1, x_2] = \alpha_1 a \in P$ implies that $x_2 \in S$ (the same is true for x_1). Moreover, one of the α_i, say α_1, is different from 0. Hence $[u, x_2] = [\alpha_1 x_1, x_2]$ implies $a = [x_1, x_2] = [u, \alpha_1^{-1} x_2] \in [u, S]$. For the last part, note that $S = \{ax \mid a \in P, x \in X\}$, and therefore S is uniquely determined by

P. Suppose now that $P = [v, S]$ for another nonzero vector v of S. If $\{u, v\}$ is linearly independent, then for any $s \in S$ and $a = [u, s]$, $v \in \mathrm{Im}(a)$ implies $\{v, s, u\}$ is linearly dependent, i.e., $\dim S = 2$. □

Lemma 5.14. *Let P be a point space of $L = \mathfrak{fo}(X, (,))$. If P contains three elements $a = [x_1, x_2]$, $b = [x_1, x_3]$ and $c = [x_2, x_3]$ such that $\{x_1, x_2, x_3\}$ is linearly independent, then $P = \mathbb{F}a \oplus \mathbb{F}b \oplus \mathbb{F}c$; equivalently, $P = eKe^*$ where $e \in \mathcal{F}(X)$ is an isotropic idempotent of rank 3. Moreover, P is of type 2 and a maximal point space.*

Proof. Let N be a point space of L containing P, and let $[u, v]$ be any nonzero element of N. By Lemma 5.11, each of the sets $\{u, v, x_1, x_2\}$, $\{u, v, x_1, x_3\}$ and $\{u, v, x_2, x_3\}$ is linearly dependent. Hence $\{u, v\} \subset S$, the linear span of $\{x_1, x_2, x_3\}$. Therefore, $N = \mathbb{F}a \oplus \mathbb{F}b \oplus \mathbb{F}c = P$. Moreover, P is of type 2 since $(\mathbb{F}x_1 \oplus \mathbb{F}x_2) \cap (\mathbb{F}x_1 \oplus \mathbb{F}x_3) \cap (\mathbb{F}x_2 \oplus \mathbb{F}x_3) = 0$. Finally, $\mathbb{F}a \oplus \mathbb{F}b \oplus \mathbb{F}c = [S, S] = eKe^*$ for any idempotent $e \in \mathcal{F}(X)$ such that $eX = S$. □

Corollary 5.15. *Suppose that X contains a totally isotropic subspace of dimension 5. Then $\mathfrak{fo}(X, (,))$ has two point spaces of dimension 3 which are not conjugate under any isomorphism of $\mathfrak{fo}(X, (,))$.*

Proof. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a linearly independent subset of X whose linear span is totally isotropic, and let S denote the linear span of x_2, x_3, x_4. Then $P = [S, S]$ and $N = [x_1, \mathbb{F}x_1 + S]$ are point spaces of dimension 3. Moreover, P is maximal by Lemma 5.14, but N is contained in the 4-dimensional point space $[x_1, \mathbb{F}x_1 + S + \mathbb{F}x_5]$. □

Theorem 5.16. *Let P be a point space of $L = \mathfrak{fo}(X, (,))$. Then either P is of type 1, or $P = eKe^*$ for some isotropic idempotent e of rank 3 and P is a point space of type 2.*

Proof. If $\dim P = 1$, then $P = \mathbb{F}[x_1, x_2]$ is of type 1. The same is true if $\dim P = 2$. Indeed, $P = \mathbb{F}a \oplus \mathbb{F}b$ implies, by Lemma 5.11, $a = [x_1, x_2]$ and $b = [x_3, \alpha_1 x_1 + \alpha_2 x_2]$. Hence $P = [u, S]$, where $u = \alpha_1 x_1 + \alpha_2 x_2$ and $S = \mathbb{F}x_1 + \mathbb{F}x_2 + \mathbb{F}x_3$.

Suppose then that $\dim P \geq 3$, and let (a, b, c) be any triple of linearly independent elements of P. By Lemma 5.11, we may write $a = [x_1, x_2]$,

$b = [x_3, y_1]$ and $c = [x_4, y_2]$, where $\{y_1, y_2\}$ is contained in the linear span of $\{x_1, x_2\}$, each of the sets $\{x_1, x_2, x_3\}$ and $\{x_1, x_2, x_4\}$ is linearly independent, and $\{x_3, x_4, y_1, y_2\}$ is linearly dependent.

If for some a, b, c as above we have that $\{y_1, y_2\}$ is linearly independent, then we can write $x_4 = \alpha x_3 + \beta y_1 + \gamma y_2$ with $\alpha \neq 0$, and hence $[x_3, y_2] \in P$. Since $[y_1, y_2]$ and $[x_3, y_1]$ also belong to P, we have by Lemma 5.14 that $P = eKe^*$ for some isotropic idempotent e of rank 3.

If on the contrary, for fixed $a = [x_1, x_2]$ we have that $\{y_1, y_2\}$ is linearly dependent for any choice of b, c, then P is a point space of type 1. □

5.17. GRADINGS INDUCED BY FINITE-DIMENSIONAL POINT SPACES

It was shown in [FGGN, Cor. 5.2] that each abelian inner ideal B of finite length in a nondegenerate Lie algebra L induces a finite $(2n+1)$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$.

In this subsection we construct the grading induced by a finite-dimensional point space P of the finitary orthogonal algebra $\mathfrak{fo}(X, (,))$. Since gradings induced by inner ideals of the form eKe^*, where e is an isotropic idempotent, were described in [FGGN, Prop. 6.7], we can assume that P is of type 1, i.e, $P = [x, S]$, where S is a finite-dimensional totally isotropic subspace of X and $x \in S$.

Proposition 5.18. *Let $P = [x, S]$ be a finite-dimensional point space of type 1 of the finitary algebra $\mathfrak{fo}(X, (,))$. Then L has a 7-grading with $L_3 = P$.*

Proof. Let $\{x = x_1, x_2, \ldots, x_n\}$ be a basis of S. As noted in the proof of Lemma 3.7 (ii), we can construct a sequence $y_1, y_2, \ldots, y_n$ of vectors of X such that $(x_i, y_j) = \delta_{ij}$ and $(y_i, y_j) = 0$ for all i, j. Set $U = \mathbb{F}x_2 \oplus \cdots \oplus \mathbb{F}x_n$, $V = \mathbb{F}y_2 \oplus \cdots \oplus \mathbb{F}y_n$, and $W = (\mathbb{F}x_1 \oplus U \oplus V \oplus \mathbb{F}y_1)^\perp$. Since $\mathbb{F}x_1 \oplus U \oplus V \oplus \mathbb{F}y_1$ is a nondegenerate subspace of X, $X = \mathbb{F}y_1 \oplus U \oplus W \oplus V \oplus \mathbb{F}x_1$. Let e_0, e_1, e_2, e_3, e_4 be the projections of $X = \mathbb{F}y_1 \oplus V \oplus W \oplus \Re U \oplus \mathbb{F}x_1$ onto $\mathbb{F}y_1$, V, W, U, $\mathbb{F}x_1$, respectively. Let $\mathcal{L}(X)$ denote the ring of all linear operators on X having an adjoint with respect to $(,)$. It is easy to see that e_0, e_1, e_2, e_3, e_4 are idempotents in $\mathcal{L}(X)$ with $e_0^* = e_4$, $e_1^* = e_3$ and $e_2^* = e_2$, which induces a 9-grading $R = R_{-4} \oplus R_{-3} \oplus R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2 \oplus R_3 \oplus R_4$ in the simple ring $R = \mathcal{F}(X)$ with $R_k = \sum_{i-j=k} e_i R e_j$ for all k. Moreover, each R_k is invariant under $*$ and $\mathrm{Skew}(R_4, *) = \mathrm{Skew}(R_{-4}, *) = 0$. Hence $L = \mathfrak{fo}(X, (\,,\,)) = \mathrm{Skew}(\mathcal{F}(X), *) = L_{-3} \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus L_3$, where $L_k = \mathrm{Skew}(R_k, *)$, is a 7-grading in L with $L_3 = \mathrm{Skew}(e_4 R e_1 + e_3 R e_0, *) = \{e_4 a e_1 + e_3 b e_0 - e_3 a^* e_0 - e_4 b^* e_1 \mid a, b \in R\} = \{e_4(a - b^*)e_1 - e_3(a^* - b)e_0 \mid a, b \in R\} = \{e_4 c e_1 - (e_4 c e_1)^* \mid c \in R\}$. Take $c = y^* x$, for $x, y \in X$ (any $c \in R$ is a sum of rank one operators of this form). Then $e_4 c e_1 = e_4(y^* x)e_1 = (e_1^* y)^* e_4 x = (e_3 y)^* e_4 x = u^*(\alpha x_1)$, with $\alpha \in \mathbb{F}$ and $u = e_3 y \in U$, implies $e_4 c e_1 - (e_4 c e_1)^* = u^*(\alpha x_1) - (\alpha x_1)^* u = [\alpha u, x_1]$. Therefore, $L_3 = [x_1, U] = [x_1, \mathbb{F} x_1 + U] = [x, S] = P$. $\square$

6. Inner ideal structure of rings with involution

We return to the case that R is a simple Artinian ring of characteristic $\neq 2, 3$ with involution $*$ and center Z such that $\dim_Z R > 16$, realized as the complete ring $\mathrm{End}_\Delta X$ of linear transformations on a vector space X over a division ring with involution $(\Delta, -)$, where the involution $*$ is the adjoint relative to a nondegenerate symmetric or skew-Hermitian form on X as in **3.2**. Also as above, $K = \mathrm{Skew}(R, *)$, the skew-symmetric elements of R relative to $*$. Everything is ready to state and prove the completed version of [B1, Thm. 5.5].

Theorem 6.1. *Let $R = \mathrm{End}_\Delta X$ be a simple Artinian ring of characteristic $\neq 2, 3$ with involution $*$ such that $\dim_Z R > 16$, and let B be a proper (equivalently, abelian) inner ideal of $L = [K, K]/Z \cap [K, K]$.*

- (i) *If there exists $a \in B$ with $a^2 \neq 0$, then $L = \mathfrak{fo}(X, (\,,\,))$ and $B = [x, H^\perp]$, where H is a hyperbolic plane of X and x is a nonzero isotropic vector of H.*
- (ii) *If every $a \in B$ has $a^2 = 0$, then*
  - (ii.1) *if $(\,,\,)$ is skew-Hermitian, $B = gKg^* = [gX, gX]$, where g is an isotropic idempotent of R,*
  - (ii.2) *if $(\,,\,)$ is symmetric, either B is a point space of type 1, or $B = gKg^*$ for some isotropic idempotent of rank greater than 2.*

Moreover, any subspace $[x, H^\perp]$ is a maximal abelian inner ideal, while gKg^* is a maximal abelian inner ideal if and only if $\mathrm{rank}\, g$ *is equal to* *the Witt index of* $(\,,\,)$*. Two point spaces are isomorphic if and only if they have the same dimension; and for any point space* P *of dimension* n*,* $\mathrm{Sub}_L\, P \cong (M_{1\times n}(\mathbb{F}), M_{n\times 1}(\mathbb{F}))$*.*

Proof. By Proposition 4.3 (i), it suffices to consider the case that $a^2 = 0$ for each $a \in B$. Then, by (ii) of Proposition 4.3, $B \subseteq eKe^*$ for a $*$-orthogonal idempotent e. If $(\,,\,)$ is skew-Hermitian (respectively, symmetric), we have by Proposition 4.4 (ii) that $\mathrm{Sub}_L\, eKe^*$ is isomorphic to the Jordan pair of Hermitian matrices $(H_r(\Delta,-), H_r(\Delta,-))$ (respectively, $\mathrm{Sub}_L\, eKe^*$ is isomorphic to the Jordan pair of alternating matrices $(A_r(\mathbb{F}), A_r(\mathbb{F}))$), where r is the Witt index in both cases. Since by **2.5** (iii) the inner ideals of L contained in eKe^* are precisely the inner ideals of $\mathrm{Sub}_L\, eKe^*$ contained in eKe^*, all we must do is to consider the inner ideal structure of the Jordan pairs $(H_r(\Delta,-), H_r(\Delta,-))$ and $(A_r(\mathbb{F}), A_r(\mathbb{F}))$.

Let us assume first that $(\,,\,)$ is skew-Hermitian. Following the notation introduced in the proof of Proposition 4.4, we have the isomorphisms

$$\begin{aligned}\mathrm{Sub}_L\, eKe^* &\cong (eKe^*, e^*Ke) \cong (\mathrm{Sym}(S,\star), \mathrm{Sym}(S,\star))\\ &\cong (H_r(\Delta,-), H_r(\Delta,-)).\end{aligned}$$

By [M, §5, Thm. 2], any inner ideal of the Jordan algebra $J := \mathrm{Sym}(S,\star)$ is of the form $B = f\cdot J\cdot f^\star = (fv)J(vf^\star)$, where f is an idempotent of S and v is invertible. Put $g := fv$. Then $g^2 = (fvf)v = fv = g$, and $g = fv \in Sv = (eRe^*)v$ implies $g^*g = (eg)^*(eg) = g^*(e^*e)g = 0$, so g is an isotropic idempotent of R. Moreover, since $ve = v$ and $v^* = -v$, we have $gKg^* = fvKvf^* = fv(eKe^*)vf^* = fvSvf^* = f\cdot J\cdot f^\star = B$.

Assume now that $(\,,\,)$ is symmetric. As in the previous case, we have the sequence of isomorphisms

$$\begin{aligned}\mathrm{Sub}_L\, eKe^* &\cong (eKe^*, e^*Ke) \cong (\mathrm{Skew}(S,\star), \mathrm{Skew}(S,\star))\\ &\cong (A_r(\mathbb{F}), A_r(\mathbb{F})).\end{aligned}$$

It follows from [N, 3.2 (e)] that every inner B of $(A_r(\mathbb{F}), A_r(\mathbb{F}))$ is of the form $A_s(\mathbb{F})$ for $s \le r$, or its subquotient is covered by a family of collinear idempotents. If the first holds, then $B = f\cdot J\cdot f^\star = (fv)J(vf^\star)$

for some idempotent f of S and some invertible v, and hence, as in the previous skew-Hermitian case, $B = gKg^*$ for an isotropic idempotent g of R. In the second case, B is a point space and therefore it is uniquely determined up to isomorphism by its dimension by Theorem 5.4. Moreover, since by Theorem 5.16 point spaces of type 2 are of the form eKe^* for an isotropic idempotent of rank 3, we may assume that B is of type 1 in (ii). This gives (ii.2) and completes the proof. □

6.2. INNER IDEALS OF THE LIE ALGEBRA $[K, K]$

In order to extend the theorem above to the non-Artinian case, we need to determine the inner ideals of the Lie algebra of $[K, K]$.

Theorem 6.3. *Let $R = \mathrm{End}_\Delta X$ be a simple Artinian ring of characteristic $\neq 2, 3$ with involution $*$ such that $\dim_Z R > 16$; let L be the Lie algebra $[K, K]$ over $\mathrm{Sym}(Z(\Delta), -)$; and let B be a proper inner ideal of L.*

(i) *If the involution is of the first kind, then either* (i.1) $B = gKg^*$, *where g is an isotropic idempotent of R, or* (i.2) $B = [x, H^\perp]$, *where H is a hyperbolic plane of X and x is a nonzero isotropic vector of H, or* (i.3) *B is a type 1 point space of dimension greater than 1. In the last two cases Δ is a field and $L = \mathfrak{fo}(X, (,))$.*

(ii) *If the involution is of the second kind, then either* (ii.1) $B = (Z \cap [K, K]) + gKg^*$, *or* (ii.2) $B = gKg^*$, *where g is an isotropic idempotent of R.*

Proof. Suppose first that $*$ is an involution of the first kind. Then, by [B1, Lem. 4.23], $b^3 = 0$ for any $b \in B$. Hence $B \cap Z = 0$, and B can be regarded as a proper inner ideal of the Lie algebra $[K, K]/Z \cap [K, K]$. Then, by Theorem 6.1, $B = gKg^*$, where g is an isotropic idempotent of R, or Δ is a field $\mathbb{F}$ with the identity as involution, $(,)$ is symmetric and either $B = [x, H^\perp]$ or B is a point space.

Suppose now that $*$ is an involution of the second kind, and set $B' := B + (Z \cap [K, K])$. Then $B'/Z \cap [K, K]$ is a proper inner ideal of $[K, K]/Z \cap [K, K]$ and hence, again by Theorem 6.1, it is of the form gKg^*, where g is an isotropic idempotent of R (the other cases cannot occur because $*$ is of the second kind). Then any $b \in B$ can be written as $b = z + c$ for some $z \in Z \cap [K, K]$ and $c \in gKg^*$. Since c is von Neumann regular and skew-symmetric, there exists $a \in K$ such that $c = cac = c(g^*ag)c$, with $g^*ag = [g^*ag, g] \in [K, K]$. Since c has zero square, $c = c(g^*ag)c = 1/2[[c, g^*ag], c] = 1/2[[b, g^*ag], b] \in B$. Therefore, both c and z belong to B, i.e, $B = (B \cap gKg^*) \oplus (B \cap Z)$. But $B \cap gKg^* = fKf^*$ for some isotropic idempotent f of R by Theorem 6.1, and since $*$ is of the second kind, if $B \cap \mathrm{Skew}(Z, *) = B \cap Z \neq 0$, then $B \cap Z = Z \cap [K, K]$. $\square$

6.4. THE NON-ARTINIAN CASE

Assume now that R is a simple ring with minimal one-sided ideals and involution $*$, i.e., R is the ring $\mathcal{F}(X)$ of all finite rank linear operators $a : X \to X$ on a vector space X over a division ring with involution $(\Delta, -)$ having an adjoint $a^* : X \to X$ with respect to a nondegenerate Hermitian or skew-Hermitian form $(,)$ (see **3.2**). Even in the case when R is not Artinian, equivalently, when X is infinite-dimensional over Δ, R can be still described as a direct limit of simple Artinian algebras R_α with the same type of involution as R. In fact, R is a *strongly local matrix ring* in the following sense: any finite subset of R is contained in an inner ideal of the form eRe for some self-adjoint finite rank idempotent e of R (see [BMM, 4.6.15]). Note that in geometric terms, $eRe \cong \mathcal{F}(V)$, where V is a finite-dimensional nondegenerate subspace of X; in fact, $V = eX$; equivalently, e is the projection on V determined by the decomposition $X = V \oplus V^\perp$. Moreover, $Z(eRe) = Z(\Delta)e$.

6.5. Suppose that R is not Artinian and set $K = \mathrm{Skew}(R, *)$ as usual. Since $Z = 0$, $L = [K, K]$ is a simple nondegenerate Lie algebra. Moreover, L is a direct limit of Lie algebras $L_\alpha = [K_\alpha, K_\alpha]$, with $K_\alpha = \mathrm{Skew}(e_\alpha R e_\alpha, *) = e_\alpha K e_\alpha$ for a self-adjoint finite rank idempotent e_α of R.

Theorem 6.6. *Let $L = [K, K]$ be as above. If B is a proper inner ideal of L, then either*

(i) *$B = [S, S]$ for a totally isotropic subspace S of X of possibly infinite dimension, or*

(ii) *B is a type 1 point space of dimension greater than* 1*, or*

(iii) *$B = [x, H^{\perp}]$, where H is a hyperbolic plane of X and x is a nonzero isotropic vector of H.*

In cases (ii) *and* (iii)*, Δ is necessarily is a field and $L = \mathfrak{fo}(X, (\,,\,))$. Moreover, $B = [S, S]$ as in* (i)*, and B is a nonzero point space of L (over $\mathbb{F} = \mathrm{Sym}(\Delta, -)$) if and only if either $(\,,\,)$ is symmetric and $\dim_{\mathbb{F}} S = 2$ or 3, or $(\,,\,)$ is skew-symmetric and $\dim_{\mathbb{F}} S = 1$.*

Proof. Choose a *directed set* $\{e_\alpha\}$ of self-adjoint idempotents of $R = \mathcal{F}(X)$ of finite rank > 4 ($e_\alpha \le e_\beta \Leftrightarrow e_\alpha R e_\alpha \subset e_\beta R e_\beta \Leftrightarrow e_\alpha X \subset e_\beta X$). Set $Z_\alpha = Z(R_\alpha) = Z(\Delta)e_\alpha$ and $L_\alpha = [K_\alpha, K_\alpha]$ as in **6.5**. If B is a proper inner ideal of L, it follows from [B1, Thm. 4.21] (if the involution $*$ is of the first kind), and from [B1, Thm. 4.26] (if the involution is of the second kind) that B is abelian. Then $B_\alpha = B \cap L_\alpha$ is an abelian inner ideal of L_α. If $a^2 \neq 0$ for some $a \in B$, then we have by [FGG2, Prop. 3.8] that (,) is symmetric and $B = [x, H^{\perp}]$, so we may assume that $a^2 = 0$ for any $a \in B$.

If the involution $*$ of R is of the first kind, then the same is true for the involution of each ring $R_\alpha = e_\alpha R e_\alpha$. Then by Theorem 6.3 we have for each index α that either B_α is a point space (in this case (,) is symmetric), or $B_\alpha = g_\alpha K_\alpha g_\alpha^* = g_\alpha K g_\alpha^*$ for some isotropic idempotent g_α of R_α (of rank > 3 if (,) is symmetric, by Theorem 5.16). If the former holds for all indices α, then B itself is a point space by Lemma 5.11. Suppose on the contrary that for some index α, $B_\alpha = g_\alpha K g_\alpha^*$, where g_α is an isotropic idempotent (of rank > 3 if $(\,,\,)$ is symmetric, c.f. Theorem 5.16). Then for every $e_\beta \ge e_\alpha$, $B_\beta = g_\beta K g_\beta^* = [g_\beta X, g_\beta X]$, for some isotropic idempotent g_β of R, c.f. Lemma 3.7 (i). Since these e_β form a directed set, the same is true for the family of corresponding subspaces $e_\beta X$. Hence $S := \bigcup e_\beta X$ is a totally isotropic subspace of X and $B = [S, S]$.

We claim that if the involution $*$ of R is of the second kind, then for each index α, $B_\alpha = g_\alpha K_\alpha g_\alpha^*$, where g_α is an isotropic idempotent of $R_\alpha = e_\alpha R e_\alpha$, and hence $B = [S, S]$ for some totally isotropic subspace S of X as in the previous case. Otherwise, by Theorem 6.3, there exists an index α such that $B_\alpha = (Z_\alpha \cap [K_\alpha, K_\alpha]) + g_\alpha K_\alpha g_\alpha^*$, with $Z_\alpha \cap [K_\alpha, K_\alpha] \neq$

0. Since $Z_\alpha \cap [K_\alpha, K_\alpha] \subset Z_\alpha \cap K_\alpha = \mathrm{Skew}(Z(\Delta), -)e_\alpha$, there exists a nonzero element $z \in \mathrm{Skew}(Z(\Delta), -)$ such that $ze_\alpha \in B$, but this leads to a contradiction. Indeed, let f be a nonzero self-adjoint finite rank idempotent orthogonal to $e := e_\alpha$ (such an idempotent always exists because X is infinite-dimensional). Since R is simple, we can take $a \in R$ such that $eaf \neq 0$. Then $eaf - fa^*e \in K$ and hence $z(eaf + fa^*e) = [ze, eaf - fa^*e]$ belongs to $[K, K] = L$ and satisfies $\mathrm{ad}^3_{ze} z(eaf + fa^*e) = z^4(eaf - fa^*e) \neq 0$, a contradiction, since $\mathrm{ad}^3_b = 0$ for all b because B is an abelian inner ideal, which proves the claims. Therefore, every abelian inner ideals of L is as in (i), (ii) or (iii).

To prove the final assertions of the theorem, note first that if L contains a nonzero point space, then it contains a one-dimensional point space by 5.7, and hence, by [DFGG1, 4.14], $(,)$ is either symmetric or skew-symmetric. Let $B = [S, S]$ be as in (i), a nonzero point space. If $(,)$ is symmetric, then we have by 5.16 that S has dimension 2 or 3, while if $(,)$ is skew-symmetric, then S is necessarily one-dimensional, i.e, $[S, S] = \mathbb{F}x^*x$ for a nonzero $x \in X$. Suppose on the contrary that $x, y \in S$ are linearly independent, and set $a = [x, y] = x^*y + y^*x$. Since a has zero square, $[[a, b], a] = 2aba$ for any $b \in L$. Then

$$aba = (x^*y + y^*x)b(x^*y + y^*x) = (by, x)[x, y] + (bx, x)y^*y + (by, y)x^*x.$$

Taking $b = z^*z$, with $z \in X$ satisfying $(z, x) = 1$, and therefore, $(bx, x) = (z, x) = 1$, we get that $[[a, b], a]$ and a are linearly independent, which is a contradiction. This completes the proof. $\square$

Remarks 6.7. The point spaces in (ii) of Theorem 6.6 were missing from the original description of the inner ideals of the finitary simple Lie algebras in [FGG2, Prop. 3.6 (iv)] and in its later extension in [FGG3, Prop. 4.7] to the Lie algebras $L = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$, where X is an infinite-dimensional vector space with a nondegenerate symmetric or skew-Hermitian form over a division algebra with involution $(\Delta, -)$ of characteristic $\neq 2, 3$ (see also [FGGN, Prop. 6.7] where [FGG3] was quoted). This does not affect the main results of [FGG3] or [FGGN]. The reason for the omission was that a direct limit approach based on the incomplete statement of the finite-dimensional case [B1, Thm. 5.5] was used in the proof. We have now remedied the situation by providing a corrected version of finite-dimensional case in Theorem 6.1 above and of the finitary case in Theorem 6.6.

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