Grothendieck's inequalities revisited

Antonio M. Peralta* and Angel Rodríguez Palacios†[a]

[a]Departamento de Análisis Matemático, Facultad de Ciencias,
Universidad de Granada, 18071 Granada, Spain.
aperalta@goliat.ugr.es and apalacio@goliat.ugr.es

Introduction

A celebrated result of A. Grothendieck [11] asserts that there is a universal constant K such that, if Ω is a compact Hausdorff space and T is a bounded linear operator from $C(\Omega)$ to a complex Hilbert space H, then there exists a probability measure μ on Ω such that

$$\|T(f)\|^2 \leq K^2\|T\|^2\left(\int_\Omega |f|^2 d\mu\right)$$

for all $f \in C(\Omega)$. This result is called "Commutative Little Grothendieck's inequality". Actually the result just quoted is a consequence of the so called "Commutative Big Grothendieck's Inequality" assuring the existence of a universal constant $M > 0$ such that for every pair of compact Hausdorff spaces (Ω_1, Ω_2) and every bounded bilinear form U on $C(\Omega_1) \times C(\Omega_2)$ there are probability measures μ_1 and μ_2 on Ω_1 and Ω_2, respectively, satisfying

$$|U(f,g)|^2 \leq M^2\|U\|^2\left(\int_{\Omega_1} |f|^2 d\mu_1\right)\left(\int_{\Omega_2} |g|^2 d\mu_2\right)$$

for all $(f,g) \in C(\Omega_1) \times C(\Omega_2)$.

Reasonable non-commutative generalizations of the original little and big Grothendieck's inequalities have been obtained by G. Pisier ([23], [24]) and U. Haagerup ([12],[13]). In these generalizations non-commutative C*-algebras replace $C(\Omega)$-spaces, norm-one positive linear functionals replace probability measures, and the module $|a|$ of an element a in a C*-algebra is defined as $(\frac{aa^*+a^*a}{2})^{\frac{1}{2}}$.

At the end of 80's, the important works of T. Barton and Y. Friedman [2] and C-H. Chu, B. Iochum, and G. Loupias [8] on Grothendieck's inequalities for the so-called complex JB*-triples appeared. Complex JB*-triples are natural generalizations of C*-algebras, although they need not have a natural order structure. One of the most important ideas contained in the Barton-Friedman paper is the construction of "natural" prehilbertian

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seminorms $\|.\|_\varphi$, associated to norm-one continuous linear functionals φ on complex JB*-triples, in order to play, in Grothendieck's inequalities, the same role as that of the prehilbertian seminorms derived from norm-one positive linear functionals in the case of C*-algebras.

Real JB*-triples have been recently introduced in [16], and their theory has been quickly developed. The class of real JB*-triples includes all JB-algebras [14], all real C*-algebras [10], all J*B-algebras [1], and all complex JB*-triples (regarded as real Banach spaces). We have studied in deep the papers [2] and [8], cited in the previous paragraph, with the aim of extending their results to the context of real JB*-triples, as well as obtaining weak* versions of Grothendieck's inequalities for the so-called real or complex JBW*-triples. The last goal follows the line of [13, Proposition 2.3] in the case of von Neumann algebras. The results obtained by us in these directions appear in [21] and [22]. In fact we have found some gaps in the proofs of the results of [2] and [8], and given partial solutions to them (see [21, Introduction] and [22, Section 1]). In words of L. J. Bunce [6], "the articles [21] and [22] provide antidotes to some subtle difficulties in [2] and subsequence works, including certain results on the important strong* topology of a JBW*-triple".

In the present paper we review the main results in [21] and [22], and prove some new related facts. Most novelties consist in getting better values of the constants involved in Grothendieck's inequalities. In some case (see for instance Theorem 2.6) such an improvement need a completely new proof. As shown in [21, Introduction] and [22, Section 1], the actual formulations of Grothendieck's inequalities for complex JB*-triples in [2] and [8] remain up to date mere conjectures. We show in Theorems 1.2, 1.8 and 2.2 that those conjectures are valid (even for real JB*-triples) whenever we allow a small enlargement of the family of prehilbert seminorms $\{\|.\|_\varphi\}$.

Notation

Let X be a normed space. We denote by S_X, B_X, X^*, and I_X the unit sphere, the closed unit ball, the dual space, and the identity operator, respectively, of X. When necessary we will use the symbol J_X for the natural embedding of X in its bidual X^{**}. If Y is another normed space, then $BL(X,Y)$ will stand for the normed space of all bounded linear operators form X to Y. Of course we write $BL(X)$ instead of $BL(X,X)$. Now assume that the normed space X is complex. A **conjugation** on X will be a conjugate-linear isometry on X of period 2. If τ is a conjugation on X, then X^τ will stand for the real normed space of all τ-fixed elements of X. Real normed spaces which can be written as X^τ, for some conjugation τ on X, are called **real forms** of X. By $X_{\mathbb{R}}$ we denote the real Banach space underlying X.

1. Little Grothendieck's inequality

A complex JB*-triple is a complex Banach space $\mathcal{E}$ with a continuous triple product $\{.,.,.\} : \mathcal{E}\times\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

1. (Jordan Identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\}-\{x,L(b,a)y,z\}+\{x,y,L(a,b)z\}$ for all a,b,c,x,y,z in $\mathcal{E}$, where $L(a,b)x := \{a,b,x\}$;

2. The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with nonnegative spectrum for all a in $\mathcal{E}$;

3. $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathcal{E}$.

Concerning condition 2 above, we recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** is $\|\exp(i\lambda T)\| = 1$, for every $\lambda \in \mathbb{R}$.

Complex JB*-triples were introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces (see [17], [18] and [29]).

By a complex JBW*-triple we mean a complex JB*-triple which is a dual Banach space. We recall that the triple product of every complex JBW*-triple is separately weak*-continuous [4], and that the bidual $\mathcal{E}^{**}$ of a complex JB*-triple $\mathcal{E}$ is a JBW*-triple whose triple product extends the one of $\mathcal{E}$ [9].

Given a complex JBW*-triple $\mathcal{W}$ and a norm-one element φ in the predual $\mathcal{W}_*$ of $\mathcal{W}$, we can construct a prehilbert seminorn $\|.\|_\varphi$ as follows (see [2, Proposition 1.2]). By the Hahn-Banach theorem there exists $z \in \mathcal{W}$ such that $\varphi(z) = \|z\| = 1$. Then $(x,y) \mapsto \varphi\{x,y,z\}$ becomes a positive sesquilinear form on $\mathcal{W}$ which does not depend on the point of support z for φ. The prehilbert seminorm $\|.\|_\varphi$ is then defined by $\|x\|_\varphi^2 := \varphi\{x,x,z\}$ for all $x \in \mathcal{W}$. If $\mathcal{E}$ is a complex JB*-triple and φ is a norm-one element in $\mathcal{E}^*$, then $\|.\|_\varphi$ acts on $\mathcal{E}^{**}$, hence in particular it acts on $\mathcal{E}$.

In [2, Theorem 1.3], Barton and Friedman established a "Little Grothendieck's inequality" for complex JB*-triples, assuring that if T is a bounded linear operator from a complex JB*-triple $\mathcal{E}$ to a complex Hilbert space $\mathcal{H}$ whose second transpose T^{**} attains its norm at a so-called "complete tripotent", then there exists a norm-one functional $\varphi \in \mathcal{E}^*$ such that

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in \mathcal{E}$. However, although assumed in the proof, the hypothesis that T^{**} attains its norm at a complete tripotent does not arise in the statement of [2, Theorem 1.3] (compare [21]). Since by [21, proof of Theorem 4.3] we know that T^{**} attains its norm at a complete tripotent whenever it attains its norm, and the set of all operators $T \in BL(\mathcal{E},\mathcal{H})$ such that T^{**} attains its norm is norm dense in $BL(\mathcal{E},\mathcal{H})$ [19, Theorem 1], we have the following theorem.

Theorem 1.1. [21, Theorem 1.1] *Let $\mathcal{E}$ be a complex JB*-triple and $\mathcal{H}$ a complex Hilbert space. Then the set of those bounded linear operators T from $\mathcal{E}$ to $\mathcal{H}$ such that there exists a norm-one functional $\varphi \in \mathcal{E}^*$ satisfying*

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in \mathcal{E}$, is norm dense in $BL(\mathcal{E},\mathcal{H})$.

Let $\mathcal{E}$ and $\mathcal{H}$ be as in Theorem 1.1. The question if for every T in $BL(\mathcal{E},\mathcal{H})$ there exists $\varphi \in S_{\mathcal{E}^*}$ satisfying

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in \mathcal{E}$, remains an open problem. In any case, if we allow a slightly enlargement of the family of prehilbert seminorms $\{\|.\|_\varphi : \varphi \in S_{\mathcal{E}^*}\}$, then, as we are showing in what follows, the answer to the above question becomes affirmative. We note that the new prehilber seminorms we are building are as naturally derived from the structure of $\mathcal{E}$ as those in the family $\{\|.\|_\varphi : \varphi \in S_{\mathcal{E}^*}\}$.

Let X be a Banach space, and u a norm-one element in X. The set of **states** of X relative to u, $D(X,u)$, is defined as the non empty, convex, and weak*-compact subset of X^* given by

$$D(X,u) := \{\Phi \in B_{X^*} : \Phi(u) = 1\}.$$

For $x \in X$, the **numerical range** of x relative to u, $V(X,u,x)$, is given by $V(X,u,x) := \{\Phi(x) : \Phi \in D(X,u)\}$. It is well known that a bounded linear operator T on a complex Banach space X is hermitian if and only if $V(BL(X), I_X, T) \subseteq \mathbb{R}$ (compare [5, Corollary 10.13]).

Let $\mathcal{E}$ be a complex JB*-triple and $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$. Since for every $x \in \mathcal{E}$, the operator $L(x,x)$ is hermitian and has non-negative spectrum, it follows from [5, Lemma 38.3] that the mapping $(x,y) \to \Phi(L(x,y))$ from $\mathcal{E}\times\mathcal{E}$ to $\mathbb{C}$ becomes a positive sesquilinear form on $\mathcal{E}$. Then we define the prehilbert seminorm $|||.|||_\Phi$ on $\mathcal{E}$ by $|||x|||_\Phi^2 := \Phi(L(x,x))$.

Let $\varphi \in S_{\mathcal{E}^*}$ and let $e \in S_{\mathcal{E}^{**}}$ such that $\varphi(e) = 1$. We consider the element $\Phi_{\varphi,e}$ of $D(BL(\mathcal{E}), I_{\mathcal{E}})$ given by $\Phi_{\varphi,e}(T) := \varphi T^{**}(e)$ for all $T \in BL(\mathcal{E})$, so that we have $|||.|||_{\Phi_{\varphi,e}} = \|.\|_\varphi$ on $\mathcal{E}$.

Theorem 1.2. *Let $\mathcal{E}$ be a complex JB*-triple, $\mathcal{H}$ a complex Hilbert space and $T : \mathcal{E} \to \mathcal{H}$ a bounded linear operator. Then there exists $\Phi \in D(BL(\mathcal{E}), \mathcal{I}_{\mathcal{E}})$ such that*

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ |||x|||_\Phi$$

for all $x \in \mathcal{E}$.

Proof. By Theorem 1.1, for every $n \in \mathbb{N}$ there is a bounded linear operator $T_n : \mathcal{E} \to \mathcal{H}$ and a norm-one functional $\varphi_n \in \mathcal{E}^*$ satisfying

$$\|T_n - T\| \leq \frac{1}{n},$$

and

$$\|T_n(x)\| \leq \sqrt{2}\|T_n\|\|x\|_{\varphi_n} = \sqrt{2}\|T_n\|\ |||x|||_{\Phi_{\varphi_n,e_n}}$$

for all $x \in \mathcal{E}$, where $e_n \in S_{\mathcal{E}^{**}}$ with $\varphi_n(e_n) = 1$ $(n \in \mathbb{N})$.

Since $D(BL(\mathcal{E}), I_{\mathcal{E}})$ is weak*-compact, we can take a weak* cluster point $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ of the sequence Φ_{φ_n,e_n} to obtain

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ |||x|||_\Phi$$

for all $x \in \mathcal{E}$. $\square$

From the previous Theorem we can now derive a remarkable result of U. Haagerup.

Corollary 1.3. [12, Theorem 3.2] *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}$ a complex Hilbert space, and $T : \mathcal{A} \to \mathcal{H}$ a bounded linear operator. Then there exist two norm-one positive linear functionals φ and ψ on $\mathcal{A}$, such that*

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)),$$

for all $x \in \mathcal{A}$.

Proof. By Theorem 1.2 there exists $\Phi \in D(BL(\mathcal{A}), I_{\mathcal{A}})$ such that

$$\|T(x)\|^2 \leq 2\|T\|^2\Phi(L(x,x))$$

for all $x \in \mathcal{A}$. Since for every $x \in \mathcal{A}$ the equality $L(x,x) = \dfrac{1}{2}(L_{xx^*} + L_{x^*x})$ holds (where, for $a \in \mathcal{A}$, L_a and R_a stands for the left and right multiplication by a, respectively), we have

$$\|T(x)\|^2 \leq \|T\|^2\Phi(L_{xx^*} + R_{x^*x})$$

for all $x \in \mathcal{A}$.

Now, denoting by $\widehat{\varphi}$ and $\widehat{\psi}$ the positive linear functionals on $\mathcal{A}$ given by $\widehat{\varphi}(x) := \Phi(L_x)$, and $\widehat{\psi}(x) := \Phi(R_x)$, respectively, and choosing norm-one positive linear functionals φ, ψ on $\mathcal{A}$ satisfying $\widehat{\varphi} \leq \varphi$ and $\widehat{\psi} \leq \psi$ (which is posible because $\widehat{\varphi}$ and $\widehat{\psi}$ belong to $B_{\mathcal{A}^*}$), we get

$$\|T(x)\|^2 \leq \|T\|^2(\varphi(x^*x) + \psi(xx^*)),$$

for all $x \in \mathcal{A}$. $\square$

Following [16], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. If $\mathcal{E}$ is a complex JB*-triple, then conjugations on $\mathcal{E}$ preserve the triple product of E, and hence the real forms of $\mathcal{E}$ are real JB*-triples. In [16] it is shown that actually every real JB*-triple can be regarded as a real form of a suitable complex JB*-triple.

By a real JBW*-triple we mean a real JB*-triple whose underlying Banach space is a dual Banach space. As in the complex case, the triple product of every real JBW*-triple is separately weak*-continuous [20], and the bidual $\mathcal{E}^{**}$ of a real JB*-triple $\mathcal{E}$ is a real JBW*-triple whose triple product extends the one of $\mathcal{E}$ [16]. Noticing that every real JBW*-triple is a real form of a complex JBW*-triple [16], it follows easily that, if W is a real JBW*-triple and if φ is a norm-one element in W_*, then, for $z \in W$ such that $\varphi(z) = \|z\| = 1$, the mapping $x \mapsto (\varphi\{x,x,z\})^{\frac{1}{2}}$ is a prehilbert seminorm on W (not depending on z). Such a seminorm will be denoted by $\|.\|_\varphi$.

The main goal of [21] is to extending Theorem 1.1 to the setting of real JB*-triples. Such an extension is actually obtained in [21, Theorem 4.5] with constant $4\sqrt{2}$ instead of $\sqrt{2}$. However, an easy final touch to the proof of Theorem 4.3 in [21] is letting us to get a better value of the constant.

Proposition 1.4. *Let E be a real JB*-triple, H a real Hilbert space, and $T : E \to H$ a bounded linear operator which attains its norm. Then there exists $\varphi \in S_{E^*}$ satisfying*

$$\|T(x)\| \leq (1 + 3\sqrt{2})\|T\|\|x\|_\varphi$$

for all $x \in E$.

Proof. Without loss of generality we can suppose $\|T\| = 1$. Then, by the proof of [21, Theorem 4.3], there exist $e \in S_{E^{**}}$ and $\psi, \xi \in D(E^{**}, e) \cap E^*$ such that

$$\|T(x)\| \leq \sqrt{8}\|x\|_\psi + (1 + \sqrt{2})\ \|x\|_\xi$$

for all $x \in E$. Setting $\rho = \dfrac{2\sqrt{2}}{1+\sqrt{2}}$ and $\varphi := \frac{1}{1+\rho}(\xi + \rho\ \psi)$, φ is a norm-one functional in E^* with $\varphi(e) = 1$, and we have

$$\|T(x)\| \leq \sqrt{(1+\sqrt{2})^2 + \frac{8}{\rho}}\sqrt{\|x\|_\xi^2 + \rho\ \|x\|_\psi^2}$$

$$= \left([(1+\sqrt{2})^2 + \frac{8}{\rho}](1+\rho)\right)^{\frac{1}{2}}\ \|x\|_\varphi = (1 + 3\sqrt{2})\ \|x\|_\varphi$$

for all $x \in E$. ☐

Keeping in mind Proposition 1.4 above, a new application of [19, Theorem 1] gives us the following improvement of [21, Theorem 4.5].

Theorem 1.5. *Let E be a real JB*-triple and H a real Hilbert space. Then the set of those bounded linear operators T from E to H such that there exists a norm-one functional $\varphi \in E^*$ satisfying*

$$\|T(x)\| \leq (1 + 3\sqrt{2})\|T\|\|x\|_\varphi$$

for all $x \in E$, is norm dense in $BL(E, H)$.

Let X be a complex Banach space and τ a conjugation on X. We define a conjugation $\widetilde{\tau}$ on $BL(X)$ by $\widetilde{\tau}(T) := \tau T \tau$. If T is a $\widetilde{\tau}$-invariant element of $BL(X)$, then we have $T(X^\tau) \subseteq X^\tau$, and hence we can consider $\Lambda(T) := T|_{X^\tau}$ as a bounded linear operator on the real Banach space X^τ. Since the mapping $\Lambda : BL(X)^{\widetilde{\tau}} \to BL(X^\tau)$ is a linear contraction sending I_X to I_{X^τ}, we get

$$V(BL(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq V(BL(X)^{\widetilde{\tau}}, I_X, T)$$

for all $T \in BL(X)^{\widetilde{\tau}}$. On the other hand, by the Hahn-Banach Theorem, we have

$$V(BL(X)^{\widetilde{\tau}}, I_X, T) = V(BL(X)_{\mathbb{R}}, I_X, T)$$

for every $T \in BL(X)^{\widetilde{\tau}}$. It follows

$$V(BL(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq \Re e\ V(BL(X), I_X, T)$$

for all $T \in BL(X)^{\widetilde{\tau}}$.

Let E be a real JB*-triple. Since $E = \mathcal{E}^\tau$ for some complex JB*-triple $\mathcal{E}$ with conjugation τ, it follows from the above paragraph that, for $x \in E$, $V(BL(E), I_E, L(x,x))$ consists only of non-negative real numbers. Therefore, for $\Phi \in D(BL(E), I_E)$, the mapping $(x, y) \to \Phi(L(x, y))$ from $E \times E$ to $\mathbb{R}$ is a positive symmetric bilinear form on E, and hence $|||x|||_\Phi^2 := \Phi(L(x,x))$ defines a prehilbert seminorm on E.

Now, when in the proof of Theorem 1.2 Theorem 1.5 replaces Theorem 1.1, we arrive at a real variant of Theorem 1.2 with constant $(1 + 3\sqrt{2})$ instead of $\sqrt{2}$. However, as we show in Theorem 1.8 below, a better result holds.

Lemma 1.6. *Let X be a complex Banach space with a conjugation τ. Denote by $\mathbf{H}$ the real Banach space of all hermitian operators on X which lie in $BL(X)^{\widetilde{\tau}}$. Then, for every $\Phi \in D(BL(X), I_X)$, there exists $\Psi \in D(BL(X^\tau, I_{X^\tau})$ such that $\Phi(T) = \Psi(\Lambda(T))$ for every T in $\mathbf{H}$.*

Proof. It is easy to see that, for T in $BL(X)^{\widetilde{\tau}}$, the inequality $\|T\| \leq 2\|\Lambda(T)\|$ holds. Now, let T be in $\mathbf{H}$. Then, for $n \in \mathbb{N}$, T^n lies in $BL(X)^{\widetilde{\tau}}$ and, by [5, Theorem 11.17], we have

$$\|T\|^n = \|T^n\| \leq 2\|\Lambda(T^n)\| = 2\|\Lambda(T)^n\| \leq 2\|\Lambda(T)\|^n.$$

By taking n-th roots and letting $n \to +\infty$, we obtain $\|T\| \leq \|\Lambda(T)\|$. It follows that Λ, regarded as a mapping from $\mathbf{H}$ to $BL(X^\tau)$, is a linear isometry. Therefore, given $\Phi \in D(BL(X), I_X)$, the composition $\Phi|_{\mathbf{H}}\ \Lambda^{-1}$ belongs to $D(\Lambda(\mathbf{H}), I_{X^\tau})$, and it is enough to choose $\Psi \in D(BL(X^\tau), I_{X^\tau})$ extending $\Phi|_{\mathbf{H}}\ \Lambda^{-1}$ to obtain

$$\Phi(T) = \Psi(\Lambda(T))$$

for all $T \in \mathbf{H}$. □

The next corollary follows straightforwardly from Lemma 1.6 above.

Corollary 1.7. *Let $\mathcal{E}$ be a complex JB*-triple with a conjugation τ, and Φ in $D(BL(\mathcal{E}), I_{\mathcal{E}})$. Then there exists $\Psi \in D(BL(\mathcal{E}^\tau), I_{\mathcal{E}^\tau})$ such that*

$$|||x|||_\Phi = |||x|||_\Psi$$

for all $x \in \mathcal{E}^\tau$.

Theorem 1.8. *Let E be a real JB*-triple, H a real Hilbert space and $T : E \to H$ a bounded linear operator. Then there exists $\Psi \in D(BL(E), I_E)$ such that*

$$\|T(x)\| \leq 2\ \|T\|\ |||x|||_\Psi$$

for all $x \in E$.

Proof. Let $\mathcal{E}$ be a complex JB*-triple with conjugation τ such that $E = \mathcal{E}^\tau$, let $\mathcal{H}$ be a complex Hilbert space with conjugation ρ such that $\mathcal{H}^\rho = H$, and let $\widehat{T} \in BL(\mathcal{E}, \mathcal{H})$ such that $\widehat{T}|_E = T$. We note that $\|\widehat{T}\| \leq \sqrt{2}\|T\|$. By Theorem 1.2 there exists $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ satisfying

$$\|T(x)\| \leq \sqrt{2}\|\widehat{T}\| |||x|||_\Phi \tag{1}$$

for all $x \in E$. By Corollary 1.7 there exists $\Psi \in D(BL(E), I_E)$ such that

$$|||x|||_\Phi = |||x|||_\Psi \tag{2}$$

for all $x \in E$. Finally combining (1) and (2) we get

$$\|T(x)\| \leq \sqrt{2}\|\widehat{T}\| |||x|||_\Psi \leq 2\|T\| |||x|||_\Psi$$

for all $x \in E$. □

Section 2 of [22] is mainly devoted to obtaining weak*-versions of the "Little Grothendieck's inequality" for real and complex JBW*-triples. In a first approach we prove the following result.

Proposition 1.9. *If $\mathcal{W}$ is a complex (respectively, real) JBW*-triple, if $\mathcal{H}$ a complex (respectively, real) Hilbert space, and if $M = \sqrt{2}$ (respectively, $M > 1 + 3\sqrt{2}$), then the set of weak*-continuous linear operators T from $\mathcal{W}$ to $\mathcal{H}$ such that there exists a norm-one functional $\varphi \in \mathcal{W}_*$ satisfying*

$$\|T(x)\| \leq M \ \|T\| \ \|x\|_\varphi$$

for all $x \in \mathcal{W}$, is norm dense in the space of all weak-continuous linear operators from $\mathcal{W}$ to $\mathcal{H}$.*

Proposition 1.9 above follows from [22, Lemma 3] (respectively, [22, Lemma 4]) and [30]. When the result in [30] is replaced with a finer principle in [25] on approximation of operator by operator attaining their norms, we get the following theorem.

Theorem 1.10. [22, Theorems 3 and 5] *Let $K > \sqrt{2}$ (respectively, $K > 1+3\sqrt{2}$), $\varepsilon > 0$, $\mathcal{W}$ a complex (respectively, real) JBW*-triple, $\mathcal{H}$ a complex (respectively, real) Hilbert space, and $T : \mathcal{W} \to \mathcal{H}$ a weak*-continuous linear operator. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that the inequality*

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}$$

holds for all $x \in \mathcal{W}$.

Of course, Theorem 1.10 above has as a corollary the next non-weak* version of "Little Grothendieck's inequality".

Corollary 1.11. *Let $K > \sqrt{2}$ (respectively, $K > 1 + 3\sqrt{2}$) and $\varepsilon > 0$. Then, for every complex (respectively, real) JB*-triple $\mathcal{E}$, every complex (respectively, real) Hilbert space $\mathcal{H}$, and every bounded linear operator $T : \mathcal{E} \to \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ such that the inequality*

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}},$$

holds for all $x \in \mathcal{E}$.

Remark 1.12. Let K, $\mathcal{W}$, $\mathcal{H}$, and T be as in Theorem 1.10. We claim that there exists Φ in $D(BL(\mathcal{W}), I_\mathcal{W})$ which lies in the natural non-complete predual $\mathcal{W} \otimes \mathcal{W}_*$ of $BL(\mathcal{W})$ and satisfies

$$\|T(x)\| \leq K \ \|T\| \ |||x|||_\Phi$$

for all $x \in \mathcal{W}$. Indeed, take $\varepsilon > 0$ such that $\dfrac{K}{\sqrt{1+\varepsilon^2}} > \sqrt{2}$ (respectively $\dfrac{K}{\sqrt{1+\varepsilon^2}} > 1 + 3\sqrt{2}$) and apply Theorem 1.10 to find $\varphi_1, \varphi_2 \in S_{\mathcal{W}_*}$ satisfying

$$\|T(x)\| \leq \frac{K}{\sqrt{1+\varepsilon^2}} \ \|T\| \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}$$

for all $x \in \mathcal{W}$. Then, choosing $e_i \in D(\mathcal{W}_*, \varphi_i)$ $(i = 1, 2)$ and putting

$$\Phi := \frac{1}{1+\varepsilon^2}(\Phi_{\varphi_2,e_2} + \varepsilon^2 \Phi_{\varphi_1,e_1}),$$

Φ lies in $D(BL(\mathcal{W}), I_{\mathcal{W}}) \cap (\mathcal{W} \otimes \mathcal{W}_*)$ and satisfies

$$\|T(x)\| \leq K \ \|T\| \ |||x|||_{\Phi}$$

for all $x \in \mathcal{W}$. It seems to be plausible that the claim just proved could remain true with $K = \sqrt{2}$ (respectively, $K = 1+3\sqrt{2}$) whenever we allow the element Φ in $D(BL(\mathcal{W}), I_{\mathcal{W}})$ to lie in the natural complete predual $\mathcal{W}\widehat{\otimes}_{\pi}\mathcal{W}_*$ of $BL(\mathcal{W})$.

The concluding section of the paper [22] deals with some applications of Theorem 1.10, including certain results on the strong*-topology of real and complex JBW*-triples. We recall that, if W is a real or complex JBW*-triple, then the strong*-topology of W, denoted by $S^*(W, W_*)$, is defined as the topology on W generated by the family of seminorms $\{\|.\|_{\varphi} : \varphi \in W_*, \ \|\varphi\| = 1\}$. It is worth mentioning that, if a JBW*-algebra $\mathcal{A}$ is regarded as a complex JBW*-triple, then $S^*(\mathcal{A}, \mathcal{A}_*)$ coincides with the so-called "algebra-strong* topology" of $\mathcal{A}$, namely the topology on $\mathcal{A}$ generated by the family of seminorms of the form $x \mapsto \sqrt{\xi(x \circ x^*)}$ when ξ is any weak*-continuous positive linear functional on $\mathcal{A}$ [26, Proposition 3]. As a consequence, when a von Neumann algebra $\mathcal{M}$ is regarded as a complex JBW*-triple, $S^*(\mathcal{M}, \mathcal{M}_*)$ coincides with the familiar strong*-topology of $\mathcal{M}$ (compare [28, Definition 1.8.7]).

For every dual Banach space X (with a fixed predual X_*), we denote by $m(X, X_*)$ the Mackey topology on X relative to its duality with X_*.

The following theorem extends to real JBW*-triples some results in [3], [26], and [27] for complex JBW*-triples, and completely solved a gap in the proof of the results of [26].

Theorem 1.13. [22, Corollary 9 and Theorem 9] *Let W be a real or complex JBW*-triple. Then we have:*

1. *The strong*-topology of W is compatible with the duality (W, W_*).*
2. *If V is a weak*-closed subtriple of W, then the inequality $S^*(W, W_*)|_V \leq S^*(V, V_*)$ holds, and in fact $S^*(W, W_*)|_V$ and $S^*(V, V_*)$ coincide on bounded subsets of V.*
3. *The triple product of W is jointly $S^*(W, W_*)$-continuous on bounded subsets of W.*
4. *The topologies $m(W, W_*)$ and $S^*(W, W_*)$ coincide on bounded subsets of W.*

Moreover, linear mappings between real or complex JBW-triples are strong*-continuous if and only if they are weak*-continuous.*

Remark 1.14. In a recent work L. J. Bunce obtains an improvement of Assertion 2 of Theorem 1.13. Indeed, in [6, Corollary] he proves that, if W is a real or complex JBW*-triple, and if V is a weak*-closed subtriple, then each element of V_* has a norm preserving extension in W_*, and hence $S^*(W, W_*)|_V = S^*(V, V_*)$

From Assertion 4 in Theorem 1.13 we derive in [22, Theorem 10] a Jarchow-type characterization of weakly compact operators from (real or complex) JB*-triples to arbitrary Banach spaces. With Theorems 1.8 and 1.2 instead of [22, Corollaries 5 and 1] in the proof, Theorem 10 of [22] reads as follows.

Theorem 1.15. *Let E be a real (respectively, complex) JB*-triple, X a real (respectively, complex) Banach space, and $T : E \to X$ a bounded linear operator. The following assertions are equivalent:*

1. *T is weakly compact.*
2. *There exist a bounded linear operator G from E to a real (respectively, complex) Hilbert space and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.
3. *There exist $\Phi \in D(BL(E), I_E)$ and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\ \||x\||_\Phi + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.

For a forerunner of the complex case of Theorem 1.15 above the reader is referred to [7] (see also the comment after [22, Theorem 10]).

2. Big Grothendieck's inequality

Big Grothendieck's inequalities for complex JB*-triples appear in the papers [2] and [8]. However, the proofs of such inequalities in both papers contain some gaps, so we are not sure if the statements of those inequalities are true. In any case, putting together facts completely proved in [2] and [8], the complex case of the following theorem follows with minor difficulties (see for instance [22, Section 1]). The real case of the following theorem has no forerunner before [22].

Theorem 2.1. [22, Theorem 1 and Corollary 8] *Let $M > 4(1 + 2\sqrt{3})\ (1 + 3\sqrt{2})^2$ (respectively, $M > 3 + 2\sqrt{3}$) and E, F be real (respectively, complex) JB*-triples. Then the set of all bounded bilinear forms U on $E \times F$ such that there exist norm-one functionals $\varphi \in E^*$ and $\psi \in F^*$ satisfying*
$$|U(x, y)| \leq M\ \|U\|\ \|x\|_\varphi\ \|y\|_\psi$$
for all $(x, y) \in E \times F$, is norm dense in the Banach space of all bounded bilinear forms on $E \times F$.

The complex case of the next theorem follows from Theorem 2.1 above by arguing as in the proof of Theorem 1.2. The real case then follows from the complex one by a suitable application of Corollary 1.7.

Theorem 2.2. *Let $\mathcal{E}$, $\mathcal{F}$ be complex (respectively, real) JB*-triples, $M = 3+2\sqrt{3}$ (respectively, $M = 2(3+2\sqrt{3})$), and let U be a bounded bilinear form on $\mathcal{E}\times\mathcal{F}$. Then there are $\Phi \in D(BL(\mathcal{E}), I_{\mathcal{E}})$ and $\Psi \in D(BL(\mathcal{F}), I_{\mathcal{F}})$ such that*

$$|U(x,y)| \leq M\|U\| |||x|||_{\Phi} |||y|||_{\Psi}$$

for all $(x,y)\in\mathcal{E}\times\mathcal{F}$.

The main goal of Section 3 in [22] is to prove weak*-versions of the "Big Grothendieck's inequality" for real and complex JBW*-triples. In this line, the main result is the following.

Theorem 2.3. [22, Theorems 6 and 7] *Let $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ (respectively, $M > 4(1+2\sqrt{3})$) and $\varepsilon > 0$. For every pair (V,W) of real (respectively, complex) JBW*-triples and every separately weak*-continuous bilinear form U on $V\times W$, there exist norm-one functionals $\varphi_1,\varphi_2 \in V_*$, and $\psi_1,\psi_2\in W_*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2 + \varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y)\in V\times W$.

Since every bounded bilinear form on the cartesian product of two real or complex JB*-triples has a separately weak*-continuous bilinear extension to the cartesian products of their biduals [22, Lemma 1], Theorem 2.3 above, has a natural non-weak* corollary. However, a better value of the constant M in the complex case of such a corollary can be got by means of an independent argument (see [22, Remark 2]). Precisely, we have the following result.

Corollary 2.4. *Let $M > 3+2\sqrt{3}$ (respectively, $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$) and $\varepsilon > 0$. Then for every pair $(\mathcal{E},\mathcal{F})$ of complex (respectively, real) JB*-triples and every bounded bilinear form U on $\mathcal{E}\times\mathcal{F}$ there exist norm-one functionals $\varphi_1,\varphi_2\in\mathcal{E}^*$ and $\psi_1,\psi_2\in\mathcal{F}^*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|y\|_{\psi_2}^2 + \varepsilon^2\ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y)\in\mathcal{E}\times\mathcal{F}$.

In view of the complex case of Corollary 2.4 above, the question whether the interval $M > 3+2\sqrt{3}$ is valid in the complex case of Theorem 2.3 naturally appears. In the rest of this paper we answer affirmatively this question. We recall that if $\mathcal{E}$ and $\mathcal{F}$ are complex JB*-triples, then every bounded bilinear form U on $\mathcal{E}\times\mathcal{F}$ has a (unique) separately weak*-continuous extension, denoted by $\widetilde{U}$, to $\mathcal{E}^{**}\times\mathcal{F}^{**}$.

Lemma 2.5. *Let $M > 3+2\sqrt{3}$ and $\varepsilon > 0$. Then for every pair $(\mathcal{E},\mathcal{F})$ of complex JB*-triples and every bounded bilinear form U on $\mathcal{E}\times\mathcal{F}$ there exist norm-one functionals $\varphi_1,\varphi_2\in\mathcal{E}^*$ and $\psi_1,\psi_2\in\mathcal{F}^*$ satisfying*

$$|\widetilde{U}(\alpha,\beta)| \leq M\ \|U\|\ \left(\|\alpha\|_{\varphi_2}^2 + \varepsilon^2\ \|\alpha\|_{\varphi_1}^2\right)^{\frac{1}{2}}\ \left(\|\beta\|_{\psi_2}^2 + \varepsilon^2\ \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(\alpha,\beta)\in\mathcal{E}^{**}\times\mathcal{F}^{**}$.

Proof. By Corollary 2.4, there are norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ and $\psi_1, \psi_2 \in \mathcal{F}^*$ satisfying

$$|\widetilde{U}(x,y)| \leq M \, \|U\| \, \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \, \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \, \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \, \|y\|_{\psi_1}^2\right)^{\frac{1}{2}} \tag{3}$$

for all $(x,y) \in \mathcal{E} \times \mathcal{F}$.

Let (α, β) be in $\mathcal{E}^{**} \times \mathcal{F}^{**}$. By Assertion 1 in Theorem 1.13, there are nets $(x_\lambda) \subseteq \mathcal{E}$ and $(y_\mu) \subseteq \mathcal{F}$ converging to α and β in the strong* topology (hence also in the weak* topology) of $\mathcal{E}^{**}$ and $\mathcal{F}^{**}$, respectively. Since, for $i \in \{1,2\}$, the seminorm $\|.\|_{\psi_i}$ is strong*-continuous on $\mathcal{E}^{**}$, by (3) and the separately weak*-continuity of $\widetilde{U}$ we have

$$|\widetilde{U}(x,\beta)| \leq M \, \|U\| \, \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \, \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \, \left(\|\beta\|_{\psi_2}^2 + \varepsilon^2 \, \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $x \in \mathcal{E}$. By taking $x = x_\lambda$ in the last inequality, and arguing similarly, the proof is concluded. □

We can now state the complex case of Theorem 2.3 with constant $M > 3 + 2\sqrt{3}$.

Theorem 2.6. *Let $M > 3 + 2\sqrt{3}$ and $\varepsilon > 0$. For every pair $(\mathcal{V}, \mathcal{W})$ of complex JBW*-triples and every separately weak*-continuous bilinear form U on $\mathcal{V} \times \mathcal{W}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$, and $\psi_1, \psi_2 \in \mathcal{W}_*$ satisfying*

$$|U(x,y)| \leq M \, \|U\| \, \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \, \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \, \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \, \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$.

Proof. Let $\widetilde{U}$ the unique separately weak*-continuous extension of U to $\mathcal{V}^{**} \times \mathcal{W}^{**}$. By Lemma 2.5 there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}^*$ and $\psi_1, \psi_2 \in \mathcal{W}^*$ satisfying

$$|\widetilde{U}(\alpha,\beta)| \leq M \, \|U\| \, \left(\|\alpha\|_{\varphi_2}^2 + \varepsilon^2 \, \|\alpha\|_{\varphi_1}^2\right)^{\frac{1}{2}} \, \left(\|\beta\|_{\psi_2}^2 + \varepsilon^2 \, \|\beta\|_{\psi_1}^2\right)^{\frac{1}{2}} \tag{4}$$

for all $(\alpha, \beta) \in \mathcal{V}^{**} \times \mathcal{W}^{**}$.

Let $\mathcal{U}$ stand for either $\mathcal{V}$ or $\mathcal{W}$. Then $(J_{\mathcal{U}_*})^* : \mathcal{U}^{**} \to \mathcal{U}$ is a weak*-continuous surjective triple homomorphism. Indeed, the map $(J_{\mathcal{U}_*})^* \, J_{\mathcal{U}}$ is the identity on $\mathcal{U}$, $J_{\mathcal{U}}$ is a triple homomorphism, and $J_{\mathcal{U}}(\mathcal{U})$ is weak*-dense in $\mathcal{U}^{**}$. Now $\mathcal{I}(\mathcal{U}) := \ker((J_{\mathcal{U}_*})^*)$ is a weak*-closed ideal of $\mathcal{U}^{**}$, and hence there exists a weak*-closed ideal $\mathcal{J}(\mathcal{U})$ such that $\mathcal{U}^{**} = \mathcal{I}(\mathcal{U}) \oplus^{\ell_\infty} \mathcal{J}(\mathcal{U})$ [15]. Denoting by $\Pi_{\mathcal{U}}$ the linear projection from $\mathcal{U}^{**}$ onto $\mathcal{J}(\mathcal{U})$ corresponding to the decomposition $\mathcal{U}^{**} = \mathcal{I}(\mathcal{U}) \oplus \mathcal{J}(\mathcal{U})$, it follows that the restriction of $(J_{\mathcal{U}_*})^*$ to $\mathcal{J}(\mathcal{U})$ is a weak*-continuous surjective triple isomorphism with inverse mapping $\Psi_{\mathcal{U}} := \Pi_{\mathcal{U}} \, J_{\mathcal{U}} : \mathcal{U} \to \mathcal{J}(\mathcal{U})$.

Now note that, since the bilinear mapping $(\alpha, \beta) \mapsto U((J_{\mathcal{V}_*})^*(\alpha), (J_{\mathcal{W}_*})^*(\beta))$, from $\mathcal{V}^{**} \times \mathcal{W}^{**}$ to $\mathbb{C}$, is separately weak*-continuous and extends U, we have $\widetilde{U}(\alpha,\beta) = U((J_{\mathcal{V}_*})^*(\alpha), (J_{\mathcal{W}_*})^*(\beta))$ for all $(\alpha, \beta) \in \mathcal{V}^{**} \times \mathcal{W}^{**}$. As a consequence, we obtain

$$U(x,y) = \widetilde{U}(\Psi_{\mathcal{V}}(x), \Psi_{\mathcal{W}}(y)) \tag{5}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$.

Since $\mathcal{V}^{**} = \mathcal{I}(\mathcal{V}) \oplus^{\ell_\infty} \mathcal{J}(\mathcal{V})$ and $\mathcal{W}^{**} = \mathcal{I}(\mathcal{W}) \oplus^{\ell_\infty} \mathcal{J}(\mathcal{W})$, we are provided with decompositions $\varphi_i = \varphi_i^1 + \varphi_i^2$ and $\psi_i = \psi_i^1 + \psi_i^2$ $(i \in \{1,2\})$, where

$$\varphi_i^1 \in (\mathcal{J}(\mathcal{V}))_*,\ \varphi_i^2 \in (\mathcal{I}(\mathcal{V}))_*,\ \|\varphi_i^1\| + \|\varphi_i^2\| = 1,$$

and

$$\psi_i^1 \in (\mathcal{J}(\mathcal{W}))_*,\ \psi_i^2 \in (\mathcal{I}(\mathcal{W}))_*,\ \|\psi_i^1\| + \|\psi_i^2\| = 1.$$

Now, choosing norm-one elements $e_i^1 \in \mathcal{J}(\mathcal{V})$ and $e_i^2 \in \mathcal{I}(\mathcal{V})$ such that $\varphi_i^j(e_i^j) = \|\varphi_i^j\|$ $(i,j \in \{1,2\})$, taking $\widetilde{\varphi}_i := \dfrac{\varphi_i^1 \Psi_{\mathcal{V}}}{\|\varphi_i^1 \Psi_{\mathcal{V}}\|} \in \mathcal{V}_*$ if $\varphi_i^1 \Psi_{\mathcal{V}} \neq 0$ and $\widetilde{\varphi}_i$ arbitrary in $S_{\mathcal{V}_*}$ otherwise, and keeping in mind that $\mathcal{J}(\mathcal{V})$ and $\mathcal{I}(\mathcal{V})$ are orthogonal, we get

$$\begin{aligned}\|\Psi_{\mathcal{V}}(x)\|_{\varphi_i}^2 &= \varphi_i^1 \left\{\Psi_{\mathcal{V}}(x), \Psi_{\mathcal{V}}(x), e_i^1\right\} + \varphi_i^2 \left\{\Psi_{\mathcal{V}}(x), \Psi_{\mathcal{V}}(x), e_i^2\right\} \\ &= \varphi_i^1 \Psi_{\mathcal{V}} \left\{x, x, \Psi_{\mathcal{V}}^{-1}(e_i^1)\right\} \leq \|x\|_{\widetilde{\varphi}_i}^2\end{aligned}$$

for all $x \in \mathcal{V}$.

Similarly we find norm-one functionals $\widetilde{\psi}_i$ in $\mathcal{W}_*$ such that

$$\|\Psi_{\mathcal{W}}(y)\|_{\psi_i}^2 \leq \|y\|_{\widetilde{\psi}_i}^2$$

for all $y \in \mathcal{W}$, $i \in \{1,2\}$.

Finally, applying (4) and (5), we get

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|_{\widetilde{\varphi}_2}^2 + \varepsilon^2\ \|x\|_{\widetilde{\varphi}_1}^2\right)^{\frac{1}{2}} \left(\|y\|_{\widetilde{\psi}_2}^2 + \varepsilon^2\ \|y\|_{\widetilde{\psi}_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$. $\square$

Remark 2.7. It is shown in [21, Theorem 3.2] that, if A is a JB-algebra, if H is a real Hilbert space, and if $T : A \to H$ is a bounded linear operator, then there is a norm-one positive linear functional φ in A^* such that

$$\|T(x)\| \leq 2\sqrt{2}\ \|T\|\ (\varphi(x^2))^{\frac{1}{2}}$$

for all $x \in A$. Keeping in mind the parallelism between the theories of JB*-triples and JB-algebras (see [14]), the spirit of the arguments in the proof of Theorem 2.6 can be applied to derive from the result in [21] just quoted the following fact which improves [22, Corollary 3].

Fact A: If A is a JBW-algebra, if H is a real Hilbert space, and if $T : A \to H$ is a weak-continuous linear operator, then there is a norm-one positive linear functional φ in A_* such that*

$$\|T(x)\| \leq 2\sqrt{2}\ \|T\|\ (\varphi(x^2))^{\frac{1}{2}}$$

for all $x \in A$.

Now, Lemma 4 in [22] can be improved as follows. Indeed, it is enough to replace in its proof [22, Corollary 3] with Fact A.

Fact B: If W is a real JBW-triple, if H is a real Hilbert space, and if $T : W \to H$ is a weak*-continuous linear operator which attains its norm, then there is a norm-one functional $\varphi \in W_*$ such that*

$$\|T(x)\| \leq (1 + 3\sqrt{2})\ \|T\|\ \|x\|_\varphi$$

for all $x \in W$.

Finally, Fact B above and [30] allow us to take $M = 1 + 3\sqrt{2}$ in the real case of Proposition 1.9.

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