Irreductible Representations of the Simple Jordan Superalgebra of Grassmann Poisson bracket

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Abstract

We obtain classification of the irreducible bimodules over the Jordan superalgebra $Kan(n)$, the Kantor double of the Grassmann Poisson superalgebra G_n on n odd generators, for all $n \geq 2$ and an algebraically closed field of characteristic $\neq 2$. This generalizes the corresponding result of C.Martínez and E.Zelmanov announced in [MZ2] for $n > 4$ and a field of characteristic zero.

Keywords: *Jordan superalgebra, Grassmann algebra, Poisson superalgebra, irreducible bimodule*
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1 Introduction

An algebra J over a field F of characteristic $\neq 2$ is called a *Jordan algebra* if it satisfies the identities

$$\begin{aligned} xy &= yx \\ (x^2y)x &= x^2(yx). \end{aligned}$$

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These algebras were introduced in [JNW] as an algebraic formalism of quantum mechanics. Since then, they have found various applications in mathematics and theoretical physics and now form an intrinsic part of modern algebra. We refer the reader to the books [Jac, Mc, ZSSS] and the survey [KS] for principal results on the structure theory and representations of Jordan algebras.

Jordan superalgebras appeared in 1977–1980 [Kap, Kac]. A *Jordan superalgebra* is a $\mathbb{Z}_2$-graded algebra $J = J_0 + J_1$ satisfiying the graded identities:

$$
\begin{gathered}
xy = (-1)^{|x||y|}yx, \\
((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x = \\
= (xy)(zt) + (-1)^{|y||z|}(xz)(yt) + (-1)^{|t|(|y|+|z|)}(xt)(yz), \qquad (1)
\end{gathered}
$$

where $|x| = i$ if $x \in J_i$. The subspaces J_0 and J_1 are referred as the even and the odd parts of J, respectively. The even part J_0 is a Jordan subalgebra of J, and the odd part J_1 is a Jordan bimodule over J_0.

In [Kac] (see also [Kan]), the simple finite dimentional Jordan superalgebras over an algebraically closed field of zero characteristic were classified. The only superalgebras in this classification whose even part is not semisimple are the Jordan superalgebras of Grassmann Poisson brackets $Kan(n)$, defined below.

A *dot-bracket superalgebra* $A = (A_0 + A_1, \cdot, \{,\})$ is an associative supercommutative superalgebra $(A, \cdot)$ together with a super-skew-symmetric bilinear product $\{,\}$. One can constract the *Kantor superalgebra* $J(A)$ via the *Kantor doubling process* as follows [KMc]: Consider the vector space direct sum $J = A \oplus \overline{A}$, where $\overline{A}$ is just A labelled, multiplication in $J(A)$ is given by:

$$
\begin{aligned}
f \bullet g &= f \cdot g, \\
f \bullet \overline{g} &= \overline{f \cdot g}, \\
\overline{f} \bullet g &= (-1)^{|g|}\overline{f \cdot g}, \\
\overline{f} \bullet \overline{g} &= (-1)^{|g|}\{f, g\},
\end{aligned}
$$

for $f, g \in A_0 \cup A_1$. Then, we have the $\mathbb{Z}_2$-grading $J(A) = J_0 + J_1$, where $J_0 = A_0 + \overline{A_1}$ and $J_1 = A_1 + \overline{A_0}$, and $J(A)$ is a supercommutative superalgebra under this grading.

Theorem ([KMc1]). *If $A = A_0 + A_1$ is a unital dot-bracket superalgebra, then $J(A)$ is a Jordan superalgebra if and only if the following identities hold:*

$$
\{f, (g \cdot h)\} = \{f, g\} \cdot h + (-1)^{|f||g|} g \cdot \{f, h\} - D(f) \cdot g \cdot h, \qquad (2)
$$

$$
\begin{gathered}
\{f, \{g, h\}\} - \{\{f, g\}, h\} - (-1)^{|f||g|}\{g, \{f, h\}\} = \\
D(f) \cdot \{g, h\} + (-1)^{|g|(|f|+|h|)} D(g) \cdot \{h, f\} + (-1)^{|h|(|f|+|g|)} D(h) \cdot \{f, g\} \qquad (3)
\end{gathered}
$$

$$
\{\{x, x\}, x\} = -\{x, x\} \cdot D(x) \qquad , \qquad (4)
$$

where $D(f) = \{f, 1\}$, $f, g, h \in A_0 \cup A_1$ and $x \in A_1$. The last identity is needed only in characteristic 3 *case.*

A dot-bracket superalgebra P is called a *Poisson superalgebra* if it satisfies the identities of the above theorem with $D \equiv 0$. The above construction was first introduced by I.Kantor [Kan] for the Grassmann Poisson superalgebras.

Let G_n be the *Grassman superalgebra* generated by $n \geq 2$ odd generators $e_1, e_2, \ldots, e_n$ over a field F, such that $e_i e_j + e_j e_i = 0$ and $e_i^2 = 0$. Define on G_n an odd superderivation $\dfrac{\partial}{\partial e_j}$ for $j = 1, 2, \ldots, n$ by the equalities

$$
\begin{aligned}
\frac{\partial e_i}{\partial e_j} &= \delta_{ij}, \\
\frac{\partial (uv)}{\partial e_j} &= \frac{\partial u}{\partial e_j} v + (-1)^{|u|} u \frac{\partial v}{\partial e_j},
\end{aligned}
$$

and then define the superbracket

$$\{f, g\} = (-1)^{|f|} \sum_{i=1}^{n} \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j}.$$

One can check that the dot-bracket superalgebra G_n is a Poisson superalgebra, and it was proved in [Kan] that the superalgebra $Kan(n) = J(G_n)$ is a simple Jordan superalgebra for all $n \geq 2$.

A *Jordan (super)bimodule* over a Jordan superalgebra J is defined in a usual way: a $\mathbb{Z}_2$-graded J-bimodule $V = V_0 + V_1$ is called a Jordan J-bimodule if the *split null extention* $E(J, V) = J \oplus V$ is a Jordan superalgebra. Recall that the multiplication in the split null extention extends the multiplication in J and the action of J on V, while the product of two arbitrary elements in V is zero.

The first main problem of the representation theory for any class of algebras is the classification of irreducible bimodules. The description of unital irreducible finite dimensional Jordan bimodules is practically finished for simple finite dimensional Jordan superalgebras over an algebraically closed field of characteristic zero [MS, MSZ, MZ1, MZ2, MZ3, T1, T2, Sht1, Sht2]. One of main tools used in these papers was the famous Tits-Kantor-Koecher functor (TKK-functor) which associates with a Jordan (super)algebra J a certain Lie (super)algebra $TKK(J)$. Using the known classification of irreducible Lie bimodules over $TKK(J)$, the authors recovered the structure of irreducible bimodules over J. Observe that this method may be used only in the characteristic zero case since classification of irreducible Lie supermodules is not known for positive characteristic.

The classification of irreducible bimodules over the Kantor superalgebra $Kan(n)$ was first obtained in [Sht1] via TKK-functor. In [MZ2], the authors pointed out that the using of the TKK-functor in [Sht1] was not quite correct, and the classification obtained there was not complete. They announced a new classification of irreducible bimodules over $Kan(n)$ for all $n > 4$ and an algebraically closed field F of characteristic zero.

In this paper, we classify the irreducible bimodules for the superalgebra $Kan(n)$ over an algebraically closed field F of characteristic $\neq 2$ and $n \geq 2$. Our proof is direct and does not use the structure of Lie modules over the Lie superalgebra $L = TKK(Kan(n))$. In order to prove that the constructed bimodule is Jordan, we give a new construction of a Jordan bracket on the tensor product of a Poisson superalgebra P with a certain generalized derivation and an associative commutative algebra with a derivation.

2 Some Properties

Recall that the Grassmann algebra G_n has a base formed by 1 and the products $e_{i_1} e_{i_2} \cdots e_{i_n}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq n$.

For an ordered subset $I = \{i_1, i_2, \ldots, i_k\} \subseteq I_n = \{1, 2, \ldots, n\}$, we denote

$$e_I := e_{i_1} e_{i_2} \cdots e_{i_k},$$

so

$$\overline{e_I} = \overline{e_{i_1} e_{i_2} \cdots e_{i_k}}, \quad e_\phi = 1, \text{ and } \overline{e_\phi} = \overline{1}.$$

Now, as $e_i e_j = -e_j e_i$, for $i, j \in I_n$, $i \neq j$, if σ is a permutation of the set I, we have

$$e_I = sgn(\sigma) e_{\sigma(I)},$$

where $sgn(\sigma)$ is the sign of the permutation σ.

For ordered subsets $I = \{i_1, \dots i_k\}$ and $J = \{j_1, \dots j_s\}$, denote by $I \cup J$ the ordered set

$$I \cup J = \{i_1, \dots i_k, j_1, \dots j_s\}.$$

Then the multiplication in $Kan(n)$ is given by:

$$e_I \bullet e_J = e_I e_J = \begin{cases} e_{I \cup J} & \text{if} \quad I \cap J = \phi, \\ 0 & \text{if} \quad I \cap J \neq \phi, \end{cases}$$

$$e_I \bullet \overline{e_J} = \overline{e_I e_J} = \begin{cases} \overline{e_{I \cup J}} & \text{if} \quad I \cap J = \phi, \\ 0 & \text{if} \quad I \cap J \neq \phi, \end{cases}$$

$$\overline{e_I} \bullet e_J = (-1)^s \overline{e_I e_J} = \begin{cases} (-1)^s \overline{e_{I \cup J}} & \text{if} \quad I \cap J = \phi, \\ 0 & \text{if} \quad I \cap J \neq \phi, \end{cases}$$

$$\overline{e_I} \bullet \overline{e_J} = (-1)^s \{e_I, e_J\} = \begin{cases} (-1)^{s+k+p+q} e_{I' \cup J'} & \text{if} \quad I \cap J = \{i_p\} = \{j_q\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $I' = \{i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_k\}$ and $J' = \{j_1, \dots, j_{q-1}, j_{q+1}, \dots, j_s\}$.

We will use the notation $\bullet$ only in the presence of other multiplications.

Let V be a Jordan bimodule over $Kan(n)$. For $a \in Kan(n)$ we denote by R_a the action of a on V: $R_a(v) = v \cdot a$. The Jordan superidentity (1) implies the following operator relations:

$$\begin{aligned} & R_y R_z R_t + (-1)^{|y||z|+|y||t|+|z||t|} R_t R_z R_y + (-1)^{|z||t|} R_{(yt)z} \\ = \; & R_y R_{zt} + (-1)^{|y||z|} R_z R_{yt} + (-1)^{|t||yz|} R_t R_{yz}, & (5) \\ & [R_{xy}, R_z]_s + (-1)^{|y||z|} [R_{xz}, R_y]_s + (-1)^{|x||yz|} [R_{yz}, R_x]_s = 0, & (6) \end{aligned}$$

where $[R_x, R_y]_s = R_x R_y - (-1)^{|x||y|} R_y R_x$ denotes the supercomutator of the operators R_x, R_y.

It is well known (see, for instance, [Jac, MZ1]) that every Jordan bimodule V over a unital Jordan (super)algebra J is decomposed into a direct sum of three subbimodules

$$V = V(0) \oplus V(1) \oplus V(1/2),$$

where $V(0)$ is a trivial bimodule, $V(1)$ is a unital bimodule, and $V(1/2)$ is a *semi-unital* or *one-sided* bimodule, that is, where the unit 1 of J acts as $\frac{1}{2}$. Moreover, for a semi-unital bimodule V, the mapping $a \mapsto 2R_a$ is a homomorphism of a Jordan (super)algebra J into the special Jordan (super)algebra $(End\, V)^+$. Therefore, a simple exceptional unital Jordan (super)algebra admits only unital irreducible bimodules.

It was shown in [Sh] that the Kantor double $J(P)$ for a Poisson superalgebra P is special if and only if it satisfies the identity $\{\{P, P\}, P\} = 0$. Evidently, the superalgebra G_n does not satisfy this condition, hence the superalgebra $Kan(n) = J(G)$ is exceptional. In particular, every irreducible Jordan bimodule over $Kan(n)$ is unital.

Below V denotes a unital Jordan bimodule over the superalgebra $Kan(n)$.

The next Lemma gives the first properies of the right operators over V.

Lemma 2.1. *Given index sets* $I = \{i_1, \dots, i_k\}$ *and* $J = \{j_1, \dots, j_s\}$ *contained in* $I_n = \{1, \dots, n\}$, *we have*

1. $[R_{e_I}, R_{e_J}]_s = 0$, *for all* I *and* J.
2. $[R_{e_I}, R_{\overline{e_J}}]_s = 0$, *if* $|J \cap I| \geq 2$.
3. $[R_{e_I}, R_{\overline{1}}]_s = 0$, *for all* $I \neq \{1, 2, \ldots, n\}$.
4. $[R_{\overline{e_I}}, R_{\overline{e_J}}]_s = 0$, *if* $I \cap J \neq \phi$.

Proof. We will leave the proof of item 1 to the end.

For item 2, the set $I \cap J$ has at least two elements, and we can assume, without loss of generality, that these elements are j_1 and j_s. Let

$$x = e_{j_1},\ y = \overline{e_{j_2} \cdots e_{j_s}},\ \text{and } z = e_I,$$

then

$$xy = e_J \text{ and } xz = yz = 0,$$

and relation (6) finishes the proof.

For item 3, as $I \neq \{1, 2, \ldots, n\}$, there is $p \notin I$, so if

$$x = \overline{e_p},\ \ y = \overline{e_p e_I} \text{ and } z = \overline{1},$$

we have

$$xy = (-1)^k e_I \text{ and } xz = yz = 0,$$

and again by (6) the item is shown.

For item 4, we take

$$x = e_I,\ \ y = \overline{1} \text{ and } z = \overline{e_J},$$

then

$$xy = \overline{e_I} \text{ and } xz = yz = 0,$$

and one more time using (6) we obtain the result.

For item 1, first we suppose that $e_I \neq e_J$, so there exists e_{j_1} such that $e_{j_1} \notin I$. Therefore, taking

$$x = \overline{e_{j_1}},\ \ y = \overline{e_{j_1} e_I} \text{ and } z = e_J,$$

we have

$$xy = (-1)^k e_I \text{ and } xz = yz = 0,$$

and (6) proves the item.

Now, if $I = J = \{e_i\}$, we take $e_j \neq e_i$, and

$$x = \overline{e_j},\ \ y = \overline{e_j e_i}\ \text{ and } z = e_i,$$

then

$$xy = e_i,\ \ xz = -\overline{e_j e_i} \text{ and } yz = 0,$$

hence by (6) and item 4,

$$[R_{e_i}, R_{e_i}]_s = [R_{xy}, R_z]_s = [R_y, R_{xz}]_s = [R_{\overline{e_j e_i}}, R_{\overline{e_j e_i}}]_s = 0.$$

Finally, if $I = J$ and $|I| \geq 2$, we take

$$x = e_{i_1},\ y = e_{i_2} \cdots e_{i_k},\ \text{and } z = e_I,$$

then

$$xy = e_I,\ xz = yz = 0,$$

and (6) finishes the proof. $\square$

As a corollary, we obtain the next lemma.

Lemma 2.2. *The following statements hold:*

1. *If $a \in Kan(n)_1$, $a = e_I$ or $\overline{e_I}$ and $a \neq \overline{1}$, then $R_a^2 = 0$.*
2. *If $a \in Kan(n)_0$, $a = e_I$ or $\overline{e_I}$ and $a \neq 1,\ \overline{e_i}$, then $R_a^3 = 0$.*
3. *$R_{\overline{e_i}}^3 = R_{\overline{e_i}}$, for all $i \in \{1, \ldots, n\}$.*
4. *If V is irreducible and F is algebrically closed then $R_{\overline{1}}^2 = \alpha$ for some $\alpha \in F$.*

Proof. By items 1 and 4 of the previous lemma, for $a \in Kan(n)_1,\ a = e_I$ or $a = \overline{e_I}$, and $a \neq \overline{1}$ we have

$$[R_a, R_a]_s = 2R_a^2 = 0,$$

which proves item 1.

Now, if $a \in Kan(n)_0$, by superidentity (5) we have

$$2R_a^3 + R_{a^3} - 3R_aR_{a^2} = 0.$$

If $a = e_I$ or $\overline{e_I}$ and $a \neq 1,\ \overline{e_i}$, then $a^2 = 0$ and $R_a^3 = 0$, proving item 2.

On the other hand, if $a = \overline{e_i}$ then $a^2 = 1$, and since V is unital, the same identity implies $2R_a^3 = 2R_a$, proving item 3.

For item 4, we recall the following identity which holds in Jordan algebras [Jac]:

$$(a, d, b)c - (a, dc, b) + d(a, b, c) = 0,$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator of the elements a, b, c. For Jordan superalgebras, the super-version of this identity holds:

$$(a, d, b)c - (-1)^{|b||c|}(a, dc, b) + (-1)^{|a||d|}d(a, b, c) = 0.$$

Now, if we take $c \in Kan(n),\ a = b = \overline{1}$, and $d = v \in V$, is easy to see that $(a, c, b) = 0$, hence

$$(\overline{1}, v, \overline{1})c = (-1)^{|c|}(\overline{1}, vc, \overline{1}).$$

Therefore, $U = (\overline{1}, V, \overline{1})$ is a subbimodule of V, and as V is irreducible, we have $U = 0$ or $U = V$.

If $U = 0$, it is clear that $R_{\overline{1}}^2 = 0 \in F$. Otherwise $R_{\overline{1}}^2$ is an authomorphim of V, and by the Schur lemma, $R_{\overline{1}}^2 = \alpha \in F$. $\square$

3 Special Element in V

Lemma 3.1. *If V is an unital Jordan bimodule over $Kan(n)$, then there exists $0 \neq v \in V_0 \cup V_1$ such that*

$$ve_I = v\overline{e_I} = 0,$$

for all $\phi \neq I \subseteq I_n = \{1, \ldots, n\}$.

Proof. For $w \in V$, denote $N_w = \{a \in Kan(n) \,|\, wa = 0\}$. We want to find $0 \neq v \in V$ such that $e_I, \overline{e_I} \in N_v$ for all $\phi \neq I \subseteq I_n$.

As $[R_{e_I}, R_{e_J}]_s = 0$ for all $I, J \subseteq I_n$, and $R^3_{e_I} = 0$ for all $I \neq \phi$, the subsuperalgebra of $End\, V$ generated by the set $\{R_{e_I} \,|\, \phi \neq I \subseteq I_n\}$ is nilpotent. Therefore, there exists $0 \neq u \in V_0 \cup V_1$ such that

$$e_I \in N_u, \text{ for all } \phi \neq I \subseteq I_n.$$

If $\overline{e_{I_n}} \notin N_u$, consider $u_1 = u\overline{e_{I_n}}$. Since $[R_{e_I}, R_{\overline{e_{I_n}}}]_s = 0$ for $|I| \geq 2$, for these I's we have $e_I \in N_{u_1}$. In order to show that $e_i \in N_{u_1}$ for all $i \in I_n$, we first substitute in the main Jordan superidentity (1) $x = \overline{e_i}$, $y = e_{I'_n}$, $z = u$, and $t = e_i$, where $I'_n = \{1, \ldots, i-1, i+1, \ldots, n\}$. Then we obtain

$$(u\overline{e_{I_n}})e_i = (u\overline{e_i})e_{I_n}.$$

Substituting now again in (1) $x = u$, $y = e_{I'_n}$, $z = \overline{e_i}$, and $t = e_i$, we get

$$(u\overline{e_i})e_{I_n} = 0,$$

hence $u_1 e_i = 0$, for all $i \in I_n$. Therefore,

$$e_I \in N_{u_1}, \text{ for all } \phi \neq I \subseteq I_n.$$

If $\overline{e_{I_n}} \notin N_{u_1}$, we consider the element $u_2 = u_1\overline{e_{I_n}}$ and again get

$$e_I \in N_{u_2}, \text{ for all } \phi \neq I \subseteq I_n.$$

Since $R^3_{\overline{e_{I_n}}} = 0$, we conclude that there exists $0 \neq w \in \{u, u_1, u_2\}$ such that

$$e_J,\ \overline{e_{I_n}} \in N_w \text{ for all } \phi \neq I \subseteq I_n.$$

For elements $\overline{e_I}$ with $2 \leq |I| < n$, substitute in (1) $x = e_{i_1}$, $y = e_{I'}$, $z = \overline{1}$, and $t = w$, where $I = \{i_1, \ldots, i_k\}$ and $I' = \{i_2, \ldots, i_k\}$; then we get

$$w\overline{e_I} = (-1)^{k+1}(w\overline{1})e_I.$$

Since $[R_{\overline{1}}, R_{e_I}]_s = 0$ and $we_I = 0$, this implies

$$\overline{e_I} \in N_w \text{ for all } I \text{ with } |I| \geq 2.$$

At this point, we need to incorporate the elements $\overline{e_i}$ for $i \in I_n$. First we show that

$$(w\overline{e_i})\overline{e_i} = (w\overline{e_j})\overline{e_j}, \text{ for all } i \neq j.$$

Substituting in (1) $x = w$, $y = \overline{e_i}$, $z = \overline{1}$, and $t = \overline{e_j}$, we obtain

$$((w\overline{e_i})\overline{1})\overline{e_j} = -((w\overline{e_j})\overline{1})\overline{e_i},$$

and continuing with $x = w\overline{e_i}$, $y = \overline{1}$, $z = \overline{e_j}$, and $t = \overline{e_ie_j}$, we get

$$(((w\overline{e_i})\overline{1})\overline{e_j})\overline{e_ie_j} - (((w\overline{e_i})\overline{e_ie_j})\overline{e_j})\overline{1} = ((w\overline{e_i})\overline{1})e_i.$$

Since $[R_{\overline{e_i}}, R_{\overline{e_ie_j}}]_s = 0$ we have

$$(((w\overline{e_i})\overline{e_ie_j})\overline{e_j})\overline{1} = (((w\overline{e_ie_j})\overline{e_i})\overline{e_j})\overline{1} = 0,$$

so

$$(((w\overline{e_i})\overline{1})\overline{e_j})\overline{e_ie_j} = ((w\overline{e_i})\overline{1})e_i.$$

Substituting again in (1) $x = w$, $y = \overline{e_i}$, $z = \overline{1}$, and $t = e_i$, we obtain

$$((w\overline{e_i})\overline{1})e_i = -(w\overline{e_i})\overline{e_i}, \tag{7}$$

hence

$$(((w\overline{e_i})\overline{1})\overline{e_j})\overline{e_ie_j} = -(w\overline{e_i})\overline{e_i}.$$

Similarly,

$$(((w\overline{e_j})\overline{1})\overline{e_i})\overline{e_ie_j} = -((w\overline{e_j})\overline{1})e_j = (w\overline{e_j})\overline{e_j},$$

and finally

$$-(w\overline{e_i})\overline{e_i} = (w\overline{e_i})\overline{1})\overline{e_j})\overline{e_ie_j} = -(((w\overline{e_j})\overline{1})\overline{e_i})\overline{e_ie_j} = -(w\overline{e_j})\overline{e_j},$$

what we wanted to prove.

Now, since $R^3_{\overline{e_i}} = R_{\overline{e_i}}$, we have

$$((w\overline{e_i})\overline{e_i})\overline{e_j} = ((w\overline{e_j})\overline{e_j})\overline{e_j} = w\overline{e_j}.$$

Therefore, if there exists $k \in I_n$ such that $w\overline{e_k} = 0$, then $w\overline{e_i} = 0$ for all $i \in I$, and we finish the proof taking $v = w$.

Suppose then that $w\overline{e_i} \neq 0$ for all $i \in I_n$, then we show that

$$(w\overline{e_i})e_i \neq 0, \text{ for all } i \in I_n,$$

In fact, by item 3 of Lemma 2.1, $R_{\overline{1}}R_{e_i} = -R_{e_i}R_{\overline{1}}$, hence by (7)

$$(w\overline{e_i})\overline{e_i} = ((w\overline{e_i})e_i)\overline{1},$$

and if $(w\overline{e_i})e_i = 0$ for some $i \in I_n$, then

$$0 \neq w\overline{e_i} = ((w\overline{e_i})\overline{e_i})\overline{e_i} = (((w\overline{e_i})e_i)\overline{1})\overline{e_i} = 0,$$

a contradiction. Therefore,

$$(w\overline{e_i})e_i \neq 0 \quad \text{for all } i \in I_n,$$

Furthermore, for $i, j \in I_n$ with $i \neq j$ we substitute in (1) $x = w$, $y = \overline{e_i}$, $z = \overline{e_ie_j}$, and $t = \overline{e_j}$, then in view of the relations $[R_{e_i}, R_{e_ie_j}]_s = [R_{e_j}, R_{e_ie_j}]_s = 0$, we obtain

$$(w\overline{e_i})e_i = (w\overline{e_j})e_j.$$

We now show that the element $v = (w\overline{e_1})e_1$ satisfies the statement of lemma. Let $i \in I_n$, then

$$ve_i = ((w\overline{e_1})e_1)e_i = ((w\overline{e_i})e_i)e_i = 0, \text{ since } R^2_{e_i} = 0.$$

Now, let $I = \{i_1, \ldots, i_k\} \subseteq I_n$ be, with $k \geq 2$, then in view of item 1 of Lemma 2.1

$$ve_I = ((w\overline{e_{i_1}})e_{i_1})e_I = \pm((w\overline{e_{i_1}})e_I)e_{i_1}.$$

Substituting in (1) $x = w$, $y = e_{I'}$, $z = \overline{e_{i_1}}$, and $t = e_{i_1}$, where $I' = \{i_2, \ldots, i_k\}$, we obtain

$$(w\overline{e_{i_1}})e_I = 0,$$

hence

$$ve_I = 0, \ \text{ for all } \phi \neq I \subseteq I_n.$$

Analogously as we showed that $w\overline{e_I} = 0$, for $I \subseteq I_n$ with $2 \leq |I| = k < n$, we can show that $v\overline{e_I} = 0$ for these I's. Furthermore, substituting in (1) $x = w$, $y = \overline{e_1}$, $z = e_1$, and $t = \overline{e_{I_n}}$, we have

$$v\overline{e_{I_n}} = ((w\overline{e_1})e_1)\overline{e_{I_n}} = 0.$$

Finally, substituting in (1) $x = w$, $y = \overline{e_i}$, $z = e_i$, and $t = \overline{e_i}$, we obtain

$$v\overline{e_i} = ((w\overline{e_i})e_i)\overline{e_i} = 0,$$

ending the proof. $\square$

4 Action of $Kan(n)$ on V.

In this section, we will assune that the bimodule V is irreducible. We will find a finite set that generates V as a vector space and will determine the action of the superalgebra $Kan(n)$ on this set.

Let us begin with notation. If $I = \{i_1, \ldots, i_k\} \subseteq I_n = \{1, \ldots, n\}$ and $w \in V$, we denote

$$w(I) := w\bar{1}\overline{e_{i_1}}\bar{1}\cdots\bar{1}\overline{e_{i_k}} := (\cdots(((w\bar{1})\overline{e_{i_1}})\bar{1})\cdots\bar{1})\overline{e_{i_k}},$$

and

$$\overline{w(I)} := w\bar{1}\overline{e_{i_1}}\bar{1}\cdots\bar{1}\overline{e_{i_k}}\bar{1} := ((\cdots(((w\bar{1})\overline{e_{i_1}})\bar{1})\cdots\bar{1})\overline{e_{i_k}})\bar{1}.$$

In particular, $w(\phi) = w$ and $\overline{w(\phi)} = w\bar{1}$.

It follows from (5) that

$$R_{\bar{e}_i}R_{\bar{1}}R_{\bar{e}_i} = 0, \tag{8}$$

and

$$R_{\bar{e}_i}R_{\bar{1}}R_{\bar{e}_j} = -R_{\bar{e}_j}R_{\bar{1}}R_{\bar{e}_i}, \quad \text{for } i \neq j, \tag{9}$$

so if σ is a permutation of I,

$$w(I) = sgn(\sigma)w(\sigma(I)),$$

where $sgn(\sigma)$ is the sign of σ.

We want to show that the subspace of V generated by the elements $v(I)$ and $\overline{v(I)}$, where I runs all the subsets of I_n and v is the element from the previous section, is closed under the action of $Kan(n)$ and hence coincides with V.

Lemma 4.1. *If $I, J \subseteq I_n$, with $J \not\subseteq I$, then*

$$v(I)e_J = v(I)\overline{e_J} = \overline{v(I)}e_J = 0.$$

Moreover, if $|J \setminus I| \geq 2$, then

$$\overline{v(I)}\overline{e_J} = 0.$$

Proof. Let $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_s\}$. We use induction on $|I| = k$. If $k = 0$, then by the properties of the element v we have

$$ve_J = v\overline{e_J} = 0, \text{ if } |J| \geq 1.$$

Moreover, by item 3 of Lemma 2.1, $[R_{\bar{1}}, R_{e_J}]_s = 0$ for $J \neq I_n$, hence

$$(v\bar{1})e_J = \pm(ve_J)\bar{1} = 0 \text{ for } J \neq I_n.$$

For $J = I_n$, we substitute in (1) $x = e_1$, $y = e_{I'}$, $z = \bar{1}$, and $t = v$, where $I' = \{2, \ldots, n\}$, then $(v\bar{1})e_{I_n} = \pm v\overline{e_{I_n}} = 0$.

Finally, substituting in (1) $x = \bar{1}$, $y = e_{j_1}$, $z = v$, and $t = \overline{e_{J'}}$, where $J' = \{j_2, \ldots, j_s\}$, we obtain

$$(v\bar{1})\overline{e_J} = 0, \text{ if } |J| \geq 2.$$

Now, suppose that the lemma is true for $|I| = m < k$. Let $I' = I \setminus \{i_k\}$, then we have by (1) and induction on $|I|$

$$\begin{aligned} v(I)e_J &= ((v(I')\bar{1})\bar{e}_{i_k})e_J \\ &= \pm((v(I')e_J)\bar{e}_{i_k})\bar{1} \pm v(I')(\bar{e}_J\,\bar{e}_{i_k}) + (v(I')\bar{1})(\bar{e}_{i_k}e_J) \pm (v(I')\bar{e}_{i_k})\bar{e}_J \\ &= \pm v(I')(\overline{e_J}\,\overline{e_{i_k}}) \pm \overline{v(I')}(\overline{e_{i_k}}e_J). \end{aligned}$$

Consider the two cases. If $i_k \in J$, then $\overline{e_J}\,\overline{e_{i_k}} = \pm e_{J\setminus\{i_k\}}$, $\overline{e_{i_k}}e_J = 0$. Since $J \not\subseteq I$, we have $J \setminus \{i_k\} \not\subseteq I'$. Therefore, by induction on $|I|$, $v(I')e_{J\setminus\{i_k\}} = 0$. Finally, if $i_k \notin J$ then $\overline{e_J}\,\overline{e_{i_k}} = 0$, $\overline{e_{i_k}}e_J = \pm\overline{e_{J\cup\{i_k\}}}$. Clearly, $|(J \cup \{i_k\}) \setminus I'| \geq 2$ hence by induction on $|I|$ we have $\overline{v(I')}\overline{e_{J\cup\{i_k\}}} = 0$. Therefore, in both cases $v(I)e_J = 0$, proving first equality of the lemma.

Similarly,

$$\begin{aligned} v(I)\overline{e_J} &= ((v(I')\bar{1})\overline{e_{i_k}})\overline{e_J} = \pm((v(I')\overline{e_J})\overline{e_{i_k}})\bar{1} \pm (v(I')\bar{1})(\overline{e_{i_k}}\,\overline{e_J}) \\ &= (\text{by induction on } |I|) = \pm\overline{v(I')}(\overline{e_{i_k}}\,\overline{e_J}). \end{aligned}$$

As in the previous case, we have $\overline{v(I')}(\overline{e_{i_k}}\,\overline{e_J}) = 0$, proving second equality of the lemma.

Since $[R_{\bar{1}}, R_{e_J}]_s = 0$ for $J \neq I_n$, we have $\overline{v(I)}e_J = (v(I)\bar{1})e_J = \pm(v(I)e_J)\bar{1} = 0$, with $J \neq I_n$, $J \not\subseteq I$. For $J = I_n$ we have by (1)

$$\begin{aligned} \overline{v(I)}e_{I_n} &= ((\overline{v(I')}\overline{e_{i_k}})\bar{1})e_{I_n} = \pm((\overline{v(I')}e_{I_n})\bar{1})\overline{e_{i_k}} + (-1)^n(\overline{v(I')}\overline{e_{i_k}})\overline{e_{I_n}} \\ &= \ (\text{by induction on } |I|)\ = (-1)^n v(I)\overline{e_{I_n}}, \end{aligned}$$

where $I' = I \setminus \{i_k\}$. Therefore, for any $I \subsetneq I_n$ we have $\overline{v(I)}e_{I_n} = \pm v(I)\overline{e_{I_n}} = 0$. We will also need later the following equality for $I = I_n$:

$$\overline{v(I_n)}\,e_{I_n} = (-1)^n v(I_n)\overline{e_{I_n}}. \tag{10}$$

Finally, we have by (1) and Lemma 4.1 for $J' = J \setminus \{j_1\}$, with $e_{j_1} \notin I$:

$$\overline{v(I)}\,\overline{e_J} = (v(I)\bar{1})(e_{j_1}\overline{e_{J'}}) = \pm\overline{e_{j_1}}(v(I)\overline{e_{J'}}) \pm ((\bar{1}e_{j_1})v(I))\overline{e_{J'}} \pm \bar{1}(v(I)\overline{e_J}).$$

By the previous cases, since $j_1 \notin I$ and $J' \not\subseteq I$, we have

$$\begin{gathered} ((\bar{1}e_{j_1})v(I))\overline{e_{J'}} = \pm(v(I)\overline{e_{j_1}})\overline{e_{J'}} = 0, \\ v(I)\overline{e_J} = v(I)\overline{e_{J'}} = 0, \end{gathered}$$

hence

$$\overline{v(I)}\overline{e_J} = 0,$$

proving the lemma. □

Lemma 4.2. *Let* $J = \{j_1, \ldots, j_s\} \subseteq I = \{i_1, \ldots, i_{k-s}, j_s, j_{s-1}, \ldots, j_1\}$. *Then*

- $v(I)e_J = v(I \setminus J)$,
- $v(I)\overline{e_J} = \overline{v(I \setminus J)}$,
- $\overline{v(I)}e_J = (-1)^{|J|}\overline{v(I \setminus J)}$,
- $\overline{v(I)}\,\overline{e_J} = (-1)^{|J|-1}\alpha(|J|-1)v(I \setminus J)$,

where $\alpha = R_{\bar{1}}^2$. *Furthermore, if* $|J \backslash I| = 1$, $I = \{i_1, \ldots, i_{k-s+1}, j_{s-1}, \ldots, j_1\}$, $J = \{j_1, \ldots, j_s\}$, *then*

$$\overline{v(I)}\,\overline{e_J} = (-1)^{s-1}\overline{v(I \setminus J)}\,\overline{e_{j_s}} = (-1)^{s-1}v((I \setminus J) \cup \{j_s\}) = (-1)^{s-1}v(\{i_1, \ldots, i_{k-s+1}, j_s\}).$$

Proof. We will use induction on $|J| = s$. If $s = 0$, we have $e_J = 1$ and all the claims are clear. For $s = 1$, consider first $v(I)e_{j_1}$. Let $I' = I \setminus \{j_1\}$, then by (1) and Lemma 4.1 we have

$$\begin{aligned} v(I)e_{j_1} &= ((v(I')\bar{1})\bar{e}_{j_1})e_{j_1} \\ &= ((v(I')e_{j_1})\bar{e}_{j_1})\bar{1} + v(I')(\bar{e}_{j_1}\,\bar{e}_{j_1}) - (v(I')\bar{e}_{j_1})\bar{e}_{j_1} = v(I'), \end{aligned}$$

which proves first equality for $s = 1$. Similarly,

$$\begin{aligned} v(I)\overline{e_{j_1}} &= ((v(I')\bar{1})\bar{e}_{j_1})\bar{e}_{j_1} \\ &= -((v(I')\bar{e}_{j_1})\bar{e}_{j_1})\bar{1} + \overline{v(I')}(\bar{e}_{j_1}\,\bar{e}_{j_1}) = \overline{v(I')}. \end{aligned}$$

Third equality is true for $s = 1$ since

$$\overline{v(I)}e_{j_1} = (v(I)\bar{1})e_{j_1} = -(v(I)e_{j_1})\bar{1} = -v(I')\bar{1} = -\overline{v(I')}.$$

Furthermore, it follows from (8) and (9) that $\overline{v(I)}\,\overline{e_{j_1}} = 0$, which proves fourth equality for $s = 1$. Finally, if $j \notin I$ then by definition $\overline{v(I)}\,\overline{e_j} = v(I \cup \{j\})$, proving the last equality for $s = 1$.

Assume now that the lemma is true if $|J| < s$. Let $I' = I \setminus \{j_1\}$ and $J' = J \setminus \{j_1\}$; then by (1), Lemma 4.1, and induction on $|J|$, we have

$$\begin{aligned} v(I)e_J &= ((v(I')\bar{1})\overline{e_{j_1}})e_J \\ &= \pm((v(I')e_J)\overline{e_{j_1}})\bar{1} - (-1)^s v(I')(\overline{e_J}\,\overline{e_{j_1}}) \pm (v(I')\overline{e_{j_1}})\overline{e_J} \\ &= v(I')e_{J'} = v(I' \setminus J') = v(I \setminus J). \end{aligned}$$

Similarly,

$$\begin{aligned} v(I)\overline{e_J} &= ((v(I')\bar{1})\overline{e_{j_1}})\overline{e_J} \\ &= \pm((v(I')\overline{e_J})\overline{e_{j_1}})\bar{1} + (v(I')\bar{1})(\overline{e_J}\,\overline{e_{j_1}}) \\ &= (-1)^{s-1}\overline{v(I')e_{J'}} = \overline{v(I' \setminus J')} = \overline{v(I \setminus J)}. \end{aligned}$$

Furthermore, if $J \neq I_n$ then

$$\overline{v(I)}e_J = (v(I)\bar{1})e_J = (-1)^{|J|}(v(I)e_J)\bar{1} = (-1)^{|J|}v(I \setminus J)\bar{1} = (-1)^{|J|}\overline{v(I \setminus J)},$$

and for $J = I_n$ we have by (10) $\overline{v(I_n)}e_{I_n} = (-1)^n v(I_n)\overline{e_{I_n}} = (-1)^n(v\bar{1})$, which proves third equality.

To prove fourth equality, observe first that

$$\begin{aligned} v(I)\overline{e_{J'}} &= v(\{i_1, \ldots, i_{k-s}, j_s, \ldots, j_2, j_1\})\overline{e_{\{j_2,\ldots,j_s\}}} \\ &= (-1)^{s-1}v(\{i_1, \ldots, i_{k-s}, j_1, j_s, \ldots, j_2\})\overline{e_{\{j_2,\ldots,j_s\}}} \\ &= (-1)^{s-1}\overline{v(\{i_1, \ldots, i_{k-s}, j_1\})} = (-1)^{s-1}\overline{v(I \setminus J')}. \end{aligned}$$

Now, applying again (1), Lemma 4.1, and induction on $|J|$, we get

$$\begin{aligned} \overline{v(I)}\overline{e_J} &= (v(I)\bar{1})(e_{j_1}\overline{e_{J'}}) \\ &= -(v(I)\overline{e_{J'}})(\bar{1}e_{j_1}) + ((v(I)\bar{1})e_{j_1})\overline{e_{J'}} - ((v(I)\overline{e_{J'}})e_{j_1})\bar{1} \\ &= \overline{v(I \setminus J')}\overline{e_{j_1}} + (\overline{v(I)}e_{j_1})\overline{e_{J'}} - (-1)^{s-1}(\overline{v(I \setminus J')}e_{j_1})\bar{1} \\ &= -\overline{v(I')}\overline{e_{J'}} + (-1)^{s-1}\overline{v(I \setminus J)}\bar{1} \\ &= -(-1)^{s-2}\alpha(s-2)v(I' \setminus J') + (-1)^{s-1}\alpha v(I \setminus J) \\ &= (-1)^{s-1}\alpha(s-1)v(I \setminus J). \end{aligned}$$

Finally, let $I = \{i_1, \ldots, i_{k-s+1}, j_{s-1}, \ldots, j_1\}$, $J = \{j_1, \ldots, j_s\}$, $J' = J \setminus \{j_s\}$. Arguing as above, we get by Lemma 4.1

$$\begin{aligned}
\overline{v(I)\overline{e_J}} &= (v(I)\bar{1})(e_{J'}\overline{e_{j_s}}) \\
&= -(v(I)\overline{e_{j_s}})(\bar{1}e_{J'}) + ((v(I)\bar{1})e_{J'})\overline{e_{j_s}} - ((v(I)\overline{e_{j_s}})e_{J'})\bar{1} \\
&= (\overline{v(I)}e_{J'})\overline{e_{j_s}} = (-1)^{s-1}\overline{v(I \setminus J')}\overline{e_{j_s}} = (-1)^{s-1}v((I \setminus J') \cup \{j_s\}).
\end{aligned}$$

This finishes the proof of the lemma. □

Lemmas 3 – 5 imply the next theorem:

Theorem 4.3. *Let V be a unital irreducible bimodule over the superalgebra $Kan(n)$ and $v \in V$ be a special element from Lemma 3.1, then V is generated as a vector space by elements of the type*

$$v(I),\ \overline{v(I)},\ \textit{where } I \subseteq I_n = \{1, \ldots, n\}. \tag{11}$$

Furthermore, let $I, J \subseteq I_n$, $J = \{j_1, \ldots, j_{s_1}, j_{s_1+1}, \ldots, j_{s_1+s_2}\}$, $I = \{i_1, \ldots, i_{k-s_1}, j_{s_1}, \cdots, j_1\}$. Then the action of $Kan(n)$ on V is defined, up to permutations of the index sets I and J, as follows:

$$v(I)e_J = \begin{cases} v(I \setminus J) & \textit{if} \quad s_2 = 0 \ (\textit{or, equivalently, } J \subseteq I), \\ 0 & \qquad \textit{otherwise;} \end{cases}$$

$$v(I)\overline{e_J} = \begin{cases} \overline{v(I \setminus J)} & \textit{if} \quad s_2 = 0, \\ 0 & \qquad \textit{otherwise;} \end{cases}$$

$$\overline{v(I)}e_J = \begin{cases} (-1)^s\overline{v(I \setminus J)} & \textit{if} \quad s_2 = 0, \\ 0 & \qquad \textit{otherwise;} \end{cases}$$

$$\overline{v(I)}\overline{e_J} = \begin{cases} (-1)^{s_1}\overline{v(I \setminus J_1)}\,\overline{e_{J_2}} & \textit{if} \quad s_2 = 1, \\ (-1)^{s-1}\alpha(s-1)\,v(I \setminus J) & \textit{if} \quad s_2 = 0, \\ 0 & \qquad \textit{otherwise,} \end{cases}$$

where $\alpha = R_{\bar{1}}^2$, and $s = s_1 + s_2 = |J|$.

Proof. Let W be the vector subspace of V spanned by the set (11). It follows from Lemmas 4.1 and 4.2 that $WKan(n) \subseteq W$, that is, W is a subbimodule of V. Clearly, $W \neq 0$, hence $W = V$. The rest of the theorem follows directly from Lemmas 4.1 and 4.2. □

Since the action of $Kan(n)$ on V depends on the choise of a special element $v \in V$ and a parameter $\alpha = R_{\bar{1}}^2 \in F$, we will denote the bimodule V as $V(v, \alpha)$.

5 Linear independence, Irreducibility, and Isomorphism problem

In this section, we will prove that the set (11), when I runs all different (non-ordered) subsets of I_n, is in fact linearly independent and hence forms a base of the bimodule V. Furthermore, we will prove that $V(v, \alpha)$ is irreducible for any $\alpha \in F$ and that the bimodules $V(v, \alpha)$ and $V(v', \alpha')$ are isomorphic if and only if $|v| = |v'|$, $\alpha = \alpha'$.

Lemma 5.1. *Given $I \subseteq I_n$, there exists an element $W = W(I)$ of the form $W = R_{a_1} \cdots R_{a_p}$ for some $a_i \in Kan(n)$ such that*

$$\begin{aligned} v(I)W &= v, \\ \overline{v(J)}W &= 0 \quad \text{for all } J \subseteq I_n, \\ v(J)W &= 0 \quad \text{for all } J \subseteq I_n \text{ such that } J \neq I \text{ as sets.} \end{aligned}$$

Similarly, there exists $W' = W'(I)$ of the form $W' = R_{b_1} \cdots R_{b_s}$ for some $b_i \in Kan(n)$ such that

$$\begin{aligned} \overline{v(I)}W' &= v, \\ v(J)W' &= 0 \quad \text{for all } J \subseteq I_n, \\ \overline{v(J)}W' &= 0 \quad \text{for all } J \subseteq I_n \text{ such that } J \neq I \text{ as sets.} \end{aligned}$$

Proof. Assume, for simplicity, that $I = \{1, \dots, k\}$. Consider $W_1 = R_{e_I} R_{\bar{1}} R_{\bar{e}_{k+1}} \cdots R_{\bar{1}} R_{\bar{e}_n}$; then we have

$$\begin{aligned} v(I)W_1 &= vR_{\bar{1}} R_{\bar{e}_{k+1}} \cdots R_{\bar{1}} R_{\bar{e}_n} = v(\{k+1, \dots, n\}), \\ \overline{v(I)}W_1 &= \pm(v\bar{1}) R_{\bar{1}} R_{\bar{e}_{k+1}} \cdots R_{\bar{1}} R_{\bar{e}_n} = \cdots = \begin{cases} 0 & \text{if } I \neq I_n, \\ \pm v\bar{1} & \text{if } I = I_n, \end{cases} \\ v(J)W_1 &= \begin{cases} 0 & \text{if } I \not\subseteq J, \\ v(J \setminus I) R_{\bar{1}} R_{\bar{e}_{k+1}} \cdots R_{\bar{1}} R_{\bar{e}_n} & \text{otherwise} \end{cases} = \cdots = 0 \text{ if } J \neq I, \\ \overline{v(J)}W_1 &= \begin{cases} 0 & \text{if } I \not\subseteq J, \\ \pm\alpha v(J \setminus I) R_{\bar{e}_{k+1}} \cdots R_{\bar{1}} R_{\bar{e}_n} & \text{otherwise} \end{cases} \\ &= \begin{cases} \pm\alpha v(\{k+2, \dots, n\}) & \text{if } I \subseteq J \text{ and } J \setminus I = \{k+1\}, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now, if $I \neq I_n$, we can take $W(I) = W_1 R_{e_{\{I_n \setminus I\}}}$, and for $I = I_n$ we can take $W(I) = W_1 R_{\bar{1}} R_{\bar{e}_1} R_{e_1}$. It is easy to check that in both cases $W(I)$ satisfies the conclusions of the first claim of lemma.

For the second claim, if $\alpha \neq 0$, it suffices to take $W'(I) = R_{\bar{1}} W(I)$. Nevetherless, we will give a general proof. Assume first that $I \neq I_n$ and let $i \notin I$. Consider $\overline{v(I)}\bar{e}_i = \pm v(I \cup \{i\})$, then the element $W' = R_{\bar{e}_i} W(I \cup \{i\})$, up to sign, satisfies the needed conditions. Finally, for $I = I_n$ one can take $W'(I_n) = -R_{e_n} W'(I_n \setminus \{n\})$.

□

Lemma 5.1 implies several corollaries.

Corollary 5.2. *Let $V = V(v, \alpha)$ be the bimodule over $Kan(n)$ with the action defined in Theorem 4.3. Then the set of elements (11), when I runs all different (non-ordered) subsets of I_n, is a base of the vector space V.*

Proof. We have already seen that the set (11) generates V. Assume that there exists a linear combination

$$\sum_{I \subseteq I_n} \beta_I v(I) + \sum_{J \subseteq I_n} \bar{\beta}_J \overline{v(J)} = 0,$$

where $\beta_I, \bar{\beta}_J \in F$. Applying the operators $W(I)$ and $W'(J)$, we get that all $\beta_I = \bar{\beta}_J = 0$.

□

Corollary 5.3. *A special element $v \in V(v, \alpha)$ is defined uniquely up to a nonzero scalar.*

Proof. Let v' be another special element in $V = V(v,\alpha)$:

$$v' = \sum_{I\subseteq I_n} \beta_I v(I) + \sum_{J\subseteq I_n} \bar{\beta}_J \overline{v(J)}$$

for some $\beta_I, \bar{\beta}_J \in F$. Observe that the operators W, W' in Lemma 5.1 do not depend on choose of the element v, and have the same form for elements v, v'. Applying the operator $W(\phi)$ to both parts of the above equality, we get $v' = \beta_\phi v$. Clearly, $\beta_\phi \neq 0$.

□

Corollary 5.4. *The bimodule $V(v,\alpha)$ is irreducible for any $\alpha \in F$.*

Proof. Suppose that M is a non-zero sub-bimodule of $V = V(v,\alpha)$, and choose $0 \neq x \in M$:

$$x = \sum_{I\subseteq I_n} \beta_I v(I) + \sum_{J\subseteq I_n} \bar{\beta}_J \overline{v(J)}, \quad \beta_I, \bar{\beta}_J \in F.$$

Since $x \neq 0$, there is some $\beta_I \neq 0$ or $\bar{\beta}_J \neq 0$. Applying operators $W(I)$ or $W'(J)$ from Lemma 5.1, we get in both cases that $v \in M$ and hence $M = V$.

□

Remark 5.5. *It is easy to check that $V(v,\alpha)$ for $\alpha = 0$ is isomorphic to the regular bimodule $Reg(Kan(n))$, hence this corollary gives an alternative proof that the superalgebra $Kan(n)$ is simple.*

Corollary 5.6. *Bimodules $V(v,\alpha)$ and $V(v',\alpha')$ are isomorphic if and only if $|v| = |v'|$ and $\alpha = \alpha'$.*

Proof. Denote $V = V(v,\alpha)$ and $V' = V(v',\alpha')$. Observe that the operators W, W' in Lemma 5.1 have the same form for elements v, v'. Assume that $\varphi : V \to V'$ is an isomorphism of bimodules over $Kan(n)$, then we have in V'

$$\varphi(v) = \sum_{I\subseteq I_n} \beta_I v'(I) + \sum_{I\subseteq I_n} \bar{\beta}_I \overline{v'(I)}, \text{ for some } \beta_I, \overline{\beta_I} \in F.$$

Applying to both parts of this equality the operator $W = W(\phi)$, we get $\varphi(v) = \beta_\phi v'$, with $0 \neq \beta_\phi \in F$. Since φ maintains parity, this is impossible if $|v| \neq |v'|$. Therefore, $|v| = |v'|$, and we have

$$\alpha\beta_\phi v' = \alpha\varphi(v) = \varphi(\alpha v) = \varphi(vR_{\bar{1}}^2) = \varphi(v)R_{\bar{1}}^2 = \alpha'\varphi(v) = \alpha'\beta_\phi v',$$

hence $\alpha = \alpha'$. □

Since, for a given $\alpha \in F$, the bimodule $V(v,\alpha)$ is defined, up to isomorphism, by the parity of v, we will denote by $V(\alpha)$ the bimodule $V(v,\alpha)$ with $|v| = |n|$.

Recall that, for a superalgebra $A = A_0 \oplus A_1$, an A-superbimodule $V^{\mathrm{op}} = V_0^{\mathrm{op}} + V_1^{\mathrm{op}}$ is called *opposite* to an A-superbimodule $V = V_0 \oplus V_1$, if $V_0^{\mathrm{op}} = V_1$, $V_1^{\mathrm{op}} = V_0$ and A acts on it by the following rule: $a \cdot v = (-1)^{|a|} av$, $v \cdot a = va$, where $v \in V^{\mathrm{op}}$, $a \in A_0 \cup A_1$.

It is easy to check that, for any superbimodule V, the identical application $V \to V^{op}$, $v \mapsto v$, is an odd isomorphism between V and V^{op}. In particular, if V is Jordan, the opposite superbimodule V^{op} is Jordan as well. We sometimes will say that the bimodule V^{op} is obtained from V by *changing of parity.*

Corollaries 5.3 and 5.6 imply

Proposition 5.7. *Every unital irreducible bimodule over $Kan(n)$ is isomorphic to a bimodule $V(\alpha)$ or to its opposite $V(\alpha)^{\mathrm{op}}$. Moreover, the bimodules $V(\alpha)$ and $V(\alpha)^{\mathrm{op}}$ are not isomorphic.*

Proof. It suffices to note that, for $|v'| = |v| + 1$, the mapping

$$\sum_{I\subseteq I_n} \beta_I v(I) + \sum_{J\subseteq I_n} \bar{\beta}_J \overline{v(J)} \mapsto \sum_{I\subseteq I_n} \beta_I v'(I) + \sum_{J\subseteq I_n} \bar{\beta}_J \overline{v'(J)}$$

defines an isomorphism of the bimodules $V(v,\alpha)^{op}$ and $V(v',\alpha)$.

$\square$

6 $V(\alpha)$ is Jordan

Finally, we will show that $V(\alpha)$ is a Jordan bimodule over $Kan(n)$ for all α. For this, we will embedd it into a Jordan superalgebra.

Recall than a linear operator E on a unital algebra A is called *a generalized derivation* of A if it satisfies the relation

$$E(ab) = E(a)b + aE(b) - abE(1).$$

Let $P = \langle P_0 \oplus P_1, \cdot, \{,\}\rangle$ be a unital Poisson superalgebra, $E : P \to P$ be a generalized derivation of P which satisfies also the condition

$$E(\{p,q\}) = \{E(p), q\} + \{p, E(q)\} + \{p,q\}E(1). \tag{12}$$

Furthermore, let (A, D) be a commutative associative algebra with a derivation D. Define the following bracket on the tensor product $P \otimes A$:

$$\langle p\otimes a, q\otimes b\rangle \;\;=\;\; \{p,q\}\otimes ab + E(p)q\otimes aD(b) - (-1)^{|p||q|}E(q)p\otimes D(a)b, \tag{13}$$

where $p, q \in P,\; a, b \in A$.

Theorem 6.1. *The bracket (13) is a Jordan bracket on the commutative and associative superalgebra* $P \otimes A = (P_0 \otimes A) \oplus (P_1 \otimes A)$.

Proof. Observe first that a commutative associative superalgebra $P \otimes A$ with a superanticommutative bracket $<,>$ satisfies graded identities (2)-(4) if and only if the Grassmann envelope $G(P\otimes A) = G_0\otimes(P_0\otimes A)+G_1\otimes(P_1\otimes A)$, with the bracket $\langle a\otimes g, b\otimes h\rangle = \langle a,b\rangle\otimes gh$, satisfies the nongraded versions of these identities. It is easy to check the isomorphism

$$G(P\otimes A) \cong G(P)\otimes A,$$

where $G(P) = G_0 \otimes P_0 + G_1 \otimes P_1$ is the Grassmann envelope of the superalgebra P. So, passing to the Grassmann envelope, we see that it sufficient to prove nongraded identities (2)–(4) for the case when P is a Poisson algebra (not a superalgebra).

Let us first check identity (2):

$$\begin{gathered}
\langle (p\otimes a)(q\otimes b), r\otimes c\rangle = \langle pq\otimes ab, r\otimes c\rangle \\
= \{pq, r\}\otimes abc + E(pq)r\otimes abD(c) - E(r)pq\otimes cD(ab) \\
= (p\{q,r\} + q\{p,r\})\otimes abc + (E(p)q + pE(q) - pqE(1))r\otimes abD(c) - E(r)pq\otimes cD(ab) \\
= (p\otimes a)(\{q,r\}\otimes bc + E(q)r\otimes bD(c) - qE(r)\otimes cD(b)) \\
+(q\otimes b)(\{p,r\}\otimes ac + E(p)r\otimes aD(c) - pE(r)\otimes cD(a)) - pqE(1)r\otimes abD(c) \\
= (p\otimes a)\langle q\otimes b, r\otimes c\rangle + (q\otimes b)\langle p\otimes a, r\otimes c\rangle - (p\otimes a)(q\otimes b)\langle 1, r\otimes c\rangle,
\end{gathered}$$

so (2) is satisfied.

Furthermore, the nongraded version of identity (3) has form

$$J(a,b,c) = S(a,b,c),$$

where $J(a,b,c) = \langle\langle a,b\rangle,c\rangle + \langle\langle b,c\rangle,a\rangle + \langle\langle c,a\rangle,b\rangle$ and $S(a,b,c) = \langle a,b\rangle\langle 1,c\rangle + \langle b,c\rangle\langle 1,a\rangle + \langle c,a\rangle\langle 1,b\rangle$.

Consider

$$\begin{gathered}\langle\langle p\otimes a, q\otimes b\rangle, r\otimes c\rangle = \langle\{p,q\}\otimes ab + E(p)q\otimes aD(b) - E(q)p\otimes bD(a), r\otimes c\rangle \\ = \{\{p,q\},r\}\otimes abc + E(\{p,q\})r\otimes abD(c) - E(r)\{p,q\}\otimes cD(ab) \\ +\{E(p)q,r\}\otimes aD(b)c + E(E(p)q)r\otimes aD(b)D(c) - E(r)E(p)q\otimes D(aD(b))c \\ -\{E(q)p,r\}\otimes bD(a)c - E(E(q)p)r\otimes bD(a)D(c) + E(r)E(q)p\otimes D(bD(a))c.\end{gathered}$$

By the properties of the bracket $\{,\}$ and the generalized derivation E, we have further

$$\begin{gathered}\langle\langle p\otimes a, q\otimes b\rangle, r\otimes c\rangle = \{\{p,q\},r\}\otimes abc + (\{E(p),q\} + \{p,E(q)\} + \{p,q\}E(1))r\otimes abD(c) \\ -E(r)\{p,q\}\otimes c(D(a)b + aD(b)) + (E(p)\{q,r\} + q\{E(p),r\})\otimes aD(b)c \\ +(E^2(p)q + E(p)E(q) - E(p)qE(1))r\otimes aD(b)D(c) \\ -E(r)E(p)q\otimes(D(a)D(b)c + aD^2(b)c) - (E(q)\{p,r\} - p\{E(q),r\})\otimes bD(a)c \\ -(E^2(q)p + E(q)E(p) - E(q)pE(1))r\otimes bD(a)D(c) \\ +E(r)E(q)p\otimes(D(b)D(a) + bD^2(a))c\end{gathered} \quad .$$

Calculating the cyclic sum, we get

$$J(p\otimes a, q\otimes b, r\otimes c) = (E(1)\otimes 1)(\{p,q\}r\otimes abD(c) + \{q,r\}p\otimes bcD(a) + \{r,p\}q\otimes caD(b)).$$

Consider now

$$\begin{gathered}\langle p\otimes a, q\otimes b\rangle\langle 1\otimes 1, r\otimes c\rangle = (\{p,q\}\otimes ab + E(p)q\otimes aD(b) - E(q)p\otimes D(a)b)(E(1)r\otimes D(c)) \\ = (E(1)\otimes 1)(\{p,q\}r\otimes abD(c) + E(p)qr\otimes aD(b)D(c) - E(q)pr\otimes D(a)bD(c)).\end{gathered}$$

Therefore, the cyclic sum

$$\begin{aligned} S(p\otimes a, q\otimes b, r\otimes c) &= (E(1)\otimes 1)(\{p,q\}r\otimes abD(c) + \{q,r\}p\otimes bcD(a) + \{r,p\}q\otimes caD(b)) \\ &= J(p\otimes a, q\otimes b, r\otimes c),\end{aligned}$$

proving (3).

Finally, observe that all partial linearizations of (4) follow from identity (3), hence to prove (4) it suffices for us to prove that for any $p\in P_1$ and $a\in A$ holds

$$\langle\langle p\otimes a, p\otimes a\rangle, p\otimes a\rangle = \langle p\otimes a, p\otimes a\rangle\langle 1\otimes 1, p\otimes a\rangle.$$

We have

$$\begin{aligned}\langle p\otimes a, p\otimes a\rangle &= \{p,p\}\otimes aa + E(p)p\otimes aD(a) + E(p)p\otimes D(a)a \\ &= \{p,p\}\otimes a^2 + E(p)p\otimes D(a^2),\end{aligned}$$

and, furthermore,

$$\begin{gathered}\langle\langle p\otimes a, p\otimes a\rangle, p\otimes a\rangle = \langle\{p,p\}\otimes a^2 + E(p)p\otimes D(a^2), p\otimes a\rangle \\ = \{\{p,p\},p\}\otimes a^3 + E(\{p,p\})p\otimes a^2D(a) - E(p)\{p,p\}\otimes D(a^2)a \\ +\{E(p)p,p\}\otimes D(a^2)a + E(E(p)p)p\otimes D(a^2)D(a) - E(p)E(p)p\otimes D^2(a^2)a.\end{gathered}$$

We have $\{\{p,p\},p\} = p^2 = E(p)^2 = 0$, therefore by (12)

$$\begin{aligned}\langle\langle p\otimes a, p\otimes a\rangle, p\otimes a\rangle &= (2\{E(p),p\} + E(1)\{p,p\})p\otimes a^2D(a) - E(p)\{p,p\}\otimes D(a^2)a\\ &+(E(p)\{p,p\} - p\{E(p),p\})\otimes D(a^2)a = E(1)\{p,p\}p\otimes a^2D(a).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle p\otimes a, p\otimes a\rangle\langle 1\otimes 1, p\otimes a\rangle &= (\{p,p\}\otimes a^2 + E(p)p\otimes D(a^2))(E(1)p\otimes D(a))\\ &= E(1)\{p,p\}p\otimes a^2D(a).\end{aligned}$$

Hence (4) is true, and the theorem is proved.

□

Corollary 6.2. *Let $P = \oplus_{i=0}^{\infty}P_i$ be a $\mathbf{Z}$-graded Poisson superalgebra such that $\{P_i, P_j\} \subseteq P_{i+j-2}$. Then the application $E : P \to P$, $E : p_i \mapsto (i-1)p_i$, $p_i \in P_i$, is a generalized derivation of P which satisfies relation (12). In particular, for any associative and commutative algebra (A, D) with a derivation D, the tensor product superalgebra $P\otimes A$ has a Jordan bracket given by (13).*

The Grassmann superalgebra G_n has a natural $\mathbf{Z}$-grading given by degrees of monomials: $e_I \in (G_n)_i$ if and only if $|I| = i$. Clearly, $\{(G_n)_i, (G_n)_j\} \subseteq (G_n)_{i+j-2}$, hence G_n satisfies the previous corollary. Consider the polynomial algebra $A = F[t]$ with a derivation D_α defined by the condition $D_\alpha(t) = -\alpha t$, then the superalgebra $G_n[t] \cong G_n\otimes F[t]$ has a Jordan bracket defined by (13) with $D = D_\alpha$. Therefore, we have

Corollary 6.3. *The Kantor double $J(G_n[t])_\alpha$ with respect to the bracket defined on $G_n[t] = G_n\otimes F[t]$ according to (13) with the derivation D_α, is a Jordan superalgebra.*

Now, we will find in the superalgebra $J(G_n[t])_\alpha$ a $Kan(n)$-subbimodule isomorphic to the bimodule $V(\alpha)$. Since $\langle a,b\rangle = \{a,b\}$ for $a,b\in G_n$, the superalgebra $Kan(n) = J(G_n)$ is a subsuperalgebra of $J(G_n[t])_\alpha$. Consider in $J(G_n[t])_\alpha$ the subspace $W = G_n\otimes t + \overline{G_n\otimes t}$. Clearly, $W \bullet G_n \subseteq G_n$, and for $a,b\in G_n$ we have

$$\begin{aligned}\overline{a\otimes 1}\bullet\overline{b\otimes t} &= (-1)^{|b|}\langle a\otimes 1, b\otimes t\rangle = (-1)^{|b|}(\{a,b\}\otimes t + E(a)b\otimes \alpha t)\\ &= (-1)^{|b|}\alpha\{a,b\}E(a)b\otimes t \in W.\end{aligned}$$

Therefore, W is a unital Jordan bimodule over $Kan(n) = J(G_n)$. Let $w = e_{I_n}\otimes t$, then it is clear that w is a specal element in W that satisfies the properties of Lemma 3.1. Moreover, one can easily check that $W = V(w,\alpha)$, in notation of Section 5; the exact isomorphism is given by the mapping

$$v(I)\mapsto (-1)^{sgn\,\sigma}e_{\{I_n\setminus I\}}\otimes t,\ \overline{v(I)}\mapsto (-1)^{sgn\,\sigma}\overline{e_{\{I_n\setminus I\}}\otimes t},$$

where, for $I = \{i_1,\dots,i_k\}$, σ is a permutation $\sigma : I_n \mapsto (I_n\setminus I)\cup\{i_k,\dots,i_1\}$.

Resuming, we can formulate our main theorem:

Theorem 6.4. *The bimodule V_α is a unital Jordan irreducible bimodule over the superalgebra $Kan(n)$, and every such a bimodule over $Kan(n)$ for $n\geq 2$ over an algebraically closed field of characteristic not 2 is isomorphic to V_α or to its opposite bimodule.*

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References

[Jac] N. Jacobson, Structure and Representation of Jordan Algebras, *Amer. Math. Soc.* Providence, R.I.,(1969).

[JNW] P. Jordan, J. V. Neumann e E. Wigner, On an Algebraic Generalization of the Quantum Mechanical Formalism, *Ann. of Math.* **(2)35** no.1, 29-64, (1934).

[Kac] V. G. Kac, Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, *Comm. in Algebra* **5** no. 13, 1375-1400, (1977).

[KMZ] V. G. Kac, C. Martínez and E. Zelmanov, Graded simple Jordan superalgebras of growth one, *Memoris of the AMS* **150** , 1-140, (2001).

[Kap] I. Kaplansky, Superalgebras, *Pacific J. Math.* **86** no.1, 93-98, (1980).

[Kan] I. Kantor, Jordan and Lie superalgebras defined by Poisson algebras, Algebra and Analysis (Tomsk, 1989), 55–79, Amer. Math. Soc. Transl. Ser. (2), 151, Amer. Math. Soc., Providence, RI, 1992.

[KMc] D. King and K. McCrimmon, The Kantor construction of Jordan superalgebras, *Comm. in Algebra* **20** no.1, 109-126, (1992).

[KMc1] D. King and K. McCrimmon, The Kantor doubling process revisited, *Comm. in Algebra* **23** no.1, 357-372, (1995)

[KS] E.N. Kuzmin and I.P. Shestakov, Non-associative structures [MR1060322 (91i:17001)]. Algebra, VI, 197–280, Encyclopaedia Math. Sci., 57, Springer, Berlin, 1995.

[MS] C. Martínez and I. Shestakov, Unital irreducible bimodules over M_{1+1}, *Preprint.*

[MSZ] C. Martínez, I. Shestakov and E. Zelmanov, Jordan bimodules over the superalgebras $P(n)$ and $Q(n)$, *Trans. Amer. Math. Soc.* **362**, 2037-2051, (2010).

[MZ1] C. Martínez and E. Zelmanov, Representation theory of Jordan superalgebras I, *Trans. Amer. Math. Soc.* **362** no.2, 815-846, (2010).

[MZ2] C. Martínez and E. Zelmanov, Representation theory of Jordan superalgebras, *Contem. Math.* **483**, 179-194, (2009).

[MZ3] C. Martínez and E. Zelmanov, Unital bimodules over the simple Jordan superalgebras $D(t)$, *Trans. Amer. Math. Soc.* **358** no.8, 3637-3649, (2006).

[Mc] K. McCrimmon, A taste of Jordan algebras, Universitext, Springer-Verlag, New York, 2004.

[Sh] I.P. Shestakov, Quantization of Poisson superalgebras and speciality of Jordan Poisson superalgebras, Algebra i Logika, 32 (1993), no.5, 571–584; English transl, in Algebra and Logic, 32 (1993), no. 5, 309–317.

[Sht1] A.S. Shtern, Representations of finite-dimensional Jordan superalgebras of Poisson brackets, *Comm. in Algebra* **23** no.5, 1815-1823, (1995).

[Sht2] A.S. Shtern, Representations an exepcional Jordan superalgebras, *Funktzional Annal. i Prilozhen* **21**, 93-94, (1987).

[T1] M. N. Trushina, Irreducible representations of a certain Jordan superalgebras, *J. Algebra Appl.* **4** no. 1, 1-14, (2005).

[T2] M. N. Trushina, Modular representations of the Jordan superalgebras $D(t)$ and K_3, *J. of Algebra.* **320** no. 4, 1327-1343, (2008).

[W] C.T.C. Wall, Graded Brauer groups, *J. Reine Angew Math.* **213**, 187-199,(1964).

[ZSSS] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, *Rings that are nearly associative* [in Russian], Nauka, Moscow (1978), English transl.: Academic Press, New York - London (1982).