Generic Jordan Polynomials

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Abstract

The universal multiplication envelope $\mathcal{UME}(J)$ of a Jordan system J (algebra, triple, or pair) encodes information about its linear actions – all of its possible actions by linear transformations on bimodules M (equivalently, on all larger split null extensions $J \oplus M$). In this paper we study all possible actions, linear and nonlinear, on larger systems. This is encoded in the universal polynomial envelope $\mathcal{UPE}(J)$, which is a system containing J and a set X of indeterminates. Its elements are generic polynomials in X with coefficients in the system J, and it encodes information about all possible multiplications by J on extensions $\tilde{J} \supseteq J$. The universal multiplication envelope is recovered as the "linear part", the elements homogeneous of degree 1 in some variable x. We are especially interested in generic polynomial identities, free Jordan polynomials $p(x_1, \ldots, x_n; y_1, \ldots, y_m)$ which vanish for particular $a_j \in J$ and all possible x_i in all $\tilde{J}$, i.e., such that the generic polynomial $p(x_1, \ldots, x_n; a_1, \ldots, a_m)$ vanishes in $\mathcal{UPE}(J)$. These represent "generic" multiplication relations among elements a_i, which will hold no matter where J is imbedded. This will play a role in the problem of imbedding J in a system of "fractions" $\tilde{J}$.

The natural domain for a fraction $Q_s^{-1}n$ is the *dominion* $K_{s \succ n} = \Phi n + \Phi s + Q_s V$ where the denominator s *dominates* the numerator n in the sense that $Q_n, Q_{n,s}$ are divisible by Q_s on the left and right. We show that by passing to subdomains we can increase the "fractional" properties of the domain, especially if s generically dominates n in $\mathcal{UPE}(\mathcal{V})$.[1]

Throughout, we consider algebraic systems over an arbitrary ring of scalars Φ. We will work primarily in the context of Jordan pairs, indicating briefly how the pair results must be modified for Jordan algebras and triple systems. A *Jordan pair* is a pair $\mathcal{V} = (V^+, V^-)$ of Φ-modules with compositions $(x, a) \mapsto Q_x(a) \in V^\sigma$ for $(x, a) \in V^\sigma \times V^{-\sigma}$, $\sigma = \pm$, which are quadratic in x and linear in a, and satisfy the following axioms *strictly* (in all scalar extensions, equivalently, all their linearizations hold in $\mathcal{V}$ itself): for all $x, y \in V^\sigma, a, b \in V^{-\sigma}$

$$\text{(JP1)} \quad D_{x,a}Q_x = Q_x D_{a,x}, \quad \text{(JP2)} \quad D_{Q_x a,a} = D_{x,Q_a(x)}, \quad \text{(JP3)} \quad Q_{Q_x a} = Q_x Q_a Q_x,$$

where as usual we set $Q_{x,y} := Q_{x+y} - Q_x - Q_y$, which gives the trilinear product $\{x, a, y\} := Q_{x,y}(a) =: D_{x,a}(y)$ with $\{V^\sigma V^{-\sigma} V^\sigma\} \subseteq V^\sigma$. Remember that quadratic identities linearize automatically, so it is only identities of degree 3 or more in a variable whose linearizations must be assumed to hold, and even these hold automatically if the ring of scalars Φ has sufficiently many invertible elements, or if the identities hold in the particular scalar extension $\mathcal{V}[t] := \mathcal{V} \otimes_\Phi \Phi[t]$ by the scalar

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polynomial ring in one variable. The only linearizations we need to assume in general are[2]

$$\begin{array}{ll} \text{(JP1)}' & D_{x,a}Q_{x,y}+D_{y,a}Qx=Q_{Q_xa,y}+Q_{Q_{x,y}a,x}=Q_{x,y}D_{a,x}+Q_xD_{a,y}, \\ \text{(JP2)}' & D_{x,Q_ay}+D_{y,Q_ax}=D_{Q_{x,y}a,a}, \quad D_{Q_xa,b}+D_{Q_xb,a}=D_{x,Q_{a,b}x}, \\ \text{(JP3)}' & Q_{Q_xa,Q_{x,y}a}=Q_xQ_aQ_{x,y}+Q_{x,y}Q_aQ_x, \\ \text{(JP3)}'' & Q_{Q_xa,Q_ya}+Q_{\{x,a,y\}}=Q_xQ_aQ_y+Q_yQ_aQ_x+Q_{x,y}Q_aQ_{x,y}. \end{array}$$

We will try to economize on superscripts and use typography instead, denoting, for a fixed $\tau=\pm$, elements of V^σ by x,y,z,w and elements of $V^{-\sigma}$ by a,b,c. Every Jordan pair $\mathcal{V}=(V^+,V^-)$ has a **dual** or **opposite** pair $\widetilde{\mathcal{V}}=(\widetilde{V}^+,\widetilde{V}^-)$ for $\widetilde{V}^\sigma:=V^{-\sigma}$ and operations $\widetilde{Q}_{\tilde{x}}\tilde{a}:=Q_ax$, $\widetilde{D}_{\tilde{x},\tilde{a}}\tilde{y}:=\{a,x,b\}$ for $\tilde{x}=a,\tilde{y}=b\in\widetilde{V}^\sigma$, $\tilde{a}=x\in\widetilde{V}^{-\sigma}$ [Loos, p.3]. We could avoid all superscripts by formulating only positive results for $x\in V^+,a\in V^-$, and applying duality for the corresponding negative results, but we won't be quite this parsimonious. Since our alphabet and our attention span are finite, we will use tildes to denote larger systems $(\tilde{x},\tilde{a})\in\widetilde{V}^\sigma\times\widetilde{V}^{-\sigma}$ for $\widetilde{\mathcal{V}}\supseteq\overline{\mathcal{V}}$ (containing a homomorphic image of $\mathcal{V}$, not necessarily $\mathcal{V}$ itself) and in §2 we will start to use $(\tilde{\tilde{x}},\tilde{\tilde{a}})$ to denote "incipient" larger elements (generic elements in the universal polynomial envelope, which can be specialized to elements in any larger system $\widetilde{\mathcal{V}}\supseteq\overline{\mathcal{V}}$).

Jordan triples correspond to Jordan pairs where $V^+=V^-=T$, $Q_{x^\sigma}a^{-\sigma}=P_xa,\{x^\sigma,a^{-\sigma},y^\sigma\}=\{x,a,y\}=L_{x,a}(y)$ satisfying analogues (JT1-3) of (JP1-3), and Jordan algebras are triples with product U_xy and an additional squaring operation x^2 with linearization $\{x,y\}=V_x(y)$ satisfying several additional axioms (equivalently, which imbed in unital Jordan algebras $\Phi 1\oplus J$ defined by 3 analogous axioms (QJ1),(QJ3) but (JP2) replaced by $U_1=\mathbf{1}$).

We will use [**?**] as reference bible for all results about Jordan pairs. The following formulas are used frequently enough in the paper for us to display them:

$$\begin{array}{ll} (0.1.1) & D_{x,a}Q_y+Q_yD_{a,x}=Q_{\{x,a,y\},y}, \\ (0.1.2) & D_{x,Q_ay}=D_{\{x,a,y\},a}-D_{y,Q_ax}=D_{x,a}D_{y,a}-Q_{x,y}Q_a, \\ & D_{Q_ay,x}=D_{a,\{y,a,x\}}-D_{Q_ax,y}=D_{a,y}D_{a,x}-Q_aQ_{y,x}, \\ (0.1.3) & Q_{Q_xa,y}=Q_{x,y}D_{a,x}-D_{y,a}Q_x=D_{x,a}Q_{x,y}-Q_xD_{a,y}, \\ (0.1.4) & Q_{\{x,a,y\}}+Q_{Q_xQ_ay,y}=Q_xQ_aQ_y+Q_yQ_aQ_x+D_{x,a}Q_yD_{a,x}, \\ (0.1.5) & Q_{Q_xQ_ay,D_{x,a}y}=Q_xQ_aQ_yD_{a,x}+D_{x,a}Q_yQ_aQ_x,\ Q_xQ_aD_{x,b}-D_{x,a}D_{Q_xa,b}+D_{Q_xQ_ax,b}=0, \\ (0.1.6) & Q_{\alpha x+Q_xa}=B_{\alpha,x,a}Q_x=Q_xB_{\alpha,a,x},\ Q_{B_{\alpha,x,a}y}=B_{\alpha,x,a}Q_yB_{\alpha,a,x}, \\ & (B_{\alpha,x,a}:=\alpha^2\mathbf{1}+\alpha D_{x,a}+Q_xQ_a), \\ (0.1.7) & D_{Q_xa,b}Q_x=Q_xD_{a,Q_xb}. \end{array}$$

The first part of (0.1.5) is (JP22) of [**?**, p.20]; the second part differs from (JP18) $Q_xQ_aD_{x,b}-D_{Q_xa,b}D_{x,a}+D_{x,Q_aQ_xb}$, but its difference is $[D_{x,a},D_{Q_xa,b}]+D_{x,Q_aQ_xb}-D_{Q_xQ_ax,b}=\big(-D_{\{Q_xa,b,x\},a}+D_{x,\{b,Q_xa,a\}}\big)+\big(-D_{Q_xb,Q_ax}+D_{\{x,a,Q_xb\},a}\big)+\big(D_{Q_xb,Q_ax}-D_{x,\{Q_ax,x,b\}}\big)$ [by (0.1.2), (JP2)′], which vanishes since $\{Q_xa,b,x\}=\{x,a,Q_xb\}$ by (JP1) and $\{b,Q_xa,a\}=\{Q_ax,x,b\}$ by (JP2).

Recall that each element $a^{-\sigma}\in V^{-\sigma}$ turns V^σ into a Jordan algebra, the *a-homotope* $(V^\sigma)^{(a)}$, via

$$(0.2)\qquad U_x^{(a)}y:=Q_xQ_ay,\ V_{x,y}^{(a)}:=D_{x,Q_ay},\ V_x^{(a)}:=D_{x,a},\ x^{(2,a)}:=Q_xa,\ \textit{so}\ x^{(n+1,a)}=Q_xa^{(n,x)}.$$

We will have occasion to use the following formulas relating homotopes $(V^\sigma)^{(a)},(V^{-\sigma})^{(x)}$; to avoid excessive superscripts, we will abbreviate the powers $x^{(n,a)},a^{(m,x)}$ simply by x^n,a^m, and always assume $n\geq m\geq 1$.

[2]We throw (JP2)′ in for future reference, though it holds automatically.

(0.2.1) (Power Shifting): $x^{k+1} = Q_x a^k, \quad Q_{x^n} a^k = x^{2n+k-1}, \quad Q_{x^n,x^m} a^k = 2x^{n+k+m-1},$
(0.2.2) (Power to Power): $x^{(n,a^k)} = x^{nk-k+1},$
(0.2.3) (D Power Shifting): $D_{x^n,a^k} = D_{x,a^{n+k-1}} = D_{x^{n+k-1},a},$
(0.2.4) (Q Power Shifting): $Q_{x^n}Q_{a^k} = Q_xQ_{a^{n+k-1}} = Q_{x^{n+k-1}}Q_a, \ Q_xQ_{a^n,a^m} = Q_{x^n,x^m}Q_a,$
(0.2.5) (Outer Triality): $D_{Q_y a^{m+2},a} - D_{Q_{y,x}a^{m+1},Q_a y} + D_{Q_x a^m,Q_aQ_y a} = 0,$
(0.2.6) (Inner Triality): $D_{Q_xQ_a y,a^{m-1}} - D_{D_{x,a}y,a^m} + D_{y,a^{m+1}} = 0,$
$D_{a^{m-1},Q_xQ_a y} - D_{a^m,D_{x,a}y} + D_{a^{m+1},y} = 0.$

Proof: **(1)** holds by induction on k; for $k=1$ as $Q_{x^n}a = (x^n)^2 = x^{2n}$, and for $k \geq 2$ as $Q_{x^n}a^k = Q_{x^n}(Q_a x^{k-1})$ [by the induction case $k-1$ with x, a switched when $n=1$] $= U_{x^n}x^{k-1} = x^{2n+k-1}$. **(2)** is trivial for $n=1$, easy for $n=2$ [$Q_x(a^k) = x^{k+1} = x^{2k-k+1}$ by (1)], and by induction $x^{n+2,a^k} = Q_xQ_{a^k}x^{n,a^k} = Q_x\big(Q_{a^k}x^{nk-k+1}\big) = Q_x\big(a^{2k+(nk-k+1)-1}\big)$ [by (1)] $= Q_x\big(a^{(n+2)k-k}\big) = x^{(n+2)k-k+1}$ [by (1) again]. For **(3)** when $k=1$, $n=1$ is trivial, for $n \geq 2, D_{x^n,a} = V^{(a)}_{x^n} = V^{(a)}_{x,x^{n-1}}$ [in Jordan algebras] $= D_{x,Q_a x^{n-1}} = D_{x,a^n}$ [by (1)], while for $k \geq 2$ $D_{x^n,a^k} = D_{x^n,Q_a x^{k-1}} = V^{(a)}_{x^n,x^{k-1}}$ equals [in Jordan algebras] both $V^{(a)}_{x,x^{n+k-2}} = D_{x,a^{n+k-1}}$ and $V^{(a)}_{x^{n+k-1}} = D_{x^{n+k-1},a}$. Similarly, for **(4)** by induction on k for $k=1$ we have $Q_{x^n}Q_a = U_{x^n} = U_xU_{x^{n-1}} = Q_xQ_aQ_{x^{n-1}}Q_a = Q_xQ_{Q_a(x^{n-1})}$ [by (JP3)] $= Q_xQ_{a^n}$ [by (1)], and similarly for the bilinear version, while for $k \geq 2$ $Q_{x^n}Q_{a^k} = Q_{x^n}Q_{Q_a x^{k-1}}$ [by (1)] $= Q_{x^n}Q_aQ_{x^{k-1}}Q_a$ [by (JP3)] $= U_{x^n}U_{x^{k-1}}$ equals both $U_{x^{n+k-1}} = Q_{x^{n+k-1}}Q_a$ and $U_xU_{x^{n+k-1}} = Q_xQ_aQ_{x^{n+k-1}}Q_a = Q_xQ_{Q_a x^{n+k-1}}$ [by (JP3)] $= Q_xQ_{a^{n+k}}$. The triality relation **(5)** is just the Jordan algebra relation $V_{U_z x^{k+1}} - V_{U_{z,x}x^k,z} + V_{x^{k+1},z^2}$ read in the a-homotope, and the first relation in **(6)** is $V^{(a)}_{U_x y,x^{m-1}} - V^{(a)}_{V_x y,x^m} + V^{(a)}_{y,x^{m+1}}$. The second relation follows dually; note that it cannot be immediately expressed in terms of an a-homotope, but we will see it is just a relation in a "dual homotope". ■

For a subpair $\mathcal{V} \subseteq \widetilde{\mathcal{V}}$, the **unital outer multiplication algebra** of $\mathcal{V}$ on $\widetilde{\mathcal{V}}$ is denoted by $\mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})$; it is generated over Φ by the identity operator $\mathbf{1}$ and all operators of the form $D_{x,a}, Q_x$ for $x, a \in \mathcal{V}$; when $\mathcal{V} = \widetilde{\mathcal{V}}$ we get the full outer multiplication algebra $\mathcal{M}(\widetilde{\mathcal{V}})$. We now turn to the abstract or "universal" concept of a multiplication algebra.

1 The Universal Multiplication Envelope

An **elemental** or **linear specialization** $\sigma = (\sigma^+, \sigma^-)$ of a Jordan pair $\mathcal{V}$ is a homomorphism $\mathcal{V} \xrightarrow{\sigma} \mathcal{V}(A)$ of $\mathcal{V}$ into a special pair coming from an associative pair or algebra A : $\sigma^\tau(Q_x a) = \sigma^\tau(x)\sigma^{-\tau}(a)\sigma^\tau(x)$. A **multiplication specialization**[3] of a Jordan pair $\mathcal{V}$ in $\mathcal{A}$ is a pair of maps $\mu = (q, d) = ((q^{+,-}, q^{-,+}), (d^{+,+}, d^{-,-}))$ into a unital associative algebra $\mathcal{A}$ with 2×2 *matrix grading* i.e., a decomposition $\mathcal{A} = \bigoplus_{\tau,\sigma \in \{\pm\}} \mathcal{A}^{\tau,\sigma}$ satisfying the matrix relations $\mathcal{A}^{\tau,\sigma}\mathcal{A}^{\rho,\nu} \subseteq \delta_{\sigma,\rho}\mathcal{A}^{\tau,\nu}$ [equivalently, with Peirce decomposition $\mathcal{A}^{\tau,\sigma} = e^\tau \mathcal{A} e^\sigma$ relative to $e^+ \in \mathcal{A}^{+,+}, e^- \in \mathcal{A}^{-,-}$ where $1 = e^+ + e^-$], where $d^{\sigma,\sigma} : (x, a) \mapsto A^{\sigma,\sigma}$ is bilinear in x, a and $q^{\sigma,-\sigma} : x \mapsto A^{\sigma,-\sigma}$ is quadratic in x, strictly satisfying the *multiplication specialization relations* for all $\tau = \pm, x, y \in V^\sigma, a, b \in V^{-\sigma}$.

[3] For algebras these were called *quadratic specializations* [?], but we now adopt the adjective *multiplication*; linear and quadratic specializations suggest specializations of linear and quadratic Jordan systems, wheras the real distinction is between representing the *elements* of $\mathcal{V}$ in an associative algebra, and representing their *multiplication operators* in an associative algebra. We will preserve the distinction between *specialization* (map into an associative algebra) and *representation* (map into an associative algebra of linear transformations acting on a space). These multiplication specializations were called *associative representations* in [Loos 2.4 p.16-17], leaving out (QS2) since it follows from (QS4,4*) via $d_{Q_x a,a} - d_{x,Q_a x} = \big(d^2_{x,a} - q_x q_{a,a}\big) - \big(d^2_{x,a} - q_{x,x}q_a\big) = 0$. (QS4) in turn usually follows by applying (QS5) with y, a replaced by m, b, acting on a, and reading the result as an operator on m. But due to the asymmetry between the pair elements x, y and a, b we cannot derive (QS4) this way and must assume it as an axiom. This contrasts with the Jordan algebra case [?, p.282] where $\tilde{\tilde{U}}_1 = 1, \tilde{\tilde{U}}_{U_x y} = \tilde{\tilde{U}}_x\tilde{\tilde{U}}_y\tilde{\tilde{U}}_x, \tilde{\tilde{U}}_{U_x y,x} = \tilde{\tilde{U}}_x\tilde{\tilde{V}}_{y,x} = \tilde{\tilde{V}}_{x,y}\tilde{\tilde{U}}_x$ suffice to define multiplication specializations.

For the sake of legibility we promote all subscripts to the main line, writing $d(x,a), q(x)$ in place of $d_{x,a}, q_x$, and write these relations as

$$\begin{aligned}
&\text{(QS1)} \quad d^{\tau,\tau}(x,a)q^{\tau,-\tau}(x) = q^{\tau,-\tau}(Q_x a, x) = q^{\tau,-\tau}(x)d^{-\tau,-\tau}(a,x),\\
&\text{(QS2)} \quad d^{\tau,\tau}(x,Q_a x) = d^{\tau,\tau}(Q_x a, a),\\
&\text{(QS3)} \quad q^{\tau,-\tau}(Q_x a) = q^{\tau,-\tau}(x)q^{-\tau,\tau}(a)q^{\tau,-\tau}(x),\\
&\text{(QS4)} \quad d^{-\tau,-\tau}(b,x)d^{-\tau,-\tau}(a,x) = d^{-\tau,-\tau}(b,Q_x a) + q^{-\tau,\tau}(b,a)q^{\tau,-\tau}(x),\\
&\text{(QS4)}^* \quad d^{\tau,\tau}(x,a)d^{\tau,\tau}(x,b) = d^{\tau,\tau}(Q_x a, b) + q^{\tau,-\tau}(x)q^{-\tau,\tau}(a,b),\\
&\text{(QS5)} \quad d^{\tau,\tau}(y,a)q^{\tau,-\tau}(x) + q^{\tau,-\tau}(x)d^{-\tau,-\tau}(a,y) = q^{\tau,-\tau}(\{y,a,x\},x).
\end{aligned}$$

These relations imply

$$\begin{aligned}
&\text{(QS6)} \quad d^{\tau,\tau}(Q_x b, a)q^{\tau,-\tau}(x) = q^{\tau,-\tau}(x)d^{-\tau,-\tau}(b, Q_x a),\\
&\text{(QS7)} \quad d^{\tau,\tau}(Q_x b, a)d^{\tau,\tau}(x,b) = q^{\tau,-\tau}(x)q^{-\tau,\tau}(b)d^{\tau,\tau}(x,a) + d^{\tau,\tau}(x, Q_b Q_x a),\\
&\text{(QS8)} \quad q^{\tau,-\tau}(x,y)d^{-\tau,-\tau}(a,x) = d^{\tau,\tau}(y,a)q^{\tau,-\tau}(x) + q^{\tau,-\tau}(Q_x a, y),\\
&\text{(QS9)} \quad q^{\tau,-\tau}(x)q^{-\tau,\tau}(a,b) + d^{\tau,\tau}(x,\{a,x,b\}) = d^{\tau,\tau}(Q_x b, a) + d^{\tau,\tau}(x,a)d^{\tau,\tau}(x,b).
\end{aligned}$$

Here (6),(7),(8) are Lemma 2.6 (4),(5),(2) in [Loos, p.17-18] ; (9) is JP6, which was not derived for specializations in Lemma 2.6, but is equivalent to (QS4) since (QS9) + (QS4) $= \big(q(x)q(a,b) + d(x,\{a,x,b\}) - d(Q_x b, a) - d(x,a)d(x,b)\big) + \big(d(x,a)d(x,b) - d(Q_x a, b) - q(x)q(b,a)\big) = d(x,\{a,x,b\}) - d(Q_x b, a) - d(Q_x a, b)$ vanishes as a linearization of (QS2).[4]

Multiplication specializations $\mu = (p,\ell)$ or (u,v) for Jordan triples and algebras are maps into an ordinary associative triple or algebra A (which can always be enlarged to a unital algebra). The conditions on (p,ℓ) for Jordan triples take the same form as pairs, deleting all superscripts. For unital Jordan algebras [**?**, Prop 15,p.298] the three relations $u(1) = 1, u(U_x y) = u(x)u(y)u(x), u(U_x y, x) = u(x)v(y,x) = v(x,y)u(x)$ suffice to define multiplication specializations (u,v) (recall that in algebras $v(x,y) = v(x)v(y) - u(x,y)$ are determined by u,v), but for general nonunital algebras the messier defining relations in terms of $u(x), v(x)$ are

$$\begin{aligned}
&\text{(QA1)} \quad v(x^2) = v(x,x),\\
&\text{(QA2)} \quad v(x^3) = v(x,x^2) = v(x^2,x),\\
&\text{(QA3)} \quad u(x^2,y) = u(x,y)v(x) - v(y)u(x) = v(x)u(x,y) - u(x)v(y),\\
&\text{(QA4)} \quad u(x^3,y) = u(x,y)v(x^2) - v(y,x)u(x) = v(x^2)u(x,y) - u(x)v(x,y),\\
&\text{(QA5)} \quad u(x^2) = u(x)^2,\\
&\text{(QA6)} \quad u(x^3) = u(x)^3.
\end{aligned}$$

Multiplication specializations $\mathcal{V} \xrightarrow{\mu} A$ can be composed with homomorphisms $A \xrightarrow{\varphi} A'$ of graded associative algebras to provide new specializations $\varphi \circ \mu$.

A **multiplication representation** or **bi-representation** is a concrete multiplication specialization in an associative algebra $A = \mathrm{End}(\mathcal{M})$ for a graded module $\mathcal{M} = (M^+, M^-)$ with grading determined by $e^\sigma = E^\sigma$, the projection on M^σ (thus $\mathcal{M}$ is the module $M = M^+ \oplus M^-$ together with a memory of where it came from, i.e., its decomposition via E^+, E^-). Multiplication representations of triples or algebras are multiplication specializations in $A = \mathrm{End}(M)$ for some Φ-module M. The archetypal example of a multiplication representation is an **outer multiplication representation**,

[4] Similarly, (QS8) is equivalent to (QS5) since (QS8)+(QS5) equals the linearization $x \to x, y$ in (QS1) $q(x)d(a,x) = q(Q_x a, x)$. Note that (QS1-3) are (JP1-3), (QS4) is (0.1.2), (QS5) is (0.1.1), (QS8) is (0.1.3). The Bimodule Theorem below shows that (QS8),(QS9) are more directly involved than (QS4),(QS5) in capturing bimodule structure, but we prefer (QS5) as a basic result ($d(x,a)$ is a Lie struction), and (QS4)*since it is the dual of (QS4).

i.e., a multiplication specialization $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})|_{\mathcal{M}}$ by outer multiplication operators

$$q^{\tau,-\tau}(x) := Q_x|_{M^{-\tau}}, \quad d^{\tau,\tau}(x,a) := D_{x,a}|_{M^\tau}$$

for $\mathcal{M} = (M^+, M^-)$ a $\mathcal{V}$-invariant subspace of a Jordan pair $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$. The *regular outer representation* is the outer multiplication representation of $\mathcal{V}$ on itself ($\mathcal{M} = \widetilde{\mathcal{V}} = \mathcal{V}$). By restriction we obtain a multiplication representation on any *outer ideal* $\mathcal{I} \subseteq \mathcal{V}$.

A **bimodule** for a pair $\mathcal{V}$ consists of a pair $\mathcal{M} = (M^+, M^-)$ of Φ-modules and a bi-representation of $\mathcal{V}$ on $\mathcal{M}$. A bimodule for a triple or algebra J consists of a bi-representation of J in $\mathrm{End}(M)$ for a single Φ-module M. Any multiplication specialization $\mathcal{V} \xrightarrow{\mu} \mathcal{A}$ becomes, via the left regular representation $\mathcal{A} \to \mathrm{End}(\mathcal{A})$, a multiplication representation of $\mathcal{V}$ in $\mathrm{End}(\mathcal{A})$, and thus turns $\mathcal{A}$ into a $\mathcal{V}$-bimodule $\mathcal{M}(\mathcal{A}, \mu) = M^+ \oplus M^-$ ($M^\sigma := \mathcal{A}e^\tau = \mathcal{A}^{\tau,\tau} \oplus \mathcal{A}^{-\tau,\tau}$). This bimodule is cyclic with generator $1_{\mathcal{A}} = e^+ \oplus e^-$ if $\mu(\mathcal{V})$ together with e^+, e^- generate $\mathcal{A}$ as algebra. Thus bimodules are the same as birepresentations (multiplication representations), which are nearly the same thing as multiplication specializations.

Every elemental specialization $\mathcal{V} \xrightarrow{\sigma} \mathcal{V}(A)$ gives rise to a multiplication specialization in A via $q(x) := 0, d(x,a) := \sigma(x)\sigma(a)$ (or representation on A via $q(x) := 0, d(x,a) := L_{\sigma(x)}L_{\sigma(a)}$) via the left regular representation of A (turning A into a "left $\mathcal{V}$-module" $M = A_L$ via $a \cdot m = am$, $m \cdot a = 0$). In particular, just because Q_s is invertible on $V^{-\sigma}$ does not imply it is injective on all bimodules (only on "unital" bimodules).

In fact, *all* $\mathcal{V}$-bimodules for Jordan systems arise as invariant subspaces of some Jordan system $\mathcal{E} \supseteq \mathcal{V}$, and *all* birepresentations $\mathcal{V} \to \mathrm{End}(\mathcal{M})$ are outer multiplication representations $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\mathcal{E})|_{\mathcal{M}}$ on a split null extension.

Bimodule Theorem [?, 2.7 p. 18] 1.1 *Any $\mathcal{V}$-bimodule $\mathcal{M}$ gives rise to a* **split null extension** $\mathcal{E} = \mathcal{V} \oplus \mathcal{M} = (V^+ \oplus M^+, V^- \oplus M^-)$, *which is a Jordan pair under the operations*

$$\widetilde{Q}_{x\oplus m}(a \oplus p) := Q_x a \oplus \big(q(x)(p) + d(x,a)(m)\big),$$
$$\widetilde{D}_{x\oplus m, a\oplus p}(y \oplus n) := D_{x,a}(y) \oplus \big(d_(x,a)(n) + q(x,y)(p) + d(y,a)(m)\big)$$

for all $x, y \in V^\sigma$, $m, n \in M^\sigma$, $a \in V^{-\sigma}$, $p \in M^{-\sigma}$, and the original birepresentation is the restriction of the regular outer representation of $\mathcal{E}$ to $\mathcal{V}$ and $\mathcal{M}$. ■

Thus bimodules and birepresentations are essentially the same thing as multiplication representations. Bimodules are inherently *outer* modules for $\mathcal{V}$, they have no inner multiplications ($\cap_V(M) = Q_M V = 0$). Thus they can reflect only outer multiplicative properties of a Jordan pair.

By taking homotopes we can convert a Jordan triple or pair into a Jordan algebra, and specializations and bimodules for the pair or triple induce specializations and bimodules for the resulting homotope algebra [**?**, 13.8 p. 146]. A little-known fact is that each such homotope bimodule has a strange dark **dual homotope** (duotope).

Duotope Theorem 1.2 *A multiplication specialization $\mu = (q, d)$ of a Jordan pair $\mathcal{V}$ in a 2×2-graded associative algebra A together with an element $a \in V^-$ induce a homotopic multiplication specialization $\mu^{(a)} := (u^{(a)}, v^{(a)})$ of the homotopic Jordan algebra $J := (V^+)^{(a)}$ in the associative subalgebra A^{++} via*

$$u^{(a)}(x) := q(x)q(a), \qquad v^{(a)}(x) := d(x,a), \qquad v^{(a)}(x,y) := d(x, Q_a y),$$

and at the same time induce a multiplication representation $\mu^{*(a)} := (u^{*(a)}, v^{*(a)})$ of J in the opposite subalgebra A^{--} via

$$u^{*(a)}(x) := (u^{(a)}(x))^* = q(a)q(x), \qquad v^{*(a)}(x) := (v^{(a)}(x))^* = d(a,x),$$
$$v^{*(a)}(x,y) := (v^{(a)}(y,x))^* = d(Q_a x, y).$$

A Jordan bimodule $\mathcal{M}$ induces a homotope bimodule $(M^+)^{(a)}$ for the homotopic Jordan algebra $J := (V^+)^{(a)}$ via

$$U_x^{(a)}(m) := q(x)q(a)m, \quad V_x^{(a)}(m) := \{x,a,m\}, \quad V_{x,y}^{(a)}(m) := \{x, Q_a y, m\}.$$

and at the same time a J-bimodule structure $(M^-)^{*(a)}$ in the opposite module M^- via

$$U^{*(a)}(x)(m) := q(a)q(x)m, \quad V_x^{*(a)}(m) := \{a,x,m\}, \quad V_{x,y}^{*(a)}(m) := \{Q_a x, y, m\}.$$

If J is a Jordan algebra or triple and $z \in J$, any bimodule M for J, or multiplication specialization $\mu = (p,\ell)$ of J in A, induces homotopic and duotopic bimodules $M^{(z)}, M^{*(z)}$ or multiplication specializations $\mu^{(z)}$, $\mu^{*(z)}$ for the Jordan algebra $J^{(z)}$ via[5]

$$u^{(z)}(x) := p(x)p(z), \quad v^{(z)}(x) = \ell(x,z), \qquad u^{*(z)}(x) := p(z)p(x), \quad v^{*(z)}(x) = \ell(z,x).$$

Proof: The result for $\mu^{(z)}$ is well-known; in the bimodule case $\mathcal{E} = \mathcal{V} \oplus \mathcal{M}$ is again a Jordan pair, so $\mathcal{E}^{*(a)} = (V^+ \oplus M^+)^{(a)} = (V^+)^{(a)} \oplus (M^+)^{(a)}$ is a Jordan algebra, and thus $(M^+)^{(a)}$ is a Jordan algebra bimodule for $(V^+)^{(a)}$. The dual bimodule situation is a special case of the multiplication specialization case, so we verify only the latter. Omitting all the superscripts on q, d (which are clear from the context $x, y \in V^+, a, b \in V^-$), we check the conditions (QA1-6). **(QA1)** is $v^{*(a)}(x^2) = d(a, Q_x a) = d(Q_a x, x) = v^{*(a)}(x,x)$ by (JP2). **(QA2)** is $v^{*(a)}(x^{(3,a)}) = d(a, Q_x Q_a x) = d(Q_a x, Q_x a)$ $[= v^{*(a)}(x, x^{(2,a)})] = d(Q_a Q_x a, x)$ $[= v^{*(a)}(x^{(2,a)}, x)]$ by D-Power Shifting (0.2.2). **(QA3)** is $u^{*(a)}(x^{(2,a)}, y) = q(a)q(Q_x(a), y)$ equals by (0.1.3) both $q(a)\big(q(x,y)d(a,x) - d(y,a)q(x)\big) = q(a)q(x,y)d(a,x) - d(a,y)q(a)q(x)\big)$ [by (JP1)] $= u^{*(a)}(x,y)v^{*(a)}(x) - v^{*(a)}(y)u^{*(a)}(x)$ and dually $q(a)\big(d(x,a)\, q(x,y) - q(x)d(a,y)\big) = d(a,x)\, q(a)\, q(x,y) - q(a)\, q(x)\, d(a,y) = v^{*(a)}(x)\, u^{*(a)}(x,y)$ $-u^{*(a)}(x)v^{*(a)}(y)$. Similarly, the condition **(QA4)** is $u^{*(a)}(x^{(3,a)}, y) = q(a)q(Q_x Q_a x, y)$ equals by (0.1.3) both $q(a)\big(q(x,y)d(Q_a x, x) - d(y, Q_a x)q(x)\big) = q(a)q(x,y)d(a, Q_x a) - d(Q_a y, x)q(a)q(x)$ [by (JP2), (0.1.7)] $= u^{*(a)}(x,y)v^{*(a)}(x^{(2,a)}) - v^{*(a)}(y,x)\, u^{*(a)}(x)$ and equals also $q(a)\,\big(d(x, Q_a x)\, q(x,y)$ $-q(x)d(Q_a x, y)\big) = q(a)\big(d(Q_x a, a)q(x,y) - q(x)\, d(Q_a x, y)\big)$ [by (JP2)] $= d(a, Q_x a)\, q(a)\, q(x,y)$ $-q(a)q(x)d(Q_a x, y)$ [by (JP1)] $= v^{*(a)}(x^{(2,a)})u^{*(a)}(x,y) - u^{*(a)}(x)v^{*(a)}(x,y)$. **(QA5)** is $u^{*(a)}(x^{(2,a)}) = q(a)q(Q_x a) = q(a)\big(q(x)q(a)q(x)\big)$ [by (JP3)] $= (q(a)q(x))^2 = u^{*(a)}(x)^2$, analogously **(QA6)** is $u^{*(a)}(x^{(3,a)}) = q(a)q(Q_x Q_a x) = q(a)\big(q(x)q(a)q(x)q(a)q(x)\big)$ [by (JP3)] $= (q(a)q(x))^3 = u^{*(a)}(x)^3$.

A similar calculation shows that when J is a Jordan algebra or triple, any multiplication specialization of J induces one of $J^{(z)}$ as stated; alternately, the multiplication specialization of J in A induces one of $\mathcal{V}(J)$ in $M_{2,2}(A)$ and then of $(V^+)^{(z)} \cong J^{(z)}$ in $A^{++} \cong A$ by the pair result [**?**, §13.8, p. 146]. ■

Notice that the J-bimodules have duals, but not the Jordan algebra J itself: the definitions $U_x^* := Q_a Q_x$, $V_x^* = D_{a,x}$ definitely *do not* yield a quadratic Jordan algebra structure $(V^-)^{(a)}$, since in the pair case $x \in V^+$ but the maps U_x^*, V_x^* map V^- to V^-, and even in Jordan algebras and triples where $V^+ = V^- = J$ the axiom (QJ3) = (JP3) fails flagrantly (as is easily seen for special algebras), though (QJ2) = (JP1) holds.

Example 1.3 *If $J \subseteq A^+$ is a special Jordan system, any J-invariant subspace M of the regular A-bimodule becomes a Jordan bimodule for J via the compound-linear multiplication specialization $U_x := L_x R_x$, $V_x = L_x + R_x, V_{x,y} = L_x L_y + R_x R_y$ of J on M. The maps $L_a, R_a :\ A^{(a)} \longrightarrow A$ are commuting homomorphisms of associative algebras, and hence induce commuting linear specializations $\ell(x) := (L \circ L_a)(x) = L_{ax}$, $r(x) := (R \circ R_a)(x) = R_{xa} :\ A^{(a)} \to A \to \mathrm{End}(A)$, yielding a compound-linear multiplication specialization $u^{*(a)}(x) := \ell(x)r(x) = L_{ax}R_{xa} = U_a U_x$, $v^{*(a)}(x) =$*

[5] The existence of dual bimodules for Jordan algebras and pairs has been a closely guarded secret, and we wish to thank Deep Throat for permission to reveal their existence.

$\ell(x)+r(x) = L_{ax}+R_{xa} = V_{a,x}$, $v^{*(a)}(x,y) = \ell(x)\ell(y)+r(x)r(y) = L_{ax}L_{ay}+R_{xa}R_{ya} = V_{axa,y}$. *Thus the dual homotope multiplication specialization and module in this case are have clear associative backgrounds.* ■

Universal gadgets for multiplication specializations of Jordan algebras and triples are well known. Jordan pairs too have a universal gadget for multiplication specializations, the **universal multiplication envelope** $\mathcal{UME}(\mathcal{V})$ (introduced by Loos [**?**, §13, pp. 141-143] as $U(\mathcal{V})$, compare [**?**, p. 289-290] for the algebra case), a unital associative $\mathcal{U}$ with 2×2 matrix grading, together with a **universal multiplication specialization** $\mu_u : \mathcal{V} \to \mathcal{U}$, having the universal property that every multiplication specialization $\mathcal{V} \xrightarrow{\mu} \mathcal{A}$ factors through the universal one

(1.4)
$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\ \mu\ } & \mathcal{A} \\ \mu_u \searrow & & \nearrow \widehat{\mu} \\ & \mathcal{UME}(\mathcal{V}) & \end{array}$$

via a unique homomorphism $\widehat{\mu}$ of unital 2×2-graded associative algebras. This implies, in particular, that $\mathcal{UME}$ is unique up to isomorphism and is generated by the universal elements $\tilde{\tilde{e}}^+, \tilde{\tilde{e}}^-$, $\tilde{\tilde{q}}^{\tau,-\tau}(x) \in \mathcal{U}^{\tau,-\tau}$, $\tilde{\tilde{d}}^{\tau,\tau}(x,a) \in \mathcal{U}^{\tau,\tau}$ for $x \in V^\sigma, a \in V^{-\sigma}$. The elements of $\mathcal{UME}(\mathcal{V})$ are to be thought of as *generic outer multiplications* by $\mathcal{V}$ acting linearly on all possible $\mathcal{V}$-bimodules, in particular, on all extensions $\tilde{\mathcal{V}} \supseteq \mathcal{V}$.

The universal property is always a two-way street: since composing the universal multiplication specialization μ_u of $\mathcal{V}$ in $\mathcal{UME}(\mathcal{V})$ with any graded homomorphism $\mathcal{UME}(\mathcal{V}) \xrightarrow{\varphi} A$ yields a multiplication specialization $\mathcal{V} \xrightarrow{\varphi\circ\mu_u} A$ with $\widehat{\varphi\circ\mu_u} = \varphi$, we see that the multiplication specializations μ of $\mathcal{V}$ are in 1-1 correspondence with the graded associative homomorphisms φ of the universal gadget $\mathcal{UME}(\mathcal{V})$. The universal multiplication specialization μ_u turns the universal envelope $\mathcal{U} = \mathcal{UME}(\mathcal{V})$ into a **universal cyclic bimodule** $\mathcal{M}(\mathcal{U},\mu_u)$; every cyclic $\mathcal{V}$-bimodule is a homomorphic image of $\mathcal{M}(\mathcal{U},\mu_u)$.

The standard model of $\mathcal{UME}$ is F/I for F the free unital associative Φ-algebra generated by all $\tilde{\tilde{e}}^\sigma$ ($\tilde{\tilde{e}}^+ + \tilde{\tilde{e}}^- = 1$), $\tilde{\tilde{q}}^{\tau,-\tau}(x)$, $\tilde{\tilde{d}}^{\tau,\tau}(x,a)$ and I is the ideal generated by $(\tilde{\tilde{e}}^+)^2 = \tilde{\tilde{e}}^+$ and all elements needed to make $\tilde{\tilde{d}}$ linear in x, a and $\tilde{\tilde{q}}$ quadratic in x [all $\tilde{\tilde{d}}(\alpha x+x',a) - \alpha\tilde{\tilde{d}}(x,a) - \tilde{\tilde{d}}(x',a)$, $\tilde{\tilde{d}}(x,\alpha a + a') - \alpha\tilde{\tilde{d}}(x,a) - \tilde{\tilde{d}}(x,a')$, $\tilde{\tilde{q}}(\alpha x) - \alpha^2\tilde{\tilde{q}}(x)$, $\tilde{\tilde{q}}(\alpha x+x',a) - \alpha\tilde{\tilde{q}}(x,a) - \tilde{\tilde{q}}(x',a)$], and insure that (QS1-5), hence also (QS6-9), and their linearizations hold [all elements $LHS - RHS$ in (QS1-5), plus the x-linearizations of the cubic relations (QS1),(QS6),(QS7) and the quartic relation (QS3)]. The defining relations (JP1-3), (0.1.1-6) show that if $\{x_i\}$ is a set of graded generators $x_i \in V^{\tau(i)}$ for $\mathcal{V}$, then the operators $\tilde{\tilde{q}}(x_i), \tilde{\tilde{q}}(x_i,x_j), \tilde{\tilde{d}}(x_i,x_j)$ is a set of generators for $\mathcal{UME}(\mathcal{V})$.

Since the set of generators for both $\mathcal{U}$ and I are homogeneous and invariant under the **reversal involution** [**?**, 13.2d p. 142] of F (determined by $(\tilde{\tilde{q}}^{\tau,-\tau}(x))^* := \tilde{\tilde{q}}^{\tau,-\tau}(x), (\tilde{\tilde{d}}^{\tau,\tau}(x,a))^* := \tilde{\tilde{d}}^{-\tau,-\tau}(a,x)$), the quotient $\mathcal{UME}$ inherits the matrix grading and involution. This leads to the **Duality Principle** [**?**, Prop. 2.9, p.19]: if a Jordan pair operator $\tilde{\tilde{m}} \in \mathcal{UME}(\mathcal{V})$ is an identity, $\tilde{\tilde{m}} = 0$ in $\mathcal{UME}$, then its reversal $\tilde{\tilde{m}}^*$ is also an identity, $\tilde{\tilde{m}}^* = 0$ in $\mathcal{UME}$. We have tacitly used this reversal involution in the second part of (0.2.6); alternately, we now can recognize this part when $m \geq 4$ as $V^{*(a)}_{x^{m-4},U_xy} - V^{*(a)}_{x^{m-3},\{x,y\}} + V^{*(a)}_{x^{m-2},y}$, which vanishes on the split null extension $(V^+)^{(a)} \oplus (A^{-,-})^{*(a)}$ of the duotope.

The involution $*$ leads naturally to dual specializations. If A is an associative algebra with 2×2 matrix grading, then its opposite algebra A^{op} has an *opposite grading* given by $(A^{op})^{\sigma,\tau} := A^{-\tau,-\sigma}$. When $A = \mathrm{End}(M_1 \oplus M_{-1})$ with matrix grading given by e^1, e^{-1}, this opposite grading is that given by $e^{\sigma*} := e^{-\sigma}$, since $e^{\sigma*} \cdot_{op} A \cdot_{op} e^{\tau*} = e^{-\sigma} \cdot_{op} A \cdot_{op} e^{-\tau} = e^{-\tau}Ae^{-\sigma} = A^{-\tau,-\sigma}$. The involution on $\mathcal{U} = \mathcal{UME}(\mathcal{V})$ is an isomorphism $\mathcal{U} \xrightarrow{*} \mathcal{U}^{op}$, and since $(A^{\sigma,\tau})^* = (e^\sigma Ae^\tau)^* = (e^\tau)^*A(e^\sigma)^* = e^{-\tau}Ae^{-\sigma} = A^{-\tau,-\sigma}$ this is an isomorphism of graded algebras. Any multiplication specialization

$\mathcal{V} \xrightarrow{\mu} A$ induces a graded homomorphism $\mathcal{U} \xrightarrow{\widehat{\mu}} A$ and hence a graded homomorphism of their opposite algebras $\mathcal{U}^{op} \xrightarrow{\widehat{\mu}^{op}} A^{op}$. Thus we obtain a **dual multiplication specialization** (cf. [**?**, 2.5 p.17]) μ^* of $\mathcal{V}$ in A^{op} via the composition $\mathcal{U} \xrightarrow{*} \mathcal{U}^{op} \xrightarrow{\widehat{\mu}^{op}} A^{op}$. The dual has the action $\mu^*(\widetilde{\widetilde{q}}^{\sigma,-\sigma}(x)) = q(x)$, $\mu^*(\widetilde{\widetilde{d}}^{\sigma,\sigma}(x,a)) = d(a,x)$, with $\widehat{\mu^*} = \widehat{\mu}^{op} \circ *$. In fact, from this action one verifies directly that $\mu^* = (q^*, d^*)$ satisfies the axioms (QS1-5) in A^{op}. Note that the dual is a specialization into the *opposite* matrix-graded algebra. For Jordan triples or algebras, the dual μ^* of $\mu = (p,\ell)$ or (u,v) has $(p^*(x), \ell^*(x,y)) = (p(x), \ell(y,x))$ or $(u^*(x), v^*(y)) = (u(x), v(y))$ [but note $v^*(x,y) := v^*(x)\cdot_{op} v^*(y) - u^*(x,y) = v^*(y)\cdot v^*(x) - u(x,y) = v(y,x)$]. If $\mathcal{M}$ is a bimodule for $\mathcal{V}$ the opposite module $\mathcal{M}^{op}$ (M regarded as a right A^{op}-module with opposite grading $(M^{op})^\sigma := M^{-\sigma}$) becomes a dual right $\mathcal{V}$-bimodule under the dual representation.

We will rapidly get tired of writing $\widetilde{\widetilde{q}}^{\tau,-\tau}(x), \widetilde{\widetilde{d}}^{\tau,\tau}(x,a)$ for the generators of $\mathcal{U}$ and simply write $\widetilde{\widetilde{q}}(x), \widetilde{\widetilde{d}}(x,a)$ when the indices are understood, keeping the $\approx$ to remind us of universality. In fact, we sometimes omit $\approx$ and just write $Q_x, D_{x,a}$ in place of their preimages ($Q_x = \widehat{\mu}_r(\widetilde{\widetilde{q}}(x)), D_{x,a} = \widehat{\mu}_r(\widetilde{\widetilde{d}}(x,a))$) under the regular representation μ_r, and say "in the universal envelope", "in $\mathcal{U}$", or just "universally".

If $\mathcal{V}$ is a subalgebra of $\widetilde{\mathcal{V}}$, we denote by $\mathcal{UME}(\mathcal{V}|\widetilde{\mathcal{V}})$ the subalgebra of $\mathcal{UME}(\widetilde{\mathcal{V}})$ generated by 1 and all $\widetilde{d}(x,a), \widetilde{q}(x)$ for $x, a \in \mathcal{V}$, and we have natural epimorphisms $\mathcal{UME}(\mathcal{V}) \to \mathcal{UME}(\mathcal{V}|\widetilde{\mathcal{V}}) \to \mathcal{M}(\mathcal{V}|\widetilde{\mathcal{V}})$ via $d(x,a), q(x) \to \widetilde{d}(x,a), \widetilde{q}(x) \to \widetilde{D}_{x,a}, \widetilde{Q}_x \in \mathrm{End}(\widetilde{\mathcal{V}})$. In particular, $\widetilde{\mathcal{V}}$ becomes a left $\mathcal{UME}(\mathcal{V})$-module, and we can form $\widetilde{\widetilde{m}}(\tilde{x})$ for any $\widetilde{\widetilde{m}} \in \mathcal{UME}(\mathcal{V})$ and any $\tilde{x} \in \widetilde{\mathcal{V}}$. We also have the **Action Principle**: If a Jordan pair operator $\widetilde{\widetilde{m}} \in \mathcal{UME}(\mathcal{V})$ is zero as a bimodule operator, $\widetilde{\widetilde{m}} = 0 \in \mathrm{End}(\mathcal{M})$ for all $\mathcal{V}$-bimodules $\mathcal{M}$, then $\widetilde{\widetilde{m}} = 0$ in $\mathcal{UME}(\mathcal{V})$; indeed if $\widetilde{\widetilde{m}}$ is zero on the universal cyclic module $\mathcal{M}(\mathcal{U}, \mu_u)$ then $0 = \widetilde{\widetilde{m}}(1_{\mathcal{U}}) = \widehat{\mu_u}(\widetilde{\widetilde{m}})1_{\mathcal{U}} = \widetilde{\widetilde{m}}$ implies $\widetilde{\widetilde{m}} = 0$ in $\mathcal{U}$ [note that $\widehat{\mu_u} = \mathbf{1}_{\mathcal{U}}$ by uniqueness in (1.2)].

Another formulation of the Action Principle is that an operator $\widetilde{\widetilde{m}} \in \mathcal{U}$ is zero iff it is zero as an operator in all extensions $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$: if $\widetilde{\widetilde{m}} = 0$ on $\widetilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{M}(\mathcal{U}, \mu_u)$ then $\widetilde{\widetilde{m}} = 0$ on $\mathcal{M}(\mathcal{U}, \mu_u)$, so $\widetilde{\widetilde{m}} = 0$ in $\mathcal{U}$. Conversely, if $\widetilde{\widetilde{m}}$ vanishes in $\mathcal{U}$ then it vanishes on all $\widetilde{m} \in \widetilde{\mathcal{V}}$ since $\mathcal{M} = \mathcal{M}(V|\widetilde{\mathcal{V}})\widetilde{m}$ is a Jordan bimodule, and $\widetilde{\widetilde{m}} = 0$ on $\mathcal{M}$ implies $\widetilde{\widetilde{m}} = 0$ on $\widetilde{m}$.

2 Universal Polynomial Envelope

Jordan algebras and pairs have linear *outer* multiplications $\cup_x : a \to U_x a, Q_x a$ and $V_{x,a}, D_{x,a} : y \to \{x,a,y\}$ which are linear operators, but they also have *inner* multiplications $\cap_x : a \to Q_a x$ mapping $V^{-\sigma} \to V^{-\sigma}$ which are quadratic rather than linear operators. We can interpret these as mappings on the associated polarized Jordan triple system $\mathcal{V}^p := V^+ \oplus V^-$ by setting $\cap_{V^\sigma}(V^\sigma) = \{V^\sigma, V^\sigma, V\} = 0$. The full **polynomial multiplication algebra** $\mathcal{PM}(\mathcal{V}) \subseteq \mathrm{Pol}(\mathcal{V})$ is the associative algebra of polynomial maps in several variables on $\mathcal{V}$ generated by the $Q_x, D_{x,a}, \cap_x$.[6] The easiest approach to these polynomials is through the free product of $\mathcal{V}$ with a free pair.

Recall from your subconscious that the free Jordan pair $\mathcal{FJP}_\Phi[X]$ over Φ on nonempty sets $X = X^+ \uplus X^-$ of graded generators is the quotient of the free Φ-module on the free pair monad $\mathcal{FPM}_\Phi[X]$ on the generators X, divided out by the ideal $\mathcal{I}_\Phi(X)$ generated by the Jordan pair identities (JP1-JP3) as well as their linearizations (JP1)′, (JP3)′, (JP3)″.[7] We call it the *free Jordan*

[6] It is not true that $\mathcal{UPE}(\mathcal{V})$ is generated by $q(x_i), q(x_i,x_j), d(x_i,x_j), \cap_{x_i}$ for generators $\{x_k\}$ of $\mathcal{V}$ ($\cap_{Q_{x_i}x_j} : a \to Q_a Q_{xi} x_j$ is not directly expressible in terms of these).

[7] If we used this collection of relations, it would suffice to consider only those relations where x, y, a are themselves monomials. The full list of linearizations would involve further linearizing all the quadratic relations: replacing $x \to x + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, $a \to a + \lambda_4 a_1$, we would have to add for (JP1) the coefficients of $\lambda_1\lambda_2$, for (JP2) those of $\lambda_1, \lambda_4, \lambda_1\lambda_4$, and for (JP3) those of $\lambda_1\lambda_2, \lambda_1\lambda_2\lambda_3, \lambda_4, \lambda_1\lambda_4, \lambda_1^2\lambda_4, \lambda_1\lambda_2\lambda_4, \lambda_1\lambda_2\lambda_3\lambda_4$. But that's too steep a price to pay!

pair $\Phi\langle X\rangle$ *on the free graded variables* $X^+ \uplus X^-$ *over* Φ, using pointy brackets to distinguish it from the scalar polynomial ring $\Phi[X]$ in ungraded *scalar variables.* Here the free pair monad consists of all pair monomials in the generators, constructed recursively by taking in degree 1 the generators $x_i^{\pm}$, and if $p^\sigma, q^\sigma, a^{-\sigma}$ of degrees d, e, f have been constructed, then $m^\sigma = Q_p a, \{p, a, q\} = \{q, a, p\}$ are monomials of degrees $2d+f, d+e+f$. The ideal $\mathcal{I}_\Phi(X)$ is generated by the relations (JP1-3),(JP1)$'$, (JP3)$'$, (JP3)$''$. The quotient Φ-module $\mathcal{FJP}_\Phi[X] := \mathcal{FPM}_\Phi[X]/\mathcal{I}_\Phi(X)$ becomes a Jordan pair by defining $Q_{\sum_i p_i}(\sum_j a_j) = \sum_{i,j} Q_{p_i} a_j + \sum_{i<k,j}\{p_i, a_j, p_k\}$. We will speak of *THE free Jordan pair* $\Phi\langle X\rangle$ over Φ when $X^\sigma = \{x_1^\sigma, x_2^\sigma, \ldots\}$ $(\tau = \pm)$ are both countably infinite sets of indeterminates; its elements may be thought of as universal Jordan pair polynomials in any (necessarily finite) number of variables.

The free pair on X is graded by degree in each variable, and agrees with the free monad up to degree 4 (the lowest-degree Jordan identities are (JP1), (JP2) of degree 5), in particular has a natural imbedding $X \xrightarrow{\text{in}} \Phi\langle X\rangle$. It enjoys the usual universal property, that every set-theoretic map $X^\pm \xrightarrow{\varphi} V^\pm$ extends uniquely to a homomorphism $\Phi\langle X\rangle \xrightarrow{\widetilde{\varphi}} \mathcal{V}$ of Jordan pairs over Φ. The universal property leads by universal nonsense to the usual properties of the free object and yields a functor from sets to Jordan pairs over Φ.

In defining the generic polynomial envelope we make a further useful but unusual move: since the generic polynomials are supposed to act on all $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$, in particular all scalar extensions, we will include in the construction the *universal scalar extension* [8] $\mathcal{V}_{\widetilde{\widetilde{\Phi}}}$ by the *universal scalars*

$$\widetilde{\widetilde{\Phi}} := \Phi[\Lambda] := \Phi[\widetilde{\widetilde{\lambda}}_1, \widetilde{\widetilde{\lambda}}_2, \ldots]$$

for countably many independent indeterminates $\widetilde{\widetilde{\lambda}}_i$. For the generic Jordan polynomial envelope for a particular Jordan pair $\mathcal{V}$ and set X we will adopt the notation $\mathcal{V}\langle X\rangle$, using pointy brackets to distinguish it from the scalar polynomial extension $\mathcal{V}[X] = \mathcal{V} \otimes_\Phi \Phi[X]$. We construct $\mathcal{V}\langle X\rangle$ as the free product over $\widetilde{\widetilde{\Phi}}$ of $\mathcal{V}_{\widetilde{\widetilde{\Phi}}} = \mathcal{V}[\Lambda]$ with $\widetilde{\widetilde{\Phi}}\langle X\rangle$ to get $\mathcal{V}\langle X\rangle$, the polynomials in the graded *free variables* $X^+ \uplus X^-$ and the *free scalar variables* Λ with coefficients in $\mathcal{V}$ [note that we do not mention $\widetilde{\widetilde{\Phi}}$ explicitly in the notation, the variables $\widetilde{\widetilde{\lambda}}_i$ will be tacitly understood]. One advantage of this convention is that if $p(x_1, \ldots, x_n) = 0 \in \mathcal{V}\langle X\rangle$ of degree d_i in the variables x_i then *automatically* all its linearizations also vanish [due to the endomorphism of $\mathcal{V}\langle X\rangle$ sending $x_i \to \sum_{j=1}^N \lambda_{Ni+j} x_{Ni+j}$, choosing an $N > n$ and $N \geq d_i$ for all i].

There is no transparent way to view this algebra of Jordan polynomials as there is in the category of associative algebras, where the elements of $A\langle X\rangle$ are just linear combinations of strings $a_0 m_1 a_1 m_2 \cdots a_n m_n a_{n+1}$, $n \geq 0$, for nonzero $a_i \in A$ (allowing a_0, a_{n+1} to be absent) and nontrivial free noncommutative monomials $m_i = m_i(X)$ in the free associative algebra $\Phi\langle X\rangle$ (with the obvious multiplication and linearity in the variables a_i).[9] The elements of $\mathcal{V}\langle X\rangle$ can be thought of as **generic polynomials** in the sense of Martindale (see, for example, [**?**, p.111f]): noncommutative nonassociative Jordan polynomials in indeterminates x_i with coefficients from $\mathcal{V}[\Lambda]$. In this paper we will not be concerned with *generalized polynomial identities* in the sense of Martindale and Amitsur, *nonzero elements* of $\mathcal{V}\langle X\rangle$ which vanish on $\mathcal{V}$ or related pairs, but rather *generic polynomial identities*, the *zero elements* themselves. These are polynomials $p(x_1, \ldots x_n, a_1, \ldots a_m) = 0 \in \mathcal{V}\langle X\rangle$ where $p(x_1, \ldots x_n, y_1, \ldots y_m) \neq 0 \in \Phi\langle Y \uplus X\rangle$ is a nontrivial Jordan polynomial which vanishes for the particular substitutions $y_i \to a_i \in \mathcal{V}$ *and all possible substitutions* $x_j \to \tilde{b}_j$ for all pairs $\widetilde{\mathcal{V}}$ containing

[8] While it is not true that every scalar extension $\mathcal{V}_\Omega$ is a homomorphic image of $\mathcal{V}_{\widetilde{\widetilde{\Phi}}} = \mathcal{V}[\Lambda]$, every finite set of elements of $\mathcal{V}_\Omega$ lies in such a homomorphic image, and $\mathcal{V}_\Omega$ itself is such an image if Ω is countably generated as Φ-algebra.

[9] As pair theorists, we can blithely ignore the complications in the category of unital algebras, where we would want $1 \in A$ to remain the unit in $A\langle X\rangle$ and therefore must face the sort of collapse $am_1 1 m_2 b - a(m_1 m_2)b$ familiar from the case of free groups. Indeed, associative ring theory wants the entire center C of A to remain the center of $A\langle X\rangle$, forming the free product over C instead of Φ.

a homomorphic image of $\mathcal{V}$. In this case we will say $p(x_1, \dots x_n, a_1, \dots a_m)$ vanishes *universally* or *generically* in the x_i, and to emphasize this genericity will write $p(\widetilde{\widetilde{x}}_1, \dots \widetilde{\widetilde{x}}_n, a_1, \dots a_m) = 0$.

The easiest way to form this free Jordan product is to present $\mathcal{V}$ in the most egregious way (take indeterminates $Y^\sigma = V^\sigma$ and write $\mathcal{V} \cong \Phi\langle X\rangle/K$ induced from the natural inclusion $Y \xrightarrow{\text{in}} \mathcal{V}$), and then form $\mathcal{V}\langle X\rangle := \widetilde{\widetilde{\Phi}}\langle X \uplus Y\rangle/K$ (dividing out by the Φ-relations K in the variables Y defining $\mathcal{V}$, but no further relations in the variables X other than those $\mathcal{I}_{\widetilde{\widetilde{\Phi}}}(X \uplus Y)$ imposed in the formation of $\widetilde{\widetilde{\Phi}}\langle X \uplus Y\rangle$). There are natural inclusions $X \xrightarrow{\sigma_u} \mathcal{V}\langle X\rangle$, $\mathcal{V} \xrightarrow{\iota_u} \mathcal{V}\langle X\rangle$. This has the universal property that any (graded) set-theoretic map $X^\sigma \xrightarrow{\sigma} \widetilde{V}^\sigma$ together with Φ-homomorphisms $\Lambda \to \Omega$ of Φ-algebras and $\mathcal{V} \xrightarrow{\varphi} \widetilde{\mathcal{V}}$ of Jordan pairs for an Ω-algebra $\widetilde{\mathcal{V}}$, extends uniquely to a Jordan pair homomorphism $\mathcal{V}\langle X\rangle \xrightarrow{\widetilde{(\varphi,\sigma)}} \widetilde{\mathcal{V}}$ of Φ-pairs.

When X is countably infinite we call $\mathcal{V}\langle X\rangle$ THE **universal polynomial envelope** $\mathcal{UPE}(\mathcal{V})$ of $\mathcal{V}$ over Φ. The universal property leads by universal nonsense to standard properties of the free object: it determines a functor from Jordan-pairs-and-sets to Jordan pairs, distinct variables can be adjoined one-by-one or in one fell swoop,

$$\mathcal{V}\langle X\rangle\langle Y\rangle \cong \mathcal{V}\langle X \uplus Y\rangle, \tag{2.1}$$

that a bijection of sets induces an isomorphism of polynomial envelopes

$$X_1 \cong X_2 \implies \mathcal{V}\langle X_1\rangle \cong \mathcal{V}\langle X_2\rangle, \tag{2.2}$$

in particular that the universal polynomial envelope is indifferent to countable extensions,

$$\mathcal{UPE}(\mathcal{V}) \cong \mathcal{UPE}(\mathcal{V}\langle Y\rangle) \qquad (Y \text{ countable}). \tag{2.3}$$

We have the **Action Principle** that $p = 0$ in $\mathcal{V}\langle X\rangle$ iff the map induced by p vanishes on all Jordan algebras $\widetilde{\mathcal{V}}$ with homomorphism (not necessarily an imbedding) $\mathcal{V} \xrightarrow{\varphi} \widetilde{\mathcal{V}}$. Certainly if $p(x_1, \dots, x_n) = 0$ in $\mathcal{V}\langle X\rangle$ then for any $\tilde{b}_1, \dots \tilde{b}_n \in \widetilde{\mathcal{V}}$ and $\sigma(x_i) = \tilde{b}_i$ we have $0 = \widetilde{(\varphi,\sigma)}(p) = p(\tilde{b}_1, \dots, \tilde{b}_n)$ and p vanishes on $\widetilde{\mathcal{V}}$. Conversely, if p vanishes on all pairs $\widetilde{\mathcal{V}}$ it certainly vanishes on the pair $\mathcal{V}\langle X\rangle$ itself, so $p = \widetilde{(\iota_u, \sigma_u)}(p) = 0$.

Any generic polynomial envelope $\mathcal{V}\langle X\rangle$ is again X-graded, with the elements in degree 0 being precisely $\mathcal{V}$. We have a graded decomposition $\mathcal{V}\langle X\rangle = \mathcal{V} \oplus \bigoplus_{x\in X} \mathcal{V}_x \oplus \mathcal{V}_2$ into homogeneous parts of degree 0,1, and ≥ 2. Importantly, *the homogeneous polynomials of degree 1 are naturally isomorphic to the universal multiplications of the universal multiplication envelope.*

Quadratic Envelope Imbedding 2.4 *Fix an even and odd variable $x_0^\pm \in X^\pm$, set $x_0 := x_0^+ \oplus x_0^-$. Then the cyclic $\mathcal{V}$-sub-bimodule $M = \mathcal{M}(\mathcal{V})x_0 = \mathcal{V}_{x_0^+} \oplus \mathcal{V}_{x_0^-} \subseteq \mathcal{V}\langle X\rangle$ is naturally isomorphic to the universal cyclic bimodule $\mathcal{M}(\mathcal{U}, \mu_u) = \mathcal{U}(1_\mathcal{U}) = \mathcal{U}$ via the inverse linear maps $\mathcal{U} \xrightarrow{\psi} M$ given by $\widetilde{\widetilde{m}} \to eval_{x_0}(\widetilde{\widetilde{m}}) = \widetilde{\widetilde{m}}(x_0)$ and $M \xrightarrow{\varphi_0} \mathcal{U}$ by $p(x_0) \to p(1_\mathcal{U})$. Under this isomorphism $\mathcal{UME}(\mathcal{V}) \cong \mathcal{V}_{x_0^+} \oplus \mathcal{V}_{x_0^-}$ and $\mathcal{UME}(\mathcal{V})^{\pm,\tau} \cong \mathcal{V}_{x_0^\tau}$ as spaces.*

Proof: We have a multiplication representation $\mathcal{V} \to \mathcal{M}(\mathcal{V}|\mathcal{V}_{x_0})$, so by the universal property of $\mathcal{U}$ this induces an algebra homomorphism $\mathcal{U} \to \mathcal{M}(\mathcal{V}|\mathcal{V}_{x_0})$, which can be followed by the evaluation map $eval_{x_0}$. Since evaluation is a $\mathcal{V}$-bimodule map, the resulting composite $\psi : \widetilde{\widetilde{m}} \mapsto \widetilde{\widetilde{m}}(x_0)$ is a homomorphism of cyclic $\mathcal{V}$-bimodules.

The recursive construction shows the polynomials in $\mathcal{V}_{x_0}$ have the form $p(x_0) = \widetilde{\widetilde{m}}(x_0)$ for a multiplication operator $\widetilde{\widetilde{m}}$: in degree 1 there is just $x_0^\sigma = e^\sigma(x_0)$, if true for degrees less than n then in degree n any homogeneous degree 1 monomial must be $Q_p q$ (where p must be constant and by recursion $q = \widetilde{\widetilde{m}}(x_0)$ for a multiplication operator $\widetilde{\widetilde{m}}$) or $\{p, q, r\}$ (where we must have two constant factors and one an operator on x_0 by recursion) so $\{p,q,\overset{\approx}{m}(x_0)\} = (D_{p,q}\overset{\approx}{m})(x_0)$ or $\{p,\overset{\approx}{m}(x_0),r\} = (Q_{p,r}\overset{\approx}{m})(x_0)$. The specializations $x_i \to 0, x_0 \to 1_{\mathcal{U}}$ (i.e., $x_0^\sigma \to e^\sigma$) induce a homomorphism $\mathcal{V}\langle X\rangle \xrightarrow{\varphi} \mathcal{V}\oplus M$ sending $f(x_0,x_1,\dots,x_n) \to f(1_{\mathcal{U}},0,\dots,0)$ by the universal property, which restricts to a $\mathcal{V}$-bimodule homomorphism $\mathcal{V}_{x_0} \xrightarrow{\varphi_0} \mathcal{U}1_{\mathcal{U}} = \mathcal{U}$ sending $p(x_0) \to p(1_{\mathcal{U}}) = p$.

These two homomorphisms are inverses since $(\varphi_0\circ\psi)(\overset{\approx}{m}) = \varphi_0(\overset{\approx}{m}(x_0)) = \overset{\approx}{m}(1_{\mathcal{U}}) = \overset{\approx}{m}$ and $(\psi\circ\varphi_0)(\overset{\approx}{m}(x_0)) = \psi(\overset{\approx}{m}(1_{\mathcal{U}})) = \psi(\overset{\approx}{m}) = \overset{\approx}{m}(x_0)$. Thus the two bimodules are isomorphic. It is clear that under this bimodule isomorphism $\mathcal{UME}(\mathcal{V})^{\sigma,\tau} = e^\sigma\mathcal{UME}(\mathcal{V})e^\tau = e^\sigma\mathcal{M}e^\tau$ corresponds to $e^\sigma\mathcal{V}_{x_0^\tau}$ and $\mathcal{UME}(\mathcal{V})^{\pm,\tau}$ to $\mathcal{V}_{x_0^\tau}$ as spaces. ∎

We remark that $\mathcal{V}\langle X\rangle$ has no involution corresponding to the powerful reversal involution on $\mathcal{UME}(\mathcal{V})$. Nevertheless some traces of duality remain. For example, making our first use of the notation $\approx$ for generic variables, if $\overset{\approx}{x},\overset{\approx}{y},\overset{\approx}{a},\overset{\approx}{b}$ are distinct free variables and for some elements $x,y\in V^+$, $a,b,c\in V^-$ the quadratic polynomial $D^+_{x,a}Q^+_yQ^-_bQ_{\overset{\approx}{x}}c$ vanishes generically in $\overset{\approx}{x}$ (in all $\widetilde{\mathcal{V}}$ over $\mathcal{V}$, equivalently in $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle$), then its linearization $D^+_{x,a}Q^+_yQ^-_bQ_{\overset{\approx}{x},\overset{\approx}{y}}c$ vanishes generically as a bilinear function of $\overset{\approx}{x},\overset{\approx}{y}$ in $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{y},\overset{\approx}{a},\overset{\approx}{b}\rangle = \mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle\langle\overset{\approx}{y},\overset{\approx}{b}\rangle$, so $\big(D^+_{x,a}Q^+_yQ^-_bD^+_{\overset{\approx}{x},c}\big)\overset{\approx}{y} = 0$ in $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle_{\overset{\approx}{y}}$, and under the isomorphism $D^+_{x,a}Q^+_yQ^-_bD^+_{\overset{\approx}{x},c} = 0$ in $\mathcal{UME}(\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle)^{+,+}$. But then its reverse $D^-_{c,\overset{\approx}{x}}Q^-_bQ^+_yD^-_{a,x}$ also vanishes in $\mathcal{UME}\big(\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle\big)^{-,-}$, leading (via the isomorphism, this time of $\mathcal{UME}(\mathcal{V})^{\pm,-}$ with $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle_{\overset{\approx}{b}}$) to $D^-_{c,\overset{\approx}{x}}Q^-_bQ^+_yD^-_{a,x}(\overset{\approx}{b}) = 0$ in $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a},\overset{\approx}{y},\overset{\approx}{b}\rangle$ and hence (via the homomorphism $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a},\overset{\approx}{y},\overset{\approx}{b}\rangle \to \mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle$ induced by $\overset{\approx}{x}\to\overset{\approx}{x},\overset{\approx}{y}\to 0,\overset{\approx}{a}\to\overset{\approx}{a},\overset{\approx}{b}\to\overset{\approx}{a}$) to an unexpected relation $D^-_{c,\overset{\approx}{x}}Q^-_bQ^+(y)\{a,x,\overset{\approx}{a}\} = 0$ back in $\mathcal{V}\langle\overset{\approx}{x},\overset{\approx}{a}\rangle$. Notice that vanishing of a function of $\overset{\approx}{x},\overset{\approx}{y}$ has led to vanishing of a function of $\overset{\approx}{x},\overset{\approx}{a}$ (which is exactly what happens in the universal multiplication envelope, where a relation like $d(x,a)=0$ as a universal map on $\overset{\approx}{x}$ in modules M^σ leads to $d(a,x)=0$ universally on $\overset{\approx}{a}$ in $M^{-\sigma}$). One suspects that vanishing of the original quadratic function of x implies some "dual" quadratic function vanishes, but I have been unable to find examples. At any rate, universal vanishing of a generic Jordan pair polynomial has powerful unexpected consequences.

In the construction of algebras of fractions [?], [?], [?] it is important to know whether certain multiplicative relations, such as a multiplication operator T being a structural transformation $Q_{Tx}(y) = TQ_xT*(y)$, hold generically on all extensions $\tilde{J}$ rather than just on J itself.

Injection Question 2.5 *If J is a subsystem of $\tilde{J}$, or more generally if $J \xrightarrow{\varphi} \tilde{J}$ is injective, is $\mathcal{UPE}(J) \xrightarrow{\mathcal{U}(\varphi)} \mathcal{UPE}(\tilde{J})$ also injective?* ∎

Though no counterexamples seem to be known, one expects the answer to be negative. It is reasonable to assume that if $\widetilde{\mathcal{V}}$ is obtained from $\mathcal{V}$ by adjoining some inverses s^{-1}, but in such a way that no elements of $\mathcal{V}$ die under the imbedding, there still might be some polynomials $\overset{\approx}{m}\in\mathcal{UPE}(\mathcal{V})$ which vanish on $\mathcal{V}$ itself, but not in some extension $\mathcal{V}'$, yet vanish on all extensions $\widetilde{\mathcal{V}}'$ of $\widetilde{\mathcal{V}}$ due to the restriction imposed by invertibility of $s^{-1}\in\widetilde{\mathcal{V}}'$.

3 Dominions

An **inner ideal** $I^{-\sigma} \lhd_{in} V^{-\sigma}$ is a subspace closed under inner multiplication, $Q_{I^{-\sigma}}V^\sigma \subseteq I^{-\sigma}$; then $\mathcal{V}(I^{-\sigma}) := (I^{-\sigma},V^\sigma)$ forms a subpair of $\mathcal{V}$. By (JP3) and (0.1.6), every element $s^{-\sigma}\in V^{-\sigma}$ determine closed and open *principal inner ideals* $K_s^{-\sigma} := \Phi s + Q_sV^\sigma$ and $I_s^{-\sigma} := Q_sV^\sigma$. In the theory of Jordan fractions an important role is played by a *sesqui-principal inner ideal* determined by a dominating pair. We say that an element s **dominates** the element n, written $s\succ n$, if there are pairs $\mathcal{N}_{s,n} = (N^{-\sigma},N^\sigma)$, $\mathcal{S}_{s,n} = (S^{-\sigma},S^\sigma)$ of globally-defined operators $M^\tau\in\mathrm{End}(V^\tau)$, $\tau=\pm$, such that

(3.1) **(Domination):** $Q_n = N^{-\sigma} Q_s = Q_s N^\sigma, \qquad Q_{n,s} = S^{-\sigma} Q_s = Q_s S^\sigma.$

Such pairs arise in the consideration of Jordan fractions $Q_s^{-1} n$ with "reduced" numerator n and denominator s. In practice (see 3.4.12 below, [2]) both S and N can be built from multiplications entirely within the original pair $\mathcal{V}$. We say the domination is **inner** if for $\sigma = \pm\tau$ both $S^\sigma \in D_{V^\sigma, V^{-\sigma}}$, $N^\sigma \in Q_{V^\sigma} Q_{V^{-\sigma}}$ are given as multiplications, and is **generic** (or that s **generically dominates** n, $s \succ_{gen} n$) if $S^\sigma \in d_{V^\sigma, V^{-\sigma}} \subseteq \mathcal{UME}(\mathcal{V})^{\sigma,\sigma}$ and $N^\sigma \in q_{V^\sigma} q_{V^{-\sigma}} \subseteq \mathcal{UME}(\mathcal{V})^{\sigma,\sigma}$ are given as generic multiplication operators with $S^- = (S^+)^*$, $N^- = (N^+)^*$ and (3.1) holding generically in $\mathcal{UPE}(\mathcal{V})$. In both cases N^σ, S^σ act on $\mathcal{V}$ as inner multiplications. Note that s automatically generically dominates all $n = \alpha s + Q_s a$ in the principal inner ideal K_s by (0.1.6), (JP1). In fact, any n dominated by s is already halfway in K_s, because such a pair (s, n) of dominator and dominatee determines an inner ideal which is almost principal.

Dominion Theorem 3.2 *If the element s dominates n, then the* **dominion**

(3.2.1) $$K^{-\sigma}_{s\succ n} := \Phi n + \Phi s + Q_s V^\sigma$$

is an inner ideal satisfying

(3.2.2) $$Q_{K^{-\sigma}_{s\succ n}} V^\sigma \subseteq Q_s V^\sigma = I_s^{-\sigma} \subseteq K_s^{-\sigma} \subseteq K^{-\sigma}_{s\succ n}.$$

The elements $x := \gamma n + \alpha s + Q_s a$, $y := \alpha s + Q_s a$, $z := Q_s a$ of the dominion have Q-operators which can be "divided by Q_s",

(3.2.3) $Q_n = N^{-\sigma} Q_s = Q_s N^\sigma$,
(3.2.4) $Q_{n,s} = S^{-\sigma} Q_s = Q_s S^\sigma$,
(3.2.5) $Q_z = Q_s Q_a Q_s$,
(3.2.6) $Q_y = B^{-\sigma} Q_s = Q_s B^\sigma \quad (B^{-\sigma} = B_{\alpha,s,a}, B^\sigma = B_{\alpha,a,s})$,
(3.2.7) $Q_{n,z} = M_a^{-\sigma} Q_s = Q_s M_a^\sigma$, $(M_a^{-\sigma} {=} S^{-\sigma} D_{s,a} {-} D_{n,a},\ M_a^\sigma {=} (M^{-\sigma})^*)$,
(3.2.8) $Q_{n,y} = G^{-\sigma} Q_s = Q_s G^\sigma \quad (G^\tau = \alpha S^\tau + M_a^\tau)$,
(3.2.9) $Q_x = X^{-\sigma} Q_s = Q_s X^\sigma \quad (X^\tau = \gamma^2 N^\tau + \gamma G^\tau + B^\tau)$

where $\tau = \pm\sigma$.

If s dominates n generically, then the **generic dominion** $\widetilde{\widetilde{K}}{}^{-\sigma}_{s\succ n} := \widetilde{\widetilde{\Phi}} n + \widetilde{\widetilde{\Phi}} s + Q_s \widetilde{\widetilde{V}}{}^\sigma$ *is likewise an inner ideal in $\mathcal{UPE}(\mathcal{V})$ satisfying $Q_{\widetilde{\widetilde{K}}{}^{-\sigma}_{s\succ n}} (\widetilde{\widetilde{V}}{}^\sigma) \subseteq Q_s(\widetilde{\widetilde{V}}{}^\sigma) = \widetilde{\widetilde{I}}{}_s^{-\sigma} \subseteq \widetilde{\widetilde{K}}{}^{-\sigma}_{s\succ n}$, and (3.2.3-9) hold in $\mathcal{UME}(\mathcal{V})$ for generic $\widetilde{\widetilde{\gamma}}, \widetilde{\widetilde{\alpha}} \in \widetilde{\widetilde{\Phi}}$ and $\widetilde{\widetilde{a}} \in \widetilde{\widetilde{V}}{}^\sigma$, with $T^{-\sigma} = (T^\sigma)^*$ for all $T = N, S, B, M_a, G, X$.*

Proof: We will omit all indices in the following arguments, since they are clear by context from the statements in the theorem. (2) will show that the dominion as defined in (1) is indeed an inner ideal, and (2) will follow from (9) since $Q_x V^\sigma = Q_s X^\sigma V^\sigma \subseteq Q_s V^\sigma$. So all that remains is to establish the formulas (3)-(9). (3),(4) are the definition (3.1) of domination $s \succ n$. The formula (5) is just (JP3), the Bergmann formula (6) is (0.1.6). For (7), we have $Q_{n,z} = Q_{n,Q_s a} = D_{s,a} Q_{s,n} - Q_s D_{a,n}$ [by (0.1.3)] $= D_{s,a}(Q_s S^\sigma) - Q_s D_{a,n}$ [by (3.1)] $= Q_s(M_a^\sigma)$ [by (JP1)] $= Q_s M_a^\sigma$, and dually $Q_{n,Q_s a} = Q_{s,n} D_{a,s} - D_{n,a} Q_s = S^{-\sigma} Q_s D_{a,s} - D_{n,a} Q_s = M_a^{-\sigma} Q_s = M_a^{-\sigma} Q_s$. Then (8) follows immediately from (3.1),(4),(7) since $Q_{n,y} = \alpha Q_{n,s} + Q_{n,z}$, and (9) follows similarly from (3),(8),(6) since $Q_x = Q_{\gamma n + y} = \gamma^2 Q_n + \gamma Q_{n,y} + Q_y$.

In the case of generic domination, (3.1) holds generically in $\mathcal{UME}(\mathcal{V})$; then $M_a^\sigma, S^\sigma, G^\sigma$ exist in $\mathcal{UME}(\mathcal{V})$ (B^σ already does), and satisfy (3.2.3-9) for $z = Q_s \widetilde{\widetilde{a}}$, $y = \widetilde{\widetilde{\alpha}} s + z$, $x = \widetilde{\widetilde{\gamma}} n + y \in \mathcal{UPE}(\mathcal{V})$. By definition of generic dominations we have $N^{-\sigma} = (N^\sigma)^*$, $S^{-\sigma} = (S^\sigma)^*$, and automatically $B^{-\sigma} = (B^\sigma)^*$, so the recipes in (3.7-9) guarantee that $T^{-\sigma} = (T^\sigma)^*$ for $T = M_a, G, X$ too. ■

This inner ideal $K_{s\succ n}$ is not bi-principal, since the formulas indicate that n is already "half in K_s", so a fraction $Q_s^{-1}n$ is really of degree -1 in s, not -2. The operator G provides important "glue" binding the two structural transformations N and B into a new structural X [**?**].

Note that $T = N, S, M_a, G, X$ *are not uniquely determined by* (3.4.3-9), though the T^σ are unique if Q_s is injective and $T^{-\sigma}$ are unique if Q_s is surjective or Q_s is generically injective and s generically dominates n. At the opposite extreme, if $Q_s = Q_n = Q_{s,n} = 0$ then *any* N^σ, S^σ will work, and need not satisfy any reasonable relation (see (3.3.1) below). Domination is thus a rather impersonal relation. A much closer relation, for forming properly "reduced" fractions [**?**],[**?**] is that of tight domination: we say s **tightly dominates** n if

(3.3.1a) $\quad Q_n = N^{-\sigma}Q_s = Q_sN^\sigma$ *for an inner multiplication* $N \in Q_VQ_V$,

(3.3.1b) $\quad Q_{n,s} = S^{-\sigma}Q_s = Q_sS^\sigma$ *for* $S \in D_{V,V}$ *an* **inner Lie struction,** *i.e.,*
$Q_{s^\tau(w),w} = S^\tau Q_w + Q_wS^{-\tau}$ *for all* $w \in V^\tau$,

(3.3.2) *there are* $q_2, q_3 \in V^\sigma$ *so that* $s_1 := s,\ s_2 := n,\ s_3 := Q_sq_2, s_4 := Q_sq_3 \in V^{-\sigma}$ *satisfy*
(Power Shifting): $S^{-\sigma}(s_i) = 2s_{i+1}, N^{-\sigma}(s_i) = s_{i+2},\ N^\sigma(q_i) = q_{i+2}$,

(3.3.3) (Two N): $\quad (S^{-\sigma})^2 = 2N^{-\sigma} + D_{s,q_2}, \quad (S^\sigma)^2 = 2N^\sigma + D_{q_2,s}$.

Multiplying (3.3.3) on the right and left by Q_s yields, via (3.3.1) and (JP1), the consequence

(3.3.4) (Two Q): $\quad S^{-\sigma}Q_{s,n} = 2Q_n + Q_{s_3,s} = Q_{s,n}S^\sigma$.

These conditions insure that when $\frac{1}{2} \in \Phi$ the domination is completely determined by the Lie struction $\mathcal{S}$. We have the obvious notion of *generic tight domination.*

Life improves the smaller our dominions get. It is easy to construct more tightly dominated subdominions inside a given dominion $K_{s\succ n}$.

Subdominion Theorem 3.4 *If s dominates n in $V^{-\sigma}$, then for any $c \in V^\sigma$ the element $s' := Q_sc$ more tightly dominates $n' := Q_sQ_cn$, inducing a* **subdominion**

(3.4.1) $$K^{-\sigma}_{s'\succ n'} = \Phi n' + \Phi s' + Q_{s'}V^\sigma = Q_s\big(\Phi c + Q_c(\Phi n + Q_sV^\sigma)\big) \subseteq Q_sK^\sigma_c.$$

(I) *If we set*

(3.4.2)
$$z' := Q_{s'}a = Q_sQ_cQ_sa, \quad y' := \alpha s' + z' = Q_s(\alpha c + Q_cQ_sa)),$$
$$x' := \gamma n' + y' = Q_s(\gamma Q_cn + \alpha c + Q_cQ_sa) \in K^{-\sigma}_{s'\succ n'} \subseteq I^{-\sigma}_s,$$
$$q'_2 := N^\sigma(c),\ q'_3 := N(Q_cn),\ q_{k+1} := N(c^{(k,n)}) \in V^\sigma \quad (k \ge 1),$$
$$s'_1 = s',\ s'_2 = n',\ s'_3 := Q_sQ_cQ_nc,\ s_k := Q_{s'}q'_k = Q_sc^{(k,n)} \in K_{s'\succ n'}\ (k \ge 1),$$

then all elements x' of the subdominion have Q-operators which can be divided by $Q_{s'}$,

(3.4.3) $\quad Q_{n'} = Q_{s'}N'^{\,\sigma} = N'^{\,-\sigma}Q_{s'}, \qquad (N'^{\,-\sigma} := Q_sQ_cN^{-\sigma}, \quad N'^{\,\sigma} := N^\sigma Q_cQ_s)$,
(3.4.4) $\quad Q_{n',s'} = Q_{s'}S'^{\,\sigma} = S'^{\,-\sigma}Q_{s'} \qquad \big(S'^{\,-\sigma} := D_{s,S(c)} - D_{n,c},\ S'^{\,\sigma} := D_{S(c),s} - D_{c,n}\big)$,
(3.4.5) $\quad Q_{z'} = Q_{s'}Q_aQ_{s'}$,
(3.4.6) $\quad Q_{y'} = Q_{s'}B^\sigma_{\alpha,a,s'} = B^{-\sigma}_{\alpha,s',a}Q_{s'}$,
(3.4.7) $\quad Q_{n',z'} = M'^{\,-\sigma}_aQ_{s'} = Q_{s'}M'^{\,\sigma}_a, \qquad \big(M'^{\,-\sigma}_a := Q_sQ_cM^{-\sigma}_a, \quad M'^{\,\sigma}_a := M^\sigma_aQ_cQ_s\big)$,
(3.4.8) $\quad Q_{n',y'} = G'^{\,-\sigma}Q_s = Q_sG'^{\,\sigma} \qquad \big(G'^{\,\tau} := \alpha S^\tau + M'^{\,\tau}_a\big)$
(3.4.9) $\quad Q_{x'} = Q_{s'}X'^{\,\sigma} = X'^{\,-\sigma}Q_{s'} \qquad \big(X'^{\,\tau} := \gamma^2N'^{\,\tau} + \gamma G'^{\,\tau} + B'^{\,\tau}\big)$.

Whenever s dominates n generically, any generic $\overset{\approx}{c}$ induces a generic $n' := Q_sQ_{\overset{\approx}{c}}n$ and a corresponding **generic subdomain** $K^{-\sigma}_{s'\succ n'} = \Phi n' + \Phi s' + Q_{s'}\overset{\approx}{V}^{-\tau}$ *satisfying* (3.4.3-9) *generically.*

(II) *Automatically* S', N' *of the derived dominion satisfy*

- $S' \in D_{\mathcal{V},\mathcal{V}} \subseteq \mathcal{UME}(\mathcal{V})$ *is an inner Lie struction* (3.3.1b);
- *Power Shifting* (3.3.2) *holds;*
- *Two* Q (3.3.4) *holds.*

(III) *Whenever* $\mathcal{S}$ *is already a Lie struction, the above operators* M_a^σ *for the subdominion coincide generically with those guaranteed by* (3.2.5),

$$M_a'^{-\sigma} := Q_s Q_c M_a^{-\sigma} = S'^{-\sigma} D_{s',a} - D_{n',a}, \quad M_a'^{\sigma} := M_a^\sigma Q_c Q_s = D_{a,s'} S'^{\sigma} - D_{a,n'} \ \text{ in } \mathcal{UME}(\mathcal{V}).$$

(IV) *Whenever* S *is already a Lie struction satisfying Power Shifting* (3.3.2) *and Two* Q (3.3.4), *and in addition satisfies the weak "gluing" identities*

$$(3.4.10) \quad N^\sigma Q_c + Q_c N^{-\sigma} + S^\sigma Q_c S^{-\sigma} = Q_{S^\sigma(c)} + Q_{N^\sigma(c),c},$$
$$(3.4.11a) \quad \Delta := D_{s,N^\sigma(c)} - D_{n,S^\sigma(c)} + D_{s_3,c} = 0,$$
$$(3.4.11b) \quad \Delta^* := D_{N^\sigma(w),s} - D_{S^\sigma(w),n} + D_{w,s_3} = 0 \ \text{ for all } w = c^{(2k-1,s)},$$

then s' *tightly dominates* n' *in the new subdominion* $K^{-\sigma}_{s'\succ n'}$: *besides* (3.3.1b), (3.3.2) *it satisfies Two* N (3.3.3) *and innernress* (3.3.1a) *since the new* N' *is inner in* $Q_V Q_V$ (*though perhaps The Innner Multiplication from the Black Lagoon*!):

$$(3.4.12) \quad \begin{aligned} N'^{-\sigma} &= Q_n Q_c + Q_{s_3,s} Q_c - Q_{s,n} Q_{S^\sigma(c),c} + Q_s Q_{S(c)} + Q_s Q_{N^\sigma(c),c} \\ &\in Q_{\Phi n+\Phi s+\Phi N^{-\sigma}(s)+\Phi S^{-\sigma}(s)} Q_{\Phi c+\Phi S^{-\sigma}(c)+N^{-\sigma}(c)} \subseteq Q_{K^{-\sigma}_{s\succ n}} Q_{V^\sigma} \subseteq \mathcal{UME}(\mathcal{V}), \\ N'^{\sigma} &= Q_c Q_n + Q_c Q_{s_3,s} - Q_{S^\sigma(c),c} Q_{s,n} + Q_{S(c)} Q_s + Q_{N^\sigma(c),c} Q_s \in Q_{V^\sigma} Q_{K^{-\sigma}_{s\succ n}}. \end{aligned}$$

Moreover, in this case (3.4.10-11) *for* S', N', s', n' *are inherited from* S, N *taking the same* $c' = c$.

(IV) *In particular, in the presence of* (3.4.10-11) *and* (3.3.4) *the element* $s'' := Q_{s'} c$ *always tightly dominates* $n'' := Q_{s'} Q_c n'$ *and the sub-subdominion* $K^{-\sigma}_{s''\succ n''}$ *is tight satisfying* (3.4.10-11).

Proof: (I): (1) follows from (3.2.1) and the definitions of s', n'. (3) follows from (JP3), (3.1) by $Q_{n'} = Q_{Q_s Q_c n} = Q_s Q_c Q_n Q_c Q_s = Q_s Q_c (Q_s N^{-\tau}) Q_c Q_s = Q_{s'} \big(N^{-\tau} Q_c Q_s\big)$, so $N'^{-\tau} = N^{-\tau} Q_c Q_s$, and dually $N'^{\tau} = Q_s Q_c N^{-\tau}$. For (4) we first note that generically we have

$$(3.4.13) \quad \begin{gathered} S'^{-\sigma} := D_{s,S(c)} - D_{n,c}, \quad S'^{\sigma} := D_{S(c),s} - D_{c,n} \\ \textit{satisfy} \quad D_{n,c} Q_s = Q_s S'^{\sigma}, \quad Q_s D_{c,n} = S'^{-\sigma} Q_s \ \text{ in } \mathcal{UME}(\mathcal{V}) \end{gathered}$$

since in $\mathcal{UME}(\mathcal{V})$ the element $S(c)$ satisfies $Q_s D_{c,n} + D_{n,c} Q_s = Q_{\{n,c,s\},s}$ [by (0.1.1)] $= Q_{Q_s S(c),s}$ [by (3.1)] $= Q_s D_{S(c),s} = D_{s,S(c)} Q_s$ [(by (JP1)]. Then $Q_{n',s'} = Q_{Q_s Q_c n, Q_s c} = Q_s Q_{Q_c n,c} Q_s = Q_s \big(Q_c D_{n,c}\big) Q_s = Q_s Q_c Q_s S'^{-\tau}$ [by (JP1) and (13)] $= Q_{s'} S'^{-\tau}$ [by (JP3)], and dually, yielding (4) generically.

Once (3-4) hold we know $s' \succ n'$ and by (3.2.3-9) that (3-9) hold for operators B, M_a, G, X. By the above, (4) holds generically, and (5), (6) clearly hold generically by (JP3), (0.1.6).

For the operators M_a' of (7), we compute $Q_{n', Q_{s'} a} = Q_{Q_s Q_c n, Q_s Q_c Q_s a} = Q_s Q_c Q_{n, Q_s a} Q_c Q_s = Q_s Q_c \big(Q_s M_a^{-\tau}\big) Q_c Q_s$ [by (3.2.7)] $= Q_{s'} \big(M_a^{-\tau} Q_c Q_s\big) = Q_{s'} M_a'^{-\tau}$, and dually. As in (3.2), (8) follows immediately from (3),(7) and (9) from (3),(7),(6).

If the domination is generic, all the formulas (3.4.1-9) thake place in $\mathcal{UME}(\mathcal{V})$.

(II) By (3.4.4) and (0.1.1), automatically $S' \in D_{\mathcal{V},\mathcal{V}}$ is inner Lie structural as in (3.3.1b), and Power Shifting (3.3.2) automatically follows: for $k \geq 1$ the elements q'_{k+1}, s'_k of (2) satisfy the relations

$$(3.4.14) \quad S'^{-\sigma}(s'_k) = 2 s'_{k+1}, \quad N'^{-\sigma}(s'_k) = s'_{k+2}, \quad N'^{\sigma}(q'_{k+1}) = q'_{k+3}, \quad s'_{k+2} = Q_{s'}(q'_{k+1}).$$

Indeed, $S'(s'_k) = (S'Q_s)c^{(k,n)} = (Q_sD_{n,c})c^{(k,n)}$ [by (13)] $= Q_sV_c^{(n)}(c^{(k,n)}) = Q_s 2c^{(k+1,n)} =: 2s_{k+1}$ and $N'(s'_k) = (Q_sQ_cN)(s'_k) = Q_sQ_cN(Q_sc^{(k,n)}) = Q_sQ_cQ_n(c^{(k,n)}) = Q_sU_c^{(n)}(c^{(k,n)}) = Q_sc^{(k+2,n)} =: s'_{k+2}$. On V^σ we have $N'^\sigma(q'_{k+1}) = (NQ_cQ_s)N(c^{(k,n)}) = N(Q_cQ_nc^{(k,n)}) = N(c^{(k+2,n)}) =: q'_{k+3}$. We have the alternate expression $s'_{k+2} := Q_sc^{(k+2,n)} = Q_sQ_cQ_nc^{(k,n)} = Q_sQ_cQ_sN(c^{(k,n)}) = Q_{s'}q'_{k+1}$. Here $s'_1 = Q_sc^{(1,n)} = Q_sc = s'$, $s'_2 = Q_sc^{(2)} = Q_sQ_cn = n'$, and $s'_3 = Q_sc^{(3,n)} = Q_sQ_cQ_nc$.[10]

Furthermore, $\mathcal{V}'$ automatically satisfies Two Q (3.3.4) since $Q_{s',n'} = Q_{Q_sc,Q_sQ_cn} = Q_sQ_{c,Q_cn}Q_s = Q_sQ_cD_{n,c}Q_s$ by (JP3), (JP1), so $S'Q_{s',n'} = S'Q_sQ_cD_{n,c}Q_s = (Q_sD_{c,n})Q_cD_{n,c}Q_s$ [by (13)] $= Q_sQ_cD^2_{n,c}Q_s = Q_sQ_c\big[D_{Q_nc,c} + 2Q_nQ_c\big]Q_s = 2Q_sQ_cQ_nQ_cQ_s + Q_sQ_{Q_cQ_nc,c}Q_s$ [by (0.1.2), (JP1)] $= 2Q_{Q_sQ_cn} + Q_{Q_sQ_cQ_nc,Q_sc} = 2Q_{n'} + Q_{s'_3,s}$ [by (JP3)], and dually $Q_{s',n'}S' = 2Q_{n'} + Q_{s'_3,s}$.

If $\mathcal{N}$ is already an inner multiplication, so is $\mathcal{N}'$. In case $\mathcal{N}$ is already principal, so is $\mathcal{N}'$: if $N^{-\sigma} = Q_sQ_q$ (with $Q_sq = n$) then $N'^{-\sigma} = Q_sQ_c(Q_sQ_q) = Q_{Q_sc}Q_q = Q_{s'}Q_q$, and dually.

(III): To help the reader through the labyrinth of verifications of the for some of the following formulas, we indicate the migration of terms via numeric-alphabetic superscripts;[11] a superscript ▲, ▼, •, ⧫ denotes a term which about to die, cancelled out by its evil twin. If S is Lie-structural, then the two versions of M_a^σ agree, since

$$
\begin{aligned}
&S^{-\sigma}D_{s',a} - D_{n',a} = \big[D^{(1)}_{s,S(c)} - D^{(2)}_{n,c}\big]D_{Q_sc,a} - D^{(3)}_{Q_sQ_cn,a}\\
&= D_{s,S(c)}\big[D_{s,c}D^{(1a)}_{s,a} - Q_sQ^{(1b)}_{c,a}\big] - D_{n,c}\big[D_{s,c}D^{(2a)}_{s,a} - Q_sQ^{(2b)}_{c,a}\big] - \big[D_{s,Q_cn}D^{(3a)}_{s,a} - Q_sQ^{(3b)}_{Q_cn,a}\big]\\
&\hfill \text{[by (0.1.2) on (1), (2), (3)]}\\
&= \big[M^{(1a1)}_{Q_sS(c)} + Q_sQ^{(1a2)}_{S(c),c}\big]D_{s,a} - Q_{Q_sS(c),s}Q^{(1b)}_{c,a} - \big[D^{(2a1)}_{n,Q_cs} + Q_{n,s}Q^{(2a2)}_c\big]D_{s,a} + D_{n,c}Q_sQ^{(2b)}_{c,a}\\
&\qquad - D_{s,Q_cn}D^{(3a)}_{s,a} + Q_sQ^{(3b)}_{Q_cn,a} \qquad \text{[by (0.1.2) on (1a),(2a), (JP1) on (1b)]}\\
&= D^{(1a1)}_{Q_{s,n}(c),c} + Q_s\big[S^{-\sigma}Q^{(1a2a)}_c + Q_cS^{-\tau(1a2b)}\big]D_{s,a} - Q_{Q_{s,n}(c),s}Q^{(1b)}_{c,a} - D^{(2a1)}_{n,Q_cs}D_{s,a} - Q_{n,s}Q^{(2a2)}_cD_{s,a}\\
&\qquad + D_{n,c}Q_sQ^{(2b)}_{c,a} - D_{s,Q_cn}D^{(3a)}_{s,a} + Q_sQ^{(3b)}_{Q_cn,a} \qquad \text{[by (3.1) on (1a1),(1b), Lie struction on (1a2)]}\\
&= \big[D^{(1a1a)\blacktriangle}_{s,Q_cn} + D^{(1a1b)\blacktriangleleft}_{n,Q_cs}\big]D_{s,a} + Q_{s,n}Q_cD^{(1a2a)\blacktriangledown}_{s,a} + Q_sQ_cS^{-\tau(1a2b)\bullet}D_{s,a} - Q_sD_{c,n}Q^{(1b/2b)}_{c,a}\\
&\qquad - D^{(2a1)\blacktriangleleft}_{n,Q_cs}D_{s,a} - Q_{s,n}Q^{(2a2)\blacktriangledown}_cD_{s,a} - D_{s,Q_cn}D^{(3a)\blacktriangle}_{s,a} + Q_sQ^{(3b)}_{Q_cn,a}\\
&\hfill \text{[by (JP2)$'$ on (1a1), (3.1) on (1a2a), (0.1.1) on (1b/2b)]}\\
&= -Q_s\big[Q^{(1b/2b1)\blacklozenge}_{Q_cn,a} + Q_cD^{(1b/2b2)}_{n,a}\big] + Q_sQ^{(3b)\blacklozenge}_{Q_cn,a} + Q_sQ_cS^{-\sigma}D^{(1a2b)}_{s,a} \qquad \text{[by (0.1.2) on (1b/2b)]}\\
&= Q_sQ_c\big[S^{-\sigma}D^{(1a2b)}_{s,a} - D^{(1b/2b2)}_{n,a}\big] = Q_sQ_cM_a^{-\sigma} =: M'^{-\sigma}
\end{aligned}
$$

as claimed. The result for $M_a^{-\tau}$ follows by a dual argument [or by the involution in the generic case, in which case the equalities hold generically].

(IV): Now assume S is Lie-structural and Two Q (3.3.2,4), (3.4.10) hold; innerness (12) of $N^{-\sigma}$ follows from $N'^{-\sigma} = Q_sQ_cN^{-\sigma} = Q_s\big[Q_cN^{-\sigma} + N^\sigma Q_c\big] - Q_nQ_c$ [by (3.1)] $= Q_s\big[-SQ_cS + Q_{S(c)} + Q_{N(c),c}\big] - Q_nQ_c$ [by (10)] $= Q_{s,n}\big[SQ_c - Q_{S(c),c}\big] + Q_sQ_{S(c)} + Q_sQ_{N(c),c} - Q_nQ_c$ [by (3.1) and Lie structurality of S] $= \big[2Q_n + Q_{s_3,s}\big]Q_c - Q_{s,n}Q_{S(c),c} + Q_sQ_{S(c)} + Q_sQ_{N(c),c} - Q_nQ_c$ [by (3.3.4)] $= Q_nQ_c + Q_{s_3,s}Q_c - Q_{s,n}Q_{S(c),c} + Q_sQ_{S(c)} + Q_sQ_{N(c),c}$, and dually for N^σ.

If also triality (11) holds (in addition to (10), (3.3.4), and Lie structionality), then Two N (3.3.3)

[10] Note that we have not required $S'(q'_{k+1}) = 2q'_{k+2}$ in our definition of Power Shifting, primarily because we have been unable to establish it here (not even $S'(q'_2) = 2q'_3$). The missing ingredient is a formula $S'^\sigma N^\sigma = N^\sigma D_{c,n}$, which holds in the case of fractions.

[11] These serve much the same function as ear-tags to track migrating wildlife.

holds for S', N' as well: for $-\sigma$ we have

$$
\begin{aligned}
&S'^{-\sigma}S'^{-\sigma} - \big[2N'^{-\sigma} + D_{s',q_2'}\big] = \big[D_{s,S(c)} - D_{n,c}\big]\big[D_{s,S(c)} - D_{n,c}\big] - \big[2Q_sQ_cN^{-\sigma} + D_{Q_sc,N^{\sigma}(c)}\big] \\
&= (D^{(1)}_{s,S(c)})^2 + (D^{(2)}_{n,c})^2 - D_{n,c}D^{(3)}_{s,S(c)} - D_{s,S(c)}D^{(4)}_{n,c} - 2Q_s\big[-NQ^{(5)}_c - SQ^{(6)}_c S + Q^{(7)}_{S(c)} + Q^{(8)}_{N(c),c}\big] \\
&\quad -\big[-D^{(9)}_{Q_sN(c),c} + D^{(10)}_{s,\{c,s,N(c)\}}\big] \qquad \text{[by (3.4.10), (JP2)}'] \\
&= \big[D^{(1a)}_{Q_sS(c),S(c)} + 2Q_sQ^{(1b)\blacktriangle}_{S(c)}\big] + \big[Q^{(2a)}_{Q_nc.c} + 2Q_nQ^{(2b)}_c\big] \\
&\quad -\big[-D_{n,S(c)}D^{(3a)}_{s,c} + D^{(3b)}_{n,\{S(c)s,c\}} + Q_{n,s}Q^{(3c)}_{c,S(c)}\big] - \big[-D_{n,S(c)}D^{(4a)}_{s,c} + D^{(4b)}_{\{s,S(c),n\},c} + Q_{n,s}Q^{(4c)}_{S(c),c}\big] \\
&\quad +2Q_nQ^{(5)}_c + 2Q_{s,n}Q_cS^{(6)} - 2Q_sQ^{(7)\blacktriangle}_{S(c)} - 2\big[D_{s,N(c)}D^{(8a)}_{s,c} - D^{(8b)}_{Q_sN(c),c}\big] + D^{(9)}_{Q_nc,c} - D^{(10)}_{s,D_{N(c),s}(c)} \\
&\qquad \text{[by (0.1.2) on (1),(2),(8); linearized (0.1.2) on (3),(4); (3.1) on (5),(6),(9)]} \\
&= \big[D^{(1a)\blacktriangleright}_{Q_sS(c),S(c)} + Q^{(2a)}_{Q_nc,c} + 2Q_nQ^{(2b)}_c + 2D_{n,S(c)}D^{(3a/4a)\blacktriangledown}_{s,c} - 2Q_{s,n}Q^{(3c/4c)}_{c,S(c)} - D^{(4b)}_{Q_{s,n}S(c),c} \\
&\quad -\big[-D^{(3b1)\blacktriangleleft}_{s,Q_{S(c),c}n} + D^{(3b2)}_{Q_{n,s}S(c),c} + D^{(3b3)\blacktriangleright}_{Q_{n,s}c,S(c)}\big] + 2Q_nQ^{(5)}_c + 2Q_{s,n}Q_cS^{(6)} \\
&\quad -2\big[D^{(8a1)\blacktriangledown}_{n,S(c)} - D^{(8a2)}_{s_3,c}\big]D_{s,c} + 2D^{(8b)}_{Q_nc,c} + D^{(9)}_{Q_nc,c} - \big[D^{(10a)\blacktriangleleft}_{s,D_{S(c),n}(c)} - D^{(10b)}_{s,D_{c,s_3}(c)}\big] \\
&\qquad \text{[by (3.1) on (1a), (JP2)}' \text{ on (3b); (3.4.11a,b) for } w = c \text{ on (8a),(10); (3.1) on (8)]} \\
&= 4D^{(2a/8b/9)}_{Q_nc,c} + 4Q_nQ^{(2b/5)}_c - 2Q_{s,n}\big[SQ^{(3c1)}_c + Q^{(3c2)\blacktriangle}_c S\big] - 2D^{(4b/3b2)}_{Q_{s,n}S(c),c} + 2Q_{s,n}Q^{(6)\blacktriangle}_c S \\
&\quad +2\big[D^{(8a2a)}_{s_3,Q_cs} + Q^{(8a2b)}_{s_3,s}Q_c\big] + 2D^{(10b)}_{s,Q_cs_3} \\
&\qquad \text{[by Lie structurality on (3c), (0.1.2) on (8a2); (3.1) on (1a),(3b3)]} \\
&= 4D^{(2a/8b/9)\blacktriangleleft}_{Q_nc,c} + 4Q_nQ^{(2b/5)\blacktriangledown}_c - 2\big[2Q^{(3c1a)\blacktriangledown}_n + Q^{(3c1b)\blacktriangle}_{s_3,s}\big]Q_c - 2\big[D^{(4b1)\blacktriangleleft}_{2Q_nc,c} + D^{(4b2)\blacktriangleright}_{Q_{s_3,s}(c),c}\big] \\
&\quad +2D^{(8a2a/10b)\blacktriangleright}_{\{s_3,c,s\},c} + 2Q_{s_3,s}Q^{(8a2b)\blacktriangle}_c = 0 \qquad \text{[by (3.3.4) on (3c1),(4b); (JP2)}' \text{ on (8a2a),(10b)].}
\end{aligned}
$$

A dual argument [or the involution in the generic case] establishes the case σ.

It is not trivial to show that the conditions (10-11) are inherited by a subdominon. For (10) we use (10-11), (3.3.4) for the original dominion to compute (using the same $c' = c$)

$$
\begin{aligned}
&N'Q_c + Q_cN' + S'Q_cS' - Q_{S'(c)} - Q_{N'(c),c} \\
&= (NQ_cQ_s)Q^{(1)}_c + Q^{(2)}_c(Q_sQ_cN) + \big[D_{S(c),s} - D_{c,n}\big]Q^{(3)}_s\big[D_{s,S(c)} - D_{n,c}\big] \\
&\quad -Q^{(4)}_{\{S(c),s,c\}-2Q_cn} - Q^{(5)}_{NQ_cQ_sc,c} \\
&= NQ_cQ_sQ^{(1)\square\surd}_c + Q^{(2)\square\surd}_c Q_sQ_cN) + D_{S(c),s}Q_sD^{(3a)\circ\surd}_{s,S(c)} +_{c,n} Q_sD^{(3b)\diamond\surd}_{n,c} \\
&\quad -D_{S(c),s}Q_cD^{(3c)\square\surd}_{n,c} - D_{c,n}Q_cD^{(3d)\square\surd}_{s,S(c)} - Q_{\{S(c),s,c\}}D_{S(c),s}Q_cD^{(4a)\blacktriangle}_{n,c} - 4Q_cQ_nQ^{(4b)\blacklozenge}_c + 2Q^{(4c)\bigstar}_{\{S(c),S,c\},Q_cn} \\
&\quad +Q^{(5a)\blacktriangleright}_{Q_cQ_nc,c} + Q^{(5b)\triangle\surd}_{SQ_cS(Q_sc),c} - Q^{(5c)\blacktriangleleft}_{Q_{S(c)}(Q_sc),c} - Q^{(5d)\triangle\surd}_{Q_{N(c),c}(Q_sc),c}
\end{aligned}
$$

using (JP3) on (4b), (3.1) on (5a).

We now expand out certain of these terms marked by a $\surd$, indicating by $\blacktriangle, \blacktriangleleft, \blacktriangleright$ etc. where they cancel out terms above or in other expansions. We have from (0.1.4)

$$
(3a)^{\circ} \quad = -Q_{S(c)}Q_sQ^{(3a1)\square}_c - Q_cQ_sQ^{(3a2)\square}_{S(c)} + Q^{(3a3)\blacktriangle}_{\{S(c),s,c\}} + Q^{(3a4)\blacktriangleleft}_{Q_{S(c)}Q_sc,c},
$$

and similarly $D_{c,n}Q_cD_{n,c} = -2Q_{c)Q_nQ_c - Q_{\{c,n,c\}} + Q_{Q_cQ_nc,c} = -2Q_cQ_{[}n}Q_c + 4Q_{Q_cn} + Q_{Q_cQ_nc,c}$ so

$(3b)^{\Diamond} \quad = Q^{\blacktriangleright}_{Q_cQ_nc,c} + 2Q_cQ_nQ^{\blacklozenge}_c.$

We have

$$\begin{array}{lllll} (5b)^{\triangle} & = Q^{\bigstar\bigstar}_{\{S(c),s,Q_cn\},c} & +Q^{\bullet}_{\{S(c),n,Q_cs\},c} & -2Q^{\blacktriangleright}_{Q_cQ_nc,c} & -Q^{\bullet\bullet}_{Q_cQ_{s_3,s}c,c} \\ (5d)^{\triangle} & = & -Q^{\bullet}_{\{S(c),n,Q_cs\},c} & & +Q^{\bullet\bullet}_{Q_cQ_{s_3,s}c,c} \end{array}$$

since for (5b) $Q_{SQ_cQ_{s,n}c,c} = Q_{\{S(c),Q_{s,n}c,c\},c} - Q_{Q_cSQ_sc,c}$ [by Lie struction] $= Q_{\{S(c),s,Q_cn\},c}$ $+Q_{\{S(c),n,Q_cs\},c} - 2Q_{Q_cQ_nc,c} - Q_{Q_cQ_{s_3,s}c,c}$ [by (JP2)′, (3.3.4)], while for (5b) we have (elevating subscripts $Q_{c,d}$ to $Q(c,d)$)

$$\begin{aligned}
&-Q_{c,[Q_{c,N(c)}Q_s]c} = -Q(c,\big[D_{c,s}D_{N(c),s}c - D_{c,Q_sN(c)}c\big]) && \text{[by (0.1.2)]}\\
&= Q(c,\big[-D_{c,s}(D_{S(c),n} - D_{c,s_3})c + D_{c,Q_nc}c\big]) && \text{[by (3.4.11b) for } w=c\text{, (3.1)]}\\
&= Q(c,\big[-D_{c,s}D_{c,n}S(c) + \big(D_{Q_cs,s_3} + Q_cQ_{s_3,s}c\big) + 2Q_cQ_nc\big]) && \text{[by (0.1.2)]}\\
&= Q(c,\big[-\big(D_{Q_cs,n} + Q_cQ_{s,n}\big)S(c) + D_{Q_cs,s_3}c + Q_cQ_{s_3,s}c + 2Q_cQ_nc\big]) && \text{[by (0.1.2)]}\\
&= Q(c,\big[-\{S(c),n,Q_cs\} - Q_c\big(2Q^{\bullet}_n + Q^{\bullet\bullet}_{s_3,s}\big]\big)c + D_{Q_cs,s_3}c + Q_cQ^{\bullet\bullet}_{s_3,s}c + 2Q_cQ^{\bullet}_nc\big]) && \text{[by (3.3.4)].}
\end{aligned}$$

By far the most complicated are the expansions of terms (1) and (2):

$$\begin{aligned}
&(1) + (3a1) + (3d) = Q_cQ_nQ_c + D_{Q_cs_3,s}Q_c + Q_{Q_cQ_nc,c} - D_{c,n}Q_{\{S(c),s,c\},c}\\
&(2) + (3a2) + (3c) = Q_cQ_nQ_c + Q_cD_{s,Q_cs_3} + Q_{c,Q_cQ_nc} - Q_{\{S(c),s,c\},c}D_{n,c}\\
&(1)^{\Box} + (2)^{\Box} + (3a1)^{\Box} + (3a2)^{\Box} + (3c)^{\Box} + (3d)^{\Box} = 2Q_cQ_nQ^{\blacklozenge}_c - Q^{\bigstar\bigstar}_{\{S(c),s,Q_cn\},c} - 2Q^{\bigstar}_{\{S(c),s,c\},Q_cn}.
\end{aligned}$$

For the expansion of (1), the three terms become

$$\begin{aligned}
&= \big[NQ_c - Q_{S(c)}\big]Q_sQ_c\big] - D_{c,n}Q_cD_{s,S(c)}\\
&= \big[-Q_cN^{(1)} - SQ^{(2)}_cS + Q^{(3)}_{N(c),c}\big]Q_sQ_c - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c} + D_{c,n}D_{S(c),s}Q^{(5)}_c && \text{[by (3.4.10),(0.1.1)]}\\
&= -Q_cQ_nQ^{(1)}_c - SQ_cQ_{s,n}Q^{(2)}_c + \big[D_{c,s}D^{(3a)}_{N(c),s} - D^{(3b)}_{c,Q_sN(c)}\big]Q_c - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}\\
&\quad + D_{c,n}D_{S(c),s}Q^{(5)}_c && \text{[by (JP3) on (1), (3.1) on (2), (0.1.2) on (3),]}\\
&= -Q_cQ_nQ^{(1)}_c - Q_{S(c),c}Q_{s,n}Q^{(2a)}_c + Q_c(SQ_{s,n})Q^{(2b)}_c + D_{c,s}\big[D^{(3a1)}_{S(c),n} - D^{(3a2)}_{c,s_3}\big]Q_c\\
&\quad - D_{c,Q_n}Q^{(3b)}_c - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c} + D_{c,n}D_{S(c),s}Q^{(5)}_c\\
&&& \text{[by Lie struction on (2), (3.4.11b) for } w=c \text{ on (3a), (3.1) on (3b)]}\\
&= -Q_cQ_nQ^{(1)}_c + Q_c(2Q^{(2b1)}_n + Q^{(2b2)\bullet}_{s_3,s})Q_c - \big[D^{(3a2a)}_{Q_cs,s_3} + Q_cQ^{(3a2b)\bullet}_{s_3,s}\big]Q_c\\
&\quad + \big[D_{c,s}D^{(3a1)}_{S(c),n} + D_{c,n}D^{(5)}_{S(c),s} - Q_{S(c),c}Q^{(2a)}_{s,n}\big]^{(6)}Q_c - Q^{(3b)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}\\
&&& \text{[by (3.3.4) on (2b), (0.1.2) on (3a2), (JP1) on (3b)]}\\
&= +Q_cQ_nQ^{(1)/(2b1)}_c - D^{(3a2a)}_{Q_cs,s_3}Q_c + \big[D_{c,Q_{s,n}S(c)}\big]^{(6)}Q_c - Q^{(3b)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}\\
&&& \text{[by linearized (0.1.2) on (6)]}\\
&= Q_cQ_nQ^{(1)/(2b1)}_c - D^{(3a2a)}_{Q_cs,s_3}Q_c + \big[2D^{(6a)}_{c,Q_nc} + D^{(6b)}_{c,Q_{s_3,s}c}\big]Q_c - Q^{(3b)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}
\end{aligned}$$

[by (3.3.4) on (6)]

$$= Q_cQ_nQ_c^{(1)/(2b1)} - D_{Q_cs,s_3}Q_c^{(3a2a)} + 2Q^{(6a)}_{Q_cQ_nc,c} + D_{c,Q_{s_3,s}c}Q_c^{(6b)} - Q^{(3b)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}$$

[by (JP1) on (6a)]

$$= Q_cQ_nQ_c^{(1)/(2b1)} + D_{Q_cs_3,s}Q_c^{(3a2a/6b)} + Q^{(3b/6a)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}$$

[by (JP2) on (3a2a),(6b)] as claimed. A dual argument establishes the expansion of (2). Adding the two together yields the combined expansion of (1)+(2) since

$$\begin{aligned}
&\big[Q_cQ_nQ_c^{(1)/(2b1)} + D_{Q_cs_3,s}Q_c^{(3a2a/6b)} + Q^{(3b/6a)}_{Q_cQ_nc,c} - D_{c,n}Q^{(4)}_{\{S(c),s,c\},c}\big]\\
&\quad+\big[Q_cQ_nQ_c^{(1)/(2b1)} + Q_cD^{(3a2a/6b)}_{s,Q_cs_3} + Q^{3b/6a}_{c,Q_cQ_nc} - Q_{\{S(c),s,c\},c}D^{(4)}_{n,c}\big]\\
&= 2Q_cQ_nQ_c^{(1)/(2b1)} + Q^{(3a2a/6b)}_{\{Q_cs_3,s,c\},c} + 2Q^{(3b/6a)}_{Q_cQ_nc,c} - Q^{(4a)}_{D_{c,n}\{c,s,S(c)\},c} - Q^{(4b)}_{\{S(c),s,c\},D_{n,c}(c)}
\end{aligned}$$

[by (0.1.1) on (3a2a/6b), (4)]

$$= 2Q_cQ_nQ_c^{(1)/(2b1)} + Q^{(3a2a/6b)}_{Q_c\{s_3,c,s\},c} + 2Q^{(3b/6a)}_{Q_cQ_nc,c} - 2Q^{(4b)}_{\{S(c),s,c\},Q_cn)} - \big[Q^{(4a1)}_{\{Q_cn,s,S(c)\},c} + Q^{(4a2)}_{Q_cQ_{s,n}S(c),c}\big]$$

[by (0.1.2) on (4a), (JP1) on (3a2a/6b)]

$$\begin{aligned}
&= 2Q_cQ_nQ_c^{(1)/(2b1)} + Q^{(3a2a/6b)\bullet}_{Q_cQ_{s_3,s}c,c} + 2Q^{(3b/6a)\bullet\bullet}_{Q_cQ_nc,c} - 2Q^{(4b)}_{\{S(c),s,c\},Q_cn)} - Q^{(4a1)}_{\{Q_cn,s,S(c)\},c}\\
&\quad-\big[2Q^{(4a2a)\bullet\bullet}_{Q_cQ_nc,c} + Q^{(4a2)\bullet}_{Q_cQ_{s_3,s}](c),c}\big] \qquad \text{[by (3.3.4) on (4a2)]}\\
&= 2Q_cQ_nQ_c^{(1)/(2b1)} - 2Q^{(4b)}_{\{S(c),s,c\},Q_cn)} - Q^{(4a1)}_{\{Q_cn,s,S(c)\},c}
\end{aligned}$$

as claimed. In view of these expansions of (3a), (3b), (5b), (5d), (1)+(2)+(3a1)+(3a2)+(3c)+(3d), all the terms in our expansion of the new (3.4.10) cancel, and the identity holds.

The verification that the subdominion inherits (11) is also quite involved. For the new Δ' we compute, for an arbitrary $w \in V^\sigma$,

$$\begin{aligned}
\Delta' &= D_{s',N'(w)} - D_{n',S'(w)} + D_{s_3',w}\\
&= D_{Q_sc,NQ_cQ_s(w)} - D_{Q_sQ_cn,S'(w)} + D_{Q_sQ_cQ_nc,w}\\
&= \big(-D_{Q_sNQ_cQ_s(w),c} + D_{s,\{c,s,NQ_cQ_s(w)\}}\big) + \big(D_{Q_sS'(w),Q_cn} - D_{s,\{Q_cn,s,S'(w)\}}\big)\\
&\quad + \big(-D_{Q_s(w),Q_cQ_nc} + D_{s,\{Q_cQ_nc,s,w\}}\big) \qquad \text{[by (0.1.2)]}
\end{aligned}$$

so that we have an expression

$$\begin{gathered}
\Delta' = D_{s,\Delta_1(w)} - \Delta_2(y) \qquad \textit{for}\\
\Delta_1(w) = \{c,s,N(Q_cQ_sw)\} - \{Q_cn,s,S'(w)\} + \{Q_cQ_nc,s,w\},\\
\Delta_2(y) = D_{Q_nQ_cy,c} - D_{D_{n,c}y,Q_cn} + D_{y,Q_cQ_nc} \qquad (y := Q_sw)
\end{gathered}$$

[using (13) $Q_sS'(w) = D_{n,c}Q_s(w) = D_{n,c}y$]. A completely dual calculation (this is one reason we have kept w arbitrary, since it plays different roles in Δ and Δ^*),

$$\Delta^{*\prime} = D_{\Delta_1(w),s} - \Delta_2^*(y)$$

where $\Delta_1(w)$ takes the alternate form $\{N(Q_cQ_s(w),s,c\} - \{S'(w),s,Q_cn\} + \{w,s,Q_cQ_nc\}$ and $\Delta_2^* = D_{c,Q_nQ_cy} - D_{Q_cn,D_{n,c}y} + D_{Q_cQ_nc,y}$ with $y := Q_sw$ again is precisely the dual of Δ_2 in $\mathcal{UME}(\mathcal{V})$. But $\Delta_2(y) = \Delta_2^*(y) = 0$ for arbitrary y (hence arbitrary w) by $m = 2$ in Inner Triality (0.2.6) [replacing $x, a \to n, c$].

We have reduced the vanishing (11a) of Δ' for the element $w = c$ to the vanishing of $\Delta_1(c)$,[12] and the vanishing of $\Delta^{*\prime}$ for all odd powers $w' = c^{(2n-1,s')}$ to the vanishing of $\Delta_1(w)$ for all such w'; but by Power Shifting (0.2.1) $w' = c^{(m,s')} = c^{(m,Q_s c)} = c^{(m,s^2)} = c^{(2m-1,s)}$ remains an odd s-power of c, as does $c = c^{(1,s)}$, thus it will suffice to prove $\Delta_1(w) = 0$ for all odd s-powers of c.

At this point we establish two further formulas before proceeding. The first formula holds automatically for all w,

$$\{c, s, S'(w)\} = \{Q_c n, s, w\} + S(Q_c Q_s w), \tag{3.4.15}$$

since

$$\begin{aligned}
\{c, s, S'(w)\} &= \{c, s\{S(c), s, w\}\} - \{c, s, \{c, n, w\}\} && \text{[by definition (4)]}\\
&= \big(\{c, Q_s S(c), w\} + Q_{S(c),c} Q_s w\big) - \big(\{Q_s, n, w\} + Q_c Q_{s,n} w\big) && \text{[by (0.1.2)]}\\
&= \{c, Q_{s,n}(c), w\} - \{Q_c s, n, w\} + \big(S(Q_c Q_s w) + Q_c S Q_s^{\blacktriangle} w\big) - Q_c Q_{s,n} w^{\blacktriangle} && \text{[by Lie struction]}\\
&= \{Q_c n, s, w\} + S(Q_c Q_s w). && \text{[by (JP2)']}
\end{aligned}$$

The second formula also holds for all w, but depends on (11a):

$$S(\{n, c, Q_s w) = \{s_3, c, Q_s w\} + Q_n\{c, s, w\}\}, \tag{3.4.16}$$

since

$$\begin{aligned}
S(\{n, c, Q_s w\}) &= \{S(n), c, Q_s w\}^{(1)} + n, c, S(Q_s w)^{(2)} - \{n, S(c), Q_s w\}^{(3)} && \text{[by Lie struction]}\\
&= 2\{s_3, c, Q_s w\}^{(1)} + \{n, c, \{n, w, s\}\}^{(2)} - \big[D^{(3a)}_{s,N(c)} + D^{(3b)}_{s_3,c}\big] Q_s w\\
& && \text{[by (3.1) on (1), (14) on (2), (11a) in (3)]}\\
&= \{s_3, c, Q_s w\}^{(1/3b)} + \{Q_n c, w, s\}^{(2a)\blacktriangle} + Q_n Q_{c,w} s^{(2b)} - \{Q_s N(c), w, s\}^{(3a)\blacktriangle}\\
& && \text{[by (0.1.2) on (2), (JP1) on (3a)]}\\
&= \{s_3, c, Q_s w\}^{(1/3b)} + Q_n Q_{c,w} s^{(2b)}.
\end{aligned}$$

With these out of the way, we can attack Δ_1 for all $w = c^{(2n-1,s)}$:

$$\begin{aligned}
\Delta_1(w) &= \{N(Q_c Q_s w), s, c\}^{(1)} - \{S'(w), s, Q_c n\}^{(2)} + \{w, s, Q_c Q_n c\}^{(3)}\\
&= \big[\{S(Q_c Q_s w), n, c\}^{(1a)} - \{Q_c Q_s w, s_3, c\}^{(1b)}\big] - \big[\{\{S'(w), s, c\}, n, c\}^{(2a)} - Q_c Q_{n,s} S'(w)^{(2b)}\big]\\
&\quad + \{w, s, Q_c Q_n c\}^{(3)} \qquad \text{[by (11b) on (1) for } w' = Q_c Q_s c^{(2n-1,s)} = c^{(2n+1,s)}\text{, (0.1.2) on (2)]}\\
&= \{S(Q_c Q_s w), n, c\}^{(1a)\blacktriangle} - Q_c\{Q_s w, c, s_3\}^{(1b)} - \big[\{\big(S(Q_c Q_s w), n, c\}^{(2a1)\blacktriangle} + \{\{w, s, Q_c n\}, n, c\}^{(2a2)}\big]\\
&\quad + Q_c(S Q_s) S'(w)^{(2b)} + \{w, s, Q_c Q_n c\}^{(3)} \qquad \text{[by (JP1) on (1b), (15) on (2a), (3.1) on (2b)]}\\
&= -Q_c\{Q_s w, c, s_3\}^{(1b)} - D_{c,n}\{w, s, Q_c n\}^{(2a2)} + Q_c S(D_{n,c} Q_s w)^{(2b)} + \{Q_c Q_n c, s, w\}^{(3)}
\end{aligned}$$

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[12] A careful examination of the proof reveals that in (11a) we only need that Δ vanish on $2V^{-\sigma}$ and $Q_s V^{\sigma}$; the vanishing of $D_{s,\Delta_1(c)} Q_s = Q_{Q_s \Delta_1, s}$ is automatic since $Q_s \Delta_1 = \{Q_s N(Q_c Q_s w), c, s\} - \{Q_s S'(w), Q_c n, s\} + \{Q_s w, Q_c Q_n c, s\}$ [by (JP1)] $= \big[D_{Q_n Q_c y, c} - D_{D_{n,c} y, Q_c n} + D_{y, Q_c Q_n c}\big](s)$ [by (13) with $y := Q_s w$] vanishes by Inner Triality (0.2.6) with $m = 2$, $x, a \to n, c$. However, we couldn't derive $2\Delta_1(c) = 0$ and more easily than $\Delta_1(c) = 0$. Similarly, we only need the vanishing (11b) of Δ^* on the space $\Phi c + Q_c V^{-\sigma}$, but that didn't simplify matters either.

[by (13) on (2b)]

$$= -Q_c\{Q_s w, c, s_3\}^{(1b)} - \big[Q_c Q_n\{c, s, w\}^{(2a2a)} + \{Q_c Q_n c, s, w\}^{(2a2b)\blacktriangledown}\big] + Q_c S(\{n, c, Q_s w\})^{(2b)}$$

$$+\{Q_c Q_n c, s, w\}^{(3)\blacktriangledown} \qquad \text{[by (0.1.5) on (2a2) with } x, a, b \to c, n, s \text{ acting on } w]$$

$$= -Q_c\big[\{s_3, c, Q_s w\}^{(1b)} + Q_n\{c, s, w\}^{(2a2a)} - S(\{n, c, Q_s w\})^{(2b)}\big] = 0. \qquad \text{[by (16)]}$$

(IV): Thus the presence of (3.3.2,4) and (10-11) guarantees that S', N' are both inner as in (3.3.1), Squaring (3.3.3) holds, and (3.3.2) still holds, so s' tightly dominates n'. For any S, N satisfying (10-11),(3.3.4) we know that S' is inner and (10-11),(3.3.2,4) still hold for S', N'; applying these results to S', N' we see S'', N'' are both inner as in (3.3.1), (3.3.2,4) always holds, and now (3.3.3) holds in addition, so s'' is tight over n''. ■

Remark 3.5 *It is not hard to check that if we take $c = q_2$ then with relations such as those in Example* 3.4*, the resulting subdominion has ${N'}^{\sigma} = Q_{q_2}Q_n$, ${N'}^{-\sigma} = Q_n Q_{q_2}$ a principal struction. However, in the theory of fractions we want only injective denominators, and $s' = Q_s q_2$ is usually not injective, so we must take some other c. The derived N', S' of* (3.4) *have more cohesion.* ■

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