FINITELY DEEP MATRICES

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Dedicated to Ivan Shestakov on the occasion of his 60th birthday

Abstract. In the algebraic study of deep matrices $\mathcal{DM}_X(\mathbb{F})$ on a finite set of indices over a field, Christopher Kennedy has recently shown that there is a unique proper ideal $\mathcal{Z}$ whose quotient is a central simple algebra. He showed that this ideal, which doesn't appear for infinite index sets, is itself a central simple algebra. In this paper we extend the result to deep matrices with a finite set of 2 or more indices over an arbitrary coordinate algebra A, showing that when the coordinates are simple there is again such a unique proper ideal, and in general that the lattice of ideals of $\mathcal{DM}_X(A)/\mathcal{Z}$ and $\mathcal{Z}$ are isomorphic to the lattice of ideals of the coordinate algebra A.

In [1, 1.13, p,179] J. Cuntz introduced C^*-algebras $\mathcal{O}_n$ of operators on a separable Hilbert space generated by a finite family of n orthogonal isometries S_i, S_i^*, subject to $S_i^* S_j = \delta_{ij}1$ and the additional condition $\sum_{i=1}^n S_i S_i^* = 1$, and showed by analytic methods that these algebras are simple. Recently, C. Kennedy [4] showed that the condition $\sum_i^n x_i x_i^* = 1$ can be removed in the finite case: there is a unique proper ideal $\mathcal{Z}$ in the deep matrix algebra $\mathcal{DM}_X(\mathbb{F})$ over any field $\mathbb{F}$, which is itself a simple algebra of finite matrices, and that imposing the condition $\sum_i^n x_i x_i^* = 1$ is equivalent to passing to the quotient $\mathcal{DM}_X(\mathbb{F})/\mathcal{Z}$.

1. Deep Matrix Algebras

We recall the definition of the algebra of deep matrices [6] with coefficients in a unital associative (but usually noncommutative) algebra A over some fixed ring of scalars Φ. For any nonempty index set X the set of **deep X-indices** or **heads** is the free monoid $H(X) := Mon(X)$ based on X, consisting of all finite n-tuples $h = (x_1, \ldots, x_n)$ of arbitrary *depth* $|h| = n \geqslant 0$ whose individual entries x_i come from X; the unit element of this monoid is the empty-tuple $\emptyset$ of depth 0. The heads carry a partial order $h \Vdash k$ if h **begins** k, i.e., $k = hk'$ **begins with** h, in which case we say k is *deeper than* h; h is a proper head of k, $h \vdash k$, if $k' \neq \emptyset$. Two heads h, k are **related** $h \sim k$ if one is deeper than the other, $h \Vdash k$ or $k \Vdash h$, otherwise they are **unrelated** $h \not\sim k$. If $h \sim k$, the question of who is deeper is strictly a matter of depth and, as a consequence, two heads of the same depth are related iff they are equal.

$$\textbf{Depth:}\quad h \sim k, |h| \leqslant |k| \iff h \Vdash k; \quad h \sim k, |h| = |k| \iff h = k.$$

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The set of **bodies** $B(X) := \prod_1^\infty X$ consists of all infinite sequences $\mathbf{b} = (y_1, y_2, \ldots)$; if $|X| = 1$ there is only one body $(x, x, x, \ldots)$, but if $|X| \geqslant 2$ there are uncountably many bodies. We say the head $h = (x_1, \ldots, x_n)$ **begins** a body $\mathbf{b}$, $h \vdash \mathbf{b}$, if $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, i.e., $\mathbf{b} = h\mathbf{b}' = (x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots)$. In this case we say that h is the n**th head** $h_n(\mathbf{b})$ of $\mathbf{b}$. If $\mathbf{b}_1, \ldots, \mathbf{b}_r$ are distinct bodies we have **Head Separation** [6, Th.20.1(iii), p.265]: there is an N so that the heads $h_N(\mathbf{b}_1), \ldots h_N(\mathbf{b}_r)$ are distinct [if $\mathbf{b}_i, \mathbf{b}_j$ differ in the N_{ij} entry let $N = \max_{i \neq j} N_{ij}$].

The deep matrix algebra based on X over A is spanned by all **deep matrix units** $E_{h,k}$ where the row and column indices are heads in $H(X)$. We will call the elements of A **scalars** and for clarity denote them by lowercase Greek α, β, γ, whereas elements of the matrix algebra (linear combinations of matrix units) will be denoted by lowercase Latin u, v, w. The notation $E_{h,k}$ [4] (E_h^k in [6]) is chosen to emphasize the analogy with ordinary matrix units $E_{i,j}$.

Rather than this matrix notation, we will usually use the C^*-algebra notation denoting the basis elements by "back-and-forth shifts" hk^* (segregated monoid products with $h \in H(X)$ at the beginning and $k^* \in H(X^*)$ at the end), which are elements of the free monoid-with-involution $H(X, *) := Mon(X \sqcup X^*)$ for set X and a disjoint copy X^*. The map $*$ induces a reversal involution on this monoid via $(x^*)^* = x$, and the unit of the monoid (the empty product of x's and also the empty product of x^*'s) is denoted simply 1 instead of $\emptyset$. We refer to the elements of $H(X)$ as "forward shifts" and those of $H(X^*)$ as "backward shifts" of "negative depth", leading us to define a general notion of **depth** for back-and-forth shifts in $H(X, *)$ (distinguished by $\|$ in place of $|$):

$$\|u\| = \|hk^*\| := |h| - |k|, \quad \|u^*\| = -\|u\|, \quad \|k^*\| := -|k|.$$

The forward and backward shifts h, k^* for $h, k \in H(X)$ are generated by primitive shifts x, y^* for $x, y \in X$: $h = (x_1 \ldots x_n) = x_1 \cdots x_n$, $k^* = (y_1 \ldots y_m)^* = y_m^* \cdots y_1^*$. The **shift monoid** $ShM(X)$ on X is the $*$-monoid-with-zero $Mon(X \sqcup X^* \sqcup 0)$ on X modulo the homogeneous relations

$$x^*x = 1, \qquad x^*y = 0, \qquad x0 = x^*0 = 0 = 0^* = 0x^* = 0x \qquad (y \neq x \in X).$$

Definition 1.1 (Deep Matrix Construction [6, Thm.2.1], [4])**.** *The* **deep matrix algebra** $\mathcal{DM}_X(A)$*(called* $\mathcal{E}(X, A)$ *in* [6]*) based on* X *over a unital associative coordinate algebra* A *consists of the free left* A*-module with the basis of all* **deep matrix units** $E_{h,k}$ *for* $h, i, j, k \in H(X)$*, determined by the* **Deep Multiplication Rules** [1, p.175] *for the basic products* $(\alpha, \beta \in A)$ *:*

$$\text{(DMI)} \quad (\alpha E_{h,i})(\beta E_{j,k}) = (\alpha E_{h,i})(\beta E_{ij',k}) = \alpha\beta E_{hj',k} \ \text{ if } \ i \Vdash j = ij',$$
$$\text{(DMII)} \quad (\alpha E_{h,i})(\beta E_{j,k}) = (\alpha E_{h,ji'})(\beta E_{j,k}) = \alpha\beta E_{h,ki'} \ \text{ if } \ j \Vdash i = ji',$$
$$\text{(DMIII)} \quad (\alpha E_{h,i})(\beta E_{j,k}) = 0 \ \text{ if } \ i \not\sim j \text{ are unrelated.}$$

In C^**-notation* $\mathcal{DM}_X(A)$ *is the free* A*-module with basis of all* hk^* *with rules*

$$\begin{aligned}
&\text{(DMI)} && (\alpha h i^*)(\beta j k^*) = (\alpha h\, i^*)(\beta i j' k^*) = \alpha\beta (h j') k^* \ \ \textit{if}\ \ i \Vdash j = i j', \\
&\text{(DMII)} && (\alpha h i^*)(\beta j k^*) = (\alpha h (j i')^*)(\beta j k^*) = \alpha\beta h (k i')^* \ \ \textit{if}\ \ j \Vdash i = j i', \\
&\text{(DMIII)} && (\alpha h i^*)(\beta j k^*) = 0 \ \ \textit{if}\ \ i \not\sim j\ \textit{are unrelated.}
\end{aligned}$$

The Deep Multiplication Rules follow from the **Orthogonality Relations** $y^* x = \delta_{y,x} 1$ since these allow us to repeatedly remove any y^* to the left of an x in any mixed product of x's and y^*'s until we reach a segregated shift:

Head Product Rules

The un-segregated product $k^* h$ is always zero or a forward or backward shift:

$$\begin{array}{lll}
k^* h = 1, & \|k^* h\| = 0, & \text{if } k = h; \\
k^* h = h', & \|k^* h\| = |h| - |k| > 0, & \text{if } k \vdash h = k h'; \\
k^* h = k'^*, & \|k^* h\| = |h| - |k| < 0, & \text{if } h \vdash k = h k'; \\
k^* h = 0, & & \text{if } k, h \text{ are unrelated.}
\end{array}$$

The deep matrices form a unital associative algebra with unit 1 (or $1_{deep} = E_{\emptyset,\emptyset}$ in matrix notation): it is just the monoid algebra over A on the monoid-with-zero $ShM(X)$. In analogy with group algebras $k[G] = kG$, this suggests a notation $\mathcal{DM}_X(A) = AHH^*$ for the deep matrix algebra. The subalgebras AH, AH^* are just the usual monoid algebras over A on the monoids $H(X), H(X^*)$.

The deep matrix construction yields a bifunctor from sets and coordinate algebras to unital associative algebras. The *transpose map* $\alpha E_{h,k} \longrightarrow \alpha E_{k,h}$ is an isomorphism of $\mathcal{DM}_X(A)$ with $\mathcal{DM}_X(A^{op})$. Thus, despite the apparent asymmetry in the indices, there is a duality: all general results for deep matrices over all A remain true if the roles of h, k are interchanged. If the coordinate algebra A carries an involution $\alpha \to \bar\alpha$ (e.g. if A is commutative, $\bar\alpha = \alpha$), then $\mathcal{DM}_X(A)$ carries a natural conjugate transpose involution uniquely determined by $(\alpha h\, k^*)^* := \bar\alpha k\, h^*$, yielding a functor from sets and coordinate $*$-algebras to unital $*$-algebras.

The associativity of deep matrix multiplication also follows from the faithful **Kennedy representation** [4] of $\mathcal{D} := \mathcal{DM}_X(A)$ on the **Kennedy module**, the free right A-module

$$V[H] = HA := \bigoplus_{j \in H(X)} jA, \qquad (\alpha h\, i^*)(j\beta) = \begin{cases} h j' \alpha\beta & \textit{if}\ i \Vdash j = i j', \\ 0 & \textit{if}\ i \not\Vdash j. \end{cases}$$

This is not a left regular representation and $HA \cong AH$ is not a $\mathcal{D}$-submodule of $\mathcal{D} = AHH^*$ since $x^*(1) = 0 \neq x^*$.

Any deep matrix algebra $\mathcal{DM}_X(A)$ has a **Frankenstein representation** [6, 20.5, p.269] on the **Frankenstein module**, the free right A-module

$$V[B] = BA := \bigoplus_{\mathbf{b} \in B(X)} \mathbf{b}A$$

(called $V(X,A)$ in [6]) with basis of all bodies, given by

$$\textit{if } |i| = n \textit{ then} \quad (\alpha\, hi^*)(\mathbf{b}\,\beta) = \begin{cases} h\mathbf{b}'\alpha\beta & \textit{if } i \Vdash \mathbf{b} = i\mathbf{b}',\ \textit{i.e.},\ i = h_n(\mathbf{b}); \\ 0 & \textit{if } i \nVdash \mathbf{b},\ \textit{i.e.},\ i \neq h_n(\mathbf{b}). \end{cases}$$

The corresponding operators $F_{h,i}$ of left multiplication by $E_{h,i} = hi^*$ chop the head i off $\mathbf{b}$ and replace it with h (killing the patient $\mathbf{b}$ if it doesn't begin with the head i). Linear combinations of these operators form the **Frankenstein algebra** $\mathcal{F}_X(A)$, a left A-module (free if X is infinite) with a canonical **Deep Matrix epimorphism**

$$\mathcal{DM}_X(A) \xrightarrow{\mathcal{F}} \mathcal{F}_X(A) \quad \textit{via} \quad E_{h,k} \to F_{h,k}.$$

However, we will see that $\mathcal{F}$ is an isomorphism (the Frankenstein representation is faithful) only when X is infinite.

The always-faithful Kennedy representation on $V[H]$ is more satisfying than the faithless Frankenstein representation on $V[B]$ for finite X. But the Frankenstein action actually becomes faithful if we move to a slightly larger set $X' := X \sqcup \{z\}$ (without going all the way to an infinite set): both the Frankenstein representation on $B(X')$ and the regular representation on $\mathcal{DM}_{X'}(A)$ contain many copies of the Kennedy representation $V[H]$ (and hence are faithful).

Theorem 1.2 (Kennedy Imbedded)**.** *For arbitrary X, set $X' := X \sqcup \{z\}$, $\mathbf{b}' \in B' := B(X'), \mathcal{D}' := \mathcal{DM}_{X'}(A), \mathcal{D} := \mathcal{DM}_X(A)$. The Kennedy representation of $\mathcal{D}$ on $V[H]$ is isomorphic to a subrepresentation of the left regular representation of $\mathcal{D}'$ on $W = AHz \subseteq \mathcal{D}'$, and to a subrepresentation of the Frankenstein representation of $\mathcal{D}'$ on $W = Hz\mathbf{b}'A \subseteq V[B']$ for each $\mathbf{b}' \in B'$.*

Proof. We imbed $V[H] \hookrightarrow AHz \subseteq \mathcal{D}'$ via the map $h\alpha = (x_1, x_2, \ldots, x_n)\alpha \mapsto \alpha hz = \alpha(x_1, x_2, \ldots, x_n, z)$ and $V[H] \hookrightarrow Hz\mathbf{b}' \subseteq V[B']$ via the mapping $h\alpha \mapsto hz\mathbf{b}'\alpha = (x_1, x_2, \ldots, x_n, z, x_1', x_2', \ldots)\alpha$. To see these W are $\mathcal{D}$-invariant with Kennedy action, note first that αhi^* kills βjz and $jz\mathbf{b}'\beta$ if $i \nVdash jz$ (i.e., $i \nVdash j$) [we never have $jz \Vdash i$ as in (DMII) because of the factor $z \notin X$]. Next, if $i \Vdash j = ij'$ then as in (DMI) we have $(\alpha hi^*)(\beta jz) = (\alpha hi^*)(\beta ij'z) = \alpha\beta hj'z$ and $(\alpha hi^*)(jz\mathbf{b}'\beta) = (\alpha hi^*)(ij'z\mathbf{b}'\beta) = hj'z\mathbf{b}'\alpha\beta$.

The standard argument [4, Th.2 p.527], [6, (3) p.271][1] that $\mathcal{D}$ acts faithfully on $V[B]$ when X is infinite merely requires a z distinct from the x's, so it shows $\mathcal{D}$ acts faithfully on AHz and $Hz\mathbf{b}'A$. $\square$

The deep matrix units commute with scalar multiplication (though A itself need not be commutative)

$$\alpha(hk^*) = (\alpha 1)hk^* = hk^*(\alpha 1).$$

This shows that we can represent deep matrices over A as a "scalar extension" of the deep matrices over the scalar ground ring Φ,

[1]If $(\sum \alpha_{h,k} hk^*)Hz = 0$ for distinct (h,k) and nonzero $\alpha_{h,k}$, choose $|k_0|$ minimal and then $|h_0|$ maximal among all $|h|$ among the (h, k_0). Then the Head Product Rules yields the contradiction $0 = h_0^*(\sum \alpha_{h,k} hk^*)k_0 z = \alpha_{h_0,k_0} 1$ [never $k \vdash k_0$, and never $h_0 \vdash h$ among the (h, k_0) and $x^* z = 0$].

$$\mathcal{DM}_X(A) \cong A \bigotimes_{\Phi} \mathcal{DM}_X(\Phi).$$

For coordinate algebras A over a field $\Phi = \mathbb{F}$, this shows that several of our results can be obtained directly from Kennedy's results for $A = \mathbb{F}$.

Proposition 1.3. *The deep matrices $\mathcal{D} := \mathcal{DM}_X(A)$ have a $\mathbb{Z}$-grading*

$$\mathcal{D} = \bigoplus_{n=-\infty}^{\infty} \mathcal{D}_n, \quad \mathcal{D}_n\mathcal{D}_m \subseteq \mathcal{D}_{n+m}, \quad for \ \ \mathcal{D}_n := \sum_{\|hk^*\|=n} A\,hk^*.$$

If H_n denotes the heads of depth n and for $N = |X_n|, M = |H_m|$, then in terms of the subalgebra $\mathcal{D}_0$ we can write $\mathcal{D}_n := H_n\mathcal{D}_0, \quad \mathcal{D}_{-n} = \mathcal{D}_0 H_n^* \quad$ if $n > 0$. The subalgebra $\mathcal{D}_0$ is a module direct sum of level-n matrix subalgebras [1, 1.4 p.175],

$$\mathcal{D}_0 = \bigoplus_{n=0}^{\infty} \mathcal{D}_0^{(n)}, \quad \mathcal{D}_0^{(n)} := \sum_{|h|=|k|=n} AE_{h,k} \cong Mat^{fin}_{N\times N}(A), \quad \mathcal{D}_0^{(n)}\mathcal{D}_0^{(m)} \subseteq \mathcal{D}_0^{(max\{n,m\})}.$$

If $|X| > 1$ the Frankenstein representation of the level-n matrix subalgebra $\mathcal{D}_0^{(n)}$ is the direct sum of uncountably many copies of the standard representation of $Mat^{fin}_{N\times N}(A)$ on $\bigoplus^N A$:

$$V[B] = H_nBA \cong \textstyle\bigoplus_{\mathbf{b}\in B(X)} V[H_n]^{\mathbf{b}}, \quad V[H_n]^{\mathbf{b}} := H_n\mathbf{b}A \ \ with \ E_{h,i}(j\mathbf{b}) = \delta_{i,j}h\mathbf{b},$$

while the $\mathcal{D}_n$ for $n \neq 0$ act as $Mat^{fin}_{M\times N}(A)$ from $\bigoplus^N A$ to $\bigoplus^M A$:

$$if \ \ |h| = m, |k| = n, \quad then \ \ E_{h,k}(V[H_n]^{\mathbf{b}}) \subseteq V[H_m]^{\mathbf{b}} \ \ with \ E_{h,i}(j\mathbf{b}) = \delta_{i,j}h\mathbf{b}.$$

Proof. The grading decomposition comes directly from the direct decomposition into matrix units $E_{h,k} = hk^*$, and $\mathcal{D}_n\mathcal{D}_m \subseteq \mathcal{D}_{n+m}$ follows since by the Head Product Rules if $jk^*\,hi^* = jui^*$ is nonzero then

$$\begin{aligned} \|jui^*\| &= |j| + |u| - |i| = |j| + (|h| - |k|) - |i| \\ &= (|j| - |k|) + (|h| - |i|) = \|jk^*\| + \|hi^*\|. \end{aligned}$$

For the description of $\mathcal{D}_n$ in terms of $\mathcal{D}_0$, note that

$$x_1\cdots x_i \cdot y_1^*\cdots y_j^* = \begin{cases} x_1\cdots x_n\big(x_{n+1}\cdots x_{n+j}\cdot y_1^*\cdots y_j^*\big) & if\ i = n+j \geqslant j \\ \big(x_1\cdots x_i\cdot y_1^*\cdots y_i^*\big)y_{i+1}^*\cdots y_{i+n}^* & if\ i \leqslant j = i+n. \end{cases}$$

That $\mathcal{D}_0^{(n)}$ is a matrix algebra follows from the Head Products since if $|h| = |i| = |j| = |k| = n$ we have $E_{h,i}E_{j,k} = \delta_{i,j}E_{h,k}$, For the products, if $|h| = |i| = n$, $|j| = |k| = m$ a nonzero $E_{h,i}E_{j,k}$ has one of the forms

$$(hi^*)(jk^*) = \begin{cases} (hj')k^* = E_{hj',k} \in \mathcal{D}_0^{(m)} \ for\ |j'| = m-n & if\ n \leqslant m \\ h(i'^*k^*) = E_{h,ki'} \in \mathcal{D}_0^{(n)} \ for\ |i'| = n-m & if\ n \geqslant m \end{cases}.$$

The action of the $E_{h,i}$ on $V[H_n]^{\mathbf{b}}$ follows directly from the action $(\alpha hi^*)(j\mathbf{b}\beta) = \delta_{i,j}h\mathbf{b}\alpha\beta$ if $|i| = |j| = n$. $\square$

The term "deep matrices" comes from the fact that $\mathcal{D}$ contains matrix subalgebras $\mathcal{D}_0^{(n)}$ isomorphic to $|X^n| \times |X^n|$ matrices with finitely many nonzero entries, and as we descend we pick up larger and larger algebras of matrices at every depth.[2]

We mention several consequences of the Deep Multiplication Rules that will be useful in what follows. The forward shifts are left-invertible, $h^*h = 1$, but not invertible: the *range projections* $e_h := hh^*$ form a nested family of idempotents where unrelated members are orthogonal, but shallow members contain the deeper members:

$$(\mathbf{D}_H)\quad \begin{array}{l} e_h h = h,\ h^* e_h = h^*, \\ h \not\sim k \Longrightarrow e_h(kj^*) = (jk^*)e_h = 0, \quad e_h e_k = e_k e_h = 0, \\ h \Vdash k = hk' \Longrightarrow e_h(kg^*) = kg^*,\ (gk^*)e_h = gk^*, \quad e_h e_k = e_k e_h = e_k \\ k \Vdash h = kh' \Longrightarrow e_h(kg^*) = h(gh')^*,\ (gk^*)e_h = (gh')h^*, \quad e_h e_k = e_k e_h = e_h. \end{array}$$

In particular the $e_x = xx^*$ for $x \in X$ form an orthogonal family of "maximal" idempotents which pick out terms beginning with x or ending with x^*: $e_x[(yh')k^*] = \delta_{xy}(yh')k^*$ and dually $[k(yh')^*]e_x = \delta_{xy}k(yh')^*$, so

$$(\mathbf{D}_X)\quad e_x e_y = \delta_{xy} e_x, \quad e_x(hk^*) = \delta_{xy} hk^*, \quad (kh^*)e_x = \delta_{xy} kh^* \qquad \textit{if } h = yh'.$$

Head Separation leads [6, Th.20.9, p.272] to :

Body Separation: *if* $\mathbf{b}_1, \ldots, \mathbf{b}_r$ *are distinct bodies there exists a projection* $e_h = hh^*$ *with* $e_h(\mathbf{b}_1) = \mathbf{b}_1$, $e_h(\mathbf{b}_i) = 0$ *for* $i > 1$.

Remark 1.4. *The principal left ideal generated by the deep matrix unit* hk^* *is the* k^{th} column ideal

$$\mathcal{D}e_k = \mathcal{D}k^* = \mathcal{D}hk^* = \sum_{i,j} Ai(kj)^* =: AHH^*k^*,$$

and the principal right ideal generated by $h\,k^*$ *is the* h^{th} row ideal

$$e_h\mathcal{D} = h\mathcal{D} = hk^*\mathcal{D} = \sum_{j,i} A(hj)i^* =: AhHH^*.$$

Any $h \in H$ *determines a "downward"* **push** *monomorphism* $\mu_h(u) = h\,u\,h^*$ *sending* 1 *to* e_h *and hence* $\mathcal{D} = 1\,\mathcal{D}\,1$ *to the Peirce subalgebra* $h\mathcal{D}h^* = e_h\mathcal{D}e_h$, *with left inverse the "upward"* **pull** *epimorphism* $\mu_h^* := \mu_{h^*}$.

Proof. Indeed, the Deep Multiplication Rules show every left or right multiple ij^*hk^* or hk^*ij^* has the above form, and conversely all such matrix units arise as multiples: $i(kj)^* = i(hj)^*hk^*$ and $(hj)i^* = hk^*(kj)i^*$. For the push and pull, any h has $h^*h = 1$, $hh^* = e_h$ so $\mu_h^* \circ \mu_h = \mu_{h^*h} = \mathbf{1}_{\mathcal{D}}$, $\mu_h \circ \mu_h^* = \mu_{hh^*} = \mu_{e_h}$. □

[2]When $\sum_i x_i x_i^* = 1$ the $\mathcal{D}_0^{(n)}$ (denoted $\mathcal{F}_n$ in [2, Prop.2.4 p.253]) have $\mathcal{D}_0^{(n)} \subseteq \mathcal{D}_0^{(n+1)}$ since $hk^* = h1k^* = \sum_i hx_i x_i^* k_i^*$.

2. The Obstacle

So far all our remarks apply to deep matrices over any set X. From now on we examine the special properties of finite index sets. When $|X| < \infty$, a nonzero element u need not have a nonzero "scalar multiple" $vuw = \alpha 1$, and it is not true that all ideals come from coordinate ideals: there is a rogue ideal of obstinately finite character, and until the deep matrix algebra is purged of this obstacle it cannot enjoy the desirable properties of its infinite brethren.

In the ordinary matrix algebra $Mat_n^{fin}(A)$ the unit is $1 = \sum_i E_{ii}$. In the finite index case, by $(\mathbf{D}_X)$

$$e := \sum_{x\in X} e_x = \sum_{x\in X} E_{x,x} = \sum_{x\in X} xx^*$$

exists in $\mathcal{DM}_X(A)$ as an idempotent, the **finite idempotent**, but it is NOT the unit 1; its evil twin $f := 1 - e$ is the **infinite idempotent**. Kennedy [4] pointed out that e should be the "true" unit (the C^*-algebraists decree $e = 1$ as part of their axioms for the algebra $\mathcal{O}_n$), and the imposter f gives rise to all the difficulties in the finite case.

Note that $eh = h, \quad h^*e = h^*$ for all nonempty heads h since if h begins with $x \neq y$ then $e_x h = h, e_y h = 0$ by $(\mathbf{D}_X)$. Consequently the element f has the following multiplication rules:

$$\begin{aligned}
&(F1) \quad eh = h,\ h^*e = h^*, \quad f(hk^*) = (kh^*)f = 0 \ \ \textit{if} \ \ h \neq 1,\\
&(F2) \quad \textit{for all bodies } \mathbf{b},\ \ e\mathbf{b} = \mathbf{b},\ f\mathbf{b} = 0,\\
&(F3) \quad f\,1\,f = f, \quad f(h\,k^*)f = 0 \ \ \textit{if} \ \ (h,k) \neq (1,1).
\end{aligned}$$

The **obstacle** is the ideal $\mathcal{Z}_X(A) := \mathcal{D}f\mathcal{D}$ generated by f. For finite X the **reduced** deep matrix algebra $\overline{\mathcal{DM}_X(A)}$ is obtained by adding the axiom $\sum_x e_x = 1$ (i.e., $f = 0$) to the Deep Multiplication Rules, equivalently, by forming the quotient

$$\overline{\mathcal{DM}_X(A)} := \mathcal{DM}_X(A)/\mathcal{Z}_X(A) : \qquad \sum_{x\in X} \overline{e_x} = \bar{1}.$$

A simple but useful property is the

$$\textbf{Nonzero } f\textbf{-Scalar Rule} \qquad \alpha \neq 0 \implies \alpha f \neq 0$$

since in fact $\alpha u = 0$ forces $u = 0$ in any free A-module as soon as u is "monic" (has one of its basic coefficients equal to 1).

We will see shortly that $\mathcal{Z}_X(A)$ is *precisely* the annihilator of the Frankenstein module $V[B]$ when $|X| > 1$, but for now it is immediate that it is *contained in* the annihilator: certainly the element f annihilates $V[B]$ by (F2), and the annihilator is always an ideal, so it contains $\mathcal{D}f\mathcal{D} = \mathcal{Z}_X(A)$. This gives us an important rule:

$$\textbf{Non-}\mathcal{Z}\textbf{ Scalar Rule} \qquad \alpha \neq 0 \implies \alpha 1 \notin \mathcal{Z}_X(A) \subseteq Ann(V[B])$$

since $\alpha \neq 0$ does not annihilate the free right module $V[B]$ [note $(\alpha 1)\mathbf{b} = \mathbf{b}\alpha$].

Theorem 2.1 (Obstacle [4])**.** *For a finite set X the obstacle $\mathcal{Z}_X(A)$ is a free A-module spanned by all*

$$\textbf{\textit{F-Matrix Units}} \qquad F_{h,k} := hfk^* = hk^* - \sum_{x\in X} h\,xx^*\,k^* \quad (h, k \in H)$$

satisfying [1, Prop. 3.1 p.183] *the*

$$\textbf{\textit{F-Matrix Unit Rules}} \qquad F_{h,i}F_{j,k} = \delta_{i,j}F_{h,k} \qquad (h,i,j,k \in H).$$

This algebra is isomorphic to $Mat_\infty^{fin}(A)$*, the countably-infinite square matrices with only a finite number of nonzero entries from* A*, under an isomorphism sending* f *to the matrix unit* E_{00}*. The lattice of ideals of the obstacle is the lattice of ideals of* A*: every ideal in* $\mathcal{Z}_X(A)$ *is* $\mathcal{Z}_X(I) = I\mathcal{Z}_X(A)$ *for an ideal* $I \lhd A$*. In particular,* $\mathcal{Z}_X(A)$ *is simple iff* A *is simple.*

The centroid of the obstacle is the center of A*:* $\Gamma(\mathcal{Z}_X(A)) = C(A)1_{\mathcal{Z}}$ *(of course* $\mathcal{Z}_X(A)$ *has no center) .*

Proof. For the F-matrix unit rules, if $i \neq j$ then $i^*j \neq 1$ by the Head Product Rule, so by (F3) $f\, i^*j\, f = 0$. If $i = j$ we have $hf\, i^*j\, fk^* = hf\,1\,fk^* = hfk^*$. Thus in all cases

$$F_{h,i}F_{j,k} = (hfi^*)(jfk^*) = \delta_{i,j}hfk^* = \delta_{i,j}F_{h,k}.$$

This is the multiplication table for the free left A-module $Mat_\infty^{fin}(A)$ with standard matrix units $E_{m,n}$ for $m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and if we label the countable number of heads as h_m, $m \in \mathbb{N}_0$ with $h_0 = 1$, there is a natural A-linear epimorphism

$$Mat_\infty^{fin}(A) \xrightarrow{\pi} \mathcal{Z}_X(A) \quad via \quad E_{m,n} \mapsto F_{h_m,h_n},$$

where $E_{0,0}$ is sent to $F_{h_0,h_0} = F_{0,0} = f$. This map is an isomorphism (equivalently, $\mathcal{Z}_X(A)$ is free with basis of $F_{h,k}$) since $\sum_{h,k} \alpha_{h,k}F_{h,k} = 0$ with some $\alpha_{i,j} \neq 0$ implies $0 = F_{0,i}\big(\sum_{h,k} \alpha_{h,k}F_{h,k}\big)F_{j,0} = \alpha_{i,j}F_{0,0} = \alpha_{i,j}f$, contrary to the Nonzero f-Scalar Rule.

From another viewpoint, the above argument shows that every ideal $\mathcal{I}$ of $Mat_\infty^{fin}(A)$ is $Mat_\infty^{fin}(I)$ for $I = \{\alpha \in A \mid \alpha E_{m,n} \in \mathcal{I}$ for some (hence all) $m, n\}$. The kernel of π must therefore be $Mat_\infty^{fin}(I)$ for some ideal $I \lhd A$, in particular it contains $IE_{0,0}$, so the image $If = \pi(IE_{0,0})$ must vanish. But by the Nonzero f-Scalar Rule this forces $I = 0$, so π is an isomorphism.

It is well-known that the center is 0 (there is no unit element in $Mat_\infty^{fin}(A)$ since the identity matrix has an infinite number of nonzero entries), and that the centroid is just multiplication by central elements of A. □

The Peirce decomposition of $\mathcal{D}$ and the obstacle $\mathcal{Z} = \mathcal{D}f\mathcal{D}$ relative to the idempotent e is

$$\mathcal{D} = \mathcal{D}_{11} \oplus \mathcal{D}_{10} \oplus \mathcal{D}_{01} \oplus \mathcal{D}_{00}, \quad \mathcal{Z}_X(A) = \mathcal{D}_{10}\mathcal{D}_{01} \oplus \mathcal{D}_{10} \oplus \mathcal{D}_{01} \oplus \mathcal{D}_{00}.$$

Here the Peirce spaces $\mathcal{D}_{10}\mathcal{D}_{01}, \mathcal{D}_{01}, \mathcal{D}_{10}, \mathcal{D}_{00}$ have A-bases $F_{11}(H', H')$, $F_{01}(H')$, $F_{10}(H')$, $F_{00}(1)$ where for $h', k' \in H' := H \setminus 1 = \{h \in H \mid |h| \geqslant 1\}$

$$\begin{aligned}
&F_{11}(h',k') := F_{h',k'} := (h')f(k')^* = h'\,k'^* - \textstyle\sum_x xh'(xk')^*,\\
&F_{01}(k') := F_{1,k'} := f(k')^* = k'^* - \textstyle\sum_x x(x^*k'^*),\\
&F_{10}(h') := F_{h',1} := h'\,f = h' - \textstyle\sum_x (h'x)x^*,\\
&F_{00}(1) := F_{1,1} = 1 - \textstyle\sum_x xx^* = 1 - e = f.
\end{aligned}$$

3. The Scalar Multiple Theorem

The proper analogue of the Scalar Multiple Theorem [1, Thm.3.4 p.184], [6, Th.20.3 p.267] for the finite case is the following.

Theorem 3.1 (Scalar Multiple). *If $2 \leqslant |X| < \infty$, then every element of the deep matrix algebra $\mathcal{DM}_X(A)$ which doesn't lie in the obstacle ideal $\mathcal{Z}_X(A)$ has a "scalar multiple": if $u \in \mathcal{DM}_X(A) \setminus \mathcal{Z}_X(A)$ then there exist a nonzero coordinate $0 \neq \alpha \in A$ and heads $h, k \in H$ with*

$$k^*uh = \alpha 1, \qquad k^*h = \delta 1 \quad (\delta = 1, 0).$$

Proof. [3] Let $\mathcal{S}$ denote the set $\mathcal{D} \setminus \mathcal{Z}$ of deep matrices outside the obstacle. For $u \in \mathcal{S}$ we define the set $[u]$ of *proper multiples* of u to be the set of all $k^*uh \in \mathcal{S}$. Note that $v \in [u] \Longrightarrow [v] \subseteq [u]$ since $k_2^*(k_1^*uh_1)h_2 = (k_1k_2)^*u(h_1h_2)$. Our goal is to show that for a fixed element $u \in \mathcal{S}$ there exists a scalar in $[u]$. The natural choice is a $v = \sum_{i=0}^r \alpha_i h_i {k_i}^*$ ($\alpha_i \neq 0$) in $[u]$ with the fewest number $r+1$ of terms (so all $h_ik_i^*$ are distinct) and shallowest $h_0{k_0}^*$ (minimal $m_0 := |k_0| + |h_0|$). We can shift v forwards or backwards by

$$\text{(Shiftability)} \qquad v \in [u] \Longrightarrow \exists\, x, y \in X \quad \textit{such that} \quad vx, y^*v \in [u].$$

Indeed, working modulo the ideal $\mathcal{Z} = \mathcal{D}f\mathcal{D}$ we have $0 \not\equiv v = ve + vf \equiv ve = \sum_{x \in X}(vx)x^*$, so $vx \not\equiv 0$ for at least one x. Analogously, $0 \not\equiv v = ev + fv \equiv ev = \sum_{y \in X} y(y^*v)$, so $y^*v \not\equiv 0$ for at least one y, and both multiples vx, y^*v remain in $\mathcal{S}$.

Since v was already shortest possible in $[u]$, vx and y^*v in $\mathcal{S}$ cannot have fewer terms. Thus $\alpha_ih_ik_i^*x,\ \alpha_iy^*h_ik_i^* \neq 0$. Then by $\mathbf{D}_X$ we know each $k_i \neq 1$ must begin with x, each $h_i \neq 1$ must begin with y. Moreover, the shallowest term must have $h_0 = k_0 = 1$: if $k_0 = xk_0'$ then $h_0k_0^*x = h_0{k_0'}^*$ for $m_0' = |h_0| + |k_0'| < |h_0| + |k_0| = m_0$, contradicting minimality, and analogously $h_0 = yh_0'$ would lead to a shallower $y^*h_0k_0^* = h_0'k_0^*$. Thus $\alpha_0 1$ is one of the $r+1$ terms of v *and all the other terms have*

$$(*) \qquad h_ik_i^* \in y\mathcal{D} + \mathcal{D}x^* \ \textit{ if } i \neq 0$$

[$h_i = k_i = 1$ would contradict distinctness of the $h_ik_i^*$]. By Shiftability there is also $z \in X$ with $z^*(vx) \in [u]$, so $z^*(\alpha_0 1)x \neq 0$ (by minimality) and therefore $z = x$. Hence $x^*(vx) = (x^*v)x \notin \mathcal{Z} \Rightarrow x^*v \notin \mathcal{Z} \Rightarrow x^*v \in [u]$. Thus we can take $y = x$ from the start, so $(*)$ becomes

$$(**) \qquad h_ik_i^* \in x\mathcal{D} + \mathcal{D}x^* \ \textit{ if } i \neq 0.$$

By our assumption $|X| \geqslant 2$ we can find $w \neq x$ in X, so $w^*(x\mathcal{D} + \mathcal{D}x^*)w = 0$ implies $w^*vw = w^*(\alpha_0 1)w = \alpha_0 1 \in [u]$ (by the Non-$\mathcal{Z}$ Scalar Rule) as claimed. Thus the minimal $r+1$ is 1 and there never were any other terms $\alpha_ih_ik_i^*$ to begin with.

[3] Compare the more elaborate proof in [1, 2, Lemma 1.8, p.176], shifting u until it becomes $k_0^*uh_0 = \alpha 1 + \sum_{h_i \neq 1} \alpha_{h_i} h_i \in [u]$ for $\alpha \neq 0$ and then isolating the scalar by a pull $\mu_{p_m}^*$ (as in Remark 1.2) [$m > |k_0|, |h_0|, |h_i|$ for all i] killing all individual h and k^* for nonempty heads $h, k \neq 1$ of length $< m$ and send the empty head 1 to 1. Cuntz created p_m using an infinite sequence $x_1x_2\ldots$ from X which was aperiodic; the simplest such sequence is $x_1^1x_2x_1^2x_2x_1^3x_2\ldots x_1^nx_2\ldots$, and $p_m = x_1^mx_2$ for suitably large m has the desired properties $\mu_{p_m}^*(hk^*) = 0$ for all $|h|, |k| \leqslant m$ except $\mu_{p_m}^*(x_1^r{x_1^r}^*) = \mu_{p_m}^*(1) = 1$ for $0 \leqslant r < m$.

Finally, if $k^*h \neq 0, 1$ then the Head Product Rules show that $k^*h = h'$ or $k^*h = h'^*$ with $|h'| \neq 0$. If $h' = xh''$ starts with $x \in X$ choose $y \neq x$ (using $|X| \geqslant 2$ again) and replace h, k by the heads hy, ky. Then $(ky)^*u(hy) = y^*(k^*uh)y = \alpha_0 y^*y = \alpha_0 1$ again, but now $(ky)^*(hy) = y^*k^*hy$ becomes $y^*xh''y = 0$ or $y^*h''^*x^*y = 0$ as claimed. □

Another aspect of the obstacle (cf. also [4]) is that it is precisely the kernel of the Frankenstein action of $\mathcal{DM}_X(A)$ on the space $V[B]$ of bodies. We have already noted that the obstacle is contained in the annihilator of the module, and now the reverse inclusion is an immediate consequence of the Scalar Multiple Theorem.

Theorem 3.2. *When the index set X is finite, $2 \leqslant |X| < \infty$, the kernel of the Frankenstein representation of $\mathcal{DM}_X(A)$ on the space $V[B]$ of bodies is precisely the obstacle $\mathcal{Z}_X(A)$, and the reduced algebra $\overline{\mathcal{DM}_X(A)} = \mathcal{DM}_X(A)/\mathcal{Z}_X(A) \cong \mathcal{F}_X(A)$ is isomorphic to the Frankenstein algebra via $\overline{\mathcal{F}}(\overline{E_{h,k}}) = F_{h,k}$.*

Proof. If u is *not* in $\mathcal{Z}_X(A)$ we know by the Scalar Multiple Theorem that there is a scalar $0 \neq \alpha 1 = k^*uh$, and since $V[B]$ is a free A-module the nonzero scalar does not act trivially on $V[B]$, so u cannot either, and u is not in the kernel. Thus the kernel cannot contain any more than $\mathcal{Z}_X(A)$, and the Frankenstein Epimorphism $\mathcal{DM}_X(A) \xrightarrow{\mathcal{F}} \mathcal{F}_X(A)$ has kernel precisely $\mathcal{Z}_X(A)$, so it induces an isomorphism $\overline{\mathcal{F}}$ of $\mathcal{DM}_X(A)/\mathcal{Z}_X(A)$ with $\mathcal{F}_X(A)$. □

A corresponding result about scalar multiples of f holds even when $|X| = 1$.

Theorem 3.3 (Scalar f-Multiple)**.** *Every nonzero element u of the deep matrix algebra $\mathcal{DM}_X(A)$ for $1 \leqslant |X| < \infty$ has a "scalar f-multiple": there exist a nonzero coordinate $0 \neq \alpha \in A$ and heads $h, k \in H$ with $fk^*uhf = \alpha f$.*

Proof. The case $|X| \geqslant 2$ follows easily from the ordinary Scalar Multiple Theorem. If $u \notin \mathcal{Z}$ then there is $k^*uh = \alpha 1 \neq 0$ for heads h, k, hence $fk^*uhf = f(\alpha 1)f = \alpha f$. So suppose $u = \sum_{h,k} \alpha_{h,k} F_{h,k} \in \mathcal{Z} \cong Mat_\infty^{fin}(A)$. If $\alpha_{h_0,k_0} = \alpha \neq 0$, then by the F-Matrix Unit Rules $\alpha f = \alpha F_{\emptyset,\emptyset} = F_{\emptyset,h_0} u F_{k_0,\emptyset} = (fh_0^*)u(k_0 f)$.

We give a general proof for all $|X| \geqslant 1$ along the lines of [6, Thm. 20.3 p.267]: some multiple v of u contains a nonzero term $\alpha\, 1$, hence $fvf = \alpha f$. Indeed, if $u = \sum_{h,k} \alpha_{h,k} hk^* \neq 0$ for distinct hk^*, let α_{h_0,k_0} be a nonzero coefficient which is *minimal* in the sense that $\alpha_{h,k} = 0$ when $h \vdash h_0$ is a proper initial segment, and also when $h = h_0$ but $k \vdash k_0$ is a proper initial segment. Then the product $fh_0^*\, u\, k_0 f$ has a unique term $\alpha_{h_0,k_0} fh_0^*(h_0k_0^*)k_0 f = \alpha_{h_0,k_0} f1f = \alpha_{h_0,k_0} f$ since any other nonzero term $\alpha_{h,k} fh_0^*(hk^*)$ vanishes unless h_0, h are related by the Head Product Rule; we can't have $h \vdash h_0$ by minimality, so we must have $h_0 \Vdash h = h_0h'$, in which case the product $fh_0^*(h_0h')k^* = f\,h'k^*$ vanishes unless $h' = 1, h = h_0$ (since f kills $h'k'^*$ if $|h'| \geqslant 1$ by (F1)). But when $h = h_0$ we have $fh_0^*(hk^*)k_0 f = fh_0^*(h_0k^*)k_0 f = f\,k^*k_0 f = 0$ unless k, k_0 are related; we can't have $\alpha_{h_0,k} \neq 0$ if $k \vdash k_0$ by minimality again, and $k \neq k_0$ by distinctness, so we must have $k_0 \vdash k = k_0k'$ for $k' \neq 1$, in which case $f\,k^*k_0\,f = f\,k'^*\,f = 0$ by (F3). □

Observe that when X is finite (so that e, f exist), on W we have $ez\mathbf{b}' = 0$, $fz\mathbf{b}' = z\mathbf{b}'$ because in $\mathcal{D}'$ we have $f = zz^* + f'$ for f' the new infinite idempotent f'.

4. The ideal lattice

For any ideal $I \lhd A$ we have an ideal $\mathcal{DM}_X(I) := I\,\mathcal{DM}_X(A) \ \lhd\ \mathcal{DM}_X(A)$ with quotient $\mathcal{DM}_X(A)/\mathcal{DM}_X(I) \cong \mathcal{DM}_X(A/I)$. For infinite X these are the only ideals of $\mathcal{DM}_X(A)$ [6, Thm 20.4 p.268] as with ordinary matrix algebras $Mat_n^{fin}(A)$ and $Mat_\infty^{fin}(A)$, but for finite index sets X this is not quite the case, due to the obstacle. Clearly an ideal in $\mathcal{DM}_X(A)$ contains $\mathcal{DM}_X(I)$ iff it contains $I1$, and contains $\mathcal{Z}_X(I')$ iff it contains $I'f$.

Theorem 4.1 (Ideal Lattice). *The ideals of $\mathcal{DM}_X(A)$ are precisely all $\mathcal{I} = \mathcal{I}_{I,I'} := \mathcal{DM}_X(I) + \mathcal{Z}_X(I')$ for nested pairs $I \subseteq I'$ of ideals in A. Here $\mathcal{I}$ uniquely determines I, I': $\quad I1 = \mathcal{I} \cap A1$, $\mathcal{Z}_X(I') = \mathcal{I} \cap \mathcal{Z}_X(A)$ (equivalently, $I'f = \mathcal{I} \cap Af$).*

In particular, if A is simple the only ideals are $\mathcal{DM}_X(A) = \mathcal{I}_{A,A}$, $\mathcal{Z}_X(A) = \mathcal{I}_{0,A}$, and $0 = \mathcal{I}_{0,0}$, so $\mathcal{DM}_X(A)/\mathcal{Z}_X(A)$ and $\mathcal{Z}_X(A)$ are simple algebras.

Proof. Given $\mathcal{I} \lhd \mathcal{D}$, $\mathcal{I} \cap A1 \lhd A1$ must be $I1$ and by the Obstacle Theorem the ideal $\mathcal{I} \cap \mathcal{Z} \lhd \mathcal{Z}$ must have the form $\mathcal{Z}_X(I')$. Clearly $I \subseteq I'$ are ideals of A, and $\mathcal{I} \supseteq I1\mathcal{D} + I'f\mathcal{Z} = \mathcal{DM}_X(I) + \mathcal{Z}_X(I') = \mathcal{I}_{I,I'}$. Note that $\mathcal{I} \cap \mathcal{Z} = \mathcal{Z}_X(I')$ implies $\mathcal{I} \cap Af = I'f$, so the two expressions for I' are equivalent. We claim we have equality $\mathcal{I} = \mathcal{I}_{I,I'}$. Passing to $\overline{\mathcal{I}} \lhd \overline{\mathcal{D}} = \mathcal{DM}_X(A)/\mathcal{DM}_X(I) \cong \mathcal{DM}_X(\overline{A})$ for $\overline{A} := A/I$, if $\overline{\mathcal{I}} \not\subseteq \overline{\mathcal{Z}}$ then by the Scalar Multiple Theorem we have a nonzero $\bar{\alpha}1 \in \overline{\mathcal{I}}$; but then $\alpha 1 \in \mathcal{I} + \mathcal{DM}_X(I) = \mathcal{I}$, forcing $\alpha \in I$ by the definition of I, hence $\bar{\alpha}1 = \bar{0}$, a contradiction. Thus we must have $\overline{\mathcal{I}} \subseteq \overline{\mathcal{Z}}$ and $\mathcal{I} \subseteq \mathcal{DM}_X(I) + \mathcal{Z}_X(A)$. By the Dedekind Modular Law $\mathcal{I} = \mathcal{I} \cap \big(\mathcal{DM}_X(I) + \mathcal{Z}_X(A)\big) = \mathcal{DM}_X(I) + \mathcal{I} \cap \mathcal{Z}_X(A) = \mathcal{DM}_X(I) + \mathcal{Z}_X(I')$ by definition of I', as claimed. $\square$

The algebra $Mat_n^{fin}(A)$ arises as the endomorphisms of a free right A-module $V_A \cong A^n$. Similarly, the Frankenstein algebra $\mathcal{F} = \mathcal{F}_X(A)$ arises as endomorphisms of the free right Frankenstein module $V[B] = BA$, which becomes an $(\mathcal{F}, A)$-bimodule. We also have a free left Frankenstein A-module $V_L[B] = AB := \bigoplus_{\mathbf{b} \in B} A\mathbf{b}$ and a representation of $\mathcal{DM}_X(A)$ as Φ-linear (but not A-linear unless A is commutative) endomorphisms of AB via $(\alpha E_{h,k})(\beta\mathbf{b}) = \alpha\beta E_{h,k}(\mathbf{b})$; again the annihilator of this module is precisely $\mathcal{Z}$ in the finite case, and $V_L[B]$ provides a faithful representation for $\mathcal{F}$. Instead of working with the left Frankenstein module $V_L[B]$ as $(\mathcal{F}, \Phi)$-bimodule, we will work with the ordinary Frankenstein module $V[B]$ as $(\mathcal{F}, A)$-bimodule.

The sub-bimodules of the Frankenstein $(\mathcal{F}, A)$-bimodule $V[B]$ are described in [6, Th.20.10, p.272]. The argument given there works for arbitrary X and for $(\mathcal{F}, \Phi)$-bimodules as well. Recall that the **tail-class** τ of a body $\mathbf{b}$ is the equivalence class of $\mathbf{b}$ where $\mathbf{b}' \sim \mathbf{b}$ if they have the same tail but different heads ($\mathbf{b} = k\mathbf{c}$, $\mathbf{b}' = h\mathbf{c}$, equivalently $\mathbf{b}' = hk^*\mathbf{b}$). Thus the hk^* act transitively on the members of a tail class, and for a fixed tail-class τ the **tail submodule** $V_\tau = \oplus_{\mathbf{b} \in \tau} \mathbf{b}A$ is a cyclic $\mathcal{F}$-module with any body $\mathbf{b} \in \tau$ as generator. The Frankenstein module splits as a direct sum $V[B] = \bigoplus_\tau V_\tau$ of these $\mathcal{F}$-invariant tail submodules. For any ideal $I \lhd A$ we obtain invariant submodules $V_\tau(I) := I\,V_\tau(B) = V_\tau(B)\,I = \oplus_{\mathbf{b} \in \tau} \mathbf{b}I$.

Theorem 4.2 (Frankenstein Submodules)**.** *The $\mathcal{F}_X(A)$-invariant right A-submodules (the $(\mathcal{F}, A)$-bimodules) of $V[B]$ are precisely all*

$$W = \bigoplus_\tau V_\tau(I_\tau) \quad \textit{for ideals } I_\tau := \{\alpha \in A \mid V_\tau \alpha \subseteq W\}.$$

The irreducible sub-bimodules are the $V_\tau(I)$ for minimal ideals $I \lhd A$. If A is simple, $V[B]$ is completely reducible with the irreducible invariant sub-bimodules precisely the $V_\tau = V_\tau(A)$.

The $\mathcal{F}_X(A)$-invariant right Φ-submodules (the $(\mathcal{F}, \Phi)$-bimodules) of $V[B]$ are precisely all

$$W = \bigoplus_\tau V_\tau(L_\tau) \quad \textit{for left ideals } L_\tau := \{\alpha \in A \mid V_\tau \alpha \subseteq W\}.$$

The irreducible submodules are the $V_\tau(L)$ for minimal left ideals $L \lhd_\ell A$. If A is completely reducible as a left A-module (a direct sum of simple left ideals) then $V[B]$ is completely reducible as a left $\mathcal{F}$-module. In particular, if $A = \Delta$ is a division algebra, $V[B]$ is completely reducible with the irreducible $\mathcal{F}$-invariant Φ-submodules precisely the $V_\tau = V_\tau(\Delta)$.

Proof. First notice that

$$\begin{aligned} I_\tau &= \{\alpha \in A \mid \mathbf{b}A\alpha \subseteq W \textit{ for all } \mathbf{b} \in \tau\} \\ &= \{\alpha \in A \mid \mathbf{b}'\alpha \in W \textit{ for some } \mathbf{b}' \in \tau\} \\ &= \{\alpha \in A \mid \mathbf{b}A\alpha A \subseteq W \textit{ for all } \mathbf{b} \in \tau\} \end{aligned} \qquad \begin{aligned} L_\tau &= \{\alpha \in A \mid \mathbf{b}A\alpha \subseteq W \textit{ for all } \mathbf{b} \in \tau\} \\ &= \{\alpha \in A \mid \mathbf{b}'\alpha \in W \textit{ for some } \mathbf{b}' \in \tau\} \end{aligned}$$

In both cases (bimodule or left module) the first equality is the definition, the second holds since $\mathbf{b}A\alpha = Ahk^*(\mathbf{b}'\alpha)$ [from $\mathbf{b} \sim \mathbf{b}'$] and W is left-invariant under $Ahk^* \subseteq \mathcal{F}$, the third holds in the bimodule case since W is a right A-module. From the first equation it is clear that I_τ is a left ideal with $W \supseteq \sum_\tau V_\tau(I_\tau) = \sum_{\mathbf{b}\in\tau} \mathbf{b}I_\tau = \sum_\tau V_\tau I_\tau$, and from the third equation in the bimodule case it is clear that I_τ is an ideal, with $\sum_{\mathbf{b}\in\tau} \mathbf{b}I_\tau = \sum_\tau I_\tau V_\tau = \sum_\tau V_\tau I_\tau$.

Conversely, every $w = \sum_i \mathbf{b}_i\alpha_i \in W$ for distinct bodies has each $\mathbf{b}\alpha_i \in W$: by Body Separation there is an e_h fixing $\mathbf{b}_i$ and killing the other $\mathbf{b}_j$, hence $\mathbf{b}_i\alpha_i = e_h(w) \in W$. Then by the second criterion $\alpha_i \in I_{\tau_i}$ for τ_i the tail-class of $\mathbf{b}_i$, so $\mathbf{b}_i\alpha_i \in V_{\tau_i}(I_{\tau_i})$ for all i, and $W \subseteq \sum_\tau V_\tau(I_\tau)$. □

Similarly, the characterization of $\mathcal{F}$-endomorphisms of the Frankenstein module [6, Th. 20.11, p.273] carries over verbatim.

Theorem 4.3 (Frankenstein Endomorphisms)**.** *The endomorphisms of $V[B]$ as a right A-module which commute with the Frankenstein action (the $(\mathcal{F}_X(A), A)$-bimodule endomorphisms) reduce to central scalar multiplications on the tail submodules:*

$$End_{\mathcal{F}_X(A)}(V[B]) = \textstyle\bigoplus_\tau Center(A)\mathbf{1}_{V_\tau},$$
$$Hom_{\mathcal{F}_X(A)}(V_\tau, V_\tau) = Center(A)\mathbf{1}_{V_\tau}, \quad Hom_{\mathcal{F}_X(A)}(V_\tau, V_\sigma) = 0 \ (\tau \neq \sigma).$$

Thus the V_τ are non-isomorphic $(\mathcal{F}_X(A), A)$-bimodules.

Proof. First, each endomorphism φ acts diagonally: $\mathbf{b}A = \cap_{n=1}^{\infty} e_{h_n(\mathbf{b})}V \implies \varphi(\mathbf{b}) \in \cap_{i=1}^{\infty} e_{h_n(\mathbf{b})}\varphi(V) \subseteq \cap_{n=1}^{\infty} e_{h_n(\mathbf{b})}V = \mathbf{b}A$ so that $\varphi(\mathbf{b}) = \mathbf{b}\alpha_b$ for each body. Since bodies with the same tail are conjugate under $\mathcal{F}$, the scalars α_b are constant on tail-classes: if $\mathbf{b}' \sim \mathbf{b}$ then $\mathbf{b}' = hk^*\mathbf{b}$ and $\mathbf{b}'\alpha_{\mathbf{b}'} = \varphi(\mathbf{b}') = \varphi(hk^*\mathbf{b}) = hk^*\varphi(\mathbf{b}) = hk^*\mathbf{b}\alpha_{\mathbf{b}} = \mathbf{b}'\alpha_{\mathbf{b}}$ implies $\alpha_{\mathbf{b}'} = \alpha_{\mathbf{b}}$. The diagonal scalars must lie in the center: $\varphi(\beta 1(\mathbf{b})) = \varphi(\mathbf{b}\beta) = \varphi(\mathbf{b})\beta$ [by right A-linearity] $= \mathbf{b}\alpha_{\mathbf{b}}\beta$ must equal $\beta 1(\varphi(\mathbf{b})) = \beta 1(\mathbf{b}\alpha_{\mathbf{b}}) = \mathbf{b}\beta\alpha_{\mathbf{b}}$, so $\alpha_{\mathbf{b}}$ must commute with all $\beta \in A$.

Since any bimodule-homomorphism in $Hom(V_\tau, V_\sigma)$ extends to an endomorphism φ of $V[B]$ by setting $\varphi = 0$ on all V_ρ for $\rho \neq \tau$; the the above diagonal result says $\varphi = 0$ if $\sigma \neq \tau$, and when $\sigma = \tau$ it is $\alpha\mathbf{1}_{V_\tau}$ for some central α. □

5. The Case $|X| = 1$

We are primarily concerned with the case $|X| \geqslant 2$, but we indicate here how the "exceptional case" [5] $X = \{x\}$ of cardinality 1 becomes more complicated: the Frankenstein representation degenerates and the Kennedy representation becomes the standard Kaplansky shift representation, but the Scalar Multiple Theorem 3.1 fails and $\mathcal{DM}_x(A)$ has a complicated ideal structure.

Theorem 5.1 (Cardinality 1)**.** *If $X = \{x\}$ has cardinality 1, then there is only one body $\mathbf{b} = (x, x, x, \ldots)$, $V[B]$ is 1-dimensional, $\mathcal{F}_X(A) = A\mathbf{1}_{V[B]}$ is also 1-dimensional, and the Frankenstein ideal lattice is that of A.*

On the other hand, $V[H] = \bigoplus_{j\geqslant 0} v_j A$ is infinite-dimensional with Kennedy representation the shift action, $x(v_j) = v_{j+1}$ the forward shift operator, $x^*(v_j) = v_{j-1}$ $(j \geqslant 1)$, $x^*(v_0) = 0$ the backward shift. Thus $\mathcal{DM}_x(A)$ is faithfully imbedded in the row-and-column-finite matrices as the standard Kaplansky shift algebra via $x \to \sum_{j=0}^{\infty} E_{j+1,j}$ with 1's on the subdiagonal,and $x^* \to \sum_{j=0}^{\infty} E_{j,j+1}$ with 1's on the superdiagonal. If A is simple this action is irreducible.

The reduced deep matrix algebra $\overline{\mathcal{DM}_x(A)} = \mathcal{DM}_x(A)/\mathcal{Z}_x(A)$ is isomorphic to the algebra $A[t, t^{-1}]$ of Laurent polynomials over A, so the lattice of ideals of $\overline{\mathcal{DM}_x(A)}$ is isomorphic to the lattice of t-cancelable ideals of the polynomial ring $A[t]$ (those $I_0 \lhd A[t]$ such that $p(t) \in A[t], tp(t) \in I_0 \implies p(t) \in I_0$).

The Scalar Multiple Theorem fails in $\mathcal{DM}_x(A)$: there are no heads h, k satisfying $k^*(x + x^*)h = \alpha 1 \neq 0$.

Proof. This situation is polycephalic, with one body but lots of heads: there is only a single body $\mathbf{b} = (x, x, x, \ldots) \in B(X)$, $V[B] = \mathbf{b}A$ is 1-dimensional, and the Frankenstein action is $\big(\sum \alpha_{m,n} x^m x^{*n}\big)(\mathbf{b}) = \big(\sum_{m,n} \alpha_{m,n}\big)\mathbf{b}$, so the kernel of the Frankenstein Epimorphism is the set of deep matrices having coefficient sum zero, with quotient the Frankenstein algebra $\mathcal{DM}_x(A)/Ker(\mathcal{F}) \cong \mathcal{F}_X(A) = A\mathbf{1}_{V[B]}$. Thus the Frankenstein algebra ideals are precisely those of A.

There are many heads: $H(X) = \{x^n \mid n \geqslant 0\}$, $V[H] = \bigoplus_{n\geqslant 0} x^n A =: \bigoplus_{n\geqslant 0} v_n A$ is an infinite-dimensional free A-module with basis of $v_n = x^n$, and under the Kennedy representation $\mathcal{DM}_x(A)$ is isomorphic to the familiar Kaplansky shift algebra [3, Example 6(2) p.35] with x the forward shift $x(v_i) = v_{i+1}$ and x^* the backward shift

$x^*(v_i) = v_{i-1}$, $x^*(v_0) = 0$. This algebra is spanned over A by the k^{th} subdiagonals $x^k = \sum_{i=0}^{\infty} E_{i+k,i}$ and superdiagonals $x^{*k} = \sum_{i=0}^{\infty} E_{i,i+k}$ (the 0^{th} sub- and super-diagonals being the identity matrix Id), plus $Mat_{\infty}^{fin}(\Phi)$ spanned by the $E_{i,j}$. Indeed, computing the action on the v_i's shows shows

$$\begin{array}{lll} x^n x^{*n} = \sum_{i=n}^{\infty} E_{i,i}, & x^{n+k} x^{*n} = \sum_{i=n}^{\infty} E_{i+k,i}, & x^m x^{*m+k} = \sum_{i=m}^{\infty} E_{i,i+k}, \\ f = 1 - xx^* = E_{0,0}, & x^{*k} E_{0,0} = E_{0,0} x^k = 0 \ (k \geqslant 1), & E_{i,j} = x^i E_{0,0} x^{*j}. \end{array}$$

These products show that $\mathcal{Z}_x(A) = AHH^*fHH^* = AHfH^* = \sum_{i,j\geqslant 0} Ax^i f x^{*j} = \sum_{i,j\geqslant 0} AE_{i,j} \cong Mat_{\infty}^{fin}(A)$, so the reduced deep matrix algebra $\mathcal{DM}_x(A)/\mathcal{Z}_x(A) = \big(\sum_{n=1}^{\infty} Ax^{*n} + A + \sum_{m=1}^{\infty} Ax^m\big) + \mathcal{Z}_x(A)$ with $x^*x = 1, xx^* = 1 - f \in 1 + \mathcal{Z}_x(A)$ is isomorphic to the algebra $A[t, t^{-1}] = \sum_{k=-\infty}^{\infty} At^k$ of Laurent polynomials over A mapping the cosets of x, x^* to t, t^{-1} with $t^{-1}t = tt^{-1} = 1$. Since $\mathcal{DM}_x(A)/Ker(\mathcal{F}) \cong A$, the Frankenstein kernel is much bigger than the obstacle $\mathcal{Z}_x(A)$.

It is easily checked that the ideals $I \lhd A[t, t^{-1}]$ are precisely all $I = \sum_{k=0}^{\infty} t^{-k} I_0$ for t-cancelable ideals $I_0 = I \cap A[t] \lhd A[t]$. But it seems difficult to describe these for arbitrary associative coordinate rings A.

For heads $h = x^n, k = x^m$, $k^*(x + x^*)h = x^{*m}x^{n+1} + x^{*m+1}x^n$ has a term $\alpha 1$ only if $m = n+1$ or $m+1 = n$, yielding $1 + x^{*2}$ or $x^2 + 1$, so we cannot reduce to a single term and we can never obtain $k^*uh = \alpha 1 \neq 0$. The trouble with a single x is lack of orthogonality: no product of x's and x^*'s is ever zero. □

References

[1] J. Cuntz: Simple C^*-algebras generated by isometries. *Comm. in Math. Phys.* 57: 173-185 (1970).

[2] J. Cuntz, W. Krieger: A class of C^*-algebras and topological Markov chains. *Inv. Math.* 56: 251-268 (1980).

[3] N. Jacobson: *Structure of Rings*, Amer. Math.Soc. Pub. vol. 37, Amer. Math. Soc., Providence, 1956.

[4] C. Kennedy: Deep matrix algebras of finite type, *Algebras and Representation Theory* 9: 525-537 (2006).

[5] L. Kennedy: Exceptional type deep matrix algebras, in progress.

[6] K. McCrimmon: Chapter 20: Deep matrices and their Frankenstein actions, in *Nonassociative Algebras and its Applications* by L. Sabinen, L. Sbitneva, I. Shestakov (eds.), Lecture Notes in Pure and Applied Mathematics, vol. 246, CRC Press, Boca Raton (2006).

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