AN EMBEDDING THEOREM FOR REDUCED ALBERT ALGEBRAS OVER ARBITRARY FIELDS

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ABSTRACT. Extending two classical embedding theorems of Albert-Jacobson and Jacobson for Albert (= exceptional simple Jordan) algebras over fields of characteristic not two to base fields of arbitrary characteristic, we show that any element of a reduced Albert algebra can be embedded into a reduced absolutely simple subalgebra of degree 3 and dimension 9 which may be chosen to be split if the Albert algebra was split to begin with.

INTRODUCTION

It is sometimes useful to embed certain elements of a given algebraic structure into a substructure with particularly nice properties. For example, any semi-simple element of a connected linear algebraic group can be embedded into a maximal torus (Borel [4, Thm. 11.10]). Or, along different lines, any element of an octonion algebra can be embedded into a quaternion subalgebra (Springer-Veldkamp [26, Prop. 1.6.4]), which may be chosen to be split if the octonion algebra was split to begin with (Jacobson [9, Lemma IX.6.2], Garibaldi-Petersson [8, Prop. 5.5]). Similar embedding theorems, due to Albert-Jacobson [2, Thm. 1] and Jacobson [9, Thm. IX.11], respectively, exist for reduced Albert algebras but are confined to base fields of characteristic not 2. It is the purpose of the present paper to remove this restriction by establishing the following result.

Embedding Theorem. *Let J be a reduced Albert algebra over an arbitrary field. Then any element of J can be embedded into a reduced absolutely simple subalgebra of degree 3 and dimension 9. Moreover, if J is split, this subalgebra may be chosen to be split as well.*

Carrying out the proof of this theorem in full generality turns out to be surprisingly delicate. In the reduced non-split case, a straightforward application of the Jacobson coordinatization theorem (Jacobson [10, 5.4.2] and [9, p. 137]) combined with the uniqueness of the coefficient algebra (Albert-Jacobson [2, Thm. 3] and Faulkner [7, Thm. 1.8]) immediately yields the embeddibility of an arbitrary element into a reduced simple subalgebra of degree 3 and dimension 9; but to show that this subalgebra may, in fact, be chosen to be *absolutely* simple (which is automatic in characteristic not 2) requires a considerable amount of effort. In the split case, on the other hand, while we will adhere rather closely to the overall strategy [9, Lemmata IX.6.1−3] employed by Jacobson in his proof of the embedding theorem, quite a few cumbersome detours have to be taken in order to include base fields of characteristic 2, and more specifically, the field with two elements.

The final statement of the embedding theorem has recently been applied by Anquela-Cortés-Petersson [3] in their study of commuting U-operators in Jordan algebras.

1. NOTATIONAL CONVENTIONS.

Many of our subsequent considerations, when phrased with appropriate care, are valid over arbitrary commutative rings; we refer to [20] for details. But since our principal objective will be the proof of the embedding theorem, we always confine ourselves to

base fields rather than rings, so throughout this paper, we let F be a field of arbitrary characteristic. The bilinearization (or polar form) of a quadratic map $Q\colon V \to W$ between vector spaces V, W over F will always be indicated by $Q(x,y) = Q(x+y) - Q(x) - Q(y)$. A quadratic form over F, i.e., a quadratic map $q\colon V \to F$ with V finite-dimensional, is said to be *non-singular* if its induced symmetric bilinear form $q(x,y)$ is non-degenerate in the usual sense of linear algebra. For standard facts about (unital quadratic) Jordan algebras, the reader is referred to Jacobson [10]. When it comes to labeling Peirce spaces relative to individual idempotents, however, we follow Loos [14, § 5]. Free use will be made of the differential calculus for polynomial maps as explained in Jacobson [9, Chap. VI] which, with appropriate care, works over arbitrary fields, not just infinite ones; see Roby [24] or Loos [14, § 18] for generalization. Deviating from the notation used in [9], we write $(DP)(x,y)$ for the directional derivative of a polynomial map P at x in the direction y.

2. Conic algebras

Conic algebras are the same as algebras of degree 2 in the sense of McCrimmon [19]. Over arbitrary commutative rings, they have been thoroughly investigated by Loos [15]. Their main properties over fields have been summarized, with appropriate references, by Garibaldi-Petersson [8]. In what follows, we confine ourselves to what is absolutely indispensable for the subsequent development.

2.1. **The concept of a conic algebra.** A (non-associative) F-algebra C is said to be *conic* if it has an identity element $1_C \neq 0$ and there exists a quadratic form $n_C\colon C \to F$, necessarily unique and called the *norm* of C, such that $x^2 - n_C(1_C, x)x + n_C(x)1_C = 0$ for all $x \in C$. The linear map $t_C\colon C \to F$ defined by $t_C(x) := n_C(1_C, x)$ is then called the *trace* of C, while we refer to $\iota_C\colon C \to C$ given by $x \mapsto \bar{x} := t_C(x)1_C - x$ as the *conjugation* of C; it is a linear map of period 2. The unital subalgebra of C generated by an element $v \in C$ will be denoted by $F[v]$; it is spanned by $1_C, v$ as a vector space over F and, in particular, has dimension at most 2. *Homomorphisms* of conic algebras are defined as algebra homomorphisms preserving norms and units, hence traces and conjugations as well.

2.2. **Identities in conic algebras.** Let C be a conic algebra over F, with norm n_C, trace t_C and conjugation ι_C. Then the following identities hold.

$$(1) \qquad\qquad n_C(1_C) = 1,$$
$$(2) \qquad\qquad t_C(1_C) = 2,$$
$$(3) \qquad\qquad t_C(x) = n_C(1_C, x)$$
$$(4) \qquad\qquad x^2 = t_C(x)x - n_C(x)1_C,$$
$$(5) \qquad\qquad \bar{x} = t_C(x)1_C - x,$$

Conic algebras are clearly stable under base field extensions.

2.3. **Conic alternative algebras.** Let C be a conic algebra over F which is also alternative, so beside the alternative laws and flexibility,

$$(1) \qquad x(xy) = x^2 y, \quad (yx)x = yx^2, \quad (xy)x = x(yx) =: xyx,$$

C satisfies the Moufang identities

$$(2) \qquad x\big(y(xz)\big) = (xyx)z, \quad \big((zx)y\big)x = z(xyx), \quad (xy)(zx) = x(yz)x.$$

Combining (1) with (2.2.4) and (2.2.5), we deduce Kirmse's identities

$$(3) \qquad\qquad x(\bar{x}y) = n_C(x)y = (y\bar{x})x.$$

Moreover, the conjugation of C is an algebra involution, we have the relations

$$(4) \qquad n_C(x,y) = t_C(x\bar{y}) = t_C(x)t_C(y) - t_C(xy),$$

and the trace is a commutative associative linear form:

$$(5) \qquad t_C(xy) = t_C(yx), \quad t_C\big((xy)z\big) = t_C\big(x(yz)\big) =: t_C(xyz).$$

Also, the norm *permits composition* in the sense that it satisfies the relation

$$(6) \qquad n_C(xy) = n_C(x)n_C(y).$$

2.4. **Inverses.** Let C be a conic alternative F-algebra. An element $x \in C$ is *invertible* in the sense that some $x^{-1} \in C$ (necessarily unique and called the *inverse* of x in C) has $xx^{-1} = 1_C = x^{-1}x$ if and only if $n_C(x) \in F$ is invertible, in which case we have

$$(1) \qquad x^{-1} = n_C(x)^{-1}\bar{x}, \quad (x^{-1})^{-1} = x, \quad x(x^{-1}y) = y = (yx^{-1})x$$

for all $y \in C$.

2.5. **Composition algebras.** We use the term *composition algebra* in order to designate Hurwitz algebras in the sense of Knus-Merkurjev-Rost-Tignol [13, § 33]. Hence composition algebras over F are the same as finite-dimensional conic alternative F-algebras such that either their norm is a non-singular quadratic form, or their dimension is 1 and F has characteristic 2; in the former case, we speak of *non-singular* composition algebras. Of particular importance in this paper are quaternion algebras (composition algebras of dimension 4, which are associative but not commutative) and octonion (= Cayley) algebras (composition algebras of dimension 8, which are alternative but not associative). For further properties of these, see also Springer-Veldkamp [26].

2.6. **Idempotents.** Let C be a composition algebra over F. An element $c \in C$ is a *non-trivial idempotent* (i.e., $c^2 = c \neq 0, 1_C$) if and only if $n_C(c) = 0$, $t_C(c) = 1$. In this case, $c' := 1_C - c$ is a non-trivial idempotent as well, and the Peirce decomposition of C relative to the complete orthogonal system (c, c') (Schafer [25, Prop. 3.4]) takes on the form $C = Fc + C_{12} + C_{21} + Fc'$ as a direct sum of subspaces. It is then easy to check that the off-diagonal Peirce components C_{12}, C_{21} are dual to each other under the bilinearized norm (or trace).

3. Cubic Jordan algebras.

In this section, we recall some basic facts about cubic Jordan algebras that will be needed frequently later on. Our main reference is McCrimmon [16].

3.1. **Cubic norm structures.** By a *cubic norm structure* over F (originally called a *cubic form with adjoint and base point*) we mean a quadruple $X = (X, N, \sharp, 1)$ consisting of a finite-dimensional vector space X over F, a cubic form $N \colon X \to F$ (the *norm*), a quadratic map $X \to X$, $x \mapsto x^\sharp$, (the *adjoint*) and a distinguished element $1 \in X$ (the *base point*) such that the following conditions are fulfilled. Writing $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ for the bilinearization of the adjoint,

$$T := -(D^2 \log N)(1) \colon X \times X \longrightarrow F$$

for the *bilinear trace* of X, which is a symmetric bilinear form and induces the *linear trace*

$$T \colon X \longrightarrow F, \quad x \longmapsto T(x) := T(x, 1),$$

the relations

$$(1) \qquad N(1) = 1, \quad 1^\sharp = 1 \qquad \text{(base point identities)},$$

$$(2) \qquad x^{\sharp\sharp} = N(x)x \qquad \text{(adjoint identity)},$$

$$(DN)(x, y) = T(x^\sharp, y) \qquad \text{(gradient identity)},$$

$$(3) \qquad 1 \times x = T(x)1 - x \qquad \text{(unit identity)}$$

hold under all base field extensions. The quadratic form

$$(4) \qquad S \colon X \longrightarrow F, \quad x \longmapsto S(x) := T(x^\sharp),$$

is then called the *quadratic trace* of X. As a companion to the base point identities (1), we have

(5) $$T(1) = 3,$$

while the quadratic trace bilinearizes to

(6) $$S(x, y) = T(x \times y) = T(x)T(y) - T(x, y).$$

Homomorphisms of cubic norm structures are defined as linear maps preserving norms, adjoints and base points in the obvious sense.

3.2. The concept of a cubic Jordan algebra.

Let $X = (X, N, \sharp, 1)$ be a cubic norm structure over F with (bi-)linear trace T and quadratic trace S. Then the vector space X together with the unit element $1 \in X$ and the U-operator

(1) $$U_x y := T(x, y)x - x^\sharp \times y \qquad\qquad (x, y \in X)$$

converts X into a Jordan algebra over F which we write as $J = J(X)$. The Jordan triple product of J is given by

(2) $$\{xyz\} = T(x, y)z + T(y, z)x - (z \times x) \times y \qquad\qquad (x, y, z \in X).$$

Jordan algebras of the form $J = J(X)$ are said to be *cubic* because the formula $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$ holds in every base field extension; moreover, the adjoint relates to the algebra structure of J by

(3) $$x^\sharp = x^2 - T(x)x + S(x)1,$$

which linearizes to

(4) $$x \times y = x \circ y - T(x)y - T(y)x + S(x, y)1.$$

If $f \colon X \to X'$ is a homomorphism of cubic norm structures, then $J(f) := f \colon J(X) \to J(X')$ is a homomorphism of Jordan algebras. Hence our construction yields a faithful functor from the category of cubic norm structures over F to the category of Jordan algebras over F which, however, is not full (Petersson-Racine [21, Remark to Prop. 2.6]). This difficulty disappears when dealing with isomorphisms of

3.3. Generically algebraic Jordan algebras of degree 3.

Let J be a Jordan algebra over F that is generically algebraic of degree 3 in the sense of Jacobson-Katz [11]. Then there is a *unique* cubic norm structure X over F satisfying $J = J(X)$, namely the one whose norm, adjoint and base point are respectively given by the generic norm [11, Theorem 2], the adjoint [11, p. 221] and the unit element of J in its capacity as a generically algebraic Jordan algebra. It follows that any isomorphism between generically algebraic Jordan algebras of degree 3 is also one of the underlying cubic norm structures, hence preserves norms, adjoints and base points.

3.4. Cubic Jordan matrix algebras.

Let C be a conic alternative F-algebra and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(F)$ an invertible 3×3 diagonal matrix with entries in F. We write $\mathrm{Her}_3(C, \Gamma)$ for the totality of 3×3-matrices over C which are Γ-hermitian in the sense that $x = \Gamma^{-1}\bar{x}^t\Gamma$ and have scalars down the diagonal, the latter condition being automatic unless the characteristic is 2. We abbreviate $\mathrm{Her}_3(C) := \mathrm{Her}_3(C, \mathbf{1}_3)$, where $\mathbf{1}_3$ stands for the 3×3 unit matrix. Writing e_{ij} for the ordinary matrix units, there is a natural set of generators for $\mathrm{Her}_3(C, \Gamma)$ as a vector space over F furnished by the *hermitian matrix units*

$$v[jl] := \gamma_l v e_{jl} + \gamma_j \bar{v} e_{lj}$$

for $v \in C$ and $j, l = 1, 2, 3$ distinct. More precisely, an element $x \in \mathrm{Mat}_3(C)$ belongs to $\mathrm{Her}_3(C, \Gamma)$ if and only if it can be written in the form, necessarily unique,

(1) $$x = \sum \big(\alpha_i e_{ii} + v_i[jl]\big) \qquad\qquad (\alpha_i \in F,\ v_i \in C,\ i = 1, 2, 3),$$

where we systematically adhere to the convention that summations like the one on the right of (1) always extend over the cyclic permutations (ijl) of (123). Similarly, for

$i = 1, 2, 3$ (resp. $p = 1, 2, 3$), indices $j, l = 1, 2, 3$ (resp. $q, r = 1, 2, 3$) will always be so chosen as to form the unique cyclic permutation (ijl) (resp. (pqr)) of (123) starting with i (resp. p).

We now put $X := \mathrm{Her}_3(C, \Gamma)$ as a vector space over F and define a cubic form $N \colon X \to F$, a quadratic map $\sharp \colon X \to X$ and a base point $1 \in X$ by observing $(2.3.5)$ and imposing the condition that, for x as in (1), the relations

$$(2) \qquad N(x) = \alpha_1 \alpha_2 \alpha_3 - \sum \gamma_j \gamma_l \alpha_i n_C(v_i) + \gamma_1 \gamma_2 \gamma_3 t_C(v_1 v_2 v_3),$$

$$(3) \qquad x^\sharp = \sum \Big(\big(\alpha_j \alpha_l - \gamma_j \gamma_l n_C(v_i) \big) e_{ii} + \big(-\alpha_i v_i + \gamma_i \overline{v_j v_l} \big) [jl] \Big),$$

$$1 = \sum e_{ii}$$

hold in all base field extensions. Then $(X, N, \sharp, 1)$ is a cubic norm structure over F, again denoted by $\mathrm{Her}_3(C, \Gamma)$, giving rise to a cubic Jordan algebra over F as in 3.2, which we designate by $\mathrm{Her}_3(C, \Gamma)$ as well, and which is, in fact, generically algebraic of degree 3. With

$$y = \sum \big(\beta_i e_{ii} + w_i[jl] \big) \qquad\qquad (\beta_i \in F, \ w_i \in C, \ i = 1, 2, 3),$$

the (bi-)linear and quadratic trace of $\mathrm{Her}_3(C, \Gamma)$ are respectively given by

$$(4) \qquad\qquad T(x, y) = \sum \big(\alpha_i \beta_i + \gamma_j \gamma_l n_C(v_i, w_i) \big),$$

$$(5) \qquad\qquad T(x) = \sum \alpha_i,$$

$$(6) \qquad\qquad S(x) = \sum \big(\alpha_j \alpha_l - \gamma_j \gamma_l n_C(v_i) \big),$$

while the adjoint bilinearizes to

$$(7) \qquad x \times y = \sum \big(\alpha_j \beta_l + \beta_j \alpha_l - \gamma_j \gamma_l n_C(v_i, w_i) \big) e_{ii}$$
$$+ \sum \big(-\alpha_i w_i - \beta_i v_i + \gamma_i \overline{v_j w_l + w_j v_l} \big) [jl].$$

As special cases of (3), (7), we record

$$(8) \quad u[jl]^\sharp = -\gamma_j \gamma_l n_C(u) e_{ii}, \ \ u[jl] \times v[jl] = -\gamma_j \gamma_l n_C(u, v) e_{ii}, \ \ v[jl] \times w[li] = \gamma_l \overline{vw}[ij].$$

3.5. Lemma. *With the notation and assumptions of 3.4, let (pqr) be a cyclic permutation of (123) and suppose the element $u \in C$ has norm 1. Then the map*

$$\varphi := \varphi_{p,u} \colon \ \mathrm{Her}_3(C, \Gamma) \xrightarrow{\ \sim\ } \mathrm{Her}_3(C, \Gamma)$$

defined by

$$(1) \quad \varphi\Big(\sum (\beta_i e_{ii} + w_i[jl]) \Big) := \sum \beta_i e_{ii} + (uw_p)[qr] + (w_q u)[rp] + (u^{-1} w_r u^{-1})[pq]$$

for $\beta_i \in F$, $w_i \in C$, $i = 1, 2, 3$, is an automorphism of $\mathrm{Her}_3(C, \Gamma)$.

Proof. Since φ is bijective, it remains to show that it preserves norms, adjoints and base points. While the last assertion is obvious, the first two follow from $(3.4.2)$, $(3.4.3)$ by a straightforward computation, involving $(2.3.5)$, $(2.3.6)$, the Moufang identities $(2.3.2)$, and $(2.4.1)$. $\qquad \square$

The proof of the following observation is even more elementary and will be totally omitted.

3.6. Lemma. *With the notation and assumptions of 3.4, put $\Gamma' := \mathrm{diag}(\gamma_3, \gamma_1, \gamma_2)$. Then the assignment*

$$\sum (\alpha_i e_{ii} + v_i[jl]) \longmapsto \sum (\alpha_l e_{ii} + v_l[jl]) \qquad (\alpha_i \in F, \ v_i \in C, \ i = 1, 2, 3)$$

defines an isomorphism $\mathrm{Her}_3(C, \Gamma) \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma')$ sending e_{ii} to e_{jj} for $i = 1, 2, 3$. $\qquad \square$

4. Elementary idempotents and co-ordinatization.

Before we will be able to tackle the embedding theorem, it will be necessary to give a brief survey, mostly without proofs, of the Jacobson co-ordinatization theorem for cubic Jordan algebras. Rather than specializing the theorem in its most general form (see [10, 5.4.2], [9, p. 137], but also McCrimmon [18, p. 260]) to the peculiar case at hand, we find it more useful adapting Racine's ad-hoc approach [23] to the somewhat more general set-up required in the present investigation. We do so by fixing a cubic norm structure $X = (X, N, \sharp, 1)$ with trace T and quadratic trace S over F and writing $J = J(X)$ for the corresponding cubic Jordan algebra.

4.1. **The concept of an elementary idempotent.** By an *elementary idempotent* of J we mean an element $e \in X$ satisfying $T(e) = 1$ and $e^\sharp = 0$. In this case, (3.2.3) and (3.1.4) show $e^2 = e$, so $e \in J$ is indeed an idempotent. Combining (3.2.1) with [10, (5.1.6)] and the proofs of [23, (28)] and Petersson-Racine [22, Lemma 5.3 (d)], it follows that the corresponding Peirce components have the form

(1) $J_2(e) = Fe,$

(2) $J_1(e) = \{x \in J \mid T(x) = 0, \ e \times x = 0\},$

(3) $J_0(e) = \{x \in J \mid e \times x = T(x)(1 - e) - x\}.$

In particular, elementary idempotents are always absolutely primitive in the sense that, even after an arbitrary base field extension, they cannot be written as an orthogonal sum of two non-zero idempotents, but the converse does not hold [21, Prop. 2.6, (2)]. An *elementary frame* of J is defined as a complete orthogonal system $(e_1, \ldots, e_r)$ of elementary idempotents. The integer $r \geq 1$ is then called the *length* of the elementary frame.

4.2. **Lemma.** *The elementary frames of J are precisely of the form (e_1, e_2, e_3), where $e_1, e_2 \in J$ are orthogonal elementary idempotents and $e_3 = e_1 \times e_2 = 1 - e_1 - e_2$. Given such an elementary frame, J is generically algebraic of degree 3, the corresponding Peirce components can be written as*

(1) $J_{ii} = Fe_i, \quad J_{jl} = \{x \in J \mid T(x) = 0, \ e_j \times x = e_l \times x = 0\} \qquad (i = 1, 2, 3),$

and we have

(2) $x_{jl} \times x_{li} = x_{jl} \circ x_{li} \qquad\qquad (x_{jl} \in J_{jl}, \ x_{li} \in J_{li}, \ i = 1, 2, 3).$

Proof. An elementary frame of length 1 in J would consist of the unit element alone. But the base point identities (3.1.1) show $1^\sharp = 1 \neq 0$, so $1 \in J$ can never be an elementary idempotent. Hence the length of an elementary frame in J is at least 2, and given orthogonal elementary idempotents $e_1, e_2 \in J$, the first assertion will follow once we have shown that $e_1 \times e_2 = 1 - e_1 - e_2$ is an elementary idempotent as well. Since $e_2 \in J_0(e_1)$ by orthogonality, (4.1.3) implies $e_1 \times e_2 = T(e_2)(1 - e_1) - e_2 = 1 - e_1 - e_2$, and from (3.1.5) we deduce $T(e_1 \times e_2) = 1$. Moreover, expanding $(1 - e_1 - e_2)^\sharp$ yields $(e_1 \times e_2)^\sharp = 0$. Thus $e_1 \times e_2$ is an elementary idempotent, and J contains an elementary frame of length 3, forcing it to be generically algebraic of degree 3. Now (1) follows immediately from (4.1.1), (4.1.2) and the standard Peirce relations $J_{ii} = J_2(e_i)$, $J_{jl} = J_1(e_j) \cap J_1(e_l)$ [14, (5.14.2), (5.14.3)]. In order to establish (2), we first combine [10, Prop. 5.1.4, (5.1.6)] with (3.2.1) and obtain $0 = U_{e_j} U_{e_j, e_l} x_{jl} = U_{e_j} x_{jl} = T(e_j, x_{jl}) e_j$, hence $T(e_j, x_{jl}) = 0$. Now (1) and (3.1.6) yield $0 = T(e_j \times x_{jl}) = T(e_j) T(x_{jl}) - T(e_j, x_{jl}) = T(x_{jl})$ and, similarly, $T(x_{li}) = 0$. On the other hand, the proof of [23, (33)] gives $T(x_{jl}, x_{li}) = 0$, which combines with (3.1.6) to imply $S(x_{jl}, x_{li}) = 0$. Now (2) may be read off from (3.2.4). $\square$

4.3. **Co-ordinate systems.** Let (e_1, e_2, e_3) be an elementary frame in J with Peirce components as in (4.2.1) and fix an index $i \in \{1, 2, 3\}$. Recall from [10, 5.3.1] that e_j, e_l are said to be *connected by* $u_{jl} \in J_{jl}$ if u_{jl} is invertible in $J_2(e_j + e_l) = J_0(e_i)$. If also e_l, e_i are connected by $u_{li} \in J_{li}$, then so are e_i, e_j by $u_{ij} := u_{jl} \circ u_{li} \in J_{ij}$ [10, 5.3.3]. The entire elementary frame is said to be *connected* if elements u_{jl}, u_{li} with the aforementioned properties exist. This holds true automatically in case the Jordan algebra is simple ([10, Thm. 6.3.1] and [23, p. 98]). By a *co-ordinate system* of J we mean a quintuple

$$(1) \qquad \mathfrak{S} = (e_1, e_2, e_3, u_{23}, u_{31}) \in J^5$$

such that (e_1, e_2, e_3) is an elementary frame of J with the corresponding Peirce decomposition

$$(2) \qquad J = \sum (Fe_i + J_{jl})$$

and e_j, e_l for $i = 1, 2$ are connected by $u_{jl} \in J_{jl}$. Note that in this case J is generically algebraic of degree 3 (Lemma 4.2).

For example, let $J = \mathrm{Her}_3(C, \Gamma)$ be a cubic Jordan matrix algebra as in 3.4. Then (3.4.3), (3.4.5) show that the diagonal idempotents form an elementary frame (e_{11}, e_{22}, e_{33}) with Peirce components

$$(3) \qquad J_{ii} = Fe_{ii}, \quad J_{jl} = C[jl] \qquad\qquad (i = 1, 2, 3).$$

In particular, e_{jj}, e_{ll} are connected by $u_{jl} := 1_C[jl]$ since (2.2.1), (3.2.3), (3.4.5), (3.4.8) imply $u_{jl}^2 = u_{jl}^\sharp - S(u_{jl})1 = -\gamma_j\gamma_l e_{ii} + \gamma_j\gamma_l 1 = \gamma_j\gamma_l(e_{jj} + e_{ll})$. Summing up,

$$\mathfrak{D} := (e_{11}, e_{22}, e_{33}, 1_C[23], 1_C[31]) \in J^5$$

is a co-ordinate system of $\mathrm{Her}_3(C, \Gamma)$, called its *diagonal co-ordinate system.*

We now derive a useful criterion for connectedness of orthogonal elementary idempotents.

4.4. **Lemma.** *Let (e_1, e_2, e_3) be an elementary frame of J with the Peirce decomposition $J = \sum(Fe_i + J_{jl})$. Fixing an index $i = 1, 2, 3$ and $u_{jl} \in J_{jl}$, the following conditions are equivalent.*

    (i) *e_j, e_l are connected by u_{jl}.*
    (ii) *$S(u_{jl}) \in F^\times$.*

Moreover, if we are given a conic alternative F-algebra C, an invertible diagonal matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(F)$ and a homomorphism $\varphi \colon X \to \mathrm{Her}_3(C, \Gamma)$ of cubic norm structures sending e_p to the diagonal idempotent e_{pp} for $p = 1, 2, 3$, then there is a unique element $u \in C$ satisfying $\varphi(u_{jl}) = u[jl]$, and the preceding conditions are also equivalent to

    (iii) *$n_C(u) \in F^\times$.*

Proof. By [10, 5.3.1], (i) holds if and only if u_{jl} is invertible in $J_2(e_j + e_l) = J_0(e_i)$. Here Faulkner's Lemma [7, Lemma 1.5], which was phrased originally in a more restrictive context but is easily seen to hold under the present conditions as well, implies that $J_0(e_i)$ agrees with the Jordan algebra of the pointed quadratic form $(V, S_i, e_j + e_l)$, where V stands for $J_0(e_i)$ as a vector space over F and S_i for the restriction of the quadratic form S to $J_0(e_i)$. But since an element v of the Jordan algebra of a pointed quadratic form (V, q, e) over F is invertible if and only if $q(v)$ is so in F (Jacobson-McCrimmon [12, p. 14]), conditions (i), (ii) are equivalent. In the remainder of the proof, existence and uniqueness of u follow from (4.3.3) and, writing S' for the quadratic trace of $\mathrm{Her}_3(C, \Gamma)$, we apply (3.4.6) to obtain $-\gamma_j\gamma_l n_C(u) = S'(u[jl]) = S'(\varphi(u_{jl})) = S(u_{jl})$. Hence (ii) and (iii) are equivalent as well. $\square$

4.5. **Explicit co-ordinates.** Let J be a cubic Jordan algebra over F and $\mathfrak{S}$ a co-ordinate system of J as in 4.3, particularly (4.3.1), (4.3.2). From Lemma 4.4 we deduce $S(u_{23}), S(u_{31}) \in F^\times$ and put

$$(1) \qquad\qquad \omega := \omega(J, \mathfrak{S}) := S(u_{23})^{-1} S(u_{31})^{-1} \in F^\times.$$

Then the F-vector space J_{12} becomes an F-algebra under the multiplication

$$(2) \qquad vw := \omega(v \times u_{23}) \times (u_{31} \times w) \qquad\qquad (v, w \in J_{12}),$$

which we denote by $C := C(J, \mathfrak{S})$. We also put

$$(3) \qquad \Gamma := \Gamma(J, \mathfrak{S}) := \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), \quad \gamma_1 := -S(u_{31}), \quad \gamma_2 := -S(u_{23}), \quad \gamma_3 := 1.$$

One can now check that the arguments developed by Racine [23, pp. 98-100] carry over mutatis mutandis to our more general setting and yield

4.6. **Theorem.** (The Jacobson co-ordinatization theorem) *With the assumptions and notation of 4.5, the following statements hold.*

(a) $C = C(J, \mathfrak{S})$ *is a conic alternative F-algebra with unit element, norm, trace given by*

$$1_C = u_{23} \times u_{31}, \quad n_C(v) = -\omega S(v), \quad t_C(v) = \omega T(1_C, v) \qquad (v \in C).$$

(b) *Writing $v \mapsto \bar{v} = \omega T(1_C, v) 1_C - v$ for the conjugation of C, the map*

$$\Phi = \Phi_{J, \mathfrak{S}} \colon \ \mathrm{Her}_3(C, \Gamma) \xrightarrow{\sim} J$$

defined by

$$\Phi\big(\sum (\alpha_i e_{ii} + v_i[jl])\big) := \sum (\alpha_i e_i + v_{jl})$$

for $\alpha_i \in F$, $v_i \in C$, $i = 1, 2, 3$, where

$$v_{23} := -S(u_{31})^{-1} u_{31} \times \bar{v}_1, \quad v_{31} := -S(u_{23})^{-1} u_{23} \times \bar{v}_2, \quad v_{12} := v_3,$$

is an isomorphism of Jordan algebras satisfying $\Phi(e_{ii}) = e_i$ for $i = 1, 2, 3$ and $\Phi(1_C[jl]) = u_{jl}$ for $i = 1, 2$. $\qquad\qquad\square$

4.7. **Absolute simplicity versus simplicity.** A Jordan algebra over F is said to be *absolutely simple* if it remains simple under all base field extensions. Absolute simplicity always implies central simplicity but the following proposition (which, though probably known to experts, seems to lack a convenient reference) shows that the converse fails in characteristic 2.

4.8. **Proposition.** *Let $\Gamma \in \mathrm{GL}_3(F)$ be a diagonal matrix.*
(a) *If C is a composition algebra over F, then $\mathrm{Her}_3(C, \Gamma)$ is absolutely simple.*
(b) *If F has characteristic 2 and $K \neq F$ is a purely inseparable extension field of F having exponent 1, then $\mathrm{Her}_3(K, \Gamma)$ is simple but not absolutely simple.*

Proof. For any conic algebra B over F, the cubic Jordan matrix algebra $\mathrm{Her}_3(B, \Gamma)$ is isomorphic to an isotope of $\mathrm{Her}_3(B)$ [20, 6.6 (c)]; moreover, it is easily verified that isotopes of an arbitrary Jordan algebra have exactly the same ideals as the original algebra itself. Hence we may assume $\Gamma = 1_3$. Since composition algebras are stable under base field extensions, (a) follows from [10, Thm. 5.2.7]. Turning to (b), K is a conic F-algebra with trivial conjugation and $\mathrm{Her}_3(K)$ is central simple by ([10, Thm. 5.2.7], McCrimmon [17, Prop. 6]) but not absolutely simple by [17, p. 302, Example]. $\qquad\square$

4.9. **Reduced Albert algebras.** By a *reduced Albert algebra* over F we mean a Jordan algebra J that is isomorphic to a cubic Jordan matrix algebra $\mathrm{Her}_3(C, \Gamma)$ as in 3.4, where C is an octonion algebra over F and $\Gamma \in \mathrm{GL}_3(F)$ is a diagonal matrix. By the Albert-Jacobson-Faulkner theorem [7, Thm. 1.8], C is uniquely determined by J up to isomorphism, allowing us to call it the *coefficient* (or *co-ordinate*) *algebra* of J. Combining this with Theorem 4.6, we conclude that, given a co-ordinate system of J as in 4.3, particularly (4.3.1), (4.3.2), there exists a diagonal matrix $\Gamma' \in \mathrm{GL}_3(F)$ and an isomorphism $\Phi \colon J \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma')$ satisfying $\Phi(e_i) = e_{ii}$ (resp. $\Phi(u_{jl}) = 1_C[jl]$) for $i = 1, 2, 3$ (resp. $i = 1, 2$). Following Prop. 4.8 (a), reduced Albert algebras are absolutely simple.

4.10. **Split Albert algebras.** An Albert algebra over F is said to be *split* if it is reduced and its coefficient algebra is split in the sense that it has zero divisors, equivalently, that it is isomorphic to $\mathrm{Zor}(F)$, the algebra of Zorn vector matrices over F [26, pp. 19-20]. Up to isomorphism, there is a unique split Albert algebra over F, namely, $J := \mathrm{Her}_3(\mathrm{Zor}(F))$; in fact, for any diagonal matrix $\Gamma \in \mathrm{GL}_3(F)$, we have an isomorphism $J \cong \mathrm{Her}_3(\mathrm{Zor}(F), \Gamma)$ matching the diagonal idempotents of both algebras in their natural order. It follows from this and 4.9 that every elementary frame (e_1, e_2, e_3) of $J = \mathrm{Her}_3(\mathrm{Zor}(F))$, being extendible to a co-ordinate system by simplicity (cf. 4.3), is conjugate to the diagonal elementary frame under an automorphism of J. We will refer to the choice of such an automorphism as a *re-co-ordinatization* of J by means of e_1, e_2, e_3.

The preceding considerations hold equally well when the split octonions are replaced everywhere by $\mathrm{Mat}_2(F)$, the split quaternion algebra of 2×2-matrices over F.

5. The reduced non-split case.

In this section, we will finally be able to prove the embedding theorem for reduced Albert algebras that are not split. Before phrasing a more detailed version of the theorem in the special set-up at hand, we derive three auxiliary results the first one of which turns out to be useful in the split case as well.

5.1. **Lemma.** *Let C be a conic alternative F-algebra and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(F)$. Given $u \in C$ making $\gamma_2\gamma_3 n_C(u) + 1 \in F$ invertible and setting*

$$(1) \qquad\qquad \nu := \gamma_2\gamma_3 n_C(u), \quad \mu := (\nu + 1)^{-1},$$

the quantities

$$(2) \qquad e_1 := e_{11}, \quad e_2 := \mu(e_{22} + \nu e_{33} + u[23]), \quad e_3 := \mu(\nu e_{22} + e_{33} - u[23])$$

form an elementary frame of J, and if $x = \sum(\beta_i e_i + x_{jl})$ is the corresponding Peirce decomposition of an arbitrary element

$$(3) \quad x = \sum \big(\alpha_i e_{ii} + v_i[jl]\big) \in J := \mathrm{Her}_3(C, \Gamma) \qquad (\alpha_i \in F, \ v_i \in C, \ i = 1, 2, 3),$$

then

$$(4) \qquad\qquad \beta_1 = \alpha_1,$$

$$(5) \qquad\qquad \beta_2 = \mu\big(\alpha_2 + \nu\alpha_3 + \gamma_2\gamma_3 n_C(u, v_1)\big),$$

$$(6) \qquad\qquad \beta_3 = \mu\big(\nu\alpha_2 + \alpha_3 - \gamma_2\gamma_3 n_C(u, v_1)\big),$$

$$(7) \qquad\qquad x_{23} = \mu^2\Big\{\big(2\nu(\alpha_2 - \alpha_3) + (\nu - 1)\gamma_2\gamma_3 n_C(u, v_1)\big)(e_{22} - e_{33})$$
$$+ \Big(\big((\nu - 1)(\alpha_2 - \alpha_3) - 2\gamma_2\gamma_3 n_C(u, v_1)\big)u + v_1\Big)[23]\Big\},$$

$$(8) \qquad\qquad x_{31} = \mu\big((v_2 - \gamma_2\overline{v_3 u})[31] + (\nu v_3 - \gamma_3\overline{u v_2})[12]\big),$$

$$(9) \qquad\qquad x_{12} = \mu\big((\nu v_2 + \gamma_2\overline{v_3 u})[31] + (v_3 + \gamma_3\overline{u v_2})[12]\big).$$

Proof. Since $\sum e_i = \sum e_{ii} = 1$ and (4.3.3) combined with [14, (5.14.2)] yields $e_2, e_3 \in J_0(e_1)$, $e_2 + e_3 = 1_{J_0(e_1)}$, the quantities in (2) will form an elementary frame of J once we

have shown that e_j is an elementary idempotent, equivalently, that $e_j^\sharp = 0$ and $T(e_j) = 1$, for $j = 2, 3$. Here the first (resp. second) relation follows from (2) combined with (3.4.3) (resp. (3.4.5)). Turning to the remaining assertion of the lemma, we first note that the Peirce spaces of any elementary frame in J are mutually orthogonal relative to the bilinear trace. Hence $\beta_i = T(e_i, x)$ for $i = 1, 2, 3$, and combining (2) with (3.4.4) we obtain (4)$-$(6). Computing the off-diagonal Peirce components of x is more complicated. First of all, combining (3.2.2) with [10, Prop. 5.1.4, (5.1.6)], 4.1 and Lemma 4.2, we conclude

$$(10) \qquad x_{jl} = U_{e_j, e_l} x = \{e_j x e_l\} = \beta_j e_l + \beta_l e_j - e_i \times x.$$

In order to complete the proof, it will therefore be necessary to determine the quantities $e_i \times x$, which will be accomplished by a routine computation involving (2), (3) and (3.4.7). We obtain

$$(11) \qquad e_1 \times x = \alpha_3 e_{22} + \alpha_2 e_{33} - v_1[23],$$

$$(12) \qquad e_2 \times x = \mu\Big(\big(\nu\alpha_2 + \alpha_3 - \gamma_2\gamma_3 n_C(u, v_1) \big) e_{11} + \nu\alpha_1 e_{22} + \alpha_1 e_{33}$$
$$- \alpha_1 u[23] + (-v_2 + \gamma_2\overline{v_3 u})[31] + (-\nu v_3 + \gamma_3\overline{u v_2})[12] \Big),$$

$$(13) \qquad e_3 \times x = \mu\Big(\big(\alpha_2 + \nu\alpha_3 + \gamma_2\gamma_3 n_C(u, v_1) \big) e_{11} + \alpha_1 e_{22} + \nu\alpha_1 e_{33}$$
$$+ \alpha_1 u[23] - (\nu v_2 + \gamma_2\overline{v_3 u})[31] - (v_3 + \gamma_3\overline{u v_2})[12] \Big).$$

With the aid of (2), (4)$-$(6), (10) and (11)$-$(13) we can now compute

$$x_{23} = \beta_2 e_3 + \beta_3 e_2 - e_1 \times x$$
$$= \mu^2\Big(\big(\alpha_2 + \nu\alpha_3 + \gamma_2\gamma_3 n_C(u, v_1) \big) \big(\nu e_{22} + e_{33} - u[23] \big)$$
$$+ \big(\nu\alpha_2 + \alpha_3 - \gamma_2\gamma_3 n_C(u, v_1) \big) \big(e_{22} + \nu e_{33} + u[23] \big) \Big)$$
$$- \alpha_3 e_{22} - \alpha_2 e_{33} + v_1[23]$$
$$= \big((\mu^2\nu + \mu^2\nu)\alpha_2 + (\mu^2\nu^2 + \mu^2 - 1)\alpha_3 + (\mu^2\nu - \mu^2)\gamma_2\gamma_3 n_C(u, v_1) \big) e_{22}$$
$$+ \big((\mu^2 + \mu^2\nu^2 - 1)\alpha_2 + (\mu^2\nu + \mu^2\nu)\alpha_3 + (\mu^2 - \mu^2\nu)\gamma_2\gamma_3 n_C(u, v_1) \big) e_{33}$$
$$+ \Big(\big((-\mu^2 + \mu^2\nu)\alpha_2 + (-\mu^2\nu + \mu^2)\alpha_3 + (-\mu^2 - \mu^2)\gamma_2\gamma_3 n_C(u, v_1) \big) u + v_1 \Big)[23].$$

Here we make use of (1) to deduce $\mu^2\nu^2 + \mu^2 - 1 = \mu^2(\nu^2 + 1 - (1 + \nu)^2) = -2\mu^2\nu$. Hence we have (7). Similarly,

$$x_{31} = \beta_3 e_1 + \beta_1 e_3 - e_2 \times x$$
$$= \mu\Big\{ \big(\nu\alpha_2 + \alpha_3 - \gamma_2\gamma_3 n_C(u, v_1) \big) e_{11} + \alpha_1\big(\nu e_{22} + e_{33} - u[23] \big)$$
$$- \Big(\big(\nu\alpha_2 + \alpha_3 - \gamma_2\gamma_3 n_C(u, v_1) \big) e_{11} + \nu\alpha_1 e_{22} + \alpha_1 e_{33}$$
$$- \alpha_1 u[23] + (-v_2 + \gamma_2\overline{v_3 u})[31] + (-\nu v_3 + \gamma_3\overline{u v_2})[12] \Big) \Big\}$$
$$= \mu\big((v_2 - \gamma_2\overline{v_3 u})[31] + (\nu v_3 - \gamma_3\overline{u v_2})[12] \big),$$

and we have (8), while a completely analogous computation yields (9).   □

5.2. **Remark.** O. Loos has asked whether, in the presence of suitable regularity conditions, the elementary frame of Lemma 5.1 is conjugate to the diagonal frame under the automorphism group of J. Using Racine's extension [23, Cor. of Lemma 3] of the Albert-Jacobson criterion [2, Thm. 9], it can be shown that, for C an octonion algebra over F, the desired conjugacy holds if and only if $\nu + 1 \in n_C(C^\times)$, hence fails for appropriate choices of Γ and u if the norm of C is not universal.

5.3. **Lemma.** *Let C be a non-singular composition division algebra over F and $0 \neq v \in C$. Then there exists an element $u \in C$ such that $n_C(u) = 1$ and $t_C(uv) \neq 0$.*

Proof. If $t_C(v) \neq 0$, then $u := 1_C$ satisfies the requirements of the lemma. Thus we may assume $t_C(v) = 0$. Since the linear forms t_C and $x \mapsto t_C(xv)$ are both non-zero, the union of their kernels cannot be all of C. Hence we find an element $w \in C$ such that $t_C(w) \neq 0 \neq t_C(wv)$. Then $u := n_C(w)^{-1}w^2 \in C$ has norm 1 and by (2.2.4) satisfies $t_C(uv) = n_C(w)^{-1}t_C(w^2v) = n_C(w)^{-1}t_C(w)t_C(wv) - t_C(v) = n_C(w)^{-1}t_C(w)t_C(wv) \neq 0$, as desired. $\square$

5.4. **Lemma.** *Let C be a split octonion algebra over F and suppose $v \in C \setminus F1_C$ has trace zero. Then there exists a non-trivial idempotent $c \in C$ such that $n_C(c, v) = 0$.*

Proof. $V := F1_C + Fv \subseteq C$ is a two-dimensional subspace with basis $1_C, v$, and since $n_C(1_C) = 1$, $n_C(1_C, v) = t_C(v) = 0$, the matrix of $n_C|_V$ relative to the basis $1_C, v$ of V is given by

$$(1) \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha := n_C(v).$$

Now let d be a non-trivial idempotent of C. Since the Peirce components C_{12}, C_{21} of C with respect to $(d, 1_C - d)$ by 2.6 are dual to each other relative to the bilinearized norm, there are elements $w_{12} \in C_{12}$, $w_{21} \in C_{21}$ satisfying $n_C(w_{12}, w_{21}) = 1$. Setting $w := w_{12} + \alpha w_{21} \in C$, we therefore have $n_C(1_C, w) = 0$, $n_C(w) = n_C(w_{12}) + \alpha n_C(w_{12}, w_{21}) + \alpha^2 n_C(w_{21}) = \alpha$, so (1) gives also the matrix of $n_C|_W$, $W := F1_C + Fw$, relative to its basis $1_C, w$. Now the Witt extension theorem [6, Thm. 8.3] yields an isometry φ from the quadratic space (C, n_C) to itself fixing 1_C and sending w to v; in particular, φ is an automorphism of the Jordan algebra C^+, forcing $c := \varphi(d)$ to be a non-trivial idempotent of C^+, hence of C, with $n_C(c, v) = n_C(\varphi(d), \varphi(w)) = n_C(d, w) = 0$. $\square$

5.5. **Theorem.** (cf. Albert-Jacobson [2, Thm. 1]) *Let C be an octonion division algebra over F, $\Gamma \in \mathrm{GL}_3(F)$ a diagonal matrix and $J := \mathrm{Her}_3(C, \Gamma)$ the corresponding reduced non-split Albert algebra over F. Given any element $x \in J$, there exist a unital subalgebra $J' \subseteq J$, a separable quadratic subfield $K \subseteq C$ and a diagonal matrix $\Gamma' \in \mathrm{GL}_3(F)$ such that*

$$x \in J' \cong \mathrm{Her}_3(K, \Gamma').$$

Proof. Writing

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3), \quad x = \sum(\alpha_i e_{ii} + v_i[jl]) \quad (\gamma_i \in F^\times, \ \alpha_i \in F, \ v_i \in C, \ i = 1, 2, 3),$$

and defining a *diagonal isomorphism* as an isomorphism $J \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma_1)$ (for some diagonal matrix $\Gamma_1 \in \mathrm{GL}_3(F)$) that matches the respective diagonal idempotents in their natural order, we perform the following steps.

1^0. *If $\phi\colon J \xrightarrow{\sim} J_1 := \mathrm{Her}_3(C, \Gamma_1)$, for some diagonal matrix $\Gamma_1 \in \mathrm{GL}_3(F)$, is any isomorphism such that the theorem holds for J_1 and $x_1 := \phi(x) \in J_1$, then it also holds for J and x.* Indeed, there exist a unital subalgebra $J_1' \subseteq J_1$, a separable quadratic subfield $K_1 \subseteq C$ and a diagonal matrix $\Gamma_1' \in \mathrm{GL}_3(F)$ satisfying $x_1 \in J_1' \cong \mathrm{Her}_3(K_1, \Gamma_1')$. Then $J' := \phi^{-1}(J_1')$ is a unital subalgebra of J such that $x \in J' \cong \mathrm{Her}_3(K_1, \Gamma_1')$.

2^0. *The theorem holds if $v_i \in F1_C$ for $i = 1, 2, 3$* since, in this case, we have $x \in J' := \mathrm{Her}_3(K, \Gamma) \subseteq J$ for *any* separable quadratic subfield $K \subseteq C$, e.g., for $K = F[v]$, where $v \in C \setminus F1_C$ satisfies $t_C(v) \neq 0$.

3^0. *The theorem holds if $v_q = v_r = 0$ for some cyclic permutation (pqr) of (123).* Indeed, if $v_q = v_r = 0$, then 2^0 allows us to assume $v_p \notin F1_C$, in particular $v_p \neq 0$. Applying Lemma 5.3, we find an element $u \in C$ of norm 1 satisfying $t_C(uv_p) \neq 0$. Now consider the trialitarian automorphism $\varphi := \varphi_{p,u}$ of Lemma 3.5. From (3.5.1) we conclude $\varphi(x) = \sum \alpha_i e_{ii} + (uv_p)[qr]$, and by 1^0, 2^0 we are reduced to the case $t_C(v_p) \neq 0$. Hence v_p belongs to the separable quadratic subfield $K := F[v_p] \subseteq C$ satisfying $x \in J' := \mathrm{Her}_3(K, \Gamma) \subseteq J$, and we are done.

4^0. *For any cyclic permutation (pqr) of (123) satisfying $v_q \neq 0 \neq v_r$, we may assume $v_q = v_r = 1_C$ up to a diagonal isomorphism. If this is so, the theorem holds provided one of the following conditions is fulfilled.*

      (i) $v_p \in F1_C$.

      (ii) $v_p \notin F1_C$ *and* $F1_C + Fv_p \nsubseteq \mathrm{Ker}(t_C)$.

Indeed, since C is a division algebra, we may combine Lemma 4.4 with 4.9 to find a diagonal isomorphism $\phi\colon J \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma_1)$ (for some diagonal matrix $\Gamma_1 \in \mathrm{GL}_3(F)$) such that $x_1 := \phi(x) = \sum(\alpha_i e_{ii} + v'_i[jl])$, where $v'_p \in C$ and $v'_q = v'_r = 1_C$. This proves the first assertion. By 1^0, 2^0, the second one is clear if (i) holds, so we may assume (ii). But then v_p generates a separable quadratic subfield $K \subseteq C$ such that $x \in \mathrm{Her}_3(C, \Gamma)$, which completes the proof. The preceding argument, essentially due to Albert-Jacobson [2, pp. 404-5], is basically all that's needed in order to establish the theorem over fields of characteristic not 2.

5^0. Writing $\bar{C}$ (resp. $\bar{J}$) for the scalar extension of C (resp. J) from F to $\bar{F}$, the algebraic closure of F, we have $\bar{J} \cong \mathrm{Her}_3(\bar{C}, \Gamma) \cong \mathrm{Her}_3(\bar{C})$ under diagonal isomorphisms and claim that *it suffices to exhibit an elementary frame (e_1, e_2, e_3) of $\bar{J}$ such that*

$$(1) \qquad\qquad\qquad T(x_{23}, x_{31} \times x_{12}) \neq 0,$$

where T is the trace of J and the x_{jl} are the off-diagonal Peirce components of x. To see this, suppose some elementary frame of $\bar{J}$ satisfies (1). Then we put $\mathbf{G} := \mathbf{Aut}(J)$, the automorphism group of J viewed as a group scheme, which may also be regarded as a simple algebraic group of type F_4 defined over F, and let $\mathbf{X}$ be the F-scheme of elementary frames in J, so $\mathbf{X}(R)$ is the totality of elementary frames in J_R, for all unital commutative associative F-algebras R. We know from Alberca-Loos-Martín [1, 4.5, Prop. 4.6] that $\mathbf{X}$ is an integral smooth affine scheme of dimension 24 over F on which $\mathbf{G}$ acts in a natural way. Moreover, $\mathbf{G}(\bar{F})$ acts transitively on $\mathbf{X}(\bar{F})$ (loc. cit.), and since J is reduced, we have $\mathbf{X}(F) \neq \emptyset$. Moreover, F is infinite (C being an octonion division algebra over F), forcing $\mathbf{G}(F)$ to be Zariski dense in $\mathbf{G}(\bar{F})$ [4, 18.3]. On the other hand, fixing $\mathfrak{E} \in \mathbf{X}(F)$, the assignment $g \mapsto g \cdot \mathfrak{E}$ gives a surjection $\varphi\colon \mathbf{G}(\bar{F}) \to \mathbf{X}(\bar{F})$ which is continuous in the Zariski topology. Hence $\mathbf{X}(F) \supseteq \varphi(\mathbf{G}(F))$ is Zariski dense in $\mathbf{X}(\bar{F})$, and since the left-hand side of (1) depends continuously on the elementary frame chosen, our claim yields an elementary frame (e_1, e_2, e_3) belonging not only to $\bar{J}$ but, in fact, to J such that (1) holds. Extending (e_1, e_2, e_3) to a co-ordinate system of J by simplicity (cf. 4.3) and applying 4.9, we find a diagonal matrix $\Delta = \mathrm{diag}(\delta_1, \delta_2, \delta_3) \in \mathrm{GL}_3(F)$ and an isomorphism $\phi\colon J \xrightarrow{\sim} \tilde{J} := \mathrm{Her}_3(C, \Delta)$ matching $e_i \in J$ with the diagonal idempotent $e_{ii} \in \tilde{J}$ for $i = 1, 2, 3$ and x with $\tilde{x} := \phi(x) = \sum(\beta_i e_{ii} + w_i[jl]) \in \tilde{J}$ for some $\beta_i \in F$, $w_i \in C$. From (4.3.3) we deduce $\phi(x_{jl}) = w_i[jl]$ for $i = 1, 2, 3$ and by 1^0, 3^0, 4^0 we may assume $w_q = w_r = 1_C$, $w_p \notin F1_C$, for some cyclic permutation (pqr) of (123). Writing $\tilde{T}$ for the trace of $\tilde{J}$ and combining (1) with (2.3.4), (2.3.5), (3.4.4), (3.4.8), we now obtain

$$\delta_1\delta_2\delta_3 t_C(w_p) = \delta_1\delta_2\delta_3 t_C(w_p w_q w_r) = \delta_1\delta_2\delta_3 t_C(w_1 w_2 w_3) = \delta_2\delta_3 n_C(w_1, \delta_1\overline{w_2 w_3})$$

$$= \tilde{T}(w_1[23], w_2[31] \times w_3[12]) = T(x_{23}, x_{31} \times x_{12}) \neq 0.$$

Thus $t_C(w_p) \neq 0$, and the theorem follows from 4^0.

6^0. *Given any cyclic permutation (pqr) of (123), we may assume up to a diagonal isomorphism that*

$$(2) \qquad\qquad v_q = v_r = 1_C, \quad v_p \notin F1_C, \quad t_C(v_p) = 0.$$

Indeed, by 3^0 we may assume $v_q \neq 0 \neq v_r$ *for some* cyclic permutation (pqr) of (123), which by 4^0 may then be assumed to satisfy (2) up to a diagonal isomorphism. This implies $v_i \neq 0$ for all $i = 1, 2, 3$, so by 4^0 again we may assume *for all* cyclic permutations (pqr) of (123) that (2) holds up to a diagonal isomorphism.

7^0. *We may assume $\alpha_2 \neq \alpha_3$.* Otherwise, by 1^0 combined with a two-fold application of Lemma 3.6, we may in fact assume that $\alpha_1 = \alpha_2 = \alpha_3 =: \alpha$. Since the elements x

and $x - \alpha 1$ belong to the same unital subalgebras of J, we are actually reduced to the case $\alpha = 0$. By 6^0, we may assume (2) for $p = 1$, which, in particular, implies $v_1 \neq 0$. Let $u \in C$ satisfy $n_C(u, v_1) \neq 0$. Replacing u by an appropriate non-zero scalar multiple if necessary (note that, C being an octonion division algebra, the base field is infinite), we can ensure in addition that $\gamma_2 \gamma_3 n_C(u) + 1 \neq 0$. Thus we are in the situation of Lemma 5.1. In particular, by (5.1.4), (5.1.5), we have $\beta_1 = 0 \neq \mu \gamma_2 \gamma_3 n_C(u, v_1) = \beta_2$. Following 4.3, 4.9 and Lemma 3.6, we find an isomorphism $\psi \colon J \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma')$ for some diagonal matrix $\Gamma' \in \mathrm{GL}_3(F)$ sending e_i to e_{jj} $(i = 1, 2, 3)$. Then

$$\psi(x) = \sum (\beta_i' e_{ii} + v_i'[jl]) \qquad\qquad (\beta_i' \in F,\ v_i' \in C,\ i = 1, 2, 3),$$

where $\beta_2' = \beta_1 \neq \beta_2 = \beta_3'$, which shows that we are indeed allowed to assume $\alpha_2 \neq \alpha_3$.

8^0. The preceding reductions $1^0 - 4^0$, 6^0, 7^0 allow us to assume for the rest of the proof that

$$(3) \qquad\qquad \alpha_2 \neq \alpha_3, \quad v_2 = v_3 = 1_C, \quad v_1 \notin F1_C, \quad t_C(v_1) = 0.$$

By 5^0, the proof of the theorem will be complete once we have exhibited an elementary frame (e_1, e_2, e_3) of $\bar{J}$ such that (1) holds. Applying Lemma 5.4, we find a non-trivial idempotent $u \in \bar{C}$ such that $n_C(u, v_1) = 0$. This implies $n_C(u) = 0$, $t_C(u) = 1$ by 2.4, so we may apply Lemma 5.1 with $\nu = 0$ and $\mu = 1$. In particular, the off-diagonal Peirce components of x relative to the elementary frame of (5.1.2) by $(5.1.7) - (5.1.9)$ have the form

$$x_{23} = \big(-(\alpha_2 - \alpha_3)u + v_1 \big)[23],$$
$$x_{31} = (1_C - \gamma_2 \bar{u})[31] - (\gamma_3 \bar{u})[12],$$
$$x_{12} = (\gamma_2 \bar{u})[31] + (1_C + \gamma_3 \bar{u})[12].$$

Now (3.4.8) yields

$$\begin{aligned}
x_{31} \times x_{12} &= \big((1_C - \gamma_2 \bar{u})[31] - (\gamma_3 \bar{u})[12]\big) \times \big((\gamma_2 \bar{u})[31] + (1_C + \gamma_3 \bar{u})[12]\big) \\
&= -\gamma_3 \gamma_1 n_C(1_C - \gamma_2 \bar{u}, \gamma_2 \bar{u})e_{22} - \gamma_1 \gamma_2 n_C(-\gamma_3 \bar{u}, 1_C + \gamma_3 \bar{u})e_{33} \\
&\quad + \gamma_1 \big(\overline{(1_C - \gamma_2 \bar{u})(1_C + \gamma_3 \bar{u})} - \overline{(\gamma_2 \bar{u})(\gamma_3 \bar{u})}\big)[23] \\
&= -\gamma_1 \gamma_2 \gamma_3(e_{22} - e_{33}) + \gamma_1\big(1_C - (\gamma_2 - \gamma_3 + 2\gamma_2 \gamma_3)u\big)[23],
\end{aligned}$$

and from (3.4.4), (3) we deduce

$$\begin{aligned}
T(x_{23}, x_{31} \times x_{12}) &= \gamma_1 \gamma_2 \gamma_3 n_C\big(-(\alpha_2 - \alpha_3)u + v_1, 1_C - (\gamma_2 - \gamma_3 + 2\gamma_2 \gamma_3)u\big) \\
&= -\gamma_1 \gamma_2 \gamma_3(\alpha_2 - \alpha_3) \neq 0.
\end{aligned}$$

Thus (1) holds and the theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. The split case.

Finally, we turn to the proof of the embedding theorem for split Albert algebras over F. While adhering rather closely to the overall strategy employed by Jacobson in [9, Chap. IX, Sec. 6], we will have to make a substantial number of non-trivial technical adjustments in order to include base fields of characteristic 2.

6.1. **Algebras with involution.** Let (A, τ) be a unital non-associative F-algebra with involution. Harmonizing the terminology of [13, § 2] with the one of present paper, we define

$$(1) \qquad \mathrm{Her}(A, \tau) := \{x + \tau(x) \mid x \in A\} \subseteq \mathrm{Sym}(A, \tau) := \{x \in A \mid \tau(x) = x\}$$

as F-subspaces of A. In (1) we have equality for $\mathrm{char}(F) \neq 2$ but not in general, see [13, Lemma 2.3] for a more precise statement if A is finite-dimensional central simple associative. We also note $x \,\mathrm{Her}(A, \tau)\tau(x) \subseteq \mathrm{Her}(A, \tau)$ for all $x \in A$ if A is an associative algebra.

6.2. **Examples.** (a) Let C be a conic alternative F-algebra and $\Gamma \in \mathrm{GL}_3(F)$ a diagonal matrix. Then

$$\tau_\Gamma\colon \ \mathrm{Mat}_3(C) \longrightarrow \mathrm{Mat}_3(C), \quad x \longmapsto \Gamma^{-1}\bar{x}^t\Gamma$$

is an involution, and one checks easily that

$$\mathrm{Her}\big(\mathrm{Mat}_3(C), \tau_\Gamma\big) = \mathrm{Her}_3(C, \Gamma)$$

in the sense of 3.4 provided the trace of C is not zero.

(b) If $\sigma\colon V \times V \to F$ is a non-degenerate alternating bilinear form on a finite-dimensional vector space V over F and

$$\tau\colon \ \mathrm{End}_F(V) \longrightarrow \mathrm{End}_F(V)$$

is the adjoint involution relative to σ, then, again, one checks easily that

$$\mathrm{Her}\big(\mathrm{End}_F(V), \tau\big) = \big\{f \in \mathrm{End}_F(V) \mid \forall x \in V : \sigma\big(f(x), x\big) = 0\big\}.$$

6.3. **Theorem.** (cf. [9, Thm. IX.11]) *Let J be a split Albert algebra over F and $x \in J$. Then there exists a unital subalgebra $J' \subseteq J$ satisfying $x \in J' \cong \mathrm{Mat}_3(F)^+$.*

Proof. By definition (see 4.10), there is an identification $J = \mathrm{Her}_3(C)$, where C stands for a split octonion algebra over F. Slightly modifying Jacobson's approach [9, Lemmata IX.6.1−3] to his proof of Thm. 6.3 in characteristic not 2, we claim that *it suffices to establish the following two technical results.*

6.4. **Lemma.** (cf. [9, Lemma IX.6.3]) *Let V be a vector space over F of even dimension $2m$, $\sigma\colon V \times V \to F$ a non-degenerate alternating bilinear form on V and τ the adjoint involution of $\mathrm{End}_F(V)$ relative to σ. Then any element of the Jordan algebra $\mathrm{Her}(\mathrm{End}_F(V), \tau)$ is contained in a unital subalgebra isomorphic to $\mathrm{Mat}_m(F)^+$.*

6.5. **Lemma.** *There exists a unital subalgebra $\tilde{J} \subseteq J$ that contains x and is isomorphic to $\mathrm{Her}_3(B)$, where $B \subseteq C$ is a split quaternion subalgebra.*

Suppose these results have been proved. Choosing $\tilde{J}, B$ as in Lemma 6.5 and identifying $B = \mathrm{Mat}_2(F)$, the conjugation of B is given by $x \mapsto s^{-1}x^t s$, $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the natural matching $\mathrm{Mat}_3(B) = \mathrm{Mat}_3(\mathrm{Mat}_2(F)) = \mathrm{Mat}_6(F)$ extends to an identification

$$\tag{1} \big(\mathrm{Mat}_3(B), \tau_{\mathbf{1}_3}\big) = \big(\mathrm{End}_F(V), \tau\big)$$

of algebras with involution, where $\tau_{\mathbf{1}_3}$ is the conjugate transpose involution on $\mathrm{Mat}_3(B)$ while the right-hand side of (1) is defined as follows: $V = F^6$, $\sigma\colon V \times V \to F$ is the non-degenerate alternating bilinear form given by $\sigma(x, y) = x^t S y$; $x, y \in F^6$, $S = \mathrm{diag}(s, s, s) \in \mathrm{Mat}_6(F)$ and $\tau\colon \mathrm{End}_F(V) \to \mathrm{End}_F(V)$ is the adjoint involution relative to σ. Thus, by 6.2 (a),

$$\mathrm{Her}_3(B) = \mathrm{Her}\big(\mathrm{Mat}_3(B), \tau_{\mathbf{1}_3}\big) = \mathrm{Her}\big(\mathrm{End}_F(V), \tau\big),$$

allowing us to apply Lemma 6.4, and the proof of Theorem 6.3 is complete.

It therefore remains to prove Lemmata 6.4, 6.5. First of all, the proof of [9, Lemma IX.6.3] can be easily converted into one of Lemma 6.4 once the obvious modifications along the lines of 6.1, 6.2 are taken into account. We omit the details and are thus left with the proof of Lemma 6.5, which will be accomplished by using the proof of [9, Lemma IX.6.1, pp. 389-391], due to McCrimmon, as a guide so as to allow base fields of arbitrary characteristic. The difficulties that have to be overcome along the way are not so much related to characteristic 2 but, more specifically, to the field with two elements. As a matter of fact, while we will not quite be able to establish the characteristic-free version of [9, Lemma IX.6.1] in full generality (see Claim (∗) and Case 2.2.2 in 6.11 below), our modifications of McCrimmon's original arguments will at least be strong enough to yield a proof of Lemma 6.5.

In order to succeed, we will need a few additional preparations the first one of which is due to Brühne [5, Prop. 3.2.2]. Its proof given below simplifies Brühne's original argument and will be included here for completeness.

6.6. Proposition. *Let $X = (X, 1, N, \sharp)$ be a cubic norm structure over F and $J = J(X)$ the corresponding cubic Jordan algebra. Then the unital subalgebra of J generated by elements $x, y \in J$ is spanned as a vector space over F by*

$$(1) \qquad 1, x, y, x^\sharp, y^\sharp, x \times y, x \times y^\sharp, x^\sharp \times y, x^\sharp \times y^\sharp.$$

Proof. We will need not only the identities of 3.1, but also a number of consequences most of which may be found in [16]:

$$(2) \qquad u^\sharp \times (u \times v) = N(u)v + T(u^\sharp, v)u,$$

$$(3) \qquad u^\sharp \times v^\sharp + (u \times v)^\sharp = T(u^\sharp, v)v + T(u, v^\sharp)u,$$

$$(4) \qquad u^\sharp \times u = \big(T(u)S(u) - N(u)\big)1 - S(u)u - T(u)u^\sharp,$$

$$(5) \qquad u \times (u^\sharp \times v) = N(u)v + T(u, v)u^\sharp.$$

Moreover, from [23, (21)] we recall

$$(6) \qquad u \times (u \times v) = \big(T(u)S(u, v) + S(u)T(v) - T(u^\sharp, v)\big)1$$
$$- S(u, v)u - S(u)v - T(u)u \times v - T(v)u^\sharp - u^\sharp \times v,$$

which linearizes to

$$(7) \qquad u \times (v \times w) + v \times (w \times u) + w \times (u \times v) =$$
$$\big(T(u)S(v, w) + T(v)S(w, u) + T(w)S(u, v) - T(u \times v, w)\big)1$$
$$- \big(S(u, v)w + S(v, w)u + S(w, u)v\big)$$
$$- \big(T(u)v \times w + T(v)w \times u + T(w)u \times v\big).$$

We must show that the subspace $M \subseteq J$ spanned by the elements assembled in (1) is stable under the adjoint map. First of all, we have $1^\sharp = 1 \in M$ by the base point identities (3.1.1) and $1 \times M \subseteq M$ by the unit identity (3.1.3). Moreover, M is spanned by the elements $1, s, t, s \times t, \ s \in \{x, x^\sharp\}, \ t \in \{y, y^\sharp\}$. Hence it will be enough to show that, for all $s, s' \in \{x, x^\sharp\}, \ t, t' \in \{y, y^\sharp\}$,

$$(8) \qquad s^\sharp, t^\sharp, (s \times t)^\sharp \in M,$$

$$(9) \qquad s \times s', s \times (s' \times t) \in M,$$

$$(10) \qquad t \times t', t \times (s \times t') \in M,$$

$$(11) \qquad (s \times t) \times (s' \times t') \in M.$$

The adjoint identity (3.1.2) implies $s^\sharp \in \{N(x)x, x^\sharp\}$, $t^\sharp \in \{N(y)y, y^\sharp\}$, hence not only the first two inclusions of (8) but also $s^\sharp \times t^\sharp \in M$, while the third one now follows from (3). In the first inclusion of (9) we may assume $s = x, \ s' = x^\sharp$, whence the assertion follows from (4). The second relation of (9) follows from (6), (8) for $s = s'$ and from (5), (2) for $s \neq s'$. Next, (10) is (9) with x and y interchanged, so we are left with (11). We first note that $1 \times M \subseteq M$ and (9), (10) imply

$$(12) \qquad s \times M + t \times M \subseteq M.$$

Then we combine (7) with (8)−(10) to conclude

$$(s \times t) \times (s' \times t') \equiv -s' \times \big(t' \times (s \times t)\big) - t' \times \big((s \times t) \times s'\big) \bmod M,$$

and (12) implies (11). $\qquad \square$

Remark. The preceding result, including its proof, carries over verbatim to arbitrary commutative base rings in place of F, see also [5, Prop. 3.2.2].

6.7. Corollary. *Let $X = (X, 1, N, \sharp)$ be a cubic norm structure over F and $J = J(X)$ the corresponding cubic Jordan algebra. Suppose $E \subseteq J$ is a three-dimensional étale subalgebra and $x \in J$ is an arbitrary element. Then the subalgebra $J_0 \subseteq J$ generated by E and x has dimension at most 9.*

Proof. J_0 is a cubic Jordan algebra on two generators, so Prop. 6.6 applies, unless $F = \mathbb{F}_2$ is the field with two elements. In the latter case, we change scalars to the separable quadratic field extension F'/F, $F' = \mathbb{F}_4$, by considering the cubic Jordan algebra $J' = J \otimes_F F'$ over F' which contains $E' = E \otimes_F F'$ as a three-dimensional étale subalgebra. Canonically identifying $J \subseteq J'$ as an F-subalgebra and writing σ for the non-trivial Galois automorphism of F'/F (which acts on J' through the second factor), the F'-subalgebra, J_0', of J' generated by E' and x has dimension at most 9 and is stable under σ since E' and x are. Hence σ acts on J_0' as a σ-linear automorphism σ' of order two. Moreover, the fixed algebra of σ' not only contains J_0 but also has F-dimension at most 9 because it becomes isomorphic to J_0' when changing scalars from F to F'. Therefore $\dim_F(J_0) \leq 9$. $\qquad\square$

6.8. Proposition. *Let r be a positive integer, $q \colon V \to F$ a possibly singular quadratic form of dimension r and $Q \colon W \to F$ a hyperbolic quadratic form of dimension $2r$ over F. Then there exists an (injective) isometry from (V, q) to (W, Q).*

Proof. We argue by induction on r. For $r = 1$, the assertion follows from the fact that Q, being hyperbolic, is universal. For $r > 1$, we pick a hyperplane $V' \subseteq V$ and put $q' := q|_{V'}$. We also let $Q' \colon W' \to F$ be a hyperbolic quadratic subform of Q having dimension $2(r-1)$. Then the induction hypothesis yields an isometry $\varphi' \colon (V', q') \to (W', Q')$. Choosing any $v \in V \setminus V'$, we have $V = V' \oplus Fv$, and since φ' is injective, non-singularity of Q' leads to an element $w' \in W'$ such that $Q'(\varphi'(v'), w') = q(v', v)$ for all $v' \in V'$. On the other hand, there is a hyperbolic plane $Q_0 \colon W_0 \to F$ satisfying $Q = Q' \perp Q_0$, and since Q_0 is universal, we find a non-zero element $w_0 \in W_0$ with $Q_0(w_0) = q(v) - Q'(w')$. Now extend $\varphi' \colon V' \to W' \hookrightarrow W$ to a linear map $\varphi \colon V \to W$ by setting $\varphi(v) := w' + w_0$. Then a straightforward verification shows that φ is an isometry from (V, q) to (W, Q), as desired. $\qquad\square$

6.9. Corollary. *Let r be a positive integer and suppose the non-singular quadratic form $q \colon V \to F$ over F has Witt index $> r$. Suppose further we are given linearly independent vectors $v_1, \ldots, v_r \in V$ and arbitrary scalars $\alpha_1, \ldots, \alpha_r, \beta \in F$. Then there exists a non-zero element $u \in V$ such that*

$$q(u, v_i) = \alpha_i, \quad q(u) = \beta \qquad\qquad (1 \leq i \leq r).$$

Proof. By hypothesis, q contains a hyperbolic quadratic subform $Q \colon W \to F$, $W \subseteq V$ an appropriate subspace of dimension $2r$, and q is isotropic, hence universal, on $W^\perp$. Combining Prop. 6.8, applied to $V' := \sum Fv_i$, $q' := q|_{V'}$, with Witt's theorem [6, Thm. 8.3], we may assume $v_1, \ldots, v_r \in W$. Since Q is non-singular, some $w \in W$ has $q(w, v_i) = \alpha_i$, $1 \leq i \leq r$. Now pick a non-zero element $y \in W^\perp$ satisfying $q(y) = \beta - q(w)$ and put $u := w + y$. The assertion follows. $\qquad\square$

6.10. Specialization. Write x as in Thm. 6.3 in the form

$$(1) \qquad\qquad x = \sum (\alpha_i e_{ii} + v_i[jl]) \qquad\qquad (\alpha_i \in F, \, v_i \in C, \, i = 1, 2, 3)$$

and suppose we are given an element $u \in C$ that is not invertible. Then we can apply Lemma 5.1 with $\nu = 0$, $\mu = 1$, so the quantities

$$(2) \qquad\qquad e_1 := e_{11}, \quad e_2 := e_{22} + u[23], \quad e_3 := e_{33} - u[23]$$

form an elementary frame in J and, writing $x = \sum(\beta_i e_i + x_{jl})$ for the corresponding Peirce decomposition of x, the following relations hold.

$$\text{(3)} \qquad\qquad \beta_1 = \alpha_1,$$

$$\text{(4)} \qquad\qquad \beta_2 = \alpha_2 + n_C(u, v_1),$$

$$\text{(5)} \qquad\qquad \beta_3 = \alpha_3 - n_C(u, v_1),$$

$$\text{(6)} \qquad\qquad x_{23} = -n_C(u, v_1)(e_{22} - e_{33})$$
$$+ \big(v_1 - [\alpha_2 - \alpha_3 + 2n_C(u, v_1)]u\big)[23],$$

$$\text{(7)} \qquad\qquad x_{31} = (v_2 - \overline{v_3 u})[31] - \overline{uv_2}[12],$$

$$\text{(8)} \qquad\qquad x_{12} = \overline{v_3 u}[31] + (v_3 + \overline{uv_2})[12].$$

Moreover, setting

$$\text{(9)} \qquad\qquad g := \mathbf{t}^2 + (\alpha_2 - \alpha_3)\mathbf{t} - n_C(v_1) \in F[\mathbf{t}],$$

we claim

$$\text{(10)} \qquad\qquad S(x_{23}) = g\big(n_C(u, v_1)\big),$$

$$\text{(11)} \qquad\qquad -S(x_{31}) = n_C(v_2) - t_C(uv_2 v_3),$$

$$\text{(12)} \qquad\qquad -S(x_{12}) = n_C(v_3) + t_C(uv_2 v_3).$$

Indeed, combining $(6)-(8)$ with $(3.4.6)$, we conclude

$$\begin{aligned}
S(x_{23}) = {}& -n_C(u, v_1)^2 - n_C(v_1) + \big(\alpha_2 - \alpha_3 + 2n_C(u, v_1)\big)n_C(u, v_1) \\
& - \big(\alpha_2 - \alpha_3 + 2n_C(u, v_1)\big)^2 n_C(u) \\
= {}& -n_C(u, v_1)^2 - n_C(v_1) + (\alpha_2 - \alpha_3)n_C(u, v_1) + 2n_C(u, v_1)^2 \\
= {}& g\big(n_C(u, v_1)\big),
\end{aligned}$$

since $n_C(u) = 0$. Thus (10) holds; moreover, $(2.3.4)$, $(2.3.5)$ yield $-S(x_{31}) = n_C(v_2 - \overline{v_3 u}) + n_C(uv_2) = n_C(v_2) - n_C(v_2, \overline{v_3 u}) + n_C(v_3)n_C(u) + n_C(u)n_C(v_2)$ giving (11), while (12) is derived analogously.

We remark in closing that, by 4.10, the preceding formalism is invariant under arbitrary permutations of the indices (123).

6.11. Proof of Lemma 6.5.
Returning to the proof of Thm. 6.3 at its very beginning, we write $x \in J = \mathrm{Her}_3(C)$ as in (6.10.1) and perform the following steps.
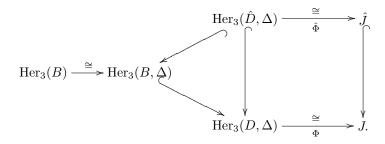
1^0. The lemma will follow as soon as we have been able to deduce the following

> **Claim (∗).** *There exists a unital subalgebra $\hat{J} \subseteq J$ that contains x and is isomorphic to $\mathrm{Her}_3(\hat{C}, \Gamma)$, where $\hat{C} \subseteq C$ is a unital subalgebra of dimension at most 2 and $\Gamma \in \mathrm{GL}_3(F)$ is a diagonal matrix.*

Suppose this claim has been established. Choosing $\hat{J}, \hat{C}, \Gamma$ accordingly, any isomorphism $\varphi\colon \mathrm{Her}_3(\hat{C}, \Gamma) \xrightarrow{\sim} \hat{J}$ maps the diagonal co-ordinate system of $\mathrm{Her}_3(\hat{C}, \Gamma)$ onto a co-ordinate system of $\hat{J}$ (cf. 4.3), which we denote by $\mathfrak{S}$ and which is a co-ordinate system of J as well. We therefore conclude from $(4.5.1)-(4.5.3)$ that $\hat{D} := C(\hat{J}, \mathfrak{S})$ is a unital subalgebra of $D := C(J, \mathfrak{S})$; moreover, $\Delta := \Gamma(\hat{J}, \mathfrak{S}) = \Gamma(J, \mathfrak{S})$. Now Theorem 4.6 produces isomorphisms

$$\hat{\Phi}\colon \mathrm{Her}_3(\hat{D}, \Delta) \xrightarrow{\sim} \hat{J}, \quad \Phi\colon \mathrm{Her}_3(D, \Delta) \xrightarrow{\sim} J$$

making a commutative square on the very right of the diagram

$$
\begin{array}{ccc}
\mathrm{Her}_3(\hat{D}, \Delta) & \xrightarrow[\hat{\Phi}]{\cong} & \hat{J} \\
& & \\
\mathrm{Her}_3(B) \xrightarrow{\cong} \mathrm{Her}_3(B, \Delta) & & \\
& & \\
\mathrm{Her}_3(D, \Delta) & \xrightarrow[\Phi]{\cong} & J.
\end{array}
$$

By 4.9, D is a split octonion algebra over F, and the subalgebra $\hat{D} \subseteq D$ along with $\hat{C} \subseteq C$ has dimension at most 2. From [8, Prop. 5.5] we therefore obtain a split quaternion subalgebra $B \subseteq D$ containing $\hat{D}$, which allows us to complete the above diagram as indicated. It yields an embedding of $\mathrm{Her}_3(B)$ as a unital subalgebra of J containing x, and Lemma 6.5 is proved

Claim $(*)$ agrees with [9, Lemma IX.6.1], hence holds if F has characteristic not two. However, among the various special cases to be discussed in the subsequent portions of the proof below, there is one instance where we have not been able to establish its validity (forcing us to reach the conclusion of Lemma 6.5 by different means). In particular, it is not clear whether Claim $(*)$ always holds in characteristic two.

2^0. *The lemma will follow if there are at least two indices $i = 1, 2, 3$ having $v_i \in C$ invertible.* To see this, we may assume by symmetry that $v_1, v_2 \in C$ are both invertible. Let $\hat{J}$ be the subalgebra of J generated by x and e_{11}, e_{22}, e_{33}. Then all $v_i[jl]$, $i = 1, 2, 3$, being Peirce components of x relative to the diagonal frame of J, belong to $\hat{J}$. Hence our hypothesis combined with Lemma 4.4 ensures that $(e_{11}, e_{22}, e_{33}, v_1[23], v_2[31])$ forms a co-ordinate system of both $\hat{J}$ and J (4.3). From 4.5, Thm. 4.6 and 4.9 we therefore obtain a unital subalgebra $\hat{C} \subseteq C$, a diagonal matrix $\Gamma \in \mathrm{GL}_3(F)$ and an isomorphism $J \xrightarrow{\sim} \mathrm{Her}_3(C, \Gamma)$ matching $\hat{J}$ with $\mathrm{Her}_3(\hat{C}, \Gamma)$. By Cor. 6.7, $\hat{J}$ has dimension at most 9, forcing $\hat{C}$ to have dimension at most 2. We have therefore established Claim $(*)$ of 1^0, and we are through.

3^0. *We may always assume that the subspace of C spanned by 1_C and v_1, v_2, v_3 has dimension at least 3* since, otherwise, it is a subalgebra $\hat{C} \subseteq C$ of dimension at most 2 such that $x \in \mathrm{Her}_3(\hat{C})$, which again implies Claim $(*)$ of 1^0.

4^0. *We may always assume that at least one index $i = 1, 2, 3$ has $v_i \in C$ invertible.* This is more delicate. By symmetry and 3^0, we may assume $v_2 \neq 0$. Then Cor. 6.9 yields an element $u \in C$ satisfying $n_C(u) = 0$, $n_C(u, v_2) = \alpha_2 - \alpha_3$, and we deduce from 6.10 that the quantities $e_2 := e_{22}$, $e_3 := e_{33} + u[31]$, $e_1 := e_{11} - u[31]$ make up an elementary frame of J with $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3 + n_C(u, v_2) = \alpha_2$, $\beta_1 = \alpha_1 - n_C(u, v_2)$ by (3)−(5). Re-co-ordinatizing J by means of e_1, e_2, e_3 (cf. 4.10), we are therefore allowed to assume $\alpha_2 = \alpha_3$. Though the condition $v_2 \neq 0$ may have got lost in the process, step 3^0 above still allows us to maintain it since the situation we have arrived at is symmetric in the indices $2, 3$. We now distinguish the following cases.

Case 1. $v_1 \neq 0$. Since $\alpha_2 = \alpha_3$, the polynomial g of (6.10.9) has the form $g = \mathbf{t}^2 - n_C(v_1)$, so $g(\alpha) \neq 0$ for some $\alpha \in F$ (even if $F = \mathbb{F}_2$ is the field with two elements). By Cor. 6.9, we find a non-zero element $u \in C$ with $n_C(u) = 0$, $n_C(u, v_1) = \alpha$. Hence $S(x_{23}) \in F^\times$ by (6.10.10), so re-co-ordinatizing J by means of the elementary frame (6.10.2) and invoking Lemma 4.4, we may assume $v_1 \in C^\times$.

Case 2. $v_1 = 0$. Then v_2, v_3 are linearly independent by 3^0. We will be through with 4^0 if v_2 or v_3 is invertible in C, so let us assume $n_C(v_2) = n_C(v_3) = 0$.

Case 2.1. $v_2 v_3 \neq 0$. By Cor. 6.9, some $u \in C$ has $n_C(u) = 0$, $t_C(u v_2 v_3) = n_C(u, \overline{v_2 v_3}) = 1$, and (6.10.11), (6.10.12) show $S(x_{31}) \neq 0 \neq S(x_{12})$, so re-co-ordinatizing J by means of elementary frame (6.10.2) , we are allowed to assume that v_2 *and* v_3 are both invertible.

Case 2.2. $v_2 v_3 = 0$. Again Cor. 6.9 yields an element $u \in C$ with $n_C(u) = n_C(u, \bar{v}_2) = n_C(u, \bar{v}_3) = 1$. Setting $c := uv_2$, $d := v_3 u$, we apply (2.3.6), (2.3.4) and obtain $n_C(c) = n_C(d) = 0$, $t_C(c) = t_C(d) = 1$, so $c, d \in C$ are non-trivial idempotents by 2.6 satisfying $cd = u(v_2 v_3)u = 0$ by one of the Moufang identities (2.3.2). Moreover, Lemma 3.5 produces a "trialitarian" automorphism, $\varphi := \varphi_{2,u}$, of J by the assignment

$$(1) \qquad \sum (\beta_i e_{ii} + w_i[jl]) \longmapsto \sum \beta_i e_{ii} + (u^{-1} w_1 u^{-1})[23] + (uw_2)[31] + (w_3 u)[12].$$

In particular, φ sends x to $x' := \varphi(x) = \sum \alpha_i e_{ii} + c[31] + d[12]$. Hence we may assume that $v_2 = c$, $v_3 = d$ *are non-trivial idempotents in C satisfying $cd = 0$.*

Case 2.2.1. $dc = 0$. Then (c, d) is a complete orthogonal system of non-trivial idempotents in C (2.4) such that $x \in \mathrm{Her}_3(\hat{C})$, where $\hat{C} = Fc \oplus Fd \cong F \oplus F$ is a two-dimensional unital subalgebra of C. Claim $(*)$ of 1^0 follows.

Case 2.2.2. $w := dc \neq 0$. This is the only instance where we do not rely on Claim $(*)$ of 1^0 but attack Lemma 6.5 directly. Since $cw = cdc = 0$ and $wc = dc^2 = w$, we conclude $w \in C_{21}$, where C_{ij}, $i, j = 1, 2$, stand for the Peirce components of C relative to (c, c'), $c' := 1_C - c$. Since C_{12} and C_{21} are dual to each other under the bilinearized trace by 2.6, one finds an element $v \in C_{12}$ such that $t_C(vw) = 1$. This and the Peirce multiplication rules [25, Prop. 3.4] imply $vw = c$, $wv = c'$, and it is readily checked that the map

$$\mathrm{Mat}_2(B) \longrightarrow C, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \alpha c + \beta v + \gamma w + \delta c'$$

is a monomorphism of composition algebras. We conclude that $B := Fc + Fv + Fw + Fc' \subseteq C$ is a split quaternion subalgebra that contains c. On the other hand, $cd = 0$ implies $d = v_{21} + \delta c'$ for some $v_{21} \in C_{21}$, $\delta \in F$, hence $dc = v_{21}$, and taking traces yields $\delta = 1$. Thus $d = w + c' \in B$, forcing $x \in \mathrm{Her}_3(B) \subseteq J$, so Lemma 6.5 holds.

5^0. *We may always assume that at least one index $i = 1, 2, 3$ has $v_i = 0$.* By symmetry and $1^0 - 4^0$, we may assume $v_2 \in C^\times$, $v_3 \neq 0$, $n_C(v_1) = n_C(v_3) = 0$. Then $u := -\bar{v}_3 v_2^{-1} \in C$ satisfies $n_C(u) = 0$, $v_3 = -\overline{uv_2}$, and (2.3.3) yields $v_3 u = -(\bar{v}_2 \bar{u})u = -n_C(u)\bar{v}_2 = 0$, hence $t_C(uv_2 v_3) = t_C(v_2 v_3 u) = 0$. Now (6.10.8), (6.10.11) show $x_{12} = 0$, $S(x_{31}) = -n_C(v_2) \neq 0$. Re-co-ordinatizing J by means of the elementary frame (6.10.2) gives what we want.

6^0. By symmetry and $1^0 - 5^0$, we are reduced to the case $v_2 \in C^\times$, $v_3 = 0 \neq v_1$, $n_C(v_1) = 0$. Note that the polynomial g of (6.10.9) has degree 2. Hence one of the following holds. (i) Some $\alpha \in F$ has $g(\alpha) \neq 0$ (e.g., if $F \neq \mathbb{F}_2$). (ii) $n_C(v_2) = 1$ (e.g., if $F = \mathbb{F}_2$). In case (i), we choose $u \in C$ with $n_C(u) = 0$, $n_C(u, v_1) = \alpha$, forcing $S(x_{23}) = g(\alpha) \neq 0$, $S(x_{31}) = -n_C(v_2) \neq 0$ by (6.10.10), (6.10.11), so re-co-ordinatizing J by means of the elementary frame (6.10.2) brings us back to 2^0. On the other hand, in case (ii), we put $u := v_2^{-1}$ and consider the automorphism φ of J defined by (1), which sends x to $\varphi(x) = \sum \alpha_i e_{ii} + (u^{-1} v_1 u^{-1})[23] + 1_C[31]$. Replacing x by $\varphi(x)$, we may therefore assume $v_2 = 1_C$, which implies $x \in \mathrm{Her}_3(\hat{C})$, $\hat{C} = k[v_1]$, brings us back to Claim $(*)$ of 1^0 and completes the proof of Lemma 6.5. $\qquad\square$

REFERENCES

1. P. Alberca, O. Loos, and C. Martín, *Derivations and automorphisms of Jordan algebras in characteristic two*, J. Algebra **285** (2005), no. 1, 146–181. MR 2119109 (2005k:17046)

2. A. A. Albert and N. Jacobson, *On reduced exceptional simple Jordan algebras*, Ann. of Math. (2) **66** (1957), 400–417. MR 0088487 (19,527b)

3. J. Anquela, T. Cortés, and H.P. Petersson, *Commuting U-operators in Jordan algebras*, Trans. Amer. Math. Soc. (to appear).

4. A. Borel, *Linear algebraic groups*, Notes taken by Hyman Bass, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR 0251042 (40 #4273)

5. P. Bruehne, *Ordnungen und die Tits-Konstruktionen von Albert-Algebren*, Ph.D. thesis, Fernuniversität in Hagen, 2000.

6. R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, Coll. Publ., vol. **56**, Amer. Math. Soc., Providence, RI, 2008. MR 2427530

7. J.R. Faulkner, *Octonion planes defined by quadratic Jordan algebras*, Memoirs of the American Mathematical Society, No. **104**, American Mathematical Society, Providence, R.I., 1970. MR 0271180 (42 #6063)

8. S. Garibaldi and H.P. Petersson, *Wild Pfister forms over Henselian fields, K-theory, and conic division algebras*, J. Algebra **327** (2011), 386–465. MR 2746044 (2012e:11065)

9. N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol. **XXXIX**, American Mathematical Society, Providence, R.I., 1968. MR 0251099 (40 #4330)

10. _____, *Structure theory of Jordan algebras*, University of Arkansas Lecture Notes in Mathematics, vol. **5**, University of Arkansas, Fayetteville, Ark., 1981. MR 634508 (83b:17015)

11. N. Jacobson and J. Katz, *Generically algebraic quadratic Jordan algebras*, Scripta Math. **29** (1973), no. 3-4, 215–227, Collection of articles dedicated to the memory of Abraham Adrian Albert. MR MR0404370 (53 #8172)

12. N. Jacobson and K. McCrimmon, *Quadratic Jordan algebras of quadratic forms with base points*, J. Indian Math. Soc. (N.S.) **35** (1971), 1–45 (1972). MR 0357531 (50 #9999)

13. M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Amer. Math. Soc. Coll. Publ., vol. 44, Amer. Math. Soc., Providence, RI, 1998. MR 2000a:16031

14. O. Loos, *Jordan pairs*, Springer-Verlag, Berlin, 1975, Lecture Notes in Mathematics, Vol. **460**. MR R0444721 (56 #3071)

15. _____, *Algebras with scalar involution revisited*, J. Pure Appl. Algebra **215** (2011), no. 12, 2805–2828. MR 2811564

16. K. McCrimmon, *The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras*, Trans. Amer. Math. Soc. **139** (1969), 495–510. MR 0238916 (39 #276)

17. _____, *The Freudenthal-Springer-Tits constructions revisited*, Trans. Amer. Math. Soc. **148** (1970), 293–314. MR 0271181 (42 #6064)

18. _____, *Homotopes of alternative algebras*, Math. Ann. **191** (1971), 253–262. MR 0313344 (47 #1899)

19. _____, *Nonassociative algebras with scalar involution*, Pacific J. Math. **116** (1985), no. 1, 85–109. MR MR769825 (86d:17003)

20. H.P. Petersson, *Albert algebras*, `http://www.fields.utoronto.ca/programs/scientific/11-12/exceptional/index.html`, 2012.

21. H.P. Petersson and M.L. Racine, *Jordan algebras of degree 3 and the Tits process*, J. Algebra **98** (1986), no. 1, 211–243. MR 87h:17038a

22. _____, *On the invariants mod 2 of Albert algebras*, J. Algebra **174** (1995), no. 3, 1049–1072. MR 96c:17045

23. M.L. Racine, *A note on quadratic Jordan algebras of degree 3*, Trans. Amer. Math. Soc. **164** (1972), 93–103. MR 0304447 (46 #3582)

24. N. Roby, *Lois polynomes et lois formelles en théorie des modules*, Ann. Sci. École Norm. Sup. (3) **80** (1963), 213–348. MR 0161887 (28 #5091)

25. R.D. Schafer, *An introduction to nonassociative algebras*, Dover Publications Inc., New York, 1995, Corrected reprint of the 1966 original. MR 1375235 (96j:17001)

26. T.A. Springer and F.D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. MR 1763974 (2001f:17006)

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