AN ALTERNATIVE DUNFORD-PETTIS PROPERTY FOR JB*-TRIPLES

MARíA D. ACOSTA AND ANTONIO M. PERALTA

Abstract

We study a property weaker than the Dunford-Pettis property, introduced by W. Freedman, in the case of a JB*-triple. It is shown that a JBW*-triple W has this property if, and only if, either W is a Hilbert space (regarded as a type 1 or 4 Cartan factor) or W has the Dunford-Pettis property. As a consequence, we get that the JBW*-triples satisfying the Kadec-Klee property are either finite-dimensional or Hilbert spaces (regarded as Cartan factor 1 or 4).

1. *Introduction*

N. Dunford and B. J. Pettis [**10**] proved that a weakly compact operator from $L^1(\mu)$ to another Banach space sends weakly Cauchy sequences into norm convergent sequences. A. Grothendieck [**13**] showed that the same conclusion holds for weakly compact operators on $C(K)$, for any compact Hausdorff space K. A Banach space X has the the *Dunford-Pettis property* (DPP for short) if any weakly compact operator from X into some other Banach space is completely continuous.

For a long time, it was unknown if every non commutative C^*-algebras had the DPP. In [**4**], C-H. Chu and B. Iochum get a characterization of the C*-algebras having the Dunford-Pettis property. Indeed, a von Neumann algebra has the DPP if, and only if, it is a finite direct sum of type I_n von Neumann algebras. The von Neumann algebras whose predual has DPP are characterized by L. Bunce [**3**]. For JB^*-triples, a class including C^*-algebras, C-H. Chu and P. Mellon have characterized those spaces with the DPP [**5**].

The following characterization of the DPP was given by Grothendieck. A Banach space X has the Dunford-Pettis property if, and only if, for any weakly null sequences $\{x_n\}$ in X and $\{f_n\}$ in X^*, it holds $f_n(x_n) \to 0$. W. Freedman [**11**] introduced a weaker version of the DPP as follows. A Banach space has the DP1 if, and only if, for any weakly convergent sequences $x_n \to x$ in X, and $f_n \to 0$ in X^*, such that $\|x_n\| = \|x\| = 1$, it holds $f_n(x_n) \to 0$. Of course, the condition $\|x_n\| = \|x\| = 1$ can be replaced by $\|x_n\| \to \|x\|$. Freedman shows that the DP1 is equivalent to the DPP for von Neumann algebras, but is is strictly weaker than the DPP for preduals of von Neumann algebras.

The DP1 property is weaker than a well-known isometric property, the *Kadec-Klee property* (KKP in the following). Recall that a Banach space has the KKP

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if any sequence in the unit sphere whose weak limit is also in the unit sphere, is indeed norm convergent.

In this paper, we characterize those JB^*-triples having the DP1 property. In the case of JBW*-triples, we describe those spaces satisfying the DP1. As a consequence, we prove that a JBW^*-triple has the Kadec-Klee property if, and only if, either it is finite-dimensional or a Hilbert space (two possible norms).

Next we recall some well known results about the DPP, the DP1, and the KKP. We refer to [**9**] as a good survey on the DPP.

Remark 1.

1. A Banach space whose dual X^* has DPP, has also the DPP. The same fact does not hold for the DP1. Indeed, for any infinite-dimensional Hilbert space H, the space of trace-class operators $\mathcal{L}^1(H) = K(H)^*$ has the DP1 property and $K(H)$ (the space of the compact operators on H) does not [**11**, Remarks 1.2].

2. The DPP and the DP1 properties are preserved by complemented subspaces and the KKP is preserved by closed subspaces. The DPP is preserved by isomorphisms, while the DP1 property and the KKP are not [**11**, Example 1.6].

3. If H is an infinite-dimensional Hilbert space and $Y \neq \{0\}$ is a Banach space, then $X := H \oplus^\infty Y$ does not have the DP1 property. It is enough to fix an element y in the unit sphere of Y and an orthonormal system $\{e_n\}$ in H. Then, the sequence $\{x_n\} = \{(e_n, y)\}$ in X clearly satisfies $\|x_n\| = 1$ and $\{x_n\} \xrightarrow{w} x = (0, y)$, an element in the unit sphere of X. Since the sequence $\{f_n\} := \{(e_n, 0)\} \xrightarrow{w} 0$ in the dual space of $H \oplus_\infty Y$ and $f_n(x_n) = 1, \forall n$, then X does not have the DP1 property.

4. The DPP, the DP1 and the KKP do not depend on the scalar field considered in the case that the space is complex.

2. *Alternative Dunford-Pettis property for JB*-triples*

The complex Banach spaces called JB*-triples play a very important role in the study of bounded symmetric domains in several complex variables. Indeed, Kaup [**17**] showed that every such domain is biholomorphic to the open unit ball of a JB*-triple. We begin by recalling the definition of a JB*-triple and referring to [**22**], [**23**] and [**6**] for recent surveys on the theory.

Definition 1. *A JB*-triple is a complex Banach space E together with a triple product $\{.,.,.\} : E \times E \times E \to E$, which is continuous, symmetric and linear in the outer variables and conjugate linear in the middle one, satisfying*

a) *Jordan Identity: for all $a, b, x, y, z \in E$*
$$L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\},$$
where $L(a,b)x := \{a,b,x\}$;

b) *For each $a \in E$ the operator $L(a,a)$ is hermitian with nonnegative spectrum, and $\|L(a,a)\| = \|a\|^2$.*

Every C*-algebra is a JB*-triple in the triple product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. Also $B(H,K)$, the space of all bounded and linear operators between two complex Hilbert spaces H and K, is a JB*-triple with the triple product defined by
$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x).$$

Every JB*-algebra is a JB*-triple in the triple product $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. The bidual of a JB^*-triple E is also a JB^*-triple that contains E as a subtriple.

For any JB*-triple E and a tripotent $e \in E$ ($\{e,e,e\} = e$), there exists a decomposition of E in terms of the eigenspaces of $L(e,e)$, i.e.

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e),$$

where $E_k(e) := \{x \in E : L(e,e)x = \frac{k}{2}x\}$. The natural projection of E onto $E_k(e)$ will be denoted by $P_k(e)$. This decomposition is called Peirce decomposition and the natural projections are called Peirce projections. It is well known that the Peirce projections associated to each tripotent e are contractive. For every tripotent $e \in E$ the equality

$$\|P_2(e)(x) + P_0(e)(x)\| = \max\{\|P_2(e)(x)\|, \|P_0(e)(x)\|\}$$

holds for all $x \in E$ (see for instance [**12**, Corollary 1.2. and Lemma 1.3]).

A tripotent e is called *minimal* if $E_2(e) = \mathbf{C}e$. Let e, f be tripotents of E, then we say that e and f are *orthogonal*, denoted by $e \perp f$, if $L(e,f) = 0$ ($\Leftrightarrow L(f,e) = 0 \Leftrightarrow f \in E_0(e)$). We say that e and f are *colinear*, denoted by $e \top f$, if $e \in E_1(f)$ and $f \in E_1(e)$. Finally we say that e *governs* f ($e \vdash f$), if $e \in E_1(f)$ and $f \in E_2(e)$.

A JBW*-triple is a JB*-triple which is a dual Banach space. Every JBW*-triple has a unique predual and its triple product is separately weak*-continuous [**2**]. By the Krein-Milman theorem and [**18**, Proposition 3.5], to each non zero weak*-continuous functional ϕ on a JBW*-triple W, there is a tripotent $u \in W$ such that $\phi = \phi P_2(u)$ and $\|\phi\| = \phi(u)$. If W is a JBW*-triple and f is a norm one element in W_*, we can define a seminorm $\|.\|_f$ on W, given by

$$\|w\|_f^2 := f\{w,w,e\},$$

where $e \in W$ is a tripotent with $f(e) = \|f\| = 1$. It is worth mentioning that $\|.\|_f$ does not depend on the support e of f (see [**1**] for more details). The strong* topology of W, noted by $S(W, W_*)$, is defined as the topology on W generated by all the seminorms $\|.\|_f$, where f is a norm one functional in W_*. If E is a JB*-triple we denote by $S(E, E^*)$, the restriction to E of the strong* topology of E^{**}.

First we show that the DP1 in JB^*-triples can be characterized in terms of the triple product. In order to do this, we will use the following result, due to C-H. Chu and P. Mellon, which is stated in [**5**, Lemma 4]. The proof given by Chu and Mellon uses as a key tool a result due to Barton and Friedman called a "Little Grothendieck's Theorem" [**1**, Theorem 1], which has a gap in the proof (see [**20**] and [**21**]). However, the Chu-Mellon statement can be proved by using their arguments and a correct "Little Grothendieck's Theorem". We include here the result with a corrected proof.

LEMMA 1. *Let W be a JBW^*-triple without summands $L^\infty(\Omega,\mu,C^5)$ and $L^\infty(\Omega,\mu,C^6)$, where C^5 and C^6 are the type 5 and type 6 Cartan factors respectively. Let $\{f_n\}$ be a $\sigma(W_*,W)$-null sequence in W_* and let $\{w_n\}$ be an $S(W,W_*)$-null sequence in W. Then we have*

$$\lim_n \sup_{k \in \mathbf{N}} \{|f_k(w_n)|\} = 0.$$

Proof. W can be embedded as a subtriple in a von Neumann algebra M such that W_* is complemented in M_*. For any $f \in M_*$, let us define

$$N = \{w \in W : f(w^*w + ww^*) = 0\},$$

which is a w^*-closed subspace of W. The quotient W/N can be equipped with the inner product given by

$$(w + N, z + N) := f(z^*w + wz^*).$$

The natural quotient map Q from W to the completion of W/N is w^*-continuous since $f \in M_*$. By [**21**, Corollary 2.2] there are norm one functionals $\phi_1, \phi_2 \in W_*$ and a constant K such that

$$(f(w^*w + ww^*))^{\frac{1}{2}} = \|Q(w)\| \leq K\|Q\|\|w\|_{\phi_1,\phi_2}, \qquad \forall w \in W,$$

where $\|w\|^2_{\phi_1,\phi_2} := \phi_1 \{w, w, e_1\} + \phi_2 \{w, w, e_2\}$ and $e_1, e_2 \in W$ are tripotents with $\phi_i(e_i) = 1$. By the assumption of the lemma, $\{w_n\}$ converges to zero in the $S(W, W_*)$-topology, so the above inequality implies that $f(w_n^* w_n + w_n w_n^*) \to 0$ and this holds for any $f \in M_*$. That is, $\{w_n^* w_n + w_n w_n^*\}$ is w^*-null and by using [**24**, Lemma III.5.5] we conclude that

$$\lim_n \sup_k \{|f_k(w_n)|\} = 0.$$

□

Theorem 5 in [**5**] gives a criteria for DPP in JB^*-triples. Indeed, a JB*-triple E has the DPP if, and only if, whenever $\{x_n\} \to 0$ weakly in E, we have $\{x_n, x_n, y\}$ tends to zero weakly for every $y \in E$. On the other hand, a C*-algebra A has the DP1 property if and only if whenever $a_n \to a$ weakly in A with $\|a_n\| = \|a\| = 1$, we have $a_n^* a_n \to a^* a$ weakly (see [**11**, Theorem 3.1.]). These results inspired the following criteria for the DP1 property in JB*-triples.

Theorem 1. *Let E be a JB*-triple. The following assertions are equivalent:*
(i) *E has the DP1 property.*
(ii) *For any weakly convergent sequence $x_n \to x$ with $\|x_n\| = \|x\| = 1$, then $\{x_n, x_n, y\}$ converges to $\{x, x, y\}$ weakly for all $y \in E$.*
(iii) *Whenever $x_n \to x$ weakly with $\|x_n\| = \|x\| = 1$, we have $x_n \to x$ in $S(E, E^*)$-topology.*

Proof. $(i) \Rightarrow (ii)$ Let $\{x_n\}$ be a sequence weakly convergent in E to an element $x \in E$ such that $\|x_n\| = \|x\| = 1$, let $y \in E$ and $f \in E^*$. Let E_c be the JB^*-triple obtained from E by changing the scalar multiplication to $(\lambda, e) \to \overline{\lambda}e$. Define the operator $T : E_c \longrightarrow E^*$ given by

$$T(w)(x) := f\{x, w, y\}.$$

Then $T : E_c \longrightarrow E^*$ is linear and by using [**4**, Lemma 5], T is weakly compact. Since E has the DP1 and $\{x_n\} \stackrel{w}{\to} x$ in the unit sphere, by [**11**, Theorem 1.4], then $\{Tx_n\} \to Tx$ in the norm topology, and this implies that

$$\{f(\{x_n, x_n, y\})\} \to f(\{x, x, y\}), \forall y \in E.$$

The rest of the proof can be made following the argument used in [**5**, Theorem 5], where Lemma 1 is used. □

By using condition ii) in Theorem 1 we get:

COROLLARY 1. *Let E be a JB*-triple with the DP1 property and let F be a subtriple of E, then F has the DP1 property.*

For the predual of a JBW*-triple W, Chu and Mellon [**6**, Lemma 15] gave a characterization of the DPP. Indeed, the predual of W has the DPP if, and only if, for every weak null sequence w_n in W, the sequence $\{w_n, w_n, w\}$ is weakly null for all $w \in W$. Freedman [**11**, Proposition 2.6] found a necessary condition to have the DP1 property for the predual of a von Neumann algebra. In the case of a JBW*-triple we get the following result.

PROPOSITION 1. *Let W be a JBW*-triple with predual W_* and suppose that W_* has the DP1 property. Let $w_n \to w$ weakly in W with $\|w_n\|, \|w\| \leq 1$, let $f \in W_*$ and $y \in W$ with $\|y\| \leq 1$ such that $\|f\| = \|f\{., w, y\}\|$, then*

$$f\{w_n, w_n, y\} \to f\{w, w, y\}.$$

Proof. For each $n \in \mathbf{N}$ and $y \in W$, let us define the functional given by

$$f_n(x) := f\{x, w_n, y\} \quad (x \in W).$$

Since the triple product is separately weak* continuous [**1**, Theorem 2.1], then $f_n \in W_*$. We are assuming that $w_n \to w$ in the weak* topology and the triple product is separate weak* continuous, then $f_n(x) = f\{x, w_n, y\} \to f\{x, w, y\}$, thus $f_n \to f\{., w, y\}$ weakly in W_*. Moreover,

$$\|f_n\| \leq \|f\|\ \|w_n\|\ \|y\| \leq \|f\| = \|f\{., w, y\}\|,$$

so $\|f_n\| \to \|f\{., w, y\}\|$. Since W_* has the DP1 property, $f_n(w_n) \to f\{w, w, y\}$, i. e., $f\{w_n, w_n, y\} \to f\{w, w, y\}$. □

3. *JBW*-triples with the alternative Dunford-Pettis property*

The aim of this section is to describe those JBW*-triples having the DP1 property. Our first goal is the study of the DP1 property in the particular case of a Cartan factor. In a finite dimensional space, the DP1 property is trivially satisfied. For this reason we focus our attention in the infinite-dimensional case.

PROPOSITION 2. *If C^1 is an infinite dimensional type 1 Cartan factor having the DP1 property, then C^1 is an infinite dimensional Hilbert space.*

Proof. Let C^1 be an infinite dimensional type I Cartan factor having the DP1 property. Then C^1 is of the form $B(H, K)$, where H and K are Hilbert spaces with at least one of them infinite dimensional, and the triple product is given by

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x).$$

Let h in H and k in K we denote by $k \otimes h$ the element in $B(H, K)$ given by $k \otimes h(x) := (x|h)k$ $(x \in H)$. Since $B(H, K)$ can be identified as a triple to $B(K, H)$, then we can assume that $\dim K \leq \dim H = \infty$. If $\dim K \geq 2$, then, let k_1, k_2 be two orthonormal elements in K and $\{h_n\}$ be an orthonormal system in H. Let us consider the operators given by

$$x_n = k_1 \otimes h_n + k_2 \otimes h_1, \quad x = k_2 \otimes h_1.$$

It is immediate to check that $\|x_n\| = \|x\| = 1$ in $B(H,K)$. Since $\{h_n\}$ converges to 0 weakly, then $\{x_n\}$ is w-convergent to x. Now take $y = k_1 \otimes h_1$, an easy computation shows that

$$\{x,x,y\} = \frac{1}{2}y, \quad \{x_n,x_n,y\} = y \quad (n \geq 2).$$

By applying Theorem 1, then $B(H,K)$ does not have the DP1. Therefore, $\dim K = 1$ and $B(H,K) = H$. □

Now we proceed with the study of the rest of the Cartan factors. For this, we will use the following definition, which has been taken from [**7**], [**19**].

Definition 2. *A family of minimal tripotents $F = \{u_{ij} : i,j \in I\}$ in a JB*-triple is called an hermitian grid if:*

(i) *For every i,j,k,l in I, we have $u_{ij} = u_{ji}$;*

$$u_{ij} \perp u_{kl} \ \textit{if} \ \{i,j\} \cap \{k,l\} = \emptyset;$$

$$u_{ii} \dashv u_{ij} \top u_{jk} \quad \textit{if} \quad i,j,k \text{ are different.}$$

(ii) *Every non trivial triple product involving just elements of F belongs to the set*

$$\{\{u_{ij}, u_{kj}, u_{kl}\} : \ i,j,k,l \in I\}.$$

(iii) *For arbitrary i,j,k,l triple products involving at least two different elements satisfy*

$$\{u_{ij}, u_{kj}, u_{kl}\} = \frac{1}{2} u_{il} \textit{ for } i \neq l, \qquad \{u_{ij}, u_{kj}, u_{ki}\} = u_{ii}.$$

We give a brief description of the Cartan factors of type 2 and 3. Let H be a complex Hilbert space equipped with a conjugation (conjugate-linear isometry of period 2) $j : H \to H$, then for any $z \in B(H)$ we can define its transpose as $z^t := j\ z^*\ j$. The type 2 Cartan factor coincides with the Banach space of all t-symmetric elements in $B(H)$ $(z^t = z)$, and the type 3 Cartan factor is defined as the Banach space of all t-anti-symmetric elements of $B(H)$ $(z^t = -z)$. The triple product of these Cartan factors is the restriction of the triple product in $B(H)$.

Each type 2 Cartan factor admits a hermitian grid whose span is weak* dense. Notice that if $\{e_i\}_{i\in I}$ is an orthonormal basis of H we can take $u_{ik} := j(e_i) \otimes e_k + j(e_k) \otimes e_i \in \{z \in B(H) : z^t = z\}$ for all $i,k \in I$.

Proposition 3. *Let C^2 be a type 2 Cartan factor having the DP1 property, then C^2 is finite dimensional.*

Proof. Let C^2 be an infinite dimensional type 2 Cartan factor. Let $F = \{u_{ij} : i,j \in \mathbf{N}\}$ be a countable hermitian grid in C^2. The set $\mathcal{G} := \{u_{2,n} : n \geq 3\}$ is a family of mutually colinear minimal tripotents in C^2, so by [**7**, Lemma p. 306] the subspace $H := \overline{Span}^{\|.\|}(\mathcal{G})$ is isometric to a Hilbert space and $\{u_{2,n} : n \geq 3\}$ is an orthonormal basis. Moreover, there exists a contractive projection P from C^2 to H.

Now since u_{12} and u_{2n} $(n \geq 3)$ are orthogonal, we have $u_{2n} \in (C^2)_0(u_{12})$, and by the continuity of the triple product $H \subseteq (C^2)_0(u_{1,2})$. In particular by [**12**, Lemma 1.3]

$$\|x + y\| = \max\{\|x\|, \|y\|\}$$

for every $x \in (C^2)_2(u_{1,2})$ and $y \in H$. Since $u_{1,2}$ is minimal ($(C^2)_2(u_{1,2}) = \mathbf{C}u_{1,2}$), it follows that C^2 contains as a complemented subspace an isometric copy of $\mathbf{C} \oplus^\infty H$. Since H is infinite-dimensional, then $\mathbf{C} \oplus^\infty H$ does not have the DP1 property in view of Remark 1, which is impossible. □

Definition 3. ([**7**, p. 317])
A family $F = \{u_{ij} : i, j \in I\}$ in a JB-triple is called a symplectic grid if $u_{ii} = 0$, u_{ij} are minimal tripotents with $u_{ij} = -u_{ji}$ for all $i \neq j$, and*

(i) *u_{ij} and u_{kl} are colinear if they share an index and are orthogonal otherwise.*
(ii) *Every triple product among elements of F which can not be brought to the form $\{u_{ij}, u_{kj}, u_{kl}\}$ vanishes.*
(iii) *For arbitrary i, j, k, l triple products involving at least two different elements satisfy*

$$\{u_{ij}, u_{kj}, u_{kl}\} = \frac{1}{2}u_{il} \text{ for } i \neq l, \qquad \{u_{ij}, u_{jk}, u_{ki}\} = 0.$$

Each Cartan factor of type 3 is the weak* closed span of a symplectic grid (just consider $u_{i,k} := j(e_i) \otimes e_k - j(e_k) \otimes e_i$, where $\{e_i\}$ is an orthonormal basis of H). By using the same argument of the proof of Proposition 3 and replacing the hermitian grid by a symplectic grid, we get:

Proposition 4. *Every type 3 Cartan factor having the DP1 property is finite dimensional.*

Finally we proceed with the study of the DP1 property in a type 4 Cartan factor. A type 4 Cartan factor is a JB*-triple which can be equipped with an inner product $(.|.)$ and a conjugation * such that the triple product satisfies

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|z^*)y^*,$$

and the norm is given by

$$\|x\|^2 := (x|x) + ((x|x)^2 - |(x|x^*)|^2)^{\frac{1}{2}}.$$

Proposition 5. *Every type 4 Cartan factor has the Kadec Klee property and so, it satisfies the DP1 property.*

Proof. Let C^4 be a type 4 Cartan factor, with inner product $(.|.)$ and conjugation $*$. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a $(.|.)$-orthonormal basis of C^4 satisfying $e_\lambda^* = e_\lambda$ for all $\lambda \in \Lambda$. If for each $x \in C^4$ we put $x(\lambda) := (x|e_\lambda)$, then the norm is given by

$$\|x\|^2 = \|x\|_2^2 + q(x) := \sum_{\lambda \in \Lambda} |x(\lambda)|^2 + q(x) \qquad (x \in C^4),$$

where

$$q(x) := \left(\left(\sum_{\lambda \in \Lambda} |x(\lambda)|^2 \right)^2 - \left| \sum_{\lambda \in \Lambda} x(\lambda)^2 \right|^2 \right)^{\frac{1}{2}}.$$

First we will check that for any set $F \subset \Lambda$ and $x \in C^4$, the inequality

$$q(x) \geq q\left(\sum_{\lambda \in F} x(\lambda)\ e_\lambda \right). \tag{1}$$

holds. For $x \in C^4$ and $F \subseteq \Lambda$ it is satisfied

$$\left(\sum_{\lambda \in F} |x(\lambda)|^2 \right)^2 + \left| \sum_{\lambda \in F} x(\lambda)^2 + \sum_{\lambda \in \Lambda \setminus F} x(\lambda)^2 \right|^2$$

$$\leq \left(\sum_{\lambda \in F} |x(\lambda)|^2 + \sum_{\lambda \in \Lambda \setminus F} |x(\lambda)|^2 \right)^2 + \left| \sum_{\lambda \in F} x(\lambda)^2 \right|^2,$$

which is equivalent to

$$q(x) \geq q\left(\sum_{\lambda \in F} x(\lambda)\ e_\lambda \right).$$

For any subset $F \subset \Lambda$, let us denote by P_F the projection on C^4 given by $P_F(x) := \sum_{\lambda \in F} x(\lambda)\ e_\lambda$. By (1) it follows that $\|P_f\| \leq 1$. We are going to prove that for any $\varepsilon > 0$ it is satisfied

$$F \subseteq \Lambda, x \in C^4, \|x\| = 1, \|P_F(x)\|^2 > 1 - \varepsilon \Rightarrow \|x - P_F(x)\|_2 < \varepsilon. \tag{2}$$

To this end, let us fix $x \in C^4$ with $\|x\| = 1$ and a subset $F \subset \Lambda$ satisfying $\|P_F(x)\|^2 > 1 - \varepsilon$. Since

$$1 = \|x\|^2 = \sum_{\lambda \in \Lambda} |x(\lambda)|^2 + q(x),$$

then

$$\varepsilon > 1 - \|P_F(x)\|^2 = \|x\|^2 - \|P_F(x)\|^2 =$$

$$= \sum_{\lambda \in \Lambda \setminus F} |x(\lambda)|^2 + q(x) - q\left(\sum_{\lambda \in F} x(\lambda) e_\lambda \right) \geq \text{by using } (1)$$

$$\geq \sum_{\lambda \in \Lambda \setminus F} |x(\lambda)|^2 = \|x - P_F(x)\|_2^2.$$

Now we show that C^4 has the KKP. Therefore, assume that $\{x_n\} \to x_0$ weakly and $\|x_n\| = \|x_0\| = 1$. For any $\varepsilon > 0$, let us choose a finite subset $G \subset \Lambda$ satisfying

$$\|x_0 - P_G(x_0)\| \leq \varepsilon \text{ and}$$

$$1 - \varepsilon < \|P_G(x_0)\|^2.$$

Since P_G has finite rank and $P_G(x_n) \xrightarrow{w} P_G(x_0)$, it follows that $P_G(x_n)$ converges to $P_G(x_0)$ in norm. Then, we can choose $m \in \mathbf{N}$ such that for all $n \geq m$

$$\|P_G(x_n) - P_G(x_0)\| \leq \varepsilon$$

and

$$1-\varepsilon < \|P_G(x_n)\|^2.$$

By using (2) we deduce that

$$\|x_n - P_G(x_n)\|_2 \le \varepsilon \quad .$$

Therefore for $n \ge m$ we have

$$\|x_n - x_0\|_2 \le \|x_n - P_G(x_n)\|_2 + \|P_G(x_n) - P_G(x_0)\|_2 + \|P_G(x_0) - x_0\|_2 \le$$

$$\le 2\varepsilon + \|P_G(x_n) - P_G(x_0)\| \le 3\varepsilon,$$

that is, $\{x_n\}$ converges to x in the norm topology. □

REMARK 2. *Let $C^1 = B(H,K)$ a type 1 Cartan factor, where H and K are complex Hilbert spaces with dimension n and m, respectively, then ℓ_2^n and ℓ_2^m embed isometrically as 1-complemented subspaces of C^1.*

If $C^2 := \{x \in B(H) : x^t = jx^*j = x\}$ (where j is a suitable conjugation on H) is a type 2 Cartan factor, then the proof of Proposition 3 shows that C^2 admits as a complemented subspace an isometric copy of ℓ_2^{n-2}. It is worth mentioning that the natural projection from C^2 onto ℓ_2^{n-2} has norm at most 2. The same conclusion holds for type 3 Cartan factors by using Proposition 4 instead of Proposition 3.

Finally, if C^4 is a type 4 Cartan factor with inner product and conjugation denoted by $(.|.)$ and $*$, respectively, then $U := \{x \in C^4 : x^* = x\}$ is a 1-complemented real subspace of C^4, which is also isometric to $\ell_2^{dim C^4}$.

Once we have determined all Cartan factors having the DP1 property, we can now deal with the same problem in any JBW*-triple. By the structure theory (see [**14**],[**15**],[**16**]), every JBW*-triple W has a decomposition into the ℓ_∞-sum

$$W = \oplus_\alpha^{\ell_\infty} L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha) \oplus^{\ell_\infty} R \oplus^{\ell_\infty} H(M,\beta),$$

where C_α is a Cartan factor, R is a weak* closed right ideal of a continuous von Neumann algebra N, and $\beta : M \to M$ is a linear period 2 *-antiautomorphism of a continuous von Neumann algebra M with $H(M,\beta) = \{a \in M : \beta(a) = a\}$ and the self-adjoint part $A = H(M,\beta)_{sa} := \{a \in H(M,\beta) : a^* = a\}$ is a continuous (real) JW-algebra under the product $a \circ b = \frac{1}{2}(ab+ba)$.

For a continuous JW-algebra A, Chu and Mellon [**5**, Corollary 13] proved that A does not have DPP. We are going to see that similar arguments show that a continuous JW-algebra does not have the DP1 property.

LEMMA 2. *Let A be a continuous JW-algebra, then A does not have the DP1 property.*

Proof. Let A be a continuous JW-algebra, by [**5**, Proposition 12], the ℓ_∞-sum $\bigoplus_{n\ge 2}^{\ell_\infty} H_{2^n}(\mathbf{R})$ embeds (isometrically) as a Jordan subalgebra of A, where $H_{2^n}(\mathbf{R})$ is the algebra of hermitian $2^n \times 2^n$ matrices. If A has the DP1 property then the ℓ_∞-sum $\bigoplus_{n\ge 2}^{\ell_\infty} H_{2^n}(\mathbf{R})$ has the DP 1 property by Corollary 1, which is impossible since $\bigoplus_{n\geq 2}^{\ell_\infty} H_{2^n}(\mathbf{R})$ is isometrically isomorphic to

$$\Big(\bigoplus_{n\geq 2}^{\ell_\infty} H_{2^{2n-1}}(\mathbf{R})\Big)\bigoplus^{\infty}\Big(\bigoplus_{n\geq 1}^{\ell_\infty} H_{2^{2n}}(\mathbf{R})\Big),$$

which contains as a complemented subspace an isometric copy of

$$\mathbf{R}\bigoplus^{\infty}\Big(\bigoplus_{n\geq 1}^{\ell_\infty} \ell_2^{2^{2n}-2}\Big),$$

(see Remark 2), but by [**26**, p. 81] and Remark 1, the later space fails to have the DP1 property because it contains as a complemented subspace an isometric copy of $\mathbf{R}\oplus^\infty \ell_2$. Therefore A does not have the DP1 property.

□

We can now characterize those JBW*-triples having the DP1 property. If a JBW*-triple W has DPP, then W has the DP1 property. We recall that, in the particular case of a von Neumann algebra, the DPP and the DP1 properties are equivalent [**11**, Theorem 3.5]. For a JBW*-triple we will not have this equivalence in general. Indeed, every infinite dimensional Hilbert space H, regarded as a type 1 or type 4 Cartan factor has the DP1 (see Propositions 2 and 5), however, H does not have the DPP. The next result shows that this is the only possible exception.

Theorem 2. *Let W be a JBW*-triple having the DP1 property, then either W has DPP or W is a Hilbert space or W is a type 4 Cartan factor.*

Proof. We know that W is of the form

$$W=\bigoplus_{\alpha}^{\ell_\infty} L^\infty(\Omega_\alpha,\mu_\alpha,C_\alpha)\oplus^\infty R\oplus^\infty H(M,\beta).$$

By Corollary 1 and Lemma 2, we know that $H(M,\beta)=0$. If $R\neq 0$, then $R=pN$ for some continuous von Neumann algebra N and some non-zero projection $p\in N$. By Corollary 1, the von Neumann algebra pNp has the DP1 property since $R=pN$ has the DP1 property. For a von Neumann algebra the DP1 property and the DPP are equivalent [**11**, Theorem 3.5], so pNp has the DPP. Therefore by [**4**, Theorem 3] pNp is a finite type I von Neumann algebra, however pNp is a continuous von Neumann algebra since N is continuous (see [**25**, Corollary 11]), then $R=0$.

Every C_α is a subtriple of W, so by Corollary 1 and Propositions 2, 3, 4 and 5, either C_α is an infinite dimensional Hilbert space or dim $C_\alpha<\infty$.

Suppose first that one of the factors, namely C_α, is an infinite dimensional Hilbert space (regarded as a type 1 or as a type 4 Cartan factor). If any of the other Cartan factors, for example C_γ, is not zero. Then $C_\alpha\bigoplus^{\ell_\infty} C_\gamma$ is a subtriple of W and does not have the DP1 property (see Remark 1), which is impossible by Corollary 1. Therefore $W=L^\infty(\Omega,\mu,H)$, where H is an infinite dimensional Hilbert space regarded as a type 1 or as type 4 Cartan factor. If there exists a μ-measurable set S such that $\mu(S),\mu(\Omega\backslash S)>0$, W is isometrically isomorphic to

$$L^\infty(S,\mu|_S,H)\bigoplus^{\infty} L^\infty(\Omega\backslash S,\mu|_{\Omega\backslash S},H).$$

Therefore, $H \bigoplus^{\ell_\infty} H$ is a subtriple of W without the DP1 property. Hence, W is a Hilbert space regarded as a type 1 or as a type 4 Cartan factor.

Finally, we suppose that $dim\ C_\alpha < \infty$ for every α. By Remark 2, each type 1, 2 , 3 or 4 Cartan factor C_α, contains a complemented (real) subspace isometric to $\ell_2^{n_\alpha}$ and $\{n_\alpha\}$ is increasing with respect to $dim\ C_\alpha$. Moreover, the natural projection from C_α onto $\ell_2^{n_\alpha}$ has norm less than 2. Suppose that $\sup_\alpha\ dim\ C_\alpha = \infty$, then $\bigoplus^{\ell_\infty} C_\alpha$ is a subtriple of W, thus has the DP1 property. However, $\bigoplus^{\ell_\infty} C_\alpha$ contains as a complemented subspace an isometric copy of

$$C_{\alpha_0} \bigoplus^{\infty} \Big(\bigoplus_{\alpha \neq \alpha_0}^{\ell_\infty} \ell_2^{n_\alpha} \Big)$$

which contains as a complemented subspace an isometric copy of $C_{\alpha_0} \bigoplus^{\infty} \ell_2$ [**26**, p. 81]. But the latter space fails to have the DP1 property (see Remark 1).

□

Since every type 4 Cartan factor satisfies the KKP, then,

Corollary 2. *A JBW*-triple W has the DP1 property if, and only if, W has the DPP or the KKP.*

Corollary 3. *A JBW^*-triple has the Kadec-Klee property if, and only if, either it is finite-dimensional or a Hilbert space (as a Cartan factor of type 1 or 4).*

Proof. We know that Kadec-Klee property implies the DP1- property in general, and so, in view of Theorem 2, either the JBW^*-triple has the DPP or it is a Hilbert space. By using the description due to Chu and Mellon [**5**], if the space has the DPP, the triple W can be decomposed as

$$W = \oplus_\alpha^{\ell_\infty} L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha),$$

where C_α is a Cartan factor and $\sup_\alpha \dim C_\alpha < +\infty$.

If for some α, the space $L^\infty(\mu_\alpha)$ is infinite-dimensional, then $L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ contains an isometric copy of ℓ_∞. Since the Kadec-Klee property is preserved by passing to subspaces, then ℓ_∞ would have the KKP, which is far from being true (see [**8**, Theorem II.7.10]). By the same argument the set of indexes is finite and so W is finite-dimensional.

On the other hand, Hilbert spaces have the KKP. Also we proved that a type 4 Cartan factor has the KKP (see Proposition 5). □

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Universidad de Granada
Facultad de Ciencias
Departamento de Análisis Matemático
18071 Granada, Spain
E-mail: dacosta@goliat.ugr.es
E-mail: aperalta@goliat.ugr.es