Jordan type structures over a set with two operations

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Abstract

In this paper we introduce the notion of Jordan di-structures, which are a generalization of the notion of Jordan algebra possessing two operations. We show that every dialgebra is a Jordan di-structure and is a noncommutative Jordan algebra. Also we make the comparison with some well known structures.

1 Introduction

In algebra there are three strongly related classical algebras: associative, Jordan and Lie algebras. It is known that any associative algebra A becomes a Jordan algebra A^+ under the product $x \cdot y := \frac{1}{2}(xy + yx)$ and becomes a Lie algebra under the Lie bracket $[x, y] := xy - yx$.

On the other hand we know from the works of Kantor, Koecher and Tits that the Jordan algebras are imbedded in the Lie algebras. In particular, Koecher showed that for a Jordan algebra $\mathfrak{U}$ there is a Lie algebra $\mathfrak{L}(\mathfrak{U})$ such that $\mathfrak{U}$ is a subspace of $\mathfrak{L}(\mathfrak{U})$ and the product of $\mathfrak{U}$ can be expressed in terms of the bracket in the Lie algebra $\mathfrak{L}(\mathfrak{U})$ (see[K2]).

It is known that the universal enveloping algebra of a Lie algebra has the structure of an associative algebra.

More recently, J.L. Loday (see [L2]) introduced the notion of a Leibniz algebra, which is a generalization of the Lie algebras, where the skew-symmetry of the bracket is dropped. J.L. Loday also showed that the relationship between Lie algebras and associative algebras translate into an analogous relationship between Leibniz algebras and the so-called dialgebras (see [L1]), which are a generalization of associative algebras possessing two operations. In particular, Loday showed that any dialgebra $D(\vdash, \dashv)$ becomes a Leibniz algebra D_{Leib} under the bracket $[x, y] := x \dashv y - y \vdash x$ and that the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra (see [L-P]).

Our aim is to generalize the notion of Jordan algebras and to show that the relationship between associative algebras and Jordan algebras translate into analogous relationship between dialgebras and this new Jordan structure. Additionally, we show that any dialgebra is a noncommutative Jordan algebra.

The essential idea in this generalization is to define algebraic structures of Jordan type over a set with two operations, for which the commutative condition is changed for a *di-commutative condition* or is dropped and the Jordan identity is changed for a *Jordan di-identity*.

This new structure is called *Jordan di-structure* and we prove that any dialgebra is a Jordan di-structure. Finally, we show some comparisons with certain structures.

This work is divided in two sections. In section 2 we introduce the notion of Jordan di-structures, study the relationship with noncommutative Jordan algebras and show some examples. In the last section we show the relationship of Jordan di-structures and dialgebras. Additionally, we prove that every dialgebra is a noncommutative Jordan algebra.

This work is the first part of the study about the relations between Jordan di-structures and Leibniz algebras. These relations will be presented in future publications.

2 Jordan type di-structures

In this section, we introduce the notions of *Jordan di-structure* and *generalized Jordan di-structure*. We study the relation of Jordan di-structures and noncommutative Jordan algebras, and show some examples of Jordan di-structures.

Definition 1 (Jordan di-structure) *Let J be a vector space over a field $\mathbb{K}$. We say that J is a* ***Jordan di-structure*** *over $\mathbb{K}$ if over J are defined two $\mathbb{K}$-bilinear operations $\circ, \square : J \times J \to J$ satisfying the conditions:*

$$x \circ y = y \square x \qquad (DJ1)$$

$$\left(x^2 \circ y\right) \square x = x^2 \circ (y \square x), \qquad (DJ2)$$

where $x^2 = x \circ x = x \square x$, for all $x, y \in J$. The condition $(DJ1)$ is called the ***di-commutative property*** *and the condition $(DJ2)$ is called the* ***Jordan di-identity.***

The Jordan di-structures include the best-known algebras: associative algebras ($x \circ y = xy$ and $x\Box y = yx$), Jordan algebras ($\cdot = \circ = \Box$) and Lie algebras ($x \circ y = [x, y]$ and $x\Box y = [y, x]$).

Example 2 *Let $\mathbb{V}$ be a n-dimensional vector space over a field $\mathbb{K}$ of characteristic zero. We take over $\mathbb{V}$ a base $\{e_1, e_2, \ldots, e_n\}$ and for all $x = \sum_{i=1}^{n} \alpha_i e_i$ and $y = \sum_{i=1}^{n} \beta_i e_i$ in $\mathbb{V}$ we define*

$$x \circ y = \left(\sum_{i=1}^{n} \alpha_i \right) y$$

and

$$x\Box y = \left(\sum_{i=1}^{n} \beta_i \right) x.$$

If we put $\langle x, y \rangle := \sum_{i=1}^{n} \alpha_i \beta_i$ and $E := \sum_{i=1}^{n} e_i$, we have that $\sum_{i=1}^{n} \alpha_i = \langle E, x \rangle$, $\sum_{i=1}^{n} \beta_i = \langle E, y \rangle$, $\langle E, x + y \rangle = \langle E, x \rangle + \langle E, y \rangle$ and $\langle E, \gamma x \rangle = \gamma \langle E, x \rangle$. Using the previous equations, one proves that $(\mathbb{V}, \circ, \Box)$ is a Jordan di-structure.

Remark 3 *If in the previous example we make $x \cdot y = x \circ y$, then in $(\mathbb{V}, \cdot)$ the product $\cdot$ satisfies the identity*

$$x \cdot \left(x^2 \cdot y\right) = x^2 \cdot (x \cdot y), \qquad x, y \in \mathbb{V}$$

From remark 3, we have the following proposition:

Proposition 4 *A Jordan di-structure is equivalent to having a vector space J over the same field equipped with a bilinear product $\cdot : J \times J \to J$ that satisfies the identity*

$$x \cdot \left(x^2 \cdot y\right) = x^2 \cdot (x \cdot y), \qquad x, y \in J, \tag{$*$}$$

where $x^2 = x \cdot x$.

Proof. Let $(\mathbb{J}, \circ, \Box)$ be a Jordan di-structure and we define the product $\cdot : J \times J \to J$ by $x \circ y = x \cdot y$. Then, $x\Box y = y \circ x = y \cdot x$ and

$$x \cdot \left(x^2 \cdot y\right) = x \circ \left(x^2 \circ y\right) = \left(x^2 \circ y\right) \Box x = x^2 \circ (y\Box x) = x^2 \cdot (x \cdot y),$$

for all $x, y \in J$, then $(J, \cdot)$ satisfies the conditions of this proposition.

On the other hand, we suppose that $(J, \cdot)$ satisfies the identity $x \cdot \left(x^2 \cdot y\right) = x^2 \cdot (x \cdot y)$. If we define $\circ, \Box : J \times J \to J$ by $x \circ y := x \cdot y$ and $x\Box y := y \cdot x$, for all $x, y \in J$, then $(T, \circ, \Box)$ is a Jordan di-structure. ■

Remark 5 *The previous proposition says that the Jordan structure is equivalent to a structure that satisfies a Jordan type identity, but in general is not commutative. The identity $(*)$ is a special form of the classical Jordan identity and is equivalent to classical Jordan identity $\left(x^2 \cdot y\right) \cdot x = x^2 \cdot (y \cdot x)$ in Jordan algebras and in noncommutative Jordan algebras (see definition 6), but the commutative condition is not true.*

Let $(J,\circ,\Box)$ be a Jordan di-structure. We have seen that there are linear transformations $L(x)$ and $R(x)$, which are linear in x, such that $x\circ y = L(x)y$ and $y\Box x = R(x)y$. The Jordan di-identity is equivalent to

$$\left[R(x), L\left(x^2\right)\right] = 0 \tag{1}$$

where $[R,L] = RL - LR$ is the classical commutator.

Linearization of (1), if the base field contains at least 3 elements, yields

$$[R(a), L(b\circ c + b\Box c)] + [R(b), L(a\circ c + a\Box c)] + [R(c), L(a\circ b + a\Box b)] = 0 \tag{2}$$

or, equivalently,

$$(a,d,b\circ c + b\Box c) + (b,d,a\circ c + a\Box c) + (c,d,a\circ b + a\Box b) = 0, \tag{3}$$

where (a,b,c) denotes the "di-associator" $(a,b,c) = a\circ(b\Box c) - (a\circ b)\Box c$.

Definition 6 (Noncommutative Jordan algebra) *Let J be a vector space over a field $\mathbb{K}$ of the characteristic different from two. We say that J is a* ***noncommutative Jordan algebra*** *if over J is defined a $\mathbb{K}$-bilinear operation $\cdot : J\times J\to J$ satisfying the conditions*

$$x\cdot(y\cdot x) = (x\cdot y)\cdot x \tag{NJ1}$$

$$\left(x^2\cdot y\right)\cdot x = x^2\cdot(y\cdot x), \tag{NJ2}$$

where $x^2 = x\cdot x$, for all $x,y\in J$. The condition (NJ1) is the so-called ***flexible law*** *and the condition (NJ2) is the classical Jordan identity.*

The flexible law is a weaker condition than the commutative condition. The commutative condition implies the flexible law, then any Jordan algebra is a noncommutative Jordan algebra. Lie algebras and associative algebras are examples of the noncommutative Jordan algebras (see [S]).

Remark 7 *In the beginning, the flexible law is not possible to obtain from the Jordan di-structure axioms. The definitions of a noncommutative Jordan algebra and Jordan di-structure aren't equivalent. But it follows from proposition 4 that any noncommutative Jordan algebra is a Jordan di-structure.*

Of the previous affirmation we have that the central condition in di-structures of the Jordan type is the Jordan di-identity, so we introduce a more general di-structure of the Jordan type.

Definition 8 (Generalized Jordan di-structure) *Let J be a vector space over a field $\mathbb{K}$ the characteristic different from two. We say that J is a* ***generalized Jordan di-structure*** *over $\mathbb{K}$ if over J are defined two $\mathbb{K}$-bilinear operations $\circ,\Box : J\times J\to J$ satisfying the identity*

$$\left(x_\circ^\Box \circ y\right)\Box x = x_\circ^\Box \circ (y\Box x),$$

where $x_\circ^\Box = \frac{1}{2}(x\circ x + x\Box x)$, for all $x,y\in J$.

Remark 9 *Associative, Jordan, Lie, noncommutative Jordan algebras and Jordan di-structures are examples of generalized Jordan di-structures.*

Let $(J, \circ, \Box)$ be a generalized Jordan di-structure. The Jordan di-identity is equivalent to

$$\left[R(x), L\left(x_{\circ}^{\Box}\right)\right] = 0. \tag{4}$$

Linearization of (4), if the base field contains at least 3 elements, yields

$$[R(a), L(b \circledast c)] + [R(b), L(a \circledast c)] + [R(c), L(a \circledast b)] = 0 \tag{5}$$

or, equivalently,

$$(a, d, b \circledast c) + (b, d, a \circledast c) + (c, d, a \circledast b) = 0, \tag{6}$$

where (a, b, c) is the di-associator $(a, b, c) = a \circ (b \Box c) - (a \circ b) \Box c$ and $a \circledast b = a \circ b + b \circ a + a \Box b + b \Box a$.

3 Jordan di-structures and dialgebras

J.L. Loday introduced the notion of dialgebra (see [L1]), which is a generalization of an associative algebra possessing two operations. In this section, we prove that any dialgebra is a Jordan di-structure and is a noncommutative Jordan algebra. We recall this notion:

Definition 10 (Dialgebra) *A* ***diassociative algebra*** *or* ***dialgebra****, over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $(D, \vdash, \dashv)$ equipped with two $\mathbb{K}$-bilinear products $\dashv, \vdash : D \times D \to D$ are associative and satisfy*

$$x \dashv (y \dashv z) = x \dashv (y \vdash z) \tag{D1}$$

$$x \vdash (y \dashv z) = (x \vdash y) \dashv z \tag{D2}$$

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z \tag{D3}$$

The products $\dashv$ and $\vdash$ are called respectively the ***left product*** *and the* ***right product****.*

Example 11 (Associative algebras) *If A is an associative algebra, then the formulas $x \dashv y = xy = x \vdash y$ define a structure of dialgebra on A.*

Example 12 *Let E be a vector space and fix $\varphi \in E'$ (the algebraic dual), then one can define a dialgebra structure on E by setting*

$$x \dashv y = \varphi(y)\, x \qquad \text{and} \qquad x \vdash y = \varphi(x)\, y.$$

Example 13 (Differential associative algebra) *Let (A, d) be a differential associative algebra. So, by hypothesis, $d(ab) = da\ b + a\ db$ (here we work in the non-graded setting) and $d^2 = 0$. Define left and right products on A by the formulas*

$$x \dashv y := x\ dy \qquad \text{and} \qquad x \vdash y := dx\ y\ .$$

It is simple to check that A equipped with these two products is a dialgebra.

Proposition 14 *Let $(D, \dashv, \vdash)$ be a dialgebra over a field $\mathbb{K}$. If over D we define the applications $\circ, \square : D \times D \to D$ for*

$$x \circ y := x \vdash y + y \dashv x \qquad \text{and} \qquad x \square y := x \dashv y + y \vdash x,$$

for all $x, y \in D$. Then $(D, \circ, \square)$ is a Jordan di-structure.

Proof. Bilinearity and di-commutativity are evident from the bilinearity of the product $\vdash$ and $\dashv$ and of the definition of $\circ$ and $\square$.

The Jordan di-identity is proved using the equations $x^2 = x \vdash x + x \dashv x$ and the dialgebra axioms, of the following form

$$\begin{aligned}
(x^2 \circ y)\square x &= [(x \vdash x + x \dashv x) \vdash y + y \dashv (x \vdash x + x \dashv x)]\, \square x \\
&= [(x \vdash x) \vdash y + (x \dashv x) \vdash y + y \dashv (x \vdash x) + y \dashv (x \dashv x)]\, \square x \\
&= 2\, [(x \vdash x) \vdash y]\, \square x + 2\, [y \dashv (x \dashv x)]\, \square x \\
&= 2\, [(x \vdash x) \vdash (y \dashv x) + (x \vdash x) \vdash (x \vdash y)] \\
&\quad +2\, [(y \dashv x) \dashv (x \dashv x) + (x \vdash y) \dashv (x \dashv x)]
\end{aligned}$$

and

$$\begin{aligned}
x^2 \circ (y \square x) &= x^2 \circ (y \dashv x + x \vdash y) \\
&= [(x \vdash x) \vdash (y \dashv x) + (x \dashv x) \vdash (y \dashv x)] \\
&\quad + [(x \vdash x) \vdash (x \vdash y) + (x \dashv x) \vdash (x \vdash y)] \\
&\quad + [(y \dashv x) \dashv (x \vdash x) + (y \dashv x) \dashv (x \dashv x)] \\
&\quad + [(x \vdash y) \dashv (x \vdash x) + (x \vdash y) \dashv (x \dashv x)] \\
&= 2\, [(x \vdash x) \vdash (y \dashv x) + (x \vdash x) \vdash (x \vdash y)] \\
&\quad +2\, [(y \dashv x) \dashv (x \dashv x) + (x \vdash y) \dashv (x \dashv x)]
\end{aligned}$$

Then $(x^2 \circ y)\square x = x^2 \circ (y \square x)$ and this implies that $(D, \circ, \square)$ is a Jordan di-structure. ■

The applications $\circ$ and $\square$, in the previous proposition, satisfy the flexible laws

$$x \circ (y \circ x) = (x \circ y) \circ x$$

$$x \square (y \square x) = (x \square y)\, \square x$$

Additionally, these applications satisfy the identities

$$\begin{aligned}
x \circ (x \Box y) &= x \Box (x \circ y) \\
(y \circ x) \Box x &= (y \Box x) \circ x \\
(x \circ y) \circ x &= x \Box (y \Box x) \\
(x \Box y) \Box x &= x \circ (y \circ x)
\end{aligned}$$

but do not satisfy the identity

$$x \circ (y \Box x) = (x \circ y) \Box x.$$

The previous identity is a possible generalization of the flexible law with two products.

On the other hand, the applications $\circ$ and $\Box$ satisfy the special Jordan identity

$$\begin{aligned}
x^2 \circ (x \circ y) &= x \circ \left(x^2 \circ y\right) \\
(y \Box x) \Box x^2 &= \left(y \Box x^2\right) \Box x.
\end{aligned}$$

It is shown in [A] that flexibility implies that the classical Jordan identity $x^2(xy) = \left(x^2 y\right) x$ is equivalent to any of the following special Jordan identities

$$\begin{aligned}
x^2 (xy) &= x \left(x^2 y\right) \\
(xy) x^2 &= x \left(y x^2\right) \\
(yx) x^2 &= \left(y x^2\right) x.
\end{aligned}$$

According to the previous discussion, we have the following lemma:

Lemma 15 *Let $(D, \vdash, \dashv)$ be a dialgebra over a field $\mathbb{K}$. If we define the applications $\circ, \Box : D \times D \to D$ for*

$$x \circ y := x \vdash y + y \dashv x \qquad \textit{and} \qquad x \Box y := x \dashv y + y \vdash x,$$

for all $x, y \in D$, then $(D, \circ)$ and $(D, \Box)$ are noncommutative Jordan algebras.

Proposition 14 and Lemma 15 say that any dialgebra is a Jordan di-structure and is a noncommutative Jordan algebra. This implies that any dialgebra is a generalized Jordan di-structure. But if we take $x = a^2$, $y = b$ and $z = a$ in the axiom (D2) of dialgebras, we have that $(D, \vdash, \dashv)$ is a generalized Jordan di-structure.

Finally, we are going to see a particular construction of dialgebras, noncommutative Jordan algebras and Jordan di-structures, beginning with the same vector space.

Let $\mathbb{V}$ be a vector space and let $GL(\mathbb{V})$ be a general linear group of $\mathbb{V}$ i.e. the group of all $T \in Hom(\mathbb{V})$ such that T is invertible. We take T and S two applications from $\mathbb{V}$ in $GL(\mathbb{V})$ such that T and S send x to T_x and S_x in $GL(\mathbb{V})$, respectively.

We define the products $\vdash, \dashv : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ by

$$\begin{aligned} x \quad \vdash \quad y &:= T_x(y) \\ x \quad \dashv \quad y &:= S_x(y). \end{aligned}$$

If we suppose that $(\mathbb{V}, \vdash, \dashv)$ is a dialgebra, we have from the associativity of the products $\vdash, \dashv$ that

$$\begin{aligned} T_x T_y &= T_{T_x(y)} \\ S_x S_y &= S_{S_x(y)}. \end{aligned}$$

The axiom (D1) the dialgebras, i.e.

$$x \dashv (y \vdash z) = x \dashv (y \dashv z),$$

implies that $T = S$.

We obtain from the other two axioms of dialgebras that

$$T = S$$

and

$$S_{T_x(y)} = T_x S_y.$$

The previous discussion implies that $(\mathbb{V}, \vdash, \dashv)$ is a dialgebra if $T = S$ and $T_{T_x(y)} = T_x T_y$, for all $x, y \in \mathbb{V}$. But the products $\vdash, \dashv$ are equal and this imply that $(\mathbb{V}, \cdot)$, for $\cdot : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ defined by $x \cdot y = T_x(y)$, is an algebra that is associative and is not commutative.

Now, if we suppose that $(\mathbb{V}, \cdot)$ is a noncommutative Jordan algebra, under the product $\cdot : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ defined by

$$x \cdot y = T_x(y),$$

we obtain from the flexible law the identity

$$(T_x T_y)(x) = T_{T_x(y)}(x)$$

and from the Jordan identity the identity

$$(T_{x^2} T_y)(x) = T_{T_{x^2}(y)}(x),$$

for all $x, y \in \mathbb{V}$.

On the other hand, we define the products $\circ, \Box : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ by

$$\begin{aligned} x \circ y &= T_x(y) \\ x \Box y &= S_x(y) \end{aligned}$$

and if we suppose that $(\mathbb{V}, \circ, \Box)$ is a Jordan di-structure, we have of the di-commutative axiom that

$$T_x(y) = S_y(x), \qquad \text{for all} \quad x, y \in \mathbb{V}.$$

In particular, if we put $y = x$ then

$$T_x(x) = S_x(x), \qquad \text{for all} \quad x \in \mathbb{V}.$$

The Jordan di-identity implies that

$$S_{T_{x^2}(y)}(x) = (T_{x^2} S_y)(x).$$

In summary, we have that $(\mathbb{V}, \circ, \Box)$ is a Jordan di-structure if the applications T and S satisfy the conditions:

$$\begin{aligned} T_x(y) &= S_y(x) \\ S_{T_{x^2}(y)}(x) &= (T_{x^2} \circ S_y)(x), \end{aligned}$$

for all $x, y \in \mathbb{V}$. In particular, the products $\circ$ and $\Box$ are not equals in general. If the products $\circ$ and $\Box$ are equals, i.e. $T = S$, we have that $(\mathbb{V}, \circ)$ is a noncommutative Jordan algebra.

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