Division pairs: a new approach to Moufang sets

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Abstract. The concept of division pairs, a non-abelian version of Jordan division pairs, is introduced and a categorical equivalence between division pairs and Moufang sets is established. This is used to explain the non-uniqueness occurring in the description of Moufang sets in terms of pairs (U, τ) initiated by De Medts and Weiss.

Key Words: Moufang set, division pair

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Introduction

Moufang sets were first introduced by J. Tits [**10**] in the context of twin buildings and have since become the object of intense study. De Medts and Weiss [**8**] have initiated a description of Moufang sets in terms of pairs (U, τ) where U is a group and τ is a bijection of the set of non-trivial elements of U onto itself. The pair (U, τ) has to satisfy suitable conditions (the so-called Hua maps have to be group homomorphisms) for the construction to yield a Moufang set, denoted $\mathbb{M}(U, \tau)$.

This is simple and elegant but there remains a vexing problem: the Moufang set $\mathbb{M}(U, \tau)$ does not determine the data (U, τ) uniquely, see, e.g., [**5**, Section 1], [**6**, Remark 3.1]. Therefore, it seems that (U, τ) contains a "hidden variable" which is responsible for this non-uniqueness. One of the aims of the present note is to uncover this hidden variable, and at the same time give a construction of Moufang sets which does not suffer from this indeterminacy.

The solution offered here describes Moufang sets in terms of algebraic objects called *division pairs*. A division pair is a pair $V = (V^+, V^-)$ of groups, not necessarily abelian but written additively, together with a pair $j = (j_+, j_-)$ of bijective maps $j_\sigma\colon V^\sigma - \{0\} \to V^{-\sigma} - \{0\}$ (for $\sigma \in \{+, -\}$) which are inverses of each other ($j_- = j_+^{-1}$) and satisfy the condition that the R-operators (an analogue of the Hua maps) be group homomorphisms. At first glance, this seems more complicated than the approach via (U, τ) inasmuch as the single group U is replaced by two groups V^+ and V^-. However, it achieves the goal of modeling Moufang sets more closely: there is a categorical equivalence between division pairs and based Moufang sets (Theorem 3.5), that is, Moufang sets in which a pair of base points has been chosen. (The choice of two base points seems quite natural in view of the fact that Moufang sets can be considered as generalized projective lines and a basis of projective n-space consists of $n + 1$ points.)

The indeterminacy of (U, τ) is now explained as follows. Pairs (U, τ) which yield a Moufang set are categorically equivalent to triples (V, j, λ) where (V, j) is a division pair and $\lambda\colon V^+ \to V^-$ is an arbitrary group isomorphism, not in any way connected to j (Theorem 4.4). Thus λ appears to be the searched-for hidden variable.

The pair approach taken here is inspired by the theory of Jordan pairs [**3**] and in fact, Jordan division pairs are examples of division pairs in the present sense (Proposition 1.11).

A second aim of the present paper is to develop the theory in a categorical framework; in particular, to stress the role of morphisms throughout. This topic does not seem to have attracted much attention so far in the literature, although sub-Moufang sets have appeared in [**9**] and there is an unpublished note by De Medts [**5**]. We adopt De Medts' first "bad" definition of morphism [**5**, (2.1)] and show that it is equivalent to his second "better" definition [**5**, Definition 3.1].

Here is a more detailed description of the contents. Section 1 introduces the concept of pre-division pairs and division pairs, the latter being characterized by the condition that the R-operators, defined in analogy to the theory of Jordan division pairs by a Hua-type formula (Lemma 1.5), be group homomorphisms. This follows, in the pair context, the idea of De Medts and Weiss. We also give a short and simple proof, using the quasi-inverse in Jordan pairs, that a Jordan division pair is a division pair in the present sense (Lemma 1.9 and Proposition 1.11).

In Section 2, we first consider pre-Moufang sets, defined by satisfying the first of Tits' axioms. Thus a pre-Moufang set is a set X together with a map $\mathscr{U}$ from X to the set of subgroups of the symmetric group of X with the property that $\mathscr{U}(p)$ is simply transitive on $X - \{p\}$ for all $p \in X$. The groups $\mathscr{U}(p)$ need not be in any way related; in fact, there are examples where these groups are pairwise non-isomorphic. Nevertheless, pre-Moufang sets are useful, not least because morphisms between them make sense. These are defined as injective maps $f\colon X \to X'$ satisfying $f \circ \mathscr{U}(p) \subset \mathscr{U}'(f(p)) \circ f$, for all $p \in X$. Moufang sets are are then singled out by the condition that all $\mathscr{U}(p)$ consist of automorphisms. It is natural that morphisms should be injective maps since Moufang sets are parametrized by "division objects" (skew fields, Jordan division algebras, division pairs) and there, too, morphisms are either trivial or injective.

It is helpful to introduce an intermediate stage between pre-Moufang and Moufang sets, called half-Moufang sets. Here we consider based pre-Moufang sets with base points b^+ and b^-, and define a half-Moufang set of type $+$ (resp. $-$) by requiring that $\mathscr{U}(b^+)$ (resp. $\mathscr{U}(b^-)$) consist of automorphisms. Then Moufang sets are characterized by being half-Moufang of both types (2.6).

In Section 3, we define functors $\mathbb{D}$ from half-Moufang sets to pre-division pairs (Lemma 3.1) and $\mathbb{P}$ from pre-division pairs to half-Moufang sets (Lemma 3.3) and show in Proposition 3.4 that $\mathbb{D}$ and $\mathbb{P}$ are, up to natural isomorphism, inverses of each other. By restricting these functors to division pairs and Moufang sets, respectively, we obtain our main result (Theorem 3.5), the categorical equivalence of Moufang sets and division pairs.

In the last Section 4, we relate our approach with the construction of Moufang sets from pairs (U, τ), as outlined above. We also discuss the opposite and mirror Moufang set of [**7**] as well as the Hua maps and μ-maps in our setting, and show that the two concepts of morphism introduced by De Medts agree.

Notation. The inner automorphism determined by an element g of a group G is $\mathrm{Int}(g)\colon h \mapsto ghg^{-1}$. The set of non-trivial elements of G is denoted $\dot{G}$ and the set of subgroups of G is $\mathrm{sbgr}(G)$. The symmetric group of a set X is denoted $\mathrm{Sym}(X)$, it acts on X on the left. The index σ takes values in $\{+, -\}$ where we regard $+$ and $-$ as abbreviations for $+1$ and -1, respectively. The meaning of $-\sigma$ is then the obvious one.

§1. Division pairs

1.1. Pre-division pairs. Let $V = (V^+, V^-)$ be a pair of groups. Even though the V^σ are not assumed to be abelian, we denote the group law by $+$, the neutral elements by 0 or 0^σ and put $\dot{V}^\sigma = V^\sigma - \{0\}$.

A *pre-division pair* is a pair V as above together with a pair of maps $j_\sigma\colon \dot{V}^\sigma \to \dot{V}^{-\sigma}$ which are inverses of each other:

$$j_{-\sigma} \circ j_\sigma = \mathrm{Id}_{V^\sigma} \quad (\sigma \in \{+, -\}). \tag{1}$$

By abuse of language, we often speak of V as of a pre-division pair, the maps j_σ being understood from the context.

It is clear that the groups V^+ and V^- of a pre-division pair have the same cardinality; however, they need not be isomorphic: just take two non-isomorphic groups V^+ and V^- of the same cardinality and let $j_+\colon \dot V^+ \to \dot V^-$ be a bijection with inverse j_-.

Let V and W be pre-division pairs. A *homomorphism* $h = (h_+, h_-)\colon V \to W$ is a pair of injective group homomorphisms $h_\sigma\colon V^\sigma \to W^\sigma$ such that, for all $\sigma \in \{+,-\}$ and all $x \in \dot V^\sigma$, $h_\sigma(x) \in \dot W^\sigma$ and

$$h_{-\sigma}(j_\sigma(x)) = j_\sigma(h_\sigma(x)). \tag{2}$$

With these definitions, pre-division pairs form a category, denoted **pre-div**. We remark that a homomorphism h is uniquely determined by h_+ (or h_-), since $h_-(0) = 0$ and $h_-(y) = j_+(h_+(j_-(y)))$ for all $y \in \dot V^-$. By the same argument, it suffices to require (2) for $\sigma = +$ or $\sigma = -$.

The *opposite* of (V, j) is the pre-division pair $(V, j)^{\rm op} = ((V^-, V^+), (j_-, j_+))$. For a morphism $h = (h_+, h_-)\colon V \to W$ of pre-division pairs, let $h^{\rm op} = (h_-, h_+)\colon V^{\rm op} \to W^{\rm op}$. Then $(\ )^{\rm op}$ is a functor from **pre-div** to itself whose square is the identity. This definition is modeled on the analogous one for Jordan pairs [**3**, 1.5].

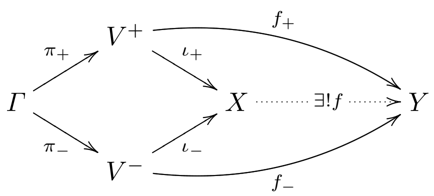
1.2. The functor $\mathbb{X}$. Let V be a pre-division pair. Denote by

$$\Gamma = \{(x, y) \in \dot V^+ \times \dot V^- : y = j_+(x)\}$$

the graph of j_+, and let $\pi_\sigma\colon \Gamma \to \dot V^\sigma \subset V^\sigma$ be the projections followed by the inclusions. Let

$$X = \mathbb{X}(V) = V^+ \sqcup_\Gamma V^-$$

be the amalgamated sum (pushout) of V^+ and V^- over Γ, with canonical maps $\iota_\sigma\colon V^\sigma \to X$. Thus $\iota_+ \circ \pi_+ = \iota_- \circ \pi_-$, and (X, ι_+, ι_-) has the following property: for every pair of set maps $f_+\colon V^+ \to Y$ and $f_-\colon V^- \to Y$ satisfying $f_+ \circ \pi_+ = f_- \circ \pi_-$, there exists a unique map $f\colon X \to Y$ such that $f \circ \iota_\sigma = f_\sigma$:



(1)

The triple (X, ι_+, ι_-) is uniquely determined up to unique isomorphism by this property. The construction is well-known: X is the quotient of the disjoint union of V^+ and V^- by the equivalence relation which is trivial on V^σ and identifies $x \in \dot V^+$ with $j_+(x) \in \dot V^-$. The maps ι_σ are then the inclusions of V^σ into the disjoint union followed by the canonical map into the quotient. They are injective and satisfy

$$\iota_\sigma(x) = \iota_{-\sigma}(y) \iff x \neq 0 \text{ and } y = j_\sigma(x) \iff y \neq 0 \text{ and } x = j_{-\sigma}(y). \tag{2}$$

We put

$$o^\sigma = \iota_{-\sigma}(0_{V^{-\sigma}}) \tag{3}$$

(note the change of sign), called the *base points* of X. Then

$$X = \iota_+(V^+) \,\dot\cup\, \{o^+\} = \iota_-(V^-) \,\dot\cup\, \{o^-\}. \tag{4}$$

Thus o^σ could be interpreted as a point at infinity added to $\iota_\sigma(V^\sigma)$.

Let $h\colon V \to W$ be a morphism of pre-division pairs and let $Y = \mathbb{X}(W)$. Then the maps $f_\sigma = \iota_\sigma \circ h_\sigma\colon V^\sigma \to W^\sigma \to Y$ make the outer diagram in (1) commutative, because for $(x,y) \in \Gamma$, we have

$$\begin{aligned}(f_+ \circ \pi_+)(x,y) &= \iota_+(h_+(x)) = \iota_-(j_+(h_+(x))) \\ &= \iota_-(h_-(j_+(x))) = \iota_-(h_-(y)) = (f_- \circ \pi_-)(x,y).\end{aligned}$$

Hence there exists a unique map $f = \mathbb{X}(h)\colon \mathbb{X}(V) \to \mathbb{X}(W)$ making (1) commutative, given by

$$f(\iota_\sigma(z)) = \iota_\sigma(h_\sigma(z)),$$

for all $z \in V^\sigma$, $\sigma = \pm$. It is clear that f preserves the base points. Also, since h_σ is injective, it follows from (4) that so is f. One sees immediately that these assignments define a functor $\mathbb{X}$ from pre-division pairs to sets.

Let $V^{\mathrm{op}} = (V^-, V^+)$ be the opposite of V as in 1.1. Then $\mathbb{X}(V) = \mathbb{X}(V^{\mathrm{op}})$ as sets. However, the base points of $\mathbb{X}(V)$ and $\mathbb{X}(V^{\mathrm{op}})$ are interchanged.

1.3. Lemma. *There is a unique homomorphism $t_\sigma\colon V^\sigma \to \mathrm{Sym}(X)$ such that*

$$t_\sigma(u) \cdot o^\sigma = o^\sigma, \qquad t_\sigma(u) \cdot \iota_\sigma(x) = \iota_\sigma(u + x), \tag{1}$$

for all $u, x \in V^\sigma$. The homomorphism t_σ is injective, and the group

$$T^\sigma := t_\sigma(V^\sigma) \subset \mathrm{Sym}(X)$$

acts simply transitively on $X - \{o^\sigma\}$. The t_σ are compatible with morphisms in the following sense: if $h\colon V \to W$ is a morphism of pre-division pairs and $f = \mathbb{X}(h)$ then

$$f \circ t_\sigma(u) = t_\sigma(h_\sigma(u)) \circ f \tag{2}$$

for all $u \in V^\sigma$.

Proof. From (1.2.4) it is clear that t_σ is uniquely determined by (1). To prove existence of t_+, define $f_+\colon V^+ \to X$ and $f_-\colon V^- \to X$ by

$$f_+(x) = \iota_+(u + x), \qquad f_-(y) = \begin{Bmatrix} o^+ & \text{if } y = 0 \\ \iota_+(u + j_-(y)) & \text{if } y \neq 0 \end{Bmatrix}.$$

Then the outer square of (1.2.1) is commutative: for $(x,y) \in \Gamma$ we have $f_+(\pi_+(x,y)) = \iota_+(u+x)$ and, since $y = j_+(x)$ and hence $x = j_-(y)$,

$$f_-(\pi_-(x,y)) = f_-(y) = \iota_+(u + j_-(y)) = \iota_+(u + x).$$

Hence there exists a unique map $t_+(u)\colon X \to X$ satisfying (1) for $\sigma = +$. Since $\iota_+\colon V^+ \to X - \{o^+\}$ is bijective, it is clear that t_+ is injective and that $t_+(V^+)$ acts simply transitively on $X - \{o^+\}$. By interchanging the roles of $+$ and $-$, we obtain the result for $\sigma = -$. Finally, (2) follows easily from the fact that the h_σ are group homomorphisms. □

1.4. Definition. Let V be a pre-division pair. We denote the group inverse in V^σ by $i(x) = -x$. For an element $x \in \dot{V}^\sigma$, we often write simply $j(x)$ instead $j_\sigma(x)$, so that formally $j(j(x)) = x$. We also define

$${}^\vee x = i(j(x)) = -j(x), \qquad x^\vee = j(i(x)) = j(-x).$$

Thus both ${}^\vee x$ and $x^\vee$ belong to $\dot{V}^{-\sigma}$, and we have

$${}^\vee(x^\vee) = -j(j(-x)) = x = j(-(-j(x))) = ({}^\vee x)^\vee.$$

Following a terminology introduced by Baumeister and Grüninger [**1**], an element $x \in V^\sigma$ is called *special* if $ij(x) = ji(x)$, equivalently, if ${}^\vee x = x^\vee$.

We define $w_\sigma(x) \in \mathrm{Sym}(X)$ by

$$w_\sigma(x) = t_{-\sigma}({}^\vee x) \circ t_\sigma(x) \circ t_{-\sigma}(x^\vee). \tag{1}$$

It follows easily from the definitions that

$$w_\sigma(x)^{-1} = w_\sigma(-x). \tag{2}$$

If $h\colon V \to V'$ is a homomorphism of pre-division pairs, then

$$h_{-\sigma}({}^\vee x) = {}^\vee h_\sigma(x), \qquad h_{-\sigma}(x^\vee) = h_\sigma(x)^\vee. \tag{3}$$

Hence (1.3.2) implies that the induced map $f\colon \mathbb{X}(V) \to \mathbb{X}(V')$ makes the diagram

$$\begin{CD} \mathbb{X}(V) @>{f}>> \mathbb{X}(V') \\ @V{w_\sigma(x)}VV @VV{w_\sigma(h_\sigma(x))}V \\ \mathbb{X}(V) @>>{f}> \mathbb{X}(V') \end{CD} \tag{4}$$

commutative.

1.5. Lemma. *Let V be a pre-division pair. We use the notations introduced in* 1.3 *and* 1.4.

(a) *$w_\sigma(x)$ interchanges o^σ and $o^{-\sigma}$.*

(b) *There exists a unique bijection $R_x\colon V^{-\sigma} \to V^\sigma$ making the diagram*

$$\begin{CD} V^{-\sigma} @>{R_x}>> V^\sigma \\ @V{\iota_{-\sigma}}VV @VV{\iota_\sigma}V \\ X @>>{w_\sigma(x)}> X \end{CD} \tag{1}$$

commutative. Explicitly, $R_x y$ is given as follows. If $y = 0$ then $R_x y = 0$. If $x^\vee + y = 0$ then

$$R_x y = j(i(j(x))) = j(x)^\vee = j({}^\vee x). \tag{2}$$

If $y \neq 0$ and $x^\vee + y \neq 0$ then also $x + j(x^\vee + y) \neq 0$ and ${}^\vee x + j(x + j(x^\vee + y)) \neq 0$, and

$$R_x y = j\Big({}^\vee x + j\big(x + j(x^\vee + y)\big)\Big). \tag{3}$$

(c) *If $h\colon V \to W$ is a morphism of pre-division pairs, then the R-maps are compatible with h in the sense that*

$$h_\sigma(R_x(y)) = R_{h_\sigma(x)}(h_{-\sigma}(y)). \tag{4}$$

Proof. (a) We have $x + j(x^\vee) = x + j(j(-x)) = x - x = 0$. Hence by Lemma 1.3 and (1.2.2),

$$\begin{aligned} w_\sigma(x) \cdot o^\sigma &= t_{-\sigma}({}^\vee x) \cdot (t_\sigma(x) \cdot (t_{-\sigma}(x^\vee) \cdot \iota_{-\sigma}(0))) = t_{-\sigma}({}^\vee x) \cdot (t_\sigma(x) \cdot \iota_{-\sigma}(x^\vee)) \\ &= t_{-\sigma}({}^\vee x) \cdot (t_\sigma(x) \cdot \iota_\sigma(j(x^\vee))) = t_{-\sigma}({}^\vee x) \cdot \iota_\sigma(x + j(x^\vee)) \\ &= t_{-\sigma}({}^\vee x) \cdot \iota_\sigma(0) = t_{-\sigma}({}^\vee x) \cdot o^{-\sigma} = o^{-\sigma}. \end{aligned}$$

This implies $w_\sigma(-x)\cdot o^{-\sigma} = w_\sigma(x)^{-1}\cdot o^{-\sigma} = o^\sigma$ by (1.4.2), so replacing x by $-x$ shows $w_\sigma(x)\cdot o^{-\sigma} = o^\sigma$.

(b) By (a), $w_\sigma(x)$ maps $\iota_{-\sigma}(\dot V^{-\sigma}) = X - \{o^+, o^-\} = \iota_\sigma(\dot V^\sigma)$ bijectively onto itself. Hence there exists a unique bijection R_x making the diagram commutative and mapping $0_{V^{-\sigma}}$ to 0_{V^σ}. To prove (2) and (3), let first $x^\vee + y = 0$. Then

$$\begin{aligned} w_\sigma(x)\cdot\iota_{-\sigma}(y) &= t_{-\sigma}({}^\vee x)\cdot(t_\sigma(x)\cdot\iota_{-\sigma}(x^\vee + y)) \\ &= t_{-\sigma}({}^\vee x)\cdot(t_\sigma(x)\cdot o^\sigma) = t_{-\sigma}({}^\vee x)\cdot o^\sigma = \iota_{-\sigma}({}^\vee x) \\ &= \iota_\sigma(j({}^\vee x)) = \iota_\sigma(j(i(j(x)))) = \iota_\sigma(R_x y). \end{aligned}$$

Now let ${}^\vee x + y \neq 0$ and assume, for a contradiction, that $x + j(x^\vee + y) = 0$. Then $-x = j(x^\vee + y)$, whence $x^\vee = j(-x) = x^\vee + y$ and therefore $y = 0$, which is impossible. We also have ${}^\vee x + j(x + j(x^\vee + y)) \neq 0$, else $-({}^\vee x) = j(x + j(x^\vee + y))$ which implies, by applying j, that $j(-({}^\vee x)) = ({}^\vee x)^\vee = x = x + j(x^\vee + y)$ and therefore $j(x^\vee + y) = 0$, contradiction. Now we compute, using (1.2.2) repeatedly:

$$\begin{aligned} w_\sigma(x)\cdot\iota_{-\sigma}(y) &= t_{-\sigma}({}^\vee x)\cdot\big(t_\sigma(x)\cdot(\iota_{-\sigma}(x^\vee + y))\big) \\ &= t_{-\sigma}({}^\vee x)\cdot\big(t_\sigma(x)\cdot\iota_\sigma(j(x^\vee + y))\big) = t_{-\sigma}({}^\vee x)\cdot\iota_\sigma(x + j(x^\vee + y)) \\ &= t_{-\sigma}({}^\vee x)\cdot\iota_{-\sigma}(j(x + j(x^\vee + y))) = \iota_{-\sigma}({}^\vee x + j(x + j(x^\vee + y))) \\ &= \iota_\sigma(j({}^\vee x + j(x + j(x^\vee + y)))) = \iota_\sigma(R_x y). \end{aligned}$$

(c) This follows easily from (1.3.2) and (1.4.4). □

1.6. Lemma. *We keep the notation introduced earlier. For an element $x \in \dot V^\sigma$ the following conditions are equivalent:*

$$\begin{aligned} &R_x\colon V^{-\sigma} \to V^\sigma \quad \text{is a group homomorphism,} && (1)\\ &w_\sigma(x)\circ t_{-\sigma}(v)\circ w_\sigma(x)^{-1} = t_\sigma(R_x v) \quad \text{for all } v \in V^{-\sigma}, && (2)\\ &\operatorname{Int}(w_\sigma(x))\,T^{-\sigma} = T^\sigma, && (3)\\ &\operatorname{Int}(t_\sigma(x))\,T^{-\sigma} = \operatorname{Int}(t_{-\sigma}(j(x)))\,T^\sigma. && (4) \end{aligned}$$

Proof. (1) $\Longrightarrow$ (2): We prove (2) in the equivalent form

$$w_\sigma(x)\circ t_{-\sigma}(v) = t_\sigma(R_x v)\circ w_\sigma(x).$$

This is an equation in $\operatorname{Sym}(X)$, so we must show that both sides yield the same result when applied to an arbitrary point $p \in X$. As $X = \{o^{-\sigma}\} \dot\cup \iota_{-\sigma}(V^{-\sigma})$ by (1.2.4), there are two cases: if $p = o^{-\sigma}$ then

$$w_\sigma(x)\cdot(t_{-\sigma}(v)\cdot o^{-\sigma}) = w_\sigma(x)\cdot o^{-\sigma} = o^\sigma$$

since $T^{-\sigma}$ fixes $o^{-\sigma}$ and $w_\sigma(x)$ switches the base points by Lemma 1.5(a). On the other hand,

$$t_\sigma(R_x v)\cdot(w_\sigma(x)\cdot o^{-\sigma}) = t_\sigma(R_x v)\cdot o^\sigma = o^\sigma$$

since $R_x v \in V^\sigma$ and T^σ fixes o^σ. If $p = \iota_{-\sigma}(y)$ for $y \in V^{-\sigma}$ then again by Lemma 1.5,

$$\begin{aligned} w_\sigma(x)\cdot(t_{-\sigma}(v)\cdot\iota_{-\sigma}(y)) &= w_\sigma(x)\cdot\iota_{-\sigma}(v + y) = \iota_\sigma(R_x(v + y)), \\ t_\sigma(R_x v)\cdot(w_\sigma(x)\cdot\iota_{-\sigma}(y)) &= t_\sigma(R_x v)\cdot\iota_\sigma(R_x y) = \iota_\sigma(R_x v + R_x y). \end{aligned}$$

(2) $\Longrightarrow$ (1) is clear from the fact that $t_\sigma\colon V^\sigma \to T^\sigma$ and conjugation by $w_\sigma(x)$ are group isomorphisms.

(2) $\Longrightarrow$ (3) holds because $R_x\colon V^{-\sigma} \to V^\sigma$ is a bijection by Lemma 1.5.

(3) $\implies$ (2): For $v \in V^{-\sigma}$ we have $w_\sigma(x) \circ t_{-\sigma}(v) \circ w_\sigma(x)^{-1} = t_\sigma(u) \in T^\sigma$. To determine u, we evaluate both sides at $o^{-\sigma}$. Then by Lemma 1.5,

$$\begin{aligned}\iota_\sigma(u) &= t_\sigma(u) \cdot o^{-\sigma} = (w_\sigma(x) \circ t_{-\sigma}(v) \circ w_\sigma(x)^{-1}) \cdot o^{-\sigma} \\ &= w_\sigma(x) \cdot (t_{-\sigma}(v) \cdot o^\sigma) = w_\sigma(x) \cdot \iota_{-\sigma}(v) = \iota_\sigma(R_x v),\end{aligned}$$

whence $u = R_x v$. Finally, (3) $\iff$ (4) follows from

$$\begin{aligned}\operatorname{Int}(w_\sigma(x))\, T^{-\sigma} = T^\sigma &\iff \operatorname{Int}(t_{-\sigma}({}^\vee x)) \operatorname{Int}(t_\sigma(x)) \operatorname{Int}(t_{-\sigma}(x^\vee)) T^{-\sigma} = T^\sigma \\ &\iff \operatorname{Int}(t_{-\sigma}({}^\vee x)) \operatorname{Int}(t_\sigma(x)) T^{-\sigma} = T^\sigma \\ &\iff \operatorname{Int}(t_\sigma(x)) T^{-\sigma} = \operatorname{Int}(t_{-\sigma}({}^\vee x)^{-1}) T^\sigma \\ &\iff \operatorname{Int}(t_\sigma(x)) T^{-\sigma} = \operatorname{Int}(t_{-\sigma}(j(x))) T^\sigma,\end{aligned}$$

since $t_{-\sigma}({}^\vee x)^{-1} = t_{-\sigma}(-({}^\vee x)) = t_{-\sigma}(-(-j(x))) = t_{-\sigma}(j(x))$. □

1.7. Definition. A *division pair* is a pre-division pair $V = (V^+, V^-)$ with the property that, for every $\sigma \in \{+,-\}$ and all $x \in \dot{V}^\sigma$ the equivalent conditions of Lemma 1.6 hold. Since (1.6.4) is invariant under the substitution $x \to j(x)$ and $\sigma \to -\sigma$, it is sufficient for V to be a division pair that R_x be a group homomorphism, for all $x \in \dot{V}^+$. We denote the full sub-category of **pre-div** whose objects are division pairs by **div**. Clearly, V is a division pair if and only if V^{op} is a division pair, so $(\ )^{\mathrm{op}}$ induces an isomorphism of the full subcategory **div** of **pre-div** onto itself.

1.8. Jordan pairs. We recall from [**3**] the notion of a Jordan pair. Let $V = (V^+, V^-)$ be a pair of modules over a commutative ring k and let $Q^\sigma\colon V^\sigma \to \operatorname{Hom}(V^{-\sigma}, V^\sigma)$ be quadratic maps. Depending on context, we write Q^σ_x or $Q^\sigma(x)$. Define bilinear maps $D^\sigma\colon V^\sigma \times V^{-\sigma} \to \operatorname{End}(V^\sigma)$ by

$$D^\sigma(x,y) \cdot z = Q^\sigma_{x+z} y - Q^\sigma_x y - Q^\sigma_z y.$$

Then V (together with the quadratic maps (Q^+, Q^-)) is called a *Jordan pair* if the following identities hold in all scalar extensions:

(JP1) $\quad D^\sigma(x,y) Q^\sigma_x = Q^\sigma_x D^{-\sigma}(y,x)$,

(JP2) $\quad D^\sigma(Q^\sigma_x y, y) = D^\sigma(x, Q^{-\sigma}_y x)$,

(JP3) $\quad Q^\sigma(Q^\sigma_x y) = Q^\sigma_x\, Q^{-\sigma}_y\, Q^\sigma_x$.

To simplify notation, we usually drop the index σ at Q^σ and D^σ. A homomorphism $h\colon V \to W$ of Jordan pairs is a pair of linear maps $h_\sigma\colon V^\sigma \to W^\sigma$ interacting correctly with the Q-operators:

$$h_\sigma(Q_x y) = Q_{h_\sigma(x)} h_{-\sigma}(y).$$

The category of Jordan pairs thus defined is denoted **jp**.

An element $x \in V^\sigma$ is called *invertible* if $Q_x\colon V^{-\sigma} \to V^\sigma$ is a module isomorphism. In this case, the inverse of $x \in V^\sigma$ is defined by

$$x^{-1} = Q_x^{-1} x \in V^{-\sigma},$$

and we have $Q_x^{-1} = Q_{x^{-1}}$, an easy consequence of (JP3). Hence $x^{-1} \in V^{-\sigma}$ is invertible with inverse $(x^{-1})^{-1} = x$. We also note

$$(Q_x y)^{-1} = Q(Q_x y)^{-1} \cdot Q_x y = Q_x^{-1} Q_y^{-1} y = Q_x^{-1} y^{-1}, \tag{1}$$

for invertible $x \in V^\sigma$ and $y \in V^{-\sigma}$. With a view towards division pairs in the sense of 1.7, we define

$$j_\sigma(x) = -x^{-1} \tag{2}$$

for invertible $x \in V^\sigma$. Then j_σ maps the invertible elements of V^σ bijectively onto the invertible elements of $V^{-\sigma}$, and $j_{-\sigma} \circ j_\sigma = \mathrm{Id}$. Also, since $Q_x = Q_{-x}$ and Q_x is a group homomorphism, we have

$$j_\sigma(-x) = -j_\sigma(x). \tag{3}$$

As before, we usually drop the index σ at j_σ.

In general, a Jordan pair will not contain any invertible elements. As a substitute, one has the notion of quasi-inverse [**3**, Section 3]. The *Bergmann operator* defined by $(x, y) \in V^\sigma \times V^{-\sigma}$ is $B(x, y) := \mathrm{Id} - D(x, y) + Q_x Q_y \in \mathrm{End}(V^\sigma)$. The pair (x, y) is said to be *quasi-invertible* if $B(x, y)$ is invertible. In this case, the *quasi-inverse* of (x, y) is

$$x^y = B(x, y)^{-1}(x - Q_x y) \in V^\sigma.$$

Quasi-invertibility is preserved under homomorphisms $h\colon V \to W$: if $(x, y) \in V^\sigma \times V^{-\sigma}$ is quasi-invertible then so is $(h_\sigma(x), h_{-\sigma}(y)) \in W^\sigma \times W^{-\sigma}$, and $h_\sigma(x^y) = h_\sigma(x)^{h_{-\sigma}(y)}$.

We now give a proof of the Hua formula for Jordan pairs using the quasi-inverse. Other proofs can be derived from the Hua formula for Jordan algebras, see for example [**4**, Section 5] and [**2**, Prop. 1.7.10]. The Hua formula for Jordan algebras was rediscovered in [**8**].

1.9. Lemma. *Let $V = (V^+, V^-)$ be a Jordan pair.*

(a) *Suppose $(x, y) \in V^\sigma \times V^{-\sigma}$ with x invertible. Then the pair (x, y) is quasi-invertible if and only if $x^{-1} - y$ is invertible, in which case*

$$x^y = (x^{-1} - y)^{-1}. \tag{1}$$

(b) *Suppose $x \in V^\sigma$ and $y \in V^{-\sigma}$ are invertible and $x^{-1} + y$ is invertible. Then also $x + j(x^{-1} + y)$ and $x^{-1} + j(x + j(x^{-1} + y))$ are invertible, and the Hua formula*

$$Q_x y = j\Big(x^{-1} + j\big(x + j(x^{-1} + y)\big)\Big) \tag{2}$$

holds.

Proof. (a) (See also [**3**, 3.13]) By [**3**, 2.12(1)],

$$B(x, y) = Q_x Q(x^{-1} - y),$$

provided x is invertible. As Q_x is invertible, we see that $B(x, y)$ is invertible if and only if $Q(x^{-1} - y)$ is invertible, that is, if and only if $x^{-1} - y$ is invertible. Furthermore,

$$\begin{aligned} x^y &= B(x, y)^{-1}(x - Q_x y) = Q(x^{-1} - y)^{-1} Q_x^{-1}(x - Q_x y) \\ &= Q(x^{-1} - y)^{-1}(x^{-1} - y) = (x^{-1} - y)^{-1}. \end{aligned}$$

(b) We use the Symmetry Principle [**3**, Prop. 3.3]: a pair $(u, v) \in V^\sigma \times V^{-\sigma}$ is quasi-invertible if and only if (v, u) is quasi-invertible, in which case

$$u^v = u + Q_u v^u. \tag{3}$$

We apply this to $u = x^{-1}$ and $v = (x^{-1} + y)^{-1}$. Here (v, u) is quasi-invertible with quasi-inverse

$$v^u = (v^{-1} - u)^{-1} = (x^{-1} + y - x^{-1})^{-1} = y^{-1}. \tag{4}$$

Hence (u, v) is quasi-invertible, so by (a), $u^{-1} - v = x - (x^{-1} + y)^{-1} = x + j(x^{-1} + y)$ is invertible, and we have

$$u^v = \big(x + j(x^{-1} + y)\big)^{-1}$$

which implies

$$-u + u^v = -\big\{x^{-1} + j\big(x + j(x^{-1} + y)\big)\big\}.$$

Then (3), (4) and (1.8.1) imply

$$Q_u v^u = Q_{x^{-1}} y^{-1} = Q_x^{-1} y^{-1} = (Q_x y)^{-1} = -\big\{x^{-1} + j\big(x + j(x^{-1} + y)\big)\big\},$$

in particular, the right hand side is invertible. Now (2) follows by taking inverses on both sides. □

Remark. The Hua formula is often formulated differently. Suppose $x \in V^\sigma$ invertible, $y \in V^{-\sigma}$ invertible and $x+j(y)$ invertible. Then also $j(-x)+j(x+j(y))$ is invertible, and the Hua formula reads

$$Q_x y = x + j\Big(j(-x) + j\big(x + j(y)\big)\Big); \tag{5}$$

see, e.g., [**4**, p. 214]. This is obtained from (2) by replacing x with x^{-1}, y with $j(y)$, applying j to both sides and observing (1.8.1).

1.10. Jordan division pairs. A *Jordan division pair* is a Jordan pair V such that every nonzero $x \in V^\sigma$ is invertible. In particular, the trivial Jordan pair $(0,0)$ is counted as a Jordan division pair.

Morphisms from a Jordan division pair are either zero or injective (or both). Indeed, let V be a Jordan division pair and let $h\colon V \to W$ a homomorphism of Jordan pairs. Suppose h_σ is not injective, so $h_\sigma(x) = 0$ for some $0 \neq x \in V^\sigma$. Since $Q_x\colon V^{-\sigma} \to V^\sigma$ is bijective, it follows that $h_\sigma(V^\sigma) = h_\sigma(Q_x V^{-\sigma}) = Q_{h_\sigma(x)} h_{-\sigma}(V^{-\sigma}) = 0$, and

$$h_{-\sigma}(V^{-\sigma}) = h_{-\sigma}(Q_{x^{-1}} Q_x V^{-\sigma}) = Q_{h_{-\sigma}(x^{-1})} Q_{h_\sigma(x)} h_{-\sigma}(V^{-\sigma}) = 0$$

as well. Hence, it makes sense to define the category of Jordan division pairs as the (non-full) subcategory **jdiv** of **jp** whose objects are Jordan division pairs and whose morphisms are *injective* Jordan pair morphisms.

1.11. Proposition. *For $V = (V^+, V^-) \in$ **jdiv** let $E(V) = (V, (j_+, j_-))$ with j_σ defined in* (1.8.2)*, and for a morphism h of* **jdiv** *let $E(h) = h$. Then $E(V)$ is a division pair in the sense of* 1.7 *and every element of $\dot V^\sigma$ is special as defined in* 1.4*. These assignments define a functor E:* **jdiv** $\to$ **div** *which is a full embedding of categories.*

Proof. It is clear that $E(V)$ is a pre-division pair and by (1.8.3) all elements are special. Let us show that the R-operators defined in (1.5.3) agree with the Q-operators of the Jordan pair V. By Lemma 1.5(b), $R_x(0) = 0$ and $Q_x(0) = 0$ as well because Q_x is a module homomorphism.

If $y \neq 0$ but ${}^\vee x + y = 0$ then $R_x y = jij(x)$ (by (1.5.2)) $= -x$ (by (1.8.3)). On the other hand, $0 = {}^\vee x + y$ implies $y = j(x) = -x^{-1}$ and therefore $Q_x y = -Q_x x^{-1} = -Q_x Q_x^{-1} x = -x$ as well. Finally, $R_x y = Q_x y$ for $y \neq 0$ and $x^{-1} + y \neq 0$ follows from (1.5.3) and (1.9.2). Since the Jordan Q-operators are in particular additive, $E(V)$ is a division pair.

Next, we show that a morphism $h\colon V \to W$ of Jordan division pairs is also a morphism of the associated division pairs, i.e., that it satisfies (1.1.2). Thus let $0 \neq x \in V^\sigma$. Since h_σ is injective by definition, $h_\sigma(x) \neq 0$ in W^σ, and since W is a Jordan division pair, it follows that $h_\sigma(x)$ is invertible. To prove $h_\sigma(x)^{-1} = h_{-\sigma}(x^{-1})$, it suffices to show that both sides yield the same result when $Q_{h_\sigma(x)}$ is applied. Now $Q_{h_\sigma(x)} h_\sigma(x)^{-1} = Q_{h_\sigma(x)} Q^{-1}_{h_\sigma(x)} h_\sigma(x) = h_\sigma(x)$, while $Q_{h_\sigma(x)} h_{-\sigma}(x^{-1}) = h_\sigma(Q_x x^{-1}) = h_\sigma(x)$, as desired.

Clearly, E is a functor which is injective on objects and morphisms. It is also full: given a morphism $h\colon E(V) \to E(W)$ of division pairs, it follows from (1.5.4) that h is a morphism of Jordan pairs. ☐

§2. Moufang sets

2.1. Pre-Moufang sets. A *pre-Moufang set* is a pair $M = (X, \mathscr{U})$ consisting of a non-empty set X and a map $\mathscr{U}\colon X \to \mathrm{sbgr}(\mathrm{Sym}(X))$ satisfying Tits's first axiom [**10**, 4.4]:

(M1) for all $p \in X$, $\mathscr{U}(p)$ acts simply transitively on $X - \{p\}$ (and therefore fixes p).

Even though the groups $\mathscr{U}(p)$ need not be abelian, we will write the group law additively and accordingly denote the neutral element of $\mathscr{U}(p)$ by 0. This makes for more readable formulas.

Since a group has at least one element, (M1) shows that X has at least two elements. On the other hand, it is clear that any two-element set has a canonical structure of a pre-Moufang set.

Let p and q be points of X and let $\zeta_{q,p}\colon \mathscr{U}(p) \to X$ be the orbit map of $\mathscr{U}(p)$ through the point q:

$$\zeta_{q,p}(u) = u \cdot q, \tag{1}$$

for all $u \in \mathscr{U}(p)$. Then (M1) shows

$$\zeta_{q,p}\colon \mathscr{U}(p) \to X - \{p\} \text{ is bijective for } q \neq p. \tag{2}$$

We denote by $\mathrm{G}(M)$ the subgroup of $\mathrm{Sym}(X)$ generated by all $\mathscr{U}(p)$, $p \in X$. It is easy to see that $\mathrm{G}(M)$ acts doubly transitively on X provided X has at least three elements.

A *morphism* $f\colon (X, \mathscr{U}) \to (X', \mathscr{U}')$ of pre-Moufang sets is an *injective* map $f\colon X \to X'$ such that, for all $p \in X$,

$$f \circ \mathscr{U}(p) \subset \mathscr{U}'(f(p)) \circ f \tag{3}$$

(as subsets of all maps from X to X'). With these definitions, pre-Moufang sets form a category, denoted **pre-mou**. Isomorphisms and automorphisms of pre-Moufang sets are now defined as in any category, and we denote by $\mathrm{Aut}(M) \subset \mathrm{Sym}(X)$ the automorphism group of M.

In more detail, (3) says: for all $p \in X$ and all $u \in \mathscr{U}(p)$, there exists $u' \in \mathscr{U}'(f(p))$ such that, for all $q \in X$,

$$f(u \cdot q) = u' \cdot f(q). \tag{4}$$

The condition that f be injective may seem unnecessarily restrictive but it only excludes constant maps. Indeed, suppose f satisfies (3) and is not injective. Then there exist $p \neq q$ in X with $f(p) = f(q)$. Let $s \in X - \{p\}$. By (M1), there exists $u \in \mathscr{U}(p)$ such that $s = u \cdot q$ and by (3), there exists $u' \in \mathscr{U}'(f(p))$ such that (4) holds. Hence $f(s) = f(u \cdot q) = u' \cdot f(q) = u' \cdot f(p) = f(p)$ since u' fixes $f(p)$.

For Moufang sets, this definition of morphism as well as the fact that a non-constant map satisfying (3) must be injective are due to De Medts [**5**, Section 2]. Although called there "a bad attempt", it turns out to be the good definition. It is also equivalent to De Medts' second definition [**5**, Section 3], see 4.8.

2.2. Lemma. *Let $f\colon (X, \mathscr{U}) \to (X', \mathscr{U}')$ be a morphism of pre-Moufang sets and let $p \in X$. Then there exists a unique injective group homomorphism $\partial_p f\colon \mathscr{U}(p) \to \mathscr{U}'(f(p))$ such that, for all $u \in \mathscr{U}(p)$ and all $q \in X$,*

$$f(u \cdot q) = (\partial_p f)(u) \cdot f(q), \tag{1}$$

equivalently, that the diagrams

$$\begin{array}{ccc} \mathscr{U}(p) & \xrightarrow{\partial_p f} & \mathscr{U}'(f(p)) \\ {\scriptstyle \zeta_{q,p}}\downarrow\cong & & \cong\downarrow{\scriptstyle \zeta'_{f(q),f(p)}} \\ X - \{p\} & \xrightarrow[f]{} & X' - \{f(p)\} \end{array} \tag{2}$$

are commutative for all $q \neq p$ in X. If $g\colon (X', \mathscr{U}') \to (X'', \mathscr{U}'')$ is a second morphism then the "chain rule"

$$\partial_p(g \circ f) = \partial_{f(p)}(g) \circ \partial_p f \tag{3}$$

holds.

Proof. For $u \in \mathscr{U}(p)$ let u' as in (2.1.4). We claim that u' is uniquely determined. Indeed, choose $q \neq p$. As f is injective, $f(p) \neq f(q)$, so uniqueness of u' follows from (2.1.4) and the fact that the orbit map $\zeta_{f(q),f(p)}$ is injective by (2.1.2). Now put $\partial_p f(u) = u'$. Then it is clear that (2) is commutative, and since f is injective, so is $\partial_p f$. It remains to show that $\partial_p f$ is a group homomorphism. Let $u, v \in \mathscr{U}(p)$. Then (1) implies

$$\begin{aligned}\partial_p f(u+v) \cdot f(q) &= f((u+v) \cdot q) = f(u \cdot (v \cdot q)) = \partial_p f(u) \cdot f(v \cdot q) \\ &= \partial_p f(u) \cdot (\partial_p f(v) \cdot f(q)) = \big(\partial_p f(u) + \partial_p f(v)\big) \cdot f(q),\end{aligned}$$

so the assertion follows again from the injectivity of $\zeta'_{f(q),f(p)}$. The chain rule is easily verified. □

Remark. For any injective map $f\colon X \to X'$, we can always define maps $f_{q,p}\colon \mathscr{U}(p) \to \mathscr{U}'(f(p))$ by the commutativity of (2) (with $\partial_p f$ replaced by $f_{q,p}$). Then f is a morphism if and only if the $f_{q,p}$ thus defined are independent of q. I am indebted to H. P. Petersson for this remark.

The following criterion for isomorphisms will be useful.

2.3. Lemma. *Let $(X, \mathscr{U})$ and $(X', \mathscr{U}')$ be pre-Moufang sets. For a map $f\colon X \to X'$, the following conditions are equivalent:*

(i) *f is an isomorphism,*
(ii) *f is a bijective morphism,*
(iii) *f is bijective and $\mathscr{U}'(f(p)) = f \circ \mathscr{U}(p) \circ f^{-1}$, for all $p \in X$.*

Proof. (i) $\Longrightarrow$ (ii): f is an isomorphism if and only if there exists a morphism in the opposite direction such that the respective compositions are the identity. In particular, f is a bijective map.

(ii) $\Longrightarrow$ (iii): If f is a bijective morphism then (2.2.1) implies that $\partial_p f(u) = f \circ u \circ f^{-1}$ for all $p \in X$ and $u \in \mathscr{U}(p)$, and by (2.2.2), $\partial_p f\colon \mathscr{U}(p) \to \mathscr{U}'(f(p))$ is a group isomorphism, so we have (iii).

(iii) $\Longrightarrow$ (i): The condition $\mathscr{U}'(f(p)) = f \circ \mathscr{U}(p) \circ f^{-1}$ implies that f and f^{-1} are morphisms, as desired. □

2.4. Moufang sets. A pre-Moufang set $M = (X, \mathscr{U})$ is called a *Moufang set* [**10**, 4.4] if it satisfies in addition to (M1) the axiom

(M2) for each $p \in X$, $\mathscr{U}(p)$ normalizes the set $\mathscr{U}(X) \subset \mathrm{sbgr}(\mathrm{Sym}(X))$.

The full subcategory of **pre-mou** whose objects are Moufang sets is denoted **mou**.

In more detail, (M2) says that for all $p, q \in X$ and all $u \in \mathscr{U}(p)$, there exists $x \in X$ such that $u \circ \mathscr{U}(q) \circ u^{-1} = \mathscr{U}(x)$. Then necessarily $x = u \cdot q$, since x is the only fixed point of $\mathscr{U}(x)$, and the only fixed point of $u \circ \mathscr{U}(q) \circ u^{-1}$ is $u \cdot q$. Since $\mathrm{G}(M)$ is generated by all $\mathscr{U}(p)$, this shows that (M2) is equivalent to the equivariance of $\mathscr{U}$ with respect to $\mathrm{G}(M)$:

$$\mathscr{U}(g \cdot p) = g \circ \mathscr{U}(p) \circ g^{-1}, \tag{1}$$

for all $p \in X$ and all $g \in \mathrm{G}(M)$. By Lemma 2.3, (1) is also equivalent to

$$\mathrm{G}(M) \subset \mathrm{Aut}(M). \tag{2}$$

In fact, $\mathrm{G}(M)$ is a normal subgroup of $\mathrm{Aut}(M)$, customarily called the little projective group. As a consequence, the automorphism group of a Moufang set is doubly transitive on X. Indeed, if X has more than two elements, even $\mathrm{G}(M)$ is doubly transitive. If X has two elements, $\mathrm{G}(M) = \{\mathrm{Id}\}$ but $\mathrm{Aut}(M) = \mathrm{Sym}(X)$.

For a pre-Moufang set to be Moufang it suffices that it satisfy the following weakening of (2):

$$\text{there exist } p \neq q \text{ in } X \text{ such that } \mathscr{U}(p) \cup \mathscr{U}(q) \subset \mathrm{Aut}(M). \tag{3}$$

Indeed, (3) implies $\mathscr{U}(x) \subset \mathrm{Aut}(M)$ for all $x \in X$. If $x = p$ this is clear, otherwise $x = u \cdot q$ for $u \in \mathscr{U}(p)$ by (M1). Since u is an automorphism of M, we have $\mathscr{U}(x) = u \circ \mathscr{U}(q) \circ u^{-1} \subset \mathrm{Aut}(M)$

2.5. Moufang sets vs. pre-Moufang sets. Trivially, any two-element set is Moufang since all $\mathscr{U}(p) = \mathrm{Id}_X$ for all $p \in X$. A Moufang set with at least three elements always has a large automorphism group by (2.4.2). In a general pre-Moufang set the groups $\mathscr{U}(p)$ are in no way related, so there may be no automorphisms different from the identity. For example, let X be a countable set and let G_p ($p \in X$) be a family of pairwise non-isomorphic countable groups. (For instance, let X be the set of prime numbers and let $G_p = \mathbb{Z}[p^{-1}]$ (additive subgroup of $\mathbb{Q}$) for all $p \in X$. The groups G_p are pairwise non-isomorphic because G_p is divisible by p and no other prime.) Since $X - \{p\}$ is still countable, we can choose a free and transitive action of G_p on $X - \{p\}$, and extend this action to all of X by declaring p a fixed point of G_p. Let $\mathscr{U}(p)$ be the subgroup of $\mathrm{Sym}(X)$ defined by this action of G_p. Then $M = (X, \mathscr{U})$ is a pre-Moufang set, and $\mathscr{U}(p) \cong \mathscr{U}(q)$ only for $p = q$. Hence M has trivial automorphism group.

2.6. Bases, half Moufang sets and the functor $(\ )^{\mathrm{op}}$. A *basis* of a pre-Moufang set is an ordered pair $b = (b^+, b^-)$ of distinct points, called the *base points.* A *based* pre-Moufang set is a pair $(M, b) = (X, \mathscr{U}, b^+, b^-)$ consisting of a pre-Moufang set $M = (X, \mathscr{U})$ and a basis b. Morphisms of based pre-Moufang sets are morphisms preserving the base points. We denote the category of based pre-Moufang sets by $\textbf{pre-mou}_b$ and let $\textbf{mou}_b$ denote the full subcategory of all (M, b) where M is a Moufang set. There are obvious forgetful functors $\textbf{pre-mou}_b \to \textbf{pre-mou}$ and $\textbf{mou}_b \to \textbf{mou}$ omitting the basis.

Let $\textbf{h-mou}_b^\sigma \subset \textbf{pre-mou}_b$ be the full subcategory whose objects (M, b) satisfy

$$\mathscr{U}(b^\sigma) \subset \mathrm{Aut}(M). \tag{1}$$

The objects of $\textbf{h-mou}_b^\sigma$ are called *half Moufang sets of type σ*, those of $\textbf{h-mou}_b^+ \cup \textbf{h-mou}_b^-$ simply half-Moufang sets. From (2.4.3) we see that

$$\textbf{mou}_b = \textbf{h-mou}_b^+ \cap \textbf{h-mou}_b^-, \tag{2}$$

so the epithet "half" refers to the fact that half Moufang sets satisfy only half of the requirements for a Moufang set, not that they constitute half of all Moufang sets.

The *opposite* of a basis $b = (b^+, b^-)$ is $b^{\mathrm{op}} = (b^-, b^+)$, and the opposite of a based pre-Moufang set $(M, b) = (X, \mathscr{U}, b^+, b^-)$ is $(M, b)^{\mathrm{op}} = (M, b^{\mathrm{op}}) = (X, \mathscr{U}, b^-, b^+)$. It is clear that $(\ )^{\mathrm{op}}$ is a functor from $\textbf{pre-mou}_b$ to itself whose square is the identity, maps $\textbf{h-mou}_b^+$ isomorphically onto $\textbf{h-mou}_b^-$ and therefore leaves $\textbf{mou}_b$ stable.

2.7. Lemma. *A based pre-Moufang set $(M, b) = (X, \mathscr{U}, b^+, b^-)$ belongs to $\textbf{h-mou}_b^\sigma$ if and only if*

$$\mathscr{U}(u \cdot b^{-\sigma}) = u \circ \mathscr{U}(b^{-\sigma}) \circ u^{-1}, \tag{1}$$

for all $u \in \mathscr{U}(b^\sigma)$.

Proof. By Lemma 2.3(iii), a permutation g of X is an automorphism of M if and only if

$$g \circ \mathscr{U}(p) \circ g^{-1} = \mathscr{U}(g \cdot p) \tag{2}$$

for all $p \in X$. For $g = u$ and $p = b^{-\sigma}$, this shows that an $(M, b) \in \textbf{h-mou}_b^\sigma$ satisfies (1). Conversely, suppose (1) holds, let $g \in \mathscr{U}(b^\sigma)$ and $p \in X$. If $p = b^\sigma$ then $g \cdot p = p$ and $g \circ \mathscr{U}(b^\sigma) \circ g^{-1} = \mathscr{U}(b^\sigma)$, so we have (2). If $p \in X - \{b^\sigma\}$ then $p = u \cdot b^{-\sigma}$ for a unique $u \in \mathscr{U}(b^\sigma)$ by (M1). Hence (1) implies

$$\begin{aligned} g \circ \mathscr{U}(p) \circ g^{-1} &= g \circ u \circ \mathscr{U}(b^{-\sigma}) \circ u^{-1} \circ g^{-1} = (g + u) \circ \mathscr{U}(b^{-\sigma}) \circ (g + u)^{-1} \\ &= \mathscr{U}((g + u) \cdot b^{-\sigma}) = \mathscr{U}(g \cdot (u \cdot b^{-\sigma})) = \mathscr{U}(g \cdot p), \end{aligned}$$

so $g \in \mathrm{Aut}(M)$. □

2.8. Lemma. *Let $(M, b) = (X, \mathscr{U}, b)$ and $(M', b') = (X', \mathscr{U}', b')$ be half-Moufang sets of the same type, and let $f\colon X \to X'$ be an injective map preserving the base points. Suppose that*

$$f \circ \mathscr{U}(b^\sigma) \subset \mathscr{U}'(o'^\sigma) \circ f \tag{1}$$

for $\sigma \in \{+, -\}$. Then f is a morphism.

Proof. By symmetry, we may assume (M, b) and (M', b') of type $+$. We have to verify (2.1.3). For $p = o^+$ this holds by assumption. Otherwise, $p = u \cdot o^-$ where $u \in \mathscr{U}(o^+)$, by (M1). Then (1) implies that $f \circ u = u' \circ f$ for some $u' \in \mathscr{U}'(o'^+)$. Hence

$$f(p) = f(u \cdot o^-) = u' \cdot f(o^-) = u' \cdot o'^-.$$

Since (M, b) and (M', b') belong to $\textbf{h-mou}_b^+$, we have $u \in \mathrm{Aut}(M)$ and $u' \in \mathrm{Aut}(M')$. This implies $\mathscr{U}(p) = u \circ \mathscr{U}(o^-) \circ u^{-1}$ and $\mathscr{U}'(f(p)) \circ u' = \mathscr{U}'(u' \cdot o'^-) \circ u' = u' \circ \mathscr{U}'(o'^-)$. Therefore,

$$\begin{aligned} f \circ \mathscr{U}(p) &= f \circ u \circ \mathscr{U}(o^-) \circ u^{-1} = u' \circ f \circ \mathscr{U}(o^-) \circ u^{-1} \\ &\subset u' \circ \mathscr{U}'(o'^-) \circ f \circ u^{-1} = \mathscr{U}'(f(p)) \circ u' \circ f \circ u^{-1} \\ &= \mathscr{U}'(f(p)) \circ f \circ u \circ u^{-1} = \mathscr{U}'(f(p)) \circ f. \end{aligned}$$ □

§3. The correspondence between division pairs and Moufang sets

3.1. Lemma. *Let $M = (X, \mathscr{U})$ be a pre-Moufang set and let $b = (b^+, b^-)$ be a basis of M.*

(a) *Put $V^\sigma = \mathscr{U}(b^\sigma)$ for $\sigma \in \{+, -\}$. Then for all $x \in \dot{V}^\sigma$, there exists a unique $y = j_\sigma(x) \in \dot{V}^{-\sigma}$ such that*

$$x \cdot b^{-\sigma} = j_\sigma(x) \cdot b^\sigma. \tag{1}$$

The map $j_+\colon \dot{V}^+ \to \dot{V}^-$ is bijective with inverse j_-. Hence the pair $V = (V^+, V^-)$ together with the maps $j_\pm$ is a pre-division pair, denoted $\mathbb{D}(M, b)$.

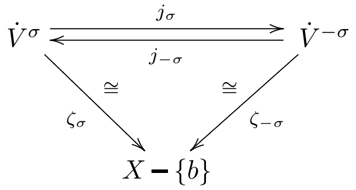
(b) *Let $f\colon (M, b) \to (M', b')$ be a morphism of based pre-Moufang sets and let $\partial_\sigma(f) := \partial_{b^\sigma}(f) : V^\sigma = \mathscr{U}(b^\sigma) \to V'^\sigma = \mathscr{U}'(o'^\sigma)$ be as in Lemma 2.2. Then the pair of group homomorphisms $\mathbb{D}(f) = (\partial_+(f), \partial_-(f))\colon \mathbb{D}(M, b) \to \mathbb{D}(M', b')$ is a morphism of pre-division pairs. These assignments define a functor*

$$\mathbb{D}\colon \textbf{pre-mou}_b \to \textbf{pre-div}$$

which commutes with the functors $(\ )^{\mathrm{op}}$ defined in 2.4 and 1.1.

(c) *If M is a Moufang set then the isomorphism class of $\mathbb{D}(M, b)$ is independent of the choice of basis.*

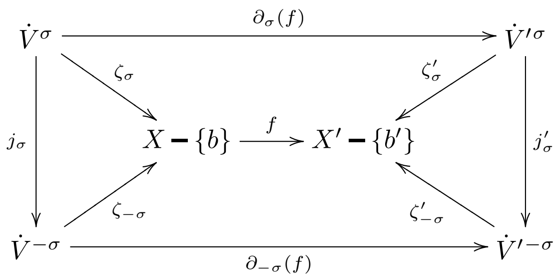
Proof. (a) Let us write $X-\{b\} = X-\{b^+, b^-\}$. Recall the maps $\zeta_{q,p}$ of (2.1.1) and put $\zeta_\sigma = \zeta_{b^{-\sigma}, b^\sigma}$ for short, so $\zeta_\sigma(x) = x \cdot b^{-\sigma}$. Then the definition of j_σ is equivalent to the commutativity of the diagram



(2)

By (2.1.2), the maps ζ_σ are bijective, so the assertion follows.

(b) Combining (2) with (2.2.2) yields the diagram



where the four subdiagrams are commutative. Hence the outer square is commutative as well, showing $j'_\sigma \circ \partial_\sigma(f) = \partial_{-\sigma}(f) \circ j_\sigma$, so $\mathbb{D}(f)$ is a morphism of pre-division pairs. The functorial properties are easily checked, and the last statement is obvious.

(c) As remarked in 2.4, the automorphism group of a Moufang set is doubly transitive on X, hence transitive on the set of bases. Since $\mathbb{D}$ is a functor, the assertion follows. $\square$

Remark. The map $\varphi_\sigma := \zeta_\sigma^{-1}\colon X-\{b^\sigma\} \to V^\sigma$ is a bijection, and the triple $c_\sigma = (X-\{b^\sigma\},\, \varphi_\sigma,\, V^\sigma)$ can be considered as a "chart" for the set X. The domains $X-\{b^\sigma\}$ of these charts cover X, and their intersection is $X-\{b\}$. Then (2) says that $j_\sigma = \varphi_{-\sigma} \circ \varphi_\sigma^{-1}\colon \dot V^\sigma \to \dot V^{-\sigma}$ is the map describing the change of coordinates.

3.2. Example. Let K be a division ring and let $X = \mathbb{P}_1(K)$ be the projective line over K, that is, the quotient of $K^2 - \{0\}$ (column vectors) by the action of $K^\times$ on the right. Let $(s:t)$ denote the equivalence class of $\binom{s}{t}$ and put $b^+ = (1:0)$ and $b^- = (0:1)$. Then X is a Moufang set with the $\mathscr{U}(p)$ the conjugates of the translation group $V^+ = \mathscr{U}(b^+)$ induced from $\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ acting by matrix multiplication on K^2 on the left, see [**10**, 4.4, Example (i)]. If we identify V^+ and V^- with K by $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $y \mapsto \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$, respectively, then $j_\sigma(x) = -x^{-1}$.

3.3. Lemma. *Let (V, j) be a pre-division pair. Recall the set $X = \mathbb{X}(V)$ and the isomorphisms $t_\sigma\colon V^\sigma \to T^\sigma \subset \mathrm{Sym}(X)$ introduced in* 1.2 *and Lemma* 1.3.

(a) *Define $\mathscr{T}\colon X \to \mathrm{sbgr}(\mathrm{Sym}(X))$ by*

$$\mathscr{T}(p) = \left\{ \begin{matrix} T^+ & \text{if } p = o^+ = \iota_-(0) \\ t_+(x) \circ T^- \circ t_+(x)^{-1} & \text{if } p = \iota_+(x) \text{ for } x \in V^+ \end{matrix} \right\}. \tag{1}$$

Then

$$\mathbb{P}(V) := (X,\, \mathscr{T},\, o^+,\, o^-) \in \textbf{h-mou}_b^+$$

is a half-Moufang set of type $+$.

(b) *For a morphism* $h = (h_+, h_-)\colon V \to V'$ *of pre-division pairs, let* $f = \mathbb{X}(h)\colon X = \mathbb{X}(V) \to X' = \mathbb{X}(V')$ *as in* 1.2*. Then* $f\colon \mathbb{P}(V) \to \mathbb{P}(V')$ *is a morphism of based pre-Moufang sets. Putting* $\mathbb{P}(h) = f$*, we obtain a functor*

$$\mathbb{P}\colon \textbf{pre-div} \to \textbf{h-mou}_b^+.$$

Proof. (a) We verify axiom (M1) of a pre-Moufang set. By Lemma 1.3, T^+ fixes o^+ and acts simply transitively on $X - \{o^+\}$ while T^- fixes o^- and is simply transitive on $X - \{o^-\}$. This together with the definition of $\mathscr{T}(p)$ easily implies that axiom (M1) is satisfied. From (1) it follows immediately that (2.7.1) holds for $\sigma = +$, so $\mathbb{P}(V) \in \textbf{h-mou}_b^+$.

(b) Let $\mathbb{P}(V') = (X', \mathscr{T}', o'^+, o'^-)$ where $\mathscr{T}'\colon X' \to \mathrm{sbgr}(\mathrm{Sym}(X'))$ is defined, mutatis mutandis, as in (1) and o'^σ are the base points of $\mathbb{P}^+(V')$. As noted in 1.2, $f = \mathbb{X}(h)$ is injective and preserves base points. By Lemma 2.8, f is a morphism of pre-Moufang sets if and only if

$$f \circ \mathscr{T}(o^\sigma) \subset \mathscr{T}'(f(o^\sigma)) \circ f \tag{2}$$

for all $\sigma = \pm$. This follows immediately from (1.3.2) and the definition of $\mathscr{T}$ and $\mathscr{T}'$ at the base points. Hence f is a morphism. The functorial properties are easily checked. ☐

3.4. Proposition. (a) *Let* V *be a pre-division pair and recall that* $\mathbb{D}(\mathbb{P}(V)) = \big(\mathscr{T}(o^+) = T^+, \mathscr{T}(o^-) = T^-\big)$ *by* (3.3.1) *and Lemma* 3.1(a)*. Then* $\eta_V = (t_+, t_-)\colon V \to \mathbb{D}(\mathbb{P}(V))$ *is an isomorphism of pre-division pairs, natural in* V*.*

(b) *Let* $(M, b) = (X, \mathscr{U}, b^+, b^-)$ *be a based half-Moufang set of type* $+$ *and* $V = \mathbb{D}(M, b) = (\mathscr{U}(b^+), \mathscr{U}(b^-))$ *the associated pre-division pair as in Lemma* 3.1*. Then the maps* $\zeta_\sigma = \zeta_{b^{-\sigma}, b^\sigma}\colon \mathscr{U}(b^\sigma) \to X$ *of* (2.1.1) *induce, by of the universal property of* $\mathbb{X}(V)$*, a map* $f\colon \mathbb{X}(V) \to X$*, and* $f = \varepsilon_{(M,b)}\colon \mathbb{P}(\mathbb{D}(M, b)) \to (M, b)$ *is an isomorphism of based pre-Moufang sets, natural in* (M, b)*.*

(c) *The isomorphisms* $\eta\colon \mathrm{Id}_{\textbf{pre-div}} \xrightarrow{\cong} \mathbb{D} \circ \mathbb{P}$ *of* (a) *and* $\varepsilon\colon \mathbb{P} \circ \mathbb{D} \xrightarrow{\cong} \mathrm{Id}_{\textbf{h-mou}_b^+}$ *of* (b) *are the unit and co-unit of an adjoint equivalence*

$$\mathrm{Mor}_{\textbf{h-mou}_b^+}\big(\mathbb{P}(V), (M, b)\big) \;\cong\; \mathrm{Mor}_{\textbf{pre-div}}\big(V, \mathbb{D}(M, b)\big).$$

In particular, the categories $\textbf{h-mou}_b^+$ *and* **pre-div** *are equivalent.*

Proof. (a) By Lemma 3.3 and Lemma 3.1,

$$\mathbb{D}(\mathbb{P}(V)) = \big((T^+, T^-), (j'_+, j'_-)\big)$$

with j'_σ defined by

$$t_\sigma(x) \cdot o^{-\sigma} = j'_\sigma(t_\sigma(x)) \cdot o^\sigma,$$

for $x \in \dot{V}^\sigma$. By Lemma 1.3, $t_\sigma\colon V^\sigma \to \mathscr{T}(o^\sigma)$ is a group isomorphism. Moreover,

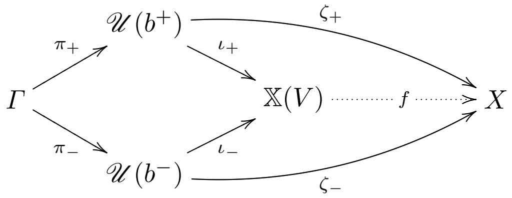
$$t_\sigma(x) \cdot o^{-\sigma} = t_\sigma(x) \cdot \iota_\sigma(0) = \iota_\sigma(x + 0) = \iota_\sigma(x)$$

and, by (1.2.2),

$$t_{-\sigma}(j(x)) \cdot o^\sigma = t_{-\sigma}(j_\sigma(x)) \cdot \iota_{-\sigma}(0) = \iota_{-\sigma}(j_\sigma(x) + 0) = \iota_{-\sigma}(j_\sigma(x)) = \iota_\sigma(x).$$

Hence $j'_\sigma(t_\sigma(x)) = t_{-\sigma}(j(x))$. Thus η_V is an isomorphism of pre-division pairs and it is easily verified that η_V is natural in V.

(b) To prove the existence of f, we must show that the outer square of the diagram



is commutative. Thus let $(x, y) \in \Gamma$, so $x \in \mathscr{U}(b^+)$, $y \in \mathscr{U}(b^-)$ and $y = j(x)$ as in (3.1.1). Hence

$$\zeta_+(\pi_+(x, y)) = \zeta_+(x) = x \cdot b^- = j_+(x) \cdot b^+ = y \cdot b^+ = \zeta_-(y) = \zeta_-(\pi_-(x, y)),$$

as desired. From $f(o^\sigma) = \zeta_{-\sigma}(0) = b^\sigma$ we see that f preserves base points. Since $\zeta_\sigma\colon \mathscr{U}(b^\sigma) \to X - \{b^\sigma\}$ is bijective by (2.1.2), f is bijective. We use Lemma 2.8 to show that f is a morphism (and hence by Lemma 2.3 an isomorphism) of pre-Moufang sets. Thus we have to show that

$$f \circ \mathscr{T}(o^\sigma) \subset \mathscr{U}(b^\sigma) \circ f \tag{1}$$

for $\sigma \in \{+, -\}$. This will follow from the relation

$$f \circ t_\sigma(u) = u \circ f : \mathbb{X}(V) \to X, \tag{2}$$

for all $u \in V^\sigma = \mathscr{U}(b^\sigma)$. By (1.2.4), $\mathbb{X}(V)$ is the union of $\iota_\sigma(V^\sigma)$ and $o^\sigma = \iota_{-\sigma}(0)$. Since $\mathscr{T}(o^\sigma)$ fixes o^σ and $\mathscr{U}(b^\sigma)$ fixes b^σ and f preserves base points, (2) holds when applied to o^σ. For the remaining points $\iota_\sigma(x)$, $x \in V^\sigma = \mathscr{U}(b^\sigma)$, we compute

$$\begin{aligned}(f \circ t_\sigma(u))(\iota_\sigma(x)) &= f(t_\sigma(u) \cdot \iota_\sigma(x)) = f(\iota_\sigma(u + x)) = \zeta_\sigma(u + x) = (u + x) \cdot b^{-\sigma} \\ &= u \cdot (x \cdot b^{-\sigma}) = u \cdot \zeta_\sigma(x) = u \cdot f(\iota_\sigma(x)) = (u \circ f)(\iota_\sigma(x)).\end{aligned}$$

Hence f is an isomorphism of based Moufang sets. Again, it is easily seen that $f = \varepsilon_{(M,b)}$ depends functorially on (M, b), i.e., defines an isomorphism $\varepsilon\colon \mathbb{P} \circ \mathbb{D} \to \mathrm{Id}_{\textbf{h-mou}_b^+}$ of functors.

(c) We show that there are natural inverse bijections

$$\mathrm{Mor}_{\textbf{h-mou}_b^+}\big(\mathbb{P}(V), (M, b)\big) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Mor}_{\textbf{pre-div}}\big(V, \mathbb{D}(M, b)\big) \tag{3}$$

given by

$$\begin{aligned}\alpha(f) &= \mathbb{D}(f) \circ \eta_V\colon V \to \mathbb{D}(M, b), \\ \beta(h) &= \varepsilon_{(M,b)} \circ \mathbb{P}(h)\colon \mathbb{P}(V) \to (M, b),\end{aligned}$$

for morphisms $f\colon \mathbb{P}(V) \to (M, b)$ and $h\colon V \to \mathbb{D}(M, b)$.

First we make $\alpha(f)$ and $\beta(h)$ explicit. Let $\partial_\sigma(f)\colon \mathscr{T}(o^\sigma) \to \mathscr{U}(b^\sigma)$ be the induced group homomorphisms as in Lemma 2.2, characterized by the commutative diagram

$$\begin{array}{ccc} \mathscr{T}(o^\sigma) & \xrightarrow{\ \partial_\sigma(f)\ } & \mathscr{U}(b^\sigma) \\ \Big\downarrow{\scriptstyle \zeta_\sigma} & & \Big\downarrow{\scriptstyle \zeta_\sigma} \\ \mathbb{X}(V) - \{o^\sigma\} & \xrightarrow[\ f\ ]{} & X - \{b^\sigma\} \end{array} \tag{4}$$

Since $\eta_V = (t_+, t_-)$, and $\alpha(f) = (\alpha(f)_+, \alpha(f)_-)$ is a pair of homomorphisms $\alpha(f)_\sigma\colon V^\sigma \to \mathscr{U}(b^\sigma)$, we have

$$\alpha(f)_\sigma = \partial_\sigma(f) \circ t_\sigma\colon V^\sigma \to \mathscr{T}(o^\sigma) \to \mathscr{U}(b^\sigma). \tag{5}$$

The homomorphism $h\colon V \to \mathbb{D}(M,b)$ has components $h_\sigma\colon V^\sigma \to \mathscr{U}(b^\sigma)$, and $\beta(h)$ is given, by what we proved in (b), on an element $\iota_\sigma(x) \in \mathbb{X}(V)$ by

$$\iota_\sigma(x) \mapsto \mathbb{X}(h)(\iota_\sigma(x)) = \iota_\sigma(h_\sigma(x)) \mapsto \varepsilon_{(M,b)}(\iota_\sigma(h_\sigma(x))) = \zeta_\sigma(h_\sigma(x)).$$

So we have

$$\beta(h) \circ \iota_\sigma = \zeta_\sigma \circ h_\sigma, \tag{6}$$

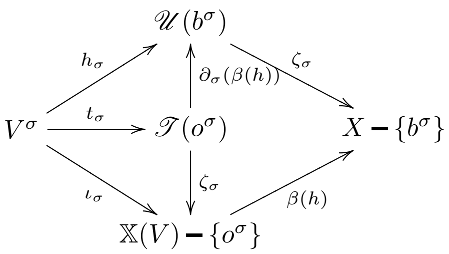
that is, the commutative diagram

$$\begin{array}{ccc} V^\sigma & \xrightarrow{\ h_\sigma\ } & \mathscr{U}(b^\sigma) \\ {\scriptstyle \iota_\sigma}\downarrow & & \downarrow{\scriptstyle \zeta_\sigma} \\ \mathbb{X}(V) - \{o^\sigma\} & \xrightarrow[\ \beta(h)\ ]{} & X - \{b^\sigma\} \end{array} \tag{7}$$

Now we show $\alpha(\beta(h)) = h$. Let us put $f = \beta(h)$ in (5). Then $\alpha(\beta(h))_\sigma = \partial_\sigma(\beta(h)) \circ t_\sigma$, so we must show

$$h_\sigma = \partial_\sigma(\beta(h)) \circ t_\sigma. \tag{8}$$

Combining (4) and (7), we obtain the diagram



(9)

so (8) is equivalent to the commutativity of the upper left hand triangle of (9). The outer square and the inner (right hand) square is commutative by (7) and (4), and the lower triangle is commutative as well because $\zeta_\sigma(t_\sigma(x)) = t_\sigma(x) \cdot o^{-\sigma} = t_\sigma(x) \cdot \iota_\sigma(0) = \iota_\sigma(x)$ for all $x \in V^\sigma$. By chasing the diagram we obtain

$$\zeta_\sigma \circ \partial_\sigma(\beta(h)) \circ t_\sigma = \beta(h) \circ \zeta_\sigma \circ t_\sigma = \beta(h) \circ \iota_\sigma = \zeta_\sigma \circ h_\sigma.$$

Since ζ_σ is bijective, it follows that $h_\sigma = \partial_\sigma(\beta(h)) \circ t_\sigma = \alpha(\beta(h))_\sigma$, as desired.

Next, we show $\beta(\alpha(f)) = f$. Putting $h = \alpha(f)$ in (6), we obtain, by (5), (2.2.2) and (9),

$$\beta(\alpha(f)) \circ \iota_\sigma = \zeta_\sigma \circ \alpha(f)_\sigma = \zeta_\sigma \circ \partial_\sigma(f) \circ t_\sigma = f \circ \zeta_\sigma \circ t_\sigma = f \circ \iota_\sigma.$$

As $\mathbb{X}(V)$ is the union of $\iota_+(V^+)$ and $\iota_-(V^-)$, this implies $\beta(\alpha(f)) = f$.

Since ε and η are natural transformations and $\mathbb{D}$ and $\mathbb{P}$ are functors, it is clear that α and β are natural in V and (M,b), more precisely, bifunctors of V and (M,b) Finally, it remains to show that η and ε are indeed the unit and co-unit of the adjunction (3), i.e., that

$$\eta_V = \alpha(\mathrm{Id}_{\mathbb{P}(V)}), \quad \varepsilon_{(M,b)} = \beta(\mathrm{Id}_{\mathbb{D}(M,b)}).$$

By (5), $\alpha(\mathrm{Id}_{\mathbb{P}(V)})_\sigma = \partial_\sigma(\mathrm{Id}_{\mathbb{P}(V)}) \circ t_\sigma = t_\sigma = (\eta_V)_\sigma$ for $\sigma = \pm$, showing the first formula. For the second, we have, by (6), $\beta(\mathrm{Id}_{\mathbb{D}(M,b)}) \circ \iota_\sigma = \zeta_\sigma = \varepsilon_{(M,b)} \circ \iota_\sigma$. $\square$

3.5. Theorem. *For a pre-division pair V, the following conditions are equivalent:*

(i) *V is a division pair,*
(ii) $\mathbb{P}(V^{\mathrm{op}}) = \mathbb{P}(V)^{\mathrm{op}}$,
(iii) *$\mathbb{P}(V)$ is a Moufang set.*

Hence the functors $\mathbb{D}$ of Lemma 3.1 and $\mathbb{P}$ of Lemma 3.3 restrict to functors $\mathbb{D}\colon \mathbf{mou}_b \to \mathbf{div}$ and $\mathbb{P}\colon \mathbf{div} \to \mathbf{mou}_b$ commuting with the opposition functors $(\ )^{\mathrm{op}}$. The natural isomorphisms η and ε of Proposition 3.4 induce natural isomorphisms

$$\eta\colon \mathrm{Id}_{\mathbf{div}} \xrightarrow{\cong} (\mathbb{D}\big|\mathbf{mou}_b) \circ (\mathbb{P}\big|\mathbf{div}) \quad \textit{and} \quad \varepsilon\colon (\mathbb{P}\big|\mathbf{mou}_b) \circ (\mathbb{D}\big|\mathbf{div}) \xrightarrow{\cong} \mathrm{Id}_{\mathbf{mou}_b}$$

which are the unit and co-unit of an adjoint equivalence

$$\mathrm{Mor}_{\mathbf{mou}_b}\big(\mathbb{P}(V), (M,b)\big) \;\cong\; \mathrm{Mor}_{\mathbf{div}}\big(V, \mathbb{D}(M,b)\big).$$

In particular, the categories $\mathbf{mou}_b$ of based Moufang sets and $\mathbf{div}$ of division pairs are equivalent.

Proof. (i) $\Longrightarrow$ (ii): Let $V^{\mathrm{op}} = \tilde{V} = (\tilde{V}^+, \tilde{V}^-)$ where $\tilde{V}^\sigma = V^{-\sigma}$. Then $\mathbb{P}(V^{\mathrm{op}}) = (\tilde{X}, \tilde{\mathscr{T}}, \tilde{o}^+, \tilde{o}^-)$ where $\tilde{X} = \mathbb{X}(V^{\mathrm{op}})$, $\tilde{\mathscr{T}}$ is defined in analogy to (3.3.1) by

$$\tilde{\mathscr{T}}(p) = \left\{ \begin{matrix} \tilde{T}^+ & \text{if } p = \tilde{o}^+ \\ \tilde{t}_+(y) \circ \tilde{T}^- \circ \tilde{t}_+(y)^{-1} & \text{if } p = \tilde{\iota}_+(y) \text{ for } y \in \dot{\tilde{V}}^+ \end{matrix} \right\}, \tag{1}$$

and $\tilde{o}^\sigma$ are the base points. We have $\tilde{X} = X$ by 1.2, but the base points are $\tilde{o}^\sigma = o^{-\sigma}$. Moreover, $\tilde{T}^\sigma = T^{-\sigma}$ and $\tilde{t}_+(y) = t_-(y)$ as well as $\tilde{\iota}_+(y) = \iota_-(y)$. On the other hand, passing from $\mathbb{P}(V)$ to $\mathbb{P}(V)^{\mathrm{op}}$ just switches the base points by 2.6, so it remains to show that $\mathscr{T}(p) = \tilde{\mathscr{T}}(p)$, for all $p \in X$. If $p = o^\sigma$, this is clear by (3.3.1). For $p \in X - \{o^+, o^-\}$, we have $p = \iota_+(x) = \iota_-(y)$ where $x \in \dot{V}^+$, $y = j(x) \in \dot{V}^-$ and, by (3.3.1) and (1),

$$\mathscr{T}(p) = t_+(x) \circ T^- \circ t_+(x)^{-1}, \qquad \tilde{\mathscr{T}}(p) = t_-(y) \circ T^+ \circ t_-(y)^{-1}.$$

Since V is a division pair, (1.6.4) holds for all $x \in \dot{V}^+$. Hence $\mathscr{T}(p) = \tilde{\mathscr{T}}(p)$, as required.

(ii) $\Longrightarrow$ (iii): By Lemma 3.3, $\mathbb{P}(V)$ and $\mathbb{P}(V^{\mathrm{op}})$ are half-Moufang sets of type $+$ while $\mathbb{P}(V)^{\mathrm{op}} \in \mathbf{h\text{-}mou}_b^-$ by 2.6. Hence (ii) implies that $\mathbb{P}(V)^{\mathrm{op}} \in \mathbf{mou}_b$ by (2.6.2) and therefore also $\mathbb{P}(V) \in \mathbf{mou}_b$, again by 2.6.

(iii) $\Longrightarrow$ (i): Let $M = (X, \mathscr{T})$ be the un-based Moufang set underlying $\mathbb{P}(V)$. Then all $t_\sigma(v)$, where $v \in V^\sigma$, are automorphisms of M, hence so is $w_\sigma(x)$, for any $x \in \dot{V}^\sigma$. Since $w_\sigma(x)$ interchanges the base points o^+ and o^- by Lemma 1.5(a), this implies $\mathrm{Int}(w_\sigma(x))\, T^{-\sigma} = T^\sigma$, so (1.6.3) of Lemma 1.6 holds for all $x \in \dot{V}^\sigma$, and therefore V is a division pair.

The remaining assertions follow now from Proposition 3.4. □

3.6. Corollary. *Let M be a pre-Moufang set and b a basis of M. Then the following conditions are equivalent:*

(i) *M is a Moufang set,*
(ii) *(M,b) is a half-Moufang set and $\mathbb{D}(M,b)$ is a division pair.*

Proof. This follows easily from Theorem 3.5 and Proposition 3.4(b). □

3.7. Corollary. *The functors $\mathbb{D}$ and $\mathbb{P}$ induce inverse bijections between the set of isomorphism classes of Moufang sets and division pairs.*

Proof. This follows from Lemma 3.1(c) and Theorem 3.5. $\square$

§4. Moufang sets from pairs (U, τ)

4.1. The categories $\mathbf{grp}_\tau$ and $\mathbf{pre\text{-}div}_\lambda$. In this section, we relate the construction of Moufang sets from pairs (U, τ) [**8**, **7**, **6**] with our approach via division pairs. Let $\mathbf{grp}_\tau$ be the category whose objects are pairs (U, τ) where U is a group, written additively, and $\tau\colon \dot{U} \to \dot{U}$ is a bijective map. A *morphism* $f\colon (U, \tau) \to (U', \tau')$ of $\mathbf{grp}_\tau$ is an injective group homomorphism satisfying $f(\tau(x)) = \tau'(f(x))$ for all $x \in \dot{U}$. Injectivity of f is necessary for this condition to make sense.

Given $(U, \tau) \in \mathbf{grp}_\tau$, we obtain a pre-division pair (V, j) by setting $V^+ = V^- = U$, $j_+ = \tau^{-1}$, $j_- = \tau$. Trivially, $\mathrm{Id}_U\colon V^+ \to V^-$ is a group isomorphism. To model this situation more closely on the side of pre-division pairs, we introduce the category $\mathbf{pre\text{-}div}_\lambda$ whose objects are triples (V, j, λ) where (V, j) is a pre-division pair and $\lambda\colon V^+ \to V^-$ is a group isomorphism. We do not require any relation between λ and the maps $j_\pm$. A *morphism* $h\colon (V, j, \lambda) \to (V', j', \lambda')$ of $\mathbf{pre\text{-}div}_\lambda$ is a morphism $h = (h_+, h_-)\colon (V, j) \to (V', j')$ of pre-division pairs compatible with the maps λ and λ' in the sense that $\lambda' \circ h_+ = h_- \circ \lambda$. There is an obvious forgetful functor $F\colon \mathbf{pre\text{-}div}_\lambda \to \mathbf{pre\text{-}div}$ omitting λ.

4.2. Lemma. *The assignments $\Phi(U, \tau) = ((U, U), (\tau^{-1}, \tau), \mathrm{Id}_U)$ on objects and $\Phi(f) = (f, f)$ on morphisms define a functor $\Phi\colon \mathbf{grp}_\tau \to \mathbf{pre\text{-}div}_\lambda$ which is an equivalence of categories. An inverse of Φ (up to isomorphism) is the functor $\Psi\colon \mathbf{pre\text{-}div}_\lambda \to \mathbf{grp}_\tau$, defined by*

$$\Psi(V, j, \lambda) = (V^+, j_- \circ \lambda)$$

on objects, and by $\Psi(h) = h_+$ on morphisms.

Proof. It follows easily from the definitions that $\Psi \circ \Phi$ is the identity on $\mathbf{grp}_\tau$. On the other hand, for $(V, j, \lambda) \in \mathbf{pre\text{-}div}_\lambda$, we have

$$(\Phi \circ \Psi)(V, j, \lambda) = \Phi(V^+, j_- \circ \lambda) = \big((V^+, V^+),\, (\lambda^{-1} \circ j_+,\, j_- \circ \lambda),\, \mathrm{Id}_{V^+}\big)$$

and a straightforward verification shows that

$$(\mathrm{Id}_{V^+}, \lambda^{-1})\colon (V, j, \lambda) \to (\Phi \circ \Psi)(V, j, \lambda)$$

is an isomorphism which depends functorially on (V, j, λ). The details are left to the reader. $\square$

4.3. The functor $\mathbb{M}$. Let $(U, \tau) \in \mathbf{grp}_\tau$. Following De Medts and Weiss [**8**], we define a based pre-Moufang set

$$\mathbb{M}(U, \tau) = (X,\, (U_x)_{x \in X},\, b^+,\, b^-) \tag{1}$$

as follows. Put $X = U \,\dot\cup\, \{\infty\}$, the disjoint union of U and a new symbol ∞, let $b^+ = \infty$, $b^- = 0$. For $x \in U$ let $\alpha(x)\colon U \to U$ be the left translation $u \mapsto x + u$, and extend $\alpha(x)$ to X by fixing the point ∞. Also extend τ to a map from X to itself by $\tau(0) = \infty$ and $\tau(\infty) = 0$. Now define a map $\mathscr{U}\colon X \to \mathrm{sbgr}(\mathrm{Sym}(X))$, $x \mapsto \mathscr{U}(x) = U_x$, by

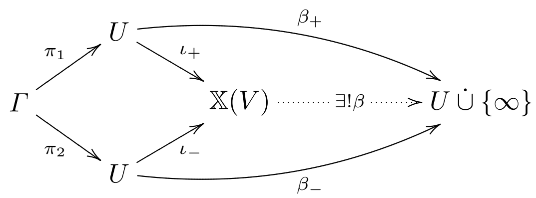
$$U_\infty = \alpha(U), \quad U_0 = \tau \circ U_\infty \circ \tau^{-1}, \quad U_x = \alpha(x) \circ U_0 \circ \alpha(x)^{-1} \quad \text{for } x \in \dot{U}. \tag{2}$$

Apart from the base points and the fact that we write maps on the left of their arguments and accordingly let U act on itself by left translations, this is the definition given in [**8**]. It is also easily seen that $\mathbb{M}(U, \tau)$ is a half-Moufang set of type $+$ depending functorially on (U, τ), so we have a functor $\mathbb{M}\colon \mathbf{grp}_\tau \to \mathbf{h\text{-}mou}_b^+$.

On the other hand, starting again from (U, τ), we have the pre-division pair $V = F(\Phi(U, \tau))$ with $V^{\pm} = U$ and $j_+ = \tau^{-1}$ as in 4.1 and Lemma 4.2, so by applying the functor $\mathbb{P}$ of Lemma 3.3 to V, we obtain a based half-Moufang set $\mathbb{P}(V) = \mathbb{P}(F(\Phi(U, \tau)))$. This is essentially the same as $\mathbb{M}(U, \tau)$, more precisely: there is an isomorphism

$$\kappa : \mathbb{P} \circ F \circ \Phi \xrightarrow{\cong} \mathbb{M} \tag{3}$$

of functors as follows. First, there is a unique bijection β: $\mathbb{X}(V) \to X = U \,\dot\cup\, \{\infty\}$ making the diagram



commutative, where π_i: $\Gamma \to U$ are the projections of the graph Γ of τ^{-1} onto the first and second factor, and $\beta_{\pm}$ is defined by

$$\beta_+(x) = x, \qquad \beta_-(y) = \begin{Bmatrix} \infty & \text{if } y = 0 \\ \tau(y) & \text{if } y \neq 0 \end{Bmatrix}. \tag{4}$$

Indeed, it is immediately verified that $\beta_+ \circ \pi_+ = \beta_- \circ \pi_-$, so the existence of β follows from the universal property of $\mathbb{X}(V)$, and from (1.2.4) we see that β is bijective. The images of the base points o^{σ} of X are $\beta(o^+) = \beta(\iota_-(0)) = \infty$ and $\beta(o^-) = \beta(\iota_+(0)) = 0$. It is straightforward to check that β is equivariant with respect to the actions t_+ of $U = V^+$ and t_- of $U = V^-$ on $\mathbb{X}(V)$ defined in Lemma 1.3 and the action α of U on X defined earlier in the following sense:

$$\beta \circ t_+(x) = \alpha(x) \circ \beta, \qquad \beta \circ t_-(y) = \tau \circ \alpha(y) \circ \tau^{-1} \circ \beta, \tag{5}$$

for $x \in U = V^+$ and $y \in U = V^-$. This implies, again by an easy verification, that $\beta = \kappa_{(U,\tau)}$: $\mathbb{P}(V) \to \mathbb{M}(U, \tau)$ is an isomorphism of based pre-Moufang sets, depending functorially on (U, τ). Hence, we have the diagram

$$\begin{array}{ccc} \mathbf{grp}_\tau & \xrightarrow{\ \Phi\ } & \mathbf{pre\text{-}div}_\lambda \\ {\scriptstyle \mathbb{M}}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathbf{h\text{-}mou}_b^+ & \xleftarrow[\ \mathbb{P}\ ]{} & \mathbf{pre\text{-}div} \end{array} \tag{6}$$

of functors, commutative up to the natural isomorphism κ.

Let us call a pair $(U, \tau) \in \mathbf{grp}_\tau$ *Moufang-admissible* if $\mathbb{M}(U, \tau)$ is a based Moufang set, and let $\mathbf{m\text{-}grp}_\tau$ be the full subcategory of $\mathbf{grp}_\tau$ whose objects are Moufang-admissible. Analogously, let $\mathbf{div}_\lambda$ be the full subcategory of $\mathbf{pre\text{-}div}_\lambda$ whose objects are all (V, j, λ) such that $F(V, j, \lambda) = (V, j) \in \mathbf{div}$ is a division pair.

4.4. Theorem. *The diagram* (4.3.6) *restricts to the diagram*

$$\begin{array}{ccc} \mathbf{m\text{-}grp}_\tau & \xrightarrow[\sim]{\ \Phi\ } & \mathbf{div}_\lambda \\ {\scriptstyle \mathbb{M}}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathbf{mou}_b & \xleftarrow[\ \mathbb{P}\ ]{\sim} & \mathbf{div} \end{array} \tag{1}$$

commutative up to natural isomorphism, where the horizontal arrows are equivalences of categories.

Proof. We first show that

$$(U,\tau) \in \mathbf{m\text{-}grp}_\tau \quad\Longleftrightarrow\quad F(\Phi(U,\tau)) \in \mathbf{div} \quad\Longleftrightarrow\quad \Phi(U,\tau) \in \mathbf{div}_\lambda. \tag{2}$$

Indeed, let $V = F(\Phi(U,\tau))$. Then

$$\begin{array}{rcll}
(U,\tau) \in \mathbf{m\text{-}grp}_\tau & \Longleftrightarrow & \mathbb{M}(U,\tau) \in \mathbf{mou}_b & \text{(by definition of } \mathbf{m\text{-}grp}_\tau) \\
& \Longleftrightarrow & \mathbb{P}(V) \in \mathbf{mou}_b & \text{(by (4.3.3))} \\
& \Longleftrightarrow & V \in \mathbf{div} & \text{(by Theorem 3.5)} \\
& \Longleftrightarrow & \Phi(U,\tau) \in \mathbf{div}_\lambda & \text{(by definition of } \mathbf{div}_\lambda).
\end{array}$$

Now (2) (together with Lemma 4.2) implies that the top arrow in (1) is an equivalence of categories, and the bottom arrow is an equivalence by Theorem 3.5. □

This result is our explanation of the hidden variable involved in describing Moufang sets in terms of (U,τ): it corresponds to the arbitrary isomorphism $\lambda\colon V^+ \to V^-$ in an object (V,j,λ) of $\mathbf{div}_\lambda$.

4.5. The functor $(\ )^{\mathrm{in}}$. By an easy verification, the assignments $(U,\tau)^{\mathrm{in}} = (U,\tau^{-1})$ on objects and $f^{\mathrm{in}} = f$ on morphisms define a functor $(\ )^{\mathrm{in}}\colon \mathbf{grp}_\tau \to \mathbf{grp}_\tau$ of period two. On the other hand, we extend the functor $(\ )^{\mathrm{op}}$ on pre-division pairs (as in 1.1) to the category $\mathbf{pre\text{-}div}_\lambda$ by defining $(V,j,\lambda)^{\mathrm{op}} = \big((V,j)^{\mathrm{op}},\ \lambda^{-1}\big)$ on objects and $h^{\mathrm{op}} = (h_+,h_-)^{\mathrm{op}} = (h_-,h_+)$ on morphisms. Then these functors correspond to each other under the categorical equivalence of Lemma 4.2 and commute with F; i.e., the diagram

$$\begin{array}{ccccc}
\mathbf{grp}_\tau & \xrightarrow[\sim]{\ \Phi\ } & \mathbf{pre\text{-}div}_\lambda & \xrightarrow{\ F\ } & \mathbf{pre\text{-}div} \\
{\scriptstyle (\ )^{\mathrm{in}}}\downarrow & & \downarrow{\scriptstyle (\ )^{\mathrm{op}}} & & \downarrow{\scriptstyle (\ )^{\mathrm{op}}} \\
\mathbf{grp}_\tau & \xrightarrow[\Phi]{\ \sim\ } & \mathbf{pre\text{-}div}_\lambda & \xrightarrow[F]{} & \mathbf{pre\text{-}div}
\end{array} \tag{1}$$

is commutative. By 1.7, V is a division pair if and only if V^{op} is a division pair, so $(\ )^{\mathrm{op}}$ induces an automorphism of the full subcategory $\mathbf{div}$ of $\mathbf{pre\text{-}div}$ and the same holds for the subcategory $\mathbf{div}_\lambda$ of $\mathbf{pre\text{-}div}_\lambda$. Now it follows from (4.4.2) and (1) that (U,τ) is Moufang-admissible if and only if (U,τ^{-1}) is, a result due to De Medts and Segev [**6**, Lemma 3.6]. So we have a commutative diagram

$$\begin{array}{ccccc}
\mathbf{m\text{-}grp}_\tau & \xrightarrow[\sim]{\ \Phi\ } & \mathbf{div}_\lambda & \xrightarrow{\ F\ } & \mathbf{div} \\
{\scriptstyle (\ )^{\mathrm{in}}}\downarrow & & \downarrow{\scriptstyle (\ )^{\mathrm{op}}} & & \downarrow{\scriptstyle (\ )^{\mathrm{op}}} \\
\mathbf{m\text{-}grp}_\tau & \xrightarrow[\Phi]{\ \sim\ } & \mathbf{div}_\lambda & \xrightarrow[F]{} & \mathbf{div}
\end{array} \tag{2}$$

We claim that there is a natural isomorphism of functors

$$(\ )^{\mathrm{op}} \circ \mathbb{M} \cong \mathbb{M} \circ (\ )^{\mathrm{in}}. \tag{3}$$

Indeed,

$$\begin{array}{rll}
(\ )^{\mathrm{op}} \circ \mathbb{M} & \cong (\ )^{\mathrm{op}} \circ (\mathbb{P} \circ F \circ \Phi) & \text{(by (4.3.3))} \\
& = \mathbb{P} \circ (\ )^{\mathrm{op}} \circ F \circ \Phi & \text{(by Theorem 3.5)} \\
& = \mathbb{P} \circ F \circ \Phi \circ (\ )^{\mathrm{in}} & \text{(by (2))} \\
& \cong \mathbb{M} \circ (\ )^{\mathrm{in}} & \text{(by (4.3.3))}
\end{array}$$

4.6. The relation with the opposite and the mirror Moufang set of [7]. In [**7**, Section 2, Section 3], the notions of opposite and mirror Moufang set are introduced in terms of the construction $\mathbb{M}(U,\tau)$. We now discuss these in our framework.

(a) For $(U,\tau) \in \mathbf{grp}_\tau$, let U° be the opposite group, with group operation $x \dot{+} y = y + x$, and let $\tau^\circ = i\tau i$ where i is inversion in the group U. Then $i\colon U \to U^\circ$ is an isomorphism, and $(U,\tau)^\circ := (U^\circ, \tau^\circ) \in \mathbf{grp}_\tau$. If $f\colon (U,\tau) \to (U',\tau')$ is a morphism of $\mathbf{grp}_\tau$, then $f^\circ = f\colon (U,\tau)^\circ \to (U',\tau')^\circ$ is a morphism as well. Hence this defines a functor $(\ )^\circ\colon \mathbf{grp}_\tau \to \mathbf{grp}_\tau$ of period two, and there is a natural isomorphism $\nu\colon \mathrm{Id}_{\mathbf{grp}_\tau} \to (\ )^\circ$ given by

$$\nu_{(U,\tau)} = i\colon (U,\tau) \to (U,\tau)^\circ. \tag{1}$$

By applying the functor $\mathbb{M}$, we obtain an isomorphism

$$\mathbb{M}(\nu_{(U,\tau)})\colon \mathbb{M}(U,\tau) \xrightarrow{\cong} \mathbb{M}((U,\tau)^\circ), \tag{2}$$

natural in (U,τ). In particular, this shows that (U,τ) is Moufang-admissible if and only if $(U,\tau)^\circ$ is.

In [**7**, Section 2], $\mathbb{M}((U,\tau)^\circ)$ is called "the opposite Moufang set". This seems to suggest that it is the opposite (in some sense) of the Moufang set $\mathbb{M}(U,\tau)$. However, $\mathbb{M}((U,\tau)^\circ)$ is not a function of the based Moufang set $\mathbb{M}(U,\tau)$ but rather results from a construction on the level of (U,τ). Evidently, the "opposite Moufang set" in this sense has nothing to do with our notion of the opposite of a based Moufang set defined in 2.6. The latter, however, is closely related to the mirror Moufang set discussed next.

(b) For $(U,\tau) \in \mathbf{grp}_\tau$, let $U^t = \big(U - \{0\}\big) \mathbin{\dot\cup} \{\infty\}$ as a set and make it into a group with neutral element ∞ and group operation $x \oplus y = \tau\big(\tau^{-1}(x) + \tau^{-1}(y)\big)$ where τ is extended from $\dot U$ to $U \mathbin{\dot\cup} \{\infty\}$ as in 4.3 by $\tau(0) = \infty$ and $\tau(\infty) = 0$. Then $(U,\tau)^t := (U^t, \tau^{-1}) \in \mathbf{grp}_\tau$. For a morphism $f\colon (U,\tau) \to (U',\tau')$, let $f^t\colon U^t \to U'^t$ be defined by $f^t(x) = f(x)$ for $x \neq \infty$ and $f^t(\infty) = \infty$. Then $f^t\colon (U,\tau)^t \to (U',\tau')^t$ is a morphism of $\mathbf{grp}_\tau$, and $(\ )^t$ is an endofunctor of $\mathbf{grp}_\tau$ of period two, isomorphic to the functor $(\ )^{\mathrm{in}}$ of 4.5: there is a natural isomorphism $\vartheta\colon (\ )^{\mathrm{in}} \xrightarrow{\cong} (\ )^t$ given by

$$\vartheta_{(U,\tau)} = \tau\colon (U,\tau)^{\mathrm{in}} \xrightarrow{\cong} (U,\tau)^t. \tag{3}$$

Indeed, it follows from the definition of the group structure of U^t that $\tau\colon U \to U^t$ is a group isomorphism, and the commutative diagram

$$\begin{array}{ccc} \dot U & \xrightarrow{\ \tau\ } & \dot U^t \\ {\scriptstyle \tau^{-1}}\downarrow & & \downarrow{\scriptstyle \tau^{-1}} \\ \dot U & \xrightarrow[\ \tau\ ]{} & \dot U^t \end{array}$$

shows that it commutes with the respective τ-maps.

By applying the functor $\mathbb{M}$ to (3), we obtain an isomorphism

$$\mathbb{M}(\vartheta_{(U,\tau)})\colon \mathbb{M}((U,\tau)^{\mathrm{in}}) \xrightarrow{\cong} \mathbb{M}((U,\tau)^t), \tag{4}$$

natural in (U,τ). From 4.5 we know that (U,τ) is Moufang-admissible if and only if $(U,\tau)^{\mathrm{in}}$ is so. Hence the same is true of $(U,\tau)^t$. Combining (4) with (4.5.3), we obtain a natural isomorphism

$$(\ )^{\mathrm{op}} \circ \mathbb{M} \cong \mathbb{M} \circ (\ )^t. \tag{5}$$

In [**7**, Section 3], $\mathbb{M}((U,\tau)^t)$ is called the "mirror Moufang set". Again, this refers to a construction on the level of (U,τ) and not on the level of the Moufang set $\mathbb{M}(U,\tau)$. However, (5) shows that the mirror Moufang set $\mathbb{M}((U,\tau)^t)$ is canonically isomorphic to the opposite in our sense of the based Moufang set $\mathbb{M}(U,\tau)$. In the framework of pairs (U,τ), the elaborate construction of U^t is required to switch the roles of U_0 and U_∞, whereas in our setting it suffices to switch V^+ and V^-.

4.7. Hua maps, R-operators and μ-maps. Let $(U,\tau) \in \mathbf{m\text{-}grp}_\tau$. We use the notations introduced in 4.3. In [**8**, Definition 3.1], De Medts and Weiss define, for $x \in \dot{U}$, the μ-map $\mu_x \in \mathrm{Sym}(X)$ by

$$\mu_x = \tau \circ \alpha(\tau^{-1}(x))^{-1} \circ \tau^{-1} \circ \alpha(x) \circ \tau \circ \alpha(\tau^{-1}(-x)) \circ \tau^{-1}.$$

By (1.4.1) and (4.3.5), this corresponds to $w_+(x)$ under the bijection $\beta\colon \mathbb{X}(V) \to X$, i.e., the right hand side of the diagram

$$\begin{array}{ccccc} V^- & \xrightarrow{\iota_-} & \mathbb{X}(V) & \xrightarrow{\beta} & X \\ {\scriptstyle R_x}\downarrow & & \downarrow{\scriptstyle w_+(x)} & & \downarrow{\scriptstyle \mu_x} \\ V^+ & \xrightarrow[\iota_+]{} & \mathbb{X}(V) & \xrightarrow[\beta]{} & X \end{array} \tag{1}$$

is commutative. Since the left hand side is commutative by definition of R_x in (1.5.1), the entire diagram is commutative. Let $y \in \dot{V}^- = \dot{U}$. Then

$$\beta(\iota_+(R_x y)) = \mu_x \cdot \beta(\iota_-(y)) = \mu_x \cdot \beta(\iota_+(j_-(y))) = \mu_x \cdot \beta(\iota_+(\tau(y))).$$

By (4.3.4), $\beta \circ \iota_+ = \beta_+$ is the embedding $a \mapsto a$ of U into $X = U \dot\cup \{\infty\}$. Hence $R_x y = \mu_x \cdot \tau(y)$ for all $y \in \dot{V}^- = \dot{U}$. We claim that the R-operators are just the Hua maps h_x:

$$R_x = h_x. \tag{2}$$

Indeed, by [**8**, Lemma 3.8(ii)] (where the order of the factors has to be reversed because we write maps on the left), $\mu(a) = \tau \circ h_{-a}$. By putting $a = -x$, taking inverses and observing $\mu(-x) = \mu_x^{-1}$ we obtain $\mu_x = h_x \circ \tau^{-1}$, or $h_x = \mu_x \circ \tau = R_x$.

4.8. Comparison of De Medts' definitions of morphism. Let (U,τ) and (U',τ') be Moufang-admissible. In [**5**, Def. 3.1], De Medts defines a morphism from $\mathbb{M}(U,\tau)$ to $\mathbb{M}(U',\tau')$ as a group homomorphism $\varphi\colon U \to U'$ satisfying

$$\varphi \circ \mu_x = \mu_{\varphi(x)} \circ \varphi, \tag{1}$$

for all $x \in \dot{U}$. Since the μ-maps are only defined for non-zero elements x, this requires φ to be injective. We show that this concept is equivalent to the definition of a morphism of Moufang sets in (2.1.3) and hence to De Medts' first definition in [**5**, (2.1)].

Let $(V,j) = F(\Phi(U,\tau))$ and $(V',j') = F(\Phi(U',\tau'))$ be the associated division pairs as in Theorem 4.4. In view of the categorical equivalence of based Moufang sets and division pairs (Theorem 3.5), it suffices to show that group homomorphisms φ satisfying (1) correspond bijectively to homomorphisms $h\colon V \to V'$ of division pairs.

Thus let φ satisfy (1). Setting $h_+ := \varphi$, we claim that there is a unique map $h_-\colon V^- \to V'^-$ making $h = (h_+,h_-)\colon V \to V'$ a morphism of division pairs. Uniqueness is clear since h_- is uniquely determined by h_+, as remarked in 1.1. To show existence, pick an element $a \in \dot{U}$, let $b = \varphi(a) \in \dot{U}'$, and define $h_-\colon V^- = U \to V'^- = U'$ by the commutative diagram

$$\begin{array}{ccc} V^+ & \xrightarrow{h_+} & V'^+ \\ {\scriptstyle R_a}\uparrow\cong & & \cong\uparrow{\scriptstyle R_b} \\ V^- & \xrightarrow[h_-]{} & V'^- \end{array}$$

Since the R-operators are group homomorphisms so is h_-. We claim that (h_+, h_-) is a homomorphism of division pairs. As remarked in 2.1, it suffices to show that $h_-(j_+(x)) = j'_+(h_+(x))$ for all $x \in \dot{V}^+ = \dot{U}$. Now by (1) and (4.7.2),

$$\begin{aligned} h_-(j_+(x)) &= R_b^{-1}\big(\varphi(R_a(\tau^{-1}(x)))\big) = R_b^{-1}\big(\varphi(\mu_a(x))\big) \\ &= R_b^{-1}\big(\mu_{\varphi(a)}(\varphi(x))\big) = R_b^{-1}\big(R_{\varphi(a)}(\tau'^{-1}(\varphi(x)))\big) \\ &= R_b^{-1}\big(R_b(\tau'^{-1}(\varphi(x)))\big) = j'_+(\varphi(x)) = j'_+(h_+(x)). \end{aligned}$$

Conversely, if $h\colon V \to V'$ is a homomorphism of division pairs then $\varphi = h_+$ satisfies (1). Indeed, since $j_+ = \tau^{-1}$, we have $\mu_x = R_x \circ j_+$, so by (1.5.4), for all $x, z \in U = V^+$,

$$\begin{aligned} \varphi(\mu_x(z)) &= h_+\big(R_x(j_+(z))\big) = R_{h_+(x)} h_-(j_+(z)) \\ &= R_{h_+(x)} j'_+(h_+(z)) = \mu_{\varphi(x)}(\varphi(z)). \end{aligned}$$

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