Derivations and automorphisms of Jordan algebras in characteristic two

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Abstract

A Jordan algebra J over a field k of characteristic 2 becomes a 2-Lie algebra $L(J)$ with Lie product $[x, y] = x \circ y$ and squaring $x^{[2]} = x^2$. We determine the precise ideal structure of $L(J)$ in case J is simple finite-dimensional and k is algebraically closed. We also decide which of these algebras have smooth automorphism groups. Finally, we study the derivation algebra of a reduced Albert algebra $J = \mathrm{H}_3(\mathcal{O}, k)$ and show that $\mathrm{Der}\, J$ has a unique proper nonzero ideal V_J, isomorphic to $L(J)/k \cdot 1_J$, with quotient $\mathrm{Der}\, J/V_J$ independent of $\mathcal{O}$. On the group level, this gives rise to a special isogeny between the automorphism group of J and that of the split Albert algebra, whose kernel is the infinitesimal group determined by V_J.

Introduction

We study finite-dimensional simple quadratic Jordan algebras J over fields of characteristic 2. This situation is of particular interest because it presents phenomena without counterpart in the linear theory:

(i) The bilinear trace form may be degenerate, giving rise to a proper outer ideal $\mathrm{Def}(J)$, the defect of J.

(ii) The algebra may be traceless, i.e., its linear trace form may be zero.

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(iii) J may not have capacity in the sense of [7].

(iv) Squaring and circle product make the vector space J into a restricted Lie algebra $L(J)$.

(v) The left multiplications $V_J = \{V_x : x \in J\}$ form an ideal in the derivation algebra $\mathrm{Der}(J)$, isomorphic to $L(J)$ modulo its centre.

(vi) The automorphism group of J is not necessarily smooth.

Our main concern is to explore (iv) – (vi), but this also necessitates a study of (i) – (iii).

Here is a more detailed description of the contents. In §1 we introduce the trace forms, the defect, and the notions of rank and primitive rank. We also classify (Prop. 1.9) the orbits of the automorphism group of a simple Jordan pair, and as an application give the classification, in a form suitable for our purposes, of the simple finite-dimensional Jordan algebras over an algebraically closed field of characteristic 2.

§2 contains a detailed study of the Lie algebra $L(J)$. We compute the derived series and show that $L(J)$ is solvable if and only if J has primitive rank ≤ 2 (Cor. 2.9). The ideal structure of $L(J)$ is completely determined in Th. 2.11, and simplicity of subquotients of $L(J)$ is studied in Cor. 2.12. We also show that some of the Lie algebras $L(J)$ occur as Lie algebras of classical algebraic groups.

The question of smoothness of the automorphism group $\mathbf{Aut}(J)$ is discussed in §3. Our method is based on the result of [14] that the structure group of a separable Jordan algebra is always smooth. This simplifies and completes work of Springer [23] who studied smoothness of $\mathbf{Aut}(J)$ in his framework of J-structures which excludes a priori traceless algebras.

The last section (§4) is devoted to the case where $J = \mathrm{H}_3(\mathcal{O}, k)$ is a reduced Albert algebra. We show (Th. 4.9) that $\mathrm{Der}(J)/V_J$ is a simple 26-dimensional Lie algebra whose isomorphism class does not depend on $\mathcal{O}$, and that V_J is the unique proper nonzero ideal of $\mathrm{Der}(J)$. Finally, using the result of Schafer-Tomber [22] that $\mathrm{Der}(J)/V_J \cong V_{J^s}$, where J^s is the split Albert algebra, we describe in Th. 4.14 a homomorphism β: $\mathbf{Aut}(J) \to \mathbf{Aut}(J^s)$ with infinitesimal kernel, which gives a concrete realization of the special isogeny between an isotropic and a split group of type F_4.

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1. Preliminaries

1.1. *Diagonalizability and defect in Jordan pairs.* Let $\mathfrak{V} = (\mathfrak{V}^+, \mathfrak{V}^-)$ be a finite-dimensional semisimple Jordan pair over an algebraically closed field k, see [13] for a general reference. We review some properties of the defect and the rank function from [15], specialized to the present situation.

Every $x \in \mathfrak{V}^\sigma$ ($\sigma \in \{+, -\}$) is von Neumann regular [13, Th. 10.17] and can therefore be embedded into a (Jordan pair) idempotent $e = (e^+, e^-)$ as $x = e^\sigma$ [13, 5.2]. The *rank* of x, denoted $\mathrm{rk}(x)$, is the capacity of the Peirce space $\mathfrak{V}_2(e)$ [15, Prop. 3]. In particular, x has rank one if and only if $\mathfrak{V}_2(e)$ is a division pair. Since k is algebraically closed, x is of rank one if and only if $Q_x\mathfrak{V}^{-\sigma} = k \cdot x$, i.e., x is reduced in the sense of [16]. Also, $\mathfrak{V}$ itself is reduced in this sense, i.e., it is spanned by its rank one elements. By [16, §1], there exists a well-defined bilinear form $f\colon \mathfrak{V}^+ \times \mathfrak{V}^- \to k$, the *Faulkner form*, with the property that

$$f(x, v)x = Q_x v, \quad f(z, y)y = Q_y z \tag{1}$$

for all rank one elements $x \in \mathfrak{V}^+$, $y \in \mathfrak{V}^-$, and arbitrary $v \in \mathfrak{V}^-$, $z \in \mathfrak{V}^+$.

An element $x \in \mathfrak{V}^\sigma$ is called *diagonalizable* if there exist orthogonal division idempotents $d_1, \ldots, d_t$ such that $x = d_1^\sigma + \cdots + d_t^\sigma$, and it is called *defective* if $Q_y x = 0$ for all rank one elements $y \in \mathfrak{V}^{-\sigma}$. Then 0 is the only element which is both diagonalizable and defective, and if $\mathfrak{V}$ is simple, every element is either diagonalizable or defective [15, Cor. 1 of Th. 1]. (For $\mathfrak{V}$ semisimple but not simple, there may be "mixed" elements which are neither diagonalizable nor defective). Letting $\mathrm{Def}(\mathfrak{V}^\sigma)$ denote the set of defective elements of $\mathfrak{V}^\sigma$, the *defect* $\mathrm{Def}(\mathfrak{V}) = (\mathrm{Def}(\mathfrak{V}^+), \mathrm{Def}(\mathfrak{V}^-))$ is either zero or a proper outer ideal of $\mathfrak{V}$, which itself has defect zero: $\mathrm{Def}(\mathrm{Def}(\mathfrak{V})) = 0$. Also, the defect can only be nonzero if k has characteristic two [15, Th. 2].

We next relate the defect to the generic trace form $m_1\colon \mathfrak{V}^+ \times \mathfrak{V}^- \to k$ [13, §16].

1.2. Lemma. *Let $\mathfrak{V}$ be a semisimple finite-dimensional Jordan pair over an algebraically closed field k.*

(a) *The Faulkner form agrees with the generic trace: $f = m_1$.*

(b) *The defect is the kernel of m_1 in the sense that*

$$x \in \mathrm{Def}(\mathfrak{V}^+) \iff m_1(x, \mathfrak{V}^-) = 0, \qquad y \in \mathrm{Def}(\mathfrak{V}^-) \iff m_1(\mathfrak{V}^+, y) = 0.$$

Proof. (a) Consider a frame $F = (d_1, \ldots, d_r)$ of division idempotents of $\mathfrak{V}$, and let $\mathfrak{C} = \sum_{i=1}^r \mathfrak{V}_{ii}$ be the Cartan subpair defined by F. Since both f and m_1 are invariant under $\mathrm{Aut}(\mathfrak{V})$, and the orbit of $\mathfrak{C}^+ \times \mathfrak{C}^-$ under $\mathrm{Aut}(\mathfrak{V})$ is Zariski-dense in $\mathfrak{V}^+ \times \mathfrak{V}^-$ [13, 15.15], it suffices to show that f and m_1 agree on $\mathfrak{C}^+ \times \mathfrak{C}^-$. We have $m_1(\sum_{i=1}^r \lambda_i d_i^+, \sum_{i=1}^r \mu_i d_i^-) = \sum_{i=1}^r \lambda_i \mu_i$ by [13, 16.15]. On the other hand, $f(d_i^+, d_j^-) = 0$ for $i \neq j$ by [16, 1.7(b)], and $f(d_i^+, d_i^-) = 1$ since the d_i are division idempotents, by 1.1.1. Hence also $f(\sum_{i=1}^r \lambda_i d_i^+, \sum_{i=1}^r \mu_i d_i^-) = \sum_{i=1}^r \lambda_i \mu_i$.

(b) This follows from (a) and [16, 1.9(a)].

1.3. *Jordan algebras.* We recall from [13, §1] the correspondence between isotopy classes of unital Jordan algebras on the one hand and isomorphism classes of Jordan pairs containing invertible elements on the other: If $v \in (\mathfrak{V}^-)^\times$ is invertible, then the vector space $\mathfrak{V}^+$ becomes a unital quadratic Jordan algebra, denoted $\mathfrak{V}_v^+$, with unit element $1 = v^{-1}$, quadratic operators $U_x = Q_x Q_v$, squaring $x^2 = Q_x v$ and circle product $x \circ y = \{xvy\}$. Conversely, if J is a unital Jordan algebra then (J, J) is a Jordan pair containing invertible elements, with quadratic operators $Q_x = U_x$. This correspondence is easily seen to preserve simplicity.

Let J be a finite-dimensional Jordan algebra over a field k, and let $\mathrm{tr}(x) = t(x)$ and $\mathrm{Tr}(x, y) = t(x, y)$ be the linear and bilinear trace forms as in [10]. By [10, (15)], we have

$$\mathrm{tr}(x) = \mathrm{Tr}(x, 1) \tag{1}$$

for all $x \in J$, and from [10, (14)] it is evident that Tr is a symmetric bilinear form on J.

Now let $J = \mathfrak{V}_v^+$ be obtained from a Jordan pair $\mathfrak{V}$ and an invertible element $v \in \mathfrak{V}^-$ as described above. Then tr and Tr are related to the generic trace form m_1 of $\mathfrak{V}$ by

$$\mathrm{tr}(x) = m_1(x, v), \qquad \mathrm{Tr}(x, y) = m_1(x, Q_v y). \tag{2}$$

Indeed, by [13, 16.3(ii)], the generic minimum polynomial of the Jordan pair $\tilde{\mathfrak{V}} := (J, J)$ is given by

$$\tilde{m}(\tau, x, y) = \sum_{i=0}^{d} (-1)^i \tilde{m}_i(x, y) \tau^{d-i} = N(x) N(\tau x^{-1} - y), \tag{3}$$

for all $y \in J$ and all invertible $x \in J^\times$. Here N is the generic norm of J and τ an indeterminate. By [13, Prop. 1.11], $(\mathrm{Id}, Q_v)\colon \tilde{\mathfrak{V}} \to \mathfrak{V}$ is an isomorphism of Jordan pairs. Hence the coefficients of the respective generic minimum polynomials are related by

$$\tilde{m}_i(x, y) = m_i(x, Q_v y).$$

From (3) we obtain by comparing coefficients at τ^{d-1} that

$$\tilde{m}_1(x, y) = N(x) \cdot \partial_y N|_{x^{-1}}.$$

By [18, Th. 3,(17)], we have $\partial_y N|_{x^{-1}} = \mathrm{Tr}((x^{-1})^\sharp, y)$, and by standard properties of the adjoint [10, Sec. 3], $(x^{-1})^\sharp = (x^\sharp)^{-1} = (N(x) x^{-1})^{-1} = N(x)^{-1} x$. Hence, $\tilde{m}_1(x, y) = N(x) \mathrm{Tr}(N(x)^{-1} x, y) = \mathrm{Tr}(x, y)$ holds for all invertible x and thus by density for all $x \in J$. (Since the generic minimum polynomial is compatible with base field extension, we may assume k algebraically closed, so density arguments involving the Zariski topology are justified) . This proves the second formula of (2), and the first follows by specializing $y = v^{-1} = 1_J$. Now [13, 16.8.2] yields the identity

$$\operatorname{Tr}(x, y \circ z) = \operatorname{Tr}(x \circ y, z). \tag{4}$$

Also, if c is any (algebra) idempotent of J and $A = J_2(c) = U_c J$ its Peirce-2-space, then the bilinear trace and the trace of A are just the restrictions of those of J to A:

$$\operatorname{Tr}_A = \operatorname{Tr}|A \times A, \quad \operatorname{tr}_A = \operatorname{tr}|A. \tag{5}$$

Indeed, this follows from the corresponding property of the Faulkner form [16, 1.7] and Lemma 1.2(a), applied to $\tilde{\mathfrak{V}} = (J, J)$ and the Jordan pair idempotent $e = (c, c)$ of $\tilde{\mathfrak{V}}$ whose Peirce-2 space is $\tilde{V}_2(e) = (J_2(c), J_2(c))$.

We define the *defect of* J to be $\operatorname{Def}(J) := \operatorname{Def}(\mathfrak{V}^+)$. From the properties of $\operatorname{Def}(\mathfrak{V})$ and Lemma 1.2 it follows easily that $\operatorname{Def}(J)$ is an outer ideal of J, connected with Tr via

$$\operatorname{Def}(J) = \{x \in J : \operatorname{Tr}(x, J) = 0\}. \tag{6}$$

Thus $\operatorname{Def}(J) = 0$ if and only if Tr is a nondegenerate bilinear form on J. The algebra J will be called *traceless* if $\operatorname{tr} = 0$, equivalently, if $v \in \operatorname{Def}(\mathfrak{V}^-)$ or $1_J \in \operatorname{Def}(J)$. Clearly, if J has zero defect then its trace will be nonzero, but not conversely.

1.4. *Jordan algebras of quadratic forms with base point.* Let k be an algebraically closed field of characteristic 2. As an example, we discuss the Jordan algebras obtained from a Jordan pair of a nondegenerate quadratic form q on k^n, where q is said to be nondegenerate if $q(x) = b(x, y) = 0$ for all $y \in k^n$ implies $x = 0$. Here $b(x, y) = q(x + y) - q(x) - q(y)$ is the bilinear form associated with q. We may then assume that q is in standard form, given by $q(x) = \sum_{i=1}^m x_{2i-1}x_{2i}$ if $n = 2m$, and $x_0^2 + \sum_{i=1}^m x_{2i-1}x_{2i}$ if $n = 2m + 1$, where $x = \sum x_i\varepsilon_i$ with respect to the standard basis (ε_i) of k^n. We always assume $n \geq 3$ because the case $n = 2$ yields a non-simple Jordan pair, and the case $n = 1$ is without interest. The quadratic operators of the Jordan pair $\mathfrak{V} = (k^n, k^n)$ determined by q are

$$Q_x y = b(x, y)x - q(x)y,$$

and $b = m_1$ is the generic trace form of $\mathfrak{V}$. Thus by Lemma 1.2, $\operatorname{Def}(\mathfrak{V}) = 0$ if n is even, while $\operatorname{Def}(\mathfrak{V}^\sigma) = k \cdot \varepsilon_0$ if n is odd.

An element v is invertible in the Jordan pair sense if and only if $q(v) \neq 0$, and it suffices to consider the case $q(v) = 1$. The isotope with respect to v is then the Jordan algebra $J = \operatorname{Jor}(k^n, q, v)$ of the quadratic form with base point (k^n, q, v), with underlying vector space k^n, unit element v, and U-operators given by

$$U_x y = b(x, \bar{y})x - q(x)\bar{y}, \quad \text{where} \quad \bar{y} = b(y, v)v - y. \tag{1}$$

Hence the squaring and the circle product are

$$x^2 = b(x, v)x - q(x)v, \quad x \circ y = b(x, v)y + b(y, v)x - b(x, y)v. \tag{2}$$

The trace form tr of J is $\operatorname{tr}(x) = b(x, v)$. A typical non-defective invertible element is $v = \varepsilon_1 + \varepsilon_2$, while in the odd-dimensional case, $v = \varepsilon_0$ is the only defective element with $q(v) = 1$. The traceless Jordan algebra $\operatorname{Jor}(k^{2m+1}, q, \varepsilon_0)$, then has $\bar{y} = y$, and hence squaring and circle product are given by

$$x^2 = q(x)\varepsilon_0, \quad x \circ y = b(x, y)\varepsilon_0. \tag{3}$$

1.5. Lemma. *Let J be a simple finite-dimensional Jordan algebra of rank r over an algebraically closed field k of characteristic 2. Then J has nonzero trace if and only if J has capacity in the sense of [9, Ch. 6], i.e., there exists an orthogonal system $(c_1, \ldots, c_r)$ of division idempotents of the algebra J such that $1 = c_1 + \cdots + c_r$. For the Peirce decomposition $J = \bigoplus_{1 \le i \le j \le r} J_{ij}$ of J with respect to such a system we have*

$$J_{ii} = k \cdot c_i \quad \text{and} \quad J_{ij} \neq 0 \quad \text{for } i \neq j. \tag{1}$$

The defect of J is

$$\operatorname{Def}(J) = \bigoplus_{i<j} J_{ij}^{\natural}, \tag{2}$$

where

$$J_{ij}^{\natural} = \{x \in J_{ij} : \operatorname{Tr}(x, J_{ij}) = 0\}. \tag{3}$$

Proof. Let $J = \mathfrak{V}_v^+$ where $\mathfrak{V}$ is a simple Jordan pair. Then $v \notin \operatorname{Def}(\mathfrak{V}^-)$ implies, by 1.1, that v is diagonalizable in $\mathfrak{V}$. Hence there exists a frame $F = (d_1, \ldots, d_r)$ of division idempotents of $\mathfrak{V}$ such that $v = d_1^- + \cdots + d_r^-$, and then $c_i = d_i^+$ are the required algebra idempotents. The properties (1) follow from well-known corresponding ones for the Peirce decomposition of $\mathfrak{V}$ with respect to F. Conversely, if J contains an algebra division idempotent c then (c, c) is a Jordan pair division idempotent and $(1 - c) \perp c$, so $1 = \operatorname{Tr}(c, c) = \operatorname{Tr}(c, 1) = \operatorname{tr}(c)$ and thus $\operatorname{tr} \neq 0$. The formula for the defect follows easily from 1.3.6, because the Peirce spaces are orthogonal with respect to Tr and $\operatorname{Tr}(c_i, c_i) = 1$.

We will call such a system of algebra division idempotents a *frame* of the Jordan algebra J. From the conjugacy of frames in the Jordan pair $\mathfrak{V}$ [13, 17.1] it follows easily that any two frames of J are conjugate by an automorphism of J.

In contrast, a traceless J cannot contain any division idempotent in the algebra sense. As a substitute for the Peirce decomposition above, we have the following result. By the rank of an element of J we mean its Jordan pair rank when considered as an element of $\mathfrak{V}^+$. The rank of J is defined as the rank of 1_J.

1.6. Lemma. *Let J be a simple finite-dimensional traceless Jordan algebra of rank r over an algebraically closed field k of characteristic 2.*

(a) *Any algebra idempotent c of J has even rank and belongs to the defect of J. In particular, $r = \mathrm{rk}(1_J) = 2s$ is even. Moreover, c is primitive if and only if it has rank 2.*

(b) *There exist orthogonal systems $c_1, \ldots, c_s$ of primitive algebra idempotents c_i with $c_1 + \cdots + c_s = 1$. Any two such systems are conjugate under an automorphism of J. In the Peirce decomposition $J = \bigoplus_{1\le i\le j\le s} J_{ij}$ with respect to such a system, J_{ii} is the Jordan algebra of a traceless nondegenerate quadratic form with base point of odd dimension ≥ 3, and $J_{ij} \neq 0$ for $i \neq j$.*

(c) *The defect of J is given by*

$$\mathrm{Def}(J) = \bigoplus_{i=1}^{s} k \cdot c_i \;\oplus\; \bigoplus_{i<j} J_{ij}. \tag{1}$$

Proof. (a) As in 1.3 we write $J = \mathfrak{V}_v^+$ where $\mathfrak{V}$ is a simple Jordan pair, and v is a defective invertible element. We put $\mathfrak{W} = \mathrm{Def}(\mathfrak{V})$. An idempotent c of the algebra J satisfies $c = c^2 = Q_c v \in \mathfrak{W}^+ = \mathrm{Def}(J)$ because $\mathfrak{W}$ is an outer ideal. Hence also $\mathrm{rk}(c) = 2\,\mathrm{rk}_{\mathfrak{W}}(c)$ (by [15, Lemma 6(b)]) is even. It follows that $\mathrm{rk}(c) = 2$ implies c is primitive, else $c = c' + c''$ could be decomposed as the sum of two orthogonal algebra idempotents, which would have $\mathrm{rk}(c') = \mathrm{rk}(c'') = 1$. Conversely, let c be a primitive idempotent of J. Then $e' = (c, Q_v c)$ and $e'' = (1 - c, Q_v(1 - c))$ are orthogonal Jordan pair idempotents in $\mathfrak{W}$ with $e' + e'' = (1, v)$. Since $\mathfrak{W}$ has defect zero, $e' = d_1 + \cdots + d_t$ is the orthogonal sum of division idempotents $d_i \in \mathfrak{W}$, so $\mathrm{rk}(d_i) = 2$. Also, $d_i \in W_2(e') = W_0(e'')$, so by the Peirce rules, $(d_i^+)^2 = Q(d_i^+)v = Q(d_i^+)e'^- = Q(d_i^+)d_i^- = d_i^+$ is an algebra idempotent. Similarly, one shows that the d_i^+ are orthogonal. Since $c = d_1^+ + \cdots + d_t^+$, it follows from primitivity of c that $t = 1$, so c has rank 2.

(b) As $\mathfrak{W}$ has defect zero, v is diagonalizable in $\mathfrak{W}$, so we can write $v = d_1^- + \cdots + d_s^-$ where $(d_1, \ldots, d_s)$ is a frame of division idempotents in $\mathfrak{W}$. In particular, the d_i have rank 1 in $\mathfrak{W}$ and therefore rank 2 in $\mathfrak{V}$. Hence the $c_i := d_i^+$ are orthogonal algebra idempotents of J with sum 1_J, and they are primitive because they have rank two. Now J_{ii} is the isotope of the Jordan pair $\mathfrak{V}_2(d_i)$ with respect to d_i^-, and the latter is simple of rank two and has d_i in its defect. By [15, p. 260, Example (a)], $\mathfrak{V}_2(d_i)$ is the Jordan pair of a nondegenerate defective quadratic form, so the structure of J_{ii} follows from 1.4. It is a general fact that Peirce-2-spaces inherit simplicity. This implies $J_{ij} \neq 0$, else $J_2(c_i + c_j) = J_{ii} \oplus J_{jj}$ would not be simple.

By the proof of (a), there is a natural bijection between frames $(d_1, \ldots, d_s)$

of $\mathfrak{W}$ with $\sum d_i^- = v$ and systems of primitive orthogonal idempotents of J with sum 1_J. Now the conjugacy statement follows easily from conjugacy of frames in $\mathfrak{W}$.

(c) By (a), all c_i belong to the defect, and hence also all $J_{ij} = c_i \circ J_{ij}$, because the defect is an outer ideal. Thus we have the inclusion from right to left in (1). On the other hand, $J_{ii} \cap \mathrm{Def}(J)$ is the defect of J_{ii} [15, Prop. 2(c)], and this is $k \cdot c_i$ by 1.4.

1.7. *The primitive rank.* Let J be a simple Jordan algebra over an algebraically closed field of characteristic 2. By Lemmas 1.5 and 1.6, it makes sense to define the *primitive rank* $\mathrm{prk}(J)$ of J by

$$\mathrm{prk}(J) = \begin{Bmatrix} \mathrm{rk}(J) & \text{if } J \text{ has nonzero trace} \\ \frac{1}{2}\,\mathrm{rk}(J) & \text{if } J \text{ is traceless} \end{Bmatrix}.$$

The primitive rank can also be characterized as the maximal length of a system of orthogonal primitive algebra idempotents of J.

1.8. *Classification.* Using the correspondence between Jordan algebras and Jordan pairs (cf. 1.3), the classification of simple unital Jordan algebras is equivalent to

- (i) the classification of simple Jordan pairs $\mathfrak{V}$ containing invertible elements,
- (ii) the determination of the orbits of $\mathrm{Aut}(\mathfrak{V})$ on the set $(\mathfrak{V}^-)^\times$ of invertible elements of $\mathfrak{V}^-$.

The classification of simple finite-dimensional Jordan algebras over algebraically closed fields of characteristic $\neq 2$ is well known, see, e.g., [8]. In the characteristic two case, the classification could be extracted from the much more general results of [19], but there seems to be no explicit handy reference (the one given in [7] is incomplete). However, under our rather restrictive assumptions, it seems simpler to use the procedure outlined above. Step (i) is well-known [13, §17], so it remains to carry out step (ii). It is actually not difficult to determine all orbits of $\mathrm{Aut}(\mathfrak{V})$ on $\mathfrak{V}^\pm$ which will be important in §3.

1.9. Proposition. *Let $\mathfrak{V}$ be a simple finite-dimensional Jordan pair of rank r over an algebraically closed field k and let $\sigma \in \{\pm\}$. Then the automorphism group* $\mathrm{Aut}(\mathfrak{V})$ *and the inner automorphism group* $\mathrm{Inn}(\mathfrak{V})$ *have the same orbits on $\mathfrak{V}^\sigma$, and these orbits are described as follows:*

(a) *If* $\mathrm{Def}(\mathfrak{V}) = 0$ *then x and $\tilde{x}$ belong to the same orbit if and only if* $\mathrm{rk}(x) = \mathrm{rk}(\tilde{x})$. *Hence there are $r+1$ orbits, corresponding to the possible values $0, \dots, r$ of the rank function.*

(b) *If* $\mathrm{Def}(\mathfrak{V}) \neq 0$ *then x and $\tilde{x}$ belong to the same orbit if and only if*

$\mathrm{rk}(x) = \mathrm{rk}(\tilde{x})$, *and both* x *and* $\tilde{x}$ *are diagonalizable or both are defective. Since the rank of a defective element is always even, there are* $r + 1 + [r/2]$ *orbits.*

Proof. The rank function and the defect are clearly invariant under automorphisms, so the conditions listed in (a) and (b) are certainly necessary for x and $\tilde{x}$ to belong to the same orbit. To prove that they are sufficient, let first $x, \tilde{x} \in \mathfrak{V}^\sigma$ both be diagonalizable of the same rank t. Then $x = d_1^\sigma + \cdots + d_t^\sigma$ and $\tilde{x} = \tilde{d}_1^\sigma + \cdots + \tilde{d}_t^\sigma$ can be embedded into frames $(d_1, \dots, d_r)$ and $(\tilde{d}_1, \dots, \tilde{d}_r)$ of division idempotents of $\mathfrak{V}$. By [13, 17.1], there exists an inner automorphism $\varphi = (\varphi_+, \varphi_-)$ of $\mathfrak{V}$ such that $\varphi(d_i) = \tilde{d}_i$, whence $x = \varphi_\sigma(\tilde{x})$. As the rank function takes all values between 0 and r, there are $r + 1$ orbits in case $\mathfrak{V}$ has defect zero.

Next, let x and $\tilde{x}$ both be nonzero and defective of the same rank, and put $\mathfrak{W} := \mathrm{Def}(\mathfrak{V})$. Then $\mathfrak{W}$ is simple and has $\mathrm{Def}(\mathfrak{W}) = 0$ [15, Th. 2], and the rank function of $\mathfrak{W}$ is given by $\mathrm{rk}_W(x) = (1/2)\,\mathrm{rk}(x)$ [15, Lemma 6(b)]. In particular, x and $\tilde{x}$ have even rank. Now x and $\tilde{x}$ are diagonalizable and of the same rank in $\mathfrak{W}$, so they are, by what we proved above, conjugate under some $\psi \in \mathrm{Inn}(\mathfrak{W})$. Here ψ is a finite product of inner automorphisms $\beta(u, v)$, where $(u, v) \in \mathfrak{W}$ is quasi-invertible in $\mathfrak{W}$. Since quasi-invertibility in $\mathfrak{W}$ implies quasi-invertibility in $\mathfrak{V}$, it follows that ψ extends to an inner automorphism φ of $\mathfrak{V}$. The formula for the number of orbits is then immediate.

1.10. Corollary. *Let* $\mathfrak{V}$ *be a simple finite-dimensional Jordan pair containing invertible elements over an algebraically closed field* k.

(a) *If* $\mathfrak{V}$ *contains no defective invertible elements then* $\mathrm{Aut}(\mathfrak{V})$ *acts transitively on* $(\mathfrak{V}^\sigma)^\times$.

(b) *If* $\mathfrak{V}$ *contains defective invertible elements then* $\mathrm{Aut}(\mathfrak{V})$ *has two orbits on* $(\mathfrak{V}^\sigma)^\times$. *In this case,* k *has characteristic* 2 *and the rank of* $\mathfrak{V}$ *is even.*

1.11. *Classification, continued.* Let k be an algebraically closed field of characteristic 2. We now carry out the classification of simple finite-dimensional Jordan algebras outlined in 1.8. Of the list of simple Jordan pairs [13, §17], precisely the following contain invertible elements: I_n, II_{2m}, III_n, IV_n, VI. From the computation of the generic trace form in [13, §17], we see that m_1 is degenerate, and hence, by Lemma 1.2, the defect of $\mathfrak{V}$ is nonzero, only in the cases III_n and IV_{2m+1}.

(a) The isotopes of the types I_n, III_n and VI with respect to the unit matrix yield the Jordan algebras of hermitian matrices with diagonal coefficients in k over $k \oplus k$, k, and the split octonions $\mathcal{O}$, respectively. It is well known that the isotope of II_{2r} with respect to the element $v = \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}$ is the Jordan algebra $\mathrm{H}_r(\mathcal{Q}, k)$ of hermitian matrices over the split quaternion algebra $\mathcal{Q} = \mathrm{Mat}_2(k)$ with respect to the involution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}$, with diagonal coefficients in k.

(b) The defect of the Jordan pair III_n ($n \times n$ symmetric matrices) is the Jordan pair II_n of alternating $n \times n$-matrices, which contains invertible elements if and only if $n = 2p$ is even. In particular, $v = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}$ is such an element, and the isotope with respect to v is easily seen to be isomorphic to $\mathrm{H}_p(\mathcal{Q}, \mathcal{Q}_0)$, hermitian matrices over $\mathcal{Q}$ as above, with diagonal coefficients in the three-dimensional fixed point set $\mathcal{Q}_0$ of $*$.

(c) The case of Jordan pairs of type IV_n, that is, Jordan pairs of a nondegenerate quadratic form, was done in 1.4.

We collect these results in the following table, which also lists the spaces $J \circ J$, determined in 2.7 below.

$\mathfrak{V}$	v	J	rk	prk	tr	$\mathrm{Def}(J)$	$J \circ J$
I_r $(r \geq 1)$	1_r	$\mathrm{H}_r(k \oplus k, k)$	r	r	$\neq 0$	0	Ker tr
II_{2r} $(r \geq 1)$	$\begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}$	$\mathrm{H}_r(\mathcal{Q}, k)$	r	r	$\neq 0$	0	Ker tr
III_r $(r \geq 1)$	1_r	$\mathrm{H}_r(k)$	r	r	$\neq 0$	$\mathrm{Alt}_r(k)$	$\mathrm{Alt}_r(k)$
III_{2p} $(p \geq 1)$	$\begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}$	$\mathrm{H}_p(\mathcal{Q}, \mathcal{Q}_0)$	$2p$	p	0	$\mathrm{H}_p(\mathcal{Q}, k)$	$\mathrm{H}_p(\mathcal{Q}, k)$
IV_{2m} $(m \geq 2)$	$\varepsilon_1 + \varepsilon_2$	$\mathrm{Jor}(k^{2m}, q, \varepsilon_1 + \varepsilon_2)$	2	2	$\neq 0$	0	Ker tr
IV_{2m+1} $(m \geq 1)$	$\varepsilon_1 + \varepsilon_2$	$\mathrm{Jor}(k^{2m+1}, q, \varepsilon_1 + \varepsilon_2)$	2	2	$\neq 0$	$k \cdot \varepsilon_0$	Ker tr
IV_{2m+1} $(m \geq 1)$	ε_0	$\mathrm{Jor}(k^{2m+1}, q, \varepsilon_0)$	2	1	0	$k \cdot \varepsilon_0$	$k \cdot \varepsilon_0$
VI	1_3	$\mathrm{H}_3(\mathcal{O}, k)$	3	3	$\neq 0$	0	Ker tr

There are the following isomorphisms in low ranks:

$$\begin{aligned}
&\mathrm{H}_1(k) \cong \mathrm{H}_1(k \oplus k, k) \cong \mathrm{H}_1(\mathcal{Q}, k) \cong k, \\
&\mathrm{H}_1(\mathcal{Q}, \mathcal{Q}_0) \cong \mathrm{Jor}(k^3, q, \varepsilon_0), \quad \mathrm{H}_2(k) \cong \mathrm{Jor}(k^3, q, \varepsilon_1 + \varepsilon_2), \\
&\mathrm{H}_2(k \oplus k, k) \cong \mathrm{Jor}(k^4, q, \varepsilon_1 + \varepsilon_2), \quad \mathrm{H}_2(\mathcal{Q}, k) \cong \mathrm{Jor}(k^6, q, \varepsilon_1 + \varepsilon_2).
\end{aligned}$$

Finally, we note that $\mathrm{H}_r(k \oplus k, k)$ is isomorphic to the full matrix algebra $\mathrm{Mat}_r(k)$, considered as Jordan algebra.

2. The Lie algebra L associated with J

2.1. *The structure Lie algebra and the derivation algebra.* Let J be a unital quadratic Jordan algebra over a commutative ring R. Recall [8, p. 437] that the structure Lie algebra $\mathrm{Strl}(J)$ of J is the set of all linear maps $A\colon J \to J$ such that there exists a linear map $A^\sharp$ from J to itself with the property that $AU_x + U_x A^\sharp = U_{x,A(x)}$ for all $x \in J$. This $A^\sharp$ is uniquely determined and given by $A^\sharp = U_{1,A(1)} - A$; in fact, the map $A \mapsto A^\vartheta := -A^\sharp$ is an automorphism of period 2 of $\mathrm{Strl}(J)$. Thus

$$A \in \mathrm{Strl}(J) \quad \Longleftrightarrow \quad U_{x,A(x)} = [A, U_x] + U_x U_{1,A(1)} \quad \text{for all } x \in J. \tag{1}$$

The structure Lie algebra is isomorphic to the derivation algebra $\mathrm{Der}(\mathfrak{V})$ of the associated Jordan pair $\mathfrak{V} = (J, J)$ via $A \mapsto (A, A^\vartheta)$. For $x, y, z \in J$ let as usual $\{x, y, z\} = V_{x,y}z = U_{x,z}y$. Then it follows from the defining identities of J that $V_{x,y} \in \mathrm{Strl}(J)$, with $V_{x,y}^\sharp = V_{y,x}$.

A *derivation of* J is an endomorphism Δ satisfying $\Delta(1) = 0$ and $[\Delta, U_x] = U_{\Delta(x),x}$, for all $x \in J$; equivalently, $\Delta \in \mathrm{Der}(J)$ if and only if $\Delta \in \mathrm{Strl}(J)$ and $\Delta(1) = 0$.

Let us specialize now to the case where $2 = 0$ in R. Then each left multiplication $V_x = V_{x,1} : y \mapsto x \circ y$ is in $\mathrm{Der}(J)$ since $x \circ 1 = 2x = 0$. Moreover, $\mathrm{Der}(J)$ is a restricted subalgebra of the restricted Lie algebra $\mathrm{End}(J)$ (with Lie product given by the commutator) because it is easy to see that for any $\Delta \in \mathrm{Der}(J)$, its square Δ^2 is again in $\mathrm{Der}(J)$. This is of course just a special case of the general fact that the Lie algebra of a linear algebraic group in characteristic p is closed under p-th powers and hence a p-Lie algebra. In particular, $\mathrm{Strl}(J)$ is also a 2-Lie algebra.

Recall the following well-known fact:

2.2. Theorem [7, Theorem 4, p. 1.28]**.** *Let J be a quadratic Jordan algebra over a ring R with $2R = 0$. Then the R-module J is a 2-Lie algebra, denoted by $L = L(J)$, with commutator $[a, b] := a \circ b$ and squaring $a^{[2]} := a^2$.*

Thus the adjoint representation of L is given by $\mathrm{ad}(x)y = x \circ y$. From the definitions it follows easily that every derivation of J is also a derivation of the restricted 2-Lie algebra L because $[\Delta, \mathrm{ad}(x)] = \mathrm{ad}(\Delta(x)) = V_{\Delta(x)}$ and $\Delta(x^2) = x \circ \Delta(x) = [x, \Delta(x)]$. This also shows that V_J is an ideal of the Lie algebra $\mathrm{Der}(J)$, and it is even an ideal in the sense of restricted Lie algebras, i.e., a 2-ideal, since the identity QJ20 of [7] says $V_x^2 - V_{x^2} = 2U_x = 0$.

2.3. Lemma. *Let J be a finite-dimensional Jordan algebra over a field k of characteristic 2 and let* tr *be its generic trace. Then* $\mathrm{tr}(x)^2 = \mathrm{tr}(x^2)$*, in particular,* $\mathrm{Ker}\,\mathrm{tr}$ *is closed under squares.*

Proof. This is a consequence of Newton's identities and probably well known. Since there seems to be no reference in the literature covering the present situation, we provide a proof. Let $\mathfrak{V} = (J, J)$ be the Jordan pair associated to J and let $N(x, y) = 1 - m_1(x, y) + m_2(x, y) \mp \cdots$ be its generic norm [13, 16.9]. Also, let $R = k(\varepsilon) \otimes k(\delta)$ be the tensor product of two copies of the dual numbers. For the moment, we don't assume that k has characteristic 2. Since $(\varepsilon+\delta)^2 = 2\varepsilon\delta$ and $(\varepsilon+\delta)^i = 0$ for $i \geq 3$ and the m_i are homogeneous of degree i in x and in y, we have in $\mathfrak{V} \otimes R$,

$$N(x, (\varepsilon+\delta)y) = 1 - (\varepsilon+\delta)m_1(x,y) + 2\varepsilon\delta m_2(x,y). \tag{1}$$

On the other hand, $(x, \varepsilon y)$ is quasi-invertible with quasi-inverse $x^{\varepsilon y} = x + \varepsilon U_x y$, so by [13, Th. 16.11],

$$\begin{aligned} N(x,(\varepsilon+\delta)y) &= N(x,\varepsilon y) \cdot N(x+\varepsilon U_x y, \delta y) \\ &= (1-\varepsilon m_1(x,y)) \cdot (1 - \delta m_1(x+\varepsilon U_x y, y)) \\ &= 1 - (\varepsilon+\delta)m_1(x,y) + \varepsilon\delta(m_1(x,y)^2 - m_1(U_x y, y)). \end{aligned} \tag{2}$$

Comparing coefficients at $\varepsilon\delta$ and putting $y = 1_J$ yields, because $m_1(x, 1_J) = \operatorname{tr}(x)$ by 1.3.1 and 1.3.2, that $\operatorname{tr}(x)^2 - \operatorname{tr}(x^2) = 2m_2(x, 1_J)$, and this vanishes in characteristic 2.

Our next aim is to determine the ideal structure of L in the finite-dimensional simple case. Recall that a simple algebra over a field k is said to be *absolutely simple* if it remains simple under any base field extension.

2.4. Lemma. *Let J be a simple Jordan algebra of primitive rank p over an algebraically closed field k of characteristic 2 and and let $c_1, \ldots, c_p$ be a complete orthogonal system of primitive idempotents, with associated Peirce decomposition $J = \bigoplus_{1 \leq i \leq j \leq p} J_{ij}$.*

(a) *We have*

$$J_{ii}^2 = k \cdot c_i \quad \text{and} \quad J_{ij}^2 = k \cdot (c_i + c_j) \quad \text{for } i \neq j. \tag{1}$$

(b) *Let $a_{ii} \in J_{ii}$. Then*

$$a_{ii} \circ J_{ii} = \left\{ \begin{matrix} 0 & \text{if } a_{ii} \in k \cdot c_i \\ k \cdot c_i & \text{if } a_{ii} \notin k \cdot c_i \end{matrix} \right\}, \tag{2}$$

$$J_{ii} \circ J_{ii} = \left\{ \begin{matrix} 0 & \text{if } \dim J_{ii} = 1 \\ k \cdot c_i & \text{if } \dim J_{ii} > 1 \end{matrix} \right\}. \tag{3}$$

(c) *Suppose J has nonzero trace. Then for $i \neq j$ and $a_{ij} \in J_{ij}$,*

$$a_{ij} \circ J_{ij} = \left\{ \begin{matrix} 0 & \text{if } a_{ij} \in J_{ij}^\natural \\ k \cdot (c_i + c_j) & \text{otherwise} \end{matrix} \right\} \tag{4}$$

where $J_{ij}^\natural$ is defined in 1.5.3. Also,

$$\dim J_{ij}^\natural = \left\{ \begin{matrix} 0 & \text{if } \dim J_{ij} \text{ is even} \\ 1 & \text{if } \dim J_{ij} \text{ is odd} \end{matrix} \right\} \tag{5}$$

and hence

$$J_{ij} \circ J_{ij} = \left\{ \begin{array}{ll} 0 & \textit{if } \dim J_{ij} = 1 \\ k \cdot (c_i + c_j) & \textit{if } \dim J_{ij} > 1 \end{array} \right\}. \tag{6}$$

Proof. J_{ii} is either $k \cdot c_i$ or the Jordan algebra of a traceless nondegenerate quadratic form q with base point c_i of dimension ≥ 3. Now the first formula of (1), formula (2) and hence (3) follow easily from 1.4.3.

Let $A := k \cdot c_i \oplus J_{ij} \oplus k \cdot c_j$. If J has nonzero trace, the c_i are division idempotents so $J_{ii} = k \cdot c_i$ and therefore $A = J_2(c_i + c_j)$. If, on the other hand, J is traceless and $K := \mathrm{Def}(J)$, then the c_i are division idempotents of K and 1.6.1 shows that $A = K_2(c_i + c_j)$. Thus in any case, A is a subalgebra which is simple of rank 2 and has nonzero trace. By the classification, $A \cong \mathrm{Jor}(k^n, q, \varepsilon_1 + \varepsilon_2)$ is the Jordan algebra of a nondegenerate quadratic form q with nonzero trace of dimension $n \geq 3$. Thus q is given by $q(\lambda c_i + \mu c_j + x_{ij}) = \lambda\mu + q(x_{ij})$ and $q|J_{ij}$ is again nondegenerate. Since J_{ij} is orthogonal to $c_i + c_j = 1_A$, formula 1.4.2 shows $x^2 = q(x)(c_i + c_j)$ for $x \in J_{ij}$. This proves the second formula of (1).

Now let J have nonzero trace. By 1.4, the bilinear trace of A is $\mathrm{Tr}_A = b$, the bilinear form associated to q, and by 1.3.5 this is also the restriction of the bilinear trace Tr of J to A. Thus we have $x \circ y = \mathrm{Tr}(x, y)(c_i + c_j)$ for all $x, y \in J_{ij}$ and therefore (4). In characteristic 2, a nondegenerate quadratic form has its associated bilinear form equal to zero if and only if it is one-dimensional. Hence we have (5) and (6).

2.5. Proposition. *Let J be a finite-dimensional absolutely simple Jordan algebra over a field k of characteristic 2. Then $V_x = 0$ for $x \in J$ if and only if $x \in k \cdot 1$, so the centre of the associated Lie algebra $L = L(J)$ is $k \cdot 1$.*

Proof. We have $V_1 = 2\mathrm{Id} = 0$ since k has characteristic two. For the converse, it is no restriction, after extending scalars, to assume k algebraically closed. We use the Peirce decompositions given in Lemmas 1.5 and 1.6. Choose an orthogonal system $c_1, \ldots, c_p$ of primitive idempotents of J such that $c_1 + \cdots + c_p = 1$, decompose $x = \sum_{1 \leq i \leq j \leq p} x_{ij}$ relative to $(c_1, \ldots, c_p)$, and put $x_{ij} = x_{ji}$ for convenience. Then for all $l = 1, \ldots, p$, using the Peirce rules and $2 = 0$ in k, we have

$$0 = V_x c_l = x_{ll} \circ c_l + \sum_{i<j} x_{ij} \circ c_l = 2x_{ll} + \sum_{i \neq l} x_{il} \circ c_l = \sum_{i \neq l} x_{il}.$$

This shows that all off-diagonal x_{il} vanish, so $x = \sum_{i=1}^{p} x_{ii}$. Furthermore, $0 = x \circ J_{ii} = x_{ii} \circ J_{ii}$ implies $x_{ii} = \lambda_i c_i$ is a scalar multiple of c_i, by 2.4.2.

Now let $i \neq j$. Since all off-diagonal Peirce spaces J_{ij} are nonzero, we may choose $0 \neq z_{ij} \in J_{ij}$ and then obtain $0 = x \circ z_{ij} = \lambda_i c_i \circ z_{ij} + \lambda_j c_j \circ z_{ij} = (\lambda_i + \lambda_j) z_{ij}$, whence $\lambda_i + \lambda_j = 0$ or $\lambda_i = \lambda_j$. It follows that $x = \lambda(c_1 + \cdots + c_p) = \lambda \cdot 1$, as asserted.

2.6. Corollary. *We keep the assumptions of* 2.5, *and assume that* Tr *is* *nondegenerate. Then* $[L,L] = \operatorname{Ker}\operatorname{tr}$ *is a 2-ideal of codimension one.*

Proof. For a subspace X of J let $X^\perp = \{y \in J : \operatorname{Tr}(X,y) = 0\}$. Then $y \in [L,L]^\perp \iff 0 = \operatorname{Tr}(J \circ J, y) = \operatorname{Tr}(J, J \circ y)$ (by 1.3.4) $\iff J \circ y = 0$ (by nondegeneracy of Tr) $\iff y \in k \cdot 1$ (by 2.5). Again by nondegeneracy of Tr, it follows that $[L,L] = [L,L]^{\perp\perp} = (k \cdot 1)^\perp = \operatorname{Ker}\operatorname{tr}$, and $\operatorname{Ker}\operatorname{tr}$ is closed under squares by Lemma 2.3.

2.7. Proposition. *Let J be a finite-dimensional simple Jordan algebra of rank $r \geq 2$ over an algebraically closed field k of characteristic 2, and L the associated Lie algebra.*

(a) *If J is traceless we have $[L,L] = L^{[2]} = \operatorname{Def}(J)$.*

(b) *If J has nonzero trace, let $c_1, \ldots, c_r$ be a frame of division idempotents of J, and let $J = \bigoplus_i k \cdot c_i \oplus \bigoplus_{i \leq j} J_{ij}$ the associated Peirce decomposition. Then*

$$[L,L] = \left\{ \begin{array}{ll} \operatorname{Def}(J) = \bigoplus_{i<j} J_{ij} & \textit{if } \dim J_{12} = 1 \\ \operatorname{Ker}\operatorname{tr} = \{\sum \lambda_i c_i : \sum \lambda_i = 0\} \oplus \bigoplus_{i<j} J_{ij} & \textit{if } \dim J_{12} > 1 \end{array} \right\}$$

and $L^{[2]} = L$.

Proof. (a) We have $1 \in \operatorname{Def}(J)$, and hence also $J \circ J = \{J, 1, J\} \subset \operatorname{Def}(J)$, because the defect is an outer ideal. To prove the reverse inclusion, we choose a frame $c_1, \ldots, c_p$ of primitive idempotents and use formula 1.6.1 for the defect: $J_{ij} = c_i \circ J_{ij} \subset J \circ J$ for $i \neq j$, and J_{ii} has dimension ≥ 3 by 1.6(b), so $c_i \in J \circ J$ by 2.4.3. Finally, $L^{[2]} = J^2 = \sum_i J_{ii}^2 + \sum_{i<j} J_{ij}^2 + J \circ J$, so $L^{[2]} = J \circ J$ follows from Lemma 2.4(a).

(b) Always $[L,L] \subset \operatorname{Ker}\operatorname{tr}$, because

$$\operatorname{tr}(x \circ y) = \operatorname{Tr}(x \circ y, 1) = \operatorname{Tr}(x, y \circ 1) = 2\operatorname{Tr}(x,y) = 0.$$

As before, $J_{ij} = c_i \circ J_{ij} \subset J \circ J$ for $i \neq j$. The only way elements in $(J \circ J) \cap \sum J_{ii}$ can arise is from $J_{ij} \circ J_{ij}$ for $i \neq j$, because $J_{ii} \circ J_{jj} = 0$, and $J_{ij} \circ J_{jl} \subset J_{il}$ for different i, j, l. From conjugacy of frames, it follows that $\dim J_{12} = \dim J_{ij}$ for all $i \neq j$. Hence there are the following two cases:

(i) $\dim J_{ij} = 1$ for all $i \neq j$. Then $J_{ij} \circ J_{ij} = 0$ by 2.4.6, so $J \circ J = \bigoplus_{i<j} J_{ij} = \operatorname{Def}(J)$, by 1.5.2 and 2.4(c).

(ii) $\dim J_{ij} > 1$ for all $i \neq j$. Then $J \circ J$ contains all J_{ij} as well as all $c_i + c_j$, for $i \neq j$, by 2.4.6. Now an element $x = \sum_{i=1}^n \lambda_i c_i$ has $\operatorname{tr}(x) = \sum_{i=1}^n \lambda_i$ because $\operatorname{tr}(c_i) = 1$, and the formula $x = \operatorname{tr}(x) c_1 + \sum_{i=2}^n \lambda_i (c_i + c_1)$ shows that every element of trace zero belongs to $J \circ J$. The final assertion is clear from the fact that all $c_i = c_i^2 \in L^{[2]}$.

It is now easy to determine the derived series of the Lie algebra L. Recall that this is defined inductively by $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. We also use the notation L', L'' etc. instead of $L^{(1)}, L^{(2)}$ etc.

2.8. Corollary. *Let J be a finite-dimensional simple Jordan algebra of rank $r \geq 2$ over an algebraically closed field k of characteristic 2, with associated Lie algebra L.*

(a) *If J has nonzero trace and $r \geq 3$ then*

$$L \supsetneqq L' = L'' \neq 0,$$

where $L' = [L, L]$ is described in 2.7(b).

(b) *Let J be traceless and $r = 2s \geq 6$, and put $K = \mathrm{Def}(J)$. Then K is a simple Jordan algebra of rank s with nondegenerate trace form Tr_K and hence $\mathrm{tr}_K \neq 0$, and the derived series of L is*

$$L \supsetneqq L' = K \supsetneqq L'' = \mathrm{Ker}(\mathrm{tr}_K) = L''' \neq 0.$$

(c) *If J is traceless of rank 2, i.e., $J = \mathrm{Jor}(k^{2m+1}, q, \varepsilon_0)$ is the Jordan algebra of a traceless quadratic form with base point of dimension $2m+1 \geq 3$, then*

$$L \supsetneqq L' = k \cdot 1_J \supsetneqq L'' = 0.$$

(d) *If J has rank 2, dimension $n \geq 4$ and nonzero trace, i.e., $J = \mathrm{Jor}(k^n, q, \varepsilon_1 + \varepsilon_2)$ is the Jordan algebra of a quadratic form with base point and nonzero trace, then*

$$L \supsetneqq L' = \mathrm{Ker}\,\mathrm{tr} \supsetneqq L'' = k \cdot 1 \supsetneqq L''' = 0.$$

(e) *If $J = \mathrm{Jor}(k^3, q, \varepsilon_1 + \varepsilon_2) \cong \mathrm{H}_2(k)$ is the Jordan algebra of a 3-dimensional quadratic form with base point and nonzero trace, then*

$$L \supsetneqq L' = k \cdot \varepsilon_0 \supsetneqq L'' = 0.$$

(f) *Let J be traceless and $r = 4$, and let $K = \mathrm{Def}(J)$. Then $K = \mathrm{Jor}(k^6, q, \varepsilon_1 + \varepsilon_2)$ is as in case (d), and the derived series is*

$$L \supsetneqq L' = K \supsetneqq L'' = \mathrm{Ker}(\mathrm{tr}_K) \supsetneqq L''' = k \cdot 1 \supsetneqq L^{(4)} = 0.$$

Proof. (a) By 2.7(b), $J_{ij} \subset [L, L]$ for $i \neq j$. Since $r \geq 3$, there exists $l \neq i, j$, and therefore $J_{ij} = J_{il} \circ J_{lj} \subset L''$. If $\dim J_{12} = 1$ then by 2.7(b), $L' = \bigoplus_{i<j} J_{ij} \subset L''$. If $\dim J_{12} > 1$ then again by 2.7(b), $[L, L] = \bigoplus_{i<j} J_{ij} + \sum_{i<j} k \cdot (c_i + c_j)$. Here $c_i + c_j \in J_{ij} \circ J_{ij} \subset L''$, so we again have $L'' = L'$.

(b) Here $L' = K$ by 2.7(a), and K has rank $s \geq 3$ and nondegenerate trace form, so the assertion follows from what we proved in (a) (applied to K) and Cor. 2.6.

(c), (d), (e) This follows by easy computations in the Jordan algebra of a quadratic form with base point. The details are left to the reader.

(f) By 2.7(a), $K = \mathrm{Def}(J)$ is a rank two algebra with nondegenerate trace form Tr_K, hence of the form $K = \mathrm{Jor}(k^{2m}, q, \varepsilon_1 + \varepsilon_2)$, and $m \geq 2$. These algebras have been dealt with in case (d), so the form of the derived series in this case follows from (d). In fact, by the classification in 1.11 we have $J = \mathrm{H}_2(\mathcal{Q}, \mathcal{Q}_0)$ and $K = \mathrm{H}_2(\mathcal{Q}, k) \cong \mathrm{Jor}(k^6, q, \varepsilon_1 + \varepsilon_2)$.

2.9. Corollary. *Under the assumptions of* 2.8*, L is a solvable Lie algebra if and only if J has primitive rank ≤ 2.*

We can now determine all ideals of the Lie algebra L, which we also call Lie ideals to distinguish them from the Jordan ideals. A Lie ideal which is closed under the squaring map will be called a 2-*ideal.* Of course, 0, L, and the centre $Z(L) = k \cdot 1$ (by 2.5) are always Lie ideals and even 2-ideals, so we will concentrate on the non-central proper ideals. Also the defect, being an outer ideal of J, is a Lie ideal (but not necessarily a 2-ideal, as the example $J = \mathrm{H}_r(k)$, $\mathrm{Def}(J) = \mathrm{Alt}_r(k)$ shows). Note that in any Lie algebra L, an arbitrary subspace containing the derived algebra $L' = [L, L]$ is always an ideal.

2.10. Lemma. *Let J be a simple finite-dimensional Jordan algebra of rank r and primitive rank p over an algebraically closed field k of characteristic 2. Let $c_1, \ldots, c_p$ be an orthogonal system of primitive idempotents of J, with associated Peirce decomposition $J = \sum_{1 \leq i \leq j \leq p} J_{ij}$. Also let $\mathfrak{a}$ be a Lie ideal of J.*

(a) *We have*

$$\mathfrak{a} = \mathfrak{a} \cap \Big(\sum_{i=1}^{p} J_{ii} \Big) \;\oplus\; \bigoplus_{i<j} (\mathfrak{a} \cap J_{ij}). \tag{1}$$

(b) *Suppose that $p \geq 2$. Then $\mathfrak{a}$ is central if and only if $\mathfrak{a} \cap J_{ij} = 0$ for all $i \neq j$.*

(c) *Suppose that $r \geq 3$. Then a non-central Lie ideal $\mathfrak{a}$ contains all J_{ij}, $i \neq j$. If $\mathfrak{a}$ is a 2-ideal then it also contains all $c_i + c_j$, $i \neq j$.*

Proof. (a) Decompose an arbitrary element $a \in \mathfrak{a}$ into its Peirce components: $a = \sum a_{ij}$. Then $c_i \circ (c_j \circ a) = a_{ij}$ for $i \neq j$ by the Peirce rules and because k has characteristic 2, so $a_{ij} \in \mathfrak{a}$, and we have (1).

(b) If $\mathfrak{a} \subset k \cdot 1$ is central then clearly $\mathfrak{a} \cap J_{ij} = 0$. Conversely, let $\mathfrak{a} \cap J_{ij} = 0$ for all $i \neq j$, so $\mathfrak{a} \subset \bigoplus_{i=1}^{p} J_{ii}$ by (1). Decompose an element $a \in \mathfrak{a}$ accordingly as $a = \sum a_{ii}$. If $a_{ii} \notin k \cdot c_i$ for some i then by 2.4.2, $a \circ J_{ii} = a_{ii} \circ J_{ii} = k \cdot c_i \subset \mathfrak{a}$, and hence $c_i \circ J_{ij} = J_{ij} \subset \mathfrak{a}$, contradiction. Thus we have $a_{ii} = \lambda_i c_i \in k \cdot c_i$ for all i. Now $a \circ J_{ij} = (\lambda_i + \lambda_j) J_{ij} \subset \mathfrak{a} \cap J_{ij} = 0$ for $i \neq j$ shows $\lambda_i + \lambda_j = 0$, so all λ_i are equal, and therefore $a \in k \cdot 1$.

(c) Let $\mathfrak{a}$ be a non-central Lie ideal. First, let J have nonzero trace. Then $p = r \geq 3$ and $J_{ii} = k \cdot c_i$ is one-dimensional. By (b) we have, say, $\mathfrak{a} \cap J_{12} \neq 0$. If J_{12} has dimension 1, this already implies $J_{12} \subset \mathfrak{a}$. Otherwise, as a consequence of the classification ($r \geq 3$ is essential here), J_{12} has even dimension ≥ 2, and it follows from 2.4(c) that $a_{12} \circ J_{12} = k \cdot (c_1 + c_2)$ for any nonzero $a_{12} \in \mathfrak{a} \cap J_{12}$, so we conclude that $c_1 + c_2 \in \mathfrak{a}$. Since $r \geq 3$, $(c_1 + c_2) \circ J_{13} = c_1 \circ J_{13} = J_{13} \subset \mathfrak{a}$. Now the well-known relations

$$J_{ij} = J_{il} \circ J_{lj} \tag{2}$$

for three distinct indices i, j, l imply that $\mathfrak{a}$ contains all J_{ij}, $i \neq j$.

Next, assume J traceless. Since the rank of a traceless algebra is even, we then have $r = 2p \geq 4$. By the classification 1.11, $J = \mathrm{H}_p(\mathcal{Q}, \mathcal{Q}_0)$ is the algebra of hermitian matrices over the split quaternion algebra $\mathcal{Q} = \mathrm{Mat}_2(k)$ with diagonal coefficients in the fixed point space $\mathcal{Q}_0 = \{\left(\begin{smallmatrix}\alpha\beta\\ \gamma\alpha\end{smallmatrix}\right) : \alpha, \beta, \gamma \in k\}$ of the involution $*$. Denote the usual matrix units by E_{ij} and put $a[ii] := aE_{ii}$ and $b[ij] := bE_{ij} + b^* E_{ji}$, for $i \neq j$, where $a \in \mathcal{Q}_0$, $b \in \mathcal{Q}$. Then $c_i = E_{ii}$ ($i = 1, \ldots, p$) is a frame of primitive idempotents of J, with Peirce spaces $J_{ii} = \mathcal{Q}_0[ii]$ and $J_{ij} = \mathcal{Q}[ij]$.

Let again $\mathfrak{a}$ be a non-central Lie ideal. Then by (b) we have, say, $M := \mathfrak{a} \cap J_{12} \neq 0$. Since $\mathfrak{a}$ is a Lie ideal, it follows from the Peirce rules that $J_{ii} \circ M \subset M$. A computation shows $a[11] \circ b[12] = (ab)[12]$ and $a[22] \circ b[12] = (ba)[12]$, for all $a \in \mathcal{Q}_0$, $b \in \mathcal{Q}$. Hence $M = B[12]$ where $B \subset \mathcal{Q}$ is a subspace with the property that $\mathcal{Q}_0 B + B\mathcal{Q}_0 \subset B$. It is an easy exercise to show that B is then a two-sided ideal of the associative algebra $\mathcal{Q}$. As $\mathcal{Q}$ is simple and $M \neq 0$, it follows that $M = J_{12} \subset \mathfrak{a}$. Now $J_{ij} \subset \mathfrak{a}$ for all $i \neq j$ follows again from (2). Finally, the last statement is immediate from the fact that $J_{ij}^2 = k \cdot (c_i + c_j)$ by 2.4.1.

2.11. Theorem. *Let J be a simple finite-dimensional Jordan algebra of rank $r \geq 2$ and primitive rank p over an algebraically closed field k of characteristic 2, and let $L = L(J)$ be the associated 2-Lie algebra with underlying vector space J, Lie product $[x, y] = x \circ y$ and squaring $x^{[2]} = x^2$. We choose a complete orthogonal system $c_1, \ldots, c_p$ of primitive idempotents of J, with associated Peirce decomposition $J = \sum_{1 \leq i \leq j \leq p} J_{ij}$.*

(a) *If J has nonzero trace and* $\dim J_{ij} = 1$*, i.e., $J \cong \mathrm{H}_r(k)$, then the proper non-central Lie ideals of L are precisely the subspaces $\mathfrak{a}$ satisfying $L \supsetneqq \mathfrak{a} \supset L' \cong \mathrm{Alt}_r(k)$, while* $\mathrm{Ker}\,\mathrm{tr}$ *is the only proper non-central 2-ideal.*

(b) *If J has nonzero trace, $r \geq 3$ and* $\dim J_{ij} > 1$ *for $i \neq j$, i.e., $J \cong \mathrm{H}_r(\mathcal{C}, k)$ where $\mathcal{C}$ is a composition algebra of dimension ≥ 2 (and $r = 3$ in case $\mathcal{C} = \mathcal{O}$) then $L' =$* $\mathrm{Ker}\,\mathrm{tr}$ *is the only non-central proper Lie ideal of L, and it is a 2-ideal.*

(c) *If J is traceless of rank $r = 2p \geq 4$, i.e., $J \cong \mathrm{H}_p(\mathcal{Q}, \mathcal{Q}_0)$ then the proper non-central Lie ideals of L are L'' and all subspaces $\mathfrak{a}$ with $L \supsetneqq \mathfrak{a} \supset L'$. Here $L' = K := \mathrm{Def}(J) \cong \mathrm{H}_p(\mathcal{Q}, k)$ and $L'' = \mathrm{Ker}\,\mathrm{tr}_K$. All Lie ideals are 2-ideals.*

(d) *Let J have nonzero trace, rank 2 and dimension $n \geq 4$, so that $J = \mathrm{Jor}(k^n, q, \varepsilon_1 + \varepsilon_2)$ is the Jordan algebra of a quadratic form with base point and nonzero trace. Then for n even, the proper non-central Lie ideals of L are precisely the subspaces $\mathfrak{a}$ with $k \cdot 1 \subsetneqq \mathfrak{a} \subset L' = \mathrm{Ker}\,\mathrm{tr}$, and in addition, for n odd, the 1-dimensional ideal $\mathrm{Def}(J) = k \cdot \varepsilon_0$. All Lie ideals are 2-ideals.*

(e) *If J is traceless of rank 2, i.e., $J = \mathrm{Jor}(k^{2m+1}, q, \varepsilon_0)$ is the Jordan algebra of a traceless quadratic form with base point, then $L' = Z(L) = k \cdot 1$ by 2.8(c), and the proper non-central ideals of L are precisely the subspaces $L \supsetneqq \mathfrak{a} \supsetneqq k \cdot 1$. All Lie ideals are 2-ideals.*

Proof. (a) We leave the case $r = 2$, which consists of simple computations with 2×2-matrices, to the reader, and assume $r \geq 3$. By Lemma 2.10(c) and 2.7(b), we have $\mathfrak{a} \supset L' = \bigoplus_{i<j} J_{ij}$, and by the remark made after Cor. 2.9, these spaces are indeed Lie ideals. If $\mathfrak{a}$ is a 2-ideal then it contains all $c_i + c_j$ by Lemma 2.4(a), and therefore $\mathfrak{a} = \mathrm{Ker}\,\mathrm{tr}$. Conversely, $\mathrm{Ker}\,\mathrm{tr}$ is closed under squares by 2.3 and hence a 2-ideal.

(b) By Lemma 2.10(c), $\mathfrak{a}$ contains all J_{ij} $(i \neq j)$, and $c_i + c_j \in J_{ij} \circ J_{ij} \subset \mathfrak{a}$ by 2.4.6, whence $\mathfrak{a} = \mathrm{Ker}\,\mathrm{tr}$.

(c) By 1.6.1, the defect is $K = \sum k \cdot c_i \oplus \bigoplus_{i<j} J_{ij}$. By (b) and (f) of 2.8, we have $L' = K$ and $L'' = \mathrm{Ker}(\mathrm{tr}_K) = \{\sum \lambda_i c_i : \sum \lambda_i = 0\} + \bigoplus_{i<j} J_{ij}$. All $J_{ij} \subset \mathfrak{a}$ for $i \neq j$, by Lemma 2.10(c). Furthermore, the classification shows that $\dim J_{ij} = 4$. Now 2.4.6, applied to K, which has nonzero trace and Peirce spaces $K_{ij} = J_{ij}$, yields $c_i + c_j \in \mathfrak{a}$ for all $i \neq j$. This shows $L'' \subset \mathfrak{a}$. Now assume $L'' \subsetneqq \mathfrak{a}$. As $\mathfrak{a}$ contains all J_{ij} for $i \neq j$, there must be an element of the form $a = \sum a_{ii} \in \mathfrak{a}$ for which either $a_{ii} \notin k \cdot c_i$ for some i, or for which all $a_{ii} = \lambda_i c_i$ but $\sum \lambda_i \neq 0$. In the first case, $a \circ J_{ii} = a_{ii} \circ J_{ii} = k \cdot c_i \subset \mathfrak{a}$ by 2.4.2, so $K = k \cdot c_i + K' \subset \mathfrak{a}$. In the second case, $a \in K$ and $\mathrm{tr}_K(a) = \sum \lambda_i \neq 0$, so again $K = k \cdot a + K' \subset \mathfrak{a}$. Since L'' is closed under squares by Lemma 2.3 and $L^{[2]} = K$ by Prop. 2.7(a), all these ideals are 2-ideals.

(d) A frame of division idempotents of J is $c_i = \varepsilon_i$ $(i = 1, 2)$, with Peirce spaces $J_{ii} = k \cdot c_i$ and $J_{12} = \sum_{j \neq 1,2} k \cdot \varepsilon_j$, of dimension $n - 2 \geq 2$. The trace is given by $\mathrm{tr}(c_i) = 1$ and $\mathrm{tr}(J_{12}) = 0$. Hence by 2.7(b), $L' = \mathrm{Ker}\,\mathrm{tr} = k \cdot 1 + J_{12}$, so that a subspace $\mathfrak{a}$ with $k \cdot 1 \subsetneqq \mathfrak{a} \subset L'$ has the form $\mathfrak{a} = k \cdot 1 \oplus M$, for an arbitrary nonzero subspace M of J_{12}. From 1.4.2 it follows easily that such $\mathfrak{a}$ are non-central proper ideals of L. Also, in the odd-dimensional case, it is clear that $\mathrm{Def}(J) = k \cdot \varepsilon_0$, which is an outer ideal of J, is an ideal of L. Conversely, let $\mathfrak{a}$ be a non-central proper ideal of L. Then by Lemma 2.10, $\mathfrak{a} = (\mathfrak{a} \cap (k \cdot c_1 + k \cdot c_2)) \oplus M$, where $M := \mathfrak{a} \cap J_{12} \neq 0$. We claim that $\mathfrak{a} \cap (k \cdot c_1 + k \cdot c_2) \subset k \cdot 1$. Indeed, suppose $\mathfrak{a}$ contains an element $a = \lambda_1 c_1 + \lambda_2 c_2$ with $\lambda_1 \neq \lambda_2$. Then $a \circ J_{12} = (\lambda_1 + \lambda_2) J_{12} = J_{12} \subset \mathfrak{a}$. As $\dim J_{12} > 1$, the restriction to J_{12} of the bilinear form b associated to q is nonzero, so by 1.4.2, $J_{12} \circ J_{12} = k \cdot 1 \subset \mathfrak{a}$, which implies $c_1, c_2 \in \mathfrak{a}$, and thus $\mathfrak{a} = L$ would not be a proper ideal. Now there are two possibilities:

Case 1: $\mathfrak{a} \cap (k \cdot c_1 + k \cdot c_2) = k \cdot 1$. Then $\mathfrak{a} = k \cdot 1 \oplus M$ has the required form.

Case 2: $\mathfrak{a} \cap (k \cdot c_1 + k \cdot c_2) = 0$. Then $M \circ J_{12} = b(M, J_{12}) \cdot 1 \subset \mathfrak{a} \cap (k \cdot c_1 + k \cdot c_2) = 0$. From $b(M, c_i) = 0$ we see that M is contained in the kernel of b. As $M \neq 0$, it follows that n is odd and $\mathfrak{a} = M = k \cdot \varepsilon_0$, the defect of J.

Finally, part (e) (where $r = 2$) is evident from 2.8(c). The fact that all Lie ideals are 2-ideals follows easily from the formulas for x^2 in 1.4.

We now discuss simplicity of subquotients of L. Since L is solvable for $p = \operatorname{prk}(J) \leq 2$, we assume $p \geq 3$.

2.12. Corollary. *Let J be a simple finite-dimensional Jordan algebra over an algebraically closed field k of characteristic 2, of rank r and of primitive rank $p \geq 3$.*

(a) *Let J have nonzero trace, so $r = p \geq 3$.*

(i) *If $\dim J_{12} = 1$ (and thus $J \cong \mathrm{H}_r(k)$) then $L' = \mathrm{Alt}_r(k)$ is simple for $r \neq 4$, and isomorphic to $\mathrm{Alt}_3(k) \times \mathrm{Alt}_3(k)$ for $r = 4$.*

(ii) *If $\dim J_{12} > 1$ then $L'/Z(L) \cap L'$ is simple.*

(b) *Let J be traceless, so $p = r/2 \geq 3$ and $J = \mathrm{H}_p(\mathcal{Q}, \mathcal{Q}_0)$ by 1.11. Then $L''/Z(L) \cap L''$ is simple.*

Proof. (a) The well-known proof about the structure of the Lie algebra $\mathrm{Alt}_r(k)$ works in any characteristic and yields (i).

Next, consider case (ii). By 2.7(b), $L' = \operatorname{Ker} \operatorname{tr}$ contains all $c_i + c_j$. Now let $\mathfrak{a} \subset L'$ be an ideal of L' with $\mathfrak{a} \not\subset k \cdot 1$. By 2.4.6, it suffices to show that $J_{ij} \subset \mathfrak{a}$ for all $i \neq j$. Decompose an element $a \in \mathfrak{a} \setminus k \cdot 1$ as $a = \sum \lambda_i c_i + \sum_{i<j} a_{ij}$ where $\operatorname{tr}(a) = \sum \lambda_i = 0$. If all $a_{ij} = 0$ then not all λ_i can be equal; say, $\lambda_1 \neq \lambda_2$. Hence $a \circ J_{12} = (\lambda_1 + \lambda_2) J_{12} = J_{12} \subset \mathfrak{a}$.

If, say, $a_{12} \neq 0$ then by 2.4(c), $c_1 + c_2 \in \mathfrak{a}$, hence $(c_1 + c_2) \circ J_{13} = J_{13} \subset \mathfrak{a}$. Now 2.10.2 implies that $\mathfrak{a}$ contains all J_{ij}.

(b) Here $K = \operatorname{Def}(J) = L'$ has nonzero trace and $\operatorname{rk}(K) = p \geq 3$. Also $\dim K_{12} = \dim J_{12} = 4$, so the assertion follows from case (ii) of (a) applied to K.

2.13. Corollary. *Let $J = H_3(\mathcal{O}, k)$ where $\mathcal{O}$ is an octonion algebra and k has characteristic 2 but is not necessarily algebraically closed. Then V_J is a simple 2-ideal of dimension 26 in $\operatorname{Der}(J)$.*

Proof. As noted after Th. 2.2, V_J is always a 2-ideal in $\operatorname{Der}(J)$, so it remains to show that V_J is simple. Since J is absolutely simple we may assume k

algebraically closed. Let $L = L(J)$ as in 2.2. We have $Z(L) = k \cdot 1_J$ by Prop. 2.5, and $\mathrm{Ker}(\mathrm{tr}) = L'$ by Prop. 2.7(b). It follows that $Z(L) \cap L' = \{0\}$, so $V_J \cong J/k \cdot 1_J \cong L' = L'/Z(L) \cap L'$ is simple by case (ii) of the previous Corollary.

For later use, we now show that the Lie algebras $L(\mathrm{H}_n(\mathcal{Q}, k))$ are isomorphic to Lie algebras of orthogonal groups of even rank, and that their trace forms correspond to the spinor trace. We begin by introducing this concept, the infinitesimal version of Bass' spinor norm [3].

2.14. *The spinor trace.* Let k be a an arbitrary field and let q be a nondegenerate quadratic form on a k-vector space X of dimension $2n$. We denote by $\mathrm{Cliff}(q)$ the Clifford algebra of q, by $\mathrm{Cliff}_0(q)$ its even part, by $\mathrm{O}(q)$ the orthogonal group, and by

$$\mathrm{Spin}(q) = \{u \in \mathrm{Cliff}_0(q)^\times : uXu^{-1} = X \text{ and } uu^* = 1\} \tag{1}$$

the spin group, cf. [11, Ch. IV]. Here $*$ denotes the main involution of $\mathrm{Cliff}(q)$, characterized by $x^* = x$ for all $x \in X$, where we identify X with its canonical image in $\mathrm{Cliff}(q)$. By "varying the base ring" we obtain the spin group and orthogonal group schemes of q, i.e.,

$$\mathbf{Spin}(q)(R) = \mathrm{Spin}(q \otimes R) \quad \text{for all } R \in k\text{-}\mathbf{alg},$$

and similarly for $\mathbf{O}(q)$. The Lie algebras of $\mathbf{Spin}(q)$ and $\mathbf{O}(q)$ are then

$$\mathfrak{spin}(q) = \{v \in \mathrm{Cliff}_0(q) : [v, X] \subset X \text{ and } v + v^* = 0\}, \tag{2}$$

$$\mathfrak{o}(q) = \{A \in \mathrm{End}(X) : b(x, A(x)) = 0 \text{ for all } x \in X\}, \tag{3}$$

where b is the bilinear form associated with q. Let $\chi\colon \mathbf{Spin}(q) \to \mathbf{O}(q)$ be the vector representation, and $\dot\chi := \mathrm{Lie}(\chi)\colon \mathfrak{spin}(q) \to \mathfrak{o}(q)$ the corresponding Lie algebra homomorphism. It is well known that $\mathrm{Cliff}(q)$ is an Azumaya algebra. From this, one sees easily that $\mathbf{Ker}(\chi) = \boldsymbol{\mu}_2 \cdot 1$ where $\boldsymbol{\mu}_2$ denotes the group of second roots of unity. Hence, if k has characteristic $\neq 2$, we have $\mathrm{Lie}(\boldsymbol{\mu}_2) = 0$ and $\dot\chi\colon \mathfrak{spin}(q) \xrightarrow{\cong} \mathfrak{o}(q)$ is injective, while $\mathrm{Ker}(\dot\chi) = \mathrm{Lie}(\mathbf{Ker}(\chi)) = k \cdot 1$ if k has characteristic 2

We now consider the question of surjectivity of $\dot\chi$. Let $k(\varepsilon)$ be the algebra of dual numbers. Every $A \in \mathfrak{o}(q)$ defines an element $g = \mathrm{Id} + \varepsilon A \in \mathbf{O}(q)(k(\varepsilon))$. Using the universal property of the Clifford algebra, one sees that g induces an automorphism α of $\mathrm{Cliff}(q) \otimes k(\varepsilon)$ such that $\alpha(x) = g(x)$, for all $x \in X$. Since the Clifford algebra is an Azumaya algebra and $k(\varepsilon)$ is a local ring, α is inner [12, IV, Cor. 1.3], given by conjugation with an element u, and it is easily seen that we may assume u in the form $u = 1 + \varepsilon v$ for some $v \in \mathrm{Cliff}_0(q)$. Then we have $[v, x] = A(x)$ for all $x \in X$, but $v + v^* \neq 0$ in general, so v is not necessarily in $\mathfrak{spin}(q)$. However, $v + v^* \in k \cdot 1$, because $[v + v^*, x] = [v, x] + [v^*, x^*] = [v, x] + [x, v]^* = [v, x] + [x, v] = 0$ for all

$x \in X$, so $v + v^* = \lambda \cdot 1$ is central in $\mathrm{Cliff}(q)$. If k has characteristic $\neq 2$, then $v' := \frac{1}{2}(v - v^*) = v - \lambda/2$ is in $\mathfrak{spin}(q)$ and satisfies $\dot\chi(v') = A$, so $\dot\chi$ is surjective and therefore an isomorphism. Now let k have characteristic 2. Then there is a linear form trs: $\mathfrak{o}(q) \to k$, called the *spinor trace*, given by $\mathrm{trs}(A) \cdot 1 = v + v^*$. This is well-defined, because if also $[w, x] = A(x)$ for all $x \in X$ then $w - v = \gamma \cdot 1$ for $\gamma \in k$, so $w + w^* = v + v^* + 2\gamma \cdot 1 = v + v^*$. By construction, it is clear that A belongs to the image of $\dot\chi$ if and only if $\mathrm{trs}(A) = 0$. Also, it is easily seen that trs is in fact a Lie algebra homomorphism, so we have the exact sequence

$$0 \longrightarrow k \cdot 1 \hookrightarrow \mathfrak{spin}(q) \xrightarrow{\dot\chi} \mathfrak{o}(q) \xrightarrow{\mathrm{trs}} k \longrightarrow 0$$

of Lie algebras. We put

$$\mathfrak{o}'(q) := \dot\chi(\mathfrak{spin}(q)) = \mathrm{Ker}(\mathrm{trs}).$$

2.15. Proposition. *Let q be a nondegenerate quadratic form on a vector space X of dimension $2n$ over a field k of characteristic 2. Let $\mathcal{Q} = \mathrm{Mat}_2(k)$ be the split quaternion algebra, and $L = L(J)$ the Lie algebra associated to $J = \mathrm{H}_n(\mathcal{Q}, k)$.*

(a) *There is an isomorphism φ: $\mathfrak{o}(q) \to L$ of 2-Lie algebras such that $\mathrm{Id}_X \mapsto 1_J$ and the spinor trace on $\mathfrak{o}(q)$ corresponds to the trace form of J.*

(b) *If $n = 2m + 1 \geq 3$ then $\mathfrak{o}(q) = k \cdot \mathrm{Id}_X \oplus \mathfrak{o}'(q)$ (direct sum of ideals) and $\mathfrak{o}'(q)$ is a simple Lie algebra, while for $n = 2m \geq 4$, we have $\mathrm{Id}_X \in \mathfrak{o}'(q)$ and $\mathfrak{o}'(q)/k \cdot \mathrm{Id}_X$ is simple.*

Proof. (a) Since k has characteristic 2, the bilinear form b associated with q is symplectic. Hence there exists a basis $e_1, \ldots, e_{2n}$ of X such that $b(e_{2i-1}, e_{2i}) = 1$ for $i = 1, \ldots, n$, while $b(e_i, e_j) = 0$ otherwise. Using the characterization 2.14.3 of an element $A \in \mathfrak{o}(q)$, it is an easy exercise to check that the matrix $\varphi(A)$ of A with respect to $e_1, \ldots, e_{2n}$ belongs to J and this yields the desired isomorphism of 2-Lie algebras.

We identify $\mathfrak{o}(q)$ and L and show that $\mathrm{tr}_J = \mathrm{trs}$. Since trs is a Lie algebra homomorphism, it vanishes on $[L, L]$. By 1.11, J has nondegenerate bilinear trace form Tr so $[L, L] = \mathrm{Ker}(\mathrm{tr}_J)$ by Cor. 2.6. Thus tr_J and trs have the same kernel, and it remains to prove that tr_J and trs take the same nonzero value on one element, say, on the matrix unit $A = E_{11} \in \mathrm{H}_n(\mathcal{Q}, k)$. Now $A(e_1) = e_1$, $A(e_2) = e_2$ while $A(e_i) = 0$ otherwise. Consider the element $v = e_1 e_2 \in \mathrm{Cliff}_0(q)$. We claim that $A(x) = [v, x]$ for all $x \in X$ (where the products are to be taken in $\mathrm{Cliff}(q)$). Indeed, $b_q(e_1, e_2) = 1$ implies $e_1 e_2 + e_2 e_1 = 1$ in $\mathrm{Cliff}(q)$, hence $ve_1 - e_1 v = e_1 e_2 e_1 - e_1 e_1 e_2 = -e_2 e_1^2 + e_1 - e_1^2 e_2 = -q(e_1)e_2 + e_1 - q(e_1)e_2 = e_1$, and similarly $ve_2 - e_2 v = e_2$, while $ve_i - e_i v = 0$ for $i > 2$ follows from $e_1 e_i + e_i e_1 = e_2 e_i + e_i e_2 = 0$, because $b_q(e_1, e_i) = b_q(e_2, e_i) = 0$. Furthermore, $v + v^* = e_1 e_2 + e_2 e_1 = 1$, so $\mathrm{trs}(E_{11}) = 1 = \mathrm{tr}_J(E_{11})$, as required.

(b) This follows from Prop. 2.5 and Cor. 2.12(a), part (ii).

We leave it to the reader to prove in a similar way the following result.

2.16. Proposition. (a) *The Lie algebra* $\mathfrak{sp}_{2n}(k)$ *of the symplectic group over a field* k *of characteristic* 2 *is isomorphic as a* 2*-Lie algebra to* $L(\mathrm{H}_n(\mathcal{Q}, \mathcal{Q}_0))$.

(b) $L(\mathrm{H}_n(k))$ *is the Lie algebra of the automorphism group of the standard bilinear form* $h(x, y) = \sum_{i=1}^{n} x_i y_i$ *on* k^n.

3. Smoothness of the automorphism group

3.1. *The structure group.* In this section, we discuss the question of smoothness of the automorphism group scheme of a separable finite-dimensional Jordan algebra. This is closely related to the structure of the orbit of the unit element under the structure group. Recall that the structure group $\mathrm{Str}(J)$ of a Jordan algebra J (over a commutative ring R) is the set of $g \in \mathrm{GL}(J)$ for which there exists $g^\sharp \in \mathrm{GL}(J)$ such that $U_{g(x)} = gU_x g^\sharp$ for all $x \in J$. Such a $g^\sharp$ is uniquely determined by g; in fact, $g^\sharp = g^{-1}U_{g(1)}$, and the map $\vartheta\colon g \mapsto g^\vartheta := (g^\sharp)^{-1}$ is an automorphism of period two of $\mathrm{Str}(J)$. Also, $\mathrm{Str}(J)$ is isomorphic to the automorphism group of the Jordan pair (J, J) associated to J under the map $g \mapsto (g, g^\vartheta)$. The automorphism group $\mathrm{Aut}(J)$ is just the isotropy group of the unit element 1_J in $\mathrm{Str}(J)$.

We establish some notation and terminology for algebraic groups. Let k be a field. Following [5], we will always embed algebraic k-groups into the category of group functors on the category k-**alg** of (commutative associative unital) k-algebras. For a k-group functor $\mathbf{G}$ and $R \in k$-**alg**, we denote by $\mathbf{G}(R)$ the associated (abstract) group. The Lie algebra of $\mathbf{G}$ is denoted by $\mathrm{Lie}(\mathbf{G})$.

For a finite-dimensional Jordan algebra J over k we have the group functors

$$\mathbf{Str}(J)(R) := \mathrm{Str}(J \otimes R), \quad \mathbf{Aut}(J)(R) := \mathrm{Aut}(J \otimes R) \quad (R \in k\text{-}\mathbf{alg}),$$

which are affine algebraic k-groups. By abuse of language, these will also be referred to simply as the structure group and the automorphism group. Their Lie algebras are then just $\mathrm{Strl}(J)$ and $\mathrm{Der}(J)$, respectively.

Example. Let $J = \mathrm{Jor}(k^n, q, 1)$ be the Jordan algebra of a nondegenerate quadratic form q with base point $1 = 1_J$ as in 1.4, and let $\mathbf{GO}(q)$ be the general orthogonal group of q, i.e., $g \in \mathbf{GO}(q)(R)$ if and only if $g \in \mathrm{GL}_n(R)$ and there exists $\lambda \in R^\times$ such that $q(g(x)) = \lambda q(x)$ for all $x \in R^n$. Note that $\lambda = \lambda(g) = q(g(1))$ is uniquely determined by g. The Lie algebra of $\mathbf{GO}(q)$ consists of all $A \in \mathrm{End}(k^n)$ for which there exists $\mu \in k$ such that $b(x, A(x)) = \mu q(x)$, for all $x \in k^n$. It is easily seen that $\mathbf{GO}(q) \subset \mathbf{Str}(J)$, with $g^\sharp = \lambda(g) g^{-1}$, and consequently, $\mathrm{Lie}(\mathbf{GO}(q)) \subset \mathrm{Strl}(J)$.

The bilinear trace Tr is invariant under the structure group and the structure Lie algebra in the following sense:

3.2. Lemma. *Let J be a finite-dimensional Jordan algebra over a field k and put $\mathbf{G} := \mathbf{Str}(J)$ and $\mathfrak{g} := \mathrm{Strl}(J)$. Then*

$$\mathrm{Tr}(g(x), y) = \mathrm{Tr}(x, g^\sharp(y)) \quad \text{for all } g \in \mathbf{G}(R),\ x, y \in J \otimes R,\ R \in k\text{-}\mathbf{alg}, \tag{1}$$
$$\mathrm{Tr}(A(x), y) = \mathrm{Tr}(x, A^\sharp(y)) \quad \text{for all } A \in \mathfrak{g},\ x, y \in J. \tag{2}$$

Consequently, the defect is stable under $\mathbf{G}$ and $\mathfrak{g}$, i.e.,

$$g(\mathrm{Def}(J) \otimes R) \subset \mathrm{Def}(J) \otimes R \qquad \text{for all } g \in \mathbf{G}(R),\ R \in k\text{-}\mathbf{alg}, \tag{3}$$
$$A(\mathrm{Def}(J)) \subset \mathrm{Def}(J) \qquad \text{for all } A \in \mathfrak{g}. \tag{4}$$

Proof. Formula (1) is a consequence of [13, Prop. 16.7] and the fact that $\mathrm{Tr}(x, y) = m_1(x, y)$ where m_1 is the generic trace of the Jordan pair (J, J) associated with J, cf. 1.3. Formula (2) then follows by letting $R = k(\varepsilon)$ (dual numbers) and $g = \mathrm{Id} + \varepsilon A$, and (3) and (4) are immediate from (1) and (2).

Recall that a Jordan algebra (or pair) over a field is said to be *separable* if any base field extension has trivial lower radical [16, 3.5]. The basic fact on $\mathbf{Str}(J)$ is

3.3. Theorem [14, Cor. 6.6]**.** *The structure group $\mathbf{Str}(J)$ of a finite-dimensional separable Jordan algebra is a reductive, hence in particular smooth, algebraic k-group.*

In contrast, the automorphism group of J is in general not smooth. However, this can only happen in characteristic 2.

3.4. *Group actions.* We recall some notions for actions of algebraic groups. Let k be a field with algebraic closure $\bar{k}$, let $\mathbf{G}$ be a smooth algebraic k-group acting on a smooth algebraic k-scheme $\mathbf{X}$ on the left, let $x \in \mathbf{X}(k)$ be a k-rational point of $\mathbf{X}$, and denote by $\mathbf{H} = \mathbf{Cent}_{\mathbf{G}}(x)$ the stabilizer of x in $\mathbf{G}$. Also let $\pi\colon \mathbf{G} \to \mathbf{X}$ be the orbit map sending $g \in \mathbf{G}(R)$ to $g \cdot x_R$ for all $R \in k\text{-}\mathbf{alg}$ (where $x_R \in \mathbf{X}(R)$ is the image of x under the map $\mathbf{X}(k) \to \mathbf{X}(R)$ induced from $k \to R$).

The *orbit of x under* $\mathbf{G}$ is the image sheaf $\mathbf{Im}(\pi)$ (in the flat topology) of π, cf. [5, III, §1, 2.3]. Denote by $\mathbf{G}\tilde{/}\mathbf{H}$ the sheaf (in the flat topology) associated to the functor $R \mapsto \mathbf{G}(R)/\mathbf{H}(R)$, cf. [5, III, §3, 1.4]. Then π induces a canonical isomorphism $\mathbf{G}\tilde{/}\mathbf{H} \cong \mathbf{Im}(\pi)$ by [5, III, §3, 1.6].

For an algebraic k-scheme $\mathbf{Y}$ and a k-rational point $y \in \mathbf{Y}(k)$, let $T_y(\mathbf{Y})$ be the Zariski tangent space of $\mathbf{Y}$ at y. Finally, let $e \in \mathbf{G}(k)$ be the unit element of $\mathbf{G}(k)$, and let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}) = T_e(\mathbf{G})$ and $\mathfrak{h} = \mathrm{Lie}(\mathbf{H})$ be the respective Lie algebras. Then we have $\mathfrak{h} = \mathrm{Ker}(d_e\pi)$, where $d_e\pi\colon \mathfrak{g} \to T_x(\mathbf{X})$ denotes the differential of π at the unit element of $\mathbf{G}(k)$.

3.5. Lemma. *In the situation of* 3.4*, assume furthermore that* $\mathbf{G}$ *acts transitively on* $\mathbf{X}$ *in the sense that* $\pi\colon \mathbf{G}(\bar{k}) \to \mathbf{X}(\bar{k})$ *is surjective.*

(a) *π induces an isomorphism* $\mathbf{G}\tilde{/}\mathbf{H} \xrightarrow{\cong} \mathbf{X}$, *and*

$$\dim \mathbf{G} = \dim \mathbf{H} + \dim \mathbf{X}. \tag{1}$$

(b) *The following conditions are equivalent:*

(i) $\mathbf{H}$ *is smooth,*
(ii) $\pi\colon \mathbf{G} \to \mathbf{X}$ *is smooth,*
(iii) $d_e\pi\colon \mathfrak{g} \to T_x(\mathbf{X})$ *is surjective,*
(iv) $\dim \mathbf{H} = \dim \mathfrak{h}$.

Proof. (a) The first statement follows from [5, III, §3, Prop. 2.1], and the dimension formula follows from [5, III, §3, 5.5(a)].

(b) (i) $\Longrightarrow$ (ii): By [5, III, §3, Cor. 2.6] the canonical morphism $\mathbf{G} \to \mathbf{G}\tilde{/}\mathbf{H}$ is smooth, and hence so is the composite $\pi\colon \mathbf{G} \to \mathbf{G}\tilde{/}\mathbf{H} \xrightarrow{\cong} \mathbf{X}$ (note that the subgroups $\mathbf{H}$ and $\mathbf{H}'$ of *loc. cit.* are $\{e\}$ and $\mathbf{H}$ in our situation).

(ii) $\Longrightarrow$ (iii): This follows from [5, I, §4, Cor. 4.14, Remark 4.15].

(iii) $\Longrightarrow$ (iv): We have $\dim \mathfrak{h} = \dim \operatorname{Ker}(d_e\pi) = \dim \mathfrak{g} - \dim \operatorname{Im}(d_e\pi) = \dim \mathbf{G} - \dim \mathbf{X} = \dim \mathbf{H}$ (by (1)).

(iv) $\Longrightarrow$ (i): See [5, II, §5, Th. 2.1(vi)].

3.6. Let J be a separable finite-dimensional Jordan algebra over a field k. We denote by $\mathbf{J}$ the affine scheme defined by the vector space J, and by $\mathbf{J}^\times$ the open dense subscheme of invertible elements of $\mathbf{J}$; thus

$$\mathbf{J}(R) = J \otimes R \quad \text{and} \quad \mathbf{J}^\times(R) = (J \otimes R)^\times \quad \text{for all } R \in k\textbf{-alg}. \tag{1}$$

For the rest of this section, we will always let

$$\mathbf{G} := \mathbf{Str}(J),$$

which is smooth by Th. 3.3, act on a suitably chosen $\mathbf{G}$-stable subscheme $\mathbf{X}$ of $\mathbf{J}$ containing the unit element $x = 1$ of J, so that

$$\mathbf{H} := \mathbf{Cent}_{\mathbf{G}}(1) = \mathbf{Aut}(J).$$

We denote the Lie algebras of $\mathbf{G}$ and $\mathbf{H}$ by

$$\mathfrak{g} = \operatorname{Lie}(\mathbf{G}) = \operatorname{Strl}(J) \quad \text{and} \quad \mathfrak{h} = \operatorname{Lie}(\mathbf{H}) = \operatorname{Der}(J).$$

The orbit map π is just evaluation of an element $g \in \mathbf{G}(R)$ at 1, and likewise, $d_e\pi\colon \mathfrak{g} \to T_x(\mathbf{X})$ is simply evaluation $A \mapsto A(1)$ of an element A in the structure Lie algebra at the unit element of J.

Clearly, $\mathbf{J}$ and $\mathbf{J}^\times$ are smooth affine schemes; in fact, a defining function for $\mathbf{J}^\times$ is the generic norm of J. The defect of J gives rise to functors $\mathbf{J}_{\mathrm{def}}$ and

$\mathbf{J}^\times_{\rm def}$ in analogy to (1) by

$$\mathbf{J}_{\rm def}(R) = \mathrm{Def}(J) \otimes R \quad \text{for all } R \in k\text{-}\mathbf{alg}, \quad \text{and} \quad \mathbf{J}^\times_{\rm def} = \mathbf{J}_{\rm def} \cap \mathbf{J}^\times. \tag{2}$$

Again, $\mathbf{J}_{\rm def}$ and $\mathbf{J}^\times_{\rm def}$ are smooth and affine, and $\mathbf{J}^\times_{\rm def}$ is open in $\mathbf{J}_{\rm def}$. We will also need the subfunctor of non-defective elements $\mathbf{J}_{\rm nd}$ of $\mathbf{J}$, defined by

$$\mathbf{J}_{\rm nd}(R) = \{x \in \mathbf{J}(R) : \text{there exists } y \in \mathbf{J}(R) \text{ such that } \mathrm{Tr}(x, y) \in R^\times\}, \tag{3}$$

for all $R \in k$-**alg**. This is an open (hence smooth) but in general not affine subscheme of $\mathbf{J}$. To see this, choose a vector space basis $v_1, \ldots, v_n$ of J and define functions f_i on $\mathbf{J}$ by $f_i(x) := \mathrm{Tr}(x, v_i)$. Then $\mathbf{J}_{\rm nd}$ is the open subscheme of $\mathbf{J}$ defined by the f_i in the sense of [5, I, §1, 3.7], i.e., $x \in \mathbf{J}_{\rm nd}(R)$ if and only if R is generated as an ideal by $f_1(x), \ldots, f_n(x)$. We finally put

$$\mathbf{J}^\times_{\rm nd} := \mathbf{J}_{\rm nd} \cap \mathbf{J}^\times. \tag{4}$$

From the fact that elements of the structure group preserve invertibility and the bilinear trace form Tr by Lemma 3.2, it follows that $\mathbf{G}$ acts on each of the schemes $\mathbf{J}^\times$, $\mathbf{J}^\times_{\rm def}$ and $\mathbf{J}^\times_{\rm nd}$.

3.7. Proposition. *If k has characteristic $\neq 2$ then the orbit of 1 under $\mathbf{G}$ is $\mathbf{J}^\times$, and $\mathbf{H}$ is smooth.*

Proof. Let $\mathbf{X} = \mathbf{J}^\times$. By Prop. 1.9, $\mathbf{G}(\bar{k})$ acts transitively on $\mathbf{X}(\bar{k}) = (J \otimes \bar{k})^\times$ so the first assertion follows from Lemma 3.5(a). Also, $T_1(\mathbf{X})$ is canonically identified with the vector space J because $\mathbf{X}$ is open in $\mathbf{J}$. For any given $a \in J$, we have $A = \frac{1}{2}V_a \in \mathfrak{g}$, and $d_e\pi(A) = A(1) = \frac{1}{2}(a \circ 1) = a$. Hence $\mathbf{H}$ is smooth by Lemma 3.5(b).

3.8. To decide smoothness of the automorphism group in characteristic 2 requires a more detailed discussion. We remark that Springer [23], in his framework of J-structures, has also studied this problem. However, his definition of J-structure is rather restrictive and in characteristic 2 rules out, a priori, the case where the orbit of 1 under the structure group is not open. For an absolutely simple Jordan algebra, this happens precisely when the algebra is traceless. Our approach includes these cases and it is simpler than Springer's since it uses the result 3.3, not available to him. — Until further notice, we assume that

J is simple and k is an algebraically closed field of characteristic 2.

We will use repeatedly the fact that $V_{J,J} \subset \mathfrak{g}$ (cf. 2.1) and hence

$$J \circ J = V_{J,J}(1) \subset \mathrm{Im}(d_e\pi). \tag{1}$$

3.9. Lemma. *If J has nondegenerate bilinear trace form* Tr, *i.e.,* $\mathrm{Def}(J) = 0$, *then the orbit of* 1 *under* $\mathbf{G}$ *is* $\mathbf{J}^\times$, *and* $\mathbf{H}$ *is smooth.*

Proof. We apply Lemma 3.5 in case $\mathbf{X} = \mathbf{J}^\times$. By Prop. 1.9, $\mathbf{G}(k)$ acts transitively on $\mathbf{X}(k)$, so the first statement follows from Lemma 3.5(a). Also, we have $T_1(\mathbf{X}) = J$ and by Cor. 2.6, $[L, L] = J \circ J = \mathrm{Ker}(\mathrm{tr})$ has codimension one in J. Hence by 3.8.1, it suffices, for $d_e\pi$ to be surjective, to find an element of trace one in the image of $d_e\pi$. We remark also that Id_J always belongs to $\mathrm{Strl}(J)$. Now we distinguish the following cases.

Case 1: $r := \mathrm{rk}(J)$ is odd. Here $\mathrm{tr}(1) = r \cdot 1_k = 1_k$ because k has characteristic 2. Hence $1 = \mathrm{Id}(1)$ is an element of trace one in the image of $d_e\pi$.

When the rank is even we use the classification.

Case 2: $r = 2$. Then $J = \mathrm{Jor}(k^{2n}, q, \varepsilon_1 + \varepsilon_2)$ is the Jordan algebra of an even-dimensional quadratic form with base point $1 = \varepsilon_1 + \varepsilon_2$, $n \geq 2$. Define $A \in \mathrm{End}(J)$ by $A(\varepsilon_i) = \varepsilon_i$ if i is odd and $A(\varepsilon_i) = 0$ if i is even. One checks easily that A belongs to the Lie algebra of the general orthogonal group of q, in fact, $b(x, A(x)) = q(x)$, and hence A belongs to the structure Lie algebra by the example in 3.1. Now $A(\varepsilon_1) = \varepsilon_1$, $A(\varepsilon_2) = 0$, and therefore $A(1) = \varepsilon_1$ which has trace one.

Case 3: r *even and* ≥ 4. Then $J = \mathrm{H}_r(\mathcal{C}, k)$ is the Jordan algebra of hermitian matrices with scalar diagonal coefficients over a composition algebra $\mathcal{C}$. The assumption $r \geq 4$ eliminates the case where $\mathcal{C}$ is an octonion algebra, and the assumption $\mathrm{Def}(J) = 0$ eliminates the case $\mathcal{C} = k$. Thus either $\mathcal{C} = k \oplus k$ or $\mathcal{C} = \mathrm{Mat}_2(k)$, the split quaternions. In both cases, $1_{\mathcal{C}} = \varepsilon_1 + \varepsilon_2$ is the sum of two primitive orthogonal idempotents which are interchanged by the involution of $\mathcal{C}$. One checks easily that every $a \in \mathrm{Mat}_r(\mathcal{C})$ defines an element A_a in the structure Lie algebra by $A_a(x) = ax + xa^*$, so all $A_a(1) = a + a^* \in \mathrm{Im}(d_e\pi)$. Now let in particular $a = \varepsilon_1 e_{11}$ where the e_{ij} are the usual matrix units. Then $a + a^* = e_{11}$ is the desired element of trace one in the image of $d_e\pi$.

3.10. Lemma. *If J is traceless, i.e.,* $1 \in \mathrm{Def}(J)$, *then the orbit of* 1 *under* $\mathbf{G}$ *is* $\mathbf{J}^\times_{\mathrm{def}}$ *and* $\mathbf{H}$ *is smooth.*

Proof. Let $\mathbf{X} = \mathbf{J}^\times_{\mathrm{def}}$ as in 3.6.2. Then $1 \in \mathbf{X}(k)$ and the tangent space of $\mathbf{X}$ at 1 is just the vector space $\mathrm{Def}(J)$. By Prop. 1.9, $\mathbf{G}(k)$ acts transitively on $\mathbf{X}(k)$, so the orbit of 1 under $\mathbf{G}$ is $\mathbf{X}$ by Lemma 3.5(a). By Prop. 2.7(a), $J \circ J = [L, L] = \mathrm{Def}(J)$, so $\mathbf{H}$ is smooth by 3.8.1 and Lemma 3.5(b).

3.11. Lemma. *Let J have* $\mathrm{tr} \neq 0$ *and* $\mathrm{Def}(J) \neq 0$, *and let* $r := \mathrm{rk}(J)$. *Then the orbit of* 1 *under* $\mathbf{G}$ *is* $\mathbf{J}^\times_{\mathrm{nd}}$ *and* $\dim \mathfrak{h} - \dim \mathbf{H} = r - 1 \geq 1$; *in particular,* $\mathbf{H}$ *is not smooth.*

Proof. Let $\mathbf{X} = \mathbf{J}_{\mathrm{nd}}^{\times}$ as in 3.6.4. Then $1 \in \mathbf{X}(k)$, and by Prop. 1.9, $\mathbf{G}(k)$ acts transitively on $\mathbf{X}(k)$ so the orbit of 1 is $\mathbf{X}$. Also, $T_1(\mathbf{X}) = T_1(\mathbf{J}) = J$ because $\mathbf{X}$ is open in $\mathbf{J}$. We determine the image of the evaluation map $d_e\pi\colon \mathfrak{g} \to J$, using the classification. There are two cases:

Case 1: $J = \mathrm{H}_r(k)$, symmetric matrices over k, $r \geq 2$, with unit element $1 = 1_r$ the $r \times r$ unit matrix, and $\mathrm{Def}(J) = \mathrm{Alt}_r(k)$, the alternating $r \times r$-matrices. We claim that

$$\mathrm{Im}(d_e\pi) = k \cdot 1 \oplus \mathrm{Alt}_r(k). \tag{1}$$

For the inclusion from right to left, let e_{ij} be the usual matrix units. Then $c_i = e_{ii}$ $(i = 1, \ldots, r)$ is a frame of division idempotents of J and the Peirce space $J_{ij} = k \cdot (e_{ij} + e_{ji})$ (for $i < j$) is 1-dimensional. Hence by the first case of Prop. 2.7(b), $\mathrm{Alt}_r(k) = J \circ J$ is contained in $\mathrm{Im}(d_e\pi)$, and so is 1. To prove the inclusion from left to right, let $A \in \mathrm{Strl}(J)$. Since now $U_x y = xyx$ (matrix product), formula 2.1.1 says

$$xyA(x) + A(x)yx = A(xyx) - xA(y)x + x(A(1)y + yA(1))x, \tag{2}$$

for all $x, y \in J$. Write $A(1) = \sum_{l,m} \alpha_{lm} e_{lm}$ and $A(x) = \sum_{l,m} \beta_{lm} e_{lm}$ as linear combinations of the matrix units. Now put $x = y = e_{ij} + e_{ji}$ (where $i \neq j$) in (2), and multiply the resulting equation with e_{ii} on the left and with e_{ji} on the right. An elementary matrix calculation then yields the relation

$$\beta_{ij} + \beta_{ij} = \beta_{ij} - \beta_{ji} + \alpha_{jj} + \alpha_{ii}.$$

Since $A(x) \in \mathrm{H}_r(k)$ is symmetric we have $\beta_{ij} = \beta_{ji}$. Hence $2 = 0$ in k implies $\alpha_{ii} = \alpha_{jj}$, so all diagonal coefficients of $A(1)$ are equal, proving the inclusion from left to right in (1).

Case 2: $J = \mathrm{Jor}(k^{2m+1}, q, \varepsilon_1 + \varepsilon_2)$ with $m \geq 1$. In fact, we can assume $m \geq 2$ because $\mathrm{Jor}(k^3, q, \varepsilon_1 + \varepsilon_2) \cong \mathrm{H}_2(k)$. Then ε_1 and ε_2 form a frame of division idempotents of J whose Peirce space J_{12} has dimension $2m - 1 \geq 3$. We claim that

$$\mathrm{Im}(d_e\pi) = \mathrm{Ker}(\mathrm{tr}). \tag{3}$$

Indeed, the inclusion from right to left holds because $\mathrm{Ker}(\mathrm{tr}) = J \circ J$ by the second case of Prop. 2.7(b), and 3.8.1.

Let us prove the inclusion from left to right. From 1.4.1 we obtain $\bar{\varepsilon}_0 = \varepsilon_0$ and $U_x\varepsilon_0 = q(x)\varepsilon_0$ whence $U_{x,z}\varepsilon_0 = b(x, z)\varepsilon_0$, for all $x, z \in J$. In particular, $U_{\varepsilon_0}\varepsilon_0 = \varepsilon_0$ (because $q(\varepsilon_0) = 1$) and $U_{\varepsilon_0,z}\varepsilon_0 = 0$ for all $z \in J$. Now let $A \in \mathrm{Strl}(J)$. By 3.2.4, $A(\varepsilon_0) \in k \cdot \varepsilon_0$, which implies $[A, U_{\varepsilon_0}]\varepsilon_0 = 0$. Hence 2.1.1 yields

$$\begin{aligned} 0 = U_{\varepsilon_0, A(\varepsilon_0)}\varepsilon_0 &= [A, U_{\varepsilon_0}]\varepsilon_0 + U_{\varepsilon_0} U_{1, A(1)}\varepsilon_0 \\ &= 0 + U_{\varepsilon_0} b(1, A(1))\varepsilon_0 = b(1, A(1))\varepsilon_0 = \mathrm{tr}(A(1))\varepsilon_0, \end{aligned}$$

which proves the inclusion from left to right in (3).

Observe that the results of the previous three lemmas also hold for an arbitrary base field k, provided J is absolutely simple. This follows by passing to the algebraic closure of k because the structure group commutes with base change, and smoothness of $\mathbf{G}$ and $\mathbf{G} \otimes \bar{k}$ are equivalent [5, I, §4, 4.1]. We collect our results in the following theorem.

3.12. Theorem. *Let J be an absolutely simple finite-dimensional Jordan algebra of rank r over a field k of arbitrary characteristic. Then $\mathbf{Aut}(J)$ is not smooth if and only if k has characteristic 2 and J has both nonzero trace and nonzero defect. In this case,* $\dim \mathrm{Der}(J) - \dim \mathbf{Aut}(J) = r - 1$.

4. The exceptional case

4.1. *Definitions and notations.* Let R be a commutative ring and let either $\mathcal{C} = R$ or let $\mathcal{C}$ be a composition algebra of constant rank $r \geq 2$ over R in the sense of [21], with norm form q and associated bilinear form b_q. Thus $\mathcal{C}$ is finitely generated and projective of rank $r = 1, 2, 4, 8$ as an R-module. If $r = 1$ and 2 is not a unit in R then b_q is singular, while in the other cases, it is nonsingular. To cover the case $r = 1$ as well, we will let $B = b_q$ if $r \geq 2$, and put $B(a, b) = ab$ in case $r = 1$, i.e., $\mathcal{C} = R$. As usual, $\bar{a}$ denotes the involution of $\mathcal{C}$. For an endomorphism $h \in \mathrm{End}(\mathcal{C})$ we define $\bar{h}$ by $\bar{h}(a) = \overline{h(\bar{a})}$. We also introduce the trilinear form $\langle a, b, c\rangle = B(ab, \bar{c})$ on $\mathcal{C}$. Then $\langle\ ,\ ,\ \rangle$ behaves as follows under permutation of its arguments:

$$\langle a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\rangle = \begin{Bmatrix} \langle a_1, a_2, a_3\rangle & \text{if } \sigma \text{ is even} \\ \langle \bar{a}_1, \bar{a}_2, \bar{a}_3\rangle & \text{if } \sigma \text{ is odd} \end{Bmatrix}. \tag{1}$$

This follows from well-known formulas for b_q (and is of course trivial in case $r = 1$).

Consider the Jordan algebra $J = \mathrm{H}_3(\mathcal{C}, R)$ of hermitian matrices over $\mathcal{C}$ with scalar diagonal coefficients which by [17] is generically algebraic of degree 3. Denoting the usual matrix units by e_{ij}, we put $e_i = [ii] = e_{ii}$ and $a[ij] = \bar{a}[ji] = ae_{ij} + \bar{a}e_{ji}$ for $i \neq j$ and $a \in \mathcal{C}$. There is an action of the symmetric group $\mathfrak{S}_3$ on J by automorphisms given by

$$\sigma(e_i) = e_{\sigma(i)}, \quad \sigma(a[ij]) = a[\sigma(i), \sigma(j)].$$

This is easily verified. Let

$$E := R \cdot e_1 \oplus R \cdot e_2 \oplus R \cdot e_3 \cong R^3$$

be the subalgebra of diagonal matrices in J. We will denote the subgroup of all $g \in \mathrm{Aut}(J)$ fixing E pointwise by $\mathrm{M} = \mathrm{M}(J)$. Then $\mathfrak{S}_3$ acts on M on the right via $g^\sigma = \sigma^{-1} \circ g \circ \sigma$. Finally, we denote by P the set of all pairs

(i,j) in $\{1,2,3\}^2$ with $i \neq j$. Clearly, $\mathfrak{S}_3$ acts simply transitively on P by $\sigma(i,j) = (\sigma(i), \sigma(j))$.

4.2. Lemma. *With the above notations, let $g \in \mathrm{M}$. For every pair $(i,j) \in P$ there exists a unique $g_{ij} \in \mathrm{O}(q)$, the orthogonal group of q, such that*

$$g(a[ij]) = g_{ij}(a)[ij], \tag{1}$$

for all $a \in \mathcal{C}$. The g_{ij} satisfy the relations

$$\overline{g_{ij}} = g_{ji}, \tag{2}$$
$$g_{ij}(a)g_{jk}(b) = g_{ik}(ab), \tag{3}$$
$$\langle g_{ij}(a), g_{jk}(b), g_{ki}(c)\rangle = \langle a,b,c\rangle, \tag{4}$$

for all $a,b,c \in \mathcal{C}$ and all pairwise different i,j,k. The map $g \mapsto (g_{ij})$ is an isomorphism η from M to the subgroup $\mathrm{S} = \mathrm{S}(\mathcal{C}) \subset \mathrm{O}(q)^P$ of all tuples $(g_{ij})_{(i,j)\in P}$ satisfying (2) and (3), or, equivalently, (2) and (4). This isomorphism is $\mathfrak{S}_3$-equivariant with respect to the natural right action of $\mathfrak{S}_3$ on S, i.e., $(g^\sigma)_{ij} = g_{\sigma(i),\sigma(j)}$.

Proof. An element $g \in \mathrm{M}$ preserves the Peirce spaces $J_{ij} = \{a[ij] : a \in \mathcal{C}\}$, so there exist unique $g_{ij} \in \mathrm{GL}(\mathcal{C})$ satisfying (1). Since g preserves squares and $a[ij]^2 = q(a)([ii]+[jj])$, we have $g_{ij} \in \mathrm{O}(q)$, and (2) follows from the fact that $a[ij] = \bar{a}[ji]$. Finally, (3) is a consequence of the formula $a[ij] \circ b[jk] = ab[ik]$. Conversely, given $(g_{ij}) \in \mathrm{S}$, define $g \in \mathrm{GL}(J)$ by $g(e_i) = e_i$ and (1). Then a simple computation shows that g preserves squares and traces in J. As J is generically algebraic of degree 3, we have $g \in \mathrm{Aut}(J)$ by [17, Th. 5.1]. The $\mathfrak{S}_3$-equivariance is clear from the definitions. The equivalence of (3) and (4) for $(g_{ij}) \in \mathrm{O}(q)^P$ satisfying (2) follows easily from the definition of $\langle\ ,\ ,\ \rangle$ and nondegeneracy of B.

Remark. In [6], A. Elduque introduces the notion of related triples $(\varphi_0, \varphi_1, \varphi_2) \in \mathrm{O}(q)^3$ by requiring $\varphi_1(ab) = \varphi_0(a)\varphi_2(b)$ for all $a,b \in \mathcal{C}$. Any element (g_{ij}) of $\mathrm{S}(\mathcal{C})$ gives rise to related triples, e.g., (g_{12}, g_{13}, g_{23}) or (g_{31}, g_{32}, g_{12}). The approach via related triples seems less natural because it involves a particular choice of indices. Also, the action of the symmetric group becomes somewhat cumbersome to describe, and the trilinear form considered in [6, p. 52] does not satisfy the equivariance property 4.1.1.

4.3. Theorem. *Let $\mathcal{O}$ be an octonion algebra over a ring R with norm form q, Clifford algebra $\mathrm{Cliff}(q)$ and even part $\mathrm{Cliff}_0(q)$. Let $\mathrm{Spin}(q) \subset \mathrm{Cliff}_0(q)^\times$ be the spin group and χ: $\mathrm{Spin}(q) \to \mathrm{O}(q)$ its vector representation.*

(a) *The maps φ, ψ: $\mathcal{O} \to \mathrm{End}(\mathcal{O} \oplus \mathcal{O})$, $a \mapsto \begin{pmatrix} 0 & l_{\bar{a}} \\ l_a & 0 \end{pmatrix}$ and $a \mapsto \begin{pmatrix} 0 & r_{\bar{a}} \\ r_a & 0 \end{pmatrix}$, where l_a and r_a denotes left and right multiplication in $\mathcal{O}$, induce isomorphisms*

$$\Phi, \Psi\colon \mathrm{Cliff}(q) \xrightarrow{\cong} \mathrm{End}(\mathcal{O} \oplus \mathcal{O})$$

which restrict to isomorphisms

$$\Phi_0, \Psi_0\colon \mathrm{Cliff}_0(q) \xrightarrow{\cong} \begin{pmatrix} \mathrm{End}(\mathcal{O}) & 0 \\ 0 & \mathrm{End}(\mathcal{O}) \end{pmatrix}.$$

(b) *For* $(i,j) \in P$*, define homomorphisms* $\varrho_{ij}\colon \mathrm{Spin}(q) \to \mathrm{GL}(\mathcal{O})$ *by*

$$\varrho_{12}(u) = \chi(u), \qquad \Phi_0(u) = \begin{pmatrix} \varrho_{23}(u) & 0 \\ 0 & \varrho_{13}(u) \end{pmatrix},$$

$$\varrho_{21}(u) = \overline{\varrho_{12}(u)}, \qquad \Psi_0(u) = \begin{pmatrix} \varrho_{31}(u) & 0 \\ 0 & \varrho_{32}(u) \end{pmatrix},$$

for all $u \in \mathrm{Spin}(q)$*. Then* $\varrho = (\varrho_{ij})_{(i,j)\in P}\colon \mathrm{Spin}(q) \to \mathrm{S}(\mathcal{O})$*,* $u \mapsto (\varrho_{ij}(u))_{(i,j)\in P}$*, is an isomorphism of groups.*

(c) *The centre of* $\mathrm{Spin}(q)$ *contains* $\boldsymbol{\mu}_2(R) \times \boldsymbol{\mu}_2(R)$*, where* $\boldsymbol{\mu}_2(R) = \{\lambda \in R : \lambda^2 = 1\}$*. More precisely, we have: For* $\lambda_1, \lambda_2 \in \boldsymbol{\mu}_2(R)$ *there exists a unique element* u *in the centre of* $\mathrm{Spin}(q)$ *such that* $\varrho_{23}(u) = \varrho_{32}(u) = \lambda_1 \mathrm{Id}_{\mathcal{O}}$*,* $\varrho_{13}(u) = \varrho_{31}(u) = \lambda_2 \mathrm{Id}_{\mathcal{O}}$*, and* $\varrho_{12}(u) = \varrho_{21}(u) = \lambda_1\lambda_2 \mathrm{Id}_{\mathcal{O}}$*.*

Proof. (a) and (b) are proved in [6, Th. 1.1] for the case of a base field, but the proof applies with slight modifications also in case of a base ring. We therefore omit the details. For part (c), define $g_{ij} \in \mathrm{O}(q)$ by $g_{23} = g_{32} = \lambda_1 \mathrm{Id}$, $g_{13} = g_{31} = \lambda_2 \mathrm{Id}$ and $g_{12} = g_{21} = \lambda_1\lambda_2 \mathrm{Id}$. Then it is easy to verify that $g = (g_{ij})$ is a central element of $\mathrm{S}(\mathcal{O})$, so the assertion follows from (b).

4.4. Corollary. *Let* $J = \mathrm{H}_3(\mathcal{O}, k)$ *be a reduced Albert algebra over a field* k*. Define group functors* $\mathbf{M} \subset \mathbf{H} = \mathbf{Aut}(J)$ *and* $\mathbf{S} \subset \mathbf{O}(q)^P$ *by* $\mathbf{M}(R) = \mathrm{M}(J \otimes R)$ *and* $\mathbf{S}(R) = \mathrm{S}(\mathcal{C} \otimes R)$ *for all* $R \in k$-**alg***, and let* $\mathbf{Spin}(q)$ *be the spin group as in* 2.14.1*. Then* $\mathbf{M}$*,* $\mathbf{S}$ *and* $\mathbf{Spin}(q)$ *are smooth (in fact, semisimple) algebraic group schemes of dimension* 28 *over* k*, and the maps* η *of Lemma* 4.2 *and* ϱ *of Th.* 4.3 *induce isomorphisms*

$$\eta\colon \mathbf{M} \xrightarrow{\cong} \mathbf{S} \quad \text{and} \quad \varrho\colon \mathbf{Spin}(q) \xrightarrow{\cong} \mathbf{S}. \tag{1}$$

Hence the Lie algebra $\mathfrak{m}$ *of* $\mathbf{M}$ *is isomorphic to the subalgebra* $\mathfrak{s} = \mathrm{Lie}(\mathbf{S})$ *of* $\mathfrak{o}(q)^P$ *consisting of all* (A_{ij}) *satisfying* $\overline{A_{ij}} = A_{ji}$ *and* $A_{il}(ab) = A_{ij}(a)b + aA_{jl}(b)$*, for all* $a, b \in \mathcal{O}$ *and all* $i, j, l \neq$*, and also to the Lie algebra* $\mathfrak{spin}(q)$ *of* $\mathbf{Spin}(q)$*, and all three are of dimension* 28*. If* k *has characteristic* $p > 0$ *then these isomorphisms are isomorphisms of restricted Lie algebras.*

Proof. Since the maps η and ϱ of 4.2 and 4.3 are compatible with base ring extension, we have the asserted isomorphisms of group functors. The rest follows from well-known facts about the spin group of a nondegenerate quadratic form.

4.5. *The scheme of frames.* With J as above, let $\mathbf{F} \subset \mathbf{J}^3$ be the functor of frames of J, i.e., for every $R \in k$-**alg**, $\mathbf{F}(R)$ is the set of complete systems $\vec{c} = (c_1, c_2, c_3)$ of orthogonal idempotents of $J \otimes R$ whose Peirce 2-spaces are R-modules of rank 1. We claim that $\mathbf{F}$ is an affine algebraic k-scheme. Indeed, it is easily seen that the conditions $c_i^2 = c_i$, $U_{c_i} c_j = \delta_{ij} c_i$ and $c_2 + c_2 + c_3 = 1$, which express the fact that $\vec{c}$ is a complete system of orthogonal idempotents, define a closed subscheme $\mathbf{F}'$ of $\mathbf{J}^3$. Now $\mathbf{F} \subset \mathbf{F}'$ is singled out by the conditions that the c_i be in addition rank 1 idempotents. Since the Peirce 2-spaces of the c_i are direct summands of $J \otimes R$ and hence finitely generated and projective R-modules, their rank functions are locally constant on $\operatorname{Spec}(R)$. From this, it easily seen that $\mathbf{F}$ is an open and closed subscheme of $\mathbf{F}'$; in particular, it is affine algebraic.

4.6. Proposition. (a) $\mathbf{F}$ *is smooth of dimension* 24.

(b) $\mathbf{H}$ *acts transitively on* $\mathbf{F}$ *and the stabilizer of* $\vec{e} = (e_1, e_2, e_3) \in \mathbf{F}(k)$ *is* $\mathbf{M}$. *Hence the dimension of* $\mathbf{H}$ *and of* $\mathfrak{h}$ *is* 52.

Proof. (a) Since $\mathbf{F}$ is open in $\mathbf{F}'$, it suffices to show that the latter is smooth. By [5, I, §4, Cor. 4.6], we must show that the map $\mathbf{F}'(R) \to \mathbf{F}'(R/I)$ induced from the canonical map $R \to R/I$ is surjective, for every ideal I of square zero of R. The kernel of the map $J \otimes R \to J \otimes (R/I)$ is a nilideal, so the assertion follows from the well-known lifting of finite orthogonal systems of idempotents through nil ideals. Now let $\vec{c} = (c_1, c_2, c_3) \in \mathbf{F}(k)$. Then $\vec{v} = (v_1, v_2, v_3) \in T_{\vec{c}}(\mathbf{F})$ if and only if $\vec{c} + \varepsilon \vec{v} \in \mathbf{F}(k(\varepsilon))$, where $k(\varepsilon)$ denotes the dual numbers. It is an easy exercise to show that this is equivalent to the conditions $v_i = x_{ij} + x_{il}$ where $\{i, j, l\} = \{1, 2, 3\}$ and $x_{ij} = -x_{ji} \in J_{ij}$, the Peirce spaces of J with respect to $\vec{c}$. Hence $T_{\vec{c}}(\mathbf{F})$ as a vector space is isomorphic to $J_{12} \oplus J_{23} \oplus J_{13}$, of dimension 24.

(b) Obviously, $\mathbf{H}$ acts on $\mathbf{F}$ via $g \cdot \vec{c} = (g(c_1), g(c_2), g(c_3))$, and the stabilizer of $\vec{e}$ is just $\mathbf{M}$. By conjugacy of frames (cf. the remark after Lemma 1.5), $\mathbf{H}(\bar{k})$ acts transitively on $\mathbf{F}(\bar{k})$, and $\mathbf{H}$ is smooth by Th. 3.12. Thus we are in the situation of Lemma 3.5 and conclude that $\dim \mathfrak{h} = \dim \mathbf{H} = \dim \mathbf{M} + \dim \mathbf{F} = 28 + 24 = 52$.

4.7. From now on, we will always assume that

$$k \textit{ is a field of characteristic } 2 \textit{ and } J = \mathrm{H}_3(\mathcal{O}, k)$$

is a reduced Albert algebra over k. As before, we let $\mathfrak{h} = \operatorname{Der}(J)$ denote the Lie algebra of $\mathbf{H} = \mathbf{Aut}(J)$. By Cor. 2.13, V_J is a simple 26-dimensional ideal of $\mathfrak{h}$. Our aim is to determine the structure of V_J as well as that of the quotient algebra $\mathfrak{h}/V_J$. Since $\operatorname{tr}(e_i) = 1$, we have $\operatorname{tr}(1_J) = 3 = 1 \neq 0$ and therefore $J = k \cdot 1_J \oplus J_0$ where $J_0 = \operatorname{Ker}(\operatorname{tr}) \cong V_J$ as a 2-Lie algebra.

The following remark will be useful: Let L and L' be p-Lie algebras over a ring R with $pR = 0$ and let $f \colon L \to L'$ be an isomorphism of Lie algebras. If

L (and hence L') has trivial centre then f is an isomorphism of restricted Lie algebras. Indeed, this follows easily from the formula $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}\, x)^p$ and the fact that the adjoint representations of L and L' are faithful.

4.8. Lemma. *We have $\mathfrak{h} = V_J + \mathfrak{m}$ and $V_J \cap \mathfrak{m} = V_E$ is a 2-dimensional central ideal of $\mathfrak{m}$.*

Proof. Let $D \in \mathfrak{h}$ be a derivation and e an idempotent of J. Then $e = e^2$ implies

$$D(e) = e \circ D(e). \tag{1}$$

Now put $D' := D + V_{D(e_1)}$ and $D'' := D' + V_{D'(e_2)}$. Then

$$D = V_{D(e_1)} + V_{D'(e_2)} + D''$$

holds because $2 = 0$ in k, and we claim that $D'' \in \mathfrak{m}$. Indeed, we have first $D'(e_1) = D(e_1) + e_1 \circ D(e_1) = 0$ by (1). Moreover, $e_1 \circ e_2 = 0$ implies, because D' is a derivation, $0 = D'(e_1) \circ e_2 + e_1 \circ D'(e_2) = e_1 \circ D'(e_2)$. Hence, $D''(e_1) = D'(e_1) + D'(e_2) \circ e_1 = 0$, and also $D''(e_2) = D'(e_2) + D'(e_2) \circ e_2 = 0$, again by (1). This shows $D'' \in \mathfrak{m}$. That $V_J \cap \mathfrak{m} = V_E$ is an easy consequence of the Peirce decomposition of J with respect to the e_i. For $D \in \mathfrak{m}$ and $x \in E$, we have $[D, V_x] = V_{D(x)} = 0$, so V_E is central in $\mathfrak{m}$.

4.9. Theorem. *Let $\mathfrak{z}$ be the centre of* $\mathrm{Cliff}_0(q)$, *which by Th.* 4.3 *is isomorphic to $k \cdot \mathrm{Id}_{\mathbb{O}} \oplus k \cdot \mathrm{Id}_{\mathbb{O}}$ under the isomorphism Φ_0.*

(a) *$\mathfrak{z} \subset \mathfrak{spin}(q)$ is the centre of $\mathfrak{spin}(q)$, V_E is the centre of $\mathfrak{m}$, and there are isomorphisms of 2-Lie algebras*

$$\mathfrak{h}/V_J \xrightarrow{\varphi_1} \mathfrak{m}/V_E \xrightarrow{\varphi_2} \mathfrak{spin}(q)/\mathfrak{z} \xrightarrow{\varphi_3} \mathfrak{o}'(q)/k\cdot \mathrm{Id}_{\mathbb{O}} \xrightarrow{\varphi_4} \mathrm{H}_4(\mathfrak{Q}^s, k)_0/k \cdot 1_4, \tag{1}$$

where $\mathrm{H}_4(\mathfrak{Q}^s, k)_0$ denotes the subspace of trace 0 elements in the Jordan algebra of 4×4 hermitian matrices with scalar diagonal entries over the split quaternion algebra $\mathfrak{Q}^s = \mathrm{Mat}_2(k)$. In particular, the quotient $\mathfrak{h}/V_J$ is a simple Lie algebra, which is up to isomorphism independent of the Cayley algebra $\mathbb{O}$.

(b) *V_J is the unique proper nonzero ideal of $\mathfrak{h}$.*

Proof. (a) The isomorphism φ_1 is clear from Lemma 4.8. We next establish φ_3. By Th. 4.3(c), $\mathbf{Spin}(q)$ contains $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ as a central subgroup, and the Lie algebra of this is $\mathfrak{z}$ because k has characteristic 2 and therefore $\mathrm{Lie}(\boldsymbol{\mu}_2) = k$. Thus $\mathfrak{z}$ is a central subalgebra of $\mathfrak{spin}(q)$. Also, $k \cdot 1 = \mathrm{Ker}(\dot\chi) \subset \mathfrak{z}$. Hence $\dot\chi(\mathfrak{z})$ is a central ideal of dimension 1 in $\dot\chi(\mathfrak{spin}(q)) = \mathfrak{o}'(q)$, cf. 2.14. By Prop. 2.15(b), $\mathfrak{o}''(q) := \mathfrak{o}'(q)/k \cdot \mathrm{Id}_{\mathbb{O}}$ is a simple Lie algebra. Hence $\mathrm{can}(\dot\chi(\mathfrak{z})) = 0$ in $\mathfrak{o}''(q)$ where can: $\mathfrak{o}'(q) \to \mathfrak{o}''(q)$ denotes the canonical map, so we have $\dot\chi(\mathfrak{z}) = k \cdot \mathrm{Id}_{\mathbb{O}}$, and therefore $\mathfrak{spin}(q)/\mathfrak{z} \cong \mathfrak{o}''(q)$. This establishes φ_3, and also shows that $\mathfrak{z}$ equals the centre of $\mathfrak{spin}(q)$. By Lemma 4.8, V_E is a 2-dimensional central ideal of $\mathfrak{m}$. By Cor. 4.4, we have a Lie algebra isomorphism $\psi = \mathrm{Lie}(\varrho^{-1} \circ \eta)$: $\mathfrak{m} \to$

$\mathfrak{spin}(q)$, which of course preserves centres, so V_E equals the centre of $\mathfrak{m}$. Now ψ induces the isomorphism φ_2. Finally, φ_4 follows from Prop. 2.15(a).

(b) Let $\mathfrak{a}$ be a proper nonzero ideal of $\mathfrak{h}$ and can: $\mathfrak{h} \to \mathfrak{h}/V_J$ the canonical map. Then $\mathrm{can}(\mathfrak{a}) \subset \mathfrak{h}/V_J$ is either zero or $\mathfrak{h}/V_J$, so either $\mathfrak{a} \subset V_J$ or $\mathfrak{a} + V_J = \mathfrak{h}$. The first case yields $\mathfrak{a} = V_J$ because V_J is simple, so we must derive a contradiction from the second case. By simplicity of V_J we have $\mathfrak{a} \cap V_J = 0$, so $\mathfrak{h} = \mathfrak{a} \oplus V_J$ (direct sum of ideals) and therefore $\dim \mathfrak{a} = 26$. We also have $\mathfrak{m} \cap \mathfrak{a} \neq 0$, otherwise $\dim \mathfrak{h} \geq \dim \mathfrak{m} + \dim \mathfrak{a} = 28 + 26$ which is impossible. Choose $0 \neq D \in \mathfrak{m} \cap \mathfrak{a}$. Because D annihilates the e_i, there exist $i \neq j$ and $x \in J_{ij}$ such that $D(x) \neq 0$, and $D(x) \in J_{ij}$, because the elements of $\mathfrak{m}$ stabilize the Peirce spaces. This implies $e_i \circ D(x) = D(x) \neq 0$. On the other hand, $[D, V_x] = V_{D(x)} \in [\mathfrak{a}, V_J] = 0$, because $\mathfrak{h} = \mathfrak{a} \oplus V_J$ is a direct sum of ideals, and therefore also $V_{D(x)}(e_i) = e_i \circ D(x) = 0$, contradiction.

Remarks. (i) The isomorphism (1) was also pointed out to us by A. Elduque with a different proof.

(ii) We remind the reader that the above theorem deals only with *reduced* Albert algebras. When J is a division algebra, it is clear, by passing to the algebraic closure, that $\mathfrak{h}/V_J$ is still a simple Lie algebra, but it is not clear whether it is isomorphic to $\mathrm{H}_4(\mathcal{Q}, k)_0/k \cdot 1_4$, independently of J. However, part (b) continues to hold in case of a division algebra, as is seen by passing to the algebraic closure of k.

4.10. *Schafer's isomorphism.* Let $\mathcal{O}^s$ be a split Cayley algebra and $\mathcal{Q}$ a quaternion algebra over k. In [22], Schafer and Tomber prove that there is a Lie algebra isomorphism Σ: $H_3(\mathcal{O}^s, k)_0 \to H_4(\mathcal{Q}, k)_0/k \cdot 1_4$, where the subscript 0 indicates the spaces of trace zero elements. We now give a more convenient description of this isomorphism.

Let $\mathcal{O}^s = \mathcal{Q} \oplus \mathcal{Q}$ be the Cayley-Dickson double of $\mathcal{Q}$, with multiplication, involution and norm given by

$$(u, v) \cdot (z, w) = (uz + w\bar{v}, \bar{u}w + zv), \quad \overline{(u, v)} = (\bar{u}, -v), \quad q(u, v) = u\bar{u} - v\bar{v}. \quad (1)$$

There is a natural embedding of $\mathrm{H}_3(\mathcal{Q}, k)_0$ into $\mathrm{H}_4(\mathcal{Q}, k)_0$ by adding a row and column of zeros. We extend this to a linear map f: $\mathrm{H}_3(\mathcal{O}^s, k) \to \mathrm{H}_4(\mathcal{Q}, k)_0$ by

$$f([ii]) := [ii] + [44], \quad f((u, v)[ij]) := u[ij] + v[k4], \quad (2)$$

where $\{i, j, k\} = \{1, 2, 3\}$. Note that this is well defined: We have $a[ij] = \bar{a}[ji]$ for $a \in \mathcal{O}^s$, but because of (1) and $2 = 0$ also $f(a[ij]) = f(\bar{a}[ji])$. Now it is easy to compute that f behaves as follows with respect to squaring and circle products, where $a = (u, v) \in \mathcal{O}^s$ and $i, j, k \in \{1, 2, 3\}$:

$$f([ii]^2) = f([ii])^2, \quad (3)$$
$$f(a[ij]^2) = f(a[ij])^2 + v\bar{v} \cdot 1_4, \quad (4)$$

$$f(a[ij] \circ b[jk]) = f(a[ij]) \circ f(b[jk]) \quad (\{i,j,k\} = \{1,2,3\}), \tag{5}$$
$$f([ii] \circ [jj]) = f([ii]) \circ f([jj]), \tag{6}$$
$$f([ii] \circ a[jk]) = f([ii]) \circ f(a[jk]). \tag{7}$$

Let $x = \sum_{i=1}^{3} \xi_i[ii] + \sum_{1\le i<j\le 3} a_{ij}[ij] \in \mathrm{H}_3(\mathcal{O}^s, k)$, where $a_{ij} = (u_{ij}, v_{ij}) \in \mathcal{Q}\oplus\mathcal{Q}$. Then (3) – (7) imply

$$f(x^2) - f(x)^2 = \Big(\sum_{1\le i<j\le 3} v_{ij}\overline{v_{ij}}\Big) \cdot 1_4.$$

Now it is clear that

$$\Sigma\colon \mathrm{H}_3(\mathcal{O}^s, k)_0 \xrightarrow{f} \mathrm{H}_4(\mathcal{Q}, k)_0 \xrightarrow{\mathrm{can}} \mathrm{H}_4(\mathcal{Q}, k)_0/k\cdot 1_4$$

is a vector space isomorphism preserving the squaring, and hence is an isomorphism of restricted Lie algebras.

4.11. Corollary. *Let $\mathcal{O}$ and $\mathcal{O}^s$ be, respectively, an octonion algebra and a split octonion algebra, and let $J = \mathrm{H}_3(\mathcal{O}, k)$ and $J^s = \mathrm{H}_3(\mathcal{O}^s, k)$ be the corresponding reduced and split Albert algebras over k. We identify the 2-Lie algebra $J_0 = \mathrm{Ker}(\mathrm{tr})$ with V_J under the map $x \mapsto V_x$ as in 4.7, and similarly for J^s.*

(a) *There is an isomorphism of restricted Lie algebras*

$$\psi := \Sigma^{-1}\circ\varphi :\ \mathfrak{h}/V_J \xrightarrow{\varphi} \mathrm{H}_4(\mathcal{Q}, k)_0/k\cdot 1_4 \xrightarrow{\Sigma^{-1}} J_0^s, \tag{1}$$

obtained by composing the isomorphism $\varphi := \varphi_4\circ\varphi_3\circ\varphi_2\circ\varphi_1$ of Th. 4.9(a) with the inverse of Schafer's isomorphism Σ.

(b) *If $\mathcal{O}$ is an octonion division algebra then $\mathfrak{h}/V_J$ is not isomorphic to V_J as a Lie algebra.*

Proof. (a) This is evident from 4.10 and Th. 4.9(a).

(b) Assume that $\mathfrak{h}/V_J \cong V_J$. Then $J_0 \cong V_J \cong \mathfrak{h}/V_J \cong J_0^s$ (by (1)) as Lie algebras. Since J_0 and J_0^s are, by Cor. 2.13, simple Lie algebras, they have trivial centres. Hence $J \cong J^s$ follows from [17, Cor. 5.3(b)]. Now the Albert-Jacobson Theorem [20] implies that $\mathcal{O} \cong \mathcal{O}^s$ is split.

4.12. Remark. The results of Th. 4.9 and Cor. 4.11(a) can also be obtained by computational algebra techniques contained in the first author's PHD thesis, see [1]. This computational approach allows a detailed description (involving root space decomposition relative to a Cartan subalgebra) of the ideal V_J and also of the quotient algebra $\mathrm{Der}(J)/V_J$ for a general octonion algebra. The split case is contained in [1]. The main routines of [1] where published in the more handy reference [2].

4.13. *The homomorphism* $\beta\colon \mathbf{H} \to \mathbf{H}^s$. We keep the notations and assumptions of Corollary 4.11. The isomorphism ψ induces an isomorphism between $\mathbf{Aut}(\mathfrak{h}/V_J)$ and $\mathbf{Aut}(J_0^s)$, where $\mathbf{Aut}$ refers to the algebraic group of Lie algebra automorphisms. In view of the remark made in 4.7 and our previous results, this is also the group of automorphisms in the restricted sense. By [17, Cor. 5.4] we have a further isomorphism

$$\mathbf{Aut}(J_0^s) \cong \mathbf{H}^s := \mathbf{Aut}(J^s), \tag{1}$$

which assigns to an automorphism g_0 of $J_0^s \otimes R$ the R-linear extension to $J^s \otimes R = R\cdot 1 \oplus (J_0^s \otimes R)$ fixing the unit element, for all $R \in k\text{-}\mathbf{alg}$. Composing these isomorphisms, we obtain an isomorphism

$$\vartheta\colon \mathbf{Aut}(\mathfrak{h}/V_J) \xrightarrow{\cong} \mathbf{H}^s. \tag{2}$$

The group $\mathbf{H}$ acts by 2-Lie algebra automorphisms on its own Lie algebra $\mathfrak{h}$ via the adjoint representation Ad, and V_J is stable under $\mathrm{Ad}\,\mathbf{H}$. Hence we have an induced homomorphism $\alpha\colon \mathbf{H} \to \mathbf{Aut}(\mathfrak{h}/V_J)$, and by composing with ϑ we obtain a homomorphism

$$\beta = \vartheta \circ \alpha\colon \mathbf{H} \xrightarrow{\alpha} \mathbf{Aut}(\mathfrak{h}/V_J) \xrightarrow{\vartheta} \mathbf{H}^s.$$

We introduce the notations

$$\mathbf{I} := \mathbf{Ker}(\beta), \quad \dot\beta := \mathrm{Lie}(\beta) \quad \text{and} \quad \mathfrak{i} := \mathrm{Lie}(\mathbf{I}) = \mathrm{Ker}(\dot\beta),$$

and recall that an algebraic group $\mathbf{G}$ over a field k is called *infinitesimal* if $\mathbf{G}(K) = \{1\}$ for every field $K \in k\text{-}\mathbf{alg}$ [5, II, §4, 7.1].

4.14. Theorem. (a) *For* $D \in \mathfrak{h} = \mathrm{Der}(J)$ *we have*

$$\dot\beta(D) = V_{\psi(\mathrm{can}(D))} \in \mathfrak{h}^s, \tag{1}$$

where $\mathrm{can}\colon \mathfrak{h} \to \mathfrak{h}/V_J$ *is the canonical map,* ψ *is as in* 4.11.1, *and* $\mathfrak{h}^s = \mathrm{Lie}(\mathbf{H}^s) = \mathrm{Der}(J^s)$. *Hence* $\mathrm{Ker}(\dot\beta) = V_J$ *and* $\mathrm{Im}(\dot\beta) = V_{J^s}$.

(b) $\mathbf{I}$ *is an infinitesimal group with Lie algebra* $\mathfrak{i} = V_J$.

(c) β *is faithfully flat, so the sequence*

$$1 \longrightarrow \mathbf{I} \hookrightarrow \mathbf{H} \xrightarrow{\beta} \mathbf{H}^s \longrightarrow 1 \tag{2}$$

is exact in the flat topology.

Proof. (a) It is a standard fact that Lie(Ad) = ad. Hence for $D, D' \in \mathfrak{h}$ we have, putting $\dot\alpha = \mathrm{Lie}(\alpha)$,

$$\dot\alpha(D)(D' + V_J) = [D, D'] + V_J = [D + V_J, D' + V_J],$$

i.e., $\dot{\alpha}(D) = \mathrm{ad}_{\mathfrak{h}/V_J}(\mathrm{can}(D))$. Since ψ is an isomorphism of Lie algebras, this implies $\psi \circ \dot{\alpha}(D) \circ \psi^{-1} = \mathrm{ad}_{J_0^s}(\psi(\mathrm{can}(D)) \in \mathrm{Der}(J_0^s)$. From the description of the isomorphism 4.13.1 it follows at once that the corresponding isomorphism $\mathrm{Der}(J_0^s) \cong \mathrm{Der}(J^s)$ on the Lie algebra level is just the k-linear extension $\tilde{\Delta}$ of $\Delta \in \mathrm{Der}(J_0^s)$ to J^s satisfying $\tilde{\Delta}(1_{J^s}) = 0$. Now $\mathrm{ad}_{J_0^s}(x) \cdot y = x \circ y = V_x(y)$ for all $x, y \in J_0^s$ and $V_x(1) = 2x = 0$, so the extension of $\mathrm{ad}_{J_0^s}(x)$ to J^s is V_x. For $x = \psi(\mathrm{can}(D))$ we obtain formula (1). The statements about kernel and image of $\dot{\beta}$ follow from the isomorphism ψ of 4.11.1 and the fact that $J^s = k \cdot 1 \oplus J_0^s$ with $J_0^s \cong V_{J^s}$ under $x \mapsto V_x$, cf. 4.7.

(b) For $\mathbf{I}$ to be infinitesimal, it suffices that $\mathbf{I}(K) = \{1\}$ for an algebraically closed field K. It is known [23, 14.20–14.25] that $\mathbf{H}$ is a group of type F_4, in particular, it is an almost simple connected algebraic k-group with trivial centre (in the sense of group schemes). Hence by [24], the group $\mathbf{H}(K)$ of K-rational points is a simple abstract group. It follows that the homomorphism $\beta_K \colon \mathbf{H}(K) \to \mathbf{H}^s(K)$ is either constant or injective. The first alternative would imply, since $\mathbf{H}$ is smooth by Th. 3.12, that $\mathrm{Lie}(\beta) = 0$ which is not the case. Hence β_K is injective and therefore $\mathbf{I}(K) = \mathrm{Ker}(\beta_K) = \{1\}$.

(c) Let $\mathbf{H}' := \mathbf{H}/\tilde{\mathbf{I}}$ be the quotient sheaf and $\pi \colon \mathbf{H} \to \mathbf{H}'$ the canonical homomorphism. By [5, III, §3, 5.6, 2.7, 2.6], $\mathbf{H}'$ is a smooth affine group scheme and π is faithfully flat. Moreover, by [5, III, §3, 1.6], β factors as $\beta = \iota \circ \pi$ where $\iota \colon \mathbf{H}' \to \mathbf{H}^s$ is a monomorphism. By Lemma 4.8 and smoothness of $\mathbf{H}$ and $\mathbf{H}^s$, both groups have dimension 52, and $\dim \mathbf{H}' = \dim \mathbf{H} - \dim \mathbf{I}$ (by [5, III, §3, 5.5(a)]) $= \dim \mathbf{H}$, because $\mathbf{I}$ as an infinitesimal group has dimension zero. Now $\mathrm{Lie}(\iota) \colon \mathrm{Lie}(\mathbf{H}') \to \mathrm{Lie}(\mathbf{H}^s)$ is injective and $\mathbf{H}'$ and $\mathbf{H}^s$ are smooth of the same dimension, so $\mathrm{Lie}(\iota)$ is bijective. It follows that ι is an open embedding [5, II, §5, 5.5(b)]. But $\mathbf{H}^s$ is connected, so ι is an isomorphism. This completes the proof.

Remark. By [4, Lemma 3.7, Cor. 3.11], β is the special isogeny between an isotropic group of type F_4 and the split group of type F_4.

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