Classification of trilinear operations and their minimal polynomial identities

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Abstract

We use the representation theory of the symmetric group to classify all multilinear operations over the field $\mathbb{Q}$ of rational numbers up to equivalence. In the case $n = 3$, we obtain explicit representatives of the equivalence classes of trilinear operations

$$[a, b, c] = x_1\, abc + x_2\, acb + x_3\, bac + x_4\, bca + x_5\, cab + x_6\, cba \ (x_i \in \mathbb{Q}).$$

From these results we obtain one-parameter families of deformations of the classical Lie, Jordan and anti-Jordan triple products and the corresponding varieties of triple systems. We use computational algebra to study the nonassociative polynomial identities satisfied by these operations in every totally associative ternary algebra. We obtain 18 new trilinear operations for which the corresponding varieties of triple systems are defined by identities of degrees 3 and 5. For 10 of these operations we classify their obvious identities in degree 3 and their minimal identities in degree 5.

Introduction

We use the representation theory of the symmetric group S_n to give a definition of equivalence for multilinear n-ary operations, and to obtain a canonical representative of each equivalence class. On the underlying vector space of a totally associative n-ary algebra, a multilinear n-ary operation defines a new structure of nonassociative n-ary algebra. In the familiar case $n = 2$, we recover the Lie bracket and the Jordan product and their basic polynomial identities: respectively, the anticommutative and Jacobi identities, and the commutative and Jordan identities. In the case $n = 3$, the representatives of the equivalence classes of trilinear operations provide one-parameter families of deformations of the classical Lie, Jordan and anti-Jordan triple products, a new fourth family of operations, and another six isolated operations. Our results clarify the position of the classical triple systems in the complete classification of triple systems arising from trilinear operations. All of our operations satisfy certain "obvious" identities in degree 3. We use computational algebra to determine which of the operations satisfy additional "minimal" identities of degree 5 which are not consequences of the obvious identities; this gives a total of 18 new trilinear operations. For 10 of these operations we explicitly determine identities which imply all the identities in degree ≤ 5. These identities define new varieties of triple systems and open up a wide territory for further investigation; our identities should be useful in developing the structure theory of these new systems. Our methods make extensive use of computational algebra, especially linear algebra on large matrices, the representation theory of the symmetric group, and a new evolutionary algorithm which will be discussed in detail in a companion paper to be published elsewhere.

1 Preliminaries

1.1 General definitions

An **n-ary algebra** is a vector space A over a field $\mathbb{F}$ together with a multilinear map

$$p\colon \overbrace{A \times \cdots \times A}^{n \text{ factors}} \to A.$$

We usually write $a_1 \cdots a_n$ instead of $p(a_1, \ldots, a_n)$. The n-ary algebra A is **totally associative** if it satisfies the polynomial identities

$$a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} = a_1 \cdots (a_j \cdots a_{j+n-1}) \cdots a_{2n-1} \text{ for } 1 \leq i < j \leq n;$$

that is, any product of $2n-1$ factors depends only on the order of the factors, not the placement of the parentheses. A **multilinear operation** on an n-ary algebra is a linear combination of permutations of the original operation:

$$[a_1, a_2, \ldots, a_n] = \sum_{\sigma \in S_n} x_\sigma \, a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \ (x_\sigma \in \mathbb{F}).$$

If we regard the arguments a_i as indeterminates then a multilinear operation can be identified with an element of the group algebra $\mathbb{F}S_n$ of the symmetric group S_n. We say that two multilinear operations $[a_1, a_2, \ldots, a_n]_1$ and $[a_1, a_2, \ldots, a_n]_2$ are **equivalent** if each can be written as a linear combination of permutations of the other:

$$[a_1, a_2, \ldots, a_n]_1 = \sum_{\sigma \in S_n} y_\sigma \, [a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)}]_2 \; (y_\sigma \in \mathbb{F}),$$
$$[a_1, a_2, \ldots, a_n]_2 = \sum_{\sigma \in S_n} z_\sigma \, [a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)}]_1 \; (z_\sigma \in \mathbb{F}).$$

We can restate this definition as follows:

Lemma 1. *Let O_1 and O_2 in $\mathbb{F}S_n$ be n-ary multilinear operations. Then O_1 and O_2 are equivalent if and only if $O_1 = YO_2$ and $O_2 = ZO_1$ for some $Y, Z \in \mathbb{F}S_n$.* □

Every multilinear operation satisfies certain **obvious identities** in its defining degree:

Lemma 2. *If $O \in \mathbb{F}S_n$ is a multilinear n-ary operation then any nonzero element $I \in \mathbb{F}S_n$ for which $IO = 0$ (the zero element) is an identity of degree n satisfied by O.* □

Example: For $n = 2$ take $O = ab-ba$ (the Lie bracket) and $I = ab+ba$ (the anticommutative identity); then $IO = 0$. If we reverse I and O, we have $O = ab + ba$ (the Jordan product) and $I = ab - ba$ (the commutative identity); and again $IO = 0$.

A **minimal identity** for an n-ary multilinear operation O is an identity of lowest degree greater than n that is satisfied by O in every totally associative n-ary algebra, but which is not a consequence of the obvious identities.

1.2 The binary case

Total associativity for a binary operation reduces to the familiar identity $(ab)c = a(bc)$. Up to equivalence, there are four bilinear operations $[a, b] = x_1 \, ab + x_2 \, ba$ $(x_i \in \mathbb{F})$, the first and last of which are trivial:

- the zero operation $[a, b] = 0$,
- the Lie bracket $[a, b] = ab - ba$,
- the Jordan product $[a, b] = ab + ba$ (we use the standard notation $a \circ b = ab + ba$),
- the original operation $[a, b] = ab$.

The minimal identity for the Lie bracket (the identity of lowest degree that is not a consequence of anticommutativity) is the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

The minimal identity for the Jordan product (the identity of lowest degree that is not a consequence of commutativity) is the Jordan identity

$$((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ a).$$

Thus the two most important varieties of nonassociative binary algebras are defined by the obvious and minimal identities for the non-trivial bilinear operations.

Our purpose in the present paper is to extend this classification to the ternary case.

1.3 The ternary case

Total associativity for a ternary operation reduces to the identities

$$(abc)de = a(bcd)e = ab(cde).$$

A trilinear operation is an expression of the form

$$[a, b, c] = x_1\, abc + x_2\, acb + x_3\, bac + x_4\, bca + x_5\, cab + x_6\, cba \ (x_i \in \mathbb{F}).$$

Three of these operations have received attention for their connections with Lie and Jordan structures. Their obvious identities in degree 3 together with their minimal identities in degree 5 define the three most important varieties of triple systems:

- The **Lie triple product** is the iterated Lie bracket $[[a, b], c]$:

  $$[a, b, c] = abc - bac - cab + cba.$$

  The obvious and minimal identities for this operation are

  $$\begin{aligned} [a, a, b] &= 0, \\ [a, b, c] + [b, c, a] + [c, a, b] &= 0, \\ [a, b, [c, d, e]] &= [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]]. \end{aligned}$$

  These identities define the variety of **Lie triple systems**; see Hodge and Parshall [11].

- The **Jordan triple product** is one-half of $(a \circ b) \circ c + (c \circ b) \circ a - b \circ (a \circ c)$:

  $$[a, b, c] = abc + cba.$$

  The obvious and minimal identities for this operation are

  $$\begin{aligned} [a, b, c] &= [c, b, a], \\ [a, b, [c, d, e]] &= [[a, b, c], d, e] - [c, [b, a, d], e] + [c, d, [a, b, e]]. \end{aligned}$$

  These identities define the variety of **Jordan triple systems**; see Zelmanov [14].

- The **anti-Jordan triple product** is the Jordan triple product on the odd subspace of a Jordan superalgebra:

  $$[a, b, c] = abc - cba.$$

  The obvious and minimal identities for this operation are

  $$\begin{aligned} [a, b, a] &= 0, \\ [a, b, [c, d, e]] &= [[a, b, c], d, e] + [c, [b, a, d], e] + [c, d, [a, b, e]]. \end{aligned}$$

  These identities define the variety of **anti-Jordan triple systems**; see Faulkner and Ferrar [9].

In this paper we will see that there are many other trilinear operations which satisfy obvious identities in degree 3 and minimal identities in degree 5.

2 Classification of multilinear operations

2.1 Structure of the group algebra

The classical structure theory for $\mathbb{F}S_n$ was discovered by Young [13] for $\mathbb{F}$ of characteristic 0; see also James and Kerber [12].

Theorem 3. *For any partition λ of n, let d_λ be the number of standard tableaux. Then $\mathbb{F}S_n$ can be represented as the direct sum over all λ of full matrix algebras of size $d_\lambda \times d_\lambda$:*

$$\pi \colon \mathbb{F}S_n \xrightarrow{\cong} \bigoplus_\lambda M_{d_\lambda}(\mathbb{F}).$$

The mapping π is an isomorphism of associative algebras. □

In the natural representation of S_n, a basis E^λ_{ij} $(1 \le i, j \le d_\lambda)$ of $M_{d_\lambda}(\mathbb{F})$ can be constructed as follows. Enumerate the standard tableaux as $T^\lambda_1, \dots T^\lambda_{d_\lambda}$. For T^λ_i let R^λ_i be the sum over the row permutations and let C^λ_i be the alternating sum over the column permutations. Let s^λ_{ij} be the permutation which interchanges T^λ_i and T^λ_j. Define $E^\lambda_{ij} \in \mathbb{F}S_n$ by

$$E^\lambda_{ii} = \frac{d_\lambda}{n!} C^\lambda_i R^\lambda_i \text{ and } E^\lambda_{ij} = E^\lambda_{ii} s^\lambda_{ij} \ (i \ne j).$$

Then for partitions λ, μ we have

$$E^\lambda_{ij} E^\mu_{k\ell} = \delta_{\lambda\mu} \epsilon^\lambda_{jk} E^\lambda_{i\ell} \text{ where } \epsilon^\lambda_{jk} \in \{-1, 0, 1\}.$$

Clifton [8] gives a modification of E^λ_{ij} which satisfies the usual matrix unit relations

$$E^\lambda_{ij} E^\mu_{k\ell} = \delta_{\lambda\mu} \delta_{jk} E^\lambda_{i\ell}.$$

The diagonal elements E^λ_{ii} are idempotent elements of $\mathbb{F}S_n$.

2.2 Equivalence classes of operations

The isomorphism of Theorem 3 allows us to study multilinear operations in terms of matrix algebra. Lemma 1 depends only on the structure of $\mathbb{F}S_n$ as a left $\mathbb{F}S_n$-module. For each λ, the E^λ_{ij} for fixed j form a basis of a simple left module: column j of $M_{d_\lambda}(\mathbb{F})$. We can therefore use arguments from linear algebra even though the E^λ_{ij} do not satisfy the usual matrix unit relations. (In fact, for $n \le 4$ we have $\epsilon_{jk} = \delta_{jk}$, so this is only a concern for $n \ge 5$.) An operation $O \in \mathbb{F}S_n$ corresponds to a sequence of matrices $\pi(O)$ in the direct sum of matrix algebras; for component λ of this sequence we write $\pi(O)_\lambda \in M_{d_\lambda}(\mathbb{F})$. An invertible element $X \in \mathbb{F}S_n$ is represented by a sequence of invertible matrices $\pi(X)_\lambda \in M_{d_\lambda}(\mathbb{F})$.

Proposition 4. *If $O_1, O_2 \in \mathbb{F}S_n$ satisfy $O_1 = YO_2$ and $O_2 = ZO_1$ for some $Y, Z \in \mathbb{F}S_n$ then we may assume that Y is invertible and $Z = Y^{-1}$.*

Proof. Theorem 3 implies that the assumptions are equivalent to

$$\pi(O_1)_\lambda = \pi(Y)_\lambda \pi(O_2)_\lambda \text{ and } \pi(O_2)_\lambda = \pi(Z)_\lambda \pi(O_1)_\lambda \text{ for all } \lambda.$$

From this it follows that rowspace$(\pi(O_1)_\lambda) \subseteq$ rowspace$(\pi(O_2)_\lambda)$ and conversely. Hence the rowspaces are equal, and so $\pi(O_1)_\lambda$ and $\pi(O_2)_\lambda$ have the same row canonical form. Therefore we may take $\pi(Y)_\lambda$ to be an invertible matrix and $\pi(Z)_\lambda = (\pi(Y)_\lambda)^{-1}$. Since this holds for all λ, the conclusion follows. □

Corollary 5. *As representatives of the equivalence classes of multilinear n-ary operations, we may take the operations $O \in \mathbb{F}S_n$ for which all the matrices $\pi(O)_\lambda \in M_{d_\lambda}(\mathbb{F})$ are in row canonical form.*

Proof. Proposition 4 implies that the operations O_1 and O_2 are equivalent if and only if for every λ there is an invertible matrix $\pi(Y)_\lambda$ for which $\pi(Y)_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$, and this condition is equivalent to the statement that for every λ the matrices $\pi(O_1)_\lambda$ and $\pi(O_2)_\lambda$ have the same row canonical form. □

Suppose that we have two operations O_1 and O_2 satisfying the condition that for every λ the matrix $\pi(O_1)_\lambda$ is the row canonical form of the matrix $\pi(O_2)_\lambda$. Then for every λ there is an invertible matrix R_λ such that $R_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$. To find an explicit element $Y \in \mathbb{F}S_n$ for which $YO_2 = O_1$, we use the inverse of the isomorphism π from Theorem 3. Since $\pi(Y)_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$ for every λ we have $\pi(Y)_\lambda = R_\lambda$ and so

$$Y = \sum_\lambda \pi^{-1}(R_\lambda).$$

For an example see Lemma 8.

Equivalent operations do not necessarily satisfy the same polynomial identities, although the equivalence gives a linear isomorphism between the spaces of identities (at least in degree n). For example, let O_1 and O_2 be equivalent multilinear n-ary operations. Then there exists an invertible $Y \in \mathbb{F}S_n$ for which $O_2 = YO_1$. Suppose that I represents an obvious identity satisfied by O_1; then $IO_1 = 0$ by Lemma 2. Since $O_1 = Y^{-1}O_2$ we obtain $IY^{-1}O_2 = 0$, and so IY^{-1} represents an obvious identity satisfied by O_2.

3 Trilinear operations

For the rest of this paper $n = 3$ and $\mathbb{F} = \mathbb{Q}$ is the field of rational numbers. We assume throughout that the underlying ternary algebra A is totally associative.

3.1 Explicit decomposition of $\mathbb{Q}S_3$

Let the symmetric group S_3 act on the set $\{a, b, c\}$. Each permutation $\sigma \in S_3$ will be represented as the word $\sigma(a)\sigma(b)\sigma(c)$. We write the partitions of 3 in the short form 3, 21, 111, and so $d_3 = 1$, $d_{21} = 2$, $d_{111} = 1$. The group algebra decomposes as the direct sum

$$\mathbb{Q}S_3 = \mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus \mathbb{Q}.$$

For $\lambda = 3$ there is one standard tableau:

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline\end{array}$$

The corresponding idempotent is the symmetric sum

$$S = \frac{1}{6}(abc + acb + bac + bca + cab + cba). \tag{1}$$

For $\lambda = 21$ there are two standard tableaux:

$$\begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array}$$

The corresponding idempotents are

$$E_{11} = \frac{1}{3}(abc - cba)(abc + bac) = \frac{1}{3}(abc + bac - bca - cba), \tag{2}$$

$$E_{22} = \frac{1}{3}(abc - bac)(abc + cba) = \frac{1}{3}(abc - bac - cab + cba). \tag{3}$$

Multiplying on the right by the transposition which interchanges the tableaux gives

$$E_{12} = \frac{1}{3}(abc + bac - bca - cba)\, acb = \frac{1}{3}(acb - bac + bca - cab), \tag{4}$$

$$E_{21} = \frac{1}{3}(abc - bac - cab + cba)\, acb = \frac{1}{3}(acb - bca + cab - cba). \tag{5}$$

The four elements E_{ij} span the subalgebra $M_2(\mathbb{Q})$ of $\mathbb{Q}S_3$ and satisfy the matrix unit relations

$$E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}.$$

For $\lambda = 111$ there is one standard tableau:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array}$$

The corresponding idempotent is the alternating sum

$$A = \frac{1}{6}(abc - acb - bac + bca + cab - cba). \tag{6}$$

The basis elements S, E_{ij} $(1 \le i, j \le 2)$, A satisfy the orthogonality relations

$$SE_{ij} = E_{ij}S = 0, \quad AE_{ij} = E_{ij}A = 0, \quad AS = SA = 0.$$

3.2 Change of basis in $\mathbb{Q}S_3$

We have two bases of $\mathbb{Q}S_3$: the first (the permutation basis) consists of the words in lexicographical order,

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba; \tag{7}$$

the second (the idempotent basis) consists of the matrix units in standard order,

$$S, \quad E_{11}, \quad E_{12}, \quad E_{21}, \quad E_{22}, \quad A. \tag{8}$$

We write M for the change of basis matrix from the latter to the former. The columns of M are the coefficients of the matrix units in terms of the words:

$$M = \frac{1}{6}\begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 1 & 2 & -2 & 0 & -2 & -1 \\ 1 & -2 & 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & 2 & -2 & 1 \\ 1 & -2 & 0 & -2 & 2 & -1 \end{pmatrix} \tag{9}$$

The columns of M^{-1} are the coefficients of the words in terms of the matrix units:

$$M^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix} \tag{10}$$

3.3 Equivalence classes of trilinear operations

By Corollary 5, for representatives of the equivalence classes of trilinear operations, we can choose ordered triples of matrices with rational entries in row canonical form:

$$\left[\; x, \; \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \; z \;\right]$$

For the first and last components the possible row canonical forms are 0 and 1. For the middle component the possible row canonical forms are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (q \in \mathbb{Q})$$

We exclude the two trivial cases: the zero operation (the zero element of $\mathbb{Q}S_3$, the zero matrix in each component), and the original totally associative operation (the identity element of $\mathbb{Q}S_3$, the identity matrix in each component). This leaves ten cases: four families of operations with a parameter q (each including the case $q = \infty$) and six isolated operations.

In the following sections we determine the identities of degree ≤ 5 for these ten (families of) operations. We order the cases by increasing rank of the middle component. We use without further comment the fact that the polynomial identities satisfied by an operation remain unchanged for scalar multiples of the operation.

4 The symmetric, alternating, and cyclic sums

These operations have the zero matrix as the middle component. Each can be expressed in two forms corresponding to the two bases of $\mathbb{Q}S_3$:

$$\begin{array}{lll} \text{symmetric sum} & \frac{1}{6}(abc + acb + bac + bca + cab + cba) & \left[1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right] \end{array} \tag{11}$$

$$\begin{array}{lll} \text{alternating sum} & \frac{1}{6}(abc - acb - bac + bca + cab - cba) & \left[0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right] \end{array} \tag{12}$$

$$\begin{array}{lll} \text{cyclic sum} & \frac{1}{3}(abc + bca + cab) & \left[1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right] \end{array} \tag{13}$$

Each of these operations satisfies obvious identities in degree 3:

$$\begin{array}{lll} \text{symmetric sum} & [\sigma(a), \sigma(b), \sigma(c)] = [a, b, c] & \text{for all } \sigma \in S_3, \end{array} \tag{14}$$

$$\begin{array}{lll} \text{alternating sum} & [\sigma(a), \sigma(b), \sigma(c)] = \varepsilon(\sigma)[a, b, c] & \text{for all } \sigma \in S_3, \end{array} \tag{15}$$

$$\begin{array}{lll} \text{cyclic sum} & [\sigma(a), \sigma(b), \sigma(c)] = [a, b, c] & \text{for all } \sigma \in A_3. \end{array} \tag{16}$$

Here $\varepsilon\colon S_3 \to \{\pm 1\}$ is the sign homomorphism and $A_3 = \ker \varepsilon$ is the alternating group. The polynomial identities of degree ≥ 5 satisfied by these operations in every totally associative ternary algebra have been studied by Bremner and Hentzel [3]; in each case the minimal identities have degree 7.

In the next four sections we consider the operations which have a matrix of rank 1 as the middle component.

5 The Lie family of operations

In this section we consider the deformation of the classical Lie triple product. This operation and its matrix form are

$$\frac{1}{3}(abc - bac - cab + cba) \qquad \left[0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0\right] \tag{17}$$

An equivalent form of operation (17) is obtained by taking the row canonical form of each matrix:

$$\frac{1}{3}(acb - bac + bca - cab) \qquad \left[0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right] \tag{18}$$

Both of these operations satisfy the Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$. But operation (17) satisfies $[a, a, b] = 0$, whereas operation (18) satisfies $[a, b, a] = 0$. In the group algebra $\mathbb{Q}S_3$, we obtain operation (18) from operation (17) by multiplying by acb (the transposition which interchanges b and c).

We replace the middle component of operation (18) with another matrix of rank 1, and obtain a deformation of the Lie triple product.

Definition 6. *The deformed Lie triple product is the trilinear operation*

$$\frac{1}{3}\Big(abc + q\,acb + (1-q)bac + (q-1)bca - q\,cab - cba\Big) \qquad \left[0, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 0\right] \qquad (q \in \mathbb{Q})$$

Operation (18) corresponds to the limiting case $q \to \infty$ in the sense of projective geometry: the ordered pair $(1, q)$ corresponds to the point $q \in \mathbb{Q}$ and the ordered pair $(0, 1)$ corresponds to the point $q = \infty$. In this section we determine the identities of degree ≤ 5 for this family of operations.

Theorem 7. *In every totally associative ternary algebra, the deformed Lie triple product satisfies the obvious identities*

$$[a, b, a] = 0, \tag{19}$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \tag{20}$$

The deformed Lie triple product satisfies no new identities in degree 5, except in two cases:

$$q = \infty\colon \qquad [a, b, c] = \frac{1}{3}(acb - bac + bca - cab), \tag{21}$$

$$q = \frac{1}{2}\colon \qquad [a, b, c] = \frac{1}{6}(2abc + acb + bac - bca - cab - 2cba). \tag{22}$$

For operation (21) we have the general ternary derivation identity:

$$[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]]. \tag{23}$$

For operation (22) multiplication by a "square" is a ternary derivation:

$$[[a, b, c], d, d] = [[a, d, d], b, c] + [a, [b, d, d], c] + [a, b, [c, d, d]]. \tag{24}$$

These identities (or the linearized form for $q = 1/2$) imply all the identities in degree 5 for operations (21, 22) which do not follow from the obvious identities in degree 3. For all other values of q the minimal identities have degree ≥ 7.

Proof. The obvious identities follow from Lemma 2. In degree 5 there are three association types for a ternary operation:

$$[[a, b, c], d, e], \quad [a, [b, c, d], e], \quad [a, b, [c, d, e]]. \tag{25}$$

The linearization of identity (19) shows that any monomial in the third type can be rewritten in the first type:

$$[a, b, [c, d, e]] = -[[c, d, e], b, a]. \tag{26}$$

Combining this with identity (20) shows that any monomial in the second type can be rewritten in the first type:

$$[a, [b, c, d], e] = -[[b, c, d], e, a] - [e, a, [b, c, d]] = -[[b, c, d], e, a] + [[b, c, d], a, e]. \tag{27}$$

Therefore any identity of degree 5 for this operation can be written as a linear combination of monomials in the first association type. The same reasoning shows that we only need to consider the monomials in which the inner triple has the property that the lexicographically first letter occurs in the leftmost position:

$$[c, b, a] = -[a, b, c], \tag{28}$$

$$[b,a,c] = -[a,c,b] - [c,b,a] = -[a,c,b] + [a,b,c]. \qquad (29)$$

This reduces the number of monomials to $5!/3 = 40$.

We now construct the expansion matrix $\mathcal{X}$ of size 120×40. The columns are labelled by the nonassociative monomials and the rows are labelled by the associative monomials (both in lexicographical order). The columns represent the expansions of the nonassociative monomials obtained by interpreting the trilinear operation as the deformed Lie triple product: $\mathcal{X}_{ij}$ is the coefficient of the i-th associative monomial in the expansion of the j-th nonassociative monomial. The expansion of (9 times) $[[a,b,c],d,e]$ appears in Table 1: the associative monomials are in lexicographical order and are preceded by their coefficients.

1	$abcde$	q	$abced$	q	$acbde$	q^2	$acbed$
$1-q$	$bacde$	$-(q-1)q$	$baced$	$q-1$	$bcade$	$(q-1)q$	$bcaed$
$-q$	$cabde$	$-q^2$	$cabed$	-1	$cbade$	$-q$	$cbaed$
$1-q$	$dabce$	$-(q-1)q$	$dacbe$	$(q-1)^2$	$dbace$	$-(q-1)^2$	$dbcae$
$(q-1)q$	$dcabe$	$q-1$	$dcbae$	$q-1$	$deabc$	$(q-1)q$	$deacb$
$-(q-1)^2$	$debac$	$(q-1)^2$	$debca$	$-(q-1)q$	$decab$	$1-q$	$decba$
$-q$	$eabcd$	$-q^2$	$eacbd$	$(q-1)q$	$ebacd$	$-(q-1)q$	$ebcad$
q^2	$ecabd$	q	$ecbad$	-1	$edabc$	$-q$	$edacb$
$q-1$	$edbac$	$1-q$	$edbca$	q	$edcab$	1	$edcba$

Table 1: Expansion of $[[a,b,c],d,e]$ using the deformed Lie triple product

Each column of $\mathcal{X}$ contains 36 nonzero entries which are polynomials in q of degree ≤ 2. The i-th row of $\mathcal{X}$ represents the linear relation on the coefficients of a polynomial identity which expresses the condition that when the identity is expanded using the deformed Lie triple product, the coefficient of the i-th associative monomial must be 0. The nonzero elements of the nullspace of $\mathcal{X}$ represent the identities satisfied by the deformed Lie triple product in every totally associative ternary algebra.

In order to determine how the nullspace of $\mathcal{X}$ depends on the parameter q, we use Maple's `linalg[smith]` procedure to compute the Smith normal form of $\mathcal{X}$ (Adkins and Weintraub [1], Section 5.3): we used this procedure rather than `LinearAlgebra[SmithForm]` since (for some unknown reason) the earlier package seemed more efficient on this particular matrix. The diagonal entries of the Smith form of $\mathcal{X}$ are

$$1 \text{ (24 times)}, \qquad q-\frac{1}{2} \text{ (16 times)}.$$

Therefore the expansion matrix has full rank (zero nullspace, no identities) except when $q = 1/2$, in which case there is a 16-dimensional nullspace of identities.

We now fix $q = 1/2$ and compute the nullspace using Maple. One of the 16 basis vectors for the nullspace represents the identity

$$[[a,c,b],d,e] + [[a,c,b],e,d] - [[a,d,e],c,d] - [[a,e,d],c,b] + [[b,d,e],c,a]$$

$$+\ [[b,e,d],c,a]-[[c,d,e],a,b]+[[c,d,e],b,a]-[[c,e,d],a,b]+[[c,e,d],b,a]=0. \tag{30}$$

This identity alternates in a, b and is symmetric in d, e. If we set $d = e$, interchange b and c, and divide by 2, we obtain the identity

$$[[a,b,c],d,d]=[[a,d,d],b,c]+[[b,d,d],a,c]-[[b,d,d],c,a]-[[c,d,d],b,a],$$

which now alternates in a, c. Using equations (26) and (27) we see that this is equivalent to the identity in the statement of the Theorem. We now apply all permutations of a, b, c, d, e to identity (30), straighten the inner triples using equations (28) and (29), and store the results in a matrix of size 120×40. We use Maple to compute the rank of this matrix, which is 16; so this identity generates the entire space of identities under the action of the symmetric group. □

We verified these results in a different way using the representation theory of the symmetric group S_5. The following description is rather brief; for more detailed discussions of these methods, see our earlier papers: Bremner and Hentzel [4, 5], Bremner and Peresi [6], Hentzel and Peresi [10].

For each partition of 5 there is a corresponding irreducible representation of S_5. The group algebra $\mathbb{Q}S_5$ decomposes as the direct sum of full matrix algebras according to Theorem 3. We consider four association types: the first represents the original totally associative operation, and the other three are the nonassociative types (25). We form the direct sum of four copies of the group algebra $\mathbb{Q}S_5$, one copy for each association type. We lift the obvious identities in degree 3 to degree 5 in every possible way (replacing one variable by a triple, or embedding the identity in a triple) and compute the rank of these identities for each representation of S_5: these identities lie entirely in the last three association types. We represent the expansions of the basic monomials (25) by storing the expansion in the associative type and the basic monomial in the appropriate nonassociative type: these identities relate the first type with the last three types. We then compute the rank of these expansions for each representation of S_5; identities which occur only in the last three types, and do not follow from the liftings of the obvious identities, will be new identities for the operation. After computing the row canonical forms of the resulting matrices, we can tell which representations of S_5 contain new identities for the operation.

		lifted and		$q=\infty, 1/2$	
λ	d_λ	expansion	lifted	all	new
5	1	3	3	3	0
41	4	12	11	11	0
32	5	15	13	14	1
311	6	18	16	17	1
221	5	15	13	14	1
2111	4	12	11	11	0
11111	1	3	3	3	0

Table 2: Ranks of representations for the Lie family

These results are summarized in Table 2: column 1 is the partition of 5, column 2 is the dimension of the corresponding irreducible representation of S_5, column 3 is the rank of the combined lifted and expansion identities (this is $3d_\lambda$, the dimension of the space of nonassociative polynomials in the last three association types), column 4 is the rank of the lifted identities alone, column 5 is the rank (in the last three types only) of the combined lifted and expansion identities, and column 6 is the difference of columns 4 and 5. There is one new identity for each of the three partitions 32, 311, 221, giving a total dimension of $5+6+5=16$, which confirms our earlier results. For $q=\infty$ (equivalent to the classical case) the ranks are the same but the identities are different.

6 The Jordan family of operations

In this section we consider the deformation of the classical Jordan triple product. This operation and its matrix form are

$$abc + cba \qquad \left[2, \left(\begin{array}{cc} 0 & -1 \\ 0 & 2 \end{array}\right), 0\right] \tag{31}$$

An operation equivalent to the classical Jordan triple product is

$$\frac{1}{6}(abc + 3acb - bac + 3bca - cab + cba) \qquad \left[1, \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), 0\right] \tag{32}$$

If we take the row canonical form of each matrix in (31) we obtain (32). Note that the obvious identity $[a,b,c]=[c,b,a]$ for operation (31) does not hold for operation (32).

Lemma 8. *Let O_1 be operation (32) and let O_2 be operation (31). Then for any nonzero $r \in \mathbb{Q}$, the group algebra element*

$$Y_r = \frac{1}{12}\Big((1{+}2r)abc + (9{-}2r)acb - (7{+}2r)bac - (3{-}2r)bca + (5{+}2r)cab + (1{-}2r)cba\Big),$$

is invertible and satisfies $Y_r O_2 = O_1$.

Proof. For each λ we need to find the inverse image under the isomorphism π from Theorem 3 of an invertible $d_\lambda \times d_\lambda$ matrix R_λ which satisfies $R_\lambda \pi(O_2)_\lambda = \pi(O_1)_\lambda$. The matrices R_λ and their inverse images $\pi^{-1}(R_\lambda)$ are as follows:

$$R_3 = \frac{1}{2} \qquad R_{21} = \left(\begin{array}{cc} -1 & 0 \\ 2 & 1 \end{array}\right) \qquad R_{111} = r \in \mathbb{Q} - \{0\}$$

$$\pi^{-1}(R_3) = \frac{1}{2}S \qquad \pi^{-1}(R_{21}) = -E_{11} + 2E_{21} + E_{22} \qquad \pi^{-1}(R_{111}) = rA$$

With respect to the standard (idempotent) basis of $\mathbb{Q}S_3$, the element Y_r has coefficients $[1/2,-1,0,2,1,r]$. Multiplying this (column) vector by the change-of-basis matrix M from equation (9) gives the coefficients of Y_r with respect to the other (permutation) basis. $\square$

Definition 9. *The deformed Jordan triple product is the trilinear operation*

$$\frac{1}{3}\Big(2q\,abc + (3{-}q)acb - q\,bac - q\,bca + (3{-}q)cab + 2q\,cba\Big) \qquad \left[2, \begin{pmatrix} -1 & -q \\ 2 & 2q \end{pmatrix}, 0\right]$$

In this section we determine the identities of degree ≤ 5 for this family of operations.

Theorem 10. *In every totally associative ternary algebra, the deformed Jordan triple product satisfies the obvious identity*

$$[a, b, c] = [c, b, a]. \tag{33}$$

In the case

$$q = \infty: \qquad [a, b, c] = abc + cba, \tag{34}$$

the deformed Jordan triple product satisfies the identity

$$[a, b, [c, d, e]] = [[a, b, c], d, e] - [c, [b, a, d], e] + [c, d, [a, b, e]], \tag{35}$$

which does not follow from the obvious identity, and which implies all the identities of degree 5 satisfied by operation (34). The deformed Jordan triple product satisfies the following identity in degree 5 for all finite values of q (from this point we will omit the commas in identities of degree 5):

$$\begin{aligned}
&[[abc]de] + [[abc]ed] + [[abd]ce] + [[abd]ec] + [[abe]cd] + [[abe]dc] \\
&+ [[cbd]ae] + [[cbd]ea] + [[cbe]ad] + [[cbe]da] + [[dbe]ac] + [[dbe]ca] \\
&- [a[bcd]e] - [a[bce]d] - [a[bdc]e] - [a[bde]c] - [a[bec]d] - [a[bed]c] \\
&- [c[adb]e] - [c[aeb]d] - [c[bad]e] - [c[bae]d] - [d[acb]e] - [d[bac]e] = 0.
\end{aligned} \tag{36}$$

This identity does not follow from the obvious identity, and it implies all the identities of degree 5 for the deformed Jordan triple product that hold for all finite values of q. The deformed Jordan triple product satisfies additional identities in degree 5 in the following three cases:

$$q = 0: \qquad [a, b, c] = acb + cab, \tag{37}$$

$$q = \frac{1}{2}: \qquad [a, b, c] = \frac{1}{6}(2abc + 5acb - bac - bca + 5cab + 2cba), \tag{38}$$

$$q = 1: \qquad [a, b, c] = \frac{1}{3}(2abc + 2acb - bac - bca + 2cab + 2cba). \tag{39}$$

These operations satisfy the identities

$$q = 0: \qquad [[acb]de] + [[ace]db] + [[bce]da] - [a[bdc]e] - [a[cde]b] - [b[adc]e] = 0, \tag{40}$$

$$\begin{aligned}
q = \frac{1}{2}: \qquad &- [[acd]be] - [[ace]bd] + 2[[adc]be] + 2[[aec]bd] - [[cad]be] - [[cae]bd] \\
&+ [[dbe]ac] + [[dbe]ca] - 2[a[dbe]c] + 2[d[abc]e] - [d[acb]e] - [d[bac]e] = 0,
\end{aligned} \tag{41}$$

$$q = 1: \quad 8[[abc]bb] + 4[[acb]bb] - 4[[bac]bb] + 4[[bbc]ab] - 8[[bbc]ba] \\ - 3[[bcb]ab] - 4[a[bbc]b] + 2[a[bcb]b] - 2[b[acb]b] + 3[b[bac]b] = 0. \tag{42}$$

These identities (or the linearized form for $q = 1$) imply (respectively) all the identities in degree 5 for operations (37, 38, 39) which do not follow from the obvious identity.

Proof. The obvious identity follows from Lemma 2. This identity shows that any monomial in the third type can be rewritten in the first type:

$$[a, b, [c, d, e]] = [[c, d, e], b, a]. \tag{43}$$

Hence any identity of degree 5 for this operation can be written as a linear combination of monomials in the first two types. For the first type, we only need the monomials in which the first letter lexicographically precedes the third letter:

$$[[c, b, a], d, e] = [[a, b, c], d, e],$$

giving $5!/2 = 60$ distinct monomials. For the second type, we only need the monomials in which the first precedes the fifth and the second precedes the fourth:

$$[e, [d, c, b], a] = [e, [b, c, d], a] = [a, [d, c, b], e] = [a, [b, c, d], e],$$

giving $5!/4 = 30$ distinct monomials. The total number of nonassociative monomials is 90, and so the expansion matrix $\mathcal{X}$ in this case has size 120×90.

As before, we use the Maple procedure `linalg[smith]` to compute the Smith normal form of $\mathcal{X}$. The diagonal entries of the Smith form of $\mathcal{X}$ are

$$1 \text{ (64 times)}, \qquad q-\frac{1}{2} \text{ (6 times)}, \qquad q\left(q-\frac{1}{2}\right)(q-1) \text{ (5 times)}, \qquad 0 \text{ (5 times)}.$$

This shows that the expansion matrix never has full rank: there is a 5-dimensional space of identities which hold for all q. For $q = 0$ and $q = 1$ this extends to a 20-dimensional space. For $q = 1/2$ it extends further to a 26-dimensional space. In each of these cases we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. For the case $q = 1$, the identities produced by this method were very complicated, with many large and apparently random coefficients. To find the much simpler but equivalent identity (42) we used an evolutionary algorithm which is described in detail in the companion paper; see Bremner [2]. For the identity which holds for all q, we used the value $q = -1$; from the Smith form we know that the identities for any value of q except $q = 0, 1/2, 1$ are the same as for q indeterminate. $\square$

We verified these results using the representation theory of S_5; the results are displayed in Table 3. The column labelled "general" has the results for $q = -1$. For $q = 0, 1$ the ranks are the same, but the identities are different.

		$q=\infty$		general		$q=0,1$		$q=1/2$	
λ	lifted	all	new	all	new	all	new	all	new
5	1	2	1	2	1	2	1	2	1
41	7	10	3	8	1	9	2	8	1
32	10	12	2	10	0	11	1	12	2
311	14	16	2	14	0	15	1	15	1
221	12	13	1	12	0	12	0	13	1
2111	11	11	0	11	0	11	0	11	0
11111	3	3	0	3	0	3	0	3	0

Table 3: Ranks of representations for the Jordan family

7 The anti-Jordan family of operations

In this section we consider the deformation of the classical anti-Jordan triple product. This operation, its matrix form, and its row canonical form, are

$$abc - cba \qquad \left[0, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, 2\right] \qquad \left[0, \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, 1\right] \tag{44}$$

(The equivalent form is obtained simply by multiplying by 1/2.) We therefore consider the following operation and its deformation:

$$\frac{1}{6}(abc + acb - 3bac + 3bca - cab - cba) \qquad \left[0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1\right] \tag{45}$$

Definition 11. *The deformed anti-Jordan triple product is the trilinear operation*

$$\frac{1}{6}\Big(3abc + (2q{-}1)acb + (1{-}2q)bac + (2q{-}1)bca + (1{-}2q)cab - 3cba\Big) \qquad \left[0, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 1\right]$$

The anti-Jordan triple product is equivalent to the deformed operation for $q = 1/2$. In this section we determine the identities of degree ≤ 5 for this family of operations.

Theorem 12. *In every totally associative ternary algebra, the deformed anti-Jordan triple product satisfies the obvious identity*

$$[a, b, a] = 0. \tag{46}$$

In the case

$$q = \infty\colon \qquad [a, b, c] = \frac{1}{6}(abc + acb - 3bac + 3bca - cab - cba), \tag{47}$$

the deformed anti-Jordan triple product satisfies the identity

$$\begin{aligned} &[[abc]de] - [[acb]de] - [[ade]bc] + [[ade]cb] + 2[[bac]de] - [[bce]da] \\ &- 2[[bec]da] + [[cbe]da] + [a[bcd]e] + 2[a[bdc]e] - [a[cbd]e] - 2[b[ade]c] = 0, \end{aligned} \tag{48}$$

which does not follow from the obvious identity, and which implies all the identities of degree 5 satisfied by operation (47). (Note that identity (48) alternates in the variables b, c.) The deformed anti-Jordan triple product satisfies the following identity in degree 5 for all finite values of q:

$$
\begin{aligned}
&- [[adb]ce] + [[adb]ec] + [[adc]be] - [[adc]eb] - [[ade]bc] + [[ade]cb] \\
&- [[bdc]ae] + [[bdc]ea] + [[bde]ac] - [[bde]ca] - [[cde]ab] + [[cde]ba] \\
&+ [a[bcd]e] - [a[bed]c] - [a[cbd]e] + [a[ced]b] - [a[dbe]c] + [a[dce]b] \\
&- [b[acd]e] + [b[aed]c] + [b[cad]e] + [b[dae]c] + [c[abd]e] - [c[bad]e] = 0. \qquad (49)
\end{aligned}
$$

This identity does not follow from the obvious identity, and it implies all the identities of degree 5 for the deformed anti-Jordan triple product that hold for all finite values of q. The deformed anti-Jordan triple product also satisfies additional identities in the following three cases:

$$
\begin{aligned}
q = -1\colon &\qquad [a,b,c] = \frac{1}{2}(abc - acb + bac - bca + cab - cba), &(50)\\
q = \frac{1}{2}\colon &\qquad [a,b,c] = \frac{1}{2}(abc - cba), &(51)\\
q = 2\colon &\qquad [a,b,c] = \frac{1}{2}(abc + acb - bac + bca - cab - cba). &(52)
\end{aligned}
$$

These operations satisfy the identities

$$
\begin{aligned}
q = -1\colon \quad & [[acb]de] - [[acb]ed] - [[acd]eb] + [[ace]db] - [[adb]ce] + [[adb]ec] \\
& [[adc]eb] - [[ade]cb] + [[aeb]cd] - [[aeb]dc] - [[aec]db] + [[aed]cb] \\
& - [[cad]be] + [[cad]eb] + [[cae]bd] - [[cae]db] - [[dae]bc] + [[dae]cb] \\
& - [c[abd]e] + [c[abe]d] + [c[adb]e] - [c[aeb]d] + [d[abc]e] - [d[acb]e] = 0, \qquad (53)\\
q = \frac{1}{2}\colon \quad & [a,b,[c,d,e]] = [[a,b,c],d,e] + [c,[b,a,d],e] + [c,d,[a,b,e]], \qquad (54)\\
q = 2\colon \quad & [[abc]de] - [[abd]ce] + [[acd]be] + [[ace]bd] - [[ace]db] - [[adc]be] \\
& - [[ade]bc] + [[ade]cb] - [[cad]be] - [[cae]bd] + [[cae]db] + [[cbd]ae] \\
& - [[cde]ab] + [[cde]ba] + [[dae]bc] - [[dae]cb] + [[dce]ab] - [[dce]ba] \\
& - [a[cbd]e] - [b[acd]e] + [b[adc]e] + [b[cad]e] + [c[abd]e] - [d[abc]e] = 0. \qquad (55)
\end{aligned}
$$

These identities imply (respectively) all the identities in degree 5 for operations (50, 51, 52) which do not follow from the obvious identity.

Proof. For $q = \infty$, the expansion matrix has size 120×90 (with scalar entries). It has rank 64, and hence its nullspace has dimension 26. One of the basis vectors of the nullspace corresponds to the identity of the Theorem. We compute the submodule generated by this identity, and find that it has dimension 26.

The rest of the proof is similar to the Jordan case. The diagonal entries of the Smith form of the expansion matrix $\mathcal{X}$ are

$$1 \text{ (50 times)}, \qquad q-\frac{1}{2} \text{ (20 times)}, \qquad (q+1)(q-2)\left(q-\frac{1}{2}\right) \text{ (15 times)}, \qquad 0 \text{ (5 times)}.$$

The expansion matrix never has full rank: there is a 5-dimensional space of identities which hold for all q. For $q = -1$ and $q = 2$ this extends to a 20-dimensional space. For $q = 1/2$ it extends further to a 40-dimensional space. In each of the non-classical cases ($q \neq 1/2$) we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. For the identity which holds for all q, we used the value $q = 1$; from the Smith form we know that the identities for any finite value of q except $q = -1, 1/2, 2$ are the same as for q indeterminate. □

We verified these results using the representation theory of S_5; the results are displayed in Table 4. The column labelled "general" has the results for $q = 1$. For $q = -1, 2$ the ranks are the same, and the identities are also the same: in other words, the identities (53) and (55) imply each other.

		$q=\infty$		general		$q=-1,2$		$q=1/2$	
λ	lifted	all	new	all	new	all	new	all	new
5	3	3	0	3	0	3	0	3	0
41	11	11	0	11	0	11	0	11	0
32	12	13	1	12	0	12	0	13	1
311	14	15	1	14	0	15	1	16	2
221	10	12	2	10	0	11	1	12	2
2111	7	8	1	8	1	9	2	10	3
11111	1	2	1	2	1	2	1	2	1

Table 4: Ranks of representations for the anti-Jordan family

8 The fourth family of operations

This is the only parametrized family which appears not to be related to the classical Lie, Jordan and anti-Jordan triple systems. The basic operation, which we call the "fourth operation" for want of a better name, is

$$\frac{1}{3}(abc + acb - bac + 2bca) \qquad \left[1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1\right] \tag{56}$$

(To get this operation we add the cyclic sum and the equivalent form of the classical Lie triple product.)

Definition 13. *The deformed fourth operation is the trilinear operation*

$$\frac{1}{3}\Big(2abc + q\,acb + (1-q)bac + q\,bca + (1-q)cab - cba\Big) \qquad \left[1, \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, 1\right]$$

This family of operations (and the three in the next section) are significantly more obscure than the operations studied in the previous sections, especially in view of their lack of connection with triple systems that have appeared in the literature of nonassociative algebra. For this reason, our discussions in this section and the next will be relatively brief.

Theorem 14. *In every totally associative ternary algebra, the deformed fourth operation satisfies the obvious identity*

$$[a,c,b]-[b,a,c]+[b,c,a]-[c,a,b]=0; \tag{57}$$

this identity implies all identities of degree 3 satisfied by operation (56) and its deformation. For the case of general finite q, and for the following special values of q, the deformed fourth operation satisfies additional identities which do not follow from the obvious identity:

$$q=\infty: \qquad [a,b,c]=\frac{1}{3}(abc+acb-bac+2bca), \tag{58}$$

$$q=0: \qquad [a,b,c]=\frac{1}{3}(2abc+bac+cab-cba), \tag{59}$$

$$q=1: \qquad [a,b,c]=\frac{1}{3}(2abc+acb+bca-cba), \tag{60}$$

$$q=-1: \qquad [a,b,c]=\frac{1}{3}(2abc-acb+2bac-bca+2cab-cba), \tag{61}$$

$$q=2: \qquad [a,b,c]=\frac{1}{3}(2abc+2acb-bac+2bca-cab-cba), \tag{62}$$

$$q=\frac{1}{2}: \qquad [a,b,c]=\frac{1}{6}(4abc+acb+bac+bca+cab-2cba). \tag{63}$$

For $q=\infty$ there is a 54-dimensional space of identities in degree 5. There is a 40-dimensional space of identities in degree 5 for the deformed fourth operation which hold for all finite q. For $q=0,1,-1,2$ this extends to a 49-dimensional space. For $q=1/2$ it extends further to a 54-dimensional space.

Proof. In identity (57), if we interchange a,b and rearrange the terms, we get

$$[a,b,c]=[a,c,b]+[b,c,a]-[c,b,a].$$

This shows that any monomial in the third association type is a linear combination of terms in the first and second types:

$$[a,b,[c,d,e]]=[a,[c,d,e],b]+[b,[c,d,e],a]-[[c,d,e],b,a].$$

We also get

$$[c,b,a]=-[a,b,c]+[a,c,b]+[b,c,a].$$

This shows that in the inner triple of any monomial of degree 5, we may assume that the first factor lexicographically precedes either the second factor or the third factor; in other words, the lexicographically greatest factor cannot be in the first position. We therefore have $(2/3)5!=80$ distinct monomials in each of association types 1 and 2, for a total of 160. Hence for this operation the expansion matrix $\mathcal{X}$ has size 120×160. From this we see that there is a nullspace of identities of dimension ≥ 40 which hold for all q. The diagonal entries of the Smith form of $\mathcal{X}$ are

$$1 \text{ (106 times)}, \quad q-\frac{1}{2} \text{ (5 times)}, \quad q(q-1)(q+1)(q-2)\left(q-\frac{1}{2}\right) \text{ (9 times)}, \quad 0 \text{ (40 times)}.$$

In each of these cases we took the expansion matrix, set q equal to the appropriate value, computed the nullspace of the matrix, and found a basis vector which generated the entire space of identities. ∎

In all these cases, the identities produced by this method were very complicated, with many large and apparently random coefficients. Since the evolutionary algorithm which we used to discover identity (42) takes a very long time to run (weeks or even months), we plan to return to these identities in a future publication; see Bremner and Peresi [7].

We verified the dimensions obtained from the Smith form by using the representation theory of S_5 and obtained the results in Table 5. The column labelled "general" has the results for $q = 5$. For $q = 0, 1$ the ranks are the same, but the identities are different; and similarly for $q = -1, 2$ the ranks are the same, but the identities are different.

		$q = \infty$		general q		$q = 0, 1$		$q = -1, 2$		$q = 1/2$	
λ	lifted	all	new	all	new	all	new	all	new	all	new
5	1	2	1	2	1	2	1	2	1	2	1
41	6	9	3	8	2	9	3	8	2	8	2
32	9	11	2	10	1	11	2	10	1	11	2
311	10	12	2	12	2	12	2	12	2	12	2
221	9	11	2	10	1	10	1	11	2	11	2
2111	6	8	2	8	2	8	2	9	3	9	3
11111	1	2	1	2	1	2	1	2	1	2	1

Table 5: Ranks of representations for the fourth family

9 The last three operations

In this section we consider the operations which have the identity matrix as the middle component. The proofs of the results in this section are similar to those in previous sections. The simplest of these operations is

$$\frac{1}{3}(2abc - bca - cab) \qquad \left[0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right] \tag{64}$$

The cyclic commutator $[a, bc]$ and its matrix form are

$$abc - bca \qquad \left[0, \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, 0 \right] \tag{65}$$

If we take the row canonical form of each matrix in (65) we obtain (64), so these two operations are equivalent.

Theorem 15. *In every totally associative ternary algebra, the cyclic commutator (65) satisfies the obvious identity*

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \tag{66}$$

This identity implies all the identities in degree 3 satisfied by operation (65). The cyclic commutator satisfies this minimal identity in degree 5:

$$[[bad]ec] - [[bec]ad] + [[dec]ba] + [a[dec]b] - [b[aec]d] = 0. \tag{67}$$

This identity is not a consequence of the obvious identity and it implies all the identities of degree 5 satisfied by operation (65). ☐

λ	cyclic commutator lifted	all	new	weakly commutative lifted	all	new	weakly anticommutative lifted	all	new
5	3	3	0	0	2	2	3	3	0
41	8	9	1	1	8	7	8	9	1
32	7	11	4	2	10	8	5	10	5
311	10	14	4	6	13	7	6	13	7
221	7	11	4	5	10	5	2	10	8
2111	8	9	1	8	9	1	1	8	7
11111	3	3	0	3	3	0	0	2	2

Table 6: Ranks of representations for the last three operations

The commutative identity $ab - ba = 0$ is the alternating sum over S_2, and the anticommutative identity $ab + ba = 0$ is the symmetric sum over S_2. If we generalize this to the ternary case, we obtain the weakly commutative and weakly anticommutative identities: the alternating sum and the symmetric sum (regarded as identities, not as operations). We use the adverb *weakly* to distinguish these identities from the *strongly* commutative and anticommutative identities (14) and (15) satisfied by the symmetric sum and the alternating sum.

The weakly commutative operation is

$$\frac{1}{6}(5abc + acb + bac - bca - cab + cba) \qquad \left[1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\right] \tag{68}$$

Theorem 16. *In every totally associative ternary algebra, every identity of degree 3 for operation (68) is a consequence of the obvious identity*

$$[a, b, c] - [a, c, b] - [b, a, c] + [b, c, a] + [c, a, b] - [c, b, a] = 0. \tag{69}$$

The weakly commutative operation satisfies a 141-dimensional space of identities in degree 5 which do not follow from the obvious identity. ☐

The weakly anticommutative operation is

$$\frac{1}{6}(5abc - acb - bac - bca - cab - cba) \qquad \left[0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right] \tag{70}$$

Theorem 17. *In every totally associative ternary algebra, every identity of degree 3 for operation (70) is a consequence of the obvious identity*

$$[a, b, c] + [a, c, b] + [b, a, c] + [b, c, a] + [c, a, b] + [c, b, a] = 0. \tag{71}$$

The weakly anticommutative operation satisfies a 141-dimensional space of identities in degree 5 which do not follow from the obvious identity. ☐

We plan to return in a future publication to the problem of using an evolutionary algorithm to simplify the very complicated identities of degree 5 satisfied by the last two operations; see Bremner and Peresi [7].

The ranks of the irreducible representations for these last three operations are displayed in Table 6.

10 Concluding remarks

10.1 Summary of results

We have identified 21 trilinear operations which satisfy obvious identities in degree 3 and minimal identities in degree 5:

- the Lie family gives 2 operations:
  - the classical Lie triple product (equivalent to the case $q = \infty$),
  - the case $q = 1/2$ of the deformation;
- the Jordan family gives 5 operations:
  - the classical Jordan triple product (the case $q = \infty$),
  - the general case of the deformation,
  - the cases $q = 0$, $q = 1$ and $q = 1/2$ of the deformation;
- the anti-Jordan family gives 5 operations:
  - the classical anti-Jordan triple product (equivalent to the case $q = 1/2$),
  - the general case of the deformation,
  - the cases $q = \infty$, $q = -1$ and $q = 2$ of the deformation;
- the fourth family gives 6 operations:
  - the general case of the deformation,
  - the cases $q = \infty$, $q = 0$, $q = -1$, $q = 1/2$ and $q = -1/2$ of the deformation;
- the cyclic commutator;
- the weakly commutative operation;
- the weakly anticommutative operation.

Of these operations, only the classical Lie, Jordan and anti-Jordan triple products have been studied in previous research on nonassociative algebra; the remaining 18 operations are new. The obvious and minimal identities for these 18 operations define new varieties of triple systems; in the first three cases, these varieties seem to be closely related to Lie and Jordan structures.

10.2 Minimal identities of degree ≥ 7

The Lie family of operations has the special feature that it has no identities in degree 5 that hold for all q. The next step in this research is to determine those values of q for which the minimal identities have degree 7. The obvious identities in degree 3 imply that we only need to consider the two association types in degree 7 which are represented by the monomials

$$[[[a,b,c],d,e],f,g], \qquad [[a,b,c],[d,e,f],g].$$

The obvious identities further imply that there are only $7!/3 = 1680$ distinct monomials for the first type and $7!/9 = 560$ distinct monomials for the second type, giving a total of 2240. The expansion matrix will therefore have size 5040×2240. It would be very interesting to see if current technology and algorithms permit the computation of the Smith normal form of a matrix of this size in a reasonable amount of time. (All the entries of this expansion matrix are polynomials in q of degree ≤ 3.) Beyond this, if there is no identity in degree 7 which holds for all values of q, we would want to determine those values of q for which the minimal identities have degree 9.

10.3 Quadrilinear operations

Another research direction is to consider operations of degree 4. There are 5 partitions of 4, namely 4, 31, 22, 211, 1111; the corresponding irreducible representations of S_4 have dimensions $d_4 = 1$, $d_{31} = 3$, $d_{22} = 2$, $d_{211} = 3$, $d_{1111} = 1$. Therefore any quadrilinear operation can be represented by a list of matrices of the form

$$\left[\; v, \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}, z \;\right]$$

The possible row canonical forms for a 3×3 matrix are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & p & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since there are $n+1$ possible ranks for an $n \times n$ matrix, we see that (even with this coarse classification) there are $2 \cdot 4 \cdot 3 \cdot 4 \cdot 2 = 192$ different families of quadrilinear operations, and many of them have two or more parameters. The only such operation that has been studied in the literature of nonassociative algebra is the Jordan tetrad $abcd + dcba$, which has the matrix form

$$\left[\; 2, \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, 2 \;\right]$$

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