SHIMURA VARIETIES mod p WITH MANY COMPACT FACTORS

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To my parents, Heinrich and Karola

Abstract. We give several new moduli interpretations of the fibers of certain Shimura varieties over several prime numbers. As a corollary we obtain that for every prescribed odd prime characteristic p every bounded symmetric domain possesses quotients by arithmetic groups whose models have good reduction at a prime divisor of p.

Contents

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1. Introduction

Let (G, X) be a Shimura datum in the sense of all five axioms [11, (2.1.1.1)-(2.1.1.5)] of Deligne. Let X^+ be a connected component of X, and let $\mathrm{Aut}(X^+)$ be the real Lie-group of biholomorphic automorphisms of X^+. Let K be a neat compact open subgroup of $G(\mathbb{A}^\infty)$. Following the ideas in, and using the notation of [11, (2.1.2)], we know that

$$G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^\infty)/K) = {}_K M(G, X) = \tag{1}$$

$$G(\mathbb{Q})_+\backslash(X^+ \times G(\mathbb{A}^\infty)/K) = \coprod_g \Gamma_g\backslash X^+ \tag{2}$$

is a smooth and quasi-projective algebraic $\mathbb{C}$-variety, where Γ_g is a bunch of torsion-free discrete subgroups of $\mathrm{Aut}(X^+)^+$, and in the special case of an anisotropic $\mathbb{Q}$-group G only, each Γ_g turns out to be cocompact, so that (1) is projective.
Moreover, one knows, that there exists a canonical variation $\mathrm{Fib}_{Hodge}(\rho)$ of pure Hodge structures over ${}_KM(G,X)$ for every $\mathbb{Q}$-rational linear representation $\rho : G \to \mathrm{GL}(V/\mathbb{Q})$. In some cases these yield so-called "moduli interpretations", more specifically if there exists an injective map $\rho : G \to \mathrm{GSp}(2g)$ with $\rho(X) \subset \mathfrak{h}_g^{\pm}$, then ${}_KM(G,X)$ is a moduli space of g-dimensional abelian varieties with additional structure, i.e. equipped with additional Hodge cycles (on suitable powers). This result is not only theoretically significant, but it also has a practical meaning, because it is this particular class of Shimura varietes of Hodge type which are, at least in principle, amenable to the methods of arithmetic algebraic geometry. For instance, it is this way that Milne has reobtained Deligne's canonical models over the reflex field $E \subset \mathbb{C}$ in [35]. Outside this class of Shimura varieties these methods fail, but the work of many people (e.g. [22], [2]) has culminated in more general results. Canonical models are finally shown to exist unconditionally in [33]. However, the proof uses a very amazing reduction to the GL(2)-cases, and again these are treated by means of abelian varieties whose endomorphism rings have a suitable structure.
However, it is also very desireable to control the integrality properties of the Shimura variety (1). So let us fix a odd rational prime p and write $\mathcal{O}_{E_\mathfrak{p}}$ for the complete valuation rings corresponding to prime divisors $\mathfrak{p}|p$. If G is anisotropic Langlands conjectured the existence of a projective and smooth $\mathcal{O}_{E_\mathfrak{p}}$-model ${}_K\mathcal{M}_\mathfrak{p}$, provided only that the group K can be written as $K^p \times K_p$, where K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, and K^p is a sufficiently small compact open subgroup of $G(\mathbb{A}^{\infty,p})$. He also points out the necessity of characterizing ${}_K\mathcal{M}_\mathfrak{p}$ (be it projective or quasi-projective), which would make the model more canonical, so that the conjecture remains meaningful in the isotropic case too [28, p.411, l.17-19]. Such a characterization was suggested by Milne, however his early attempt (in [34, Definition(2.5)]) had to be modified (cf. [54] and [36, Definition(3.3)]). Finally, integral canonical models were shown to exist for Shimura varieties of abelian type, see [24], [23] and the references therein.

Our results: In this paper we focus on Langlands original conjecture for Shimura data (G,X) which do not allow any embeddings of (G^{ad},X^{ad}) into $(\mathrm{Sp}(2g)/\{\pm 1\}, \mathfrak{h}_g^{\pm 1})$. Such non-preabelian Shimura data are rare, but they do exist. If X^+ is an irreducible tube domain for instance, there are but two families of examples, namely the $\mathbb{Q}$-groups of trialitarian type D_4 whose sole non-compact simple real factor is isomorphic to $\mathrm{SO}(6,2)/\{\pm 1\}$, and the exceptional simple $\mathbb{Q}$-groups of type E_7 whose sole non-compact simple real factor has signature -25. In these cases there exists a simple formally real Jordan algebra $\mathbb{J}$ such that X^+ can be described as a domain in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{J}$ of the shape:

$$\{X + \sqrt{-1}Y | X, Y \in \mathbb{J}, Y > 0\},$$

Here is $\mathbb{J}$ a 6-dimensional spin-factor in the former case, and the 27-dimensional exceptional simple Jordan-algebra in the latter, there exist other descriptions of X^+, cf. [7]. Observe that the occurrence of a single compact factor enforces already the anisotropicity of G. At last, if X^+ is reducible or not a tube domain, there are more complicated examples most notably $\mathbb{Q}$-forms of $\mathrm{SO}(8,2)/\{\pm 1\} \times_{\mathbb{R}} \mathrm{SO}^*(10)/\{\pm 1\}$ or the exceptional simple $\mathbb{Q}$-groups of type E_6 all of whose non-compact simple real factors have signature -14. One of the outcomes of the present paper is the following result:

Theorem 1.1. *Suppose that (G, X) is a Shimura datum, such that G^{der} is simply connected. Suppose that $f > 1$ is an integer, and $p > 2$ a prime, such that and G splits over $K(\mathbb{F}_{p^f})$ (i.e. $\mathbb{Q}_p[\exp \frac{2i\pi}{p^f-1}]$) while G^{ad} is simple and quasisplit over $\mathbb{Q}_p$. Suppose in addition that G satisfies at least one of the following conditions:*

(i) *$G^{ad} \times_{\mathbb{Q}} \mathbb{R}$ possesses more than three times as many compact simple real factors than non-compact ones and is of type B_l or C_l.*
(ii) *$G^{ad} \times_{\mathbb{Q}} \mathbb{R}$ possesses more than four times as many compact simple real factors than non-compact ones and is of type E_7.*

Fix an embedding $\iota : K(\mathbb{F}_{p^f}) \to \mathbb{C}$. Then there exists a compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ and a scheme $\mathcal{M} = \lim_{K^p \to 1} {}_{K^p}\mathcal{M}$ with a right $G(\mathbb{A}^{\infty,p})$-action over $W(\mathbb{F}_{p^f})$, such that each ${}_{K^p}\mathcal{M}$ is projective and smooth and ${}_{K_p}M(G, X) \cong \mathcal{M} \times_{W(\mathbb{F}_{p^f}),\iota} \mathbb{C}$ holds.

Under the assumptions of the theorem p is unramified and inert in the totally real number field L^+ which is needed in order to write G^{der} as a restriction of scalars of some simply-connected absolutely simple L^+-group, and without any loss of generality this field is an extension of degree f over $\mathbb{Q}$. Observe also that the choice of ι induces a continuous embedding of $E_{\mathfrak{p}}$ into $K(\mathbb{F}_{p^f})$, where $\mathfrak{p}$ is a prime of E over p, and notice that the degree of $E_{\mathfrak{p}}$ over $\mathbb{Q}_p$ is at least five, in fact if X^+ is irreducible, then E contains the whole splitting field R^+ of L^+, so that $E_{\mathfrak{p}} = K(\mathbb{F}_{p^f})$.

The unramified inertness of p in L^+ is equivalent to $\mathrm{Gal}(R^+/\mathbb{Q}) = \{\mathrm{id}_{R^+}, \vartheta, \dots, \vartheta^{f-1}\}\,\mathrm{Gal}(R^+/L^+)$, where $\vartheta \in \mathrm{Gal}(R^+/\mathbb{Q})$ stands for a Frobenius element of the $\mathbb{Q}$-extension R^+ (N.B.: this is independent of choosing $L^+ \hookrightarrow R^+$). Thus, on the one hand Chebotarev's theorem implies that under the assumptions of theorem 1.1 we can apply it also to a set of positive density of other rational primes, but on the other hand our proof of theorem 1.1 does not make it easy to compare the results for two different primes. Nevertheless, we do expect that our models ${}_{K^p}\mathcal{M}$ agree with the scalar extensions to $W(\mathbb{F}_{p^f})$ of the conjectured integral canonical models ${}_{K^pK_p}\mathcal{M}_\mathfrak{p}$ for all such $\mathfrak{p}$, we hope to come back to this problem in a future paper.

Our Methods: Very much unlike Kisin's and Vasiu's work, our construction of ${}_{K^p}\mathcal{M}$ sheds no light on its conjectural but natural interpretation as a moduli space of "polarized motives with G-structure", but instead relates its special fiber

(3) $${}_{K^p}\overline{\mathcal{M}} = {}_{K^p}\mathcal{M} \times_{W(\mathbb{F}_{p^f})} \mathbb{F}_{p^f},$$

to the subject of pathological additional structures on abelian varieties, which was pioneered e.g. in [21, Chapter 9]. More specifically, our roundabout proof of theorem 1.1 is preceded by the construction of a family of intermediary moduli schemes

(4) $${}_{K^p}\tilde{M} \to {}_{K^p}\mathfrak{M} \to {}_{\tilde{K}^p}\mathcal{U} \to \mathcal{A}_{g,n},$$

whose definitions depend on many choices. At least in principle ${}_{K^p}\mathfrak{M}$ is an intersection of a carefully chosen PEL-type Shimura subvariety of $\mathcal{A}_{g,n}$ with the canonical integral model ${}_{\tilde{K}^p}\mathcal{U}$ of an auxiliary Shimura subdatum $(\tilde{G}, \tilde{X}) \hookrightarrow (\mathrm{GSp}(2g), \mathfrak{h}_g^\pm)$ of Hodge type. The classification of the endomorphism algebras of a point on a Shimura subvariety is a classical but still unexplored subject, cf. [42], [39]. The scheme ${}_{K^p}\tilde{M}$ plays the role of a provisional candidate for the variety (3), and the morphism on the left hand-side of (4) turns out to be radicial and universally closed, moreover one can show that the generic fiber of ${}_{K^p}\mathfrak{M}$ is emtpy. An analogous moduli problem can be put in the category of p-divisible groups, which is used to introduce a certain fpqc-stack $\mathfrak{B}$ together with a p-adically formally étale 1-morphism ${}_{K^p}\mathfrak{M} \to \mathfrak{B}$, which is easily defined by the 'passage to the underlying p-divisible group'. The number g is quite large, in the special case of a group G of type E_7 our construction uses $g = 25.650f$, and due to the condition (ii) the number f is at least $[E_\mathfrak{p} : \mathbb{Q}_p] \geq 6$.

Before we give further details we would like to remark that integral canonical models and their special fibers are expected to satisfy many other nice properties. For every $\mathbb{Q}_p$-rational representation $\rho : G\times_{\mathbb{Q}}\mathbb{Q}_p \to \mathrm{GL}(V/\mathbb{Q}_p)$, for instance, there should exist a F-isocrystal $\mathrm{Fib}_{Cris}(\rho)$ (in the sense of e.g. [48, VI.3.1.3]) over ${}_{K^p}\overline{\mathcal{M}}$, and every K_p-invariant lattice in V should determine a natural non-degenerate F-crystal (in the sense of e.g. [48, VI.3.1.1]) in a suitable Tate twist of $\mathrm{Fib}_{Cris}(\rho)$. At least for every closed point $x \in {}_{K^p}\overline{\mathcal{M}}$ one expects that the completed local $W(\mathbb{F}_{p^f})$-algebra $\hat{\mathcal{O}}_{{}_{K^p}\overline{\mathcal{M}},x}$ is determined by such crystalline data, and should ideally prorepresent a functor of "crystals with additional structure", see e.g. [12, Théorème(2.1.7), Théorème(2.1.14)] for the ordinary SO(19, 2)-case, but also [54, 3.2.7 Remarks 8b)] and [36, Proposition(4.9)], for the general case. In this optic it seems to be reasonable to try to construct ${}_{K^p}\mathcal{M}$ together with the most general crystalline objects that might exist over it. It is one of the standpoints of the present paper that such should be decoded in a p-adically formally étale 1-morphism

$$ {}_{K^p}\hat{\mathcal{M}} \to \mathcal{B}(\mathfrak{G}_p, \mu_p), \tag{5}$$

where the left hand-side is the $W(\mathbb{F}_{p^f})$-functor $R \mapsto \begin{cases} \emptyset & \mathbb{Q}\otimes R \neq 0 \\ {}_{K^p}\mathcal{M}(R) & \text{otherwise} \end{cases}$ and the right hand-side is a certain fpqc-stack over $W(\mathbb{F}_{p^f})$, which should only depend on the reductive $\mathbb{Z}_p$-model with $K_p = \mathfrak{G}_p(\mathbb{Z}_p)$, and on the minuscule cocharacter associated with the Shimura-datum. In subsection 3.2 of this paper, we suggest a natural definition for $\mathcal{B}(\mathfrak{G}_p, \mu_p)$, which seems to work well at least for those pairs $(\mathfrak{G}_p, \mu_p)$ for which every simple factor of $G^{ad} \times_{\mathbb{Q}} \mathbb{Q}_p$ contains a simple factor of $G^{ad} \times_{\mathbb{Q}} E_{\mathfrak{p}}^{ac}$ in which μ_p is trivial (N.B.: this implies that every simple factor of G^{ad} contains a compact simple factor of $G^{ad} \times_{\mathbb{Q}} \mathbb{R}$ and the only PEL-cases with this property are unitary groups).

We next introduce a map

$$\mathrm{Flex} : \mathcal{B}(\mathfrak{G}_p, \mu_p) \times_{W(\mathbb{F}_{p^f})} \mathbb{F}_{p^f} \to \mathfrak{B} \times_{W(\mathbb{F}_{p^f})} \mathbb{F}_{p^f} \tag{6}$$

which is, rougly speaking a formal analog of the left hand-side morphism in (4). This adds a lot of content to the whole picture, and it gives us a clue to construct ${}_{K^p}\overline{\mathcal{M}}$ as the largest $\mathbb{F}_{p^f}$-variety fitting into a 2-commutative diagram

$$\begin{array}{ccc}\mathcal{B}(\mathfrak{G}_p,\mu_p)\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f} & \longleftarrow & {}_{K^p}\overline{\mathcal{M}} \\ \downarrow{\scriptstyle \mathrm{Flex}} & & \downarrow \\ \mathfrak{B}\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f} & \longleftarrow & {}_{K^p}\mathfrak{M}\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f}\end{array}$$

with a formally étale map to $\mathcal{B}(\mathfrak{G}_p,\mu_p)\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f}$ and a radicial map to ${}_{K^p}\mathfrak{M}\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f}$. With an eye towards (5) we move on to describe ${}_{K^p}\mathcal{M}$ as a lift of ${}_{K^p}\overline{\mathcal{M}}$. In the holomorphic category there exists a period map from the universal covering space to the associated symmetric Hermitian domain of compact type. Our investigation of this map (and its image) follows [52] very closely and completes the proof of theorem 1.1, see also [40]. In a future paper we hope to compute the group K_p.

Displays: In section 3 we introduce the stacks $\mathcal{B}(\mathfrak{G}_p,\mu_p)$ for a certain class of cocharacters of smooth affine group $\mathbb{Z}_p$-schemes $\mathfrak{G}_p$ with connected fibers. The idea is to incorporate "additional structures" into Zink's notion of a $3n$-display [60] in such a way that one recovers the displays (P,Q,F,V^{-1}) of height h and dimension d for the special case of the cocharacter

$$\mu_p:\mathbb{G}_m\to\mathrm{GL}(h); z\mapsto \mathrm{diag}(\overbrace{z,\dots,z}^{d},\overbrace{1,\dots,1}^{h-d}).$$

At least over a local ring a $3n$-display (P,Q,F,V^{-1}) possesses a normal decomposition $P=T\oplus L$ and bases $e_1,\dots,e_d$ of T and $e_{d+1},\dots,e_h$ of L relative to which there are structural equations $\sum_{i=1}^h\alpha_{i,j}e_i = \begin{cases}Fe_j & j\leq d\\ V^{-1}e_j & j>d\end{cases}$, and hence an invertible $h\times h$-display matrix $(\alpha_{i,j})=U$, cf. [60, Lemma 9]. Suppose that U' is the matrix of another display (P',Q',F,V^{-1}) of possibly different height h' and dimension d', again taken with respect to a normal decomposition, and a choice of its bases. A linear map from P' to P can be visualized as $h\times h'$-matrix over the ring of Witt vectors with a block decomposition $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ of which the upper left block A has d rows and d' columns. This map is a morphism $P'\stackrel{k}{\to}P$ in the category of $3n$-displays if and only if:

(i) the entries in B are Witt vectors with vanishing 0th ghost component, and
(ii) $U^{-1}\begin{pmatrix}A & B\\ C & D\end{pmatrix}U'=\begin{pmatrix}{}^FA & {}^{V^{-1}}B\\ p^FC & {}^FD\end{pmatrix}$ holds.

It is enough to check the commutation of k with the maps V^{-1} on Q and Q' and these are given by U and U' precomposed with $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} {}^{V^{-1}}x \\ {}^{F}y \end{pmatrix}$. In particular the isomorphism classes of displays of height h and dimension d are simply the Witt vector-valued elements of $\mathrm{GL}(h)$ modulo the equivalence relation

$$k^{-1}U\Phi^{\mu_p}(k) \sim U,$$

here we write $\Phi^{\mu_p}(k)$ for the right hand side of (ii), which is well-defined if the condition (i) on the upper right block of k is valid. In subsection 3.1 we show that a similar map Φ^{μ_p} exists on the inverse image in the Witt-vector valued elements of $\mathfrak{G}_p$ of the parabolic subgroup which is defined by μ_p. This gives rise to a definition of a fpqc-stack $\mathcal{B}(\mathfrak{G}_p, \mu_p)$ of $3n$-displays with $\mathfrak{G}_p$-structure, in a rather mechanical manner. An object which is globally represented by a Witt-vector valued element of $\mathfrak{G}_p$ is called a banal $3n$-display with $\mathfrak{G}_p$-structure (and every $3n$-display is fpqc-locally banal). For a frame (A, J, τ) we also introduce certain groupoids $\hat{B}_{A,J}(\mathfrak{G}_p, \mu_p)$, which we think of companion notions to Zink's theory of windows [59]. Do notice however, that the functor $\mathcal{BT}$ of [60, 3.1] has absolutely no analog in any of these settings.

In subsection 3.5 we introduce a p-adic deformation theory for a certain full subcategory $\mathcal{B}'(\mathfrak{G}_p, \mu_p)$ of displays with additional structure: It turns out that there is a canonical vector-bundle over the whole of $\mathcal{B}(\mathfrak{G}_p, \mu_p)$ such that the lifts of some fixed $S/\mathfrak{a}$-valued point of $\mathcal{B}'(\mathfrak{G}_p, \mu_p)$ (where $\mathfrak{a}^2 = 0$ and $\mathbb{Q} \otimes S = 0$) form a non-empty principal homogeneous space under the group of $\mathfrak{a}$-valued points of that vector bundle, see corollary 3.27. Along the same lines it follows that the diagonal of $\mathcal{B}'(\mathfrak{G}_p, \mu_p)$ is representable by affine p-adically formally unramified morphisms, in fact this kind of rigidity result has to be checked before the previous one makes any sense, see corollary 3.26 and the earlier lemma 3.18. As another consequence of this one obtains most easily that the deformation functor of $\mathcal{B}(\mathfrak{G}_p, \mu_p)$ to any point of $\mathcal{B}'(\mathfrak{G}_p, \mu_p)$ with values in a perfect field k (with $\mathrm{char}(k) = p$), is "effectively prorepresentable" by a universal deformation over a power series ring $W(k)[[t_1, \dots, t_d]]$ (cf. corollary 3.29). The second part of section 3.5 computes the generic Newton-polygon of the universal equicharacteristic deformations over $k[[t_1, \dots, t_d]]$, and it applies this to obtain another technical result, which is later on needed for the determination of the monodromy of the period map.

The subsection 8.2 constructs a provisional version ${}_{K^p}\tilde{M}$ of the envisaged special fiber ${}_{K^p}\overline{\mathcal{M}}$. The passage to the actual special fiber and its lift ${}_{K^p}\mathcal{M}$ are explained in subsection 8.5, the former makes significant use of the whole of the lengthy section 5, and the latter is a simple consequence of our deformation theory (in particular subsubsection 3.5.2). The relation between ${}_{K^p}\overline{\mathcal{M}}$ and ${}_{K^p}\mathfrak{M}$ is reminiscent of the work [57], which identifies the good reduction of several GL(2)-cases, as strata in the special fiber of a Hilbert-Blumenthal-style moduli problem of bad reduction.

Generalization: While ${}_{K^p}\mathcal{M}$ uses the machinery of the whole paper, we find it worthy of remark, that in the construction of ${}_{K^p}\tilde{M}$ it is enough to work with slightly weaker properties, allowing us to treat some cases of non-minuscule cocharacters μ_p (cf. subsubsection 8.2.2). Perhaps unsurprisingly ${}_{K^p}\tilde{M}$ does not seem to be proper in these cases, and we hope to be able to exhibit a (perhaps even $G(\mathbb{A}^{\infty,p})$-equivariant?) compactification in a future paper (cf. remark 8.17). Suppose for example, that $f \geq 5$ and that G is a $\mathbb{Q}_p$-unramified and simple $\mathbb{Q}$-group of type G_2 with f simple factors over $K(\mathbb{F}_{p^f})$ and suppose that μ_p is trivial in $f-1$ of them, but lies in the principal SL(2) of the remaining one, then I would expect the existence of a compactification, of which the boundary has codimension one in ${}_{K^p}\tilde{M}$ and might look like G/P, where P is the parabolic subgroup associated to μ_p (please consult remark 6.14 for the number $\dim {}_{K^p}\tilde{M}$). Beyond this one might raise question such as:

- Is "Deligne's philosophy" applicable to ${}_{K^p}\tilde{M}$, i.e. do its k-valued points parametrize (some subset of the set of) levelled and polarized k-motives with G-structure?
- Can one give a description of ${}_{K^p}\tilde{M}(\mathbb{F}_{p^{af}})$ in the spirit of the Langslands-Rapoport conjecture?

The ℓ- monodromy group of the universal abelian scheme over ${}_{K^p}\tilde{M}$ seems to be $G \times_{\mathbb{Q}} \mathbb{Q}_\ell$, cf. [5].

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2. Preliminaries

This section contains most basic notions and should be consulted only when needed.

2.1. **Functors.** We let *Set* be the category of sets, and we let Alg_X be the usual category structure on the class of pairs consisting of a commutative ring R together with a morphism from its spectrum to some fixed base scheme X. Any covariant functor from Alg_X to *Set* will simply be called a X-functor. Notice that every morphism ϕ from a (not necessarily affine) scheme Y to the scheme X gives rise to an X-functor, and that the set of X-morphisms between any two X-schemes can be recovered from the functorial transformations between their X-functors. In this optic every X-scheme "is" (i.e. represents) a X-functor. We use the term X-category synonymously with fibrations over Alg_X^{op}, and a X-groupoid is a X-category none of whose fibers have any non-invertible morphisms, and for an arbitrary X-category C we write C^* for the X-groupoid, all of whose fibers are obtained from $C(R)$ by discarding all non-invertible morphisms. A X-groupoid C is called discrete if no object of $C(R)$ possesses non-trivial automorphisms (i.e. "is" a X-functor).
Moreover, for any Alg_Y-object R we let $R_{[\phi]}\in\mathtt{Ob}_{Alg_X}$ stand for the altered algebra-structure on the same underlying ring, and for any X-category C we denote the pull-back along a morphism ϕ (i.e. $C\times_{Alg_X^{op},[\phi]^{op}} Alg_Y^{op}$) by: $C\times_{X,\phi} Y$, we suppress the "ϕ" in the notation whenever the context allows to do that. For every $R\in\mathtt{Ob}_{Alg_X}$ we shall denote the fiber of a X-category C over R by $C(R)$, while C_R should stand for the $\operatorname{Spec} R$-category $C\times_X\operatorname{Spec} R$. The same notation applies to any X-functor $\mathcal{F}$, we write $\mathcal{F}\times_{X,\phi} Y$ for the Y-functor defined by $Alg_Y\to Set; R\mapsto\mathcal{F}(R_{[\phi]})$, while $\mathcal{F}_R$ stands for $\mathcal{F}\times_X\operatorname{Spec} R$.

By a fpqc-covering we simply mean any faithfully flat morphism in the category Alg_X^{op}. By the big fpqc-site X_{fpqc} we mean Alg_X^{op} equipped with the Grothendieck topology induced by the above class of coverings. For instance, a fpqc-sheaf on X is a X-functor $\mathcal{F}$ such that the sequence $\mathcal{F}(R) \to \mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_R S)$ is exact for all faithfully flat X-algebra morphisms $R \to S$. Moreover, the notions of X-stacks, and the stacks in groupoids over the site X_{fpqc} will be used synonymously.

2.2. **Witt-vectors and Greenberg Transforms.** By means of the usual addition and multiplication one can regard $\mathbb{A}^1$ as a ring scheme over $\mathbb{Z}$. Now pick a prime number p and recall the ring scheme of Witt vectors: Its underlying scheme is $\operatorname{Spec}\mathbb{Z}[x_0, x_1, \dots]$, on which there exists a unique ring schemes structure W which makes each of the so-called ghost coordinates

$$w_k : \operatorname{Spec}\mathbb{Z}[x_0, x_1, \dots] \to \mathbb{A}^1; (x_0, x_1, \dots) \mapsto \sum_{i=0}^{k} p^i x_i^{p^{k-i}}$$

into a homomorphism from the ring scheme W to the ring scheme $\mathbb{A}^1$. The additive map $V : W \to W; (x_0, x_1, \dots) \mapsto (0, x_0, \dots)$ is called the Verschiebung, and there also exists a unique Frobenius homomorphism:$F : W \to W$, such that $w_k \circ F = w_{k+1}$ for all $k \in \mathbb{N}_0$. Notice that $F(V(x)) = px$ and $V(xF(y)) = yV(x)$ hold for any two Witt-vectors x and y. We will write $I_m \subset W$ for the closed subscheme defined by the equations $x_0 = \dots = x_{m-1} = 0$, and I for I_1. One also writes $[x]$ for the Teichmüller lift, which is the Witt-vector $(x, 0, \dots)$. We will need the following:

Lemma 2.1. *Suppose that R is a commutative ring. If $p \in \operatorname{rad}(R)$ then $\operatorname{rad}(W(R))$ is equal to the inverse image of $\operatorname{rad}(R)$ via the 0th ghost coordinate.*

Proof. By a limit process it is enough to prove that $I_m(R)/I_{m+1}(R)$ is contained in $\operatorname{rad}(W(R)/I_{m+1}(R))$ for all m, which is implied by

$$(1 +^{V^m} [x])(1 -^{V^m} [\frac{x}{1 + p^m x}]) \in 1 + I_{m+1}(R)$$

for all $x \in R$. □

If C is a $\operatorname{Spec} W(\mathbb{F}_{p^r})$-category, then we introduce further $\operatorname{Spec} W(\mathbb{F}_{p^r})$-categories ${}^F C$ and ${}^W C$ by defining their fibers over an arbitrary $W(\mathbb{F}_{p^r})$-algebra R to be:

$$ {}^F C(R) = C(R_{[F]})$$
$$ {}^W C(R) = C(W(R)),$$

where the $W(\mathbb{F}_{p^r})$-algebra structure on $W(R)$ is induced by the diagonal $\Delta : W(\mathbb{F}_{p^r}) \to W(W(\mathbb{F}_{p^r}))$. Likewise a fibered $\operatorname{Spec} W(\mathbb{F}_{p^r})$-functor $\rho : C \to E$ gives rise to fibered $\operatorname{Spec} W(\mathbb{F}_{p^r})$-functors ${}^F\rho : {}^F C \to {}^F E$, and ${}^W\rho : {}^W C \to {}^W E$. The $\operatorname{Spec} W(\mathbb{F}_{p^r})$-categories ${}^{FW}C$ and ${}^{WF}C$ are canonically $\operatorname{Spec} W(\mathbb{F}_{p^r})$-equivalent, which is due to the commutativity of:

$$\begin{array}{ccc} W(W(\mathbb{F}_{p^r})) & \xrightarrow{F} & W(W(\mathbb{F}_{p^r})) \\ {\scriptstyle\Delta}\uparrow & & {\scriptstyle\Delta}\uparrow \\ W(\mathbb{F}_{p^r}) & \xrightarrow{F} & W(\mathbb{F}_{p^r}) \end{array}.$$

Finally, notice that there are canonical $\operatorname{Spec} W(\mathbb{F}_{p^r})$-functors

$$F : {}^W C \to {}^{WF} C \tag{7}$$

$$w_0 : {}^W C \to C \tag{8}$$

induced by the $W(\mathbb{F}_{p^r})$-linear maps $F : W(R) \to W(R)_{[F]}$ and $w_0 : W(R) \to R$. Both F and W commute with base change along $W(\mathbb{F}_{p^r}) \to W(\mathbb{F}_{p^f})$, where f is a multiple of r, in particular it does not cause confusion to denote ${}^W(C_{W(\mathbb{F}_{p^f})}) = ({}^W C)_{W(\mathbb{F}_{p^f})}$ by ${}^W C_{W(\mathbb{F}_{p^f})}$ while ${}^W C_{\mathbb{F}_{p^f}}$ is shorthand for nothing but $({}^W C)_{\mathbb{F}_{p^f}}$. The following representability results are relevant in this paper:

- The special fiber of ${}^W \mathit{Tors}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})})$ coincides with $\mathit{Tors}({}^W\mathrm{GL}(n)_{\mathbb{F}_{p^r}})$.
- If C is representable by a $\operatorname{Spec} W(\mathbb{F}_{p^r})$-scheme, then so is ${}^W C_{\mathbb{F}_{p^r}}$, in fact the latter is relatively affine over the special fiber of C (via the special fiber of (8)).
- If C is representable by an affine $\operatorname{Spec} W(\mathbb{F}_{p^r})$-scheme, then so is ${}^W C$.

The last of these statements results from applying Freyd's adjoint functor theorem to $\mathit{Alg}_{\operatorname{Spec} W(\mathbb{F}_{p^r})} \to \mathit{Alg}_{\operatorname{Spec} W(\mathbb{F}_{p^r})}; R \mapsto W(R)$, the first statement results from Zink's theory of Witt descent ([60, Proposition 33]) and the one in the middle follows from lemma 2.1. The assignment $C \mapsto {}^W C$ seems to have originated in [16].

2.3. **Weil Restriction.** Let X be a $W(\mathbb{F}_{p^r})$-scheme, and suppose that Δ is an abstract group acting on X from the right. By an equivariant right Δ-action ϕ on some X-functor P we mean a family of maps $\phi_R(g) : P(R) \to P(R_{[g]})$ for every $g \in \Delta$ and $R \in \mathtt{Ob}_{\mathit{Alg}_X}$ which satisfy $\phi_{R_{[g]}}(h) \circ \phi_R(g) = \phi_R(gh)$ for every $h \in \Delta$ and are compatible with the restriction maps, in the sense that $|_{S_{[g]}} \circ \phi_R(g) = \phi_S(g) \circ |_S$ holds for every R-algebra S (N.B.: If P is representable by a scheme, this just means that P and X have compatible Δ-actions, from the right). In addition, suppose that Δ acts from the right on some fpqc-sheaf of groups $\mathcal{G}$ over $\operatorname{Spec} W(\mathbb{F}_{p^r})$. We say that P is a Δ-equivariant formal principal homogeneous space for $\mathcal{G}$ over X, if we are given a Δ-equivariant map of fpqc-sheaves $P \times_{W(\mathbb{F}_{p^r})} \mathcal{G} \to P$ endowing P with the structure of a formal principal homogeneous space for $\mathcal{G}$ over X in the usual sense (N.B.: If P and $\mathcal{G}$ are representable by $W(\mathbb{F}_{p^r})$-schemes, this just means that the Δ-action on the former extends to an action of the semidirect product $\mathcal{G} \rtimes \Delta$ such that every non-empty $P(S)$ becomes a principal homogeneous $\mathcal{G}(S)$-set).

We also need some notation for Weil restriction: Let us fix a finite field extension $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^s}$, and recall that any $W(\mathbb{F}_{p^s})$-functor $\mathcal{F}$ gives rise to its Weil restriction $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{F}$, which is the $W(\mathbb{F}_{p^r})$-functor given by $R \mapsto \mathcal{F}(W(\mathbb{F}_{p^s}) \otimes_{W(\mathbb{F}_{p^r})} R)$, and similarly any $W(\mathbb{F}_{p^s})$-functorial transformation $\rho : \mathcal{E} \to \mathcal{F}$ gives rise to a corresponding transformation $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \rho$ between the $W(\mathbb{F}_{p^r})$-functors $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{E}$ and $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{F}$. Finally notice the canonical isomorphisms:

$$
\begin{aligned}
{}^F(\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{F}) &\stackrel{\cong}{\to} \operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} {}^F\mathcal{F} \\
{}^W(\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{F}) &\stackrel{\cong}{\to} \operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})}({}^W\mathcal{F}),
\end{aligned}
$$

which are induced from the two natural isomorphisms:

$$W(\mathbb{F}_{p^s}) \otimes_{W(\mathbb{F}_{p^r})} R_{[F]} \stackrel{\cong}{\to} (W(\mathbb{F}_{p^s}) \otimes_{W(\mathbb{F}_{p^r})} R)_{[F]};\ a \otimes x \mapsto F(a) \otimes x,$$

as well as

$$W(\mathbb{F}_{p^s}) \otimes_{W(\mathbb{F}_{p^r})} W(R) \stackrel{\cong}{\to} W(W(\mathbb{F}_{p^s}) \otimes_{W(\mathbb{F}_{p^r})} R).$$

Let Δ be the cyclic group $r\mathbb{Z}/s\mathbb{Z}$. If $\mathcal{F}$ commutes with products, then there are canonical isomorphisms:

$$(\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{F}) \times_{W(\mathbb{F}_{p^r})} W(\mathbb{F}_{p^s}) \cong \prod_{\sigma \in \Delta} {}^{F^{-\sigma}}\mathcal{F}$$

Composing $\times_{W(\mathbb{F}_{p^r})} W(\mathbb{F}_{p^s})$ and $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})}$ in the other order yields Δ-equivariant $W(\mathbb{F}_{p^r})$-functors $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} P_{W(\mathbb{F}_{p^s})}$ equipped with a transformation $P \to \operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} P_{W(\mathbb{F}_{p^s})}$ from arbitrary $W(\mathbb{F}_{p^r})$-functors P. A $\operatorname{Spec} W(\mathbb{F}_{p^r})$-group, $\mathcal{G}$ gives rise to a Δ-equivariant $\operatorname{Spec} W(\mathbb{F}_{p^r})$-group $\operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} \mathcal{G}_{W(\mathbb{F}_{p^s})}$ containing $\mathcal{G}$ as a closed subgroup, and we have the following lemma:

Lemma 2.2. *For every flat and affine $W(\mathbb{F}_{p^r})$-group $\mathcal{G}$, the functor*

$$P \mapsto \operatorname{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})} P_{W(\mathbb{F}_{p^s})}$$

defines an equivalence from the category of locally trivial principal homogeneous spaces for $\mathcal{G}$ over X to the category of Δ-equivariant locally *trivial principal homogeneous spaces for* $\mathrm{Res}_{W(\mathbb{F}_{p^s})/W(\mathbb{F}_{p^r})}\mathcal{G}_{W(\mathbb{F}_{p^s})}$ *over* X*, where* Δ *acts trivially on* X*.*

Proof. This follows from the mere fact, that $Tors(\mathcal{G})$ is a $\operatorname{Spec} W(\mathbb{F}_{p^r})$-stack. □

3. Windows and $3n$-Displays with additional structure

In this section we introduce the groupoids of windows and $3n$-displays with additional structure. For utmost comfort of the reader we recall the following:

Definition 3.1. *A directed graph consists of set* $\mathcal{A}$ *with an automorphism* r *satisfying* $r = r^{-1}$ *and two endomorphisms* s *and* t *satisfying* $s = r \circ s = s \circ s = t \circ s = t \circ r$ *and* $t = r \circ t = t \circ t = s \circ t = s \circ r$*. A groupoid is a directed graph together with a commutative diagram*

$$\begin{array}{ccc} \mathcal{A} \times_{s,\mathcal{A},t} \mathcal{A} & \xrightarrow{p} & \mathcal{A} \\ {\scriptstyle t\times s}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{A}}} \\ \mathcal{A} \times \mathcal{A} & \xleftarrow{t\times s} & \mathcal{A} \times \mathcal{A} \end{array}$$

satisfying $s(x) = p(r(x), x)$*,* $t(x) = p(x, r(x))$ *for all* $x \in \mathcal{A}$ *and* $p(p(x, y), z) = p(x, p(y, z))$ *whenever* $s(x) = t(y)$ *and* $s(y) = t(z)$ *hold. A groupoid is called discrete if* $\mathcal{A} \to \mathcal{A} \times \mathcal{A}; x \mapsto (s(x), t(x))$ *is an injection.*

In category theoretic terms, every element of $\mathtt{Ob}_{\mathcal{A}} = \{x \in \mathcal{A} | x = r(x) = s(x) = t(x)\}$ plays the role of an object, while every $x \in \mathcal{A}$ plays the role of an isomorphism from $s(x)$ to $t(x)$, of which the inverse is $r(x)$, and any small categroy without non-invertible morphisms arises from at least one groupoid. Notice also, that the r, s, t, and p preserving maps from $\mathcal{A}$ to another groupoid are nothing but the covariant functors between the respective categories. Finally, notice that every group is a groupoid.

Example 3.2. Let M be an abstract monoid, let Γ be a subgroup thereof, and let $\phi : \Gamma \to M$ be a morphism of monoids. On the set $\mathcal{A} = M \times \Gamma$ we consider the functions:

$$s : \mathcal{A} \to \mathcal{A}; (U, k) \mapsto (k^{-1}U\phi(k), 1)$$
$$t : \mathcal{A} \to \mathcal{A}; (U, k) \mapsto (U, 1)$$
$$r : \mathcal{A} \to \mathcal{A}; (U, k) \mapsto (k^{-1}U\phi(k), k^{-1})$$

so that one finds $\mathcal{A}^2 \supset \{((U, k), (k^{-1}U\phi(k), l)) | U \in M, k, l \in \Gamma\} = \mathcal{A} \times_{s,\mathcal{A},t} \mathcal{A}$ on which we decree p to be given by the pair $(U, kl) \in$

$\mathcal{A}$. In short: Every element $k \in \Gamma$ plays the role of an isomorphism from $k^{-1}U\phi(k)$ to U. The above will be refered to as the groupoid corresponding to the diagram $M \supset \Gamma \stackrel{\phi}{\to} M$, notice that the projection to the second coordinate yields an essentially surjective and faithful covariant functor $q : \mathcal{A} \to \Gamma$ (where the latter group is regarded as a groupoid), and that $\mathtt{Ob}_{\mathcal{A}}$ is equal to the set $M \times \{1\}$. Finally, suppose that $q' : \mathcal{A}' \to \Gamma'$ is another such functor, where $\mathcal{A}'$ corresponds to, say, $M' \supset \Gamma' \stackrel{\phi'}{\to} M'$. Then giving a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}' \\ {\scriptstyle q}\downarrow & & {\scriptstyle q'}\downarrow \\ \Gamma & \longrightarrow & \Gamma' \end{array}$$

is equivalent to giving maps $\gamma : \Gamma \to \Gamma'$ and $m : M \to M'$, satisfying $\gamma(kl) = \gamma(k)\gamma(l)$ and $m(k^{-1}U\phi(k)) = \gamma(k^{-1})m(U)\phi'(\gamma(k))$. Again, the functors between the groupoids in this paper will usually be given in this language, notice that the underlying map from $\mathtt{Ob}_{\mathcal{A}}$ to the class of objects $\mathtt{Ob}_{\mathcal{A}'}$ is described by $(U, 1) \mapsto (m(U), 1)$

As a warm-up, we are now going to explain two simple prototypes of a whole hierarchy of groupoids, that will play a role in this paper. Fix a torsionfree $\mathbb{Z}_p$-algebra A:

Definition 3.3. *If $\tau : A \to A$ is a $\mathbb{Z}_p$-linear endomorphism, and $\mathfrak{G}$ a smooth and affine $\mathbb{Z}_p$-group, then we define $\mathbf{B}_{A,\tau}(\mathfrak{G})$ to be the groupoid corresponding to the diagram*

$$\mathfrak{G}(A[\frac{1}{p}]) \supset \mathfrak{G}(A) \stackrel{\tau}{\to} \mathfrak{G}(A[\frac{1}{p}])$$

(in the sense of example 3.2). Consider a A-rational cocharacter $\mu : \mathbb{G}_{m,A} \to \mathfrak{G}_A$:

- *We define $\mathbf{B}_{A,\tau}(\mathfrak{G}, \mu)$ to be the groupoid corresponding to the diagram*

$$\mathfrak{G}(A) \supset \mathfrak{G}(A) \cap \mu(p)\mathfrak{G}(A)\mu(\frac{1}{p}) \stackrel{\phi}{\to} \mathfrak{G}(A),$$

  with $\phi(k) := \tau(\mu(\frac{1}{p})k\mu(p))$.
- *We define $\mathbf{h}_\mu : \mathbf{B}_{A,\tau}(\mathfrak{G}, \mu) \to \mathbf{B}_{A,\tau}(\mathfrak{G})$ (resp. $\mathbf{h}^0_\mu : \mathbf{B}_{A,\tau}(\mathfrak{G}, \mu) \to \mathbf{B}_{A[\frac{1}{p}],\tau}(\mathfrak{G})$) to be the functor described by the pair (γ, m) (in the sense of 3.2) with $\gamma(k) = \mu(\frac{1}{p})k\mu(p)$ and $m(U) = \mu(\frac{1}{p})U$.*

Notice that $\mathbf{h}_\mu$ is fully faithful. From now on we require that $\tau : A \to A$ is a lift of the absolute Frobenius, which means that $x^p \equiv \tau(x)$

(mod p) holds for every $x \in A$. Before we embark in more serious examples to 3.2 we have to state and prove some lemmas:

Lemma 3.4. *Let A be a ring without p-torsion, and fix a Frobenius lift τ, let $J \subset A$ be a pd-ideal with $p \in J$, and consider the function*

$$l_J : A \to \mathbb{N}_0 \cup \{\infty\}; a \mapsto \inf\{n | \tau^n(a) \notin p^n J\},$$

so that $l_J(a) = \begin{cases} 1 + l_J(\frac{\tau(a)}{p}) & a \in J \\ 0 & a \notin J \end{cases}$ *for every $a \in A$. For $a, b \in A$ one has $l_J(a+b) \geq \min\{l_J(a), l_J(b)\}$ and $l_J(ab) \geq l_J(a) + l_J(b)$.*

Proof. The first inequality is clear, and for the second it suffices to consider the case $l_J(a) \neq 0 \neq l_J(b)$, in fact one only has to check the following assertion: If $\tau^{j-1}(a) \in p^{j-1}J$ and $\tau^{k-1}(b) \in p^{k-1}J$ hold for any two positive integers j and k, then $\tau^{k+j-1}(ab) \in p^{k+j-1}J$ holds too. However, the former implies $\tau^j(a) \in p^j A$ and $\tau^k(b) \in p^k A$, being due to $\tau(J) \subset pA$, while the latter is implied by $\tau^{k+j-1}(ab) \in \tau^{k-1}(p^j A)\tau^{j-1}(p^k A) \subset p^{k+j}A$, due to $pA \subset J$. □

We will call $l_J(a)$ the J-length of the element a. Notice, that $J_0 = \tau^{-1}(pA)$ is the largest pd-ideal of A, giving rise to

$$\hat{l}_A(a) := l_{J_0}(a) = \inf\{n | \tau^n(a) \not\equiv 0 \pmod{p^n}\} - 1,$$

which we just call the length of a. Also, if A/pA is reduced, then the length is just the usual p-adic valuation. The significance of the function $\hat{l}_A$ is explained by the following:

Lemma 3.5. *Let A be a flat $W(\mathbb{F}_{p^f})$-algebra. Suppose that the homomorphism $\tau : A \to A_{[F]}$ is $W(\mathbb{F}_{p^f})$-linear and lifts the absolute Frobenius. Let $\mathcal{G}$ be a smooth and affine $W(\mathbb{F}_{p^f})$-group, fix a positive integer h, and let $\mu : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to \mathcal{G}$ be a cocharacter of triangular type all of whose weights are less than or equal to h (in the sense of definition C.2).*

- *The image of the subgroup $\hat{U}^0_{\mu^{-1}}(A, \hat{l}_A)$ of $\mathcal{G}(A[\frac{1}{p}])$ under the homomorphism defined by:*

  $$\hat{\Phi}_A^{\mu,h} : \mathcal{G}(A[\frac{1}{p}]) \to ({}^{F^h}\mathcal{G})(A[\frac{1}{p}]); g \mapsto \tau^h(\mu(\frac{1}{p})g\mu(p))$$

  is contained in the subgroup $({}^{F^h}\mathcal{G})(A)$ of $({}^{F^h}\mathcal{G})(A[\frac{1}{p}])$.
- *If A/pA is reduced, then $\{g \in \mathcal{G}(A) | \hat{\Phi}_A^{\mu,h}(g) \in ({}^{F^h}\mathcal{G})(A)\} = \hat{U}^0_{\mu^{-1}}(A, \hat{l}_A)$ holds.*

Proof. Notice that $A = A[\frac{1}{p}] \times_{(1+pA)^{-1}A[\frac{1}{p}]} (1+pA)^{-1}A$, for example because $A \to A[\frac{1}{p}] \oplus (1+pA)^{-1}A$ is faithfully flat. Furthermore, it is easy to see that:

$$\hat{U}^0_{\mu^{-1}}(A, \hat{l}_A) =$$
$$\mathcal{G}(A) \times_{\mathcal{G}((1+pA)^{-1}A)} \hat{U}^0_{\mu^{-1}}((1+pA)^{-1}A, \hat{l}_A) =$$
$$\mathcal{G}(A[\frac{1}{p}]) \times_{\mathcal{G}((1+pA)^{-1}A[\frac{1}{p}])} \hat{U}^0_{\mu^{-1}}((1+pA)^{-1}A, \hat{l}_A).$$

So we may assume $p \in \mathrm{rad}(A)$, without loss of generality. This puts us into a position where we are allowed to deduce the first statement from the theorem C.4.
Under the additional assumption of the reducedness of A/pA one has $\mathcal{G}(A) = \{g \in \mathcal{G}(A[\frac{1}{p}]) | \tau^h(g) \in ({}^{F^h}\mathcal{G})(A)\}$. Thus, in order to prove the second statement, all we have to check is $\mathcal{G}(A) \cap \mu(p)\mathcal{G}(A)\mu(\frac{1}{p}) = \hat{U}^0_{\mu^{-1}}(A, \hat{l}_A)$, which can easily be deduced (cf. lemma C.1) from the special case $\mathrm{GL}(n)$. □

3.1. **Definition of $\mathcal{I}^\mu$ and Φ^μ and $\overline{\mathcal{I}}^{\mu,h}$ and $\overline{\Phi}^{\mu,h}$.** For an ideal $\mathfrak{a}$ in a ring R, one denotes the kernel of $W(R) \to W(R/\mathfrak{a})$ by $W(\mathfrak{a})$. One would like to understand the structure of $W(\mathfrak{a})$ as a $W(R)$-module, and it turns out that this is possible, provided that $\mathfrak{a}$ is equipped with divided powers $\gamma_i : \mathfrak{a} \to \mathfrak{a}$ ($i \in \mathbb{N}$): In this case Zink defines logarithmic ghost coordinates $w'_k := x_k + \sum_{i=0}^{k-1}(p^{k-i}-1)!\gamma_{p^{k-i}}(x_i)$ on $W(\mathfrak{a})$, and proves that the map:

$$W(\mathfrak{a}) \stackrel{\cong}{\to} \prod_{k \in \mathbb{N}_0} \mathfrak{a}_{[w_k]};\ (x_0, x_1, \dots) \mapsto (w'_0, w'_1, \dots), \tag{9}$$

is an isomorphism of $W(R)$-modules, see [60, section 1.4] for more details (and [41, Section 1] for the dual number case). By slight abuse of notation, we will still write $\mathfrak{a}_{[w_i]}$ and $\bigoplus_{k \in \mathbb{N}_0} \mathfrak{a}_{[w_k]}$, etc. for any $W(R)$-subideals of $W(\mathfrak{a})$ that arise by transport of structure via (9) from the respective $W(R)$-submodules of $\prod_{k \in \mathbb{N}_0} \mathfrak{a}_{[w_k]}$. In particular, we will use the canonical splitting

$$W(\mathfrak{a}) = \mathfrak{a} \oplus I(\mathfrak{a}),$$

derived from that, as well as the very interesting map $V_{\mathfrak{a}}^{-1} : W(\mathfrak{a}) + I(R) \twoheadrightarrow W(R)$ defined by V^{-1} on $I(R)$ and by 0 on $\mathfrak{a}$ (cf. [60, Lemma 38])

Lemma 3.6. *Let $\mathfrak{a} \subset R$ be a pd-ideal containing the element p, and consider the function*

$$\tilde{v}_{\mathfrak{a}} : W(R) \to \mathbb{N}_0 \cup \{\infty\}; (x_0, x_1, \dots) \mapsto \inf\{n | x_n \notin \mathfrak{a}\},$$

so that $\tilde{v}_{\mathfrak{a}}(x) = \begin{cases} 1 + \tilde{v}_{\mathfrak{a}}(V_{\mathfrak{a}}^{-1}x) & x \in W(\mathfrak{a}) + I(R) \\ 0 & \text{otherwise} \end{cases}$ *holds for every $x \in W(R)$. For any $x, y \in W(R)$ one has $\tilde{v}_{\mathfrak{a}}(x + y) \geq \min\{\tilde{v}_{\mathfrak{a}}(x), \tilde{v}_{\mathfrak{a}}(y)\}$ and $\tilde{v}_{\mathfrak{a}}(xy) \geq \tilde{v}_{\mathfrak{a}}(x) + \tilde{v}_{\mathfrak{a}}(y)$*

We will call $\tilde{v}_{\mathfrak{a}}(x)$ the $V_{\mathfrak{a}}$-adic valuation of x. The corresponding chain of ideals is just $W(\mathfrak{a}) + I_m(R) = \{x \in W(R) | m \leq \tilde{v}_{\mathfrak{a}}(x)\}$. This defines a filtration on $W(R)$ in the sense of part C of the appendix, only if and if $p1_R$ is an element of $\mathfrak{a}$.

Remark 3.7. If $W(R)$ has no p-torsion, then we have $W(\mathfrak{a}) + I_{m+1}(R) = \{x \in W(\mathfrak{a}) + I_m(R) |^{F^m} x \in p^m(W(\mathfrak{a}) + I(R))\}$ for any m, so that $\tilde{v}_{\mathfrak{a}} = l_{W(\mathfrak{a})+I(R)}$ holds for any pd-ideal $\mathfrak{a}$ containing p.

For the special case $p1_R = 0_R$ only, we would also like to define the V-adic valuation of x being $\overline{v}_R(x) := \tilde{v}_{0_R}(x)$. The following corollary elucidates the significance of the function $\overline{v}_R$, please see subsection C.1 for the definition of the endomorphism $L_{\mu^{-1}}(p)$:

Corollary 3.8. *Let the assumptions on the $W(\mathbb{F}_{p^f})$-group $\mathcal{G}$, on the cocharacter $\mu : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to \mathcal{G}$ and on $h \geq 1$ be as in lemma 3.5, and choose a μ-basis of additive 1-parameter subgroups $\epsilon_1, \dots, \epsilon_d : \mathbb{G}_{a,W(\mathbb{F}_{p^f})} \to \mathcal{G}$ of weights $0 < h_1 \leq \cdots \leq h_{d-1} \leq h_d \leq h$ as in definition C.2. Let us write $\overline{\mathcal{I}}^{\mu}$ for the $\mathbb{F}_{p^f}$-scheme representing the closed subgroup functor of ${}^W\mathcal{G}_{\mathbb{F}_{p^f}}$ defined by $R \mapsto \hat{U}^0_{\mu^{-1}}(W(R), \overline{v}_R)$. Then there exists a unique homomorphism $\overline{\Phi}^{\mu,h} : \overline{\mathcal{I}}^{\mu} \to {}^{WF^h}\mathcal{G}_{\mathbb{F}_{p^f}}$ which restricts to $F^h \circ {}^W L_{\mu^{-1}}(p)$ on the special fiber of the subgroup ${}^W U^0_{\mu^{-1}}$ and satisfies $\overline{\Phi}^{\mu,h} \circ {}^W\epsilon_i|_{I_{h_i}} = ({}^{WF^h}\epsilon_i) \circ F^{h-h_i}V^{-h_i}$ for each i.*
Assume in addition that $h = 1$, and let us write $\mathcal{I}^{\mu} = U^0_{\mu^{-1}} \times_{\mathcal{G}} {}^W\mathcal{G}$ for the closed $W(\mathbb{F}_{p^f})$-subgroup scheme of ${}^W\mathcal{G}$ obtained as the inverse image of the subgroup $U^0_{\mu^{-1}}$ of $\mathcal{G}$. Then there exists a unique homomorphism $\Phi^{\mu} : \mathcal{I}^{\mu} \to {}^{WF}\mathcal{G}$ which restricts to $F \circ {}^W L_{\mu^{-1}}(p)$ on the subgroup ${}^W U^0_{\mu^{-1}}$ and satisfies $\Phi^{\mu} \circ {}^W\epsilon_i|_I = ({}^{WF}\epsilon_i \circ V^{-1})$ for each i. Finally, one has $\mathcal{I}^{\mu}_{\mathbb{F}_{p^f}} = \overline{\mathcal{I}}^{\mu}$ and the restriction of Φ^{μ} to the special fibre coincides with $\overline{\Phi}^{\mu,1}$.

Proof. Whenever $pR = 0$ (resp. $h \leq 1$), holds we denote the effect of the envisaged transformations $\overline{\Phi}^{\mu,h}$ (resp. Φ^μ) on the group of R-valued points by $\overline{\Phi}^{\mu,h}_R : \overline{\mathcal{I}}^\mu(R) \to ({}^{F^h}\mathcal{G})(W(R))$ (resp. $\Phi^\mu_R : \mathcal{I}^\mu(R) \to ({}^F\mathcal{G})(W(R))$). The asserted representabilities of the functors $\overline{\mathcal{I}}^\mu$ and $\mathcal{I}^\mu$ are immediate consequences of theorem C.4, in fact the description given there shows, that both of these group schemes are reduced and that $\mathcal{I}^\mu$ is flat over $W(\mathbb{F}_{p^f})$. Therefore, it is enough to construct $\overline{\Phi}^{\mu,h}_R$ (resp. Φ^μ_R) on the full subcategory of all reduced $\mathbb{F}_{p^f}$-algebras (resp. torsionfree $W(\mathbb{F}_{p^f})$-algebras). In either setting this is accomplished by lemma 3.5 and remark 3.7. □

Remark 3.9. Note that we have ${}^W\mathrm{Z}^{\mathcal{G}}_{\mathbb{F}_{p^f}} \subset \overline{\mathcal{I}}^\mu$ for $h \geq 1$, and ${}^W\mathrm{Z}^{\mathcal{G}} \subset \mathcal{I}^\mu$ in case $h = 1$, moreover the restriction of $\overline{\Phi}^{\mu,h}$ (resp. Φ^μ) to these subgroups is equal to F^h.

Remark 3.10. Apart from reducedness and flatness, the theorem C.4 reveals a little bit more about the structure of the groups $\overline{\mathcal{I}}^\mu$ and $\mathcal{I}^\mu$: Let us write $\mathcal{H}_m$ for the $W(\mathbb{F}_{p^f})$-scheme representing the closed subgroup functor of ${}^W\mathcal{G}$ defined by $R \mapsto \mathcal{G}(I_m(R))$. For every $m \geq h$ there are short exact sequences:

$$1 \to \mathcal{H}_{m,\mathbb{F}_{p^f}} \to \overline{\mathcal{I}}^\mu \to \overline{\mathcal{I}}^\mu_m \to 1$$
$$1 \to \mathcal{H}_m \to \mathcal{I}^\mu \to \mathcal{I}^\mu_m \to 1,$$

with, say $\overline{\mathcal{I}}^\mu_m$ reading: $U^0_{\mu^{-1}}(W(R)/I_m(R)) \prod_{i=1}^d \epsilon_i(I_{h_i}(R)/I_m(R))$ (again using the language of theorem C.4). It follows that $\overline{\mathcal{I}}^\mu$ and $\mathcal{I}^\mu$ can be written as the limits of projective systems consisting of our smooth group schemes $\overline{\mathcal{I}}^\mu_m$ and $\mathcal{I}^\mu_m$, together with the natural smooth transition maps $\overline{\mathcal{I}}^\mu_{m+1} \to \overline{\mathcal{I}}^\mu_m$ and $\mathcal{I}^\mu_{m+1} \to \mathcal{I}^\mu_m$. In any case, the canonical projections $\overline{\mathcal{I}}^\mu \to \overline{\mathcal{I}}^\mu_m$ and $\mathcal{I}^\mu \to \mathcal{I}^\mu_m$ are flat and formally smooth morphisms of flat and formally smooth schemes. Note that neither $\overline{\mathcal{I}}^\mu$ nor $\mathcal{I}^\mu$ is of finite type.

We need to spell out the analogous facts applying to the structure of some locally trivial principal homogeneous space E for $\overline{\mathcal{I}}^\mu$ (resp. $\mathcal{I}^\mu$) over a scheme X, we write E_m for the level-m truncation, i.e. $E \times^{\overline{\mathcal{I}}^\mu} \overline{\mathcal{I}}^\mu_m$ (resp. $E \times^{\mathcal{I}^\mu} \mathcal{I}^\mu_m$):

(i) The canonical maps $E_{m+1} \to E_m$ are smooth morphisms of smooth X-schemes for all $m \geq h$.
(ii) We have a canonical isomorphism $E \cong \lim_{\leftarrow} E_m$.
(iii) The canonical projections $E \to E_m$ are flat and formally smooth morphisms of flat and formally smooth X-schemes.

Statement (i) follows from [19, Proposition (2.7.1.iv)] in conjunction with [19, Proposition (17.7.1)]. Statement (ii) follows from [19, Proposition (8.2.5)]. Statement (iii) follows from an elementary argument, and [19, Proposition (2.7.1.viii)].
One more piece of terminology will prove useful: We denote the $W(\mathbb{F}_{p^f})$-group $U^0_{\mu^{-1}}$ by $\mathcal{I}^\mu_0$, we write $\overline{\mathcal{I}}^\mu_0$ for its special fiber, and we call $E\times^{\overline{\mathcal{I}}^\mu}\overline{\mathcal{I}}^\mu_0$ (resp. $E\times^{\mathcal{I}^\mu}\mathcal{I}^\mu_0$) the level-0 truncation of the X-valued point E of $Tors(\overline{\mathcal{I}}^\mu)$ (resp. $Tors(\mathcal{I}^\mu)$).

3.1.1. *Extensions of Φ^μ and $\overline{\Phi}^{\mu,h}$.* Let $\mathcal{G}$ be a smooth and affine $W(\mathbb{F}_{p^f})$-group, and let $\mu:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathcal{G}$ be a cocharacter of triangular type all of whose weights are less than or equal to h (cf. definition C.2). Let $\mathfrak{a}$ be a pd-ideal in a $W(\mathbb{F}_{p^f})$-algebra R. If $h\leq 1$ or $p\in\mathfrak{a}$ holds, we may define a group $\mathcal{J}^\mu_\mathfrak{a}\subset\mathcal{G}(W(R))$ to be the inverse image of whichever of $\mathcal{I}^\mu(R/\mathfrak{a})$ or $\overline{\mathcal{I}}^\mu(R/\mathfrak{a})$ is defined (N.B.: $\overline{\mathcal{I}}^\mu$ is the special fiber of $\mathcal{I}^\mu$, if it exists). For the deformation theory of displays with additional structure we need to discuss certain extensions of the maps Φ^μ_R and $\overline{\Phi}^{\mu,h}_R$, our starting point for this is the following:

Corollary 3.11. *Let $\mathcal{G}$ be a smooth affine $W(\mathbb{F}_{p^f})$-group, and let $\mu:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathcal{G}$ be a cocharacter of triangular type, all of whose weights on $\mathfrak{g}$ are less than or equal to h, and let $\mathfrak{a}$ be a pd-ideal in a $W(\mathbb{F}_{p^f})$-algebra R.*

- *If $h\leq 1$, then the restriction of Φ^μ_R to $\mathcal{I}^\mu(R)\cap\mathcal{G}(\mathfrak{a})$ is trivial.*
- *If $pR=0$, then the restriction of $\overline{\Phi}^{\mu,h}_R$ to $\overline{\mathcal{I}}^\mu(R)\cap\prod_{i=0}^{h-1}\mathcal{G}(\mathfrak{a}_{[F^i]})$ is trivial.*

Proof. Just as in the proof of lemma 3.5, it is harmless to assume $p\in\mathrm{rad}(R)$, in which case we have $\mathfrak{a}_{[F^j]}\subset\mathrm{rad}(W(R))$ too (cf. lemma 2.1). We infer $\mathcal{G}(\mathfrak{a}_{[F^j]})=U^0_{\mu^{-1}}(\mathfrak{a}_{[F^j]})U^1_\mu(\mathfrak{a}_{[F^j]})$, and so we obtain that $\overline{\mathcal{I}}^\mu(R)\cap\prod_{i=0}^{h-1}\mathcal{G}(\mathfrak{a}_{[F^i]})$ is generated by the groups $U^0_{\mu^{-1}}(\mathfrak{a}_{[F^j]})$ (for any $j\in[0,h-1]$) together with $\epsilon_i(\mathfrak{a}_{[F^j]}))$ (for any $j\in[h_i,h-1]$) while $\mathcal{I}^\mu(R)\cap\mathcal{G}(\mathfrak{a})=U^0_{\mu^{-1}}(\mathfrak{a})$ (for $h=1$ only). In all cases one can check the requested vanishing of $\overline{\Phi}^{\mu,h}_R$ and Φ^μ_R from the properties mentioned in the corollary 3.8, here notice that $F^{h-h_i}V^{-h_i}$ vanishes on the ideal $\bigoplus_{j=h_i}^{h-1}\mathfrak{a}_{[F^j]}$. □

Remark 3.12. Let $\mathfrak{g}$ be the Lie-algebra of $\mathcal{G}$. If the square of the ideal $\mathfrak{a}$ in the previous lemma vanishes, then one has the isomorphisms:

$$\prod_{i=0}^{h-1}\mathcal{G}(\mathfrak{a}_{[F^i]})\cong\bigoplus_{i=0}^{h-1}\mathfrak{a}\otimes_{F^i,W(\mathbb{F}_{p^f})}\mathfrak{g}\cong\mathfrak{a}\otimes_{W(\mathbb{F}_{p^f})}\bigoplus_{i=0}^{h-1}{}^{F^i}\mathfrak{g},$$

where ${}^{F^i}\mathfrak{g}$ denotes $W(\mathbb{F}_{p^f})\otimes_{F^i,W(\mathbb{F}_{p^f})}\mathfrak{g}$. In view of this, one recovers the subgroup $\overline{\mathcal{I}}^{\mu}(R)\cap\prod_{i=0}^{h-1}\mathcal{G}(\mathfrak{a}_{[F^i]})$ to be given by $\mathfrak{a}\otimes_{W(\mathbb{F}_{p^f})}\bigoplus_{i=0}^{h-1}{}^{F^i}\operatorname{Fil}_{\mu^{-1}}^{-i}\mathfrak{g}$. Furthermore, if $\mathfrak{f}_i:=\mathfrak{g}/\operatorname{Fil}_{\mu^{-1}}^{-i}\mathfrak{g}$, then we obtain a canonical isomorphism from $\mathcal{G}(W(\mathfrak{a}))/(\overline{\mathcal{I}}^{\mu}(R)\cap\mathcal{G}(W(\mathfrak{a})))$ to $\mathfrak{a}\otimes_{W(\mathbb{F}_{p^f})}\bigoplus_{i=0}^{\infty}{}^{F^i}\mathfrak{f}_i$.

Definition 3.13. *Let $\mathcal{G}$ be a smooth and affine $W(\mathbb{F}_{p^f})$-group, and let $\mu:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathcal{G}$ be a cocharacter of triangular type all of whose weights are less than or equal to $h>0$. Let $\mathfrak{a}$ be a pd-ideal in a $W(\mathbb{F}_{p^f})$-algebra R. Assume that at least one of the conditions:*

(i) $h=1$
(ii) $pR=0$
(iii) *$p\in\mathfrak{a}$ and R is torsionfree.*

holds. Then we define a homomorphism

$$\Psi_{\mathfrak{a}}^{\mu,h}:\mathcal{J}_{\mathfrak{a}}^{\mu}\to({}^{F^h}\mathcal{G})(W(R))$$

as follows: If $h=1$, we let $\Psi_{\mathfrak{a}}^{\mu,h}$ be the unique extension of Φ_R^{μ} which vanishes on $\mathcal{G}(\mathfrak{a})$. If $pR=0$, we let $\Psi_{\mathfrak{a}}^{\mu,h}$ be the unique extension of $\overline{\Phi}_R^{\mu,h}$ which vanishes on $\prod_{i=0}^{h-1}\mathcal{G}(\mathfrak{a}_{[F^i]})$. Finally, if R is torsionfree and $p\in\mathfrak{a}$ holds, then we let $\Psi_{\mathfrak{a}}^{\mu,h}$ be the restriction of $\hat{\Phi}_{W(R)}^{\mu,h}$ to the subgroup $\hat{U}_{\mu^{-1}}^{0}(W(R),\tilde{v}_{\mathfrak{a}})$ (cf. lemma 3.5).

Notice that in the case (i) (resp. (ii)) the formula $\mathcal{I}^{\mu}(R)\mathcal{G}(W(\mathfrak{a}))=\mathcal{J}_{\mathfrak{a}}^{\mu}$ (resp. $\overline{\mathcal{I}}^{\mu}(R)\mathcal{G}(W(\mathfrak{a}))=\mathcal{J}_{\mathfrak{a}}^{\mu}$) holds, but that we have $\hat{U}_{\mu^{-1}}^{0}(W(R),\tilde{v}_{\mathfrak{a}})=\mathcal{J}_{\mathfrak{a}}^{\mu}$ in the (iii) case (while none of "$\mathcal{I}^{\mu}(R)$" or "$\overline{\mathcal{I}}^{\mu}(R)$" are well-defined).

Remark 3.14. Subsubsection 3.5.1 is going to use Lie-theoretic analogs of $\mathcal{J}_{\mathfrak{a}}^{\mu}$ and $\Psi_{\mathfrak{a}}^{\mu,h}$, namely

$$\mathfrak{j}_{\mathfrak{a}}^{\mu}:=\hat{\operatorname{Fil}}^{0}_{\operatorname{Ad}^{\mathcal{G}}\circ\mu^{-1}}(\mathfrak{g},W(R),\tilde{v}_{\mathfrak{a}}) \tag{10}$$

$$\psi_{\mathfrak{a}}^{\mu,h}:\mathfrak{j}_{\mathfrak{a}}^{\mu}\to W(R)\otimes_{F^h,W(\mathbb{F}_{p^f})}\mathfrak{g}, \tag{11}$$

which we define by $F^{h-m}V_{\mathfrak{a}}^{-m}$ on $(W(\mathfrak{a})+I_m(R))\otimes_{W(\mathbb{F}_{p^f})}\mathfrak{g}_m$ for every non-negative weight m, and simply by $p^{-m}F^h$ on all summands of μ-weight $m\leq0$. One recovers the map $\mathfrak{j}_{\mathfrak{a}}^{\mu}\rtimes\mathcal{J}_{\mathfrak{a}}^{\mu};(X,k)\mapsto(\psi_{\mathfrak{a}}^{\mu,h}(X),\Psi_{\mathfrak{a}}^{\mu,h}(k))$ when working with $\underline{\mathfrak{g}}\rtimes\mathcal{G}$ in the definition 3.13.

3.1.2. *Crystals.* At last, we wish to spell out the sheaf theoretic character of the previous constructions: A triple $(R,\mathfrak{a},\gamma)$ is called a pd-thickening (of $R/\mathfrak{a}$) if γ is a divided-power structure on a nilpotent ideal $\mathfrak{a}$ in a ring R, in which p is nilpotent. Fix a $W_\nu(\mathbb{F}_{p^f})$-scheme X, for some $\nu\in\mathbb{N}\cup\{\infty\}$. As in [1, 1.1], we define $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ to be the opposite of the obvious category structure on the class of pd-thickenings $(R, \mathfrak{a}, \gamma)$ that are equipped with the following additional data:

- a morphism from $W_\nu(\mathbb{F}_{p^f})$ to R preserving the canonical divided powers on $pW(\mathbb{F}_{p^f})$ (with respect to a necessarily unique extension of γ to $pR + \mathfrak{a}$),
- a morphism from $\operatorname{Spec} R/\mathfrak{a}$ to X, such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} R/\mathfrak{a} & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Spec} W_\nu(\mathbb{F}_{p^f}) \end{array}$$

  commutes.

We choose to endow $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ with the crystalline fpqc-topology, which is induced from the big fpqc-site $(\operatorname{Spec} W_\nu(\mathbb{F}_{p^f}))_{fpqc}$, by means of the important functor (cf. [1, Lemme 1.1.2]):

$$(X/W_\nu(\mathbb{F}_{p^f}))_{cris} \to (\operatorname{Spec} W_\nu(\mathbb{F}_{p^f}))_{fpqc}; (R, \mathfrak{a}, \gamma) \mapsto R. \tag{12}$$

This has the consequence that the composition of any fpqc-sheaf $\mathcal{F}$ on $\operatorname{Spec} W_\nu(\mathbb{F}_{p^f})$ with the said functor gives automatically a sheaf on $(W_\nu(\mathbb{F}_{p^f})/W_\nu(\mathbb{F}_{p^f}))_{cris}$, and we will denote it by: $\mathcal{F}_{cris}$. Recall the concept of a crystal $\mathbb{H}$ over $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ taking values in some fibration, say $C \stackrel{G}{\to} (\operatorname{Spec} W(\mathbb{F}_{p^f}))_{fpqc}$, being a functor from $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ to C such that:

(i) $H := G \circ \mathbb{H}$ is isomorphic to (12), and
(ii) the image under $\mathbb{H}$ of any $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$-morphism w from V to U induces an isomorpism $\mathbb{H}(V) \stackrel{\cong}{\to} \mathbb{H}(U) \times_{H(U), H(w)} H(V)$.

The last condition is very essential and means that $\mathbb{H}$ is a $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$-valued cartesian section (cf. [30, Chapter III, Definition(3.6)]), observe that it entails the implication $H(w') = H(w) \Rightarrow \mathbb{H}(w') = \mathbb{H}(w)$ for all other $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$-morphisms $w' : V \to U$. Finally note that a crystal over $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ in $Tors(\mathcal{G})$ is just a locally trivial principal homogeneous space for $\mathcal{G}_{W_\nu(\mathbb{F}_{p^f}),cris}$ over the big crystalline site $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$. Whenever $1 \leq \nu' \leq \nu \leq \infty$, sheaves on $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ as well as crystals in C thereon, can be pulled back to $(X'/W_{\nu'}(\mathbb{F}_{p^f}))_{cris}$, where X' is a $X_{W_{\nu'}(\mathbb{F}_{p^f})}$-scheme. In a somewhat different direction every sheaf $\mathcal{E}$ on $(X/W_\nu(\mathbb{F}_{p^f}))_{cris}$ has an inverse image $\mathcal{F} = i^*_{X/W_\nu(\mathbb{F}_{p^f})}(\mathcal{E})$ on X_{fpqc}, defined by $\mathcal{F}(R) := \mathcal{E}(R, 0, 0)$ (please see [1, 1.1.4.a)] for this and [1, 1.1.4.b)] for its right-adjoint).

Throughout most of the paper we are concerned with one site at a time, namely $(X/W(\mathbb{F}_{p^f}))_{cris}$ (resp. $(X/\mathbb{F}_{p^f})_{cris}$ in case $p\mathcal{O}_X = 0$): We let $\mathcal{J}^\mu \subset {}^W\mathcal{G}_{cris}$ (resp. $\overline{\mathcal{J}}^\mu \subset {}^W\mathcal{G}_{\mathbb{F}_{p^f},cris}$) be the subgroup sheaves on $(\operatorname{Spec} W(\mathbb{F}_{p^f})/W(\mathbb{F}_{p^f}))_{cris}$ (resp. $(\operatorname{Spec} \mathbb{F}_{p^f}/\mathbb{F}_{p^f})_{cris}$) given in either case by $(R, \mathfrak{a}, \gamma) \mapsto \mathcal{J}^\mu_\mathfrak{a}$, and we let $\Psi^\mu : \mathcal{J}^\mu \to {}^{WF}\mathcal{G}_{cris}$ (resp. $\overline{\Psi}^{\mu,h} : \overline{\mathcal{J}}^\mu \to {}^{WF^h}\mathcal{G}_{\mathbb{F}_{p^f},cris}$ be the natural transformations given in either case by $(R, \mathfrak{a}, \gamma) \mapsto \Psi^{\mu,h}_\mathfrak{a}$.

3.2. **$\overline{\Phi}$-data.** In this subsection we fix a positive divisor $r \mid f$, for any non-empty set $\Sigma \subset \mathbb{Z}/r\mathbb{Z}$ we will use the following notations:

$$\begin{aligned}
&\mathbf{d}^+_\Sigma(\sigma) := \min\{h \geq 0 | h + \sigma \in \Sigma\},\\
&\mathbf{d}_\Sigma(\sigma) := \sigma + \mathbf{d}^+_\Sigma(\sigma)\\
&r_\Sigma(\sigma) := \operatorname{Card}(\mathbf{d}^{-1}_\Sigma(\{\sigma\}))
\end{aligned}$$

and we define the "canonical cyclic permutation" $\varpi_\Sigma : \Sigma \to \Sigma$ to be the function $\sigma \mapsto \mathbf{d}_\Sigma(1+\sigma)$ (which is the inverse of $\sigma \mapsto \sigma - r_\Sigma(\sigma)$). Also, for a smooth affine $W(\mathbb{F}_{p^r})$-group $\mathcal{G}$ with connected fibers, we need to consider the group

$$\mathcal{G}^\Sigma := \prod_{\sigma\in\Sigma} {}^{F^{-\sigma}}\mathcal{G}.$$

Finally, we let $\mathfrak{g}$ and $\mathfrak{g}^\Sigma$ be the respective Lie-algebras of $\mathcal{G}$ and $\mathcal{G}^\Sigma$.

Definition 3.15. *Let $\mathcal{G}/W(\mathbb{F}_{p^r})$ be as above. The pair $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ is called a $\overline{\Phi}$-datum if:*

(D1) $\emptyset \neq \Sigma \subset \mathbb{Z}/r\mathbb{Z}$.

(D2) *Each $\mu_\sigma : \mathbb{G}_m \to {}^{F^{-\sigma}}\mathcal{G}_{W(\mathbb{F}_{p^f})}$ is a cocharacter of triangular type all of whose weights are less than or equal to $r_\Sigma(\sigma)$*

If, in addition, $\Sigma = \mathbb{Z}/r\mathbb{Z}$ holds, then the pair $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ is called a Φ-datum.

For some fixed $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ we need to introduce further notions and notations, which we will use in the whole paper without further notice. Consider the cocharacter

$$\mu_\Sigma : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to \mathcal{G}^\Sigma_{W(\mathbb{F}_{p^f})},$$

of which the components are given by μ_σ, for every $\sigma \in \Sigma$. Occasionally we need to work with the scalar restriction

$$\mathfrak{G} := \operatorname{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p} \mathcal{G}, \tag{13}$$

notice that $\mathcal{G}^\Sigma$ is canonically contained in $\mathfrak{G}_{W(\mathbb{F}_{p^r})}$, being $\prod_{\sigma=0}^{r-1} {}^{F^{-\sigma}}\mathcal{G}$. At last, we let μ stand for the composition of μ_Σ with the natural inclusion $\mathcal{G}^{\Sigma}_{W(\mathbb{F}_{p^f})} \hookrightarrow \mathfrak{G}_{W(\mathbb{F}_{p^f})}$. Now observe that $\overline{\mathcal{I}}^{\mu_\Sigma}$ is equal to the product $\prod_{\sigma\in\Sigma}\overline{\mathcal{I}}^{\mu_\sigma}$, on the factors of which we defined homomorphisms

$$\overline{\Phi}^{\mu_\sigma, r_\Sigma(\sigma)} : \overline{\mathcal{I}}^{\mu_\sigma} \to {}^{WF^{r_\Sigma(\sigma)-\sigma}}\mathcal{G}_{\mathbb{F}_{p^f}}. \tag{14}$$

Since the composition with ϖ_Σ yields an isomorphism:

$$\prod_{\sigma\in\Sigma} {}^{F^{r_\Sigma(\sigma)-\sigma}}\mathcal{G} \xrightarrow{\cong} \mathcal{G}^{\Sigma} \tag{15}$$

one obtains a homomorphism

$$\overline{\Phi} : \overline{\mathcal{I}}^{\mu_\Sigma} \to {}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}},$$

by composing (15) with the product of the homomorphisms (14), and we will refer to

$${}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \hookleftarrow \overline{\mathcal{I}}^{\mu_\Sigma} \xrightarrow{\overline{\Phi}} {}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}$$

as the (diagram of the) reduced Frobenius map. In the special case $\Sigma = \mathbb{Z}/r\mathbb{Z}$ we have additional homomorphisms

$$\Phi^{\mu_\sigma} : \mathcal{I}^{\mu_\sigma} \to {}^{WF^{1-\sigma}}\mathcal{G}_{W(\mathbb{F}_{p^f})}. \tag{16}$$

at our disposal, and again composing the map

$$\prod_{\sigma=0}^{r-1} {}^{F^{1-\sigma}}\mathcal{G} \xrightarrow{\cong} \mathfrak{G}_{W(\mathbb{F}_{p^r})}; (U_0, U_1, \ldots, U_{r-1}) \mapsto (U_1, \ldots, U_{r-1}, U_0) \tag{17}$$

with the product of the homomorphisms (16) yields the homomorphism

$$\Phi^{\mu} : \mathcal{I}^{\mu} \to {}^{W}\mathfrak{G}_{W(\mathbb{F}_{p^f})},$$

and again, we will refer to

$${}^{W}\mathfrak{G}_{W(\mathbb{F}_{p^f})} \hookleftarrow \mathcal{I}^{\mu} \xrightarrow{\Phi^{\mu}} {}^{W}\mathfrak{G}_{W(\mathbb{F}_{p^f})}$$

as the (diagram of the) non-reduced Frobenius map (N.B.: $\Sigma = \mathbb{Z}/r\mathbb{Z}$ implies $\mathcal{G}^{\Sigma} = \mathfrak{G}$, $\mu_\Sigma = \mu$, and $\overline{\mathcal{I}}^{\mu_\Sigma} = \mathcal{I}^{\mu}_{\mathbb{F}_{p^f}}$).

Definition 3.16. *If* ${}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \hookleftarrow \overline{\mathcal{I}}^{\mu_\Sigma} \xrightarrow{\overline{\Phi}} {}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}$ *is the reduced Frobenius map of some* $W(\mathbb{F}_{p^f})$*-rational* $\overline{\Phi}$*-datum* $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$*, then we let* $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ *be the* $\mathbb{F}_{p^f}$*-stack rendering the diagram:*

$$\begin{array}{ccc} Tors({}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}) & \longleftarrow & \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \\ \Big\downarrow{\scriptstyle \Delta_{Tors({}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}})}} & & \Big\downarrow{\scriptstyle q} \\ Tors({}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}) \times_{\mathbb{F}_{p^f}} Tors({}^{W}\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}) & \xleftarrow{Tors(\overline{\Phi}\times \mathrm{id})\circ\Delta_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma})}} & Tors(\overline{\mathcal{I}}^{\mu_\Sigma}) \end{array} \tag{18}$$

2-cartesian. Moreover, if

$$^W\mathfrak{G}_{W(\mathbb{F}_{p^f})} \hookleftarrow \mathcal{I}^\mu \stackrel{\Phi^\mu}{\to} {}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}$$

is the non-reduced Frobenius map of a Φ*-datum* $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ *we let* $\mathcal{B}(\mathfrak{G}, \mu) = \mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ *be the* $W(\mathbb{F}_{p^f})$*-stack rendering the diagram:*

(19)

$$\begin{array}{ccc} Tors({}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}) & \longleftarrow & \mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \\ \Big\downarrow{\scriptstyle \Delta_{Tors({}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})})}} & & \Big\downarrow{\scriptstyle q} \\ Tors({}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}) \times_{W(\mathbb{F}_{p^f})} Tors({}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}) & \xleftarrow{Tors(\Phi^\mu\times\mathrm{id})\circ\Delta_{Tors(\mathcal{I}^\mu)}} & Tors(\mathcal{I}^\mu) \end{array}$$

2-cartesian.

Do notice that the generic fiber of $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ is 1-isomorphic to the $K(\mathbb{F}_{p^f})$-stack $Tors(U^0_{\mu^{-1}})_{K(\mathbb{F}_{p^f})}$, and that the special fiber of $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ is just the $\mathbb{F}_{p^f}$-stack $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$. In the rest of this subsection we sum up elementary properties which are valid for both $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$. However, for notational convenience we focus our attention on $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and leave some of the analogs for $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ to the reader.
At last, for a fixed arbitrary $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$, the following convention turns out to be quite handy: By saying that $\mathcal{P}$ was a $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over a $W(\mathbb{F}_{p^f})$-scheme X we mean that at least one of the following two statements holds:

- p vanishes in $\mathcal{O}_X$, and $\mathcal{P}$ is a 1-morphism from X to $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$, where X has to be regarded as a $\mathbb{F}_{p^f}$-scheme.
- $\Sigma = \mathbb{Z}/r\mathbb{Z}$ holds, and $\mathcal{P}$ is a 1-morphism from X to $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$.

In each of these two cases we wish to define the underlying locally trivial principal homogeneous space of $\mathcal{P}$ to be $q(\mathcal{P})$, which is a X-valued point in one of $Tors(\overline{\mathcal{I}}^{\mu_\Sigma})$ or $Tors(\mathcal{I}^\mu)$. Let us denote the level-0 truncation of $q(\mathcal{P})$ by $q_0(\mathcal{P})$, which is a X-valued point in $Tors(\mathcal{I}_0^{\mu_\Sigma})$. Finally let us say that $\mathcal{P}$ is a banal $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over X if and only if $q(\mathcal{P})$ possesses a global section.

Remark 3.17. Using the notation of definition 3.16 we have canonical 1-isomorphisms:

$$^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}} \stackrel{\cong}{\to} Tors({}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}) \times_{\Delta_{Tors({}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}})}, Tors({}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}})^2, \mathbf{b}({}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}})^2} \operatorname{Spec}\mathbb{F}_{p^f}$$

$$\stackrel{\cong}{\to} \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \times_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma}), \mathbf{b}(\overline{\mathcal{I}}^{\mu_\Sigma})} \operatorname{Spec}\mathbb{F}_{p^f}$$

of which the composition with the canonical projection:

$$\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\times_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma}),\mathbf{b}(\overline{\mathcal{I}}^{\mu_\Sigma})}\operatorname{Spec}\mathbb{F}_{p^f}\to\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$$

will be denoted by:

$$\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma}):{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\to\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma}).$$

Notice that there is a canonical 2-cartesian diagram:

$$\begin{array}{ccc}{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\times_{\mathbb{F}_{p^f}}\overline{\mathcal{I}}^{\mu_\Sigma} & \xrightarrow{\mathrm{pr}_1} & {}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\\ \mathrm{pr}_2\downarrow & & \downarrow\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\\ {}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}} & \xrightarrow{\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})} & \overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\end{array}\tag{20}$$

where pr_1 (resp. pr_2) are the morphisms from ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\times_{\mathbb{F}_{p^f}}\overline{\mathcal{I}}^{\mu_\Sigma}$ to ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}$, given by $(U,k)\mapsto U$ (resp. $(U,k)\mapsto k^{-1}U\overline{\Phi}(k)$). Moreover, if pr_{12} (resp. pr_{13} or pr_{23}) are the morphisms from ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\times_{\mathbb{F}_{p^f}}\overline{\mathcal{I}}^{\mu_\Sigma}\times_{\mathbb{F}_{p^f}}\overline{\mathcal{I}}^{\mu_\Sigma}$ to ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\times_{\mathbb{F}_{p^f}}\overline{\mathcal{I}}^{\mu_\Sigma}$, given by: $(U,k,l)\mapsto(U,k)$ (resp. $(U,k,l)\mapsto(U,kl)$ or $(U,k,l)\mapsto(k^{-1}U\overline{\Phi}(k),l)$), and if

$$\alpha:\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\circ\mathrm{pr}_2\to\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\circ\mathrm{pr}_1$$

is the 2-morphism implicit in the diagram (20), then $\mathrm{pr}_{13}^*(\alpha)$ agrees with the composition of $\mathrm{pr}_{12}^*(\alpha)$ and $\mathrm{pr}_{23}^*(\alpha)$.

Lemma 3.18. *The canonical projection*

$$q:\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\to Tors(\overline{\mathcal{I}}^{\mu_\Sigma}),$$

as well as the diagonal

$$\Delta_{\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$$

are schematic and affine (hence quasicompact and separated) 1-morphisms.

Proof. The former assertion follows simply from the diagram (18), together with the fact, that $\Delta_{Tors({}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}})}$ is schematic and affine. Now this implies already that the relative diagonal 1-morphism

$$\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\to\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\times_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma})}\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$$

is schematic and closed, so that the latter assertion follows once we show that

$$\begin{aligned}&\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\times_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma})}\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\\&\to\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\times\overline{\mathcal{B}}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\end{aligned}$$

is a schematic and affine 1-morphism, which can be deduced from the same property of $\Delta_{Tors(\overline{\mathcal{I}}^{\mu_\Sigma})}$. □

3.2.1. *Banality.* In this subsubsection we fix a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$, and a $W(\mathbb{F}_{p^f})$-algebra R. Again, we assume that one of the following holds:

- $pR = 0$
- $\Sigma = \mathbb{Z}/r\mathbb{Z}$.

In the former case we constructed $\mathcal{G}^\Sigma(W(R)) \hookleftarrow \overline{\mathcal{I}}^{\mu_\Sigma}(R) \overset{\overline{\Phi}_R}{\to} \mathcal{G}^\Sigma(W(R))$ and in the latter case we constructed $\mathfrak{G}(W(R)) \hookleftarrow \mathcal{I}^\mu(R) \overset{\Phi_R^\mu}{\to} \mathfrak{G}(W(R))$, and in both cases we will write $B_R(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ for the category which results from either diagram according to example 3.2. Observe that the category of banal $3n$-displays with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over some $W(\mathbb{F}_{p^f})$-scheme X is canonically equivalent to $B_{\Gamma(X,\mathcal{O}_X)}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$. For later use we note a lemma, which is mediately tied to the concept of banality:

Lemma 3.19. *Let $\mathcal{P}$ be a $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over* Spec R*.*

(i) *If $I \subset R$ is a nilpotent ideal, then $\mathcal{P}$ is banal if and only if $\mathcal{P}_{R/I}$ is banal.*
(ii) *If pR is nilpotent then there exists an étale and faithfully flat R-algebra over which $\mathcal{P}$ becomes banal.*

Proof. The first of these two assertions follows from remark 3.10. In order to prove the second assertion we may assume $p1_R = 0$, due to [19, Théorème 18.1.2] and (i). We proceed by invoking the truncations E_m of the underlying locally trivial principal homogeneous space E (cf. remark 3.10). We can clearly find a section of the smallest possible truncation E_h over an étale and faithfully flat R-algebra S, due to [19, Corollaire (17.16.3.ii)]. However, notice that there exists a short exact sequence

$$0 \to \underline{\mathbb{F}_{p^f} \otimes_{W(\mathbb{F}_{p^r})} \mathfrak{g}^\Sigma} \hookrightarrow \overline{\mathcal{I}}^{\mu_\Sigma}_{m+1} \twoheadrightarrow \overline{\mathcal{I}}^{\mu_\Sigma}_m \to 1.$$

By induction on $\mathbb{N} \ni m \geq h$ we deduce the global triviality of all of the level m-truncations E_m, due to [32, Chapter III, Proposition 3.7]. Finally it follows that $\mathcal{P}_S$ is banal. □

Lemma 3.20. *Let $\mathcal{P}$ be a $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over* Spec R*, and let k be an element in the kernel of the map* $\mathrm{Aut}(q(\mathcal{P})) \to \mathrm{Aut}(q_0(\mathcal{P}_{R/J}))$*, where J is an ideal of R. Suppose that $I \subset J$ is another ideal with $JI = pI = 0$, and such that the image of the map* $\mathrm{Aut}(\mathcal{P}_{R/I}) \to \mathrm{Aut}(q(\mathcal{P}_{R/I}))$ *contains the* mod *I-reduction of k. Then k^p lies in the image of the map* $\mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(q(\mathcal{P}))$.

Proof. We may assume that $\mathcal{P}$ is banal, and a choice of global section of $q(\mathcal{P})$ gives rise to an element $U \in \mathcal{G}^{\Sigma}(W(R))$, moreover the element k satisfies $k^{-1}U\overline{\Phi}(k) = NU$ (resp. $k^{-1}U\Phi^{\mu}(k) = NU$) for some $N \in \mathcal{G}^{\Sigma}(W(I))$. Applying the description (9) to the specific pd-ideal I yields the mutual annihilation of the ideals $W(I)$ and $W(J) + I(R)$. We deduce $k^{-p}U\overline{\Phi}(k^p) = N^pU = U$ (resp. $k^{-p}U\Phi^{\mu}(k^p) = U$), since the elements N and k commute with each other. □

Remark 3.21. Note that the remark 3.10 implies the surjectivity of $\mathrm{Aut}(q(\mathcal{P})) \to \mathrm{Aut}(q(\mathcal{P}_{R/I}))$. It follows that all pth powers of elements in the kernel of the map $\mathrm{Aut}(\mathcal{P}_{R/I}) \to \mathrm{Aut}(q_0(\mathcal{P}_{R/J}))$ lift to automorphisms of $\mathcal{P}$.

3.3. **Functoriality.** Fix f and let $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and $(\tilde{\mathcal{G}}, \{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}})$ be two $\overline{\Phi}$-data. Suppose we are given a pair consisting of a group homomorphism

$$\gamma : \overline{\mathcal{I}}^{\mu_\Sigma} \to \overline{\mathcal{I}}^{\tilde{\mu}_{\tilde{\Sigma}}}$$

together with a morphism

$$m : {}^W\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \to {}^W\tilde{\mathcal{G}}^{\tilde{\Sigma}}_{\mathbb{F}_{p^f}},$$

that renders the diagram

$$\begin{array}{ccc} {}^W\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \times_{\mathbb{F}_{p^f}} \overline{\mathcal{I}}^{\mu_\Sigma} & \xrightarrow{\mathrm{pr}_2} & {}^W\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \\ {\scriptstyle m\times\gamma}\downarrow & & {\scriptstyle m}\downarrow \\ {}^W\tilde{\mathcal{G}}^{\tilde{\Sigma}}_{\mathbb{F}_{p^f}} \times_{\mathbb{F}_{p^f}} \overline{\mathcal{I}}^{\tilde{\mu}_{\tilde{\Sigma}}} & \xrightarrow{\tilde{\mathrm{pr}}_2} & {}^W\tilde{\mathcal{G}}^{\tilde{\Sigma}}_{\mathbb{F}_{p^f}} \end{array}$$

commutative (where pr_2 and $\tilde{\mathrm{pr}}_2$ are as in (20)). Then one obtains a canonical 2-commutative diagram:

$$\begin{array}{ccccc} {}^W\mathcal{G}^{\Sigma}_{W(\mathbb{F}_{p^f})} & \xrightarrow{\mathbf{b}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})} & \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) & \longrightarrow & Tors(\overline{\mathcal{I}}^{\mu_\Sigma}) \\ {\scriptstyle m}\downarrow & & \downarrow & & {\scriptstyle Tors(\gamma)}\downarrow \\ {}^W\tilde{\mathcal{G}}^{\tilde{\Sigma}}_{W(\mathbb{F}_{p^f})} & \xrightarrow{\mathbf{b}(\tilde{\mathcal{G}},\{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}})} & \overline{\mathcal{B}}(\tilde{\mathcal{G}}, \{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}}) & \longrightarrow & Tors(\overline{\mathcal{I}}^{\tilde{\mu}_{\tilde{\Sigma}}}) \end{array}.$$

Three instances of this procedure will occur in this paper and we collect them in this subsection (in all examples we fix r too):

3.3.1. *A Covariant Functoriality in* $\mathcal{G}$. Let $i : \mathcal{G} \to \tilde{\mathcal{G}}$ be a homomorphism of $W(\mathbb{F}_{p^r})$-groups, and assume

$$\Sigma = \tilde{\Sigma}$$
$$\forall \sigma \in \Sigma : {}^{F^{-\sigma}}i \circ \mu_\sigma = \tilde{\mu}_\sigma$$

(in the sequel we will refer to this as a morphism from $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ to the $\overline{\Phi}$-datum $(\tilde{\mathcal{G}}, \{\tilde{\mu}_\sigma\}_{\sigma\in\Sigma}))$. Consider the homomorphism $i^\Sigma : \mathcal{G}^\Sigma \to \tilde{\mathcal{G}}^\Sigma$ which is given by the cartesian product of the maps ${}^{F^{-\sigma}}i : {}^{F^{-\sigma}}\mathcal{G} \to {}^{F^{-\sigma}}\tilde{\mathcal{G}}$. It is easy to see that the homomorphism ${}^W i^\Sigma_{\mathbb{F}_{p^f}} =: m$ from ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}$ to ${}^W\tilde{\mathcal{G}}^\Sigma_{\mathbb{F}_{p^f}}$ restricts to a homomorphism γ from $\overline{\mathcal{I}}^{\mu_\Sigma}$ to $\overline{\mathcal{I}}^{\tilde{\mu}_\Sigma}$. The canonical 1-morphism which results from the pair (γ, m) will be denoted by:

$$\overline{\mathcal{B}}(i) : \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \to \overline{\mathcal{B}}(\tilde{\mathcal{G}}, \{\tilde{\mu}_\sigma\}_{\sigma\in\Sigma}).$$

3.3.2. *A Covariant Functoriality in* $\{\mu_\sigma\}_{\sigma\in\Sigma}$. Consider a family $\{g_\sigma\}_{\sigma\in\Sigma} = g \in \mathcal{G}^\Sigma(W(\mathbb{F}_{p^f}))$, and assume

$$\begin{aligned} &\Sigma = \tilde{\Sigma} \\ &\mathcal{G} = \tilde{\mathcal{G}} \\ &\tilde{\mu}_\Sigma = \mathrm{Int}^{\mathcal{G}^\Sigma}(g/W(\mathbb{F}_{p^f})) \circ \mu_\Sigma. \end{aligned}$$

Let $\gamma : \overline{\mathcal{I}}^{\mu_\Sigma} \to \overline{\mathcal{I}}^{\tilde{\mu}_\Sigma}$ be the homomorphism which is obtained by restriction of the inner automorphism $\mathrm{Int}^{{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}}(g/\mathbb{F}_{p^f})$. Finally consider the functions $\varpi^+_\Sigma(\sigma) := \mathbf{d}^+_\Sigma(\sigma+1)+1$ and $\varpi_\Sigma(\sigma) = \sigma + \varpi^+_\Sigma(\sigma)$ to describe a map m from ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}$ to itself as follows:

$$\{U_\sigma\}_{\sigma\in\Sigma} \mapsto \{g_\sigma^{-1} U_\sigma{}^{F^{\varpi^+_\Sigma(\sigma)}} g_{\varpi_\Sigma(\sigma)}\}_{\sigma\in\Sigma}.$$

The canonical 1-morphism which results from this pair (γ, m) will be denoted by:

$$\overline{\mathcal{B}}(g) : \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \to \overline{\mathcal{B}}(\mathcal{G}, \{\tilde{\mu}_\sigma\}_{\sigma\in\Sigma}).$$

3.3.3. *A contravariant Functoriality in* Σ. Let us say that a function $\mathbf{d}^+ : \mathbb{Z}/r\mathbb{Z} \to \mathbb{N}_0$ is a $\pmod r$-multidegree if $\sigma < \varsigma$ implies $\sigma+\mathbf{d}^+(\sigma) < \varsigma+\mathbf{d}^+(\varsigma)$ for any $\sigma, \varsigma \in \mathbb{Z}$. The last type of functoriality that we discuss arises in the following scenario: Consider the functions

$$\begin{aligned} &\mathbf{d} : \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z} :\ \sigma \mapsto \sigma + \mathbf{d}^+(\sigma) \\ &\tilde{\mathbf{d}}^+ : \mathbb{Z}/r\mathbb{Z} \to \mathbb{N}_0;\ \sigma \mapsto \mathbf{d}^+(\sigma) - \min\{\mathbf{d}^+(\omega) | \mathbf{d}(\omega) = \mathbf{d}(\sigma)\} \\ &\tilde{\mathbf{d}} : \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}; \sigma \mapsto \sigma + \tilde{\mathbf{d}}^+(\sigma), \end{aligned}$$

where $\mathbf{d}^+$, and hence $\tilde{\mathbf{d}}^+$, is a $\pmod r$-multidegree. Assume that we have:

$$\Sigma = \{\mathbf{d}(\omega)|\omega \in \mathbb{Z}/r\mathbb{Z}\} \tag{21}$$

$$\tilde{\Sigma} = \{\tilde{\mathbf{d}}(\omega)|\omega \in \mathbb{Z}/r\mathbb{Z}\} \tag{22}$$

$$\mathcal{G} = \tilde{\mathcal{G}} \tag{23}$$

$$\forall \omega \in \tilde{\Sigma} : \tilde{\mu}_\omega = {}^{F^{\mathbf{d}^+(\omega)}}\mu_{\mathbf{d}(\omega)}. \tag{24}$$

More specifically, let $z = \mathrm{Card}(\Sigma) = \mathrm{Card}(\tilde{\Sigma})$ and $\sigma_1 < \cdots < \sigma_z < \sigma_1 + r$ be the elements of a set of representatives for the $\pmod r$ congruence classes in Σ. Then $\tilde{\Sigma}$ consists of the $\pmod r$-congruence classes of the elements $\tilde{\sigma}_j := \max\{\sigma|\sigma + \mathbf{d}^+(\sigma) \leq \sigma_j\}$. The 1-morphism

$$\mathrm{Flex}^{\mathbf{d}^+} : \overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \to \overline{\mathcal{B}}(\mathcal{G}, \{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}})$$

is defined by writing down a pair (γ, m) as follows:

$$m : \{U_\sigma\}_{\sigma\in\Sigma} \mapsto \{{}^{F^{\mathbf{d}^+(\sigma)}}U_{\mathbf{d}(\sigma)}\}_{\sigma\in\tilde{\Sigma}} \tag{25}$$

$$\gamma : \{k_\sigma\}_{\sigma\in\Sigma} \mapsto \{{}^{F^{\mathbf{d}^+(\sigma)}}k_{\mathbf{d}(\sigma)}\}_{\sigma\in\tilde{\Sigma}}. \tag{26}$$

For the significance for all of this please see e.g. subsubsection 6.1.3.

3.4. **Weil restriction revisited.** For every subset $\Sigma \subset \mathbb{Z}/r\mathbb{Z}$, and for every $m \in \mathbb{N}_0$ we let $\Sigma^{(m)} \subset \mathbb{Z}/mr\mathbb{Z}$ be the inverse image of Σ under the canonical projection $\mathbb{Z}/mr\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$.
Recall that we fixed a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and assume that s is a number with $r \mid s \mid f$. Observe that the $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G}_{W(\mathbb{F}_{p^s})}, \{\mu_\sigma\}_{\sigma\in\Sigma^{(\frac{s}{r})}})$ is equipped with an action of the cyclic group $\Delta := r\mathbb{Z}/s\mathbb{Z}$, in the sense of subsubsection 3.3.1. The following is a restatement of lemma 2.2:

Lemma 3.22. *For any $W(\mathbb{F}_{p^f})$-scheme X, there is a natural equivalence from the category of $3n$-displays with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over X to the category of Δ-equivariant $3n$-displays with $(\mathcal{G}_{W(\mathbb{F}_{p^s})}, \{\mu_\sigma\}_{\sigma\in\Sigma^{(\frac{s}{r})}})$-structure over X, where Δ acts trivially on X.*

3.5. **Deformations of displays with additional structure.** Recall $\mathfrak{f}_i = \mathfrak{g}/\mathrm{Fil}^{-i}_{\mu^{-1}}\mathfrak{g}$ (cf. remark 3.12). Consider the direct sum

$$\bigoplus_{i=0}^{\infty} \overline{\theta}_i^\mu =: \overline{\theta}^\mu \in Rep_0(\overline{\mathcal{I}}_0^\mu),$$

where each summand $\overline{\theta}_i^\mu : \overline{\mathcal{I}}_0^\mu \to \mathrm{GL}(\mathbb{F}_{p^f} \otimes_{F^i, W(\mathbb{F}_{p^f})} \mathfrak{f}_i / W(\mathbb{F}_{p^f}))$ arises from composing the i-th iterate of the Frobenius with the $\pmod p$-reduction of the natural adjoint action:

$$\theta_i^\mu : \mathcal{I}_0^\mu \to \mathrm{GL}(\mathfrak{f}_i / W(\mathbb{F}_{p^f})).$$

The representation $\theta_0^\mu \in Rep_0(\mathcal{I}_0^\mu)$ as well as $\overline{\theta}^\mu \in Rep_0(\overline{\mathcal{I}}_0^\mu)$ have a great significance in this paper: For every $3n$-display $\mathcal{P}$ with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over X, we define a vector bundle on X by means of $T_\mathcal{P} = \omega_{q_0(\mathcal{P})}(\theta_0^\mu)$ (resp. $T_\mathcal{P} = \omega_{q_0(\mathcal{P})}(\overline{\theta}^{\mu_\Sigma})$).
One of the aims in this section is to show that the assignment $\mathcal{P} \mapsto T_\mathcal{P}$ behaves like a tangent bundle on $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ (resp. $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$). We have to start with a substitute for the result [60, Theorem 44] in the language of our stack $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ (resp. $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$). For the rest of this subsection we fix an auxiliary monotone bijection $\mathbb{Z} \to \Sigma^{(0)}; j \mapsto \sigma_j$, so that there exists $z \in \mathbb{N}$ with $r + \sigma_j = \sigma_{z+j}$ for all integers j (let $r_j := \sigma_j - \sigma_{j-1} = r_\Sigma(\sigma_j)$). The following definition is very similar, but not exactly equivalent to [60, Definition 11/Definition 13].

Definition 3.23. *Let $(\mathcal{G}/W(\mathbb{F}_{p^r}), \{\mu_\sigma\}_{\sigma\in\Sigma})$ be a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum, and let us write*

$$\pi_j \circ \pi_j = \pi_j : \omega^\mathcal{G}(\mathrm{Ad}^\mathcal{G}) \twoheadrightarrow \omega^\mathcal{G}_{{}^{F^{\sigma_j}}\mu_{\sigma_j}}(r_j, \mathrm{Ad}^\mathcal{G})$$

for the projector killing all summand of ${}^{F^{\sigma_j}}\mu_{\sigma_j}$-weight strictly less than r_j (N.B.: Due to condition (D2) all ${}^{F^{\sigma_j}}\mu_{\sigma_j}$-weights of $\omega^\mathcal{G}(\mathrm{Ad}^\mathcal{G})$ are less than or equal to r_j.).
If S is a $\mathbb{F}_{p^f}$-algebra, then a z-tuple $(U_1, \dots, U_z) \in \mathcal{G}^\Sigma(S)$ is said to satisfy the nilpotence condition, if

$$\pi_0 \circ \mathrm{Ad}^\mathcal{G}({}^{F^{\sigma_1}}U_1) \circ \cdots \circ \mathrm{Ad}^\mathcal{G}({}^{F^{\sigma_N}}U_N) \circ \pi_N = 0,$$

for all sufficiently large numbers N.

It is clear that the full fibered subcategory of $\mathcal{B}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ (resp. $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$) whose objects satisfy the $\pmod p$-nilpotence condition is a stack, and we will denote it by $\mathcal{B}'(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ (resp. $\overline{\mathcal{B}}'(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$). Its objects are called displays with additional structure, and likewise $B'_R(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ denotes the category of banal displays with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over $\operatorname{Spec} R$.

3.5.1. *Frobenius maps revisited.* Let $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ be as above, let $\mathfrak{a}$ be a pd-ideal in a $W(\mathbb{F}_{p^f})$-algebra S, and assume that one of the following holds:

(i) S is p-adically separated and complete, $\mathfrak{a}$ is p-adically closed and topologically nilpotent, and Σ is equal to $\mathbb{Z}/r\mathbb{Z}$
(ii) $\mathfrak{a}$ is nilpotent and $pS = 0$
(iii) $\mathfrak{a}$ is p-adically topologically nilpotent and contains p, and S is torsionfree and p-adically separated and complete

We define $\Psi_{\mathfrak{a}} : \mathcal{J}_{\mathfrak{a}}^{\mu_\Sigma} \to \mathcal{G}^\Sigma(W(S))$ to be the composition of the canonical cyclic permutation in (15) with the product of the extended Frobenius maps: $\Psi_{\mathfrak{a}}^{\mu_\sigma, r_\Sigma(\sigma)} : \mathcal{J}_{\mathfrak{a}}^{\mu_\sigma} \to {}^{F^{r_\Sigma(\sigma)-\sigma}}\mathcal{G}(W(S))$ (cf. definition 3.13).

Lemma 3.24. *Under the above assumptions (i), (ii) or (iii), and with the above notations every congruence $O \equiv U \mod \mathfrak{a}$ between any two $O, U \in \mathcal{G}^\Sigma(W(S))$ satisfying the nilpotence condition, implies the existence of a unique element $h \in \mathcal{G}^\Sigma(W(\mathfrak{a}))$ with $O = h^{-1}U\Psi_{\mathfrak{a}}(h)$.*

Proof. We will write $\psi_{\mathfrak{a}} : \mathfrak{j}_{\mathfrak{a}}^{\mu_\Sigma} \to W(S) \otimes_{W(\mathbb{F}_{p^f})} \mathfrak{g}^\Sigma$ for the map that is induced by the canonical cyclic permutation composed with the product of the Lie-theoretic extended Frobenius maps

$$\psi_{\mathfrak{a}}^{\mu_{\sigma_j}, r_j} : \mathfrak{j}_{\mathfrak{a}}^{\mu_{\sigma_j}} \to W(S) \otimes_{F^{-\sigma_{j-1}}, W(\mathbb{F}_{p^f})} \mathfrak{g}$$

as in (11). We endow $\mathcal{G}^\Sigma(W(\mathfrak{a}))$ with the topology coming from the restriction of the p-adic topology of S, with respect to which it is separated and complete, so what about the fixed points of $h \mapsto K_{\mathfrak{a}}(h) := U\Psi_{\mathfrak{a}}(h)O^{-1}$, as a function from $\mathcal{G}^\Sigma(W(\mathfrak{a}))$ to itself? Under either of the two assumptions (i) or (ii) we are allowed to use the $pS + \mathfrak{a}$-adic separated completeness of S too, so let us consider the chain of ideals defined by $\mathfrak{a}_n = \mathfrak{a} \cap (pS + \mathfrak{a})^n$ for any $n \in \mathbb{N}$. It is enough to prove the unique existence of the requested fixed points for each $K_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}$. Furthermore, we have natural isomorphisms

$$\mathcal{G}^\Sigma(W(\mathfrak{a}_n))/\mathcal{G}^\Sigma(W(\mathfrak{a}_{n+1})) \cong \mathcal{G}^\Sigma(W(\mathfrak{a}_n/\mathfrak{a}_{n+1})) \cong W(\mathfrak{a}_n/\mathfrak{a}_{n+1}) \otimes_{W(\mathbb{F}_{p^f})} \mathfrak{g}^\Sigma,$$

the equation $K_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}(h + X) = K_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}(h) + (\mathrm{Ad}^{\mathcal{G}^\Sigma}(U) \circ \psi_{\mathfrak{a}_n/\mathfrak{a}_{n+1}})(X)$ holds, and thus it suffices to check the nilpotence of the endomorphism $\mathrm{Ad}^{\mathcal{G}^\Sigma}(U) \circ \psi_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}$ on $W(\mathfrak{a}_n/\mathfrak{a}_{n+1}) \otimes_{W(\mathbb{F}_{p^f})} \mathfrak{g}^\Sigma$. Since $W(\mathfrak{a}_n/\mathfrak{a}_{n+1})$ can be regarded as a $S/pS + \mathfrak{a}$-module(!) we let

$$\psi_j^* : (S/pS + \mathfrak{a}) \otimes_{F^{-\sigma_j}, W(\mathbb{F}_{p^f})} \mathfrak{g} \to (S/pS + \mathfrak{a}) \otimes_{F^{-\sigma_{j-1}}, W(\mathbb{F}_{p^f})} \mathfrak{g}$$

be the composition of the r_jth iterate of the absolute Frobenius from $(S/pS+\mathfrak{a}) \otimes_{F^{-\sigma_j}, W(\mathbb{F}_{p^f})} \mathfrak{g}$ to $(S/pS+\mathfrak{a}) \otimes_{F^{-\sigma_{j-1}}, W(\mathbb{F}_{p^f})} \mathfrak{g}$ with the projection of $W(\mathbb{F}_{p^f}) \otimes_{F^{-\sigma_j}, W(\mathbb{F}_{p^r})} \mathfrak{g}$ onto the direct summand of the $\mathbb{G}_{m, W(\mathbb{F}_{p^f})}$-representation $\mathrm{Ad}^{F^{-\sigma_j}\mathcal{G}} \circ \mu_{\sigma_j}$ of degree r_j. It is easy to see that the map $\psi_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}$ can be written as the ($S/pS+\mathfrak{a}$-linear) tensor product of $\mathrm{id}_{W(\mathfrak{a}_n/\mathfrak{a}_{n+1})}$ and the map

$$\psi^*:(S/pS+\mathfrak{a})\otimes_{W(\mathbb{F}_{p^f})}\mathfrak{g}^\Sigma\to(S/pS+\mathfrak{a})\otimes_{W(\mathbb{F}_{p^f})}\mathfrak{g}^\Sigma,$$

which we obtain by composing the product of the endomorphisms ψ_j^* on $\bigoplus_{j=0}^{z-1}(S/pS+\mathfrak{a})\otimes_{F^{-\sigma_j},W(\mathbb{F}_{p^f})}\mathfrak{g}$ with the cyclic permutation thereon given by: $(X_0,X_1,\dots,X_{z-1})\mapsto(X_1,\dots,X_{z-1},X_0)$. It follows that $\mathrm{Ad}^{\mathcal{G}^\Sigma}(U)\circ\psi_{\mathfrak{a}_n/\mathfrak{a}_{n+1}}$ is nilpotent, notice that this proof is quite similar to the one of [60, Lemma 42], as is the (iii)-case, which we leave to the reader. □

Remark 3.25. Under the above assumption (iii) one can strengthen the uniqueness assertion in the previous lemma: For the extended Frobenius maps $\Psi_{\mathfrak{a}}^{\mu_{\sigma_j},r_j}$ are just the restrictions to $\mathcal{J}_{\mathfrak{a}}^{\mu_{\sigma_j}}$ of $\hat{\Phi}_{W(S)}^{\mu_{\sigma_j},r_j}:g\mapsto{}^{F^{r_j}}(\mu_{\sigma_j}(\frac{1}{p})g\mu_{\sigma_j}(p))$, in which case we have that

$$(27)\qquad {}^{F^{-\sigma_j}}\mathcal{G}(\mathbb{Q}\otimes W(\mathfrak{a}))\ni h_j=U_j{}^{F^{r_{j+1}}}(\mu_{\sigma_{j+1}}(\frac{1}{p})h_{j+1}\mu_{\sigma_{j+1}}(p))U_j^{-1}$$

for all j implies $h_j=1$ for all j. This can be seen as follows. If s is a multiple of f we deduce that the same equation holds for the elements ${}^{F^s}U_j$ and ${}^{F^s}h_j$, and the latter is contained in ${}^{F^{-\sigma_j}}\mathcal{G}(W(\mathfrak{a}))$ provided only that s is sufficiently large. We infer ${}^{F^s}h_j=1$, so that all sufficiently large ghost components of h_j vanish, and by downward induction using (27) again, we obtain the vanishing of all other ghost components of h_j too.

In complete analogy to subsubsection 3.1.2 we denote the effect of the $\Psi_{\mathfrak{a}}^{\mu_\sigma,r_\Sigma(\sigma)}$'s on the crystalline fpqc-sheaf $\overline{\mathcal{J}}^{\mu_\Sigma}$ (resp. $\mathcal{J}^\mu$) by

$$\overline{\Psi}:\overline{\mathcal{J}}^{\mu_\Sigma}\to{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f},cris}$$

(resp. by $\Psi^\mu:\mathcal{J}^\mu\to{}^W\mathfrak{G}_{W(\mathbb{F}_{p^f}),cris}$). Let $\mathcal{P}$ be a display with $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over X, and let $q(\mathcal{P})$ be the underlying principal homogeneous space for $\overline{\mathcal{I}}^{\mu_\Sigma}$ (resp. $\mathcal{I}^\mu$) over X. Note that we have a canonical isomorphism

$$(28)\qquad \imath_{\mathcal{P}}:q(\mathcal{P})\times^{\overline{\mathcal{I}}^{\mu_\Sigma},\overline{\Phi}}{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}\to q(\mathcal{P})\times^{\overline{\mathcal{I}}^{\mu_\Sigma}}{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f}}$$

(resp. $\imath_{\mathcal{P}}:q(\mathcal{P})\times^{\mathcal{I}^\mu,\Phi^\mu}{}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}\to q(\mathcal{P})\times^{\mathcal{I}^\mu}{}^W\mathfrak{G}_{W(\mathbb{F}_{p^f})}$). Our next object is the description of a canonical principal homogeneous space $\mathcal{E}_{\mathcal{P}}$ for $\overline{\mathcal{J}}^{\mu_\Sigma}$ over $(X/\mathbb{F}_{p^f})_{cris}$ (resp. $\mathcal{J}^\mu$ over $(X/W(\mathbb{F}_{p^f}))_{cris}$), together with a canonical isomorphism

$$(29)\qquad \jmath_{\mathcal{P}}:\mathcal{E}_{\mathcal{P}}\times^{\overline{\mathcal{J}}^{\mu_\Sigma},\overline{\Psi}}{}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f},cris}\to\mathbb{H}_{\mathcal{P}}$$

(resp. $\jmath_{\mathcal{P}} : \mathcal{E}_{\mathcal{P}} \times^{\mathcal{J}^\mu, \Psi^\mu} {}^W\mathfrak{G}_{W(\mathbb{F}_{p^f}),cris} \to \mathbb{H}_{\mathcal{P}}$), where $\mathbb{H}_{\mathcal{P}}$ is the locally principal homogeneous space for ${}^W\mathcal{G}^\Sigma_{\mathbb{F}_{p^f},cris}$ (resp. ${}^W\mathfrak{G}_{W(\mathbb{F}_{p^f}),cris}$) over $(X/\mathbb{F}_{p^f})_{cris}$ (resp. $(X/W(\mathbb{F}_{p^f}))_{cris}$) arising by extension of the structure group from $\mathcal{E}_{\mathcal{P}}$.
By a glueing argument we can assume that $\mathcal{P}$ is banal, so choose an element $U \in {}^W\mathcal{G}^\Sigma(X)$ representing $\mathcal{P}$. We define the value $\mathcal{E}_{\mathcal{P}}(S, \mathfrak{a}, \gamma)$ of $\mathcal{E}_{\mathcal{P}}$ on the pd-thickening $(S, \mathfrak{a}, \gamma)$ to be $\mathcal{J}_{\mathfrak{a}}^{\mu_\Sigma}$. In order to define the restriction maps for the sheaf we proceed as follows: For every pd-thickening $(S, \mathfrak{a}, \gamma)$ we choose a lift $U_S \in \mathcal{G}^\Sigma(W(S))$ of the image of U in $\mathcal{G}^\Sigma(W(S/\mathfrak{a}))$. Now consider some morphism $(S, \mathfrak{a}, \gamma) \leftarrow (T, \mathfrak{b}, \delta)$ in $(X/\mathbb{F}_{p^f})_{cris}$ (resp. $(X/W(\mathbb{F}_{p^f}))_{cris}$). By lemma 3.24 there exists $k_{T,S} \in \mathcal{G}^\Sigma(W(\mathfrak{b})) \subset \mathcal{J}_{\mathfrak{b}}^{\mu_\Sigma}$ such that the image of U_S in $\mathcal{G}^\Sigma(W(T))$ agrees with $k_{T,S}^{-1} U_T \Psi_{\mathfrak{b}}(k_{T,S})$. We define $\mathcal{E}_{\mathcal{P}}(S, \mathfrak{a}, \gamma) \to \mathcal{E}_{\mathcal{P}}(T, \mathfrak{b}, \delta)$ by left-multiplication with $k_{T,S}$, and we define the evaluation of the map (29) on the thickening $(S, \mathfrak{a}, \gamma)$ to be the left-multiplication with U_S. It is clear that (28) is the result of (29) under the functor $i^*_{X/\mathbb{F}_{p^f}}$ (resp. $i^*_{X/W(\mathbb{F}_{p^f})}$).

3.5.2. *Local and global lifts.* The next result for automorphisms can be regarded as an analog of [60, Proposition 40]:

Corollary 3.26. *Let $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ be a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum, let A be an $\mathfrak{a}$-adically separated $W(\mathbb{F}_{p^f})$-algebra, and assume that one of the following holds:*

(i) $pA = 0$.
(ii) $\mathfrak{a}$ *contains a power of p and Σ is equal to $\mathbb{Z}/r\mathbb{Z}$.*

Let ϕ be an automorphism of a display $\mathcal{P}$ with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over $\operatorname{Spec} A$. *Then $\phi_{A/\mathfrak{a}}$ is the identity if and only if ϕ is the identity.*

Proof. All we have to do is prove, that $\phi_{A/\mathfrak{a}^n}$ is the identity for every n, since the diagonals of our stacks of displays with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure are separated (cf. lemma 3.18). By a straightforward induction argument it suffices to do this under the additional assumption that $\mathfrak{a}^2 = 0$. Furthermore, after passage to some affine fpqc covering we can also assume that $\mathcal{P}$ is banal. However, in this case the result follows immediately from one of the alternatives (i) or (ii) of the lemma 3.24, simply because ideals of vanishing square can be endowed with a pd-structure. □

We fix a $W(\mathbb{F}_{p^f})$-scheme X such that one of the following assumptions holds:

(i) $p\mathcal{O}_X = 0$

(ii) $X_{K(\mathbb{F}_{p^f})}$ is empty and Σ is equal to $\mathbb{Z}/r\mathbb{Z}$

We fix a nilimmersion $X_0 \hookrightarrow X$ (i.e. a closed immersion defined by a Zariski-locally nilpotent sheaf of ideals). For any $R \in Alg_X$ we let R_0 be the Alg_{X_0}-object whose spectrum is $X_0 \times_X \operatorname{Spec} R$, and we let $\mathfrak{A}(R)$ be the ideal that defines the nilimmersion $\operatorname{Spec} R_0 \hookrightarrow \operatorname{Spec} R$. Notice that $\mathfrak{A}$ is a sheaf on X_{fpqc}. By a lift of some display $\mathcal{Q}$ with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over X_0 to $R \in \mathtt{Ob}_{Alg_X}$ we mean a pair $(\mathcal{P}, \delta)$ where $\mathcal{P}$ is a display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over R, and δ is an isomorphism from $\mathcal{Q}_{R_0}$ to $\mathcal{P}_{R_0}$. Isomorphisms between lifts are expected to preserve the respective δs, in particular no lift has any automorphisms other than the identity, thanks to the previous corollary 3.26. Let $\mathcal{D}_{\mathcal{Q},X}(R)$ be the set of isomorphism classes of lifts over R. By pull-back of lifts this is a presheaf on X_{fpqc}. Let us check that $\mathcal{D}_{\mathcal{Q},X}$ satisfies the sheaf axiom for a faithfully flat map $R \to Q$: If the pull-backs $\mathrm{pr}_1^*(\mathcal{P})$ and $\mathrm{pr}_2^*(\mathcal{P})$ of some $(\mathcal{P}, \delta) \in \mathcal{D}_{\mathcal{Q},X}(Q)$ agree for the two coordinates $\mathrm{pr}_1, \mathrm{pr}_2 : Q \to Q \otimes_R Q$, then this means that there exists $\alpha : \mathrm{pr}_1^*(\mathcal{P}) \overset{\cong}{\to} \mathrm{pr}_2^*(\mathcal{P})$ which restricts to the identity on $Q_0 \otimes_{R_0} Q_0$. We easily deduce the cocycle condition for α, because any equality of isomorphisms of displays over $Q \otimes_R Q \otimes_R Q$ can be checked over $Q_0 \otimes_{R_0} Q_0 \otimes_{R_0} Q_0$, by corollary 3.26. This shows that $\mathcal{P}$ (resp. δ) descend to R (resp. R_0). It follows that $\mathcal{D}_{\mathcal{Q},X}$ is a fpqc-sheaf on X. Let $D_{\mathcal{Q},X} = \Gamma(X, \mathcal{D}_{\mathcal{Q},X})$ denote the set of isomorphism classes of global lifts.

Corollary 3.27. *Let $\mathcal{Q}$ and $X_0 \hookrightarrow X$ be as in the beginning of this subsubsection.*

(i) *$\mathcal{D}_{\mathcal{Q},X}(R)$ is non-empty for every $R \in Alg_X$*

(ii) *Suppose that the ideal sheaf defining X_0 has vanishing square. Then $\mathcal{D}_{\mathcal{Q},X}$ possesses the structure of a locally trivial principal homogeneous space for the fpqc-sheaf:*

$$R \mapsto T_{\mathcal{Q}_{R_0}} \otimes_R \mathfrak{A}(R) = \mathrm{Hom}_{R_0}(\check{T}_{\mathcal{Q}_{R_0}}, \mathfrak{A}(R)),$$

in particular $D_{\mathcal{Q},X}$ is either empty or it is a principal homogeneous space for the group $\Gamma(X, T_{\mathcal{Q}} \otimes_{\mathcal{O}_X} \mathfrak{A})$.

Proof. Fpqc-locally the existence of lifts is evident, so that the part (i) of the lemma is a straightforward consequence of [32, Chapter III, Proposition 3.7], together with part (ii). By a gluing argument we are allowed to assume that $\mathcal{Q}$ is a banal display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over an affine scheme $X = \operatorname{Spec} S$, so that it can be represented by an element $O \in \mathcal{G}^\Sigma(W(S/\mathfrak{a}))$, where $\mathfrak{a} = \mathfrak{A}(S)$, as $X_0 = \operatorname{Spec} S/\mathfrak{a}$. Indeed any lift of $\mathcal{Q}$ over any S-algebra R is going to be banal too, and more specifically it is determined by a pair (U, h) with $U \in \mathcal{G}^\Sigma(W(R))$

and $h \in \overline{\mathcal{I}}^{\mu_\Sigma}(R_0)$ (resp. $h \in \mathcal{I}^\mu(R_0)$), with $O = h^{-1}U_0\overline{\Phi}(h)$ (resp. $O = h^{-1}U_0\Phi^\mu(h)$), where $U_0 \in \mathcal{G}^\Sigma(W(R_0))$ stands for the $\mod \mathfrak{a}$-reduction of U. Another such pair (U', h') determines the same element in $\mathcal{D}_{\mathcal{Q},X}(R)$ if and only if $U' = k^{-1}U\overline{\Phi}(k)$ (resp. $U' = k^{-1}U\Phi^\mu(k)$) holds for some $k \in \overline{\mathcal{I}}^{\mu_\Sigma}(R)$ (resp. $k \in \mathcal{I}^\mu(R)$) with $k_0h' = h$ (where k_0 stands for the $\mod \mathfrak{a}$-reduction of k). Recall that there are natural inclusions

$$\mathfrak{a}R \otimes_{W(\mathbb{F}_{p^r})} \bigoplus_{i=0}^{\infty} {}^{F^i}\mathfrak{g}^\Sigma \subset \mathcal{G}^\Sigma(W(\mathfrak{a}R)) \subset \mathcal{J}^{\mu_\Sigma}_{\mathfrak{a}R} \subset \mathcal{G}^\Sigma(W(R)).$$

It is now evident that for $N \in \mathfrak{a}R \otimes_{W(\mathbb{F}_{p^r})} \bigoplus_{i=0}^{\infty} {}^{F^i}\mathfrak{g}^\Sigma$ (resp. $N \in \mathfrak{a}R \otimes_{W(\mathbb{F}_p)} \mathfrak{g}$) the assignment $(U, h) \mapsto ((\mathrm{Ad}^{\mathcal{G}^\Sigma}(h)N)U, h)$ defines an action on $\mathcal{D}_{\mathcal{Q},X}(R)$, which is transitive by lemma 3.24. Furthermore all isotropy groups are equal to

$$\mathfrak{a}R \otimes_{W(\mathbb{F}_{p^f})} \bigoplus_{i=0}^{\infty} {}^{F^i}\mathrm{Fil}^{-i}_{\mathrm{Ad}^{\mathcal{G}^\Sigma} \circ \mu_\Sigma^{-1}}(W(\mathbb{F}_{p^f}) \otimes_{W(\mathbb{F}_{p^r})} \mathfrak{g}^\Sigma)$$

(resp. $\mathfrak{a}R \otimes_{W(\mathbb{F}_{p^f})} \mathrm{Fil}^0_{\mathrm{Ad}^{\mathfrak{G}} \circ \mu^{-1}}(W(\mathbb{F}_{p^f}) \otimes_{\mathbb{Z}_p} \mathfrak{g})$) again from lemma 3.24. □

Corollary 3.28. *Suppose that A is a sub-$W(\mathbb{F}_{p^f})$-algebra of B and that $\mathfrak{a}$ is an ideal of A containing some power of p. Assume that one of the following holds:*

(i) *$\mathfrak{a}$ is nilpotent.*
(ii) *B is finite over A, and A is a $\mathfrak{a}$-adically separated and complete noetherian ring.*

Then for each pair $\mathcal{Q}$ and $\mathcal{P}$ of displays over $\operatorname{Spec} A$ *with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure every two compatible isomorphisms $\mathcal{Q}_B \xrightarrow{\psi} \mathcal{P}_B$ and $\mathcal{Q}_{A/\mathfrak{a}} \xrightarrow{\overline{\phi}} \mathcal{P}_{A/\mathfrak{a}}$ are scalar extensions of some unique $\mathcal{Q} \xrightarrow{\phi} \mathcal{P}$.*

Proof. Due to the lemma 3.18 and the Artin-Rees lemma it is enough to prove the first assertion.
However, under the assumption of (i) we can use the map $\overline{\phi}$ in order to view $\mathcal{P}$ as a lift of $\mathcal{Q}_0 := \mathcal{Q}_{A/\mathfrak{a}}$. In fact it does no harm to assume $\mathfrak{a}^2 + p\mathfrak{a} = 0$, so that there exists a well-defined element $N \in T_{\mathcal{Q}_0} \otimes_{A/\mathfrak{a}} \mathfrak{a}$ that measures the difference between the elements $\mathcal{P}$ and $\mathcal{Q}$ of $D_{\mathcal{Q}_0,\operatorname{Spec} A}$. Its image in $T_{\mathcal{Q}_0} \otimes_{A/\mathfrak{a}} \mathfrak{a}B$ has to vanish, according to the existence of ψ. Now the injectivity of $T_{\mathcal{Q}_0} \otimes_{A/\mathfrak{a}} \mathfrak{a} \to T_{\mathcal{Q}_0} \otimes_{A/\mathfrak{a}} \mathfrak{a}B$ yields $N = 0$. □

Corollary 3.29. *Let K be a perfect field containing $\mathbb{F}_{p^f}$ and let $\mathcal{Q}$ be a display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over K. Let $Comp_K$ be the category of complete local noetherian K-algebras whose residue field is K. Let $\hat{D}_{\mathcal{Q}} : Comp_K \to Set$ be the functor sending any complete local* *noetherian K-algebra R whose residue field is K to the set of isomorphism classes of pairs $(\mathcal{P}, \delta)$ where $\mathcal{P}$ is a $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over* $\operatorname{Spec} R$*, and δ is an isomorphism from $\mathcal{Q}$ to $\mathcal{P}_K$. Then $\hat{D}_{\mathcal{Q}}$ is representable by a $3n$-display $\mathcal{Q}_{uni}$ with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over a powerseries algebra R_{uni} over K whose the Zariski tangent space is $T_{\mathcal{Q}}$.*

Proof. Consider the K-algebra $R_1 = K \oplus \check{T}_{\mathcal{Q}}$, and let $\mathcal{Q}_1 \in D_{\mathcal{Q},\operatorname{Spec} R_1}$ be the deformation whose difference to the trivial deformation is measured by the identity. If $\mathcal{Q}$ is banal, then $\mathcal{Q}_1$ is banal too, and it is clear that one obtains a universal deformation from a lift of $\mathcal{Q}_1$ to a powerseries algebra with the same Zariski tangent space.
In the general case we still have the prorepresentability of $\hat{D}_{\mathcal{Q}}$ over a canonical powerseries algebra R_{uni}. There exists a finite Galois extension l of K such that $\mathcal{Q}_l$ is banal, according to part (ii) of lemma 3.19. This completes the proof because one can use étale descent along the morphism $R_{uni} \to l \otimes_K R_{uni}$. □

3.6. **Windows and Cartier's Diagonal.** Fix a torsionfree, p-adically complete and separated commutative ring A. In this case the ghost map $W(A) \to A^{\mathbb{N}_0}$ is injective. Recall also that for any $s \in \mathbb{N}$ any two Witt vectors $x, x' \in W(A)$ have the same image in $W(A/p^sA)$ if and only if the components of their respective ghost images $w, w' \in A^{\mathbb{N}_0}$ satisfy $w_i \equiv w_i' \pmod{p^{i+s}}$, cf. [60, Lemma 4]. Now let $\tau : A \to A$ be a lift of the Frobenius endomorphism, and consider Cartier's associated diagonal homomorphism $\hat{\delta} : A \to W(A)$, the composition of which with the nth ghost coordinate $w_n : W(A) \to A$ yields the nth iterate of τ. Let us point out two more related facts:

- If we continue to denote the coordinatewise action of τ on $W(A)$ by τ, then the subring $\{w \in W(A) | F(w) = \tau(w)\}$ coincides with the image of $\hat{\delta}$.
- The image of the ghost map $W(A) \hookrightarrow A^{\mathbb{N}_0}$ agrees with the subring which is defined by the condition $w_n \equiv \tau(w_{n-1}) \mod p^n$ for all n.

Finally, observe that $F(x) \equiv \tau(x) \mod W(pA)$ holds for all $x \in W(A)$. Recall that the definition of frame from [59, Definition 1], we need the following lemma:

Lemma 3.30. *Let $J \subset A$ be a pd-ideal containing p, and recall that $W(J)+I(A)$ is a pd-ideal of $W(A)$. If $\hat{\delta}$ is as above, and if $l_J : A \to \mathbb{N}_0$ is as in definition 3.4, then $l_J = l_{W(J)+I(A)} \circ \hat{\delta}$ holds.*

For the rest of this section we fix a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum, $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$, we assume that A is a $W(\mathbb{F}_{p^f})$-algebra, furthermore we request that one of the following two assumptions must be in force:

(i) $\Sigma=\mathbb{Z}/r\mathbb{Z}$ and J is p-adically closed and topologically nilpotent in A

(ii) J is p-adically topologically nilpotent and contains p

In the next two subsubsections we introduce windows with $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$-structure and study their groups of self-isogenies. One of the aims is to prove an analog of the main result in [59], but before we do that we need another definition and another lemma:

Definition 3.31. *Consider the valuation $\tilde{l}_J:A\to\mathbb{N}_0$, that is given by v_J in case (i) holds, and that is given by the J-length l_J in case (ii) holds, and let Γ be the group $\hat{U}^0_{\mu_\Sigma^{-1}}(A,\tilde{l}_J)=\prod_{\sigma\in\Sigma}\hat{U}^0_{\mu_\sigma^{-1}}(A,\tilde{l}_J)$.*

- *Following the language of example 3.2, we define $\hat{B}_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$ to be the groupoid corresponding to the diagram:*
$$\mathcal{G}^\Sigma(A)\supset\Gamma\overset{\hat{\Phi}_A}{\to}\mathcal{G}^\Sigma(A),$$
where $\hat{\Phi}_A$ is the composition of $\prod_{\sigma\in\Sigma}\hat{\Phi}_A^{\mu_\sigma,r_\Sigma(\sigma)}$ with the cyclic permutation $(U_0,U_1,\dots,U_{z-1})\mapsto(U_1,\dots,U_{z-1},U_0)$ (as in (15)).
- *We define $\mathrm{Fib}_{A[\frac{1}{p}],\tau}:\hat{B}_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\to\mathbf{B}_{A[\frac{1}{p}],\tau}(\mathfrak{G})$ (resp. $\mathrm{Fib}_{A,\tau}:\hat{B}_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\to\mathbf{B}_{A,\tau}(\mathfrak{G})$ to be the functor described by the pair (γ,m) (again in the sense of 3.2) with*
$$\gamma:\Gamma\to\mathfrak{G}(A);\{k_\sigma\}_{\sigma\in\Sigma}\mapsto\{\tau^{\mathbf{d}^+_\Sigma(\sigma)}(k_{\mathbf{d}_\Sigma(\sigma)})\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}$$
and $m(U)=U\prod_{\omega\in\Sigma}{}^{F^{\varpi^+_\Sigma(\omega)}}\mu_{\varpi_\Sigma(\omega)}(\frac{1}{p})$.
- *Consider $\hat{B}_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$-objects $U',U\in\mathcal{G}^\Sigma(A)$. In case (i) holds (i.e. $\Sigma=\mathbb{Z}/r\mathbb{Z}$) we define an isogeny from U' to U to be an element $k\in\hat{U}^0_{\mu^{-1}}(A[\frac{1}{p}],v_{\mathbb{Q}\otimes J})$ satisfying $U'=k^{-1}U\hat{\Phi}^{\mu,1}_{A[\frac{1}{p}]}(k)$. In case (ii) holds (i.e. $p\in J$) we define an isogeny from U' to U to be a $\mathbf{B}_{A[\frac{1}{p}],\tau}(\mathfrak{G})$-isomorphism from $\mathrm{Fib}_{A[\frac{1}{p}],\tau}(U')$ to $\mathrm{Fib}_{A[\frac{1}{p}],\tau}(U)$. In both cases we write $\mathrm{Hom}^0(U',U)$, for the set of isogenies $k:U'\dashrightarrow U$ from U' to U.*

We also write $\hat{B}'_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$ for the full subcategory of $\hat{B}_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$ consisting of those objects which fulfill the (mod p)*-nilpotence condition in the sense of definition 3.23.*

Note the inclusion $\mathrm{Hom}^0(U',U)\subset\mathrm{Hom}(\mathrm{Fib}_{A[\frac{1}{p}],\tau}(U'),\mathrm{Fib}_{A[\frac{1}{p}],\tau}(U))$ (resp. $\mathrm{Hom}(U',U)\subset\mathrm{Hom}(\mathrm{Fib}_{A,\tau}(U'),\mathrm{Fib}_{A,\tau}(U))$) in both cases (i) and (ii), in particular $\mathrm{Hom}^0(U,U)=\mathrm{Aut}^0(U)$ is a subgroup of $\mathfrak{G}(A[\frac{1}{p}])$ and $\mathrm{Aut}(U)\subset\mathfrak{G}(A)$. In the sequel we write d for the $\mathcal{G}^\Sigma(K(\mathbb{F}_{p^f}))$-element $\prod_{\omega\in\Sigma}{}^{F^{\varpi^+_\Sigma(\omega)}}\mu_{\varpi_\Sigma(\omega)}(\frac{1}{p})$. The lemma 3.24 entails an immediate equivalence:

$$\hat{B}'_{W(A),W(J)+I(A)}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})\stackrel{\cong}{\to}B'_{A/J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma}),$$

by reduction mod J. In fact an even nicer statement holds:

Lemma 3.32. *Assume that (i) holds, so that $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ is a Φ-datum: The canonical functor*

$$\hat{B}'_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\stackrel{\hat{\delta}}{\to}\hat{B}'_{W(A),W(J)+I(A)}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \tag{30}$$

which is induced from Cartier's diagonal map $\hat{\delta}:A\to W(A)$ is an equivalence of categories.

Proof. The faithfulness of the functor (30) is clear, so we begin with the full faithfulness, and fix some $O,U\in\mathfrak{G}(A)$ satisfying the nilpotence condition along with some $k\in\hat{U}^0_{\mu^{-1}}(W(A),\upsilon_{W(J)+I(A)})$ satisfying $\hat{\delta}(O)=k^{-1}\hat{\delta}(U)\hat{\Phi}^{\mu,1}_{W(A)}(k)$. Due to the above lemma 3.30 we have $\hat{U}^0_{\mu^{-1}}(A,\upsilon_J)=\hat{\delta}^{-1}(\hat{U}^0_{\mu^{-1}}(W(A),\upsilon_{W(J)+I(A)}))$, so we only have to prove that k lies in the image of $\hat{\delta}$, which in turn has nothing to do with the choice of J. We proceed by envoking the natural functor

$$\hat{B}'_{W(A),W(J_0)+I(A)}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\to\hat{B}'_{W(A),W(J_0)+I(A)}(\mathcal{G},\{{}^F\mu_{\sigma+1}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}),$$

defined by the Frobenius: Both of $O':=\tau(O)=F(O)$ and $U':=\tau(U)=F(U)$ are elements of $\mathfrak{G}(A)$, and $\tau(k)$ and $F(k)$ are both 2-isomorphisms from O' to U'. Since one has evidently $\tau(k)\equiv F(k)\pmod p$ it follows from the alternative (iii) of lemma 3.24 that $\tau(k)=F(k)$ holds, which means that k lies indeed in the image of $\hat{\delta}$.
The determination of the essential image runs quite similarly: Given that we have a 2-commutative diagram

$$\begin{array}{ccc}\hat{B}'_{A,J}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \stackrel{\hat{\delta}}{\longrightarrow} & \hat{B}'_{W(A),W(J)+I(A)}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\\ \uparrow & & \uparrow\\ \hat{B}'_{A,J\cap pA}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \stackrel{\hat{\delta}}{\longrightarrow} & \hat{B}'_{W(A),W(J\cap pA)+I(A)}(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\end{array}$$

we can assume $J\subset pA$. Let us begin with the special case $J=pA$. Any $U\in\mathfrak{G}(W(A))$ satisfies automatically $\tau(U)\equiv F(U)\pmod p$ we find that a nilpotence condition on U implies the existence of some $k\in\mathfrak{G}(W(pA))$ with $\tau(U)=k^{-1}F(U)\hat{\Phi}^{\mu,1}_{W(A)}(k)$, again by the alternative

(iii) of the lemma 3.24. It is easy to see that there exists an element $h \in \mathfrak{G}(I(A)) \subset \hat{U}^0_{\mu^{-1}}(W(A), v_{W(J)+I(A)})$ with $F(h)\tau(h)^{-1} = k$. Now it follows that $h^{-1}U\hat{\Phi}^{\mu,1}_{W(A)}(h)$ lies in the image of $\hat{\delta}$. The case of arbitrary $J \subset pA$ can be covered by the following steps: Find an element $U' \equiv U \pmod p$ lying in the image of $\hat{\delta}$, find an element $k' \in \mathfrak{G}(W(pA))$ with $U' = k'^{-1}U\hat{\Phi}^{\mu,1}_{W(A)}(k')$, and notice that k' can be written as a product of an element of $\hat{U}^0_{\mu^{-1}}(W(A), v_{I(A)})$ with an element in the image of $\hat{\delta}$. □

We conclude this subsection with the following final result on windows:

Lemma 3.33. *Assume that (A, pA, τ) is a frame over $W(\mathbb{F}_{p^f})$, and that A/pA is reduced. Let $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ be as above, and let $\mathfrak{G} := \mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G}$ and its cocharacter $\mu : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to \mathfrak{G}_{W(\mathbb{F}_{p^f})}$ be as in subsection 3.2. Then the pair of functions (γ, m) given by:*

$$\gamma : \hat{U}^0_{\mu_\Sigma^{-1}}(A, \hat{l}_A) \to \mathfrak{G}(A);$$

$$\{k_\sigma\}_{\sigma\in\Sigma} \mapsto \{\tau^{\mathbf{d}^+_\Sigma(\sigma)}(\mu_{\mathbf{d}_\Sigma(\sigma)}(\frac{1}{p})k_{\mathbf{d}_\Sigma(\sigma)}\mu_{\mathbf{d}_\Sigma(\sigma)}(p))\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}$$

together with $m : \mathcal{G}^\Sigma(A) \to \mathfrak{G}(A[\frac{1}{p}]); U \mapsto \mu(\frac{1}{p})U$ defines a fully faithful functor

$$\hat{\mathbf{h}}_\mu : \hat{B}_{A,pA}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \to \mathbf{B}_{A,\tau}(\mathfrak{G}),$$

(of categories in the sense of example 3.2). We let $\hat{\mathbf{h}}^0_\mu$ be the precomposition of $\hat{\mathbf{h}}_\mu$ with the forgetful functor $\mathbf{B}_{A,\tau}(\mathfrak{G}) \to \mathbf{B}_{A[\frac{1}{p}],\tau}(\mathfrak{G})$.

Proof. This follows from lemma 3.5. □

Remark 3.34. Since the fully faithful functors $\hat{\mathbf{h}}_\mu$ and $\mathbf{h}_\mu$ have the same essential image, we obtain an equivalence between $\hat{B}_{A,pA}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and $\mathbf{B}_{A,\tau}(\mathfrak{G}, \mu)$, whenever A/pA is reduced.

3.7. **Newton points and their specializations.** Let k be a perfect field of characteristic p. We have to recall and apply some of the key-concepts in [45]: On the set of $K(k)$-valued points of a reductive algebraic $\mathbb{Q}_p$-group G one defines the important equivalence relation of F-conjugacy by requiring that $b' \sim b$ if and only if $b' = g^{-1}b^Fg$, for some $K(k)$-valued point g, write $B_k(G) := G(K(k))/\sim$ for the set of F-conjugacy classes, and $B(G) := B_{\mathbb{F}_p^{ac}}(G)$. If k is algebraically closed we have $B_k(G) = B(G)$ by [45, Lemma 1.3], and the same independence result is valid for the set of Newton points that we introduce next:

Write $\mathbf{D}$ for the pro-algebraic torus whose character group is $\mathbb{Q}$ and put

$$\mathcal{N}(G) := (G(K(k))\backslash \operatorname{Hom}(\mathbf{D}_{K(k)}, G_{K(k)}))^{<F>},$$

where the left $G(K(k))$-action is defined by composition with interior automorphisms, and where the action of the infinite cyclic group $< F >$ is defined by $\nu \mapsto {}^F\nu$. Every element $b \in G(K(k))$ gives rise to an interesting representable $\operatorname{Spec}\mathbb{Q}_p$-group

(31)

$$\mathtt{Ob}_{Alg_{\operatorname{Spec}\mathbb{Q}_p}} \ni R \mapsto J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K(k)) | (\mathrm{id}_R \otimes F)g = b^{-1}gb\},$$

and to the so-called slope homomorphism $\nu_b : \mathbf{D}_{K(k)} \to G_{K(k)}$, for which we refer the reader to loc.cit. Its formation is canonical in the sense that $\operatorname{Int}^G(g/K(k))^{-1} \circ \nu_b = \nu_{g^{-1}b^Fg}$, and the centralizer of ν_b is equal to $J_{b,K(k)}$. Consequently, one is in a position to introduce the Newton-map:

$$\overline{\nu} : B(G) \to \mathcal{N}(G); \overline{b} \mapsto \overline{\nu}(\overline{b}) := \overline{\nu_b},$$

here notice that the fractionary cocharacters ${}^F\nu_b$ and ν_b are lying in the same conjugacy class.

Let Ω be the Weyl group of a Levi section T of a $\mathbb{Q}_p^{ac}$-rational Borel group $B \subset G_{\mathbb{Q}_p^{ac}}$. The group $\Omega \rtimes \operatorname{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)$ acts on the associated lattice of cocharacters $X_*(T)$ from the left, and there are natural inclusions

$$\mathcal{N}(G) \subset (\Omega\backslash X_*(T)_{\mathbb{Q}})^{\operatorname{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)} \subset \Omega\backslash X_*(T)_{\mathbb{Q}}.$$

Moreover, the set $\Omega\backslash X_*(T)_{\mathbb{R}}$ is partially ordered in a very natural way, as one puts $\Omega x \prec \Omega x'$ if and only if x lies in the convex hull of the Ω orbit of x', cf. [45, Lemma 2.2(i)]. The same notation is used for the partial orders on $\mathcal{N}(G)$ and $B(G)$ which are immediately inherited via the Newton map.

Now suppose that G is unramified, so that one can fix a reductive $\mathbb{Z}_p$-model $\mathfrak{G}$. It does no harm to assume that T is the generic fiber of the normalizer $\mathfrak{T}$ of some maximal split torus of $\mathfrak{G}$, and that B is $\mathbb{Q}_p$-rational. Let us fix a $\mathbf{B}_{W(k),F}(\mathfrak{G})$-object, i.e. a $K(k)$-valued element b of G, where k is assumed to be an algebraically closed extension of $\mathbb{F}_{p^f}$.

- The cocharacter $\mu : \mathbb{G}_{m,\mathbb{Q}_p^{ac}} \to T$ is called the Hodge point of b if and only if it is dominant with respect to B and $b \in \mathfrak{G}(W(k))\mu(p)\mathfrak{G}(W(k))$ holds.
- The $\mathbf{B}_{W(k),F}(\mathfrak{G})$-object, represented by the element b is called $\mathfrak{G}$-ordinary if and only if ν_b lies in the conjugacy class of the average $\overline{\mu}$ of μ over the group $\operatorname{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)$ (the action of which factors through $\operatorname{Gal}(\mathbb{F}_p^{ac}/\mathbb{F}_p)$).

Observe that in the situation above we have $\nu_b \prec \overline{\mu}$ under no condition whatsoever, simply by Mazur's inequality ([45, Theorem 4.2(ii)]). Thus b is $\mathfrak{G}$-ordinary if and only if $\overline{\mu} \prec \nu_b$ holds. An element of $B_k(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ is called $\mathcal{G}$-ordinary if and only if its image under the functor $\hat{\mathbf{h}}_\mu$ is $\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G}$-ordinary. For the purposes of this paper it is necessary to translate the outcome of [56] to our fpqc-stack $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$, see also [4]:

Corollary 3.35. *Let k be a perfect field containing $\mathbb{F}_{p^f}$, and let $\mathcal{P}$ be a display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over k. Let $\mathcal{P}_{uni}/k[[t_1, \dots, t_d]]$, be its universal formal equicharacteristic deformation. Then the geometric generic fiber of $\mathcal{P}_{uni}$ is $\mathcal{G}$-ordinary.*

Proof. We prove this in several steps:

Step 1. There exists a $\mathcal{G}$-ordinary $\mathbb{F}_{p^f}$-valued point $\mathcal{P}'$ of $\overline{\mathcal{B}}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$: This follows immediately from subsubsection 3.3.2.

Step 2. For a suitable Galois extension l of k, we claim that there exists some $3n$-display $\tilde{\mathcal{P}}$ with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over some open neighbourhood $U \subset \mathrm{Spec}\, l[t]$ of $\{0, 1\}$, that specializes to $\mathcal{P}_l$ and $\mathcal{P}'_l$ in the points 0 and 1: In order to check this pick an étale map v from $\mathrm{Spec}\, W(\mathbb{F}_{p^r})[x_1, \dots, x_n, g^{-1}]$ to $\mathcal{G}^\Sigma$, where g is a polynomial which is not contained in $pW(\mathbb{F}_{p^r})[x_1, \dots, x_n]$. Consider the polynomial $f(x_1, \dots, x_{2n}) := g(x_1, \dots, x_n)g(x_{n+1}, \dots, x_{2n})$, and the map

$$u : \mathrm{Spec}\, S \to \mathcal{G}^\Sigma; (x_1, \dots, x_{2n}) \mapsto v(x_1, \dots, x_n)v(x_{n+1}, \dots, x_{2n}),$$

where $S = W(\mathbb{F}_{p^r})[x_1, \dots, x_{2n}, f^{-1}]$. The map u is smooth and surjective. Now choose representatives U and U' for $\mathcal{P}$ and $\mathcal{P}'$. Over a suitable extension of the field, these can be lifted to S, in order to achieve $U = u(x_1, \dots, x_{2n})$, and $U' = u(x'_1, \dots, x'_{2n})$, where $x_1, \dots, x_{2n}, x'_1, \dots, x'_{2n} \in W(l)$. Put $z_i := x_i + [t](x'_i - x_i) \in W(l[t])$, and write $\overline{h} \in l[t]$ for the (mod p) reduction of the polynomial $h(t) := f(x_1 + t(x'_1 - x_1), \dots, x_{2n} + t(x'_{2n} - x_{2n})) \in W(l)[t]$. Observe that $z := (z_1, \dots, z_{2n})$ is a $W(l[t, \overline{h}^{-1}])$-valued point of S (the 0th ghost coordinate of $f(z_1, \dots, z_{2n})$ is equal to $\overline{h}$). Note $\overline{h}(0) \neq 0 \neq \overline{h}(1)$, and let $\tilde{\mathcal{P}}$ be the banal $3n$-display with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over $\mathrm{Spec}\, l[t, \overline{h}^{-1}]$ which is represented by the element $\tilde{U} := u(z) \in \mathcal{G}^\Sigma(W(l[t, \overline{h}^{-1}]))$.

Step 3. By Grothendieck's specialization theorem (cf. [45, Theorem 3.6(ii)]) the generic fiber of $\tilde{\mathcal{P}}$ is $\mathcal{G}$-ordinary as well.

Step 4. The pull-back $\tilde{\mathcal{P}}_{l[[t]]}$ descents to $\mathrm{Spec}(k + tl[[t]])$, and thus provides a classifying morphism $k[[t_1, \dots, t_d]] \to k + tl[[t]]$.

□

3.8. **Automorphisms of windows.** In this subsection we study a characteristic 0 analog of the group (31). Let N be a discretely valued complete field containing $K(\mathbb{F}_{p^f})$, and let us assume that the residue field $l = \mathcal{O}_N/\mathfrak{m}_N$ is algebraically closed. By means of Witt's diagonal $\Delta : W(l) \to W(W(l))$ one can regard $W(\mathcal{O}_N)$ as an augmented $W(l)$-algebra, we have to start with the following observation:

Corollary 3.36. *Let us write $O \in \mathcal{G}^\Sigma(W(l))$ for the $\mod \mathfrak{m}_N$-reduction of an element $\hat{O} \in \mathcal{G}^\Sigma(W(\mathcal{O}_N))$ that satisfies the $(\mathrm{mod}\ p)$-nilpotence condition (so that $\hat{O}$ and O represent objects in the window categories say $\hat{B}'_{W(\mathcal{O}_N),pW(\mathcal{O}_N)+I(\mathcal{O}_N)}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$ and $B'_l(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$). For every pair $(n_1, n_2) \in \mathbb{N}_0 \times \mathbb{Z}$, there exists an element m of $\mathfrak{G}(N)$ such that the diagram*

$$\begin{array}{ccccc}
\mathrm{Aut}^0(\hat{O}) & \xrightarrow{\mod \mathfrak{m}_N} & \mathrm{Aut}^0(O) & \longrightarrow & \mathfrak{G}(K(l)) \\
\downarrow & & & & \downarrow {\scriptstyle \mathrm{int}^{\mathfrak{G}(N)}(m)\circ F^{n_2}} \\
\mathfrak{G}(W(\mathcal{O}_N)[\frac{1}{p}]) & & \xrightarrow{\quad w_{n_1} \quad} & & \mathfrak{G}(N)
\end{array}$$

commutes.

Proof. The content of the corollary does not depend on the particular pair (n_1, n_2), this is due to any $\hat{k} \in \mathrm{Aut}^0(\hat{O})$ satisfying $w_{n_1}(\hat{O}d) = w_{n_1}(\hat{k})w_{n_1}(\hat{O}d)w_{n_1+1}(\hat{k})$ and to a similar formula relating any $k \in \mathrm{Aut}^0(O)$ to its image under F, furthermore if $\hat{O}$ is replaced by ${}^{F^f}\hat{O}$ the content of the corollary does not change either, because we have ${}^{F^f}d = d$. We conclude that we may assume $\Delta(O) \equiv \hat{O} \mod p\mathcal{O}_N$, because their images under sufficiently large iterates of the Frobenius would satisfy this. Lemma 3.24 allows us to pick a unique $h \in \mathfrak{G}(W(p\mathcal{O}_N))$ with $\Delta(Od) = h^{-1}(\hat{O}d)^F h$, in fact remark 3.25 tells us that $h\Delta(k)h^{-1} = \hat{k}$ holds whenever $k \in \mathrm{Aut}^0(O)$ is the $\mod \mathfrak{m}_N$-reduction of $\hat{k} \in \mathrm{Aut}(\hat{O})$. This implies that ${}^{F^n}k$ and $w_n(\hat{k})$ are conjugated as homomorphisms on $\mathrm{Aut}^0(\hat{O})$ and the corollary is proven. □

Remark 3.37. In general the above elements m might not be in $\mathfrak{G}(\mathcal{O}_N)$. Nevertheless, their existence implies also that the $\mod \mathfrak{m}_N$-reduction induces an injective map from $\mathrm{Aut}^0(\hat{O})$ to $\mathrm{Aut}^0(O)$, note that this is very similar to corollary 3.26.

For the last lemma in this subsection we assume $\Sigma = \mathbb{Z}/r\mathbb{Z}$, so that we are in the case (i) and $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ is a Φ-datum.

Lemma 3.38. *Consider some $\hat{B}'_{W(\mathcal{O}_N),I(\mathcal{O}_N)}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-object given by $U \in \mathcal{G}^\Sigma(W(\mathcal{O}_N))$. Then there exist*

- *a $\mathbb{Q}_p$-algebraic group $\hat{\Gamma}_U$ in which $\hat{\Gamma}_U(\mathbb{Q}_p)$ is Zariski-dense.*
- *a N-rational inclusion $j_U : \hat{\Gamma}_{U,N} \hookrightarrow U^0_{\mu^{-1},N}$*

such that j_U maps $\hat{\Gamma}_U(\mathbb{Q}_p)$ onto $w_0(\mathrm{Aut}^0(U))$. Moreover $j_U^{-1}(w_0(\mathrm{Aut}(U))$ is a compact open subgroup of $\hat{\Gamma}_U(\mathbb{Q}_p)$, and neither $(\hat{\Gamma}_U, j_U)$ nor $\mathrm{Aut}(U)$ change upon enlarging N.

Proof. The condition on the Zariski density is imposed only to achieve the uniqueness of $\hat{\Gamma}_U$, and can be ignored for proving its existence. The $\mod \mathfrak{m}_N$-reduction of U gives rise to a $B'_l(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-object $O \in \mathcal{G}^\Sigma(W(l))$. Consider the group $\mathrm{Aut}(\mathrm{Fib}_{W(\mathcal{O}_N)[\frac{1}{p}],F}(U)) = \overline{\Gamma}^0$ (of self-quasi-isogenies of the $\mod p$-reduction of U). Corollary 3.36 and the commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}^0(U) & \longrightarrow & \overline{\Gamma}^0 & \xrightarrow{\mod \mathfrak{m}_N} & \mathrm{Aut}^0(O) \\ {\scriptstyle w_0}\downarrow & & {\scriptstyle w_0}\downarrow & & \downarrow \\ U^0_{\mu^{-1}}(N) & \longrightarrow & \mathfrak{G}(N) & \xleftarrow{\mathrm{int}^{\mathfrak{G}(N)}(m)} & \mathfrak{G}(K(l)) \end{array}$$

confirm that j_U is the restriction of $\mathrm{Int}^{\mathfrak{G}}(m/N)$ to $\hat{\Gamma}_U$ being the largest $\mathbb{Q}_p$-rational subgroup of J_{Od} (cf. (31)) of which the base change to N is contained in the N-rational subgroup $U^0_{\tilde{\mu}^{-1}}$, where $\tilde{\mu} = \mathrm{Int}^{\mathfrak{G}}(m/N) \circ \mu$. The openness of $w_0(\mathrm{Aut}(U))$ in the image of j_U (i.e. $w_0(\mathrm{Aut}^0(U))$) is a little bit subtle: Let us consider the group $\overline{\Gamma}$ of automorphisms of U, when regarded as a $\hat{B}'_{W(\mathcal{O}_N),pW(\mathcal{O}_N)+I(\mathcal{O}_N)}(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-object (i.e. the group of its $\mod p$-automorphisms). We have a cartesian diagram:

$$\begin{array}{ccc} \mathrm{Aut}(U) & \longrightarrow & \overline{\Gamma} \\ {\scriptstyle w_0}\downarrow & & {\scriptstyle w_0}\downarrow \\ U^0_{\mu^{-1}}(\mathcal{O}_N) & \longrightarrow & \mathfrak{G}(\mathcal{O}_N) \end{array}$$

From the definition of J_U it follows, that it is enough to check the p-adic openness of the natural image of $\overline{\Gamma}$ in $\mathrm{Aut}(O)$, which however follows from (possibly several applications of) lemma 3.20. The independence of N is clear. □

4. Auxiliary results on GL(n)

The units of $\mathrm{Mat}(n \times n, R)$ define the $\mathbb{Z}$-group $\mathrm{GL}(n)$, on which we define the standard involution to be the automorphism

$$\mathrm{GL}(n) \overset{\cong}{\to} \mathrm{GL}(n);\ \alpha_{i,j} \mapsto \check{\alpha}_{i,j}, \tag{32}$$

where $\check{\alpha}_{i,j}$ are the entries of the inverse to the $n \times n$-matrix having the element $(-1)^{i+j}\alpha_{n-j+1,n-i+1}$ in its ith row and jth column, for $1 \leq i, j \leq n$. The image of the injection

$$C : \mathbb{G}_m \hookrightarrow \mathrm{GL}(n); a \mapsto \begin{pmatrix} a & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a \end{pmatrix}. \tag{33}$$

coincides with $\mathrm{Z}^{\mathrm{GL}(n)}$. By the $W(\mathbb{F}_{p^r})$-similarity group we mean the group representing the $W(\mathbb{F}_{p^r})$-functor:

(34)

$$\mathcal{G}(R) = \{(m, A) | m \in R^\times,\ A \in \mathrm{GL}(n, R \otimes_{W(\mathbb{F}_{p^r})} W(\mathbb{F}_{p^{2r}})),\ \bar{A} = m\check{A}\},$$

where the conjugation $^-$ stands for the R-linear extension of the rth iterate of the absolute Frobenius on $W(\mathbb{F}_{p^{2r}})$. Notice that $\mathrm{Z}^{\mathcal{G}}$ agrees with the image of the injection

$$\zeta : \mathrm{Res}_{W(\mathbb{F}_{p^{2r}})/W(\mathbb{F}_{p^r})} \mathbb{G}_{m,W(\mathbb{F}_{p^{2r}})} \hookrightarrow \mathcal{G}; a \mapsto (a\bar{a}, a), \tag{35}$$

which we will call the dilatation homomorphism. The canonical surjection

$$\chi : \mathcal{G} \twoheadrightarrow \mathbb{G}_{m,W(\mathbb{F}_{p^r})}; (m, A) \mapsto m$$

is called the multiplier character, and the canonical homomorphism

$$\rho : \mathcal{G}_{W(\mathbb{F}_{p^{2r}})} \to \mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}; (m, A) \mapsto A$$

is called the tautological representation, notice that $\chi_{W(\mathbb{F}_{p^{2r}})} \oplus \rho$ defines an isomorphism from $\mathcal{G}_{W(\mathbb{F}_{p^{2r}})}$ to $\mathbb{G}_{m,W(\mathbb{F}_{p^{2r}})} \times \mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}$ and that $\chi \otimes \check{\bar{\rho}} \cong \rho$ (and $\overline{\chi} \cong \chi$) holds. For a $\overline{\Phi}$-datum of the form $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$, we will often have to consider two associated $\overline{\Phi}$-data, namely $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}, \{\upsilon_\sigma^{(2)}\}_{\sigma\in\Sigma^{(2)}})$ and $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^{(1)}\}_{\sigma\in\Sigma})$, where $\upsilon_\sigma^{(2)} := ({}^{F^{-\sigma}}\rho)\circ\upsilon_\sigma$ and $\upsilon_\sigma^{(1)} := ({}^{F^{-\sigma}}\chi)\circ\upsilon_\sigma$. We call $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}, \{\upsilon_\sigma^{(2)}\}_{\sigma\in\Sigma^{(2)}})$ (resp. $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^{(1)}\}_{\sigma\in\Sigma})$) the tautological linear (resp. multiplicative) $\overline{\Phi}$-datum of $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$, notice that giving a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum for the standard unitary group $\mathcal{G}/W(\mathbb{F}_{p^r})$ is actually equivalent to giving $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}, \{\upsilon_\sigma^{(2)}\}_{\sigma\in\Sigma^{(2)}})$, subject to the condition that $\frac{\check{\upsilon}_{\sigma+r}^{(2)}}{\upsilon_\sigma^{(2)}}$ be a homothety (namely $\upsilon_\sigma^{(1)}$). Here are some $\overline{\Phi}$-data, which will acquire a special importance in this paper:

Definition 4.1. *Let r be a divisor of f.*

(i) *The pair $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ is called a standard linear (resp. multiplicative) $\overline{\Phi}$-datum if the weights of υ_σ are contained in the interval $[0, r_\Sigma(\sigma)]$ (resp. are equal to $r_\Sigma(\sigma)$).*

(ii) *Two pairs* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ *and* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon^t_\sigma\}_{\sigma\in\Sigma})$ *of standard linear* $\overline{\Phi}$*-data are Cartier duals of each other if and only if*

$$\upsilon^t_\sigma = \delta_\sigma \check{\upsilon}_\sigma$$

holds for all $\sigma \in \Sigma$*, where* $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\delta_\sigma\}_{\sigma\in\Sigma})$ *is the standard multiplicative* $\overline{\Phi}$*-datum.*

(iii) *The pair* $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ *is called a standard unitary* $\overline{\Phi}$*-datum if,* $\mathcal{G}$ *is the* $W(\mathbb{F}_{p^r})$*-similarity group (34),* $2r$ *is a divisor of* f*, and the aforementioned tautological linear and multiplicative* $\overline{\Phi}$*-data* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^{2r}})}, \{\upsilon^{(2)}_\sigma\}_{\sigma\in\Sigma^{(2)}})$ *and* $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\upsilon^{(1)}_\sigma\}_{\sigma\in\Sigma})$ *are standard linear and multiplicative ones.*

4.1. **The functor** $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$**.** The purpose of this subsection is to give a more conceptual approach to [5, 2.2.1], by using the fpqc-stacks $\overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ of our defnition 3.16.

Theorem 4.2. *Fix integers* $l_1 \geq \cdots \geq l_n$*, let* $h := l_1 - l_n$*. Consider the cocharacter* $\upsilon : \mathbb{G}_m \to \mathrm{GL}(n)$ *given by*

$$z \mapsto \begin{pmatrix} z^{l_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z^{l_n} \end{pmatrix}.$$

Then the group of R*-valued points of* $\overline{\mathcal{I}}^\upsilon$ *consists of all matrices* $A = (a_{i,j})_{1\leq i,j\leq n}$ *with* $a_{i,j} \in I_{\max\{l_i - l_j, 0\}}(R)$*, and for every such* $A \in \overline{\mathcal{I}}^\upsilon(R) = \hat{U}^0_{\upsilon^{-1}}(W(R), \overline{\upsilon}_R)$ *one has* $\overline{\Phi}^{\upsilon,h}(A) = B = (b_{i,j})_{1\leq i,j\leq n} \in \mathrm{GL}(n, W(R))$ *with*

$$b_{i,j} = \begin{cases} {}^{F^h}a_{i,j}p^{l_j - l_i} & l_i \leq l_j \\ {}^{F^{h+l_j-l_i}V^{l_j-l_i}}a_{i,j} & \text{otherwise} \end{cases}.$$

Furthermore, if

$$\upsilon' : \mathbb{G}_m \to \mathrm{GL}(n); z \mapsto \begin{pmatrix} z^{l'_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z^{l'_n} \end{pmatrix}$$

is another cocharacter with $l'_1 \geq \cdots \geq l'_n$ *and* $h' := l'_1 - l'_n$*, then* $\overline{\Phi}^{\upsilon,h}(\overline{\mathcal{I}}^{\upsilon\upsilon'}(R)) \subset \overline{\mathcal{I}}^{\upsilon'}(R)$ *and* $\overline{\Phi}^{\upsilon',h'}(\overline{\Phi}^{\upsilon,h}(A)) = \overline{\Phi}^{\upsilon\upsilon',h+h'}(A)$ *holds for every* $A \in \overline{\mathcal{I}}^{\upsilon\upsilon'}(R)$*.*

Proof. One can reduce all statements to the special case of reduced base $\mathbb{F}_p$-algebras R, in which case they follow from the equation ${}^{F^h}(\upsilon(\frac{1}{p})A\upsilon(p)) = \overline{\Phi}^{\upsilon,h}(A)$, which holds in $\mathrm{GL}(n, W(R)[\frac{1}{p}])$. □

4.1.1. *Schematicness of* $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$. Let r be a divisor of f, and let $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$ be a standard linear $\overline{\Phi}$-datum. Let $\mathbf{j}:\mathbb{Z}/r\mathbb{Z}\to\mathbb{N}_0$ be a function with

(i) $0\leq\mathbf{j}(\sigma)\leq r_\Sigma(\mathbf{d}_\Sigma(\sigma))-1$ for every σ, and
(ii) the assignment $\mathbb{Z}/r\mathbb{Z}\ni\sigma\mapsto(\mathbf{d}_\Sigma(\sigma),\mathbf{j}(\sigma))$ is injective.

Define new cocharacters by means of:

$$H_0\Big(\frac{{}^{F^{\mathbf{d}_\Sigma^+(\sigma)}}\upsilon_{\mathbf{d}_\Sigma(\sigma)}}{C^{\mathbf{j}(\sigma)}}\Big)=:\tilde{\upsilon}_\sigma:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathrm{GL}(n)_{W(\mathbb{F}_{p^f})}\tag{36}$$

Note that $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ is a standard linear Φ-datum, as the weights of $\tilde{\upsilon}_\omega$ are contained in $\{0,1\}$, while

$$\upsilon_\sigma=\prod_{\mathbf{d}_\Sigma(\omega)=\sigma}{}^{F^{-\mathbf{d}_\Sigma^+(\omega)}}\tilde{\upsilon}_\omega\tag{37}$$

holds. In this section we are going to establish a certain 2-commutative diagram, namely:

$$\begin{array}{ccc}
{}^W\mathrm{GL}(n)^\Sigma_{W(\mathbb{F}_{p^f})} & \xrightarrow{m} & {}^W\mathrm{GL}(n)^r_{W(\mathbb{F}_{p^f})}\\
\Big\downarrow{\scriptstyle \mathbf{b}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\upsilon_\sigma\}_{\sigma\in\Sigma})} & & \Big\downarrow{\scriptstyle \mathbf{b}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})}\\
\overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}} & \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q}\\
\mathit{Tors}(\overline{\mathcal{I}}^{\upsilon_\Sigma}) & \xrightarrow{\mathit{Tors}(\gamma)} & \mathit{Tors}(\overline{\mathcal{I}}^{\tilde{\upsilon}})
\end{array}.$$

We follow subsection 3.3, starting with a description of $\gamma:\overline{\mathcal{I}}^{\upsilon_\Sigma}\to\overline{\mathcal{I}}^{\tilde{\upsilon}}$, to this end consider the cocharacters

$$\prod_{\omega=\sigma+1}^{\mathbf{d}_\Sigma(\sigma)}{}^{F^{-\mathbf{d}_\Sigma^+(\omega)}}\tilde{\upsilon}_\omega=:\Upsilon_\sigma:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathrm{GL}(n)_{W(\mathbb{F}_{p^f})},$$

and let $\gamma:\prod_{\sigma\in\Sigma}\overline{\mathcal{I}}^{\upsilon_\sigma}\to\prod_{\omega=0}^{r-1}\overline{\mathcal{I}}^{\tilde{\upsilon}_\omega}$ be the homomorphism whose ωth coordinate is given by the formula $\gamma_\omega(\{k_\sigma\}_{\sigma\in\Sigma})=\overline{\Phi}^{\Upsilon_\omega,\mathbf{d}_\Sigma^+(\omega)}(k_{\mathbf{d}_\Sigma(\omega)})$. Let $m:{}^W\mathrm{GL}(n)^\Sigma_{W(\mathbb{F}_{p^f})}\to{}^W\mathrm{GL}(n)^r_{W(\mathbb{F}_{p^f})}$ be the inclusion.

Proposition 4.3. *The* 1*-morphism* $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$ *is schematic, quasi-compact and separated.*

Proof. Notice that for every $\omega \in \Sigma$ we have $\mathbf{d}_\Sigma(\omega) = \omega$, $\Upsilon_\omega = 1$ and a commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{I}}^{\upsilon_\Sigma} & \xrightarrow{\gamma} & \overline{\mathcal{I}}^{\tilde{\upsilon}} \\ \uparrow & & \downarrow \\ \overline{\mathcal{I}}^{\upsilon_\omega} & \xrightarrow{\mathrm{id}} & \overline{\mathcal{I}}^{\tilde{\upsilon}_\omega} \end{array}$$

It follows that $\gamma : \overline{\mathcal{I}}^{\upsilon_\Sigma} \to \overline{\mathcal{I}}^{\tilde{\upsilon}}$ is a closed immersion, moreover the quotient of $\overline{\mathcal{I}}^{\tilde{\upsilon}}$ by the image of γ is representable by the $\mathbb{F}_{p^f}$-scheme:

$$\prod_{\omega \notin \Sigma} \overline{\mathcal{I}}^{\tilde{\upsilon}_\omega} \times \prod_{\omega \in \Sigma} \overline{\mathcal{I}}^{\tilde{\upsilon}_\omega} / \overline{\mathcal{I}}^{\upsilon_\omega}.$$

This proves that $Tors(\gamma)$ is schematic, quasicompact and separated. Lemma 3.18 implies that the same holds for the oblique map from $\overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ to $Tors(\overline{\mathcal{I}}^{\tilde{\upsilon}})$, which is obtained by composing $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$ with the projection $q : \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \to Tors(\overline{\mathcal{I}}^{\tilde{\upsilon}})$. To finish the proof one only needs to observe that the diagonal of the canonical projection q is again schematic, quasicompact and separated (in fact it is a closed immersion). □

Remark 4.4. In the standard multiplicative situation, the above functor does not depend on the choice of $\mathbf{j}$, so that it does no harm to denote it by: $\mathrm{Flex}^{\{\upsilon_\sigma\}_{\sigma\in\Sigma}} : \overline{\mathcal{B}}(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \overline{\mathcal{B}}(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{C\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$

4.1.2. *Composition with the modular character.* Later on we need to consider the character $\overline{\chi}_\sigma^{\mathbf{j}}$ arising from precomposing γ with the level-0 truncation to $\overline{\mathcal{I}}_0^{\tilde{\upsilon}}$ followed by the character $\prod_{\omega=0}^{r-1} \chi_{\tilde{\upsilon}_\omega}^{\mathrm{GL}(n)}(1, \mathrm{std})$, i.e.:

$$\begin{array}{ccc} \overline{\mathcal{I}}_0^{\tilde{\upsilon}} & \xrightarrow{\chi_{\tilde{\upsilon}}^{\mathrm{GL}(n)^r}(1,\mathrm{std})} & \mathbb{G}_{m,\mathbb{F}_{p^f}} \\ \uparrow & & \Big\uparrow{\scriptstyle \overline{\chi}_\sigma^{\mathbf{j}}} \\ \overline{\mathcal{I}}^{\tilde{\upsilon}} & \xleftarrow{\gamma|_{\overline{\mathcal{I}}^{\upsilon_\sigma}}} & \overline{\mathcal{I}}^{\upsilon_\sigma} \end{array},$$

please see subsection C.1 for the definition of $\chi_{\tilde{\upsilon}_\omega}^{\mathrm{GL}(n)}(1, \mathrm{std})$. Furthermore, it turns out that $\overline{\chi}_\sigma^{\mathbf{j}}$ factors through the $\pmod p$-reduction of a certain character

$$\prod_{l\in\mathbb{Z}} \chi_{\upsilon_\sigma}^{\mathrm{GL}(n)}(l, \mathrm{std})^{\tilde{d}_{\sigma,l}} = \chi_\sigma^{\mathbf{j}} : \mathcal{I}_0^{\upsilon_\sigma} \to \mathbb{G}_{m,W(\mathbb{F}_{p^f})},$$

in which the exponents are given by the nifty formulae:

$$\tilde{d}_{\sigma,l} := \sum_{\mathbf{d}(\omega)=\sigma, \mathbf{j}(\omega)<l} p^{\mathbf{d}^+(\omega)},$$

for every $\sigma \in \Sigma$.

4.1.3. *Compatibility with direct sums.* If $\upsilon_\sigma = \begin{pmatrix} \upsilon_\sigma^{(1)} & 0 \\ 0 & \upsilon_\sigma^{(2)} \end{pmatrix}$ is a decomposition into matrix blocks of size $n^{(1)} \times n^{(1)}, \ldots, n^{(2)} \times n^{(2)}$, with each $\upsilon_\sigma^{(i)}$ being a cocharacter of $\mathrm{GL}(n^{(i)})_{W(\mathbb{F}_{p^f})}$, then so is $\tilde{\upsilon}_\sigma = \begin{pmatrix} \tilde{\upsilon}_\sigma^{(1)} & 0 \\ 0 & \tilde{\upsilon}_\sigma^{(2)} \end{pmatrix}$, with each $\tilde{\upsilon}_\sigma^{(i)}$ being the cocharacter of $\mathrm{GL}(n^{(i)})_{W(\mathbb{F}_{p^f})}$ gotten from the family $\upsilon_\sigma^{(i)}$ by using (36). We have a canonical 2-commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \longleftarrow & \prod_{i=1}^2 \overline{\mathcal{B}}(\mathrm{GL}(n^{(i)})_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^{(i)}\}_{\sigma\in\Sigma}) \\ {\scriptstyle \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}}\Big\downarrow & & {\scriptstyle \prod_{i=1}^2 \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma^{(i)}\}_{\sigma\in\Sigma}}}\Big\downarrow \\ \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \longleftarrow & \prod_{i=1}^2 \overline{\mathcal{B}}(\mathrm{GL}(n^{(i)})_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma^{(i)}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \end{array},$$

where the horizontal arrows are induced from morphisms of $\overline{\Phi}$-data

$$((\mathrm{GL}(n^{(1)}) \times \mathrm{GL}(n^{(2)}))_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to (\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$$
$$((\mathrm{GL}(n^{(1)}) \times \mathrm{GL}(n^{(2)}))_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \to (\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}).$$

4.1.4. *Compatibility with Cartier-Duality.* Consider the homomorphism $i : \mathbb{G}_m \times \mathrm{GL}(n); (m, A) \mapsto m\check{A}$, and let $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\delta_\sigma\}_{\sigma\in\Sigma})$ be a standard multiplicative $\overline{\Phi}$-datum. Notice that i induces maps between two pairs of $\overline{\Phi}$-data, namely:

$$((\mathbb{G}_m \times \mathrm{GL}(n))_{W(\mathbb{F}_{p^r})}, \{(\delta_\sigma, \upsilon_\sigma)\}_{\sigma\in\Sigma}) \to (\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^t\}_{\sigma\in\Sigma})$$
$$((\mathbb{G}_m \times \mathrm{GL}(n))_{W(\mathbb{F}_{p^r})}, \{(C, \tilde{\upsilon}_\sigma)\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \to (\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma^t\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$$

where $\{\upsilon_\sigma^t\}_{\sigma\in\Sigma}$ and $\{\tilde{\upsilon}_\sigma^t\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}$ are the duals of $\{\upsilon_\sigma\}_{\sigma\in\Sigma}$ and $\{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}$ in the sense of part (ii) of definition 4.1. Consider the function

$$\check{\mathbf{j}}(\sigma) := r_\Sigma(\mathbf{d}_\Sigma(\sigma)) - \mathbf{j}(\sigma) - 1.$$

Since $\tilde{\upsilon}_\sigma$ is the cocharacter of $\mathrm{GL}(n)_{W(\mathbb{F}_{p^f})}$ obtained from the family υ_σ by using (36), we find that each $\tilde{\upsilon}_\sigma^t$ is obtained similarly from the family υ_σ^t, provided that one uses $\check{\mathbf{j}}$ instead of $\mathbf{j}$. We have a canonical 2-commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^t\}_{\sigma\in\Sigma}) & \longleftarrow & \overline{\mathcal{B}}((\mathbb{G}_m \times \mathrm{GL}(n))_{W(\mathbb{F}_{p^r})}, \{(\delta_\sigma, \upsilon_\sigma)\}_{\sigma\in\Sigma}) \\ {\scriptstyle \mathrm{Flex}^{\check{\mathbf{j}},\{\upsilon_\sigma^t\}_{\sigma\in\Sigma}}}\Big\downarrow & & {\scriptstyle \mathrm{Flex}^{\{\delta_\sigma\}_{\sigma\in\Sigma}} \times \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}}\Big\downarrow \\ \overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma^t\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \longleftarrow & \overline{\mathcal{B}}((\mathbb{G}_m \times \mathrm{GL}(n))_{W(\mathbb{F}_{p^r})}, \{(C, \tilde{\upsilon}_\sigma)\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \end{array},$$

where the horizontal arrows are the ones that are induced from i. At last we wish to explain the unitary version of $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$, so suppose that $2r$ is a divisor of f, that $(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$ is a standard unitary $\overline{\Phi}$-datum, and that $\mathbf{j}$ is a $\mathbb{N}_0$-valued function on $\mathbb{Z}/2r\mathbb{Z}$ with $\mathbf{j}(r+\sigma)=r_\Sigma(\mathbf{d}_\Sigma(\sigma))-\mathbf{j}(\sigma)-1$ (and (i) and (ii)). Whence it follows that there is a canonical 2-commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\chi_{W(\mathbb{F}_{p^{2r}})}\oplus\rho} & \overline{\mathcal{B}}((\mathbb{G}_m\times\mathrm{GL}(n))_{W(\mathbb{F}_{p^{2r}})},\{(\upsilon_\sigma^{(1)},\upsilon_\sigma^{(2)})\}_{\sigma\in\Sigma^{(2)}}) \\ {\scriptstyle \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}}\downarrow & & {\scriptstyle \mathrm{Flex}^{\mathbf{j},\{(\upsilon_\sigma^{(1)},\upsilon_\sigma^{(2)})\}_{\sigma\in\Sigma}}}\downarrow \\ \overline{\mathcal{B}}(\mathcal{G},\{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \xrightarrow{\chi_{W(\mathbb{F}_{p^{2r}})}\oplus\rho} & \overline{\mathcal{B}}((\mathbb{G}_m\times\mathrm{GL}(n))_{W(\mathbb{F}_{p^{2r}})},\{(\tilde{\upsilon}_\sigma^{(1)},\tilde{\upsilon}_\sigma^{(2)})\}_{\sigma\in\mathbb{Z}/2r\mathbb{Z}}) \end{array},$$

according to lemma 3.22.

4.1.5. *On the isogeny class of the functor* Flex*:* Fix a $\pmod r$-multidegree $\mathbf{d}^+$, for some $r\mid f$. As in (21), we let Σ be the set of $\pmod r$-congruence classes of elements of the form $\sigma+\mathbf{d}^+(\sigma)$. As before, we pick a monotone bijection $\mathbb{Z}\to\Sigma^{(0)}; j\mapsto\sigma_j$, and we denote $\max\{\omega\in\mathbb{Z}|\omega+\mathbf{d}^+(\omega)\le\sigma_j\}$ by ω_j. Let z be the cardinality of Σ, and let Ω be the set of $\pmod r$-congruence classes of $\omega_1<\dots<\omega_z<r+\omega_1$. Now suppose that $(\mathcal{G}/W(\mathbb{F}_{p^r}),\{\upsilon_\omega\}_{\omega\in\Omega})$ is a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum, along with:

$$\prod_{j=0}^{z-1}{}^{F^{\omega_j-\sigma_j}}\upsilon_{\omega_j}=\mu:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\prod_{\sigma=0}^{r-1}{}^{F^{-\sigma}}\mathcal{G} \tag{38}$$

(i.e. ${}^{F^{\omega_j-\sigma_j}}\upsilon_{\omega_j}$ is the σ_jth component μ_{σ_j} of μ and all other ones are trivial). For a point U of $\mathfrak{G}$ with values in any $W(\mathbb{F}_{p^f})$-algebra we let $U_0,\dots,U_{r-1}$ denote its various components, when the former is regarded as an element of $\prod_{\sigma=0}^{r-1}{}^{F^{-\sigma}}\mathcal{G}$. Let l be a perfect field containing $\mathbb{F}_{p^f}$. By slight abuse of notation we let

$$\mathrm{Flex}_l^{\mathbf{d}^+}:\mathbf{B}_{W(l),F}(\mathfrak{G},\mu)\to B_l(\mathcal{G},\{\upsilon_\omega\}_{\omega\in\Omega})\cong\mathbf{B}_{W(l),F}(\mathfrak{G},\upsilon) \tag{39}$$

be the 1-morphism introduced by the pair of z-tuples of functions $\gamma=(\dots,\gamma_j,\dots)$ and $m=(\dots,m_j,\dots)$, which are defined by

$$\gamma_j(k):={}^{F^{\sigma_j-\omega_j}}k_{\sigma_j}$$
$$m_j(U):={}^{F^{\sigma_j-\omega_j}}U_{\sigma_j}\cdot\dots\cdot{}^{F^{\sigma_{j+1}-\omega_j-1}}U_{\sigma_{j+1}-1},$$

where the order of multiplication matters and is the one indicated, the element U lies in $\mathfrak{G}(W(l))$, the element k lies in $\mathfrak{G}(W(l))\cap\mu(p)\mathfrak{G}(W(l))\mu(\frac{1}{p})$ (N.B.: we do not require that $(\mathcal{G},\{\mu_\omega\}_{\omega\in\Sigma})$ is a $\overline{\Phi}$-datum).
We begin our analysis with a sober examination of the $\mathcal{G}=\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}$-case, $\mathbf{j}$ denotes a function with properties as outlined at the beginning of subsubsection 4.1.1:

Lemma 4.5. *Let* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Omega})$ *be a standard linear* $\overline{\Phi}$*-datum, and let* μ *be as above. Then there exists a canonical* 2*-commutative diagram:*

$$\begin{array}{ccc} B_l(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\omega\}_{\omega\in\Omega}) & \xleftarrow{\mathrm{Flex}_l^{\mathbf{d}^+}} & \mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu) \\ {\scriptstyle \mathrm{Flex}^{\mathbf{j},\{\upsilon_\omega\}_{\omega\in\Omega}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{h}_\mu^0} \\ B_l(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \xrightarrow{\hat{\mathbf{h}}_{\tilde{\upsilon}}^0} & \mathbf{B}_{K(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}) \end{array}.$$

Proof. We think of $\mathbf{B}_{K(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})})$-objects as isogeny classes of F-crystals with $W(\mathbb{F}_{p^r})$-operation over $W(l)$ with eigenspaces of rank n, and we think of $\mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu)$-objects as r-tuples $U = (U_0, \dots, U_{r-1}) \in \mathrm{GL}(W(l))^r$. Let us write

$$\mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu) \xrightarrow{G} \mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})})$$

for the composition $\hat{\mathbf{h}}_{\tilde{\upsilon}} \circ \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Omega}} \circ \mathrm{Flex}^{\mathbf{d}^+}$. So we have to construct a family of quasi-isogenies, say:

$$g_U : \mathbf{h}_\mu(U) \dashrightarrow G(U),$$

associated to, and functorial in $\mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu)$-objects U. Using the oblique functor

$$\begin{aligned} &\hat{\mathbf{h}}_\upsilon : B_l(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\omega\}_{\omega\in\Omega}) \\ &\to \mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}) \end{aligned}$$

as an intermediary stepping stone, we may construct g_U in two stages, namely:

$$j_{\mathcal{P}} : \hat{\mathbf{h}}_\upsilon(\mathcal{P}) \dashrightarrow J(\mathcal{P}) \tag{40}$$

$$d_U : \mathbf{h}_\mu(U) \dashrightarrow D(U), \tag{41}$$

where J (resp. D) denotes the composition of $\hat{\mathbf{h}}_{\tilde{\upsilon}}$ (resp. $\hat{\mathbf{h}}_\upsilon$) with $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\omega\}_{\omega\in\Omega}}$ (resp. with $\mathrm{Flex}_l^{\mathbf{d}^+}$). The definition of (41) follows subsubsection (6.1.3) almost word for word, it is left to the reader. It remains to construct $j_{\mathcal{P}}$: Let $\{U_\omega\}_{\omega\in\Omega}$ represent a $B_l(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\omega\}_{\omega\in\Omega})$-object $\mathcal{P}$, and let $(B_0, \dots, B_{r-1})$ and $(\tilde{B}_0, \dots, \tilde{B}_{r-1})$ be the r-tuples of matrices that result when applying the two functors $\hat{\mathbf{h}}_\upsilon$ and J. The former looks like:

$$\begin{aligned} &(\dots, B_{\omega_{j-1}}, B_{\omega_{j-1}+1}, \dots, B_{\omega_j}, \dots) = \\ &(\dots, \upsilon_{\omega_{j-1}}(\frac{1}{p})U_{\omega_{j-1}}, 1, \dots, \upsilon_{\omega_j}(\frac{1}{p})U_{\omega_j}, \dots), \end{aligned}$$

(where the "1" stands in the $\omega_{j-1}+1$st position) and the latter looks like:

$$(\dots,\tilde{B}_{\omega_{j-1}},\tilde{B}_{\omega_{j-1}+1},\dots,\tilde{B}_{\omega_j},\dots)=$$
$$(\dots,\tilde{\upsilon}_{\omega_{j-1}}(\frac{1}{p})U_{\omega_{j-1}},\tilde{\upsilon}_{\omega_{j-1}+1}(\frac{1}{p}),\dots,\tilde{\upsilon}_{\omega_j}(\frac{1}{p})U_{\omega_j},\dots).$$

Let us define a sequence of matrices by $k_\sigma=\prod_{\omega=\omega_{j-1}+1}^{\sigma-1}{}^{F^{\omega-\sigma}}\tilde{\upsilon}_\omega(p)$ for $\sigma\in[\omega_{j-1}+1,\omega_j]$, and notice that $\upsilon_{\omega_j}=\prod_{\omega=\omega_{j-1}+1}^{\omega_j}{}^{F^{\omega-\omega_j}}\tilde{\upsilon}_\omega$ holds for all j, so that $k_\sigma B_\sigma=\tilde{B}_\sigma{}^F k_{\sigma+1}$ is true for every $\sigma\in\mathbb{Z}/r\mathbb{Z}$ and we are done. Notice that g_U is actually an isogeny. □

Example 4.6. The said quasi-isogenies are fairly complicated, so let us present an example: Let κ be the $W(\mathbb{F}_{p^f})$-rational cocharacter of the $\mathbb{Z}_p$-group $\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathbb{G}_{m,W(\mathbb{F}_{p^r})}$ corresponding to the standard multiplicative $\overline{\Phi}$-datum $(\mathbb{G}_{W(\mathbb{F}_{p^r})},\{\delta_\sigma\}_{\sigma\in\Omega})$ in the sense of (38). Let $\mathcal{T}$ be the $B_l(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})},\{\delta_\omega\}_{\omega\in\Omega})$-object arising from applying the functor $\mathrm{Flex}_l^{\mathbf{d}^+}$ to the $\mathbf{B}_{W(l)}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathbb{G}_{m,W(\mathbb{F}_{p^r})},\kappa)$-object given by $O:=(1,\dots,1)\in\mathbb{G}_m(W(l))^r$. Let $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}}K_\sigma=K\in\mathtt{Ob}_{\mathit{Cris}_l^{W(\mathbb{F}_{p^r})}}$ be the $\mathbb{Z}/r\mathbb{Z}$-graded F-crystal arising from the image of O under the functor $\mathbf{h}_\kappa$ (to thecategory $\mathbf{B}_{W(l)}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})})$). More specifically, the σ-eigenspaces of K are given by $K_\sigma=W(l)$ for all $\sigma\in\mathbb{Z}/r\mathbb{Z}$, while its graded Frobenii $F_{K,\sigma}^\sharp:\mathbb{Q}\otimes K_{\sigma+1}\to\mathbb{Q}\otimes K_\sigma$ satisfy $F_{K,\sigma}^\sharp(1)=p^{-\mathrm{Card}(\mathbf{d}^{-1}(\{\sigma\}))}$. Notice that $J(\mathcal{T})$ is $W(\mathbb{F}_{p^r})(1)$. Now g_O is just the precomposition of the canonical isogeny from K to $D(O)$, of which the effect on the σth eigenspace is given by the formula:

$$K_\sigma\ni 1\mapsto p^{\omega_{j-1}-\omega_{k-1}}\in D(O)_\sigma={}^{F^{\mathbf{d}^+(\sigma)}}K_{\mathbf{d}(\sigma)},$$

with the isogeny from $D(O)$ to $W(\mathbb{F}_{p^r})(1)$, of which the effect on the σth eigenspace is given by:

$$D(O)_\sigma\ni 1\mapsto p^{\sigma-\omega_{j-1}-1}\in W(l),$$

whenever $\sigma\in[\omega_{j-1}+1,\omega_j]\cap[\sigma_{k-1}+1,\sigma_k]$. In particular $g_O(K_\sigma)=p^{\sigma-\omega_{k-1}-1}W(l)$ holds for $\sigma\in[\sigma_{k-1}+1,\sigma_k]$.

For the remainder of this subsubsection we retain the notations in the proof of the previous lemma, and we assume $r\equiv 0 \pmod 2$ together with $\upsilon_{\frac{r}{2}+\omega}=\upsilon_\omega^t$ for all ω (in the sense of part (ii) of definition 4.1).

Notice that this implies $\frac{\mu_{\frac{r}{2}+\sigma}}{\check{\mu}_\sigma} = C^{\mathrm{Card}(\mathbf{d}^{-1}(\{\sigma\}))}$, so that there is a self-explanatory involution

$$\begin{aligned} * : &\mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p} \mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu) \\ &\to \mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p} \mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu); \\ &(U_0, \ldots, U_{r-1}) \mapsto (\check{U}_{\frac{r}{2}}, \ldots, \check{U}_{\frac{r}{2}-1}). \end{aligned}$$

We assume $\mathbf{d}^+(\omega) = \mathbf{d}^+(\frac{r}{2}+\omega)$ and $\mathrm{Card}(\mathbf{d}^{-1}(\{\mathbf{d}(\omega)\})) - \mathbf{j}(\omega) - 1 = \check{\mathbf{j}}(\omega) = \mathbf{j}(\frac{r}{2}+\omega)$ (so that we have $\tilde{v}_{\frac{r}{2}+\omega} = \tilde{v}_\omega^t$ too). For the last result of this subsubsection, note that $\mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p} \mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \mu)$-objects U give rise to functorial isomorphisms not only from $\mathbf{h}_\mu(U^*)$ to $K \otimes_{W(\mathbb{F}_{p^r})} \overline{\check{\mathbf{h}_\mu(U)}}$ but also from $G(U^*)$ to $\overline{G(U)}^t$ (as similar dualities are preserved by each of the three factors $\mathrm{Flex}_l^{\mathbf{d}^+}$, $\mathrm{Flex}_l^{\mathbf{j},\{v_\sigma\}_{\sigma\in\Omega}}$, and $\hat{\mathbf{h}}_{\tilde{v}}$ of G). Furthermore, it is straightforward to check that the aforementioned isomorphism carries the inverse of g_{U^*} to $g_O \otimes \check{\overline{g}}_U$, where O is as in example 4.6. At last fix an r-tuple $(a_0, \ldots, a_{r-1})$ satisfying

$$\forall\sigma:\ 1 - \mathrm{Card}(\mathbf{d}^{-1}(\{\sigma\})) = a_\sigma + a_{\frac{r}{2}+\sigma}$$

$$\mathrm{Card}(\mathbf{d}^{-1}(\{\frac{r}{2}, \ldots, r-1\}) \cap \{0, \ldots, \frac{r}{2}-1\}) \equiv \sum_{\sigma=0}^{\frac{r}{2}-1} a_\sigma \pmod 2,$$

and let $L \in \mathtt{Ob}_{Cris_l^{W(\mathbb{F}_{p^r})}}$ be the $\mathbb{Z}/r\mathbb{Z}$-graded F-crystal of which the σ-eigenspaces are given by $L_\sigma = W(l)$ for all $\sigma \in \mathbb{Z}/r\mathbb{Z}$, while its graded Frobenii $F_{L,\sigma}^\sharp : \mathbb{Q} \otimes L_{\sigma+1} \to \mathbb{Q} \otimes L_\sigma$ satisfy $F_{L,\sigma}^\sharp(1) = p^{-a_\sigma}$. We have $K \cong \overline{K}$, and $L \otimes_{W(\mathbb{F}_{p^r})} \overline{L}$ isomorphic to $K^t = \check{K}(1)$, furthermore the above parity implies the existence of a unique r-tuple of integers $(x_0, \ldots, x_{r-1})$ such that

$$\begin{aligned} &\forall\sigma:\ x_{\sigma+1} - x_\sigma = a_\sigma \\ &\forall\sigma \in [\sigma_{k-1}+1, \sigma_k]:\ x_\sigma + x_{\frac{r}{2}+\sigma} = \sigma - \omega_{k-1} - 1, \end{aligned}$$

which gives rise to a quasi-isogeny $h : W(\mathbb{F}_{p^r})(0) \dashrightarrow L$, with $h \otimes \overline{h} = g_O$ (this is meaningful as $L \otimes_{W(\mathbb{F}_{p^r})} \overline{L} \cong \check{K}(1)$ and g_O is an isogeny from K to $W(\mathbb{F}_{p^r})(1)$). This has two consequences for the scenario in the proof of lemma 4.5: First, we have another canonical isomorphism

$$L \otimes_{W(\mathbb{F}_{p^r})} \mathbf{h}_\mu(U^*) \cong \overline{L \otimes_{W(\mathbb{F}_{p^r})} \mathbf{h}_\mu(U)}^t,$$

second, we have another family of quasi-isogenies $g_U \circ (h^{-1} \otimes \mathrm{id}_{\mathbf{h}_\mu(U)}) =: f_U$ from $L \otimes_{W(\mathbb{F}_{p^r})} \mathbf{h}_\mu(U)$ to $G(U)$. Furthermore, it is straightforward to check that the aforementioned isomorphism carries the inverse of f_{U^*} to $\overline{f}_U^t$. In doing so we obtain the following:

Corollary 4.7. *Let* $(\mathcal{G}, \{\upsilon_\omega\}_{\omega\in\Omega})$ *be a standard unitary* $\overline{\Phi}$*-datum, and let* μ *be defined by the above equation* (38). *Then there exists a canonical* 2*-commutative diagram:*

$$\begin{array}{ccc} B_l(\mathcal{G}, \{\upsilon_\omega\}_{\omega\in\Omega}) & \xleftarrow{\mathrm{Flex}^{\mathbf{d}^+}} & \mathbf{B}_{W(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G}, \mu) \\ {\scriptstyle \mathrm{Flex}^{\mathbf{j},\{\upsilon_\omega\}_{\omega\in\Omega}}}\downarrow & & \downarrow{\scriptstyle L\otimes_{W(\mathbb{F}_{p^r})}\mathbf{h}^0_\mu} \\ B_l(\mathcal{G}_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) & \xrightarrow{\hat{\mathbf{h}}^0_{\tilde{\upsilon}}} & \mathbf{B}_{K(l),F}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G}) \end{array}.$$

4.2. **Graded** $3n$**-displays with skew-Hermitian structure.** We give a brief sketch of an example: Fix a p-adically separated and complete ring R, recall that $3n$-displays are quadruples $\mathcal{P} = (M, N, F, V^{-1})$:

- M is a finitely generated projective $W(R)$-module
- $N \subset M$ is a submodule which contains $I(R)M$
- M/N is projective, when regarded as a module over R
- $F : M \to M$ and $V^{-1} : N \to M$ are ^{F}-linear maps, satisfying $V^{-1}({}^V ax) = aF(x)$ for all $x \in M$ and $a \in W(R)$ (of which the Verschiebung is an element in $I(R)$).
- the image of V^{-1} generates M as a $W(R)$-module.

The number $\mathrm{rank}_{W(R)}(M)$ is called the height of (M, N, F, V^{-1}). A $3n$-display is called multiplicative (resp. étale) if $N = I(R)M$ (resp. $N = M$) holds. A morphism between the $3n$-displays (M', N', F, V^{-1}) and (M, N, F, V^{-1}) is a $W(R)$-linear mapping $M' \to M$ sending N' to N and such that the map V^{-1} is preserved. The dual of the $3n$-display $\mathcal{P}$ is the quadruple $\mathcal{P}^t = (\check{M}, N^\perp, F, V^{-1})$, where $\check{M} = \mathrm{Hom}_{W(R)}(M, W(R))$, $N^\perp = \{x \in \check{M} | (x, N) \subset I(R)\}$, and $V^{-1}(x, y) = (V^{-1}x, V^{-1}y)$ holds for all $x \in N$ and $y \in N^\perp$.

Definition 4.8. *Let* $\mathcal{R}$ *be a not necessarily commutative finite flat* $\mathbb{Z}_p$*-algebra. A* $3n$*-display with* $\mathcal{R}$*-operation is a pair* $(\mathcal{P}, \iota)$ *where* $\mathcal{P}$ *is a display, and where* $\iota : \mathcal{R} \to \mathrm{End}(\mathcal{P})$ *is a* $\mathbb{Z}_p$*-linear homomorphism.*
Fix a involution $* : \mathcal{R} \to \mathcal{R}^{op}$*. Then one defines the dual of* $(\mathcal{P}, \iota)$ *to be the pair* $(\mathcal{P}^t, \iota^t)$ *where* ι^t *is the composition* $\mathcal{R} \stackrel{*}{\to} \mathcal{R}^{op} \stackrel{\iota^{op}}{\to} \mathrm{End}(\mathcal{P})^{op} = \mathrm{End}(\mathcal{P}^t)$.
Let B *be a commutative and finite étale* $\mathbb{Z}_p$*-algebra, fix a* B*-algebra structure on* $\mathcal{R}$*, and assume that* $* : \mathcal{R} \to \mathcal{R}$ *is a* B*-linear. Then a* B*-valued skew-Hermitian form on* $(\mathcal{P}, \iota)$ *is a pair* $(\mathcal{K}, \Psi)$*, where* $\mathcal{K}$ *is a étale display of height* $\mathrm{rank}_{\mathbb{Z}_p} B$ *with* B*-operation and* $\Psi : \mathcal{P} \to \mathcal{K} \otimes_{B\otimes_{\mathbb{Z}_p} W(R)} \mathcal{P}^t$ *is a* R*-linear isomorphism such that* $\Psi = -\mathrm{id}_{\mathcal{K}} \otimes \Psi^t$ *holds.*

Let R be a $W(\mathbb{F}_{p^f})$-algebra, and let r be a divisor of f. It is folklore that $\mathcal{R} = W(\mathbb{F}_{p^r})$-operations on (M, N, F, V^{-1}) are nothing but compatible $\mathbb{Z}/r\mathbb{Z}$-gradations on N and M with respect to which V^{-1} and F are of degree -1. In fact the following theorem holds:

Theorem 4.9. *Let* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ *be a* $W(\mathbb{F}_{p^f})$*-rational standard linear* Φ*-datum, and let* R *be a* p*-adically separated and complete* $W(\mathbb{F}_{p^f})$*-algebra. Then the category of* $\mathbb{Z}/r\mathbb{Z}$*-graded* $3n$*-displays* $(\{M_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ *such that*

$$\operatorname{rank}_{W(R)}(M) = nr \tag{42}$$

$$\forall\sigma \in \mathbb{Z}/r\mathbb{Z} : \operatorname{rank}_R(M_\sigma/N_\sigma) = d_{\upsilon_\sigma}(1, \mathrm{std}) \tag{43}$$

hold is canonically equivalent to the groupoid of R*-valued points of* $\mathcal{B}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$*, furthermore, the equivalence preserves Cartier-duality.*

Proof. We may assume υ_σ to be the cocharacter

$$\mathbb{G}_m \to \mathrm{GL}(n); z \mapsto \mathrm{diag}(\overbrace{z, \dots, z}^{d_\sigma}, \overbrace{1, \dots, 1}^{n-d_\sigma}),$$

where $d_\sigma = d_{\upsilon_\sigma}(1, \mathrm{std})$. We write $\mathrm{Co}_{\underline{d},n}$ for the equivalence that we are going to exhibit: By the theory of Witt-descent it suffices to construct the restriction of $\mathrm{Co}_{\underline{d},n}$ to the subcategories $B_R(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$ of banal R-displays. Whenever some $B_R(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-object is represented by a r-tuple of matrices $(U_0, \dots, U_{r-1}) = U \in \mathrm{GL}(n, W(R))^r$ the effect of $\mathrm{Co}_{\underline{d},n}$ on it is defined as follows: Write

$$(U_0, \dots, U_{r-1}) \overset{\mathrm{Co}_{\underline{d},n}}{\mapsto} (\{M_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$$

where:

$$M_\sigma := \mathrm{Mat}(n \times 1, W(R)) = W(R)^{\oplus n}$$

$$F : M_\sigma \to M_{\sigma-1}; \begin{pmatrix} \vdots \\ x_{d_\sigma} \\ y_1 \\ \vdots \end{pmatrix} \mapsto U_{\sigma-1} \begin{pmatrix} \vdots \\ {}^F x_{d_\sigma} \\ p^F y_1 \\ \vdots \end{pmatrix}$$

$$N_\sigma := I(R)^{\oplus d_\sigma} \oplus W(R)^{\oplus n-d_\sigma}$$

$$V^{-1} : N_\sigma \to M_{\sigma-1}; \begin{pmatrix} \vdots \\ x_{d_\sigma} \\ y_1 \\ \vdots \end{pmatrix} \mapsto U_{\sigma-1} \begin{pmatrix} \vdots \\ {}^{V^{-1}} x_{d_\sigma} \\ {}^F y_1 \\ \vdots \end{pmatrix}$$

Whenever $(k_0, \dots, k_{r-1}) = k$ represents a morphism between two banal $3n$-displays which are represented by the r-tuples of matrices $k_\sigma^{-1} U_\sigma \Phi^{\upsilon_{\sigma+1}}(k_{\sigma+1}) = U'_\sigma$ and U_σ, then the effect of $\mathrm{Co}_{\underline{d},n}$ on it is defined as follows: Write k_σ as a block matrix $\begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix}$ with $B_\sigma \in \mathrm{Mat}(d_\sigma \times n - d_\sigma, I(R))$, and let $\mathrm{Co}_{\underline{d},n}(k)$ act on $\mathrm{Co}_{\underline{d},n}(U') = \bigoplus_{\sigma \in \mathbb{Z}/r\mathbb{Z}} \mathrm{Mat}(n \times 1, W(R))$ by plain matrix multiplication.

The functor $\mathrm{Co}_{\underline{d},n}$ is fully faithful. In order to determine the set of $\mathbb{Z}/r\mathbb{Z}$-graded $3n$-displays $(\{M_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ which lie in the essential image one has to observe that the ring $W(R)$ is separated and complete with respect to the $I(R)$-adic topology, so that the modules N_σ can always be written as $I(R)T_\sigma \oplus L_\sigma$ for suitable graded normal decompositions $M_\sigma = T_\sigma \oplus L_\sigma$. This allows one to consider the map

$$F \oplus V^{-1} : M_\sigma \to M_{\sigma-1}; x + y \mapsto F(x) + V^{-1}(y) \tag{44}$$

where $x \in T_\sigma$, and $y \in L_\sigma$. Following the ideas in, and using the language of [60, Lemma 9] one proves (44) to be a ^{F}-linear isomorphism. □

By slight abuse of language we say that $(\{M_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ is a display with $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}})$-structure over R if the above holds, this requires the base ring R to be a $W(\mathbb{F}_{p^f})$-algebra.

Assume that r is even, and that $*$ is the $\frac{r}{2}$th iterate of the absolute Frobenius. It is folklore that the $W(\mathbb{F}_{p^{\frac{r}{2}}})$-valued skew-Hermitian structures on a $\mathbb{Z}/r\mathbb{Z}$-graded display $(\{M_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ are given by a $\mathbb{Z}/r\mathbb{Z}$-graded multiplicative display $(\{K_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, \{I(R)K_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ together with families of perfect pairings $\Psi_\sigma : M_\sigma \times M_{\sigma+\frac{r}{2}} \to K_\sigma$ such that $N_{\sigma+\frac{r}{2}} = \{x \in M_{\sigma+\frac{r}{2}} | \Psi_\sigma(M_\sigma, x) \subset I(R)K_\sigma\}$, and $(V^{-1}x, V^{-1}y) = {}^{V^{-1}}(x, y)$ holds for all $x \in N_\sigma$ and $y \in N_{\sigma+\frac{r}{2}}$.

Corollary 4.10. *Let $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma \in \mathbb{Z}/\frac{r}{2}\mathbb{Z}})$ be a $W(\mathbb{F}_{p^f})$-rational standard unitary Φ-datum, and let R be a p-adically separated and complete $W(\mathbb{F}_{p^f})$-algebra. The category of R-valued points of $\mathcal{B}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma \in \mathbb{Z}/\frac{r}{2}\mathbb{Z}})$ is equivalent to the category of $\mathbb{Z}/r\mathbb{Z}$-graded $3n$-displays $(\{M_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ with $W(\mathbb{F}_{p^{\frac{r}{2}}})$-valued skew-Hermitian structure $(\{K_\sigma\}_{\sigma \in \mathbb{Z}/\frac{r}{2}\mathbb{Z}}, \{\Psi_\sigma\}_{\sigma \in \mathbb{Z}/\frac{r}{2}\mathbb{Z}})$ over R, where $\mathrm{rank}_{W(R)}(M)$ and $\mathrm{rank}_R(M_\sigma/N_\sigma)$ satisfy (42) and (43) (with respect to $\{\upsilon_\sigma^{(2)}\}_{\sigma \in \mathbb{Z}/r\mathbb{Z}}$).*

Remark 4.11. Readers, who are familiar with the paper [27] may raise the question of whether or not the stacks $\overline{\mathcal{B}}(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma \in \Sigma})$ of our definition 3.16 may allow a characterization as moduli for $\mathbb{Z}/r\mathbb{Z}$-graded Langer-Zink displays (of the correct rank and with the correct

Hodge numbers). If υ_σ is not minuscule the answer seems to be 'no', however one should expect that there is still a 1-morphism from the latter to the former, and a similar statement should hold under presence of additional structures.

4.3. **Deformations with constant Newton polygon.** Let k be an algebraically closed field of characteristic p, let r be a positive even integer, let $\mathcal{P} = (\{M_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ be a $\mathbb{Z}/r\mathbb{Z}$-graded display over k, and let $\{\Psi_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}$ be a skew-Hermitian structure thereon (N.B.: It does no harm to take $K_\sigma = W(k)$, so that Ψ_σ is an isomorphism from M_σ to $\check{M}_{\sigma+\frac{r}{2}}$). Let us write $s_1 \geq \cdots \geq s_n$ for the V-slopes of M, in the sense of [58], notice that $1 - s_{n-i+1} = s_i \in]0,1[$ holds, and let us write $\mathbb{Q}\otimes M_\sigma = \bigoplus_{\mathbb{Q}\ni l} M_{l,\sigma}$ for the V-slope decomposition of $\mathbb{Q}\otimes M$. One knows that for some integer $0 < s \equiv 0 \pmod r$ and suitable elements $e_{i,\sigma} \in M_\sigma$ the following assertions hold for all $\sigma \in \{0, \dots, \frac{r}{2}-1\}$, and every $i \in \{1, \dots, n\}$:

- $e_{1,\sigma}, \dots, e_{n,\sigma}$ is a $W(k)$-basis for M_σ
- $F^d(M_\sigma) \supset p^{(1-s_i)s}(M_\sigma \cap \bigoplus_{l\geq s_i} M_{l,\sigma})$
- $F^d(e_{i,\sigma}) - p^{(1-s_i)s} e_{i,\sigma} \in \bigoplus_{\mathbb{Q}\ni l>s_i} M_{l,\sigma}$
- $\Psi_\sigma(e_{i,\sigma}, e_{j,\sigma+\frac{r}{2}}) = (-1)^i \delta_{i+j,n+1}$

With respect to a choice of bases as above, we let $A_\sigma \in \mathrm{Mat}(n \times n, W(k))$ be the matrix describing the homomorphism $V^\sharp|_{M_\sigma}$ from $M_\sigma = \bigoplus_{i=1}^n W(R)e_{i,\sigma}$ to $W(R)\otimes_{F,W(R)} M_{\sigma+1} = \bigoplus_{i=1}^n W(R)\otimes e_{i,\sigma+1}$, due to the slope filtration it has the shape of an upper triangular block matrix, and the same is true for

$$^{F^{s-1}}A_{s-1} \dots {}^{F}A_1 A_0 =: A$$

moreover, one knows that A can be written as a product $A = L\alpha(p)$, where we let $\alpha : \mathbb{G}_{m,W(k)} \to \mathrm{GL}(M_0/W(k))$ stand for the cocharacter with $\alpha(z)e_{i,0} = z^{s_i s} e_{i,0}$ and where L is an element of the group $U_\alpha^1(W(k))$ (i.e. the integral upper triangular block matrices whose diagonal entries are equal to 1). From now on we assume that M has at least two different slopes, say $s_m > s_{m+1}$ for some fixed index m, and we want to pay attention to the three corresponding submatrices $B_\sigma \in \mathrm{Mat}(b \times b, W(k))$ (resp. $D_\sigma \in \mathrm{Mat}(d \times d, W(k))$ and $C_\sigma \in \mathrm{Mat}(b \times d, W(k))$), such that the family of block matrices $\begin{pmatrix} B_\sigma & C_\sigma \\ 0 & D_\sigma \end{pmatrix}$ just describe the restrictions of the homomorphism $V^\sharp$ to the subquotients

$$(M_\sigma \cap \bigoplus_{l\geq s_{m+1}} M_{l,\sigma}))/(M_\sigma \cap \bigoplus_{l>s_m} M_{l,\sigma}), \tag{45}$$

as σ varies, furthermore the same applies to A giving rise to a block matrix $\begin{pmatrix} p^{s_m s} & C \\ 0 & p^{s_{m+1} s} \end{pmatrix}$ given by

$$ {}^{F^{s-1}}B_{s-1} \dots {}^{F}B_1 C_0 + \dots + {}^{F^{s-1}}C_{s-1} {}^{F^{s-2}}D_{s-2} \dots D_0 = C $$

(and $p^{s_{m+1}s}$ divides C). Consider the $W(k)[[t]]$-module $W(k)[[t]] \otimes_{W(k)} M =: \tilde{M}$ and the frame $(W(k)[[t]], pW(k)[[t]], \tau)$ where τ is defined by $\tau(t) = t^p$. We will use the technique of Norman to define (a family of) $\mathbb{Z}/r\mathbb{Z}$-graded skew-Hermitian $W(k)[[t]]$-windows whose underlying $W(k)[[t]]$-module is $\tilde{M}$, and whose special fiber is M. Let $H = (H_0, \dots, H_{\frac{r}{2}-1})$ be a $\frac{r}{2}$-tuple of $b \times d$-matrices over $W(k)$ and proceed further as follows: Define a $W(k)[[t]]$-valued element U_σ of $\mathrm{GL}(M_\sigma/W(k))$ by decreeing $U_\sigma - \mathrm{id}_{\tilde{M}_\sigma}$ to be an upper triangular block matrix with a single non-zero block, namely tH_σ (sitting in the same position as C_σ, being the $b \times d$-block lying above the $m+1$st row and to the right of the mth column). For the time being we fix H and declare U_σ to be $\check{U}_{\sigma-\frac{r}{2}}$ for all $\sigma \in \{\frac{r}{2}, \dots, r-1\}$. Note that $U_\sigma \equiv \mathrm{id}_{\tilde{M}_\sigma} \mod t$, so that precomposing F and V^{-1} with U_σ yields some $\mathbb{Z}/r\mathbb{Z}$-graded skew-Hermitian $W(k)[[t]]$-window ${}_H\tilde{M}$, whose special fiber is M. Also, note that the corresponding matrix for its Verschiebung $V^\sharp|_{\tilde{M}_\sigma}$ from ${}_H\tilde{M}_\sigma$ to $W(R) \otimes_{F, W(R)} {}_H\tilde{M}_{\sigma+1}$ is $A_\sigma U_\sigma^{-1} = \tilde{A}_\sigma \in \mathrm{Mat}(n \times n, W(k)[[t]])$, of which the restriction to (45) gives rise to some block matrix $\begin{pmatrix} B_\sigma & \tilde{C}_\sigma \\ 0 & D_\sigma \end{pmatrix}$, and $C_\sigma - \tilde{C}_\sigma = tB_\sigma H_\sigma$ holds at least for $\sigma \in \{0, \dots, \frac{r}{2} - 1\}$.
Similarly ${}^{F^{s-1}}\tilde{A}_{s-1} \dots {}^{F}\tilde{A}_1 \tilde{A}_0 =: \tilde{A}$ gives us yet another block matrix $\begin{pmatrix} p^{s_m s} & \tilde{C} \\ 0 & p^{s_{m+1} s} \end{pmatrix}$, and $p^{s_{m+1}s}$ divides $\tilde{C}$. Next we study criterions for ${}_H\tilde{M}$ to be an isotrivial deformation of M, so suppose that some $\mathbb{Z}/r\mathbb{Z}$-graded homomorphism $h_\sigma : {}_H\tilde{M}_\sigma \to {}_0\tilde{M}_\sigma$ lifts $p^c \mathrm{id}_M$ for some $c \in \mathbb{N}_0$. Again we want to look at the matrix of h_σ: Due to slope reasons this is a upper triangular block matrix, and due to the rigidity of unit root crystals, its diagonal entries are equal to p^c (cf. [60, lemma 42]), furthermore the $\mathbb{Z}/r\mathbb{Z}$-graded homomorphism that is induced on (45) is again described by a family of block matrices $\begin{pmatrix} p^c & k_\sigma \\ 0 & p^c \end{pmatrix}$ for certain $k_\sigma \in \mathrm{Mat}(b \times d, tW(k)[[t]])$ and an inspection h_0 shows ${}^{F^s}k_0 - p^{(s_m - s_{m+1})s} k_0 = p^c \frac{C - \tilde{C}}{p^{s_{m+1}s}}$. It is easy to see that this equation has no solution unless $H_0 = \dots = H_{\frac{r}{2}-1} = 0$, in which case it has a unique solution being $k_0 = 0$. In a later section we need the following important lemma:

Lemma 4.12. *Let $\gamma : \mathcal{P}_1 \to \mathcal{P}_2$ be a $\mathbb{Z}/r\mathbb{Z}$-graded isogeny between two $\mathbb{Z}/r\mathbb{Z}$-graded displays (possibly equipped with skew-Hermitian structures) over an algebraically closed field k. If $\mathcal{P}_1$ (and hence $\mathcal{P}_2$) possesses more than one Newton slope, then there exists at least one non-isotrivial deformation $\tilde{\gamma} : \tilde{\mathcal{P}}_1 \to \tilde{\mathcal{P}}_2$ with a constant Newton-polygon over $k[[t]]$.*

Proof. We deform $\mathcal{P}_1$ by following the algorithm explained above, but we choose $0 \neq H \equiv 0$ modulo a very large power of p, so that one can use transport of structure to find a corresponding deformation $\tilde{\mathcal{P}}_2$ of $\mathcal{P}_2$. □

4.4. **Interlude on $\mathfrak{B}$.** In this subsection we study some examples for morphisms between Φ-data, in the sense of subsubsection 3.3.1:

Definition 4.13. *A family of standard linear Φ-data*

$$\{(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\}_{i\in\pi}$$

is called multicompact if the following holds:

- (C1) *For every σ the cardinalities of the sets $\{i \in \pi | \upsilon_{i,\sigma}(z) = z\}$ and $\{i \in \pi | \upsilon_{i,\sigma}(z) = 1\}$ are both at least: $\frac{\mathrm{Card}(\pi)-1}{2}$,*
- (C2) $\mathrm{Card}(\pi)$ *is odd, and*
- (C3) *there exists at least one $\varsigma \in \pi$ for which the cardinality of at least one of the two aforementioned sets is at least: $\frac{\mathrm{Card}(\pi)+1}{2}$.*

A family of standard unitary Φ-data

$$\{(\mathcal{G}_i, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\}_{i\in\pi}$$

is called multicompact if the family of their tautological linear Φ-data $\{(\mathcal{G}_{iW(\mathbb{F}_{p^{2r}})}, \{\upsilon^{(2)}_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\}_{i\in\pi}$ is multicompact.

Let us explain the relevance of multicompactness: Let $\{(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\}_{i\in\pi}$ be a multicompact family of standard linear Φ-data, assume we are given a $\mathbb{Z}/r\mathbb{Z}$-graded $3n$-display

$$\mathcal{S}_i = (\{M_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$$

with $(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-structure over R, for each $i \in \pi$. We wish to define another $\mathbb{Z}/r\mathbb{Z}$-graded R-display $(\{M_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_\sigma\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$, of which the $\mathbb{Z}/r\mathbb{Z}$-graded Hodge filtration $M_\sigma \supset N_\sigma$ looks like:

$$\bigotimes_{i\in\pi} M_{i,\sigma} \supset \bigotimes_{i\in\pi}{}' N_{i,\sigma} = \sum_{\mathrm{Card}(I)=\frac{\mathrm{Card}(\pi)+1}{2}} (\bigotimes_{i\in I} N_{i,\sigma}) \otimes_{W(R)} (\bigotimes_{i\in\pi-I} M_{i,\sigma})$$

and whose ^{F}-linear map $V^{-1}: N_\sigma \to M_{\sigma-1}$ of degree -1 reads:

$$V^{-1}(\bigotimes_{i\in\pi} x_i) := (\bigotimes_{i\in I} V^{-1}(x_i)) \otimes (\bigotimes_{i\in\pi-I} F(x_i)) \tag{46}$$

here it is understood that $x_i \in \begin{cases} N_{i,\sigma} & i \in I \\ M_{i,\sigma} & i \notin I \end{cases}$ holds for some $\frac{\mathrm{Card}(\pi)+1}{2}$-element set $I \subset \pi$. The well-definedness of V^{-1} and the unique F that goes with it can be checked along the lines of [5, Proposition 4.3], by choosing a normal decomposition. Observe that

$$p^{\frac{\mathrm{Card}(\pi)-1}{2}} F(\bigotimes_{i\in\pi} x_i) = \bigotimes_{i\in\pi} F(x_i) \tag{47}$$

holds for all $x_i \in M_{i,\sigma}$.

Definition 4.14. *With the notation and under the assumptions at the beginning of this subsection we define the restricted tensorproduct of the multicompact family* $\mathcal{S}_i = (\{M_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{N_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1})$ *of displays with* $(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$*-structure over R, to be the $\mathbb{Z}/r\mathbb{Z}$-graded R-display which is given by the quadruple*

$$\dot{\bigotimes}_{i\in\pi}\mathcal{S}_i := (\{\bigotimes_{i\in\pi} M_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, \{\bigotimes_{i\in\pi}{}' N_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}, F, V^{-1}), \tag{48}$$

where F and V^{-1} are defined as above.

If $\{\mathcal{S}_i\}_{i\in\pi}$ is as above then $\dot{\bigotimes}_{i\in\pi}\mathcal{S}_i^t$ is the Cartier dual of (48), and if r is even, then it is easy to see that a $W(\mathbb{F}_{p^{\frac{r}{2}}})$-valued skew-Hermitian structure on each $\mathcal{S}_i$ gives rise to a $W(\mathbb{F}_{p^{\frac{r}{2}}})$-valued skew-Hermitian structure on $\dot{\bigotimes}_{i\in\pi}\mathcal{S}_i$ (we leave it to the reader to define the $\mathbb{Z}/r\mathbb{Z}$-graded tensorproducts of étale displays).

Definition 4.15. *Fix a family of standard linear (resp. unitary) Φ-data $\{(\mathcal{G}_i, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})\}_{i\in\Lambda}$, and assume that Π is a fixed set of multicompact subsets with $\bigcup\Pi = \Lambda$.*
Let $\{\mathcal{R}_\pi\}_{\pi\in\Pi}$ (resp. $\{(\mathcal{R}_\pi, *)\}_{\pi\in\Pi}$) be a family of not-necessarily commutative finite flat $W(\mathbb{F}_{p^r})$-algebras (resp. $W(\mathbb{F}_{p^{2r}})$-algebras equipped with a ${}^-$-equivariant involution $*$).
Suppose that we are given a display with $(\mathcal{G}_i, \{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-structure $\mathcal{S}_i$ for every $i \in \Lambda$. By an $\{\mathcal{R}_\pi\}_{\pi\in\Pi}$-operation (resp. $\{(\mathcal{R}_\pi, *)\}_{\pi\in\Pi}$-operation) on the family $\{\mathcal{S}_i\}_{i\in\Lambda}$ we mean a family $\{s_\pi\}_{i\in\Pi}$, where s_π is a $W(\mathbb{F}_{p^r})$-linear (resp. $*$-preserving $W(\mathbb{F}_{p^{2r}})$-linear) homomorphism from $\mathcal{R}_\pi$ to $\mathrm{End}_{W(\mathbb{F}_{p^{2r}})}(\dot{\bigotimes}_{i\in\pi}\mathcal{S}_i)$.

Let $\mathfrak{B}^{\{\mathcal{R}_\pi\}_{\pi\in\Pi}}$ (resp. $\mathfrak{B}^{\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi}}$) be the stack whose objects are those pairs $(\{\mathcal{S}_i\}_{i\in\Lambda},\{s_\pi\}_{\pi\in\Pi})$, with isomorphisms being tuples $\zeta_i : \mathcal{S}_i' \overset{\cong}{\to} \mathcal{S}_i$ such that the diagrams:

$$\begin{array}{ccc} \dot{\bigotimes}_{i\in\pi}\mathcal{S}_i & \xleftarrow{\dot{\bigotimes}_{i\in\pi}\zeta_i} & \dot{\bigotimes}_{i\in\pi}\mathcal{S}_i' \\ {\scriptstyle s_\pi(a)}\uparrow & & {\scriptstyle s_\pi(a)}\uparrow \\ \dot{\bigotimes}_{i\in\pi}\mathcal{S}_i & \xleftarrow{\dot{\bigotimes}_{i\in\pi}\zeta_i} & \dot{\bigotimes}_{i\in\pi}\mathcal{S}_i' \end{array}$$

commute.) We will usually study this together with the map

$$\mathfrak{B} \to \prod_{i\in\Lambda} \mathcal{B}(\mathcal{G}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$$

5. On $\mathrm{Flex}^{\mathbf{d}^+}$

Let A be a commutative ring satisfying $pA=0$. For every non-negative integer m we will write $\sqrt[p^m]{A}$ for the A-algebra whose underlying ring is A, and whose A-algebra structure is defined by the homomorphism $x\mapsto x^{p^m}$, the mth iterate of the absolute Frobenius, notice that their direct limit

$$\lim_{\to} \sqrt[p^m]{A} =: \sqrt[p^\infty]{A}$$

is the usual perfection of the reduction of A. For an element x of A we use the notation $\sqrt[p^m]{x} \in \sqrt[p^m]{A}$ to express the alteration of A-algebra structure on the same ring (the p^m-th power of $\sqrt[p^m]{x}$ agrees with the image of x under the structural morphism $A\to\sqrt[p^m]{A}$). In this section we study the functor $\mathrm{Flex}^{\mathbf{d}^+}$ of subsubsection 3.3.3. Unlike many other functors in this paper, it does not seem to be schematic, so that we need remedies to cope with it. We will need the following lemma:

Lemma 5.1. *Let R be the ring $\sqrt[p^\infty]{k}\otimes_k\sqrt[p^\infty]{k}$, where k is a field of characteristic p. Let J_0 be the ideal $\{x\in R|x^p=0\}$. Then R is J_0-adically separated.*

Proof. In order to control the situation we use a p-basis $\{x_i|i\in I\}$, so that every element of k has a unique representation as a sum $x=\sum_{\underline{n}} a_{\underline{n}}^p \underline{x}^{\underline{n}}$ where $\underline{n}=(\dots,n_i,\dots)$ runs through the set of multiindices with $p-1\geq n_i\geq 0$ and $n_i=0$ for almost all i. It is easy to see that $\sqrt[p^\infty]{k}$ is the quotient of the polynomial algebra $k[\{t_{i,e}|i\in I, e=1,\dots\}]$ by the ideal which is generated by $t_{i,e+1}^p-t_{i,e}$ and $t_{i,1}^p-x_i$. Now consider

$$s_{i,e} := t_{i,e}\otimes 1 - 1\otimes t_{i,e} \in \sqrt[p^\infty]{k}\otimes_k\sqrt[p^\infty]{k}.$$

It follows that $\sqrt[p^\infty]{k} \otimes_k \sqrt[p^\infty]{k}$ is the quotient of the polynomial algebra $\sqrt[p^\infty]{k}[\{s_{i,e}|i \in I\,, e = 1, \dots\}]$ by the ideal which is generated by $s_{i,e+1}^p - s_{i,e}$ and $s_{i,1}^p$. We deduce that there exist ring endomorphisms $\theta_{I_0} : R \to R$, defined by

$$s_{i,e} \mapsto \begin{cases} s_{i,e} & i \in I_0 \\ 0 & i \notin I_0 \end{cases}$$

for every finite subset $I_0 \subset I$. Notice that each θ_{I_0} preserves J_0, and that $\theta_{I_0} = \theta_{I_0} \circ \theta_{I_0}$.
Now J_0 is generated by the elements $s_{i,1}$, so that J_0^ν is generated by the set $\{\prod_i s_{i,1}^{\nu_i} | \sum_i \nu_i = \nu\}$. The only multiindices which give rise to non-zero products are bounded by $\nu_i \leq p-1$, and these are seen to involve factors indexed by at least $\frac{\nu}{p-1}$ many elements of I. Consequently one has $\theta_{I_0}(J_0^\nu) = 0$ provided that $\mathrm{Card}(I_0) < \frac{\nu}{p-1}$. Now consider some $x \in \bigcap_\nu J_0^\nu$, and choose a large finite set I_0 with $\theta_{I_0}(x) = x$. We deduce $x \in \theta_{I_0}(J_0^{1+(p-1)\,\mathrm{Card}(I_0)}) = 0$. □

5.1. **Faithfulness properties.** From now onwards we fix $(\mathcal{G}/W(\mathbb{F}_{p^r}), \{\mu_\sigma\}_{\sigma\in\Sigma})$. In this subsection we further the study of faithfulness properties of

$$\overline{\mathcal{B}}'(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma}) \overset{\mathrm{Flex}^{\mathbf{d}^+}}{\to} \overline{\mathcal{B}}'(\mathcal{G}, \{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}}),$$

please recall ${}^{F^{\mathbf{d}^+(\sigma)}}\mu_{\mathbf{d}(\sigma)} = \tilde{\mu}_\sigma$. Let us begin with the following triviality:

Lemma 5.2. *Let A be a $\mathbb{F}_{p^f}$-algebra which is separated with respect to the J_0-adic topology where $J_0 = \{x \in A | x^p = 0\}$. Then the functor* $\mathrm{Flex}^{\mathbf{d}^+}|_{\overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$ *is faithful (i.e. injective on $\overline{\mathcal{B}}'(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-morphisms).*

Proof. In view of corollary 3.26 this is clear, because the kernel of $\mathrm{Flex}^{\mathbf{d}^+}$ is killed by a power of Frobenius. □

Due to lemma 5.1 we can draw the following conclusion:

Corollary 5.3. *If k is any field extension of $\mathbb{F}_{p^f}$, then the functor* $\mathrm{Flex}^{\mathbf{d}^+}$ *is faithful over the ring $R = \sqrt[p^\infty]{k} \otimes_k \sqrt[p^\infty]{k}$.*

5.2. **Separation properties of** $\mathrm{Flex}^{\mathbf{d}^+}|_{\overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$**.** In the following result $\times$ denotes the fiber product of rings:

Proposition 5.4. *Let $\mathbb{F}_{p^f} \subset A \subset B$ be a ring extension. Let $\mathcal{P}$ and $\mathcal{Q}$ be displays with $(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over* $\mathrm{Spec}\,A$ *and let ψ be an isomorphism between* $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{P})$ *and* $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{Q})$*. Assume that one of the following two assumptions is in force:*

(i) *The nilradical of B is trivial and*

$$A \cong B \times_{\sqrt[p]{B}} \sqrt[p]{A}$$

holds.

(ii) *The nilradical of B is nilpotent and A is noetherian.*

Then the 2-isomorphism ψ has a preimage under $\mathrm{Flex}^{\mathbf{d}^+}|_{\overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$ if and only if the same statement holds for the scalar extension ψ_B

Proof. The functor $\mathrm{Flex}^{\mathbf{d}^+}$ is faithful over B and over A, in both cases (i) and (ii), in particular a preimage over A is always unique and it exists if and only if the given preimage $\phi_B \in \mathrm{Hom}(\mathcal{P}_B, \mathcal{Q}_B)$ descents to an isomorphism between $\mathcal{P}$ and $\mathcal{Q}$ (for example by using the separetedness of the diagonal). It seems to be elementary to check the case (i) of the assertion directly from the formula (26). More general, let us say that ψ becomes flexed over some A-algebra B' if one has found a preimage of $\psi_{B'}$. We do the proof of (ii) in several steps:

Step 1. A is a product of finitely many fields.
It is clear that we can reduce the situation to the case of a single field, in which case B can be assumed to be a field as well, in fact B can be assumed to be equal to the perfection of A, according to part (i) of our proposition. Now we merely have to prove that the scalar extensions by means of the two maps $B \rightrightarrows B \otimes_A B$ give the same answer when applied to the isomorphism ϕ_B. As these are still preimages of $\psi_{B\otimes_A B}$ we conclude by using corollary 5.3.

Step 2. A is a reduced complete local noetherian ring.
By the previous step and part (i) of proposition 5.4 we know that ψ becomes flexed over the normalization A'. Now part (ii) of corollary 3.28 and [19, Théorème (23.1.5)] tell us that we only need to know that ψ becomes flexed over $A/\mathrm{rad}(A)$ which is dealt with by another application of the previous step.

Step 3. A is a complete local noetherian ring.
This case follows from the previous step and part (i) of corollary 3.28.

Step 4. A is a reduced local noetherian ring.
By step 1 we do know that ψ becomes flexed over $K(A)$. By the previous step we have the same result over the completion $\hat{A}$. Due to $A = \hat{A} \cap K(A)$ this is sufficient.

Step 5. A is a general noetherian ring.
As before we can assume reducedness. By (flatness of $A_\mathfrak{p}$ and) the previousstep we have that ψ becomes flexed over the local rings $A_\mathfrak{p}$. Now use $A = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\,A} A_\mathfrak{p}$, the intersection taking place in $K(A)$.

□

As a consequence we get the full faithfulness of $\mathrm{Flex}^{\mathbf{d}^+}|_{\overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$ over all reduced noetherian $\mathbb{F}_{p^f}$-algebras. Here is a separatedness property of $\mathrm{Flex}^{\mathbf{d}^+}|_{\overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})}$, as promised in the title of this subsection:

Theorem 5.5. *Let A be a noetherian $\mathbb{F}_{p^f}$-algebra and let $\mathcal{P}$ and $\mathcal{Q}$ be displays with $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over $\operatorname{Spec} A$ and let ψ be a 2-isomorphism between $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{P})$ and $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{Q})$. Then there exists a nilpotent ideal $I_\psi \subset A$, such that the following two assertions are equivalent for all A-algebras B with nilpotent nilradical:*

(i) *There exists $\mathcal{P}_B \overset{\phi}{\to} \mathcal{Q}_B$ such that $\mathrm{Flex}^{\mathbf{d}^+}(\phi) = \psi_B$ holds.*
(ii) $BI_\psi = 0$

Proof. It is clear that the set $\mathcal{I}_\psi = \{I \subset A | \psi_{A/I} \text{ is flexed}\}$ is stable under intersections of finitely many ideals, and it is also clear that $\bigcap \mathcal{I}_\psi =: I_\psi \in \mathcal{I}_\psi$ is all we need to know. One can apply part (ii) of proposition 5.4 to the ring extension $A_{red} \hookrightarrow \sqrt[p^\infty]{A}$, and so we know $\sqrt{0_A} \in \mathcal{I}_\psi$. The nilradical of $\prod_{\mathcal{I}_\psi \ni I \subset \sqrt{0_A}} A/I$ is nilpotent and so we know that $I_\psi \in \mathcal{I}_\psi$. □

Definition 5.6. *Let A be a regular noetherian $\mathbb{F}_{p^f}$-algebra and let $\tilde{\mathcal{P}}$ be a display with $(\mathcal{G},\{\tilde{\mu}_\sigma\}_{\sigma\in\tilde{\Sigma}})$-structure over $\operatorname{Spec} A$. Suppose we are given*

- *another regular noetherian $\mathbb{F}_{p^f}$-algebra B, together with a $\mathbb{F}_{p^f}$-linear injection $A \to B$ such that, B is radicial and finite (hence faithfully flat) over A,*
- *a display $\mathcal{P}$ with $(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$-structure over $\operatorname{Spec} B$, and*
- *an isomorphism*

$$\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{P}) \overset{\cong}{\to} \tilde{\mathcal{P}}_B. \tag{49}$$

Then we call this a diagonalization if the kernel of the diagonal morphism $B \otimes_A B \to B$ agrees with the ideal I_ψ where $\psi : \mathrm{Flex}^{\mathbf{d}^+}(\mathrm{pr}_1^* \mathcal{P}) \to \mathrm{Flex}^{\mathbf{d}^+}(\mathrm{pr}_2^* \mathcal{P})$ results from (49) (with $\mathrm{pr}_i : B \to B \otimes_A B$ being the coordinates).

We are now able to state our first serious result:

Theorem 5.7. *Let A be a regular noetherian $\mathbb{F}_{p^f}$-algebra, such that $\sqrt[p]{A}$ is finite over A. Let $\operatorname{Spec} A \overset{\tilde{\mathcal{P}}}{\to} \overline{\mathcal{B}}'(\mathcal{G},\{\mu_\sigma\}_{\sigma\in\Sigma})$ be as in definition 5.6. Between any two diagonalizations there exists a unique isomorphism. A diagonalization exists in each of the following cases:*

(i) *A is a field (of finite degree of imperfection)*

(ii) *A is a complete local ring with perfect residue field, and $\mathcal{P}$ is the universal formal deformation of its special fiber*

(iii) *A is a smooth algebra over a perfect field, and the pull back of $\mathcal{P}$ to the completed local ring $\hat{A}_{\mathfrak{m}}$ agrees with the universal formal deformation of its special fiber, for every maximal ideal $\mathfrak{m}$.*

Moreover, in case (ii) the diagonalization is also the universal formal deformation of its special fiber and in case (iii) the diagonalization also satisfies the property given in (iii).

Proof. It is clear, that under no assumption whatsoever, one can find an isomorphism $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{P}') \stackrel{\cong}{\to} \tilde{\mathcal{P}}_{\sqrt[p^n]{A}}$ for a sufficiently large integer n and a $\sqrt[p^n]{A}$-valued point $\mathcal{P}'$ of $\overline{\mathcal{B}}'(\mathcal{G}, \{\mu_\sigma\}_{\sigma\in\Sigma})$. Under the above assumptions the functor $\mathrm{Flex}^{\mathbf{d}^+}$ is fully faithful over any ring B containing A and contained in $\sqrt[p^n]{A}$. So some preimage $\mathcal{P}$ of $\tilde{\mathcal{P}}_B$ is unique up to a unique isomorphism, provided that it exists, in particular we have a canonical isomorphism between $\mathcal{P}'$ and $\mathcal{P}_{\sqrt[p^n]{A}}$.

Thus, in order to study diagonalizations in the sense of definition 5.6, one must pay particular attention to the ring B. Furthermore, let $\psi' : \mathrm{Flex}^{\mathbf{d}^+}(\mathrm{pr}_1^*\mathcal{P}') \to \mathrm{Flex}^{\mathbf{d}^+}(\mathrm{pr}_2^*\mathcal{P}')$ be the canonical 2-isomorphism resulting from $\mathrm{Flex}^{\mathbf{d}^+}(\mathcal{P}') \cong \tilde{\mathcal{P}}_{\sqrt[p^n]{A}}$ (with $\mathrm{pr}_i : \sqrt[p^n]{A} \to \sqrt[p^n]{A} \otimes_A \sqrt[p^n]{A}$ being the coordinates). However, the cocartesianism of the diagram

$$\begin{array}{ccc} \sqrt[p^n]{A} \otimes_A \sqrt[p^n]{A} & \longrightarrow & \sqrt[p^n]{A} \otimes_B \sqrt[p^n]{A} \\ \uparrow & & \uparrow \\ B \otimes_A B & \longrightarrow & B \end{array}$$

shows that there exists a diagonalization over a regular ring B if and only if $B = \{x \in \sqrt[p^n]{A} | x \otimes 1 - 1 \otimes x \in I_{\psi'}\}$ holds, because of the exactness of

$$B \to \sqrt[p^n]{A} \rightrightarrows \sqrt[p^n]{A} \otimes_B \sqrt[p^n]{A}.$$

The rest of the proofs of (i), (ii) and (iii) are obvious. □

6. Realizations of $3n$-displays with additional structure

In this section we need a few category theoretic preliminaries. Suppose that Q is a projective and finitely generated module over a commutative ring B, and let M be an object in an additive and Karoubian B-linear category $\mathfrak{C}$. We let $Q \otimes_B M \in \mathtt{Ob}_{\mathfrak{C}}$ be the object representing the covariant functor $N \mapsto \mathrm{Hom}_B(Q, \mathrm{Hom}(M, N))$. By a module in $\mathfrak{C}$ over a B-algebra C we mean some $M \in \mathtt{Ob}_{\mathfrak{C}}$ together with a B-linear homomorphism $\iota_M : C \to \mathrm{End}(M)$. The class of C-modules in $\mathfrak{C}$ is an additive and Karoubian category in its own right, and we denote it

by $\mathfrak{C}^C$. From now on we always assume that $\mathfrak{C}$ has the structure of an associative, commutative and unital B-linear $\otimes$-category. If C is commutative as a ring and projective as a $C\otimes_B C$-module, then the formula $C\otimes_{C\otimes_B C}(M\otimes N) =: M\otimes_C N$ defines an associative, commutative and unital C-linear $\otimes$-structure on the category $\mathfrak{C}^C$ of C-modules in $\mathfrak{C}$.
Suppose in addition, that C is an étale and finite B-algebra of degree 2 over B, and that we are given a commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{*} & \mathcal{R} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\bar{\ }} & C \end{array}, \tag{50}$$

in which $*$ stands for a B-linear involution on a fixed C-algebra $\mathcal{R}$, while the horizontal map at the bottom is the involution on C given by $x\mapsto \bar{x} := \mathrm{tr}_{C/B}(x) - x$. Accordingly we decree for any $M \in \mathtt{Ob}_{\mathfrak{C}^{\mathcal{R}}}$ (resp. $M \in \mathtt{Ob}_{\mathfrak{C}^C}$) the $\mathcal{R}^{op}$-module M^* (resp. the C-module $\bar{M}$) to have the conjugated $\mathcal{R}^{op}$-operation $\mathcal{R}^{op} \xrightarrow{*} \mathcal{R} \xrightarrow{\iota_M} \mathrm{End}(M)$ (resp. C-operation $C \xrightarrow{\bar{\ }} C \xrightarrow{\iota_M} \mathrm{End}(M)$) on the same underlying $\mathfrak{C}$-object M. By a C-skew-Hermitian $(\mathcal{R},*)$-module in $\mathfrak{C}$ we mean a triple (M,K,Ψ), where $M\in\mathfrak{C}^{\mathcal{R}}$, $K\in\mathfrak{C}$, and Ψ is a $\mathfrak{C}^C$-morphism from the $\mathcal{R}\otimes_C\mathcal{R}^{op}$-module $M\otimes_C M^*$ to $C\otimes_B K$ satisfying the following properties:

- The composition of Ψ with the $\mathfrak{C}^C$-endomorphisms on $M\otimes_C M^*$ induced by the elements $1_{\mathcal{R}}\otimes a - a\otimes 1_{\mathcal{R}^{op}} \in \mathcal{R}\otimes_C\mathcal{R}^{op}$ vanish for all $a\in\mathcal{R}$.
- The composition of Ψ with the $\mathfrak{C}$-endomorphism on $M\otimes_C M^*$ given by $x\otimes y\mapsto y^*\otimes x^*$ agrees with $-\overline{\Psi}$, which is the precomposition of Ψ with the $\mathfrak{C}$-endomorphism $^{-}\otimes(-\,\mathrm{id}_K)$ on $C\otimes_B K$ given by $x\otimes y\mapsto -\bar{x}\otimes y$.

If $\mathfrak{C}$ is a rigid $\otimes$-category, then so is $\mathfrak{C}^C$, and in this case we will say that a triple (M,K,Ψ) as above is perfect, if Ψ is induced from a $\mathfrak{C}^{\mathcal{R}}$-isomorphism $M\xrightarrow{\cong} K\otimes_C \check{M}^*$.
It goes without saying that by "skew-Hermitian C-module in $\mathfrak{C}$" we mean a C-skew-Hermitian $(C,^{-})$-module in $\mathfrak{C}$. We need the following construction: Fix a family $\{(M_i,K_i,\Psi_i)\}_{i\in\Lambda}$ of skew-Hermitian C-modules in $\mathfrak{C}$ which is indexed by a finite set Λ. For every $\pi\subset\Lambda$ one can consider tensor products

$$\bigotimes_{i\in\pi} M_i =: M^\pi \in \mathtt{Ob}_{\mathfrak{C}^C} \tag{51}$$

$$\bigotimes_{i\in\pi} K_i =: K^\pi \in \mathtt{Ob}_{\mathfrak{C}}. \tag{52}$$

Notice that $\bar{M}^\pi \cong \bigotimes_{i\in\pi} \bar{M}_i$, so that every π may give rise to a $\mathfrak{C}^C$-morphism Ψ^π, defined by the tensor product of the Ψ_i's:

$$\Psi^\pi : M^\pi \otimes_C \bar{M}^\pi \to C \otimes_B K^\pi; \tag{53}$$

$$(\bigotimes_{i\in\pi} x_i) \otimes (\bigotimes_{i\in\pi} y_i) \mapsto \bigotimes_{i\in\pi} \Psi_i(x_i, y_i), \tag{54}$$

thus defining a skew-Hermitian C-module (M^π, K^π, Ψ^π) in $\mathfrak{C}$, provided that the cardinality of the set π is an odd number.

Definition 6.1. *Let B_0 be a commutative ring, and let $\mathfrak{C}$ be an additive and Karoubian, associative, commutative and unital B_0-linear $\otimes$-category. Let B be a commutative B_0-algebra which is projective as a $B \otimes_{B_0} B$-module, and let C be a finite and étale B-algebra of degree 2 over B. Let Π be a set of subsets of Λ, with the cardinality of each member of which being a odd number. Let $\{(\mathcal{R}_\pi, *)\}_{\pi\in\Pi}$ be a family of C-algebras with involution in the above sense (of* (50)*). Then we let $\mathfrak{C}^{\Lambda,\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi}}$ be the isomorphism category of pairs of families $(\{(M_i, K_i, \Psi_i)\}_{i\in\Lambda}, \{s_\pi\}_{\pi\in\Pi})$ such that the following is true:*

- *The triple (M_i, K_i, Ψ_i) is a skew-Hermitian C-module in $\mathfrak{C}^B$, for every $i \in \Lambda$.*
- *The homomorphism $s_\pi : \mathcal{R}_\pi \to \mathrm{End}_C(M^\pi)$ defines a C-skew-Hermitian $(\mathcal{R}_\pi, *)$-module structure on the skew-Hermitian C-module (M^π, K^π, Ψ^π) (cf.* (51), (52), (53), (54)*), for every $\pi \in \Pi$.*

In this section we are primarily interested in rings B which are either fields of characteristic 0 or Dedekind rings of which the fraction field has characteristic 0. Consider a smooth and affine B-group $\mathcal{G}$ with connected fibers. Observe that the perfect skew-Hermitian C-modules in $Rep_0(\mathcal{G})$ are precisely given by homomorphisms $\rho : \mathcal{G} \to \mathrm{GU}(\mathcal{V}/C, \Psi)$, where Ψ is a perfect skew-Hermitian form on a finitely generated torsionfree C-module $\mathcal{V}$. In this paper, a very typical class of $\mathfrak{C}^{\Lambda,\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi}}$-objects arises in the following way: Fix a pair of families $\mathfrak{T} = (\{(\mathcal{V}_i, \rho_i, \Psi_i)\}_{i\in\Lambda}, \{(\mathcal{R}_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$, where

- $\{(\mathcal{V}_i, \rho_i, \Psi_i)\}_{i\in\Lambda}$ is a perfect skew-Hermitian C-module in $Rep_0(\mathcal{G})$, for every $i \in \Lambda$, and
- $\iota_\pi : \mathcal{R}_\pi \to \mathrm{End}_C(\mathcal{V}^\pi)$ is a C-skew-Hermitian $(\mathcal{R}_\pi, *)$-module structure on $\bigotimes_{i\in\pi} \mathcal{V}_i$, for every $\pi \in \Pi$,

and observe that any B-linear $\otimes$-functor $M : Rep_0(\mathcal{G}) \to \mathfrak{C}^B$ may simply be evaluated on the representations ρ_i (resp. their multiplier characters or their skew-Hermitian forms) to obtain $M_i \in \mathtt{Ob}_{\mathfrak{C}^C}$ (resp. $K_i \in$

$\mathtt{Ob}_{\mathfrak{C}^B}$ or Ψ_i). Each ι_π yields a C-skew-Hermitian $(\mathcal{R}_\pi, *)$-module structure on the skew-Hermitian C-module (M^π, K^π, Ψ^π) in a similar way. We will say that the $\mathfrak{C}^{\Lambda,\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi}}$-object $(\{(M_i, K_i, \Psi_i)\}_{i\in\Lambda}, \{s_\pi\}_{\pi\in\Pi})$ arises from the B-linear $\otimes$-functor M by restriction to $\mathfrak{T}$, and denote it by: $M|_{\mathfrak{T}}$.

6.1. **Graded Realizations functors.** Let us complement the corollary 3.8 with the following result, whose proof follows along identical lines.

Lemma 6.2. *Let the assumptions on a $W(\mathbb{F}_{p^f})$-group $\mathcal{G}$, on a cocharacter $\upsilon : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to \mathcal{G}$ and on $h \geq 1$ be as in proposition 3.5. Let $\rho : \mathcal{G} \to \mathrm{GL}(n)_{W(\mathbb{F}_{p^f})}$ be a representation, such that $\rho \circ \upsilon$ has no positive weight, so that there exists an effective cocharacter $\beta : \mathbb{A}^1_{W(\mathbb{F}_{p^f})} \to \mathrm{Mat}(n \times n)_{W(\mathbb{F}_{p^f})}$ with $\beta|_{\mathbb{G}_{m,W(\mathbb{F}_{p^f})}} = \rho \circ \upsilon^{-1}$. Then, one has an equality*

$$F^h \circ (\beta(p)^W \rho|_{\overline{\mathcal{I}}^\upsilon}) = ({}^{WF^h}\rho \circ \overline{\Phi}^{\upsilon,h})^{F^h} \beta(p)$$

of functions from $\overline{\mathcal{I}}^\upsilon$ to ${}^W\mathrm{Mat}(n \times n)_{\mathbb{F}_{p^f}}$.

The previous result is going to enter into the construction of certain realization functors, which we define in this subsection. We want to begin with a short description of their target categories:

Definition 6.3. *Let A be a p-adically complete and separated ring A, and let $\tau : A \to A$ be endomorphism satisfying $\tau(x) \equiv x^p \pmod p$ for all $x \in A$.*

- *By a F-module (resp. V-module) over (A, τ) we mean a pair $(M, F^\sharp)$ (resp. $(M, V^\sharp)$), consisting of a finitely generated and projective A-module M, together with an A-linear map $F^\sharp : A \otimes_{\tau,A} M \to M$ (resp. $V^\sharp : M \to A \otimes_{\tau,A} M$). For every $0 \geq n \in \mathbb{Z}$ (resp. $n \in \mathbb{N}_0$) we let $A(n)$ be the F-module (resp. V-module) over (A, τ) which is given by the pair $(A, \frac{\mathrm{id}_A}{p^n})$ (resp. by $(A, p^n \mathrm{id}_A)$).*
- *By a τ-crystal over A, we mean a triple $(M, F^\sharp, V^\sharp)$ consisting of a finitely generated and projective A-module M, together with a pair of mutually invers A-linear bijections $F^\sharp : A[\frac{1}{p}] \otimes_{\tau,A} M \to \mathbb{Q} \otimes M$ and $V^\sharp : \mathbb{Q} \otimes M \to A[\frac{1}{p}] \otimes_{\tau,A} M$*
- *The category $\mathbf{F}Mod_{A,\tau}$ (resp. $\mathbf{V}Mod_{A,\tau}$) consisting of all F-modules over (A, τ) (resp. V-modules over (A, τ)) is defined by decreeing its morphisms to be the class of A-linear maps*

$h : N \to M$ rendering the diagram

$$\begin{array}{ccc} A\otimes_{\tau,A} M & \xrightarrow{F_M^\sharp} & M \\ {\scriptstyle \mathrm{id}_A\otimes h}\uparrow & & {\scriptstyle h}\uparrow \\ A\otimes_{\tau,A} N & \xrightarrow{F_N^\sharp} & N \end{array} \qquad \left(\textit{resp.}\quad \begin{array}{ccc} M & \xrightarrow{V_M^\sharp} & A\otimes_{\tau,A} M \\ {\scriptstyle h}\uparrow & & {\scriptstyle \mathrm{id}_A\otimes h}\uparrow \\ N & \xrightarrow{V_N^\sharp} & A\otimes_{\tau,A} N \end{array}\right)$$

commutative, for some objects $(M, F_M^\sharp)$ and $(N, F_N^\sharp)$ of $\mathbf{F}Mod_{A,\tau}$ (resp. $(M, V_M^\sharp)$ and $(N, V_N^\sharp)$ of $\mathbf{V}Mod_{A,\tau}$).The category $Cris_{A,\tau}$ consisting of all τ-crystals over A is defined by decreeing its morphisms to be the class of A-linear maps $h : N \to M$ such that any of the above diagrams commutes upon tensorization with $\mathbb{Q}$.

It is clear that all of $\mathbf{F}Mod_{A,\tau}$, $\mathbf{V}Mod_{A,\tau}$ and $Cris_{A,\tau}$ are additive and Karoubian categories. Also, notice that there exists a bi-additive, natural, associative, commutative and unital $\otimes$-structure on each of them. Observe that $(Cris_{A,\tau}, \otimes)$ is rigid, while $(\mathbf{F}Mod_{A,\tau}, \otimes)$ and $(\mathbf{V}Mod_{A,\tau}, \otimes)$ are not. Instead, passage to the dual underlying module yields a natural anti-equivalence between $(\mathbf{F}Mod_{A,\tau}, \otimes)$ and $(\mathbf{V}Mod_{A,\tau}, \otimes)$. Finally notice that there exist natural faithful forgetful $Vec(\operatorname{Spec} A)$-valued $\otimes$-functors on $\mathbf{F}Mod_{A,\tau}$, $\mathbf{V}Mod_{A,\tau}$ and $Cris_{A,\tau}$, and it will not cause confusion to denote all of them by ω^A.
In this paper the rings A will usually have the structure of a $W(\mathbb{F}_{p^f})$-algebra, for some fixed positive integer f, in which case we want to write the composition $\mathbf{F}Mod_A^{W(\mathbb{F}_{p^r})} \to \mathbf{F}Mod_A^{W(\mathbb{F}_{p^r})} \overset{\omega^A}{\to} Vec(\operatorname{Spec} A)$ (which is faithful but not a $\otimes$-functor) as a direct sum $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} \omega_\sigma^A$, provided that r divides f. Each of these (non-faithful $\otimes$-functors) $\omega_\sigma^A(M)$ arise as the largest subspace of $\omega^A(M)$ on which the $W(\mathbb{F}_{p^r})$-operation agrees with the scalar multiplication composed with the map $W(\mathbb{F}_{p^r}) \overset{F^{-\sigma}}{\to} W(\mathbb{F}_{p^r})$. A particular class of $\mathbf{F}Mod_A^{W(\mathbb{F}_{p^r})}$-objects arises as follows: Pick a non-negative integer n. We want to regard $M_\sigma := \mathrm{Mat}(n\times 1, A)$ as a $W(\mathbb{F}_{p^r}) \otimes_{\mathbb{Z}_p} A$-module, by letting A act according to the obvious multiplication, while letting $W(\mathbb{F}_{p^r})$ act via the embedding $W(\mathbb{F}_{p^r}) \to A; a \mapsto F^{-\sigma}(a)$. Let M be the $W(\mathbb{F}_{p^r})\otimes_{\mathbb{Z}_p} W(R)$-module $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} M_\sigma$. Now, pick arbitrary $B_\sigma \in \mathrm{Mat}(n\times n, A)$, and consider the map

$$A \otimes_{\tau,A} M_{\sigma+1} \to M_\sigma; a\otimes x \mapsto aB_\sigma\tau(x)$$

(this means: compose the effect of plain matrix multiplication with the map obtained by applying the absolute Frobenius to each of the entries in the $n\times 1$-matrix x). The sum of these maps defines a F-module structure on M. It is clear that a F-module with $W(\mathbb{F}_{p^r})$-operation over A arises in this way if and only if all eigenspaces of the $W(\mathbb{F}_{p^r})$-operation are free A-modules of the same rank. If M'_σ denotes another such F-module with $W(\mathbb{F}_{p^r})$-operation over A, that is gotten in the same way from a second bunch of $n \times n$-matrices B'_σ, having again all of its entries in A, then it is easy to see that the isomorphisms $M \to M'$ are given by families $h_\sigma \in \mathrm{GL}(n, A)$ with $B_\sigma = h_\sigma^{-1} B'_\sigma \tau(h_{\sigma+1})$. In short: The groupoid of F-modules with $W(\mathbb{F}_{p^r})$-operation over A, all of whose eigenspaces are free of rank n is equivalently given by the diagram

$$\mathrm{Mat}(n \times n, A)^r \supset \mathrm{GL}(n, A)^r \xrightarrow{\phi} \mathrm{Mat}(n \times n, A)^r,$$

(in the sense of example 3.2) where ϕ is the map $(U_0, U_1, \dots, U_{r-1}) \mapsto (\tau(U_1), \dots, \tau(U_{r-1}), \tau(U_0))$ (while the groupoid of τ-crystals with $W(\mathbb{F}_{p^r})$-operation over A, all of whose eigenspaces are free of rank n is equivalently given by the diagram

$$\mathrm{GL}(n, A[\frac{1}{p}])^r \supset \mathrm{GL}(n, A)^r \xrightarrow{\phi} \mathrm{GL}(n, A[\frac{1}{p}])^r,$$

where ϕ is defined in the same way). If F stands for the absolute Frobenius on the Witt ring over a commutative $\mathbb{F}_{p^f}$-algebra R, then we denote $\mathit{Cris}_{W(R),F}^{W(\mathbb{F}_{p^r})}$ (resp. $\mathbf{V}\mathit{Mod}_{W(R),F}^{W(\mathbb{F}_{p^r})}$ or $\mathbf{F}\mathit{Mod}_{W(R),F}^{W(\mathbb{F}_{p^r})}$) by $\mathit{Cris}^{W(\mathbb{F}_{p^r})}(R)$ (resp. $\mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}(R)$ or $\mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}(R)$). At last, as R varies within the category of $\mathbb{F}_{p^f}$-algebras the formation of $\mathit{Cris}^{W(\mathbb{F}_{p^r})}(R)$ (resp. $\mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}(R)$ or $\mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}(R)$) builds up natural $\mathrm{Spec}\,\mathbb{F}_{p^f}$-fibered $\mathbb{Z}_p$-linear $\mathrm{Spec}\,\mathbb{F}_{p^f}$-$\otimes$-categories $\mathit{Cris}^{W(\mathbb{F}_{p^r})}$ (resp. $\mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$ or $\mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$), in the sense of [48, I.4.5.5], and again, there are natural forgetful $\mathrm{Spec}\,\mathbb{F}_{p^f}$-fibered ${}^W\mathit{Vec}$-valued fiber functors defined on each of $\mathit{Cris}^{W(\mathbb{F}_{p^r})}$, $\mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$, and $\mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$, and it will not cause confusion to denote all of them by ω_σ, for every $\sigma \in \mathbb{Z}/r\mathbb{Z}$.

Proposition 6.4. *Let $(\mathcal{G}/W(\mathbb{F}_{p^r}), \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ be a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum. Let $\rho : \mathcal{G} \to \mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}$ be a representation and assume that all weights of all of the cocharacters $\rho \circ \upsilon_\sigma^{-1}$ are non-negative numbers, so that they extend to effective cocharacters $\beta_\sigma : \mathbb{A}^1_{W(\mathbb{F}_{p^f})} \to \mathrm{Mat}(n \times n)_{W(\mathbb{F}_{p^f})}$. Let $\gamma : \overline{\mathcal{I}}^{\upsilon_\Sigma} \to {}^W\mathrm{GL}(n)^r_{\mathbb{F}_{p^f}}$ be the function whose ωth component is given by*

$$\gamma_\omega : \{k_\sigma\}_{\sigma\in\Sigma} \mapsto ({}^{WF^{-\omega}}\rho)({}^{F^{\mathbf{d}^+_\Sigma(\omega)}}k_{\mathbf{d}_\Sigma(\omega)}),$$

for any $\omega \in \mathbb{Z}/r\mathbb{Z}$, and let $m : {}^W\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}} \to {}^W\mathrm{Mat}(n \times n)^r_{\mathbb{F}_{p^f}}$ be the function whose ωth component is given by

$$m_\omega : \{U_\sigma\}_{\sigma\in\Sigma} \mapsto \begin{cases} ({}^{WF^{-\omega}}\rho)(U_\omega)^{F^{\varpi^+_\Sigma(\omega)}}\beta_{\varpi_\Sigma(\omega)}(p) & \omega \in \Sigma \\ 1 & \textit{otherwise} \end{cases}$$

for any $\omega \in \mathbb{Z}/r\mathbb{Z}$. Then there exists a unique $\operatorname{Spec}\mathbb{F}_{p^f}$*-fibered functor*

$$\mathrm{Fib}^-(\rho) : \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$$

of which the restriction to the groupoid $B_R(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ is given by the pair of functions (γ_R, m_R) in the sense of example 3.2, for any $\mathbb{F}_{p^f}$-algebra R.

Proof. We choose a monotone bijection $\mathbb{Z} \to \Sigma^{(0)}; j \mapsto \sigma_j$. In view of $\gamma_\omega = F \circ \gamma_{\omega+1}$ for all $\omega \in [\sigma_j+1, \sigma_{j+1}-1]$, all we have to do is prove that $\gamma_{\sigma_j}(k)^{-1} m_{\sigma_j}(U)^{F^{\sigma_{j+1}-\sigma_j}}(\gamma_{\sigma_{j+1}}(k)) = m_{\sigma_j}(k^{-1}U\overline{\Phi}(k))$ holds for elements k of $\overline{\mathcal{I}}^{\upsilon_\Sigma}$ and U of ${}^W\mathcal{G}^{\Sigma}_{\mathbb{F}_{p^f}}$. This follows, if we apply the lemma 6.2 to $\beta_{\sigma_{j+1}}$. The reduction to the banal situation is achieved by Witt descent ([60, Proposition 33]). □

6.1.1. *Two Variants.* We also have to work with the following convention: If the dual $\check{\rho}$, rather than ρ satisfies the assumptions of the previous proposition, then we will write

$$\mathrm{Fib}^+(\rho) : \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$$

for the composition of the canonical self-antiequivalence of $\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$, which is defined by reversing the isomorphisms while being the identity on the objects, the covariant functor $\mathrm{Fib}^-(\check{\rho}) : \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})}$, and the canonical contravariant functor $\mathbf{F}\mathit{Mod}^{W(\mathbb{F}_{p^r})} \to \mathbf{V}\mathit{Mod}^{W(\mathbb{F}_{p^r})}; (M, F^\sharp) \mapsto (\check{M}, \check{F}^\sharp)$, which is defined by passage to the dual object. Finally, let (A, J, τ) be a frame with $p \in J$. In the same vein, we associate to an arbitrary representations $\rho \in \mathit{Rep}_0(\mathcal{G})$ a covariant functor

$$\mathrm{Fib}_{A,\tau}(\rho) : \hat{B}_{A,J}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \mathit{Cris}^{W(\mathbb{F}_{p^r})}_{A,\tau}, \tag{55}$$

by using the same fomulae for γ and m.

6.1.2. *Compatibility with* ω_σ *and* $\otimes$. For every $\sigma \in \Sigma$ there is a natural commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\mathrm{Fib}^-(\rho)} & \mathbf{F}Mod^{W(\mathbb{F}_{p^r})} \\ \downarrow & & \downarrow{\scriptstyle \omega_\sigma} \\ Tors({}^{WF^{-\sigma}}\mathcal{G}_{\mathbb{F}_{p^f}}) & \xrightarrow{{}^{WF^{-\sigma}}\rho} & {}^W Vec \end{array}$$

and similar ones for Fib^+ and Fib. Also, there are natural isomorphisms

$$\mathrm{Fib}^-(\rho \otimes_{W(\mathbb{F}_{p^r})} \rho', \mathcal{P}) \cong \mathrm{Fib}^-(\rho, \mathcal{P}) \otimes_{W(\mathbb{F}_{p^r})\otimes_{\mathbb{Z}_p} W(R)} \mathrm{Fib}^-(\rho', \mathcal{P}),$$

whenever one of the two sides (hence both of them) are well-defined, and similarly for Fib^+ and Fib.

6.1.3. *Compatibility with* $\mathrm{Flex}^{\mathbf{d}^+}$. Let $\mathbf{d}^+$ be a (mod r)-multidegree. The purpose of this subsubsection is to define a certain $\otimes$-endofunctor $\mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}$ on the $\otimes$-category $\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ (and on $\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ and $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$ too). So let (A, τ) be as in definition 6.3, and observe that for each $\omega \in \mathbb{Z}/r\mathbb{Z}$ the $\mathbf{d}^+(\omega+1) - \mathbf{d}^+(\omega) + 1$-fold iterate of the Frobenius on some a $\mathbb{Z}/r\mathbb{Z}$-graded Frobenius module $M = \bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} M_\sigma$ over (A, τ) yields a map

$$A \otimes_{\tau^{\mathbf{d}^+(\omega+1)-\mathbf{d}^+(\omega)+1}, A} M_{\mathbf{d}(\omega+1)} \to M_{\mathbf{d}(\omega)},$$

of which the pull-back along the $\mathbf{d}^+(\omega)$-fold iterate of τ reads:

$$A \otimes_{\tau^{\mathbf{d}^+(\omega+1)+1}, A} M_{\mathbf{d}(\omega+1)} \to A \otimes_{\tau^{\mathbf{d}^+(\omega)}, A} M_{\mathbf{d}(\omega)}.$$

These very maps constitute the requested homogeneous A-linear map $F^\sharp$ of degree -1 on a $\mathbb{Z}/r\mathbb{Z}$-graded A-module $\tilde{M}$ given by:

$$\tilde{M}_\omega := A \otimes_{\tau^{\mathbf{d}^+(\omega)}, A} M_{\mathbf{d}(\omega)},$$

so that $\mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}(M) := \bigoplus_{\omega\in\mathbb{Z}/r\mathbb{Z}} \tilde{M}_\omega$ is a well-defined object in $\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$. The ωth component of the image under $\mathrm{Flex}_{A,\tau}$ of some $\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$-morphism h is clearly defined to be the pull-back along the $\mathbf{d}^+(\omega)$-fold iterate of τ of the $\mathbf{d}(\omega)$ th component of h. One proceeds analogously for $\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ and $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$, and in either of the three $\otimes$-categories $\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$, $\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ or $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$ there exist natural isomorphisms

$$\mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}(M \otimes N) \cong \mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}(M) \otimes \mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}(N),$$

and similarily for duality (which swaps $\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ and $\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$). There exists a natural transformation from the endofunctor $\mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}$ on

$\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$ to $\mathrm{id}_{\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}}$ and from $\mathrm{id}_{\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}}$ to the endofunctor $\mathrm{Flex}_{A,\tau}^{\mathbf{d}^+}$ on $\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}$. We denote the former one by $\mathrm{flex}_M^{\mathbf{d}^+} : \tilde{M} \to M$, which we define to be the $\mathbf{d}^+(\omega)$-fold iterate of the Frobenius on the $\mathbf{d}(\omega)$th eigenspace of any $M \in \mathtt{Ob}_{\mathbf{F}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}}$, and we denote the latter one by $\mathrm{flex}_M^{\mathbf{d}^+} : M \to \tilde{M}$, which we define to be the $\mathbf{d}^+(\omega)$-fold iterate of the Verschiebung on the ωth eigenspace of any $M \in \mathtt{Ob}_{\mathbf{V}Mod_{A,\tau}^{W(\mathbb{F}_{p^r})}}$. If F stands for the absolute Frobenius on the Witt ring over a commutative $\mathbb{F}_{p^f}$-algebra R, then we denote $\mathrm{Flex}_{W(R),F}^{\mathbf{d}^+}$ by $\mathrm{Flex}_R^{\mathbf{d}^+}$. At last, as R varies within the category of $\mathbb{F}_{p^f}$-algebras the formation of $\mathrm{Flex}_R^{\mathbf{d}^+}$ builds up natural $\mathrm{Spec}\,\mathbb{F}_{p^f}$-$\otimes$-endofunctors $\mathrm{Flex}^{\mathbf{d}^+}$ on $\mathbf{F}Mod^{W(\mathbb{F}_{p^r})}$ and $\mathbf{V}Mod^{W(\mathbb{F}_{p^r})}$ in the sense of [48, I.4.5.3]. The promised compatibility, that we need in this paper, is expressed by a canonically 2-commutative diagram

$$\begin{array}{ccc}
\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\mathrm{Fib}^-(\rho)} & \mathbf{F}Mod^{W(\mathbb{F}_{p^r})} \\
{\scriptstyle \mathrm{Flex}^{\mathbf{d}^+}}\big\downarrow & & {\scriptstyle \mathrm{Flex}^{\mathbf{d}^+}}\big\downarrow \\
\overline{\mathcal{B}}(\mathcal{G}, \{\tilde{\upsilon}_\sigma\}_{\sigma\in\tilde{\Sigma}}) & \xrightarrow{\mathrm{Fib}^-(\rho)} & \mathbf{F}Mod^{W(\mathbb{F}_{p^r})}
\end{array},$$

and another similar one for $\mathrm{Fib}^+(\rho)$ and $\mathrm{Flex}^{\mathbf{d}^+} : \mathbf{V}Mod^{W(\mathbb{F}_{p^r})} \to \mathbf{V}Mod^{W(\mathbb{F}_{p^r})}$.

6.1.4. *Compatibility with* $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$.

Proposition 6.5. *There exists a canonical family of natural transformations:*

$$\mathrm{flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}} : \mathrm{Fib}^+(\mathrm{std}) \circ \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}} \to \mathrm{Fib}^+(\mathrm{std})$$

(indexed by the set of all standard linear $\overline{\Phi}$*-data* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$ *together with a function* $\mathbf{j}$ *as in subsection 4.1) such that the following properties hold, for any* $3n$*-display* $\mathcal{P}$ *with* $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$*-structure over any* $\mathbb{F}_{p^f}$*-algebra:*

- *Whenever* $\mathcal{P} = \mathcal{P}^{(1)} \times \mathcal{P}^{(2)}$ *holds for two displays* $\mathcal{P}^{(i)}$ *with* $(\mathrm{GL}(n^{(i)})_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma^{(i)}\}_{\sigma\in\Sigma})$*-structure, where* $\begin{pmatrix} \upsilon_\sigma^{(1)} & 0 \\ 0 & \upsilon_\sigma^{(2)} \end{pmatrix}$ *is a matrix block decomposition of* υ_σ*, then*

$$\begin{array}{ccc}
\mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}) & \longleftarrow & \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}^{(1)}) \oplus \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}^{(2)}) \\
{\scriptstyle \mathrm{flex}_{\mathcal{P}}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}}\big\downarrow & & {\scriptstyle \mathrm{flex}_{\mathcal{P}^{(1)}}^{\mathbf{j},\{\upsilon_\sigma^{(1)}\}_{\sigma\in\Sigma}} \oplus \mathrm{flex}_{\mathcal{P}^{(2)}}^{\mathbf{j},\{\upsilon_\sigma^{(2)}\}_{\sigma\in\Sigma}}}\big\downarrow \\
\mathrm{Fib}^+(\mathrm{std}, \mathcal{P}) & \longleftarrow & \mathrm{Fib}^+(\mathrm{std}, \mathcal{P}^{(1)}) \oplus \mathrm{Fib}^+(\mathrm{std}, \mathcal{P}^{(2)})
\end{array}$$

commutes, where $\tilde{\mathcal{P}} = \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}(\mathcal{P})$ *and* $\tilde{\mathcal{P}}^{(i)} = \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma^{(i)}\}_{\sigma\in\Sigma}}(\mathcal{P}^{(i)})$ *(for* $i \in \{1,2\}$*).*

- *For all standard multiplicative* $\overline{\Phi}$*-data* $(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\delta_\sigma\}_{\sigma\in\Sigma})$ *and all* $\overline{\mathcal{B}}(\mathbb{G}_{m,W(\mathbb{F}_{p^r})}, \{\delta_\sigma\}_{\sigma\in\Sigma})$*-objects* $\mathcal{T}$ *the map* $\mathrm{flex}_{\mathcal{T}}^{\mathbf{j},\{\delta_\sigma\}_{\sigma\in\Sigma}}$ *is independent of the choice of* $\mathbf{j}$ *(and we will suppress it in the notation).*
- *Whenever* $\{\delta_\sigma\}_{\sigma\in\Sigma}$ *and* $\mathcal{T}$ *are as above and* $\mathcal{P}'$ *is the* $\mathcal{T}$*-dual of* $\mathcal{P}$*, then the diagram*

$$\begin{array}{ccc} \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{T}}) & \longleftarrow & \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}) \otimes_{W(\mathbb{F}_{p^r})} \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}') \\ \downarrow {\scriptstyle \mathrm{flex}_{\mathcal{T}}^{\{\delta_\sigma\}_{\sigma\in\Sigma}}} & & \downarrow {\scriptstyle \mathrm{flex}_{\mathcal{P}}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}} \otimes \mathrm{flex}_{\mathcal{P}'}^{\check{\mathbf{j}},\{\upsilon'_\sigma\}_{\sigma\in\Sigma}}} \\ \mathrm{Fib}^+(\mathrm{std}, \mathcal{T}) & \longleftarrow & \mathrm{Fib}^+(\mathrm{std}, \mathcal{P}) \otimes_{W(\mathbb{F}_{p^r})} \mathrm{Fib}^+(\mathrm{std}, \mathcal{P}') \end{array}$$

commutes, where $\tilde{\mathcal{T}} = \mathrm{Flex}^{\{\delta_\sigma\}_{\sigma\in\Sigma}}(\mathcal{T})$ *and* $\tilde{\mathcal{P}}' = \mathrm{Flex}^{\check{\mathbf{j}},\{\upsilon'_\sigma\}_{\sigma\in\Sigma}}(\mathcal{P}')$ *(for* $\upsilon'_\sigma = \delta_\sigma\check{\upsilon}_\sigma$*).*

Finally, the image of $\mathrm{flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$ *under* ω_σ *is an isomorphism from* $\omega_\sigma \circ \mathrm{Fib}^+(\mathrm{std}) \circ \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$ *to the functor* $\omega_\sigma \circ \mathrm{Fib}^+(\mathrm{std})$ *for every* $\sigma \in \Sigma$*.*

Proof. It is clear that is is enough to do the banal case, so let $U \in \mathcal{G}^\Sigma(W(R))$ stand for a $3n$-display $\mathcal{P}$ with $(\mathrm{GL}(n)_{W(\mathbb{F}_{p^r})}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure over an affine $\mathbb{F}_{p^f}$-scheme $\operatorname{Spec} R$, and let $\tilde{\mathcal{P}}$ be $\mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}(\mathcal{P})$. Again we choose a monotone bijection $\mathbb{Z} \to \Sigma^{(0)}; j \mapsto \sigma_j$, so that U can be written as a z-tuple $(U_0, \dots, U_{z-1})$ with $U_j \in {}^{WF^{-\sigma_j}}\mathcal{G}(R)$. We switch to the dual situation $\mathrm{Fib}^-(\check{\mathrm{std}}) \to \mathrm{Fib}^-(\check{\mathrm{std}}) \circ \mathrm{Flex}^{\mathbf{j},\{\upsilon_\sigma\}_{\sigma\in\Sigma}}$, and we write β_σ and $\tilde{\beta}_\sigma$ for the two r-tuples of effective cocharacters that come up when applying the proposition 6.4 to $\mathcal{P}$ and $\tilde{\mathcal{P}}$. It does no harm to think of $\check{\mathrm{std}}$ as the standard involution (32), in particular one sees that the cocharacters $\tilde{\beta}_\sigma$ arise from the cocharacters β_σ by the procedure described in (36). Finally, let B_σ and $\tilde{B}_\sigma$ be the two r-tuples of matrices that come up when applying the proposition 6.4 to $\mathcal{P}$ and $\tilde{\mathcal{P}}$. The former looks like:

$$B_\sigma = \dots, \check{U}_{j-1}({}^{F^{\sigma_j-\sigma_{j-1}}}\beta_{\sigma_j})(p), \dots, 1, \check{U}_j({}^{F^{\sigma_{j+1}-\sigma_j}}\beta_{\sigma_{j+1}})(p), \dots,$$

where the "1" is in the $\sigma_j - 1$th position, and the latter looks like:

$$\tilde{B}_\sigma = \dots, \check{U}_{j-1}({}^F\tilde{\beta}_{\sigma_{j-1}+1})(p), \dots, ({}^F\tilde{\beta}_{\sigma_j})(p), \check{U}_j({}^F\tilde{\beta}_{\sigma_j+1})(p), \dots.$$

Let us define r-tuples of matrices by $k_\sigma = \prod_{\omega=\sigma+1}^{\sigma_j}({}^{F^{\omega-\sigma}}\tilde{\beta}_\omega)(p)$ whenever $\sigma_{j-1} + 1 \leq \sigma \leq \sigma_j$, and notice that $\beta_{\sigma_j} = \prod_{\omega=\sigma_{j-1}+1}^{\sigma_j}({}^{F^{\omega-\sigma_j}}\tilde{\beta}_\omega)$ holds, because of (37). It follows that $k_\sigma B_\sigma = \tilde{B}_\sigma {}^F k_{\sigma+1}$ is true and we are done, the requested bijectivity of $\omega_{\sigma_j}(\mathrm{flex}^{\mathbf{j},\{v_\sigma\}_{\sigma\in\Sigma}})$ follows from $k_{\sigma_j}=1$. □

6.2. **Faithfulness of some realizations.** Let $\mathfrak{G}$ be a affine and smooth $\mathbb{Z}_p$-group. The category of τ-crystals with $\mathfrak{G}$-structure over A is defined in the usual way: The objects are $\mathbb{Z}_p$-linear $\otimes$-functors from $Rep_0(\mathfrak{G})$ to $Cris_{A,\tau}$, and the morphisms are $\mathbb{Z}_p$-linear, invertible and $\otimes$-preserving transformations. Let us call a τ-crystal $M : Rep_0(\mathfrak{G}) \to Cris_{A,\tau}$ banal if the $\otimes$-functors $A \otimes_{\mathbb{Z}_p} \omega^{\mathfrak{G}}$ and $\omega^A \circ M$ are $\otimes$-isomorphic, i.e. if and only if there exists at least one 2-commutative diagram of $\otimes$-functors:

$$\begin{array}{ccc} Rep_0(\mathfrak{G}) & \xrightarrow{M} & Cris_{A,\tau} \\ \omega\big\downarrow & & \omega^A\big\downarrow \\ Vec(\operatorname{Spec}\mathbb{Z}_p) & \longrightarrow & Vec(\operatorname{Spec} A), \end{array}$$

where the bottom horizontal functor sends $\mathcal{V}$ to $A\otimes_{\mathbb{Z}_p}\mathcal{V}$ ('base change'). The full subcategory of banal τ-crystals with $\mathfrak{G}$-structure over A is canonically equivalent to the groupoid $\mathbf{B}_{A,\tau}(\mathfrak{G})$ as every $b \in \mathfrak{G}(A[\frac{1}{p}])$ represents a τ-crystal structure M_b on the functor $A \otimes_{\mathbb{Z}_p} \omega^{\mathfrak{G}}$ given by the composition of the $\varrho(b)$-action on $A \otimes_{\mathbb{Z}_p} \omega^{\mathfrak{G}}(\varrho)$ with $\tau \otimes \mathrm{id}_{\omega^{\mathfrak{G}}(\varrho)}$ for varying $\varrho \in \mathsf{Ob}_{Rep_0(\mathfrak{G})}$.

Definition 6.6. *Let $\mathfrak{T} = (\{(\mathcal{V}_i, \Psi_i, \rho_i)\}_{i\in\Lambda}, \{(\mathcal{R}_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$, and $\mathcal{G}/B$, and $(\Lambda, \{\mathcal{R}_\pi\}_{\pi\in\Pi})$ and C be as in the beginning of this section. Then $\mathfrak{T}$ is called a C-linear metaunitary collection for $\mathcal{G}/B$, if the following hold:*

(S1) *The product*

$$\rho := \prod_{i\in\Lambda} \rho_i : \mathcal{G} \to \prod_{i\in\Lambda} \mathrm{GU}(\mathcal{V}_i/C, \Psi_i)$$

is a closed immersion.

(S2) *The (pointwise) stabilizer of $\bigcup_{\pi\in\Pi} \mathcal{R}_\pi$ in the generic fiber of $\prod_{i\in\Lambda} \mathrm{GU}(\mathcal{V}_i/C, \Psi_i)$ is contained in the image of ρ.*

In the above scenario we write $\chi_i : \mathcal{G} \to \mathbb{G}_{m,B}$ for the composition of the multiplier character of $\mathrm{GU}(\mathcal{V}_i/C, \Psi_i)$ with ρ_i. It is easy to see that $\mathcal{G}^1 := \bigcap_{i\in\Lambda} \ker(\chi_i)$ is a smooth B-group, in fact we will frequently use that $\mathrm{Res}_{C/B}\,\mathbb{G}_{m,C}^\Lambda$ is canonically immersed into the center of $\mathcal{G}$, while $\mathbb{G}_{m,B}^\Lambda$ intersects $\mathcal{G}^1$ in $\{\pm1\}^\Lambda$, so that $\mathcal{G}$ is canonically isomorphic to $(\mathbb{G}_{m,B}^\Lambda \times_B \mathcal{G}^1)/\{\pm1\}^\Lambda$. Furthermore (S2) implies that $\mathcal{G}$ (resp. $\mathcal{G}_C^1$) agrees with the Zariski-closure of its generic fiber in $\prod_{i\in\Lambda} \mathrm{GU}(\mathcal{V}_i/C, \Psi_i)$ (resp. in $\prod_{i\in\Lambda} \mathrm{GL}(\mathcal{V}_i/C)$), which in turn implies that one can pin down a constant c such that $\mathcal{G}$ (resp. $\mathcal{G}_C^1$)

is the stabilizer in $\prod_{i\in\Lambda}\mathrm{GU}(\mathcal{V}_i/C,\Psi_i)$ (resp. in $\prod_{i\in\Lambda}\mathrm{GL}(\mathcal{V}_i/C)$) of $\mathcal{O}_{Z_c}=\mathrm{End}_G(\bigotimes_{i\in\Lambda}\mathcal{V}_i^{\otimes c})$, cf. [23, Proposition(1.3.2)].

Remark 6.7. If one assumes $\frac{1}{2}\in B$ and that $\mathcal{G}$ is reductive, then $\prod_{i\in\Lambda}\rho_i$ is a closed immersion if and only if its generic fiber has that property ([44, Corollary 1.3]).

We focus on the case $B=W(\mathbb{F}_{p^r})$ with C being one of $W(\mathbb{F}_{p^{2r}})$ or $W(\mathbb{F}_{p^r})\oplus W(\mathbb{F}_{p^r})$. It is practical to put these cases on a equal footing.

Lemma 6.8. *Let A be a torsionfree, p-adically separated and complete $W(\mathbb{F}_{p^f})$-algebra, and let $\tau:A\to A_{[F]}$ be a Frobenius lift. Let $\mathcal{G}$ be a smooth and affine $W(\mathbb{F}_{p^r})$-group with connected fibers, and let the pair of families of triples $\mathfrak{T}=(\{(\mathcal{V}_i,\Psi_i,\rho_i)\}_{i\in\Lambda},\{(\mathcal{R}_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$ be a C-linear metaunitary collection for $\mathcal{G}$. The self-explanatory "restriction to $\mathfrak{T}$" defines a fully faithful functor*

$$\mathbf{B}_{A,\tau}(\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G})\to \mathit{Cris}_{A,\tau}^{\Lambda,\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi}};M\mapsto M|_{\mathfrak{T}}$$

on the category of banal τ-crystals with $\mathrm{Res}_{W(\mathbb{F}_{p^r})/\mathbb{Z}_p}\mathcal{G}$-structure over A.

6.3. **Definition of $\mathrm{Flex}^{\mathbf{T}}$.** From now on, and for the rest of this section we fix a $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-datum $(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$, choose a monotone bijection $\mathbb{Z}\to\Sigma^{(0)};j\mapsto\sigma_j$, and let C be as in the end of the previous subsection.

Definition 6.9. *Let $\{\mathbf{j}_i\}_{i\in\Lambda}$ be a family of functions from $\mathbb{Z}$ to $\mathbb{N}_0$, such that every $\mathbf{j}_i$ maps the interval $[\sigma_{j-1}+1,\sigma_j]$ onto $[0,\sigma_j-\sigma_{j-1}-1]$ for each j, and satisfies* $\mathbf{j}_i(r+\sigma)=\begin{cases}\mathbf{j}_i(\sigma) & C=W(\mathbb{F}_{p^r})\oplus W(\mathbb{F}_{p^r})\\ r_\Sigma(\mathbf{d}_\Sigma(\sigma))-\mathbf{j}_i(\sigma)-1 & C=W(\mathbb{F}_{p^{2r}})\end{cases}$
for every $\sigma\in\mathbb{Z}/r\mathbb{Z}$ (cf. subsubsection 4.1.1). Consider a C-linear metaunitary collection $(\{(\mathcal{V}_i,\Psi_i,\rho_i)\}_{i\in\Lambda},\{(\mathcal{R}_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$ for the group $\mathcal{G}$. We say that

$$\mathbf{T}=(\{(\mathcal{V}_i,\Psi_i,\rho_i,\mathbf{j}_i)\}_{i\in\Lambda},\{(\mathcal{R}_\pi,*,\iota_\pi)\}_{\pi\in\Pi}),$$

is a C-linear gauged metaunitary collection for $(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$ if the following additional requirements are fulfilled:

- *If one puts ${}^{F^{-\sigma}}\rho_i\circ\upsilon_\sigma=:\upsilon_{i,\sigma}$, then one obtains standard $\overline{\Phi}$-data (cf. definition 4.1) $(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma})$, for every $i\in\Lambda$.*
- *Every element of Π is multicompact (cf. definition 4.13) with respect to the family of standard Φ-data $(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\tilde{\upsilon}_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$, which is derived from $\upsilon_{i,\sigma}$ and $\mathbf{j}_i$ via (36).*
- *If $R_\Sigma:=\max\{r_\Sigma(\sigma)|\sigma\in\mathbb{Z}/r\mathbb{Z}\}$, then one has $\mathcal{R}_\pi\subset C+p^{(R_\Sigma-\frac{1}{2})\mathrm{Card}(\pi)+\frac{1}{2}}\mathrm{End}_C(\mathcal{V}^\pi)$.*

The C-linear tensor products $\mathcal{V}^\pi = \bigotimes_{i\in\pi}\mathcal{V}_i$ carry natural skew-Hermitian structures Ψ^π, and consider the map

$$g^\pi : \prod_{i\in\pi}\mathrm{GU}(\mathcal{V}_i/C,\Psi_i) \to \mathrm{GU}(\mathcal{V}^\pi/C,\Psi^\pi),$$

obtained by the formation of the tensor product. The purpose of introducing gauged metaunitary collections lies in a certain 2-commutative diagram

$$\begin{array}{ccc} \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}} & \longrightarrow & \prod_{i\in\Lambda}\overline{\mathcal{B}}(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\tilde{\upsilon}_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}}) \\ {\scriptstyle \mathrm{Flex}^{\mathbf{T}}}\uparrow & & \uparrow{\scriptstyle \prod_{i\in\Lambda}\mathrm{Flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}} \\ \overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\prod_{i\in\Lambda}\overline{\mathcal{B}}(\rho_i)} & \prod_{i\in\Lambda}\overline{\mathcal{B}}(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}) \end{array},$$

where $\mathbf{R} = (\Lambda,\{(\mathcal{R}_\pi,*)\}_{\pi\in\Pi})$. Here is the construction of $\mathrm{Flex}^{\mathbf{T}}$: We start out from $\overline{\mathcal{B}}(\rho_i,\mathcal{P}) =: \mathcal{P}_i \in \mathtt{Ob}_{\overline{\mathcal{B}}(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma})(R)}$ and $\overline{\mathcal{B}}(\chi_i,\mathcal{P}) =: \mathcal{K}_i \in \mathtt{Ob}_{\overline{\mathcal{B}}(\mathbb{G}_{m,W(\mathbb{F}_{p^r})},\{\delta_\sigma\}_{\sigma\in\Sigma})(R)}$, which are gotten from some fixed $\mathrm{Spec}\,R \xrightarrow{\mathcal{P}} \overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$, by extensions of its structure group. On the one hand each of the representations ρ_i and χ_i gives rise to a graded realization within the categories $\mathbf{V}Mod^{W(\mathbb{F}_{p^{2r}})}(R)$ or $\mathbf{V}Mod^{W(\mathbb{F}_{p^r})}(R)$, together with the canonical $\mathrm{Fib}^+(\chi_i,\mathcal{P})$-valued sesquilinear perfect pairing on $\mathrm{Fib}^+(\rho_i,\mathcal{P})$, which is induced from the isomorphism $\chi_i\otimes\check{\bar{\rho}}_i \cong \rho_i$. On the other hand, we may look at the regularizations

$$\tilde{\mathcal{P}}_i := \mathrm{Flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}(\mathcal{P}_i)$$
$$\tilde{\mathcal{K}}_i := \mathrm{Flex}^{\{\delta_\sigma\}_{\sigma\in\Sigma}}(\mathcal{K}_i),$$

which are $3n$-displays in the usual sense whose underlying graded modules are given by $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}_i)$ and $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{K}}_i)$. As we have seen already the results are related by specific isogenies

$$\mathrm{flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}_{\mathcal{P}_i} : \mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}_i) \to \mathrm{Fib}^+(\rho_i,\mathcal{P})$$
$$\mathrm{flex}^{\{\delta_\sigma\}_{\sigma\in\Sigma}}_{\mathcal{K}_i} : \mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{K}}_i) \to \mathrm{Fib}^+(\chi_i,\mathcal{P}),$$

preserving the V-actions, the gradations, and the canonical sesquilinear perfect pairings, which are defined in both of the two scenarios. It is easy to see that there exists a map from $\mathrm{Fib}^+(\chi_i,\mathcal{P})$ to $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{K}}_i)$ of which the composition with $\mathrm{flex}^{\{\delta_\sigma\}_{\sigma\in\Sigma}}_{\mathcal{K}_i}$ in any order is equal to the multiplication by $p^{R_\Sigma-1}$. Whence one obtains a map from $\mathrm{Fib}^+(\rho_i,\mathcal{P})$ back to $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}_i)$ of which the composition and precomposition with $\mathrm{flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}_{\mathcal{P}_i}$ yields the $p^{R_\Sigma-1}$th multiples of the identities of $\mathrm{Fib}^+(\rho_i,\mathcal{P})$ and $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}_i)$. The same reasoning can be applied to tensor products of these maps giving rise to gradation and V-preserving maps:

$$\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}^\pi)(\frac{\mathrm{Card}(\pi)-1}{2}) \rightleftarrows \mathrm{Fib}^+(\rho^\pi,\mathcal{P})$$

whose products are $p^{(R_\Sigma-1)\,\mathrm{Card}(\pi)}$, where $\tilde{\mathcal{P}}^\pi$ is defined to be the restricted tensor product $\dot{\bigotimes}_{i\in\pi}\tilde{\mathcal{P}}_i$ and $\rho^\pi := \bigotimes_{i\in\pi}\rho_i$. In total there is an action of $p^{(R_\Sigma-\frac{1}{2})\,\mathrm{Card}(\pi)-\frac{1}{2}}\,\mathrm{End}_{\mathcal{G}}(\mathcal{V}^\pi)$ on $\mathrm{Fib}^+(\mathrm{std},\tilde{\mathcal{P}}^\pi)$ and hence there is an action of $\mathcal{R}_\pi$ on $\tilde{\mathcal{P}}^\pi$, since we have the following:

Lemma 6.10. *Let $\mathcal{P}=(M,N,F,V^{-1})$ be a $3n$-display over a $\mathbb{F}_p$-algebra R, and suppose that h is a $W(R)$-linear endomorphism of M, that renders at least one of the diagrams*

$$\begin{array}{ccc} W(R)\otimes_{F,W(R)}M & \xrightarrow{F^\sharp} & M \\ {\scriptstyle \mathrm{id}_{W(R)}\otimes h}\uparrow & & \uparrow{\scriptstyle h} \\ W(R)\otimes_{F,W(R)}M & \xrightarrow{F^\sharp} & M \end{array} \quad or \quad \begin{array}{ccc} M & \xrightarrow{V^\sharp} & W(R)\otimes_{F,W(R)}M \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_{W(R)}\otimes h} \\ M & \xrightarrow{V^\sharp} & W(R)\otimes_{F,W(R)}M \end{array}$$

commutative. Then the $W(R)$-linear map $s:M\to M; x\mapsto ph(x)$ is an endomorphism of $\mathcal{P}$.

Proof. By duality we are allowed to assume that h commutes with F, it is clear that s preserves N, and the proof is finished by $V^{-1}(sx)=F(hx)=hF(x)=sV^{-1}(x)$ for every $x\in N$. □

6.4. **Local properties of** $\mathrm{Flex}^{\mathbf{T}}$**.**

Lemma 6.11. *The* 1*-morphism* $\mathrm{Flex}^{\mathbf{T}}$ *is schematic, quasicompact and radicial for every gauged metaunitary collection* $\mathbf{T}$.

Proof. The quasicompact schematicness of $\prod_{i\in\Lambda}\overline{\mathcal{B}}(\rho_i):\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})\to\prod_{i\in\Lambda}\overline{\mathcal{B}}(\mathrm{GU}(\mathcal{V}_i/C,\Psi_i),\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma})$ follows immediately from the representability results at the end of subsection 2.2, in fact the attentive reader might observe that the paper [47] could be used to show its affineness, but this will not be needed.
For the quasicompact schematicness of the morphisms $\mathrm{Flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}$ please see proposition 4.3. At last, the statement on the radiciality follows from the lemmas 6.8 and 3.33 together with the observation that the diagram

$$\begin{array}{ccc} \hat{B}_{W(R),I(R)}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) & \xrightarrow{\mathrm{Fib}_{W(R),F}(\rho)} & Cris^{W(\mathbb{F}_{p^r})}(R) \\ {\scriptstyle \mathrm{Flex}_R^{\varpi_\Sigma^+}}\downarrow & & \uparrow \\ \hat{B}_{W(R),I(R)}(\mathcal{G},\{{}^{F^{\varpi_\Sigma^+(\sigma)}}\upsilon_{\varpi_\Sigma(\sigma)}\}_{\sigma\in\Sigma}) & \xrightarrow{\hat{\mathbf{h}}_\mu} & \mathbf{B}_{W(R),F}(\mathfrak{G}) \end{array}$$

commutes for all perfect rings R (notice that $\hat{B}_{W(R),I(R)}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})\cong B_R(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$). □

We wish to introduce another definition, which is mainly used for an intersting corollary: Fix some 1-morphism

$$X\stackrel{\mathcal{P}}{\to}\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}),$$

and let X' be the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{O}_X\oplus\Omega^1_{X/\mathbb{F}_{p^f}}$. The inclusion $\mathcal{O}_X\subset\mathcal{O}_X\oplus\Omega^1_{X/\mathbb{F}_{p^f}}$, (resp. the map $x\mapsto x+d_{X/\mathbb{F}_{p^f}}(x)$) gives rise to a projection $\overline{pr}_1$ (resp. $\overline{pr}_2$) from X' to X: Consequently there exists a unique $K_{\mathcal{P}}\in\operatorname{Hom}_{\mathcal{O}_X}(T_{\mathcal{P}},\Omega^1_{X/\mathbb{F}_{p^f}})$ which measures the difference between $\mathcal{P}\times_{X,\overline{pr}_2}X'$ and $\mathcal{P}\times_{X,\overline{pr}_1}X'$ when regarded as global lifts in $D_{\mathcal{P},X'}$. We will call $K_{\mathcal{P}}$ the Kodaira-Spencer element of $\mathcal{P}$.

Remark 6.12. A 1-morphism $X\stackrel{\mathcal{P}}{\to}\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$ is formally étale (resp. formally smooth) if and only if X is formally smooth over $\mathbb{F}_{p^f}$, and $K_{\mathcal{P}}$ is a bijection (resp. a Zariski-locally split injection).

Corollary 6.13. *Suppose that X is a $\mathbb{F}_{p^f}$-scheme. If $X\stackrel{\mathcal{S}}{\to}\mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}$ is a formally étale 1-morphism, then*

$$Y:=\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})\times_{\mathrm{Flex}^{\mathbf{T}},\mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}},\mathcal{S}}X$$

is a formally smooth $\mathbb{F}_{p^f}$-scheme whose sheaf of Kaehler differentials $\Omega^1_{Y/\mathbb{F}_{p^f}}$ is locally free of rank equal to $\dim_{\mathbb{F}_{p^f}}\overline{\theta}^{\upsilon_\Sigma}$.

Remark 6.14. The dimension of $\overline{\theta}^{\upsilon_\Sigma}$ is equal to the υ_Σ-weight of the $U^0_{\upsilon_\Sigma}$-character $\det\circ\operatorname{Lie}U^0_{\upsilon_\Sigma}$.

6.5. Finiteness properties of $\mathrm{Flex}^{\mathbf{T}}$.

Lemma 6.15. *Fix a $\mathbb{F}_{p^f}$-scheme X, and X-scheme Y. Assume that*

$$\begin{array}{ccc} Y & \longrightarrow & \overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Flex}^{\mathbf{T}}} \\ X & \stackrel{\mathcal{S}}{\longrightarrow} & \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}} \end{array},$$

is a 2-commutative diagram.

(i) *Assume in addition, that Y is reduced. Then there is at most one such 2-commutative diagram (i.e. there exist unique isomorphisms to any other ones)*

(ii) *Assume in addition, that Y is faithfully flat and quasicompact over X. If $Y \times_X Y$ is reduced, then $\mathcal{S}$ factors through $\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$.*

Proof. If $Z \to X$ is a radicial morphisms of schemes, then its relative diagonal induces an isomorphism from Z_{red} to $(Z \times_X Z)_{red}$, as $\Delta_{Z/X} : Z \to Z \times_X Z$ is a surjection. Therefore part (i) follows easily and part (ii) follows from descent theory and part (i). □

Our next result in this subsection deals with fields:

Proposition 6.16. *Consider an extension $\mathbb{F}_{p^f} \subset k$ of fields with finite degree of imperfection, and fix a 1-morphism*

$$\operatorname{Spec} k \xrightarrow{\mathcal{S}} \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}.$$

If the k-scheme $\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \times_{\mathrm{Flex}^{\mathbf{T}}, \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}, \mathcal{S}} \operatorname{Spec} k$ is non-empty, then it possesses a point over a finite extension of k.

Proof. By lemma 6.15, it is clear that we may assume k to be separably closed, and due to the same reason we are allowed to assume that there exists a display $\mathcal{P}^\circ$ with $\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure over the algebraic closure k^{ac}, together with a 2-isomorphism

$$\zeta^\circ : \mathcal{S}_{k^{ac}} \xrightarrow{\cong} \mathrm{Flex}^{\mathbf{T}}(\mathcal{P}^\circ).$$

By lemma 3.22 we may assume $C = W(\mathbb{F}_{p^r}) \oplus W(\mathbb{F}_{p^r})$, in which case a simple application of Morita equivalence tells us that our "unitary" representations reads $\rho_i \oplus \chi_i \otimes_{W(\mathbb{F}_{p^r})} \check{\rho}_i$ for some $W(\mathbb{F}_{p^r})$-rational homomorphisms $\rho_i : \mathcal{G} \to \mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}$ and characters $\chi_i : \mathcal{G} \to \mathbb{G}_{m,W(\mathbb{F}_{p^r})}$, so that $\mathrm{GU}(\mathcal{V}_i/C, \Psi_i) = (\mathbb{G}_m \times \mathrm{GL}(n_i))_{W(\mathbb{F}_{p^r})}$. In the proof we will work directly with ρ_i and χ_i rather than $\rho_i \oplus \chi_i \otimes_{W(\mathbb{F}_{p^r})} \check{\rho}_i$. Likewise, a k-valued point of $\mathfrak{B}_{\mathbb{F}_{p^f}}$ is determined by a family s_π of $\mathcal{R}_\pi$-operations on the family of restricted tensor products $\mathcal{S}^\pi := \dot{\bigotimes}_{i\in\pi} \mathcal{S}_i$ coming from some other family $\mathcal{S}_i$ of displays with $(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\tilde{\upsilon}_{i,\sigma}\}_{\sigma\in\mathbb{Z}/r\mathbb{Z}})$-structure over k, together with some multiplicative family of no significance. In this optic ζ° is given by a family of $\{s_\pi\}_{\pi\in\Pi}$-preserving isomorphisms:

$$\zeta_i^\circ : \mathcal{S}_{i,k^{ac}} \xrightarrow{\cong} \tilde{\mathcal{P}}_i^\circ := \mathrm{Flex}^{\mathbf{j}_i, \{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}(\mathcal{P}_i^\circ),$$

where $\mathcal{P}_i^\circ$ stands for the image of $\mathcal{P}^\circ$ under the canonical 1-morphism

$$\overline{\mathcal{B}}(\rho_i) : \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \to \overline{\mathcal{B}}(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})}, \{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}).$$

Recall that $\mathrm{Fib}^+(\mathrm{std}, \mathcal{P}_i^\circ) = \mathrm{Fib}^+(\rho_i, \mathcal{P}^\circ) =: N_i^\circ = \bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} N_{i,\sigma}^\circ$ is a $\mathbb{Z}/r\mathbb{Z}$-graded V-module over $W(k^{ac})$ canonically associated to $\mathcal{P}^\circ$, and in the category of $\mathbb{Z}/r\mathbb{Z}$-graded V-modules over $W(k^{ac})$ we have a canonical $\mathbb{Z}/r\mathbb{Z}$-graded isogeny:

$$\mathrm{flex}_{\mathcal{P}_i^\circ}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}} : \mathrm{Fib}^+(\mathrm{std}, \tilde{\mathcal{P}}_i^\circ) \to N_i^\circ.$$

Let τ be a choice of Frobenius lift on a choice of Cohen ring A for the (in general non-perfect) field k, so that (A, pA, τ) is a frame over $W(\mathbb{F}_{p^f})$. We have a canonical commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & W(k) & \longrightarrow & W(k^{ac}) \\ {\scriptstyle\tau}\uparrow & & {\scriptstyle F}\uparrow & & {\scriptstyle F}\uparrow \\ A & \longrightarrow & W(k) & \longrightarrow & W(k^{ac}) \end{array},$$

of which the horizontal arrows are $W(\mathbb{F}_{p^f})$-linear inclusions. Let M_i denote the $\mathbb{Z}/r\mathbb{Z}$-graded (A, pA, τ)-windows to $\mathcal{S}_i$. Composition produces further isogenies:

$$W(k^{ac}) \otimes_A M_i \stackrel{u_i^\circ}{\to} N_i^\circ,$$

paving the way for a transport of structure, that turns the natural action ι of $\mathcal{O}_{Z_c}$ on the $Cris^{W(\mathbb{F}_{p^r})}(k^{ac})$-object $\mathrm{Fib}^+((\bigotimes_{i\in\Lambda}\rho_i)^{\otimes_{W(\mathbb{F}_{p^r})}c}, \mathcal{P}^\circ) =: N^\circ$, whose σ-eigenspaces are $N_\sigma^\circ = (\bigotimes_{i\in\Lambda} N_{i,\sigma}^\circ)^{\otimes_{W(k^{ac})}c}$, into its analog $s : Z_c \to \mathbb{Q} \otimes \mathrm{End}_{W(\mathbb{F}_{p^r})}(W(k^{ac}) \otimes_A M)$ for the $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$-object $M = \bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} M_\sigma$ defined by the formula

$$M_\sigma := (\bigotimes_{i\in\Lambda} M_{i,\sigma})^{\otimes_A c}$$

(within which M_i is to be interpreted in $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$). We are going to use, and will now have to check, that s yields an action on $\mathbb{Q}\otimes M$, rather than $K(k^{ac}) \otimes_A M$: To this end we write $H_\sigma/A[\frac{1}{p}]$ for the common stabilizer of the family of $\mathcal{R}_\pi$-actions s_π, this is a smooth and affine $A[\frac{1}{p}]$-group contained in the product of the groups $\mathrm{GL}(\mathbb{Q} \otimes M_i/A[\frac{1}{p}])$. If $\mathcal{Z}_\sigma$ denotes the A-algebra $\mathrm{End}_{H_\sigma}(M_\sigma)$ of H_σ-invariant, A-linear endomorphisms of M_σ, then $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} \mathcal{Z}_\sigma$ forms a $\mathbb{Z}/r\mathbb{Z}$-graded τ-crystal over A, and the composition of endomorphisms makes a $Cris_{A,\tau}^{W(\mathbb{F}_{p^r})}$-morphism. In contrast, $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} \mathrm{End}_{F^{-\sigma}\mathcal{G}_{W(k^{ac})}}(N_\sigma^\circ)$ is a $Cris^{W(\mathbb{F}_{p^r})}(k^{ac})$-object of slope 0, in fact it is equal to $W(k^{ac}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{Z_c}$ with Frobenius acting in the obvious way. Comparing these, we find the existence of at least one Frobenius-invariant A-lattice in $\mathbb{Q} \otimes \bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} \mathcal{Z}_\sigma$, leading the latter to be generated by its skeleton $S := (\mathbb{Q} \otimes \bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} \mathcal{Z}_\sigma)^{F=\mathrm{id}}$ (N.B.: the residue field of A is separably closed). This shows that the image of s is contained in $\mathbb{Q} \otimes \mathrm{End}_{W(\mathbb{F}_{p^r})}(M)$, as the sceleton of

$K(k^{ac}) \otimes_A \bigoplus_{\sigma \in \mathbb{Z}/r\mathbb{Z}} \mathcal{Z}_\sigma$ is clearly also equal to S.
Notice, that every element $\{U_\sigma\}_{\sigma\in\Sigma} = U \in \mathcal{G}^\Sigma(W(k^{ac}))$ representing the object $\mathcal{P}^\circ$ (i.e. a choice of banalization) will induce specific isomorphisms $W(k^{ac})^{n_i} \cong N_{i,\sigma}^\circ$, and due to the algebraic closedness of k^{ac}, there exist a lot of banalizations. Moreover observe, that the images of A^{n_i} under those very isomorphisms provide us with non-canonical A-module structures $N_{i,\sigma} \subset N_{i,\sigma}^\circ$ on the $W(k^{ac})$-modules $N_{i,\sigma}^\circ$, which are canonically associated to $\mathcal{P}^\circ$. Also notice, that a switch between two different banalizations, has the effect of altering the artificial A-structures on the canonical $W(k^{ac})$-modules $N_{i,\sigma}^\circ$ according to the formulae in proposition 6.4. When looking more specifically at $\sigma \in \Sigma$ a switch from, say U, to an alternative representative given by, say $O := h^{-1}U\overline{\Phi}(h)$, has the effect of switching from $N_{i,\sigma}$ to $({}^{F^{-\sigma}}\rho_i)(h_\sigma)^{-1}N_{i,\sigma}$, where $h = \{h_\sigma\}_{\sigma\in\Sigma}$. For the time being we do fix U and the artificial A-structures $N_{i,\sigma}$ going with it.
Recall that the isogeny u_i° induces isomorphisms, say $u_{i,\sigma}^\circ : M_{i,\sigma} \stackrel{\cong}{\to} N_{i,\sigma}^\circ$ for all $\sigma \in \Sigma$ being due to an analogous property of $\mathrm{flex}_{\mathcal{P}_i^\circ}^{\mathbf{j}_i,\{v_{i,\sigma}\}_{\sigma\in\Sigma}}$ (cf. proposition 6.5). This demonstrates already, that the Z_c-action s on the isocrystal $\mathbb{Q} \otimes M$ can be restricted to an integral action of $\mathcal{O}_{Z_c}$ on the A-module M_σ, provided that σ is contained in Σ. We will shortly derive an even stronger result, but before we do that we have to introduce a certain principal homogeneous space $\mathfrak{S}_\sigma$ for ${}^{F^{-\sigma}}\mathcal{G}^1$ over A, for every $\sigma \in \Sigma$: If Q is a A-algebra, then we let $\mathfrak{S}_\sigma(Q)$ be the set of families $\{q_i\}_{i\in\Lambda}$ with the following two properties:

- Each q_i is a Q-linear isomorphism from $Q \otimes_A M_{i,\sigma}$ to $Q \otimes_A N_{i,\sigma}$.
- The isomorphism $(\bigotimes_{i\in\Lambda} q_i)^{\otimes c} = q : Q \otimes_A M_\sigma \to Q \otimes_A N_\sigma$ is $Q \otimes_{W(\mathbb{F}_{p^r})} \mathcal{O}_{Z_c}$-linear

Clearly, one has $\{u_{i,\sigma}^\circ\}_{i\in\Lambda} \in \mathfrak{S}_\sigma(W(k^{ac}))$ for every $\sigma \in \Sigma$. Since $W(k^{ac})$ is faithfully flat over A, we get the local triviality of the $\mathfrak{S}_\sigma$'s. As A is strictly henselian we get the existence of global sections, i.e. some $A \otimes_{W(\mathbb{F}_{p^r})} \mathcal{O}_{Z_c}$-preserving family of A-linear isomorphisms:

$$u_{i,\sigma} : M_{i,\sigma} \stackrel{\cong}{\to} N_{i,\sigma},$$

for each $\sigma \in \Sigma$. As i varies the family $u_{i,\sigma}^\circ \circ u_{i,\sigma}^{-1}$ constitutes a $W(k^{ac})$-valued point of ${}^{F^{-\sigma}}\mathcal{G}$. Let h_0 be sufficiently large in the sense of remark 3.10. One can choose a subfield $l \subset A/pA$ of finite degree over k, together with elements $\{h_\sigma^\circ\}_{\sigma\in\Sigma} = h^\circ \in \mathcal{G}^\Sigma(W(l))$ such that

$$u_{i,\sigma}^\circ \circ u_{i,\sigma}^{-1} \equiv ({}^{F^{-\sigma}}\rho_i)(h_\sigma^\circ) \pmod{p^{h_0}}$$

and let $h_\sigma \in {}^{F^{-\sigma}}\mathcal{G}(p^n W(k^{ac}))$ be the elements with $({}^{F^{-\sigma}}\rho_i)(h_\sigma^{-1} \circ h_\sigma^\circ) = u_{i,\sigma}^\circ \circ u_{i,\sigma}^{-1}$. Observe that the oblique map in the diagram:

$$\begin{array}{ccc} W(k^{ac}) \otimes_A N_{i,\sigma} & \xrightarrow{{}^{F^{-\sigma}}\rho_i(h_\sigma)} & W(k^{ac}) \otimes_A N_{i,\sigma} \\ {\scriptstyle u_{i,\sigma}^\circ}\uparrow & & \uparrow{\scriptstyle {}^{F^{-\sigma}}\rho_i(h_\sigma^\circ)} \\ M_{i,\sigma} & \xrightarrow{u_{i,\sigma}} & N_{i,\sigma} \end{array}$$

carries $M_{i,\sigma}$ to $W(l) \otimes_A N_{i,\sigma}$, furthermore by the aforementioned remark on banalizations, we may assume $h_\sigma = 1$ without loss of generality. We deduce that $u_{i,\sigma}^\circ$ carries $M_{i,\sigma}$ to $W(l) \otimes_A N_{i,\sigma}$, so that the vertical arrows in the commutative diagram

$$\begin{array}{ccc} W(k^{ac}) \otimes_A N_{i,\sigma} & \xrightarrow{({}^{F^{\varpi_\Sigma^+(\sigma)}}\upsilon_{\varpi_\Sigma(\sigma)})(p)({}^{F^{-\sigma}}\rho_i)(U_\sigma^{-1})} & W(k^{ac}) \otimes_{\tau^{\varpi_\Sigma^+(\sigma)}, A} N_{i,\varpi_\Sigma(\sigma)} \\ {\scriptstyle u_{i,\sigma}^\circ}\uparrow & & \uparrow{\scriptstyle u_{i,\varpi_\Sigma(\sigma)}^\circ} \\ W(l) \otimes_A M_{i,\sigma} & \xrightarrow{(V^\sharp)^{\varpi_\Sigma^+(\sigma)}} & W(l) \otimes_{\tau^{\varpi_\Sigma^+(\sigma)}, A} M_{i,\varpi_\Sigma(\sigma)} \end{array}$$

preserve the $W(l)$-structure, forcing its upper horizontal arrow to preserve that $W(l)$-structure too. For elements $\sigma \in \Sigma$ and for every i it follows, that $({}^{F^{-\sigma}}\rho_i)(U_\sigma^{-1}) \in \mathrm{GL}(n_i, W(\sqrt[p^m]{l}))$ holds for a sufficiently large integer m, this is because of $W(k^{ac}) \cap \frac{W(l)}{p^m} = W(\sqrt[p^m]{l})$. Due to property (S1) we finally deduce $U_\sigma \in ({}^{F^{-\sigma}}\mathcal{G})(W(\sqrt[p^m]{l}))$, i.e. $\{U_\sigma\}_{\sigma\in\Sigma}$ represents indeed a display $\mathcal{P}$ with $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure over the field extension $\sqrt[p^m]{l}$ of k. The same type of argument settles the $\sqrt[p^m]{l}$-rationality of the ζ_i°'s, thus culminating in a $\sqrt[p^m]{l}$-valued point of

$$\overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \times_{\mathrm{Flex}^{\mathbf{T}}, \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}, \mathcal{S}} \operatorname{Spec} k.$$

□

Proposition 6.17. *Let* $\mathbf{T}$ *be a gauged metaunitary collection for a* $W(\mathbb{F}_{p^f})$*-rational* $\overline{\Phi}$*-datum* $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$*, and let us assume that* $\mathcal{G}$ *is reductive, and that every* υ_σ *is minuscule. Suppose that the field* k *is an algebraically closed extension of* $\mathbb{F}_{p^f}$*, and fix a* 1*-morphism*

$$\operatorname{Spec} k[[t]] \xrightarrow{\mathcal{S}} \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}.$$

If the $k((t))$-scheme $\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})\times_{\mathrm{Flex}^{\mathbf{T}},\mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}},\mathcal{S}}\mathrm{Spec}\,k((t))$ is non-empty, then there exists a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\,k[[\sqrt[p^m]{t}]] & \longrightarrow & \overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma}) \\ \downarrow & & {\scriptstyle \mathrm{Flex}^{\mathbf{T}}}\downarrow \\ \mathrm{Spec}\,k[[t]] & \xrightarrow{\mathcal{S}} & \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}} \end{array},$$

for some sufficiently large number m.

Proof. Let $\mathcal{P}^\circ$ be a display with $(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure over $k((t))$, and let

$$\zeta^\circ:\mathcal{S}_{k((t))}\stackrel{\cong}{\to}\mathrm{Flex}^{\mathbf{T}}(\mathcal{P}^\circ)$$

be a 2-isomorphism. Just as in the proof of the previous proposition we may and we will assume that $C=W(\mathbb{F}_{p^r})\oplus W(\mathbb{F}_{p^r})$, furthermore adopting the previous notation, we have to start out from a family $(\{\mathcal{S}_i\}_{i\in\Lambda},\{s_\pi\}_{\pi\in\Pi})$ of $\mathcal{R}_\pi$-operations s_π on a family of displays $\mathcal{S}_i$ with $(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})},\{\tilde{\upsilon}_{i,\sigma}\}_{\sigma\in\Sigma})$-structure over $k[[t]]$ (along with an insignificant multiplicative family). In the same vein ζ° is given by a family:

$$\zeta_i^\circ:\mathcal{S}_{i,k((t))}\stackrel{\cong}{\to}\tilde{\mathcal{P}}_i^\circ:=\mathrm{Flex}^{\mathbf{j}_i,\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}}(\mathcal{P}_i^\circ),$$

where $\mathcal{P}_i^\circ$ stands for the image of $\mathcal{P}^\circ$ under the canonical 1-morphism

$$\overline{\mathcal{B}}(\rho_i):\overline{\mathcal{B}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})\to\overline{\mathcal{B}}(\mathrm{GL}(n_i)_{W(\mathbb{F}_{p^r})},\{\upsilon_{i,\sigma}\}_{\sigma\in\Sigma}).$$

We set up Frobenius lifts on both $A:=W(k)[[t]]$ and the p-adic completion of $A[\frac{1}{t}]$, by decreeing $t\mapsto t^p$, and it does not cause confusion to denote both of these lifts by τ. Note that (A,pA,τ) is a frame over $W(\mathbb{F}_{p^f})$ and dito for $(A_{\{t\}},pA_{\{t\}},\tau)$. In view of proposition 6.16 we may assume the banality of $\mathcal{P}^\circ$, so that it is induced by a unique isomorphism class in the category $\hat{B}_{A_{\{t\}},pA_{\{t\}}}(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$ of $(A_{\{t\}},pA_{\{t\}},\tau)$-windows with $(\mathcal{G},\{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure, by lemma 3.32. Its image under the functor $\mathrm{Fib}_{A_{\{t\}},\tau}(\rho_i)$ (cf. (55)) shall be denoted by $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}}N_{i,\sigma}^\circ=N_i^\circ\in\mathtt{Ob}_{\mathbf{V}Mod^{W(\mathbb{F}_{p^r})}_{A_{\{t\}},\tau}}$. Any specific choice of $\{U_\sigma\}_{\sigma\in\Sigma}=U\in\mathcal{G}^\Sigma(A_{\{t\}})$ which represents the window for $\mathcal{P}^\circ$ determines coordinate systems $A_{\{t\}}^{n_i}\cong N_{i,\sigma}^\circ$, and we denote the images of A^{n_i} under our ("non-canonical") coordinate systems by: $N_{i,\sigma}\subset N_{i,\sigma}^\circ$. Notice that the isomorphism class of N_i° depends merely on $\mathcal{P}^\circ$ while different representatives give rise to different A-structures on $N_{i,\sigma}$, and at least for the elements of Σ, one can describe the transition from the representative U to an alternative, say $O:=h^{-1}U\hat{\Phi}_{A_{\{t\}}}(h)$ by replacing $N_{i,\sigma}$ with $({}^{F^{-\sigma}}\rho_i)(h_\sigma)^{-1}N_{i,\sigma}$.

Let $\bigoplus_{\sigma\in\mathbb{Z}/r\mathbb{Z}} M_{i,\sigma} = M_i$ denote the $\mathbb{Z}/r\mathbb{Z}$-graded (A, pA, τ)-windows to $\mathcal{S}_i$, and notice that just as in the proof of proposition 6.16 there are canonical $Cris^{W(\mathbb{F}_{p^r})}_{A_{\{t\}},\tau}$-isogenies

$$u_i^\circ : A_{\{t\}} \otimes_A M_i \to N_i^\circ,$$

induced by our $(A_{\{t\}}, pA_{\{t\}}, \tau)$-window to $\mathcal{P}_i^\circ$ and proposition 6.5. By construction, their σth components $u_{i,\sigma}^\circ$ are isomorphisms between the $A_{\{t\}}$-modules $A_{\{t\}} \otimes_A M_{i,\sigma}$ and $N_{i,\sigma}^\circ$, for $\sigma \in \Sigma$ only. Let M be the $\mathbb{Z}/r\mathbb{Z}$-graded τ-crystal over A given by the formula $M_\sigma := (\bigotimes_{i\in\Lambda} M_{i,\sigma})^{\otimes_{A^c}}$ for every $\sigma \in \mathbb{Z}/r\mathbb{Z}$. As soon as having noticed, that transport of structure establishes an action $s : Z_c \to \mathbb{Q} \otimes \mathrm{End}_{W(\mathbb{F}_{p^r})}(A_{\{t\}} \otimes_A M)$ satisfying:

(i) By restriction s induces an action of $\mathcal{O}_{Z_c}$ on the $A_{\{t\}}$-module $A_{\{t\}} \otimes_A M_\sigma$, for all $\sigma \in \Sigma$.
(ii) The image of s is contained $\mathbb{Q} \otimes \mathrm{End}_{W(\mathbb{F}_{p^r})}(M)$
(iii) By restriction s induces an action of $\mathcal{O}_{Z_c}$ on the A-module M_σ, for all $\sigma \in \Sigma$.

it is about time to invoke the principal homogeneous space $\mathfrak{S}_\sigma$ for ${}^{F^{-\sigma}}\mathcal{G}^1$ over A, of which the Q-valued points are the $\mathcal{O}_{Z_c}$-preserving families of Q-linear isomorphisms $Q \otimes_A M_{i,\sigma} \xrightarrow{\cong} Q \otimes_A N_{i,\sigma}$, for each $\sigma \in \Sigma$ of course. Here are the reasons for (i),(ii), and (iii): (i) follows from the aforementioned bijectivity of $u_{i,\sigma}^\circ$, (ii) is an application of the results of [31], [8], and (iii) follows from (i) together with (ii). Over each of $K(k)\{\{t\}\}$ and $A_{\{t\}}$ there exist rational points of $\mathfrak{S}_\sigma$, the former is due to Dwork's trick and the latter is due to the existence of the family $\{u_{i,\sigma}^\circ\}_{i\in\Lambda}$. Following the ideas in [23] we see that we obtain the local triviality of $\mathfrak{S}_\sigma|_{\mathrm{Spec}\,A - \mathrm{Spec}\,k}$, and hence the global triviality of $\mathfrak{S}_\sigma$ by [6] together with the algebraic closedness of k: Just as in the proof of proposition 6.16, any global section $\{u_{i,\sigma}\}_{i\in\Lambda}$ of $\mathfrak{S}_\sigma$ can be used for a careful adjustment of $N_{i,\sigma}$ in order to achieve the integrality of $u_{i,\sigma}^\circ$ for each $\sigma \in \Sigma$. Finally, we get $\{U_\sigma\}_{\sigma\in\Sigma} = U \in \mathcal{G}^\Sigma(A)$ from (S1) and we are done, notice $A_{\{t\}} \cap \frac{A}{p} = A$. □

Proposition 6.18. *Let $X \to \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}$ be a 1-morphism, where X is a noetherian $\mathbb{F}_{p^f}$-scheme. Then there exists a X-scheme Y and a 2-cartesian diagram*

$$\begin{array}{ccc} Y & \longrightarrow & \overline{\mathcal{B}}(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Flex}^{\mathbf{T}}} \\ X & \longrightarrow & \mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}} \end{array},$$

furthermore, Y is radicial, affine and integral over X. Assume in addition that:

- *X is a Nagata scheme*
- *The degree of imperfection of the residue fields of each maximal point of X is finite.*

Then Y_{red} is finite over X.

Proof. Since all assertions are local on X we may assume $X = \operatorname{Spec} A$, recall that we proved that $\mathrm{Flex}^{\mathbf{T}}$ is schematic, quasicompact and separated, hence the 2-cartesian diagram for some quasicompact and separated A-scheme Y. First of all we note that Y is universally closed over $\operatorname{Spec} A$, simply because the proposition 6.17 gives us the valuative criterion (N.B.: Y is quasicompact and A is noetherian). Without loss of generality we may assume the surjectivity of $Y \to \operatorname{Spec} A$, otherwise there was an ideal $\sqrt{I} = I \subset A$ such that $\operatorname{Spec} A/I$ was the image of Y and it did no harm to replace A by A/I. We continue with the affineness: Let $U_i \subset Y$ be an open and affine covering of Y, and let V_i be the image of U_i in $\operatorname{Spec} A$. Since bijectivity and closedness implies the openness of a morphism we know that the V_i's form an open covering of $\operatorname{Spec} A$, so let us choose a refinement consisting of sets of the form $D(f_i)$, for certain elements $f_i \in A$. Let us define the set of sections $g_i \in \Gamma(Y, \mathcal{O}_Y)$ as the pull-backs of f_i, and let us note that Y_{g_i} is clearly an open covering of Y. Moreover, every Y_{g_i} has to be affine: Just choose some j with $D(f_i) \subset V_j$, so that $Y_{g_i} \subset U_j$ and $Y_{g_i} = \operatorname{Spec} \Gamma(U_j, \mathcal{O}_Y)_{g_i|_{U_j}}$ as desired. Hence $Y = \operatorname{Spec} B$, and the integrality of B over A is clear. Notice that the finiteness assertion was already shown for the special case of a field (of finite degree of imperfection). In order to prove the asserted greater generality we may and we will assume without loss of generality that A is reduced (so that the map from A to B is injective). Let S be the set of non-zerodivisors of A. The elements of S are still non-zerodivisors of B, here notice, that a universal homeomorphism maps points of height 0 to points of height 0. Now, since the fiber of $\mathrm{Flex}^{\mathbf{T}}$ over $\operatorname{Spec} A$ is $\operatorname{Spec} B$, its fiber over $\operatorname{Spec} S^{-1}A$ is clearly $\operatorname{Spec} S^{-1}B$ (being the fiber of $\operatorname{Spec} B \to \operatorname{Spec} A$ over $\operatorname{Spec} S^{-1}A$). However $S^{-1}A$ is a finite product of fields, say $\prod_{i=1}^n k_i$, so that $S^{-1}B_{red} = \prod_{i=1}^n l_i$, where l_i is a (purely inseparable and) finite field extension of k_i. It follows that B_{red} is contained in the integral closure of A in $\prod_{i=1}^n l_i$, and this is finite over the Nagata ring A. □

Remark 6.19. Proposition 6.18 has a remarkable consequence for the universal formal equicharacteristic deformations: Let $\mathcal{P}_0$ be a display with $(\mathcal{G}, \{\upsilon_\sigma\}_{\sigma\in\Sigma})$-structure over a perfect field k, let us write $\mathcal{S}_0$ for the image in $\mathfrak{B}^{\mathbf{R}}_{\mathbb{F}_{p^f}}$, let $\mathcal{P}_{uni}/R$ and $\mathcal{S}_{uni}/A$ be their universal formal equicharacteristic deformations over the complete noetherian rings R and A (cf. corollary 3.29). Let $\operatorname{Spec} B$ be the fiber of $\mathrm{Flex}^{\mathbf{T}}$ over $\operatorname{Spec} A$, which is a local scheme. It is clear that R is the pro-artinian completion of B. However, since the former happens to be reduced (it's a power series algebra) we obtain a factorization $B \to B_{red} \to R$. According to the previous proposition this entails the completeness of B_{red}, and thus $B_{red} = \hat{B} = R$.

7. Connections

In this section we fix a $W(\mathbb{F}_{p^f})$-rational Φ-datum $(\mathfrak{G}, \mu)$, where $\mathfrak{G}$ is a $\mathbb{Z}_p$-group, and we assume $p \neq 2$. Let $\mathcal{P}$ be a display with $(\mathfrak{G}, \mu)$-structure over a $W(\mathbb{F}_{p^f})$-scheme X on which p is nilpotent. Recall that it gives rise to locally trivial principal homogeneous spaces $q(\mathcal{P})$ and $q(\mathcal{P}) \times^{\mathcal{I}^\mu} {}^W\mathfrak{G}$ for the groups $\mathcal{I}^\mu$ and ${}^W\mathfrak{G}$ over X. Note also that we defined a crystal $\mathbb{H}_\mathcal{P}$ over $(X/W(\mathbb{F}_{p^f}))_{cris}$ taking values in the fibration $Tors_{{}^W\mathfrak{G}}$. We are going to use a certain canonical connection on $q(\mathcal{P}) \times^{\mathcal{I}^\mu} {}^W\mathfrak{G}$, which can be described as follows: Assume w.l.o.g. $X = \operatorname{Spec} R$, so that the two coordinate maps $\overline{pr}_i : R \to R \oplus \Omega^1_{R/W(\mathbb{F}_{p^f})}$ (cf. part B of the appendix) yield $(X/W(\mathbb{F}_{p^f}))_{cris}$-morphisms, say

$$(R \oplus \Omega^1_{R/W(\mathbb{F}_{p^f})}, \Omega^1_{R/W(\mathbb{F}_{p^f})}, 0) =: V' \stackrel{v_i}{\to} V := (R, 0, 0)$$

for $i \in \{1, 2\}$. By the crystalline nature of $\mathbb{H}_\mathcal{P}$ there are associated isomorphisms:

$$H_1 \stackrel{\alpha_1}{\leftarrow} \mathbb{H}_\mathcal{P}(V') \stackrel{\alpha_2}{\to} H_2, \tag{56}$$

where $H_i := \mathbb{H}_\mathcal{P}(V) \times_{V, v_i} V'$. In order to compute the curvature of the connection $\alpha := \alpha_1 \circ \alpha_2^{-1}$ we have to invoke further pull-backs H_i' of the locally trivial principal homogeneous space $q(\mathcal{P}) \times^{\mathcal{I}^\mu} {}^W\mathfrak{G}$ along the three projections w_i from

$$V'' = (R \oplus \Omega^{1,\oplus 2}_{R/W(\mathbb{F}_{p^f})} \oplus \Omega^2_{R/W(\mathbb{F}_{p^f})}, \Omega^{1,\oplus 2}_{R/W(\mathbb{F}_{p^f})} \oplus \Omega^2_{R/W(\mathbb{F}_{p^f})}, 0).$$

to V, which are given by $(x, 0, 0, 0)$, $(x, d_R(x), 0, 0)$, and $(x, d_R(x), d_R(x), 0)$ (in the notation of proposition B.1), here observe that V'' is an object of $(X/W(\mathbb{F}_{p^f}))_{cris}$, and that w_1, w_2, and w_3 are morphisms in $(X/W(\mathbb{F}_{p^f}))_{cris}$. Let $\alpha_{i,j} : H_j' \to H_i'$ denote the pull-backs of α along $\overline{pr}_{i,j}$ (cf. part B of the appendix). The diagram (56) together with the transitivity of pull-back (i.e. the crystalline nature of $\mathbb{H}_\mathcal{P}$) allow to infer that these morphisms coincide with $\beta_i \circ \beta_j^{-1}$, where $\beta_i : \mathbb{H}_\mathcal{P}(V'') \to H_i'$

is induced from w_i. Whence it follows

$$\alpha_{12}\circ\alpha_{23}\circ\alpha_{13}^{-1}=\beta_1\circ\beta_2^{-1}\circ\beta_2\circ\beta_3^{-1}\circ(\beta_1\circ\beta_3^{-1})^{-1}=1,$$

and we will say that α is the Witt-connection of $\mathcal{P}$. For a banal display with representative $O\in\mathfrak{G}(W(R))$ one may argue more explicitely: If O' is the image of O under the map $\overline{pr}_2: R\to R\oplus\Omega^1_{R/W(\mathbb{F}_{p^f})}$, and if $x_O\in\mathfrak{G}(W(\Omega^1_{R/W(\mathbb{F}_{p^f})}))$ is an element with

$$O\Psi^{\mu,1}_{\Omega^1_{R/W(\mathbb{F}_{p^f})}}(x_O)O^{-1}=x_OO'O^{-1}$$

then x_O describes indeed the Witt-connection of $\mathcal{P}$ (please see definition 3.13 for the meaning of $\Psi^{\mu,1}_{\Omega^1_{R/W(\mathbb{F}_{p^f})}}$). When thinking of $\mathfrak{G}(W(\Omega^1_{R/W(\mathbb{F}_{p^f})}))$ as the additive group $W(\Omega^1_{R/W(\mathbb{F}_{p^f})})\otimes_{\mathbb{Z}_p}\mathfrak{g}$, the equation metamorphoses into:

$$\mathrm{Ad}^{\mathfrak{G}}(O)\psi^{\mu,1}_{\Omega^1_{R/W(\mathbb{F}_{p^f})}}(x_O)-x_O=O'O^{-1},$$

where $\psi^{\mu,1}_{\Omega^1_{R/W(\mathbb{F}_{p^f})}}$ is the map described in remark 3.14. Similarily, if $U\in\mathfrak{G}(W(R))$ is an alternative representative, so that $O=k^{-1}U\Phi^\mu(k)$ holds for some element $k\in\mathcal{I}^\mu(R)$, then $\mathrm{Ad}^{\mathfrak{G}}(k)(x_O)$ is the sum of x_U and $k'k^{-1}$ in $\mathfrak{G}(W(\Omega^1_{R/W(\mathbb{F}_{p^f})}))$.

Our next result presents an alternative approach to connections, which is more classical: Fix a Frobenius lift τ on a torsionfree, $\mathfrak{a}$-adically separated and complete $W(\mathbb{F}_{p^f})$-algebra A, where $\mathfrak{a}$ is an ideal containing some power of p. We consider the module of formal m-forms

$$\hat{\Omega}^m_{A/W(\mathbb{F}_{p^f})}:=\varprojlim\bigwedge^m_{A/\mathfrak{a}^n}\Omega^1_{A/\mathfrak{a}^n},$$

which is isomorphic to the $\mathfrak{a}$-adic completion of $\bigwedge^m_A\Omega^1_{A/W(\mathbb{F}_{p^f})}$. Observe that τ acts on each $\hat{\Omega}^m_{A/W(\mathbb{F}_{p^f})}$, and let τ_1 denote the unique $\mathfrak{a}$-adically continuous τ-linear endomorphism on $\bigoplus_{m\neq0}\hat{\Omega}^m_{A/W(\mathbb{F}_{p^f})}$ satisfying

$$\tau_1(d_Ax\wedge\omega):=(d_A\frac{\tau(x)-x^p}{p}+x^{p-1}d_Ax)\wedge\tau(\omega).$$

The Lie-algebra of $\mathfrak{G}$ is equipped with a decomposition $\bigoplus_{l\in\mathbb{Z}}\mathfrak{g}_l=W(\mathbb{F}_{p^f})\otimes_{\mathbb{Z}_p}\mathfrak{g}$, according to the weights of $\mu:\mathbb{G}_{m,W(\mathbb{F}_{p^f})}\to\mathfrak{G}_{W(\mathbb{F}_{p^f})}$. On the space

$$\bigoplus_{m\geq1\geq l}\hat{\Omega}^m_A\otimes_{W(\mathbb{F}_{p^f})}\mathfrak{g}_l=\bigoplus_{m\geq1}\hat{\Omega}^m_A\otimes_{\mathbb{Z}_p}\mathfrak{g}$$

of $\mathfrak{g}$-valued formal differential forms of positive degree we consider yet another endomorphism given by:

$$\psi_A^\mu(\omega \otimes x) = \begin{cases} \tau_1(\omega) \otimes \tau(x) & l = 1 \\ p^{-l}\tau(\omega) \otimes \tau(x) & \text{otherwise} \end{cases}$$

for all $x \in \mathfrak{g}_l$, and then we have:

Lemma 7.1. *If $U \in \mathfrak{G}(A)$ satisfies the $\mod \mathfrak{a}$-nilpotence condition, then there exists a unique element $D_U \in \hat{\Omega}_A^1 \otimes_{\mathbb{Z}_p} \mathfrak{g}$ with*

$$\mathrm{Ad}^{\mathfrak{G}}(U)(\psi_A^\mu(D_U)) - D_U = \eta_{\mathfrak{G}} \circ U,$$

and the following further properties hold:

(i) *The curvature vanishes, i.e. $d_A(D_U) + \frac{[D_U, D_U]}{2} = 0$.*

(ii) *Let n be a positive integer and let O be the image of $\hat{\delta}(U) \in \mathfrak{G}(W(A))$ in $\mathfrak{G}(W(A/\mathfrak{a}^n))$, where $\hat{\delta}$ is as in subsubsection 3.6. Then O represents a banal display with $(\mathfrak{G}, \mu)$-structure over $\operatorname{Spec} A/\mathfrak{a}^n$ whose Witt-connection x_O is described by the element of $\mathfrak{G}(W(\Omega_{A/\mathfrak{a}^n}^1)) = (\prod_{k=0}^\infty \Omega^1_{A/\mathfrak{a}^n}{}_{[w_k]}) \otimes_{\mathbb{Z}_p} \mathfrak{g}$, of which the kth component is equal to the image of $\tau_1^k(D_U)$ in $\Omega^1_{A/\mathfrak{a}^n} \otimes_{\mathbb{Z}_p} \mathfrak{g}$.*

Proof. This is analogous to [5, Lemma 2.8]. The nilpotence condition implies that $D \mapsto \mathrm{Ad}^{\mathfrak{G}}(U)(\psi_A^\mu(D)) - \eta_{\mathfrak{G}} \circ U$ is a contractive map for the $\mathfrak{a}$-adic topology, so there exists a unique fixedpoint D_U. The curvature of D_U satisfies $\mathrm{Ad}^{\mathfrak{G}}(U)(\psi_A^\mu(R_U)) = R_U \in \hat{\Omega}_A^2 \otimes_{\mathbb{Z}_p} \mathfrak{g}$, again this has a unique solution, namely $R_U = 0$. □

By slight abuse of terminology we will call the said element D_U of the previous lemma the Dieudonné connection of U.

7.1. **Formal Connections.** We want to specialize to the case $A := W(k_0)[[t_1, \dots, t_d]]$ and $\mathfrak{a} := pA + \sum_{i=1}^d t_i A$, where the field k_0 is an algebraically closed or an algebraic extension of $\mathbb{F}_{p^f}$ and $d := \mathrm{rank}_{W(\mathbb{F}_{p^f})} \mathfrak{g}_1$. Let us fix the Frobenius lift determined by $\tau(t_i) := t_i^p$. As in [20] we need to invoke an important subalgebra of the power series algebra in d indeterminates over $K(k_0)$:

$$K(k_0)\{\{t_1, \dots, t_d\}\} := \{\sum_{\underline{n}} a_{\underline{n}} \underline{t}^{\underline{n}} \,|\, |a_{\underline{n}}|_p C^{n_1 + \dots + n_d} \to 0 \,\forall C < 1\}$$

One has $A[\frac{1}{p}] \subset K(k_0)\{\{t_1, \dots, t_d\}\} \subset K(k_0)[[t_1, \dots, t_d]]$, and it is very straightforward to extend the endomorphism τ to each of these $K(k_0)$-algebras. We write $\mathrm{Aut}(A/W(k_0))$ for the $W(k_0)$-linear automorphisms of A. This group acts naturally on $K(k_0)\{\{t_1, \dots, t_d\}\}$ and on $\bigoplus_{i=1}^d K(k_0)\{\{t_1,\dots,t_d\}\}dt_i$ from the left (in fact it is well-known that the latter objects allow more canonical descriptions without explicit variables, e.g. as spaces of rigid functions or Kaehler differentials on the generic fiber of $\operatorname{Spf} A$). From this $\operatorname{Aut}(A/W(k_0))$-action one derives an important semi-direct product

$$\mathbb{T} := \mathfrak{G}(K(k_0)\{\{t_1,\dots,t_d\}\}) \rtimes \operatorname{Aut}(A/W(k_0)).$$

Any display $\mathcal{P}_0$ with $(\mathfrak{G},\mu)$-structure over $\operatorname{Spec} k_0$ is automatically banal, and hence represented by some $U_0 \in \mathfrak{G}(W(k_0))$. Choose a μ-basis of d additive 1-parameter subgroups $\epsilon_1,\dots,\epsilon_d : \mathbb{G}_a \to \mathfrak{G}_{W(\mathbb{F}_{p^f})}$ of μ-weights 1, as in definition C.2. Notice that the element $U_1 := (\prod_{i=1}^d \epsilon_i(t_i))U_0$ satisfies the nilpotence condition $\mod \mathfrak{a}$, so that it possesses a well-defined Dieudonné connection $D_1 \in \hat{\Omega}_A^1 \otimes_{\mathbb{Z}_p} \mathfrak{g}$, in the sense of and according to lemma 7.1. Our prime tool is the trick of Dwork, which provides us with an element $\Theta \in \mathfrak{G}(K(k_0)\{\{t_1,\dots,t_d\}\})$ satisfying:

$$\Theta(0,\dots,0) = 1$$
$$\Theta^{-1}b_1\tau(\Theta) = b_0$$
$$\eta_{\mathfrak{G}} \circ \Theta = -D_1$$

where $b_0 = {U_0}^F\mu(\frac{1}{p})$ and $b_1 = {U_1}^F\mu(\frac{1}{p})$ (N.B.: the last equation has to be interpreted in $\bigoplus_{i=1}^d K(k_0)\{\{t_1,\dots,t_d\}\}dt_i$). Notice that the subgroup $\Theta(\mathfrak{G}(K(k_0)) \times \operatorname{Aut}(A/W(k_0)))\Theta^{-1} \subset \mathbb{T}$ consists of exactly all solutions to the differential equation:

$$\operatorname{Ad}(u)s(D_1) - D_1 = \eta_{\mathfrak{G}} \circ u, \tag{57}$$

here the right-hand side is again the image of $\eta_{\mathfrak{G}}$ under the map $\Omega_{\mathfrak{G}}^1 \to \bigoplus_{i=1}^d K(k_0)\{\{t_1,\dots,t_d\}\}dt_i$ induced by u, and the left-hand side uses the natural left action of $\operatorname{Aut}(A/W(k_0))$. Also, notice that the element $U_{uni} := \hat{\delta}(U_1) \in \mathtt{Ob}_{\mathcal{B}(\mathfrak{G},\mu)(A)}$ represents the universal formal mixed characteristic deformation $\mathcal{P}_{uni}$ of $\mathcal{P}_0$, which has a rich amount of symmetry: For every $\gamma \in \operatorname{Aut}(\mathcal{P}_0)$, there exists a unique pair $(h_\gamma, s_\gamma) \in \mathcal{I}^\mu(A) \rtimes \operatorname{Aut}(A/W(k_0))$ with:

- $s_\gamma(U_{uni}) = h_\gamma^{-1}U_{uni}\Phi^\mu(h_\gamma)$
- h_γ is a lift of $\gamma \in \mathcal{I}^\mu(k_0)$

Passage to the level-0 truncation $u_\gamma := w_0(h_\gamma)$ yields elements $(u_\gamma, s_\gamma) \in \mathcal{I}_0^\mu(A) \rtimes \operatorname{Aut}(A/W(k_0))$ which are horizontal in the sense that (57) holds (N.B.: Part (ii) of lemma 7.1 shows that D_1 depends only on $\mathcal{P}_{uni}$ and not on the choice of τ). After these preparatory remarks we are able to state and prove the very important technical fact that D_1 tends to be complicated as can be, here $K(k_0)^{ac}$ (resp. $K(k_0)^{ac}\{\{t_1,\dots,t_d\}\}$, A^{ac},

$\hat{\Omega}^1_{A^{ac}/K(k_0)^{ac}}$, etc.) denotes an algebraic closure of $K(k_0)$ (resp. the p-adically incomplete tensor products $K(k_0)^{ac} \otimes_{K(k_0)} K(k_0)\{\{t_1, \dots, t_d\}\}$, $K(k_0)^{ac} \otimes_{K(k_0)} A$, $K(k_0)^{ac} \otimes_{W(k_0)} \hat{\Omega}^1_{A/W(\mathbb{F}_{p^f})}$, etc.).

Lemma 7.2. *Let $G^{spc} \lhd \mathfrak{G}_{\mathbb{Q}_p}$ be the smallest $\mathbb{Q}_p$-rational normal subgroup, such that no μ-weight of $\operatorname{Lie} \mathfrak{G}_{\mathbb{Q}_p} / \operatorname{Lie} G^{spc}$ is positive. Then $D_1 \in \hat{\Omega}^1_A \otimes_{\mathbb{Z}_p} \operatorname{Lie}(G^{spc})$ holds. Assume in addition that the following holds:*

(i) *The group $\mathfrak{G}$ is reductive and of adjoint type.*
(ii) *The field k_0 is finite.*
(iii) *There exists some $s \in \mathbb{N}$ with $b_0 {}^F b_0 \cdots {}^{F^{s-1}} b_0 = 1$*

Then there do not exist any proper subgroups $H \subset G^{spc}_{K(k_0)^{ac}}$ for which there are elements $u \in \mathfrak{G}(A^{ac})$ with:

$$\operatorname{Ad}^{\mathfrak{G}}(u)(D_1) - \eta_{\mathfrak{G}} \circ u \in \hat{\Omega}^1_A \otimes_{W(k_0)} \operatorname{Lie} H$$

Proof. By the right-invariance of the Cartan-Maurer form one has:

$$\eta_{\mathfrak{G}} \circ U_1 = \eta_{\mathfrak{G}} \circ (\prod_{i=1}^d \epsilon_i(t_i)) \in \hat{\Omega}^1_A \otimes_{\mathbb{Z}_p} \operatorname{Lie}(G^{spc}).$$

Hence the p-adically contractive map $D \mapsto \operatorname{Ad}^{\mathfrak{G}}(U_1)(\psi^\mu_A(D)) - \eta_{\mathfrak{G}} \circ U_1$ preserves $\hat{\Omega}^1_A \otimes_{\mathbb{Z}_p} \operatorname{Lie}(G^{spc})$, because $\operatorname{Lie}(G^{spc})$ is a Lie ideal.
Note that $K(k_0)$ will allow a sufficiently large Galois extension $N \subset K(k_0)^{ac}$ over which both of H and u are defined, furthermore we can pick an extension of the absolute Frobenius on $K(k_0)$ to N (hence an extension of τ to $N\{\{t_1, \dots, t_d\}\}$). In order to simplify the notation we may also assume $k_0 = \mathcal{O}_N / \mathfrak{m}_N$, so that N is totally ramified over $K(k_0)$. Let $u_0 \in \mathfrak{G}(N)$ be the evaluation of u at the specific point $t_1 = \dots = t_d = 0$. Consider the elements $\operatorname{Ad}^{\mathfrak{G}}(u)(D_1) - \eta_{\mathfrak{G}} \circ u =: \tilde{D} \in \hat{\Omega}^1_A \otimes_{W(k_0)} \operatorname{Lie} H$, $\tilde{b} = u b_1 \tau(u^{-1})$, and $\tilde{\Theta} := u \Theta u_0^{-1}$. Since $\tilde{\Theta}$ is neutral at the origin and satisfies the differential equation $-\tilde{D} = \eta_{\mathfrak{G}} \circ \tilde{\Theta}$, we are allowed to deduce $\tilde{\Theta} \in H(N\{\{t_1, \dots, t_d\}\})$.
Observe $\tilde{\Theta}^{-1} \tilde{b} \tau(\tilde{\Theta}) = u_0 b_0 \tau(u_0^{-1})$, and if the assumption (iii) is satisfied by some positive multiple s of the degree of k_0 we get: $\tilde{\Theta} \tau^s(\tilde{\Theta}^{-1}) = \tilde{b} \tau(\tilde{b}) \cdots \tau^{s-1}(\tilde{b}) \in H(N \otimes_{W(k_0)} W(k_0)[[t_1, \dots, t_d]])$. Towards shifting our attention to the Witt ring we introduce the map $\hat{\delta}_N := \operatorname{id}_N \otimes \hat{\delta}$ from $N \otimes_{W(k_0)} A$ to $N \otimes_{W(k_0)} W(A/pA)$, where $\hat{\delta}$ is Cartier's diagonal

associated to the Frobenius lift τ. If we let

$$\begin{aligned}\tilde{b}_{uni} &:= \hat{\delta}_N(\tilde{b})\\ \tilde{h}_\gamma &:= \hat{\delta}_N(u)h_\gamma\hat{\delta}_N(s_\gamma(u^{-1}))\\ \tilde{u}_\gamma &:= uu_\gamma s_\gamma(u^{-1})\end{aligned}$$

for each $\gamma \in \mathrm{Aut}(\mathcal{P}_0)$, we find

$$s_\gamma(\tilde{b}_{uni}) = \tilde{h}_\gamma^{-1}\tilde{b}_{uni}F(\tilde{h}_\gamma).$$

The key to the proof is the slope homomorphism $\tilde{\nu}$ of $\tilde{b}_{uni}$. Since H is s_γ-invariant, we infer that each $\tilde{h}_\gamma^{-1}\tilde{\nu}\tilde{h}_\gamma$ is a cocharacter of H. The h_γ's are Zariski-dense by inspection of their special fibers, observe that $\mathrm{Aut}(\mathcal{P}_0)$ is a p-adically open subgroup in a certain $\mathbb{Q}_p$-form of $\mathfrak{G}$ by [45, Proposition 1.12]). It follows that H contains a normal subgroup containing $\tilde{\nu}$, and corollary 3.35 finishes the proof. □

7.2. **Existence of lifts.**

Lemma 7.3. *Let A be a $W(\mathbb{F}_{p^f})$-algebra in which p is nilpotent, and let $\mathfrak{a} \subset A$ be an ideal such that $\mathfrak{a}^2 + p\mathfrak{a} = 0$. Let $\mathcal{P}$ be a display with $(\mathfrak{G}, \mu)$-structure over A, and denote its pull-back to $\overline{A} := A/pA$ by $\overline{\mathcal{P}}$. Fix a A-linear homomorphism $t : \Omega^1_{\overline{A}/\mathbb{F}_{p^f}} \to \mathfrak{a}$ and write N for the composition*

$$\check{T}_{\overline{\mathcal{P}}} \stackrel{K_{\overline{\mathcal{P}}}}{\to} \Omega^1_{\overline{A}/\mathbb{F}_{p^f}} \stackrel{t}{\to} \mathfrak{a}.$$

Furthermore, the automorphism $\alpha : A \to A; x \mapsto x + t(d(x))$ preserves $\mathfrak{a}$ and induces the identity on $A_0 := A/\mathfrak{a}$, so that $\mathcal{P} \times_{\mathrm{Spec}\,A,\alpha} \mathrm{Spec}\,A$ and $\mathcal{P}$ are naturally elements of $D_{\mathcal{P}_0,\mathrm{Spec}\,A}$, where $\mathcal{P}_0$ denotes the pull-back of $\mathcal{P}$ to $\mathrm{Spec}\,A_0$. Then, the difference of these lifts is described by the map N above (when regarded as an element of $\mathrm{Hom}_{\overline{A}}(\check{T}_{\overline{\mathcal{P}}}, \mathfrak{a}) \cong \mathrm{Hom}_{A_0}(\check{T}_{\mathcal{P}_0}, \mathfrak{a})$).

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} \overline{A} \oplus \Omega^1_{\overline{A}/\mathbb{F}_{p^f}} & \longleftarrow & A \oplus \Omega^1_{\overline{A}/\mathbb{F}_{p^f}} & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ \overline{A} & \longleftarrow & A & \longrightarrow & A_0 \end{array},$$

in which the vertical arrows are surjections whose kernels are p-torsion ideals with vanishing square. The three applicable spaces of infinitesimal deformations are $\mathrm{Hom}_{\overline{A}}(\check{T}_{\overline{\mathcal{P}}}, \Omega^1_{\overline{A}/\mathbb{F}_{p^f}})$, and $\mathrm{Hom}_A(\check{T}_{\mathcal{P}}, \Omega^1_{\overline{A}/\mathbb{F}_{p^f}})$, and $\mathrm{Hom}_{A_0}(\check{T}_{\mathcal{P}_0}, \mathfrak{a})$. Let us write $\overline{pr}'_2$ for the map from A to $A \oplus \Omega^1_{\overline{A}/\mathbb{F}_{p^f}}$ with $\overline{pr}'_2(x) = x + d(x)$, and let us write $\tilde{\mathcal{P}}$ for the pull-back of $\mathcal{P}$ via $\mathrm{Spec}\,\overline{pr}'_2$, and let $\tilde{N} \in \mathrm{Hom}_A(\check{T}_{\mathcal{P}}, \Omega^1_{A/\mathbb{F}_{p^f}})$ measure the difference between $\tilde{\mathcal{P}}$ and $\mathcal{P}$. By the mere functoriality of the $T_?$-actions in the corollary 3.27, we know that the element $K_{\overline{\mathcal{P}}}$ comes from the element $\tilde{N}$ under the natural map

$$\mathrm{Hom}_{\overline{A}}(\check{T}_{\overline{\mathcal{P}}}, \Omega^1_{\overline{A}/\mathbb{F}_{p^f}}) \stackrel{\cong}{\leftarrow} \mathrm{Hom}_A(\check{T}_{\mathcal{P}}, \Omega^1_{\overline{A}/\mathbb{F}_{p^f}}).$$

Due to the same reason it is clear that the series of morphisms

$$\mathrm{Hom}_A(\check{T}_{\mathcal{P}}, \Omega^1_{\overline{A}/\mathbb{F}_{p^f}}) \stackrel{\circ t}{\to} \mathrm{Hom}_A(\check{T}_{\mathcal{P}}, \mathfrak{a}) \stackrel{\cong}{\to} \mathrm{Hom}_{A_0}(\check{T}_{\mathcal{P}_0}, \mathfrak{a})$$

sends $\tilde{N}$ to the desired difference between $\mathcal{P} \times_{\mathrm{Spec}\, A, \alpha} \mathrm{Spec}\, A$ and $\mathcal{P}$. □

Lemma 7.4. *Let ν be a positive integer, fix a smooth $W_\nu(\mathbb{F}_{p^f})$-algebra A, and let $\overline{\mathcal{P}}$ be a display with $(\mathfrak{G}, \mu)$-structure over $\overline{A} := A/pA$. Let Γ be the group of $W_\nu(\mathbb{F}_{p^f})$-algebra automorphisms of A which induce the identity on A/pA. Then Γ acts on $D_{\overline{\mathcal{P}}, \mathrm{Spec}\, A}$, and if the Kodaira-Spencer element of $\overline{\mathcal{P}}$ is a split injection (resp. bijective) then this action is transitive (resp. simply transitive).*

Proof. By induction on ν this follows immediately from lemma 7.3. □

Corollary 7.5. *Let ν be a positive integer, and let $\overline{\mathcal{P}}$ be a display with $(\mathfrak{G}, \mu)$-structure over a smooth $\mathbb{F}_{p^f}$-variety $\overline{X}$, assume that $\overline{\mathcal{P}}$ is formally étale in the sense of remark 6.12. Then there exists a display $\mathcal{P}^{(\nu)}$ with $(\mathfrak{G}, \mu)$-structure over a smooth $W_\nu(\mathbb{F}_{p^f})$-variety $\mathcal{X}^{(\nu)}$ together with isomorphisms*

$$\overline{X} \cong \mathcal{X}^{(\nu)} \times_{W_\nu(\mathbb{F}_{p^f})} \mathbb{F}_{p^f}$$
$$\overline{\mathcal{P}} \cong \mathcal{P}^{(\nu)} \times_{\mathcal{X}^{(\nu)}} \overline{X}$$

moreover the quadruple consisting of $\mathcal{X}^{(\nu)}$, $\mathcal{P}^{(\nu)}$ together with the two isomorphisms above is unique up to a unique isomorphism.

Proof. Let $\bigcup_l U_l$ be an open affine covering, and let $\overline{\mathcal{P}}_l$ be the restriction of $\overline{\mathcal{P}}$ to U_l. Choose smooth affine $W_\nu(\mathbb{F}_{p^f})$-schemes $U_l^{(\nu)}$ lifting the smooth affine $\mathbb{F}_{p^f}$-schemes U_l, and let $U_{l;m,n}^{(\nu)} \subset U_{l;m}^{(\nu)}$ be the open subschemes of $U_l^{(\nu)}$ whose underlying point sets are $U_l \cap U_m \cap U_n$ and $U_l \cap U_m$. Let us pick lifts $\mathcal{P}_l^{(\nu)} \in D_{\overline{\mathcal{P}}_l, U_l^{(\nu)}}$. Their existence is due to part (i) of corollary 3.27. Let $\psi_{l;m}^{(\nu)} : U_{l;m}^{(\nu)} \to U_{m;l}^{(\nu)}$ be a lift of the identity which pulls back $\mathcal{P}_m^{(\nu)}|_{U_{m;l}^{(\nu)}}$ to $\mathcal{P}_l^{(\nu)}|_{U_{l;m}^{(\nu)}}$. It exists due to lemma 7.4. Moreover, we have

$$\psi_{l;n}^{(\nu)}|_{U_{l;n,m}^{(\nu)}} = \psi_{m;n}^{(\nu)}|_{U_{m;n,l}^{(\nu)}} \circ \psi_{l;m}^{(\nu)}|_{U_{l;m,n}^{(\nu)}},$$

as both sides of the equation pull back $\mathcal{P}_n^{(\nu)}|_{U^{(\nu)}_{n;l,m}}$ to $\mathcal{P}_l^{(\nu)}|_{U^{(\nu)}_{l;n,m}}$, and such a map is unique, again by lemma 7.4. Thus we defined a formally smooth lift along with a family of displays $\mathcal{P}^{(\nu)}$ with $(\mathfrak{G},\mu)$-structure over it. □

Corollary 7.6. *Let $\overline{\mathcal{P}}$ be a formally étale (cf. remark 6.12) display with $(\mathfrak{G},\mu)$-structure over a proper and smooth $\mathbb{F}_{p^f}$-variety $\overline{X}$. Let $\overline{\mathcal{F}} = q_0(\overline{\mathcal{P}}) \in \mathtt{Ob}_{Tors(\mathcal{I}_0^\mu)(\overline{X})}$ be the level-0 truncation. Suppose that there exists a character $\chi : \mathcal{I}_0^\mu \to \mathbb{G}_{m,W(\mathbb{F}_{p^f})}$ such that $\overline{\mathcal{L}} := \omega_{\overline{\mathcal{F}}}(\chi)$ is an ample line bundle on $\overline{X}$. Then there exists a smooth and proper $W(\mathbb{F}_{p^f})$-scheme $\mathcal{X}$, together with a sequence of displays $\{\mathcal{P}^{(\nu)}\}_{\nu\in\mathbb{N}}$ with $(\mathfrak{G},\mu)$-structure over $\mathcal{X}^{(\nu)} := \mathcal{X}\times_{W(\mathbb{F}_{p^f})} W_\nu(\mathbb{F}_{p^f})$, and isomorphisms*

$$\overline{\delta} : \overline{X} \cong \mathcal{X}^{(1)}$$
$$\delta^{(0)} : \overline{\mathcal{P}} \cong \mathcal{P}^{(1)} \times_{\mathcal{X}^{(1)},\overline{\delta}} \overline{X}$$
$$\delta^{(\nu)} : \mathcal{P}^{(\nu)} \cong \mathcal{P}^{(1+\nu)} \times_{\mathcal{X}^{(1+\nu)}} \mathcal{X}^{(\nu)}$$

moreover, the triple consisting of the lift $\overline{\delta} : \overline{X} \overset{\cong}{\to} \mathcal{X}\times_{W(\mathbb{F}_{p^f})}\mathbb{F}_{p^f}$ together with the two sequences $\{\mathcal{P}^{(\nu)}\}_{\nu\in\mathbb{N}}$, and $\{\delta^{(\nu)}\}_{\nu\in\mathbb{N}_0}$ is unique up to a unique isomorphism.

Proof. This follows from the above together with [18, Théorème (5.4.5)], observe that the lift of $\overline{\mathcal{L}}$ is automatic, since we can use the $\mathcal{P}^{(\nu)}$'s. □

8. Moduli spaces of abelian varieties with additional structure

Let L be a totally imaginary extension of a totally real number field L^+ and write $^-$ for the non-trivial element of $\mathrm{Gal}(L/L^+)$. By a skew-Hermitian L-vector space we mean

- a finite-dimensional L-vector space V together with
- a L-linear isomorphism $\Psi : V \to \check{\overline{V}}$, such that $-\overline{\Psi(x,y)} = \Psi(y,x)$.

Fix a pair (V,Ψ) as above. A place of L is called inert, if it is fixed by $^-$. By localisation, one obtains skew-Hermitian L_v-vector spaces $(L_v\otimes_L V, \Psi_v)$ for every inert place v of L. Let $\mathrm{U}(V/L,\Psi)$ represent the group functor

$$R \mapsto \{g \in \mathrm{GL}_{L\otimes_{L^+}R}(V\otimes_{L^+} R) | \Psi(gx,gy) = \Psi(x,y)\,\forall x,y \in V\otimes_{L^+} R\}.$$

In this section we need a few Hodge-theoretic preliminaries, recall that the real algebraic group $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ is called the Deligne torus, where $\mathbb{C}$ stands for an algebraic closure of $\mathbb{R}$. Let us also write $\mathbb{S}^1$ for the kernel of the norm $\mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$. Notice that $\mathbb{S} = (\mathbb{G}_{m,\mathbb{R}} \times_{\mathbb{R}} \mathbb{S}^1)/\{\pm 1\}$ holds, and we also want to choose $\sqrt{-1} \in \mathbb{C}$ once for all. A $\mathbb{Q}$-Hodge structure of weight -1 on V is called skew-Hermitian if and only if:

- the restriction of the associated homomorphism $h : \mathbb{S} \to \mathrm{GL}(V/\mathbb{Q})$ to the subgroup $\mathbb{S}^1$ factors through the subgroup $\mathrm{Res}_{L^+/\mathbb{Q}} \mathrm{U}(V/L, \Psi)$ of $\mathrm{GL}(V/\mathbb{Q})$, and
- the symmetric form $(\mathrm{tr}_{L/\mathbb{Q}} \Psi)(\rho(h(\sqrt{-1}))x, y)$ is positive definite on $\mathbb{R} \otimes V$.

Fix a triple (V, Ψ, h) as above, and note that there are Hodge decompositions $V_\iota = \bigoplus_{p+q=-1} V_\iota^{p,q}$, where $V_\iota = \mathbb{C} \otimes_{\iota,L} V$ stands for the eigenspace as ι is running through the set $L_{an} = \mathrm{Spec}\, L(\mathbb{C})$ of embeddings of L into $\mathbb{C}$, observe that L_{an} carries a natural left $\mathrm{Gal}(R/\mathbb{Q})$-action commuting with the complex conjugation which could be viewed as acting from the right, the subfield R stands for the normal closure.Let us denote the Hodge numbers $\dim_{\mathbb{C}} V_\iota^{p,q}$ by $h_\iota^{p,q}$. The $\mathbb{C}$-vector spaces V_ι carry natural skew-Hermitian forms Ψ_ι, obtained by extension of scalars. Notice that $\overline{V}_\iota^{p,q} = V_{\iota\circ *}^{q,p}$, so that $h_\iota^{p,q} = h_{\iota\circ *}^{q,p}$.

One more piece of terminology will prove useful: Consider some $\mathbb{Q}$-group that can be written in the form $G = (\mathbb{G}_m \times \mathrm{Res}_{L^+/\mathbb{Q}} G^1)/\{\pm 1\}$ for some connected L^+-group G^1, which is assumed to be an inner form of a totally compact form, and some $\mathbb{Q}$-rational normal subgroup $\{\pm 1\}$ of $\mathbb{G}_m \times \mathrm{Res}_{L^+/\mathbb{Q}} G^1$ not contained in $\mathbb{G}_m$ nor $\mathrm{Res}_{L^+/\mathbb{Q}} G^1$. The reciprocal function defines a $\mathbb{Q}$-rational injection $w : \mathbb{G}_m \hookrightarrow G$ simply called 'the weight'. Consider a further homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ such that $h(\sqrt{-1})$ is a Cartan-involution of $G_{\mathbb{R}}/w(\mathbb{G}_{m,\mathbb{R}})$ and $h|_{\mathbb{G}_{m,\mathbb{R}}} = w_{\mathbb{R}}^{-1}$. If these properties hold we will say that the pair

$$(G^1, h)$$

is a Hodge datum with coefficients in L^+. As usual we write $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ for the restriction of the complexification of h to the first factor in the canonical decomposition $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}^2$. By slight abuse of language we call (G^1, h) Shimura datum with coefficients in L^+, if and only if μ_h is minuscule (cf. [10]). Notice that $G_{\mathbb{R}}$ is contained in the product of the groups $(\mathbb{G}_{m,\mathbb{R}} \times_{\mathbb{R}} G_\iota^1)/\{\pm 1\}$ where $G_\iota^1 := G^1 \times_{L^+,\iota} \mathbb{R}$ as ι runs through L_{an}^+. Accordingly we let $\mu_\iota : \mathbb{G}_{m,\mathbb{C}} \to (\mathbb{G}_{m,\mathbb{C}} \times_{\mathbb{C}} G_{\iota,\mathbb{C}}^1)/\{\pm 1\}$ be obtained as image of μ in the factor that corresponds to ι. Let $E \subset \mathbb{C}$ be the smallest field over which the conjugacy class of μ_h is defined, observe that the conjugacy class of each μ_ι is defined over some subfield of the composite ER.

At last, suppose that (V, Ψ) is a skew-Hermitian L-vector space. Then we will say a L^+-homomorphism $\rho : G^1 \to \mathrm{U}(V/L, \Psi)$ is a unitary representation (of type $\{(-b_\iota, b_\iota - 1), \dots, (-a_\iota, a_\iota - 1)\}$), if the following holds:

(U1) $\rho(w(-1)) = -\operatorname{id}_V$

(U2) If $\varrho : G \to \mathrm{GL}(V/\mathbb{Q})$ denotes the unique extension of $\mathrm{Res}_{L^+/\mathbb{Q}} \rho$ that restricts to the identity on the subgroup $\mathbb{G}_m$, then $(V, \Psi, \varrho_{\mathbb{R}} \circ h)$ is a skew-Hermitian Hodge structure (of type $\{(-b_\iota, b_\iota - 1), \dots, (-a_\iota, a_\iota - 1)\}$).

Here, it is understood that $\{a_\iota\}_{\iota \in L_{an}}$ is some family of integers satisfying $a_\iota \leq b_\iota := 1 - a_{\iota \circ *}$. We define the character χ_ρ of a unitary representation ρ to be the composition $G^1_L \stackrel{\rho}{\to} \mathrm{GL}(V/L) \stackrel{\mathrm{tr}}{\to} \mathbb{A}^1_L$, it is a L-rational class function satisfying $\chi_\rho(\gamma) = \overline{\chi}_\rho(\gamma^{-1})$.

8.1. **ϑ-gauged representations.** In order to describe an important operation on the set of isometry classes of skew-Hermitian Hodge structures, we need to introduce certain combinatorial data: Fix an element $\vartheta \in \mathrm{Gal}(R/\mathbb{Q})$. Minimal non-empty ϑ-invariant subsets of L_{an} are called cycles. A cycle is called inert if it is invariant under composition with $*$, and otherwise it is called split. A set $S \subset L_{an}$ is called a semi-cycle if and only if

- the sets $\{\iota | \iota \circ * \in S\}$ and S are disjoint, and
- the union $S \cup \{\iota | \iota \circ * \in S\}$ is a (necessarily inert) cycle.

A skew-Hermitian Hodge structure (V, Ψ, h) is called ϑ-compact if every cycle contains at least one element ι for which all but at most one of its Hodge numbers $h^{p,q}_\iota$ vanish. A function $\mathbf{d}^+ : L_{an} \to \mathbb{Z}$ with $\mathbf{d}^+(\vartheta \circ \iota) \leq \mathbf{d}^+(\iota) + 1$ and $\mathbf{d}^+(\iota \circ *) = \mathbf{d}^+(\iota) \geq 0$ will be called a ϑ-multidegree.

Definition 8.1. *Consider the function $\mathbf{d}(\iota) := \vartheta^{-\mathbf{d}^+(\iota)} \circ \iota$, for some fixed ϑ-multidegree $\mathbf{d}^+$.*

- *We say that a family of integers $\{a_\iota\}_{\iota \in L_{an}}$ satisfying $1 - a_{\iota \circ *} := b_\iota \geq a_\iota$ is normalized if the following properties hold:*
  (N1) *For every $\iota \in L_{an}$ one has $b_\iota - a_\iota = |\mathbf{d}^{-1}(\iota)|$.*
  (N2) *If $\Theta \subset L_{an}$ is a cycle, then one has $0 = \sum_{\iota \in \Theta} a_\iota$.*
  (N3) *For every semi-cycle $S \subset L_{an}$ the congruence:*
  $$\mathrm{Card}(\{\kappa \in S | \mathbf{d}(\kappa) \notin S\}) \equiv \sum_{\iota \in S} a_\iota \pmod 2$$
  holds.
- *We say that a function $\mathbf{j} : L_{an} \to \mathbb{Z}$ is called a ϑ-gauge of type $\{(-b_\iota, b_\iota - 1), \dots, (-a_\iota, a_\iota - 1)\}$ if the following properties hold:*
  (G1) $\mathbf{j}(\iota \circ *) = -\mathbf{j}(\iota)$,

(G2) *For each* $l \in [a_\iota, b_\iota - 1]$, *there exists a unique* $\kappa \in L_{an}$, *with* $\mathbf{d}(\kappa) = \iota$ *and* $\mathbf{j}(\kappa) = l$.

Suppose that the condition (N1) holds. Then (N2) holds if and only if it holds for all split cycles, while (N3) holds if and only if it holds for one arbitrary choice of semi-cycle S, within each inert cycle. The remarkable parity condition (N3) already entered into a p-adic consideration (namely in the proof of corollary 4.7), and now it is going to enter into a real analytic consideration in the proof of following.

Lemma 8.2. *Fix integers* $a_\iota \leq b_\iota = 1 - a_{\iota\circ *}$, *and a pair* $(\mathbf{d}^+, \mathbf{j})$ *satisfying the conditions (N1), (G1) and (G2) in definition 8.1. Consider a skew-Hermitian Hodge structure* (V, Ψ, h), *such that the Hodge decomposition of* V_ι *is of type contained in* $\{(-b_\iota, b_\iota - 1), \dots, (-a_\iota, a_\iota - 1)\}$, *and let* $h_\iota^{p,q}$ *be its Hodge numbers. Then there exists a skew-Hermitian Hodge structure* $(\tilde{V}, \tilde{\Psi}, \tilde{h})$ *such that:*

(i) *For every finite inert place* v *of* L *there exists a* L_v*-linear similarity from* $(L_v \otimes_L V, \Psi_v)$ *to the skew-Hermitian* L_v*-vector space* $(L_v \otimes_L \tilde{V}, \tilde{\Psi}_v)$.

(ii) *The Hodge numbers of* $(\tilde{V}, \tilde{\Psi}, \tilde{h})$ *are given by*

$$\tilde{h}_\kappa^{\tilde{p},\tilde{q}} = \begin{cases} \sum_{p < -\mathbf{j}(\kappa)} h_{\mathbf{d}(\kappa)}^{p,q} & (\tilde{p}, \tilde{q}) = (-1, 0) \\ \sum_{p \geq -\mathbf{j}(\kappa)} h_{\mathbf{d}(\kappa)}^{p,q} & (\tilde{p}, \tilde{q}) = (0, -1) \\ 0 & (\tilde{p}, \tilde{q}) \notin \{(-1,0),(0,-1)\} \end{cases} \tag{58}$$

for every $\kappa \in L_{an}$.

In addition, suppose that the conditions (N2) and (N3) hold. Then $(\tilde{V}, \tilde{\Psi}, \tilde{h})$ *may be chosen such that the skew-Hermitian* L_v*-vector spaces* $(L_v \otimes_L V, \Psi_v)$ *and* $(L_v \otimes_L \tilde{V}, \tilde{\Psi}_v)$ *are isometric for every finite inert place* v *of* L.

Proof. Recall that the signatures (resp. discriminants) of the envisaged forms $\sqrt{-1}\tilde{\Psi}_\kappa$ have to be equal to $\tilde{h}_\kappa^{-1,0} - \tilde{h}_\kappa^{0,-1}$ (resp. $(-1)^{\tilde{h}_\kappa^{0,-1}}$), while the signatures (resp. discriminants) of the given forms $\sqrt{-1}\tilde{\Psi}_\iota$ are equal to $\sum_{p+q=-1} (-1)^q h_\iota^{p,q}$ (resp. $\prod_{p+q=-1} (-1)^{q h_\iota^{p,q}}$). We begin the proof of the lemma with the strengthened version of (i). Choose a disjoint union $L_{an} = S \cup \{\iota | \iota \circ * \in S\}$. In fact the only issue is the existence of a skew-Hermitian space $(\tilde{V}, \tilde{\Psi})$, and all we have to do is check the congruence

$$\sum_{\iota \in S} \sum_{p+q=-1} q h_\iota^{p,q} \equiv \sum_{\kappa \in S} \tilde{h}_\kappa^{0,-1} \pmod 2.$$

It is easy to see that we have

$$-a_{\iota\circ *}\dim_L V=\sum_{p+q=-1} q h_\iota^{p,q}+\sum_{\mathbf{d}(\kappa)=\iota}\tilde{h}_\kappa^{0,-1},$$

for every ι. Let us write T for the preimage of S under $\mathbf{d}$. Summation over all $\iota\in S$ yields:

$$(-\sum_{\iota\circ *\in S} a_\iota)\dim_L V=\sum_{\iota\in S}\sum_{p+q=-1} q h_\iota^{p,q}+\sum_{\kappa\in T}\tilde{h}_\kappa^{0,-1}.$$

When calculating $\pmod 2$ we find that the left-hand side agrees with $\mathrm{Card}(S-T)\dim_L V$, according to the conditions (N2) and (N3). It remains to show that $\sum_{\kappa\in S}\tilde{h}_\kappa^{0,-1}+\sum_{\kappa\in T}\tilde{h}_\kappa^{0,-1}$ agrees with $\mathrm{Card}(S-T)\dim_L V$ too. In these sums one can ignore $S\cap T$ and $L_{an}-(S\cup T)$, and the contribution from each element κ in the difference set is precisely $\tilde{h}_\kappa^{0,-1}+\tilde{h}_{\kappa\circ *}^{0,-1}=\dim_L V$.

In the weakened version of the lemma, the existence of $(\tilde{V},\tilde{\Psi})$ is an issue only if $\dim_L V\equiv 0 \pmod 2$, in which case one can avoid the use of the conditions (N2) and (N3). ☐

Remark 8.3. For each pair (i,ι), let $[a'_{i,\iota},b'_{i,\iota}]$ be the smallest interval such that the set of bi-weights occuring in the Hodge decomposition of $V_{i,\iota}$ is contained in

$$\{(-b'_{i,\iota},b'_{i,\iota}-1),\dots,(-a'_{i,\iota},a'_{i,\iota}-1)\},$$

where V is a skew-Hermitian Hodge structure of weight -1, as in the previous lemma, so that we have $a_{i,\iota}\le a'_{i,\iota}\le b'_{i,\iota}\le b_{i,\iota}$. If $\mathbf{j}$ is as in lemma 8.2, we always have $\mathbf{j}(\iota)\in[a_{i,\iota},b_{i,\iota}-1]$, due to condition (N1). However, the corresponding modified Hodge numbers $\tilde{h}_\kappa^{-1,0}$ and $\tilde{h}_\kappa^{0,-1}$ of the said lemma depend merely on the function

$$\mathbf{j}'(\iota):=\begin{cases} a'_{i,\iota}-1 & \mathbf{j}(\iota)\in[a_{i,\iota},a'_{i,\iota}-1]\\ \mathbf{j}(\iota) & \mathbf{j}(\iota)\in[a'_{i,\iota},b'_{i,\iota}-1]\\ b'_{i,\iota} & \mathbf{j}(\iota)\in[b'_{i,\iota},b_{i,\iota}-1]\end{cases},$$

furthermore $\mathbf{j}'$ is still a ϑ-gauge of type $\{(-b'_\iota,b'_\iota-1),\dots,(-a'_\iota,a'_\iota-1)\}$. Unfortunately, the passage from $\mathbf{j}$ and the $[a_{i,\iota},b_{i,\iota}]$'s to the seemingly more canonical $\mathbf{j}'$ and the $[a'_{i,\iota},b'_{i,\iota}]$'s would contradict our condition (N1) and it would obscure the parity condition too.

We fix a ϑ-multidegree $\mathbf{d}^+$. A quadruple $(V,\rho,\Psi,\mathbf{j})$ is called a ϑ-gauged L-unitary representation of type $\{(-b_\iota,b_\iota-1),\dots,(-a_\iota,a_\iota-1)\}$ if the following holds:

(GU1) (V,ρ,Ψ) is a unitary representation of type $\{(-b_\iota,b_\iota-1),\dots,(-a_\iota,a_\iota-1)\}$

(GU2) $\mathbf{j}$ is a ϑ-gauge for the type $\{(-b_\iota, b_\iota - 1), \dots, (-a_\iota, a_\iota - 1)\}$, i.e. it satisfies the conditions (G1) and (G2) of definition 8.1

Observe that for any $\iota \in L_{an} - \mathbf{d}(L_{an})$, the two conditions (U2) and (N1) imply immediately that $(-a_\iota, a_\iota - 1) = (-b_\iota, b_\iota - 1)$ is the sole bi-weight occuring in the Hodge decomposition of V_ι, while for general elements $\iota \in \mathbf{d}(L_{an})$ the endpoints $a_\iota \leq b_\iota$ for which the conditions (U2) and (G2) hold simultaneously, are certainly not always unique. Let us say that $(V, \rho, \Psi, \mathbf{j})$ is a ϑ-gauged L-unitary representation if (GU1) and (GU2) holds for a family of integers $a_\iota \leq 1 - a_{\iota\circ *} =: b_\iota$ satisfying the condition (N1). In this case the intervals $[a_\iota, b_\iota]$ are maximal in the sense that

$$\max\{\mathbf{j}(\kappa) | \mathbf{d}(\kappa) = \iota\} = b_\iota - 1$$
$$\min\{\mathbf{j}(\kappa) | \mathbf{d}(\kappa) = \iota\} = a_\iota$$

holds for all $\iota \in \mathbf{d}(L_{an})$, and in particular the family $\{a_\iota\}_{\iota \in L^{an}}$ is already uniquely determined (namely by the Hodge numbers of (V, h) and the pair $(\mathbf{d}^+, \mathbf{j})$ together with ϑ).

Remark 8.4. From now onwards we always assume that condition (N1) is fulfilled. By the preceding comments this will allow the convention of saying "$(V, \rho, \Psi, \mathbf{j})$ satisfies (N2) or/and (N3)" if and only if $\{a_\iota\}_{\iota \in L_{an}}$ satisfies (N2) or/and (N3).

8.2. **Provisional construction of $\mathbb{F}_{p^f}$-schemes ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$.** Suppose that some odd prime p is unramified in L, let $\mathfrak{r} \in \operatorname{Spec} \mathcal{O}_R$ be a divisor of p, and let $\vartheta \in \operatorname{Gal}(R/\mathbb{Q})$ be the (unique) element which fixes $\mathfrak{r}$ and induces the absolute Frobenius on $\mathcal{O}_R/\mathfrak{r}$. The following serves as input datum for poly-unitary moduli problems:

Definition 8.5. *Let (G^1, h) be a Hodge datum with coefficients in L^+, as in the beginning of this section. A pair of families*

$$\mathbf{P} = (\{(V_i, \Psi_i, \rho_i, \tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i^{\infty,p})\}_{i \in \Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi \in \Pi}),$$

is called a L-poly-unitary Shimura datum over R for $\mathfrak{r}$ if it enjoys the following properties:

(P0) *The index set Λ has finite cardinality, for each of its elements i the triple $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i)$ is a skew-Hermitian Hodge structure, the triple (V_i, ρ_i, Ψ_i) is a L-unitary representation of (G^1, h), the localizations $(\mathbb{Q}_p \otimes \tilde{V}_i, \tilde{\Psi}_{i,p})$ and $(\mathbb{Q}_p \otimes V_i, \Psi_{i,p})$ are unramified and $\varsigma_i^{\infty,p} : \mathbb{A}^{\infty,p} \otimes \operatorname{End}_L(V_i) \to \mathbb{A}^{\infty,p} \otimes \operatorname{End}_L(\tilde{V}_i)$ is a $\mathbb{A}^{\infty,p} \otimes L$-linear $*$-preserving isomorphism (i.e. a $\mathbb{A}^{\infty,p} \otimes L^+$-valued "projective similarity" from (V_i, Ψ_i) to $(\tilde{V}_i, \tilde{\Psi}_i)$).*

(P1) *Each element of the set* Π *is a subset* $\pi \subset \Lambda$ *of odd cardinality, such that the triples* $(\tilde{V}^\pi, \tilde{\Psi}^\pi, \tilde{h}^\pi)$ *are* ϑ*-compact skew-Hermitian Hodge structures of type* $\{(-1,0),(0,-1)\}$*, where* $\tilde{h}^\pi$ *(resp.* $\tilde{\Psi}^\pi$*) denote the canonical Hodge (resp. skew-Hermitian) structure on the tensorproduct:*

$$\tilde{V}^\pi := (\bigotimes_{i\in\pi} \tilde{V}_i)(\frac{1-\mathrm{Card}(\pi)}{2}),$$

which is formed in the L*-linear* $\otimes$*-category of skew-Hermitian Hodge structures with coefficients in* L*, cf.* [5, section 3.1].

(P2) $\{(R_\pi, *)\}_{\pi\in\Pi}$ *is a family of* L*-algebras with positive involution of the second kind, and* ι_π *is a* L*-linear* $*$*-preserving monomorphism from* R_π *to the* L*-algebra of* L*-linear endomorphisms of the* L*-unitary representation* $(V^\pi, \rho^\pi, \Psi^\pi)$ *which is defined to be the* L*-linear tensor product of the family* $\{(V_i, \rho_i, \Psi_i)\}_{i\in\pi}$*. The (pointwise) stabilizer of* $\bigcup_{\pi\in\Pi} R_\pi$ *in* $\prod_{i\in\Lambda} \mathrm{U}(V_i/L, \Psi_i)$ *is equal to the image of the product*

$$\rho := \prod_{i\in\Lambda} \rho_i : G^1 \to \prod_{i\in\Lambda} \mathrm{U}(V_i/L, \Psi_i).$$

Furthermore, all singletons are elements of Π*, and the* L*-algebra of* L*-linear endomorphisms of the* L*-unitary representation* (V_i, ρ_i, Ψ_i) *is equal to the image of* $\iota_{\{i\}}$ *for every* $i \in \Lambda$*.*

By saying that

$$\mathbf{T} = (\mathbf{d}^+, \{(V_i, \Psi_i, \rho_i, \mathbf{j}_i, l_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi}),$$

is a L*-metaunitary Shimura datum we mean that there exists a family* $\{(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)\}_{i\in\Lambda}$*, for which (P0)-(P2) together with the following three additional requirements hold:*

(P3) *Each* l_i *is an element of* $(\mathbb{A}^{\infty,p} \otimes L^+)^\times / \mathbb{N}_{L/L^+}(\mathbb{A}^{\infty,p} \otimes L)^\times$*, and* $\mathbf{d}^+$ *is a* ϑ*-multidegree, each quadruple* $(V_i, \Psi_i, \rho_i, \mathbf{j}_i)$ *is a* ϑ*-gauged* L*-unitary representation in the sense that the conditions (GU1), (GU2) and (N1) of the previous subsection 8.1 hold.*

(P4) *If* ϱ_i *stands for the extension of* $\mathrm{Res}_{L^+/\mathbb{Q}} \rho_i$*, as in (U2), then the Hodge numbers of* $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i)$ *are obtained from the Hodge numbers of* $(V_i, \Psi_i, \varrho_{i,\mathbb{R}} \circ h)$ *by means of the formula* (58)*, when using the* ϑ*-gauge* $\mathbf{j}_i$ *together with* $\mathbf{d}$*.*

(P5) *The multiplier of some (and hence of any)* $\mathbb{A}^{\infty,p} \otimes L$*-linear similarity* $\varepsilon_i^{\infty,p} : \mathbb{A}^{\infty,p} \otimes V_i \stackrel{\cong}{\to} \mathbb{A}^{\infty,p} \otimes \tilde{V}_i$ *by which the conjugation agrees with* $\varsigma_i^{\infty,p}$ *lies in the coset* $l_i \mathbb{N}_{L/L^+}(\mathbb{A}^{\infty,p} \otimes L)^\times$*.*

We say that $\mathbf{T}$ *is normalised if each* $(V_i, \Psi_i, \rho_i, \mathbf{j}_i)$ *is normalised in the sense of remark 8.4. The elements* $l_i \in (\mathbb{A}^{\infty,p}\otimes L^+)^\times/\mathbb{N}_{L/L^+}(\mathbb{A}^{\infty,p}\otimes L)^\times$ *are called the scale factors of* $\mathbf{T}$.
Finally, we write $\mathbf{T} \simeq \mathbf{S}$ *whenever another* ϑ*-gauged* L*-metaunitary Shimura datum* $\mathbf{S}$ *arises from* $\mathbf{T}$ *by multiplying its scale factors by totally positive units of* $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L$, *and we write* $\mathbf{T} \approx \mathbf{S}$ *if they agree up to any change of the scale factors.*

The condition (P2) implies that $(\{(V_i, \Psi_i, \rho_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$ is a L-linear metaunitary collection for the L^+-group $(\mathbb{G}_{m,L^+}^\Lambda \times_{L^+} G^1)/\{\pm1\}^\Lambda$. Furthermore, we will say that the collection is unramified if there exist hyperspecial subgroups $U_{i,p}^1$ of $\mathrm{U}(V_i/L, \Psi_i)(\mathbb{Z}_p \otimes L^+)$ such that

$$U_p^1 = \bigcap_{i\in\Lambda} \rho_i^{-1}(U_{i,p}^1) \tag{59}$$

is a hyperspecial subgroup of $G^1(\mathbb{Q}_p \otimes L^+)$. By an integral structure we mean a family of self-dual $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L$-lattices

$$V_i \supset \mathfrak{V}_{i,p} = \{x \in V_i | \Psi_i(x, \mathfrak{V}_{i,p}) \subset \mathbb{Z}_{(p)} \otimes \mathcal{O}_L\}$$

that are stabilized by those groups. Letting $\mathbb{H}(m)$ denote the Hermitian $\mathbb{A}^{\infty,p}\otimes L$-module of rank one, whose sesquilinear perfect pairing is given by $(x, y) = mxy^*$ (for some $m \in (\mathbb{A}^{\infty,p} \otimes L^+)^\times$) allows us to view the said similarity as in (P5) as an $\mathbb{A}^{\infty,p} \otimes L$-linear isometry from

$$\mathbb{H}(m_i) \otimes_L V_i, \tag{60}$$

to $\mathbb{A}^{\infty,p} \otimes \tilde{V}_i$, for a suitable preimage m_i of l_i under the canonical projection $(\mathbb{A}^{\infty,p} \otimes L^+)^\times \twoheadrightarrow (\mathbb{A}^{\infty,p} \otimes L^+)^\times/\mathbb{N}_{L/L^+}(\mathbb{A}^{\infty,p} \otimes L)^\times$.

Remark 8.6. If some family of quadruples $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)$ satisfies the conditions (P3)-(P5) for a given L-metaunitary Shimura datum $\mathbf{T}$, then all other ones can be written in the form:

$$(\tilde{V}_i, \tilde{m}_i\tilde{\Psi}_i, \mathrm{int}(g_{i,\infty}) \circ \tilde{h}_i, \mathrm{int}(g_i^{\infty,p}) \circ \varsigma_i^{\infty,p})$$

for units $\tilde{m}_i$ of $\mathbb{Z}_{(p)} \otimes \mathcal{O}_{L^+}$ and isometries $g_{i,\infty}$ and $g_i^{\infty,p}$ from $(\tilde{V}_i, \tilde{m}_i\tilde{\Psi}_i)$ to $(\tilde{V}_i, \tilde{\Psi}_i)$ over $\mathbb{R}$ and $\mathbb{A}^{\infty,p}$ respectively. This is because the similarity class of $(\tilde{V}_i, \tilde{\Psi}_i)$ as well as the isometry classes of $(\mathbb{R} \otimes \tilde{V}_i, \tilde{\Psi}_{i,\infty})$ and $(\mathbb{Q}_p \otimes \tilde{V}_i, \tilde{\Psi}_{i,p})$ depend only on $(V_i, \Psi_i, \rho_i, \mathbf{j}_i)$.

From now on we do fix choices of $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)$, and just before getting started we also choose an isometry $\varepsilon_{i,p} : \mathbb{Q}_p \otimes V_i \stackrel{\cong}{\to} \mathbb{Q}_p \otimes \tilde{V}_i$. By formation of $\bigotimes_{i\in\pi}$ and $\bigoplus_{i\in\Lambda}$ one obtains further isometries ε_p^π :

$\mathbb{Q}_p \otimes V^\pi \stackrel{\cong}{\to} \mathbb{Q}_p \otimes \tilde{V}^\pi$ and $\varepsilon_p : \mathbb{Q}_p \otimes V \stackrel{\cong}{\to} \mathbb{Q}_p \otimes \tilde{V}$, when using the notations of (P2) and (P1) along with $V := \bigoplus_{i\in\Lambda} V_i$ and $\tilde{V} := \bigoplus_{i\in\Lambda} \tilde{V}_i$, of course.

8.2.1. *First Step.* We need to associate a couple of further structures to a L-poly-unitary Shimura datum $\mathbf{P}$ equipped with integral structure $\{\mathfrak{V}_{i,p}\}_{i\in\Lambda}$: Let us write $\tilde{G}^\pi$ for the $\mathbb{Q}$-subgroup of $\mathrm{GL}(\tilde{V}^\pi/\mathbb{Q})$ generated by its center (i.e. $\mathbb{G}_{m,\mathbb{Q}}$) and $\mathrm{Res}_{L^+/\mathbb{Q}}\,\mathrm{U}(\tilde{V}^\pi/L, \Psi^\pi)$, moreover let $\tilde{X}^\pi$ be the $\tilde{G}^\pi(\mathbb{R})$-conjugacy class of $\tilde{h}^\pi$. This sets up a family of canonical PEL-type Shimura data $(\tilde{G}^\pi, \tilde{X}^\pi)$. Moreover, regarding $\tilde{V}$ as a skew-Hermitian module over the $*$-algebra $L^\Lambda = \underbrace{L \oplus \cdots \oplus L}_{\Lambda}$ yields yet another PEL-type Shimura datum $(\tilde{G}, \tilde{X})$, where $\tilde{G}$ is the $\mathbb{Q}$-group of L^Λ-linear similitudes of $\tilde{V}$. For later reference we put $\mathfrak{e}_i \subset \mathcal{O}_L^\Lambda$ for the ideal generated by the idempotent $(1, \dots, 1, 0, 1, \dots, 1)$ with the "0" in the i th position, and we let $\mathfrak{e}_\iota \subset \mathcal{O}_R \otimes \mathcal{O}_L$ be the kernel of $\mathcal{O}_R \otimes \mathcal{O}_L \stackrel{\mathrm{id}_R \otimes \iota}{\to} \mathcal{O}_R$. Finally observe that there exist canonical morphisms of Shimura data $g^\pi : (\tilde{G}, \tilde{X}) \to (\tilde{G}^\pi, \tilde{X}^\pi)$, and hence canonical $\tilde{G}(\mathbb{A}^\infty)$-equivariant morphisms of Shimura varieties:

$$M(\tilde{G}, \tilde{X}) \stackrel{g^\pi}{\to} M(\tilde{G}^\pi, \tilde{X}^\pi) \tag{61}$$

We need to collect further facts on integrality: Observe that the projective similarities $\varsigma_i^{\infty,p}$ produce a canonical embedding $\varsigma^{\infty,p} : G_{\mathbb{A}^{\infty,p}} \to \tilde{G}_{\mathbb{A}^{\infty,p}}$, and that any compact open subgroup of $G(\mathbb{A}^{\infty,p})$ can be written in the for $U^p = \varsigma^{\infty,p-1}(\tilde{K}^p)$ for some compact open subgroup $\tilde{K}^p \subset \tilde{G}(\mathbb{A}^{\infty,p})$. The previously introduced ε_p^π's and ε_p yield specific hyperspecial subgroups $\tilde{K}_p^\pi \subset \tilde{G}^\pi(\mathbb{Q}_p)$ and $\tilde{K}_p \subset \tilde{G}(\mathbb{Q}_p)$ by "transport of structure". Observe that $\tilde{K}_p^\pi$ contains $g^\pi(\tilde{K}_p)$, and let $\tilde{K}^\pi$ be any compact open subgroup of $\tilde{G}^\pi(\mathbb{A}^\infty)$ containing $g^\pi(\tilde{K}^p)\tilde{K}_p^\pi$. Let $\tilde{K} \subset \tilde{G}(\mathbb{A}^\infty)$ be the product of $\tilde{K}^p$ and $\tilde{K}_p$, so that (61) induces a morphism from ${}_{\tilde{K}}M(\tilde{G}, \tilde{X})$ to ${}_{\tilde{K}^\pi}M(\tilde{G}^\pi, \tilde{X}^\pi)$. Let ${}_{\tilde{K}}\mathcal{U}/\mathcal{O}_{R_\mathfrak{r}}$ and ${}_{\tilde{K}^\pi}\mathcal{U}^\pi/\mathcal{O}_{R_\mathfrak{r}}$ be the usual moduli interpretations for these unitary group Shimura varieties, which are smooth and proper over $\mathcal{O}_{R_\mathfrak{r}}$ according to [26] and [38] (by the above property (P2)). We write ${}_{\tilde{K}}\overline{\mathcal{U}}/\mathcal{O}_R/\mathfrak{r}$ and ${}_{\tilde{K}^\pi}\overline{\mathcal{U}}^\pi/\mathcal{O}_R/\mathfrak{r}$ for their respective special fibers. According to [26, chapter 8] there is a disjoint union:

$${}_{\tilde{K}}\mathcal{U}_\mathbb{C} \cong \coprod_i {}_{\tilde{K}}M(\tilde{G}^{(i)}, \tilde{X}),$$

where i runs through the elements of the finite set $\text{Ш}(\tilde{G})$, while the $\tilde{G}^{(i)}$'s stand for corresponding twists of $\tilde{G}$ (N.B.: In the case at hand $\text{Ш}(\tilde{G})$ has more than one element, but all $\tilde{G}^{(i)}$'s are non-canonically isomorphic to $\tilde{G}$). Following [5, Theorem 4.8], we write Y^π for the pull-back of the universal abelian scheme on ${}_{\tilde{K}^\pi}\mathcal{U}^\pi$ to ${}_{\tilde{K}}\mathcal{U}$ by means of the canonical extension

$$(62) \qquad {}_{\tilde{K}}\mathcal{U} \stackrel{g^\pi}{\to} {}_{\tilde{K}^\pi}\mathcal{U}^\pi,$$

of (61). We tacitly omit the mentioning of level structures, but do notice that $\mathbb{Z}_{(p)} \otimes \mathrm{End}_L(Y_S^\pi)$ is well-defined for every S-valued point of ${}_{\tilde{K}}\mathcal{U}$. Let $\mathcal{S}_\mathfrak{q}^\pi$ be the possibly skew-Hermitian, graded ${}_{\tilde{K}}\overline{\mathcal{U}}$-display to $Y^\pi[\mathfrak{q}^\infty] \times_{\mathcal{O}_{R_\mathfrak{r}}} \mathcal{O}_R/\mathfrak{r}$, here note that the methods of loc.cit. are applicable only because the p-rank of the $\mod \mathfrak{r}$-reductions of at least one of $Y^\pi[\mathfrak{q}^\infty]$ or $Y^\pi[\mathfrak{q}^{*\infty}]$ vanishes for every $\mathfrak{q} \in S_p$, we denote $Y^{\{i\}} =: Y_i$ and $\mathcal{S}_\mathfrak{q}^{\{i\}} =: \mathcal{S}_{i,\mathfrak{q}}$. Moreover, there exist canonical comparison isomorphisms:

$$(63) \qquad m_\mathfrak{q}^\pi : \dot{\bigotimes}_{i\in\pi} \mathcal{S}_{i,\mathfrak{q}} \to \mathcal{S}_\mathfrak{q}^\pi$$

of possibly skew-Hermitian, graded ${}_{\tilde{K}}\overline{\mathcal{U}}$-displays. Finally, the reduced induced subscheme structure on the Zariski closed subset of ${}_{\tilde{K}}\overline{\mathcal{U}}$-points all of whose $\mathcal{S}_{i,\mathfrak{q}}$s are isoclinal is denoted by ${}_{\tilde{K}}\mathcal{U}^{basic}$. Now and again we need to invoke the projective limit ${}_{\tilde{K}_p}\mathcal{U} = \lim_{\tilde{K}^p \to 1} {}_{\tilde{K}}\mathcal{U}$, which is a scheme with a right $\tilde{G}(\mathbb{A}^{\infty,p})$-action. At last, we need to introduce a family of certain orders

$$\iota_\pi^{-1}(\mathbb{Z}_{(p)} \otimes \mathcal{O}_L + \mathfrak{f}\,\mathrm{End}_{\mathcal{O}_L}(\mathfrak{V}_p^\pi)) =: \mathcal{R}_{\mathfrak{f},\pi} \subset R_\pi,$$

associated to ideals $\mathfrak{f} \subset \mathbb{Z}_{(p)} \otimes \mathcal{O}_{L^+}$, where the self-dual $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L$-lattice $\mathfrak{V}_p^\pi$ stands for the $\mathcal{O}_L$-linear tensor product of the lattices $\mathfrak{V}_{i,p} \subset V_i$. For a family of ideals $\mathfrak{f}_\pi \subset \mathbb{Z}_{(p)} \otimes \mathcal{O}_{L^+}$ we let the $\mathcal{O}_{R_\mathfrak{r}}$-scheme ${}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$ represent the functor that sends a connected pointed base scheme (S, s_0) to the set of $4 + \mathrm{Card}(\Pi)$-tuples $(Y, \lambda, \iota, \overline{\eta}, \{y_\pi\}_{\pi\in\Pi})$ with the following properties:

(i) $(Y, \lambda, \iota, \overline{\eta})$ is a $\mathbb{Z}_{(p)}$-isogeny class of: homogeneously p-principally polarized abelian S-schemes (Y, λ) together with a $*$-invariant action $\iota : \mathcal{O}_L^\Lambda \to \mathbb{Z}_{(p)} \otimes \mathrm{End}(Y)$ satisfying the determinant condition with respect to the skew-Hermitian L^Λ-module $\tilde{V}$, and a $\pi_1^{\acute{e}t}(S, s_0)$-invariant $\tilde{K}^p$-orbit $\overline{\eta}$ of $\mathcal{O}_L^\Lambda$-linear similitudes

$$\tilde{\eta} : \mathbb{A}^{\infty,p} \otimes \tilde{V} \stackrel{\cong}{\to} H_1^{\acute{e}t}(Y_{s_0}, \mathbb{A}^{\infty,p}).$$

(ii) $y_\pi : \mathcal{R}_{\mathfrak{f}_\pi,\pi} \to \mathbb{Z}_{(p)} \otimes \mathrm{End}_L(Y_S^\pi)$ is a $\mathcal{O}_L$-linear $*$-preserving homomorphism such that $\overline{\eta}$ contains at least one element $\tilde{\eta}$ rendering the diagrams

$$\begin{array}{ccc} \mathbb{A}^{\infty,p} \otimes \bigotimes_{i\in\pi} \mathrm{End}_L(V_i) & \longleftarrow & \mathcal{R}_{\mathfrak{f}_\pi,\pi} \\ \downarrow & & \downarrow y_\pi \\ \mathrm{End}(\bigotimes_{i\in\pi} H_1^{\acute{e}t}(Y_{i,s_0}, \mathbb{A}^{\infty,p})(\frac{1-\mathrm{Card}(\pi)}{2})) & \xleftarrow{H_1^{\acute{e}t}} & \mathrm{End}_L^0(Y_{s_0}^\pi) \end{array}$$

commutative, simultaneously for all $\pi \in \Pi$ where the $\pi_1^{\acute{e}t}(S, s_0)$-invariant vertical map on the left is given by:

$$\bigotimes_{i\in\pi} \phi_i \mapsto \bigotimes_{i\in\pi} \tilde{\eta}_i \circ \varsigma_i^{\infty,p}(\phi_i) \circ \tilde{\eta}_i^{-1},$$

for some family of endomorphisms $\phi_i \in \mathbb{A}^{\infty,p} \otimes \mathrm{End}_L(V_i)$.

The projective limit

$$\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}} = \lim_{U^p\to 1} {}_{U^p}\mathfrak{M}_{\mathbf{R}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$$

is equipped with a right $G(\mathbb{A}^{\infty,p})$-action. Drawing on the above canonical embedding $\varsigma^{\infty,p}$, there is a $G(\mathbb{A}^{\infty,p})$-equivariant morphism

$$\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}} \to {}_{\tilde{K}_p}\mathcal{U}, \tag{64}$$

which at the finite levels recovers the tautological forgetful morphisms $(Y, \lambda, \iota, \overline{\eta}, \{y_\pi\}_{\pi\in\Pi}) \mapsto (Y, \lambda, \iota, \overline{\eta})$, from ${}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$ to ${}_{\tilde{K}}\mathcal{U}$.

Remark 8.7. These notations are justified as the above moduli problem depends merely on $U^p = \varsigma^{\infty,p-1}(\tilde{K}^p)$ and not on the choice of $\tilde{K}^p$. In fact one could have avoided any explicit recourse to the $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)$'s altogether, quite simply by using the direct sum of the skew-Hermitian $\mathbb{A}^{\infty,p}\otimes L$-modules (60) instead of $\tilde{V}$ in (i), at the cost of having to make a choice for the introduction of the map (64). We will tacitly omit the mentioning of the $(\tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)$'s as long as the map to ${}_{\tilde{K}}\mathcal{U}$ plays no particular role.

We fix a set S_p of extensions to L of the primes of L^+ over p, and for each $\mathfrak{q} \in S_p$ we let $r_\mathfrak{q}$ be the degree of $\mathfrak{q}^+ := \mathfrak{q} \cap \mathcal{O}_{L^+}$, and we fix an embedding $\iota_\mathfrak{q} : L \hookrightarrow R$ with $\iota_\mathfrak{q}(\mathfrak{q}) \subset \mathfrak{r}$, so that

$$\vartheta^{r_\mathfrak{q}} \circ \iota_\mathfrak{q} = \begin{cases} \iota_\mathfrak{q} \circ * & \mathfrak{q}^* = \mathfrak{q} \\ \iota_\mathfrak{q} & \text{otherwise} \end{cases}.$$

Finally, all finite layers ${}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$ are proper, and if

$$\det(\check{\mathrm{Lie}} Y_i[\mathfrak{e}_{\vartheta^{-\sigma}\circ\iota_\mathfrak{q}}]) =: \mathcal{L}_{i,\mathfrak{q},\sigma} \in \mathrm{Pic}({}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}), \tag{65}$$

then $\mathcal{L} := \bigotimes_{i\in\Lambda} \bigotimes_{\mathfrak{q}\in S_p} \bigotimes_{\sigma=0}^{r_\mathfrak{q}-1} \mathcal{L}_{i,\mathfrak{q},\sigma}$ is ample. This follows from the finiteness of the aforementioned forgetful maps, as the corresponding facts hold for ${}_{\tilde{K}}\mathcal{U}$ by [38] and [37]. The idea of L-poly-unitary moduli problems was implicitely present in [5, Definition 5.3].

Remark 8.8. Notice that ${}_{U^p}\mathfrak{M}_{\mathbf{R}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$ has a natural action of the center $Z^G(\mathbb{A}^{\infty,p})$, and that its congruence subgroup

$$\mathfrak{Z} = \{g \in Z^G(\mathbb{Q}) | \forall i \in \Lambda : \mathrm{id}_{\mathfrak{V}_{i,p}} \equiv \varrho_i(g) \mod \mathfrak{f}_{\{i\}}\}$$

acts trivially thereon. Using the conventions of remark 8.7 this can be seen as follows: Let $\xi = (Y, \lambda, \iota, \eta, \{y_\pi\}_{\pi\in\Pi})$ be a (S, s_0)-valued point on $\mathfrak{M}_{\mathbf{R}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$. To any $g \in \mathfrak{Z}$, the last sentence in the property (P2) grants the existence of elements $g_i \in \mathcal{R}_{\mathfrak{f}_{\{i\}},\{i\}}$ with $\varrho_i(g) = \iota_{\{i\}}(g_i)$. Consider the p'-quasi-isogeny γ from Y to Y given by $\gamma_i = y_{\{i\}}(g_i)$ on the ith factor Y_i. Using the commutativity of the above diagram (ii) one shows that γ is an isomorphism from ξ to $\xi.g = (Y, \lambda, \iota, \eta \circ g, \{y_\pi\}_{\pi\in\Pi})$.

8.2.2. *Second Step.* Once and for all we fix a ϑ-multidegree $\mathbf{d}^+ : L_{an} \to \mathbb{N}_0$ and a tuple

$$(\{(V_i, \rho_i, \Psi_i, \mathbf{j}_i, \tilde{V}_i, \tilde{\Psi}_i, \tilde{h}_i, \varsigma_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi}),$$

that satisfies (P0)-(P4) together with an integral structure $\{\mathfrak{V}_{i,p}\}_{i\in\Lambda}$. Observe that the L^+-group $\mathrm{Res}_{L/L^+} \mathbb{G}_{m,L}^\Lambda$ is canonically contained in $(\mathbb{G}_{m,L^+}^\Lambda \times_{L^+} G^1)/\{\pm 1\}^\Lambda$, and we shall use this scenario to introduce a cocharacter

$$\alpha : \mathbb{G}_{m,\mathbb{C}} \to (\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L}^\Lambda)_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}^{\Lambda\times L_{an}}$$

by decreeing its (i, ι)-component to be given by $\alpha_{i,\iota}(z) = z^{a_{i,\iota}}$. Later on we need a family of $\pmod{r_\mathfrak{q}}$-multidegrees related to our ϑ-multidegree by means of the formula

$$\mathbf{d}_\mathfrak{q}^+(\sigma) := \mathbf{d}^+(\vartheta^{-\sigma} \circ \iota_\mathfrak{q}),$$

and we also put $r_\mathfrak{q}(\sigma) := \mathrm{Card}(\mathbf{d}^{-1}(\{\vartheta^{-\sigma} \circ \iota_\mathfrak{q}\})) = \mathrm{Card}(\mathbf{d}_\mathfrak{q}^{-1}(\{\sigma\}))$, where $\mathbf{d}_\mathfrak{q}(\sigma) = \sigma + \mathbf{d}_\mathfrak{q}^+(\sigma)$ (so that $\mathbf{d}(\vartheta^{-\sigma} \circ \iota_\mathfrak{q}) = \vartheta^{-\mathbf{d}_\mathfrak{q}(\sigma)} \circ \iota_\mathfrak{q}$). Choose a prime $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_{ER}$ lying above $\mathfrak{r} \in \mathrm{Spec}\, \mathcal{O}_R$. We write $\mathbb{F}_{p^r} \subset \mathcal{O}_{ER}/\mathfrak{p}$ for the subfield of cardinality p^r, for any divisor $\mathbb{N} \ni r \mid [\mathcal{O}_{ER}/\mathfrak{p} : \mathbb{F}_p] =: f$. Moreover, for arbitrary $\mathfrak{q} \in S_p$ we let $\mathcal{G}_\mathfrak{q}^1$ be the reductive $W(\mathbb{F}_{p^{r_\mathfrak{q}}})$-model of $G_\mathfrak{q}^1 = G^1 \times_{L^+,\iota_\mathfrak{q}} K(\mathbb{F}_{p^{r_\mathfrak{q}}})$ determined by the hyperspecial subgroup $U_\mathfrak{q}^1 := U_p^1 \cap G^1(L_\mathfrak{q}^+)$, where U_p^1 is as in (59). Let us explain how to recover the cocharacter α in these local settings: For every $i \in \Lambda$ and $\mathfrak{q}^* = \mathfrak{q} \in S_p$ (resp. $\mathfrak{q}^* \neq \mathfrak{q} \in S_p$), there are canonical inclusions

$$\frac{1}{{}^{F^{r_\mathfrak{q}}}\zeta_{i,\mathfrak{q}}^1} = \zeta_{i,\mathfrak{q}}^1 : \mathbb{G}_{m,W(\mathbb{F}_{p^{2r_\mathfrak{q}}})} \hookrightarrow \mathcal{G}_{\mathfrak{q},W(\mathbb{F}_{p^{2r_\mathfrak{q}}})}^1,$$

(resp. $\zeta^1_{i,\mathfrak{q}} : \mathbb{G}^\Lambda_{m,W(\mathbb{F}_{p^{r_\mathfrak{q}}})} \hookrightarrow \mathcal{G}^1_\mathfrak{q}$) which are obtained by composing the respective scalar extension of the dilatation homomorphisms (cf. (35)) with $\mathbb{G}_m \to \mathbb{G}^2_m; z \mapsto (z, \frac{1}{z})$. Putting $a_{i,\mathfrak{q},\sigma} := a_{i,\vartheta^{-\sigma}\circ\iota_\mathfrak{q}}$, and $b_{i,\mathfrak{q},\sigma} := b_{i,\vartheta^{-\sigma}\circ\iota_\mathfrak{q}}$ determines cocharacters

$$\alpha_{\mathfrak{q},\sigma} : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to (\mathbb{G}_{m,W(\mathbb{F}_{p^f})} \times_{W(\mathbb{F}_{p^f})} {}^{F^{-\sigma}}\mathcal{G}^1_{\mathfrak{q},W(\mathbb{F}_{p^f})})/\{\pm 1\};$$
$$z \mapsto \pm(z^{\frac{1-r_\mathfrak{q}(\sigma)}{2}}, \prod_{i\in\Lambda} {}^{F^{-\sigma}}\zeta^1_{i,\mathfrak{q}}(z^{\frac{a_{i,\mathfrak{q},\sigma}+b_{i,\mathfrak{q},\sigma}-1}{2}})).$$

For any $\mathfrak{q} \in S_p$ and any $\sigma \in \mathbb{Z}/r_\mathfrak{q}\mathbb{Z}$ we require the cocharacters

$$\mu_{\mathfrak{q},\sigma} : \mathbb{G}_{m,W(\mathbb{F}_{p^f})} \to (\mathbb{G}_{m,W(\mathbb{F}_{p^f})} \times_{W(\mathbb{F}_{p^f})} \mathcal{G}^1_{\mathfrak{q},W(\mathbb{F}_{p^f})})/\{\pm 1\}$$

to lie in the conjugacy class of the previously introduced $\mu_{\vartheta^{-\sigma}\circ\iota_\mathfrak{q}}$, and starting out from $\mathbf{d}^+_\mathfrak{q}$ we form subsets $\Sigma_\mathfrak{q}$ and $\tilde{\Sigma}_\mathfrak{q}$ of $\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}$ according to the formulae (21) and (22). Notice, that $\mu_{\mathfrak{q},\sigma} = \alpha_{\mathfrak{q},\sigma}$ for all $\sigma \notin \Sigma_\mathfrak{q}$. The $\overline{\Phi}$-datum we wish to work with is actually:

$$(\mathcal{G}_\mathfrak{q}, \{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}}), \tag{66}$$

where $(\mathbb{G}^\Lambda_{m,W(\mathbb{F}_{p^{r_\mathfrak{q}}})} \times_{W(\mathbb{F}_{p^{r_\mathfrak{q}}})} \mathcal{G}^1_\mathfrak{q})/\{\pm 1\}^\Lambda =: \mathcal{G}_\mathfrak{q}$, and ${}^{F^{\mathbf{d}^+_\mathfrak{q}(\sigma)}}(\frac{\mu_{\mathfrak{q},\mathbf{d}_\mathfrak{q}(\sigma)}}{\alpha_{\mathfrak{q},\mathbf{d}_\mathfrak{q}(\sigma)}}) =: \upsilon_{\mathfrak{q},\sigma}$ for all $\sigma \in \tilde{\Sigma}_\mathfrak{q}$. Consider the rings

$$W(\mathbb{F}_{r_\mathfrak{q}}) \otimes_{\iota_\mathfrak{q},\mathcal{O}_{L^+}} \mathcal{O}_L = C_\mathfrak{q} = \begin{cases} W(\mathbb{F}_{p^{2r_\mathfrak{q}}}) & \mathfrak{q}^* = \mathfrak{q} \\ W(\mathbb{F}_{p^{r_\mathfrak{q}}}) \oplus W(\mathbb{F}_{p^{r_\mathfrak{q}}}) & \text{otherwise} \end{cases}.$$

Next we will construct a family of $C_\mathfrak{q}$-linear gauged metaunitary collections

$$\mathbf{T}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi}) =$$
$$(\{(\mathcal{V}_{i,\mathfrak{q}}, \Psi_{i,\mathfrak{q}}, \rho_{i,\mathfrak{q}}, \mathbf{j}_{i,\mathfrak{q}})\}_{i\in\Lambda}, \{(W(\mathbb{F}_{p^{r_\mathfrak{q}}}) \otimes_{\iota_\mathfrak{q},\mathcal{O}_{L^+}} \mathcal{R}_{\mathfrak{f}_\pi,\pi}, *, \iota_{\pi,\mathfrak{q}})\}_{\pi\in\Pi})$$

for each of the aforementioned $W(\mathbb{F}_{p^f})$-rational $\overline{\Phi}$-data (66). Let $\Psi_{i,\mathfrak{q}}(x,y) = -\Psi_{i,\mathfrak{q}}(y,x)^*$ denote the perfect pairings on the selfdual $C_\mathfrak{q}$-lattices $\mathcal{V}_{i,\mathfrak{q}} := W(\mathbb{F}_{r_\mathfrak{q}}) \otimes_{\iota_\mathfrak{q},\mathcal{O}_{L^+}} \mathfrak{V}_{i,p}$ and ditto for $\Psi^\pi_\mathfrak{q}(x,y) = -\Psi^\pi_\mathfrak{q}(y,x)^*$ on the $C_\mathfrak{q}$-linear tensor products $\mathcal{V}^\pi_\mathfrak{q} := \bigotimes_{i\in\pi} \mathcal{V}_{i,\mathfrak{q}}$. The relation between the local gauges and the global ones is given by:

$$\mathbf{j}_{i,\mathfrak{q}}(\sigma) = \mathbf{j}_i(\vartheta^{-\sigma} \circ \iota_\mathfrak{q}) - a_{i,\mathfrak{q},\mathbf{d}_\mathfrak{q}(\sigma)}, \tag{67}$$

(notice that $a_{i,\mathfrak{q},\mathbf{d}_\mathfrak{q}(\sigma)} = a_{i,\mathbf{d}(\vartheta^{-\sigma}\circ\iota_\mathfrak{q})}$). We are now in a position to appeal to corollary 6.13 in order to introduce the provisional formally smooth

$\mathbb{F}_{p^f}$-scheme ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ rendering the diagram

$$\begin{array}{ccc} \prod_{\mathfrak{q}\in S_p}\overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}}) & \xleftarrow{\prod_{\mathfrak{q}\in S_p}\tilde{\mathcal{P}}_\mathfrak{q}} & {}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}} \\ \downarrow{\scriptstyle \prod_{\mathfrak{q}\in S_p}\mathrm{Flex}^{\mathbf{T}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}} & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\mathfrak{B}_{\mathbb{F}_{p^f}}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})} & \longleftarrow & {}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}} \end{array}$$

2-cartesian. We take $\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})$ to be:
$(\{(\mathrm{GU}(\mathcal{V}_{i,\mathfrak{q}}/C_\mathfrak{q},\Psi_{i,\mathfrak{q}}),\{\tilde{\upsilon}_{i,\mathfrak{q},\sigma}\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}}\}_{i\in\Lambda},\{(W(\mathbb{F}_{p^{r_\mathfrak{q}}})\otimes_{\iota_\mathfrak{q},\mathcal{O}_{L^+}}\mathcal{R}_{\mathfrak{f}_\pi,\pi},*)\}_{\pi\in\Pi})$, where $(\mathrm{GU}(\mathcal{V}_{i,\mathfrak{q}}/C_\mathfrak{q},\Psi_{i,\mathfrak{q}}),\{\tilde{\upsilon}_{i,\mathfrak{q},\sigma}\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}})$ is the standard Φ-datum arising from the standard $\overline{\Phi}$-datum $(\mathrm{GU}(\mathcal{V}_{i,\mathfrak{q}}/C_\mathfrak{q},\Psi_{i,\mathfrak{q}}),\{({}^{F^{-\sigma}}\rho_{i,\mathfrak{q}})\circ\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}})$ by plugging the function (67) into our formalism (37) of subsubsection 4.1.1.

Remark 8.9. The canonical diagrams

$$\begin{array}{ccc} \prod_{\mathfrak{q}\in S_p}\mathfrak{B}_{\mathbb{F}_{p^f}}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})} & \longleftarrow & {}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r},\mathbb{F}_{p^f}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}} \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\mathfrak{B}_{\mathbb{F}_{p^f}}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}'_\pi\}_{\pi\in\Pi})} & \longleftarrow & {}_{U^p}\mathfrak{M}_{\mathbf{P},\mathfrak{r},\mathbb{F}_{p^f}}^{\{\mathfrak{f}'_\pi\}_{\pi\in\Pi}} \end{array}$$

are 2-cartesian, whenever $\mathfrak{f}_\pi\subset\mathfrak{f}'_\pi$, so that ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ is indeed independend of the choice of $\mathfrak{f}_\pi$ and our suppressing of $\mathbf{P}$ in the notation is in accordance with the convention in remark 8.7. If $\mathbf{S}\simeq\mathbf{T}$, then ${}_{U^p}\tilde{M}_{\mathbf{S},\mathfrak{p}}$ and ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ are non-canonically isomorphic.

The 1-morphism

$${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}\to\prod_{\mathfrak{q}\in S_p}\overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}})$$

is formally étale (cf. remark 6.12), and this property is shared by the limit $\tilde{M}_{\mathbf{T},\mathfrak{p}}=\lim_{U^p\to1}{}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$. In addition the $G(\mathbb{A}^{\infty,p})$-equivariance of the natural 1-morphism from $\mathfrak{M}_{\mathbf{P},\mathfrak{r}}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}$ to $\prod_{\mathfrak{q}\in S_p}\mathfrak{B}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}$ is inherited by the displays $\tilde{\mathcal{P}}_\mathfrak{q}$ with $(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}})$-structure over the $G(\mathbb{A}^{\infty,p})$-scheme $\tilde{M}_{\mathbf{T},\mathfrak{p}}$.

8.3. **Set-theoretic properties.** In this subsection we study quasi-isogenies: Let us write $\gamma:Y'\dashrightarrow Y$ for an element of $\mathrm{Hom}^0(Y',Y)$, whenever Y and Y' are abelian schemes over some base S. The smallest integer n such that $\gamma\in\frac{1}{n}\mathrm{Hom}(Y',Y)$ is called the denominator of γ. If Y and Y' are equipped with polarizations $\lambda:Y\to\check{Y}$ and $\lambda':Y'\to\check{Y}'$ we say that γ is a quasi-isogeny provided that the pull-back of λ along say $n\gamma : Y' \to Y$ agrees with λ' up to a multiple. We say that γ is a p'-quasi-isogeny if the denominators of γ and γ^{-1} are coprime to p. Let $\gamma : Y_1 \dashrightarrow Y_2$ and $\gamma' : Y_1' \dashrightarrow Y_2'$ be quasi-isogenies over an algebraically closed field k. We say that they possess a common generization if there exists

- two k-valued points ξ and ξ' on some irreducible normal k-variety S,
- a quasi-isogeny $\gamma'' : Y_1'' \dashrightarrow Y_2''$ over S

such that $\gamma'' \times_{S,\xi} k \cong \gamma$ and $\gamma'' \times_{S,\xi'} k \cong \gamma'$ holds (N.B.: The denominator of γ'' could be strictly larger than the smallest common multiple of the denominators of γ and γ').

If $\mathrm{char}(k) = p$ we write $\mathbb{D}(Y)$ for the covariant Grothendieck-Messing crystalline Dieudonné theory of an abelian variety Y over k, and we let $\mathbb{D}^0(Y)$ be the (non-effective) F-isocrystal whose underlying $K(k)$-space is $\mathbb{Q}\otimes\mathbb{D}(Y)$, and whose Frobenius operator $K(k)\otimes_{F,K(k)}\mathbb{D}^0(Y) \to \mathbb{D}^0(Y)$ is the inverse of $\mathbb{D}(F_{Y/k})$, where $F_{Y/k} : Y \to Y \times_{k,F} k; y \mapsto y^p$ is the relative Frobenius. Suppose that k is an algebraically closed extension of $\mathcal{O}_R/\mathfrak{r}$. Note that to any k-valued point ξ of ${}_{\tilde{K}}\overline{\mathcal{U}}$ one can associate an isomorphism:

$$m^{\pi}_{\xi,ét} : \bigotimes_{\mathbb{A}^{\infty,p}\otimes L} H_1^{ét}(Y_{i,\xi}, \mathbb{A}^{\infty,p}) \stackrel{\cong}{\to} H_1^{ét}(Y^{\pi}_{\xi}, \mathbb{A}^{\infty,p})(\frac{\mathrm{Card}(\pi)-1}{2}),$$

and as a consequence of Deligne's theory of absolute Hodge cycles, [13] this assignment is functorial in k (Sketch: Once one chooses a $\mathbb{C}$-lift κ of ξ one obtains an $\mathcal{O}_L$-linear isomorphism between the Hodge structures $\bigotimes_{\mathcal{O}_L} H_1(Y_{i,\kappa}(\mathbb{C}), \mathbb{Z})$ and $H_1(Y^{\pi}_{\kappa}(\mathbb{C}), \mathbb{Z})(\frac{\mathrm{Card}(\pi)-1}{2})$ and Y^{π}_{ξ} is the reduction of Y^{π}_{κ}, see [5, section 4.4] for some more details). The special fiber of (63) over ξ sets up a further canonical isomorphism:

$$m^{\pi}_{\xi,cris} : \bigotimes_{K(k)\otimes L} \mathbb{D}(Y_{i,\xi}) \stackrel{\cong}{\to} \mathbb{D}(Y^{\pi}_{\xi})(\frac{\mathrm{Card}(\pi)-1}{2}).$$

We need two more functoriality properties which are slightly less immediate consequences of Deligne's theory of absolute Hodge cycles:

Theorem 8.10. *Let ξ and π be as above, then the involution preserving algebra maps*

$$op^{\pi}_{\xi,ét} : \bigotimes_{\mathbb{A}^{\infty,p}\otimes L} \mathrm{End}_{\mathbb{A}^{\infty,p}\otimes L}(H_1^{ét}(Y_{i,\xi}, \mathbb{A}^{\infty,p})) \to \mathrm{End}_{\mathbb{A}^{\infty,p}\otimes L}(H_1^{ét}(Y^{\pi}_{\xi}, \mathbb{A}^{\infty,p}))$$

and

$$op^{\pi}_{\xi,cris} : \bigotimes_{K(k)\otimes L} \mathrm{End}_{K(k)\otimes L}(\mathbb{D}^0(Y_{i,\xi})) \to \mathrm{End}_{K(k)\otimes L}(\mathbb{D}^0(Y^{\pi}_{\xi}))$$

that are defined by $(\dots, f_i, \dots) \mapsto \bigotimes_{i\in\pi} f_i$ *with the help of the above comparison isomorphisms, send* $\bigotimes_L \operatorname{End}_L^0(Y_{i,\xi})$ *into* $\operatorname{End}_L^0(Y_\xi^\pi)$*, moreover the two maps thus induced are equal.*

Theorem 8.11. *To any two points* $\xi_1, \xi_2 \in {}_{\tilde{K}}\overline{\mathcal{U}}(k)$*, and to every* $\mathcal{O}_L^\Lambda$*-linear quasi-isogeny* $\gamma : Y_{\xi_1} \dashrightarrow Y_{\xi_2}$ *there is a canonical quasi-isogeny* $g^\pi(\gamma) : Y_{\xi_1}^\pi \dashrightarrow Y_{\xi_2}^\pi$*, such that application of* $\mathbb{D}^0$ *and* $H_1^{\acute{e}t}(\dots, \mathbb{A}^{\infty,p})$ *recovers the usual tensor-products (with coefficients in* L*).*

Remark 8.12. None of the proofs for these two results are easy, however their analogs in characteristic zero do follow directly from the definition of Y^π. In particular, one knows already that the theorem 8.11 holds for p'-quasi-isogenies, because one can lift them.

Proof. We begin with the proof of theorem 8.10 by following [5, proof of Proposition 5.1] very closely: In order to check the assertion for an arbitrary family of endomorphisms $f_i \in \mathbb{Z}_{(p)} \otimes \operatorname{End}_L(Y_{i,\xi})$ it suffices to restrict to the eigenspaces under $*$, i.e. $f_i^* = (-1)^{c_i} f_i$ for some $c_i \in \{0,1\}$. Now pick an auxiliary element with $-d^* = d \in \mathcal{O}_L \backslash \{0\}$ and observe that $\gamma_i := \frac{1+pd^{1-c_i}f_i}{1-pd^{1-c_i}f_i}$ is a $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L^\Lambda$-linear isogeny, as is its inverse. According to the preceding remark we are allowed to use the assertion for the family γ_i, by using appropriate lifts γ_i' of γ_i between two possibly different lifts Y_{i,κ_1} and Y_{i,κ_2} of $Y_{i,\xi}$. We may deduce the assertion for the family $f_i = \frac{\gamma_i - 1}{pd^{1-c_i}(\gamma_i+1)}$, and the proof of theorem 8.10 is complete.

Before we embark in the proof of theorem 8.11 we note that it holds at least under the following two additional assumptions:

(i) ξ_1 and ξ_2 factor through ${}_{\tilde{K}}\mathcal{U}^{basic}$ and,
(ii) $Y_{\xi_1}[p^\infty] \cong Y_{\xi_2}[p^\infty]$ (as homogeneously p-principally polarized p-divisible groups with $\mathcal{O}_L^\Lambda$-operation)

The reason being: If some family of isomorphisms $u_i : Y_{i,\xi_1}[p^\infty] \stackrel{\cong}{\to} Y_{i,\xi_2}[p^\infty]$ preserves the $\mathcal{O}_L$-actions together with the polarization, then $h_i := u_i^{-1} \circ \gamma_i[p^\infty]$ constitutes a $\mathbb{Q}_p$-valued point in the group scheme $I_{i,\xi}/\mathbb{Z}_{(p)}$ that represents the functor

$$Alg_{\operatorname{Spec}\mathbb{Z}_{(p)}} \ni R \mapsto \{f \in R \otimes \operatorname{End}_L(Y_{i,\xi}) | f^* f \in R^\times\},$$

simply because $\operatorname{End}_L(\mathbb{D}^0(Y_{i,\xi})) = \mathbb{Z}_p \otimes \operatorname{End}_L(Y_{i,\xi})$ holds for all k-valued points of ${}_{\tilde{K}}\mathcal{U}^{basic}$ as a mediate consequence of [50]. Whence it follows that there exist factorizations $h_i = v_i \circ \beta_i$ with $v_i \in I_{i,\xi}(\mathbb{Z}_p)$ and $\beta_i \in I_{i,\xi}(\mathbb{Q})$, as $I_{i,\xi}(\mathbb{Z}_p)$ is open while $I_{i,\xi}(\mathbb{Q})$ is dense in $I_{i,\xi}(\mathbb{Q}_p)$. The assertion follows by applying the said remark to the p'-quasi-isogenies $\gamma_i \circ \beta_i^{-1} = u_i \circ v_i$ and applying theorem 8.10 to $\beta_i \in \operatorname{End}_L^0(Y_{i,\xi_1})$.

Our proof of the general case of theorem 8.11 does not follow [5, proof of Theorem 4.10], instead we appeal to the two lemmas 8.13 and 8.14 below, so that it only remains to check the special case of quasi-isogenies satisfying (i) alone. We argue as follows: Methods of [55, page 321, line 16- page 323, line 13] provide us with a k-valued point ξ_0 with:

- ξ_2 and ξ_0 are lying in the same connected component of ${}_{\tilde{K}}\mathcal{U}^{basic}$ and,
- $Y_{\xi_1}[p^\infty] \cong Y_{\xi_0}[p^\infty]$ (as homogeneously p-principally polarized p-divisible groups with $\mathcal{O}_L^\Lambda$-operation)

Pick a sequence of say n points ξ_2, ξ_3,...,$\xi_{n+1} = \xi_0$, such that the points in each of the consecutive pairs $\{\xi_2, \xi_3\}$,..., $\{\xi_n, \xi_{n+1}\}$ factor through common irreducible normal subvarieties say S_2,...,S_n of ${}_{\tilde{K}}\mathcal{U}^{basic}$. By induction one obtains a sequence of n quasi-isogenies $Y_{\xi_1} \dashrightarrow Y_{\xi_2}, \dots, Y_{\xi_{n+1}}$ of which each consecutive pair possesses a common $\mathcal{O}_L^\Lambda$-linear generization (note that the abelian scheme Y is isotrivial over each of S_2,...,S_n). We know already that $Y_{\xi_1} \dashrightarrow Y_{\xi_0}$ satisfies the theorem, so we get it for $Y_{\xi_1} \dashrightarrow Y_{\xi_2}$ from a $n-1$-fold application of lemma 8.13. □

The following can be regarded as an analog of a special case of Deligne's principle B [13, Theorem 2.12]:

Lemma 8.13 (principle B)**.** *Let S be a irreducible normal algebraic variety over $\mathcal{O}_R/\mathfrak{r}$, and let ξ_1 and ξ_2 be two morphisms from S to ${}_{\tilde{K}}\mathcal{U}$, and let $\gamma : Y_{\xi_1} \dashrightarrow Y_{\xi_2}$ be a quasi-isogeny over S. If $\gamma \times_{S,\xi} k : Y_{\xi_1 \circ \xi} \dashrightarrow Y_{\xi_2 \circ \xi}$ satisfies the claim in theorem 8.11 for some geometric point ξ, then it does so for all other ones.*

Proof. From the Galois equivariance of the comparison isomorphisms $m_{\acute{e}t}^\pi$ and the theorem of Mori-Zarhin [37, Chapitre XII, Th. 2.5] we obtain an adelic $\mathbb{A}^{\infty,p} \otimes L$-linear map $\gamma_{\acute{e}t}^\pi \in \mathbb{A}^{\infty,p} \otimes \mathrm{Hom}_L^0(Y_{\xi_1}^\pi, Y_{\xi_2}^\pi)$, with the desired properties. The lemma follows, as any $\xi : \operatorname{Spec} k \to S$ satisfies

$$\mathrm{Hom}_L^0(Y_{\xi_1}^\pi, Y_{\xi_2}^\pi) = \mathrm{Hom}_L^0(Y_{\xi_1\circ\xi}^\pi, Y_{\xi_2\circ\xi}^\pi) \cap (\mathbb{A}^{\infty,p} \otimes \mathrm{Hom}_L^0(Y_{\xi_1}^\pi, Y_{\xi_2}^\pi)),$$

the intersection taking place in the group $\mathbb{A}^{\infty,p} \otimes \mathrm{Hom}_L^0(Y_{\xi_1\circ\xi}^\pi, Y_{\xi_2\circ\xi}^\pi)$. Note that the compatibility of $\gamma \mapsto g^\pi(\gamma)$ with $\mathbb{D}^0$ follows from the rigidity result [60, Proposition 40], as

$$\begin{array}{ccc} \dot{\bigotimes}_{i\in\pi}\mathcal{S}_{i,\mathfrak{q}} \times_{\xi_1} S & \xrightarrow{\bigotimes_{i\in\pi} g_i(\gamma)[\mathfrak{q}^\infty]} & \dot{\bigotimes}_{i\in\pi}\mathcal{S}_{i,\mathfrak{q}} \times_{\xi_2} S \\ {\scriptstyle m_{\xi_1,\mathfrak{q}}^\pi}\downarrow & & {\scriptstyle m_{\xi_2,\mathfrak{q}}^\pi}\downarrow \\ \mathcal{S}_{\mathfrak{q}}^\pi \times_{\xi_1} S & \xrightarrow{g^\pi(\gamma)[\mathfrak{q}^\infty]} & \mathcal{S}_{\mathfrak{q}}^\pi \times_{\xi_2} S \end{array}$$

commutes only if and if any of its specialisations does that. □

The following lemma is well-known:

Lemma 8.14. *Let ξ_1 and ξ_2 be k-valued points of ${}_{\tilde{K}}\overline{\mathcal{U}}\backslash{}_{\tilde{K}}\mathcal{U}^{basic}$ and consider a quasi-isogeny $\gamma : Y_{\xi_1} \dashrightarrow Y_{\xi_2}$. Then there exists another quasi-isogeny $\gamma' : Y_{\xi_1'} \dashrightarrow Y_{\xi_2'}$ enjoying the following properties:*

- *The Newton-polygon of $Y_{\xi_1'}$ lies strictly above the Newton-polygon of Y_{ξ_1}*
- *γ and γ' possess a common $\mathcal{O}_L^\Lambda$-linear generization.*

Proof. The lemma 4.12 gives us some non-isotrivial deformation $\tilde{\gamma} : Y_{\tilde{\xi}_1} \dashrightarrow Y_{\tilde{\xi}_2}$ over $k[[t]]$. The generic fiber of the triple $(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\gamma})$ is certainly definable over some subfield $N \subset k((t))$ finitely generated over, and containing k. Let S be the normalization of ${}_{\tilde{K}}\overline{\mathcal{U}}_k^2$ in N, we will write $(\xi_1'', \xi_2'', \gamma'')$ for the extension to S of our chosen model of $(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\gamma})$ over N. Recall that the properness of ${}_{\tilde{K}}\overline{\mathcal{U}}$ is conveniently implied by the overall assumptions on the Shimura datum $(\tilde{G}, \tilde{X})$, as introduced in subsubsection 8.2.1. It follows that S is proper over k, because it is finite over ${}_{\tilde{K}}\overline{\mathcal{U}}_k^2$. From the construction of S it is clear, that the Newton-polygons of Y_{ξ_1} and Y_{ξ_2} agree with the generic one. The existence of some $\xi' \in S(k)$ with strictly larger Newton-polygon follows from the well-known fact, due to Raynaud, that over a proper, normal and irreducible basis the constancy of the Newton-polygon implies étale locally the isotriviality of the abelian scheme. □

We fix a homogeneously polarized L^Λ-abelian k-variety (Y, λ, ι) up to $\mathbb{Q}$-isogeny, say in the sense of [26, chapter 9], and we write $Y[\mathfrak{e}_i]$ for its homogeneously polarized L-abelian factor corresponding to $i \in \Lambda$. We will say that (Y, λ, ι) is of type $\tilde{V}$ if the homogeneously polarized $\mathbb{Q}$-isogeny class of Y contains at least one member for which

(K1) the polarization $\lambda : Y \to \check{Y}$ is p-principal, and
(K2) the homomorphism $L^\Lambda \to \mathrm{End}^0(Y)$ is p-integral and satisfies the determinant condition with respect to the skew-Hermitian L^Λ-module $\tilde{V}$.

Verifying that ι be of type $\tilde{V}$ is certainly equivalent to finding a choice of $\mathcal{O}_L^\Lambda$-invariant selfdual Dieudonné lattice in $\mathbb{D}^0(Y)$, with the correct determinant condition. Furthermore, any choice of a full $\tilde{K}^p$-level structure $\tilde{\eta}^p$ completes this to a PEL-quadruple, say $(Y, \lambda, \iota, \tilde{\eta})$ constituting a k-valued point $\xi \in {}_{\tilde{K}}\mathcal{U}$. From now onwards we will write $\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i]$ to denote the homogeneously p-principally polarized $\mathbb{Z}_{(p)}$-isogeny class of abelian k-varieties derived by discarding the level structure on the homogeneously p-principally polarized $\mathbb{Z}_{(p)}$-isogeny class of

Y^π_ξ, this is meaningful, since the formation of $\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i]$ is independent of the choice of $\tilde{\eta}^p$. Notice that the canonical $*$-invariant action $\iota^\pi : \mathcal{O}_L \to \mathbb{Z}_{(p)} \otimes \mathrm{End}(\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i])$ satisfies the determinant condition with respect to the skew-Hermitian L-module $\tilde{V}^\pi$.
However, the theorem 8.11 tells us that the $\mathbb{Q}$-isogeny class of $\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i]$ depends merely on the homogeneously polarized L-abelian $\mathbb{Q}$-isogeny class of the family $\{Y[\mathfrak{e}_i]\}_{i\in\pi}$, it has an action $\iota^\pi : L \to \mathrm{End}^0(\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i])$ (of type $\tilde{V}^\pi$), and it is canonically homogeneously polarized too. These observations give sense to the following definition:

Definition 8.15. *Recall, that we have fixed a L-metaunitary Shimura datum*

$$\mathbf{T} = (\mathbf{d}^+, \{(V_i, \rho_i, \Psi_i, \mathbf{j}_i, l_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$$

for a Hodge datum with L^+-coefficients (G^1, h), and let k be an algebraically closed extension of $\mathbb{F}_{p^f}$: A homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties of type $\mathbf{T}$ *is a quadruple $(Y, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi})$, where*

- (T1) *(Y, λ, ι) is a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties (Y, λ, ι) of type $\tilde{V}$, and*
- (T2) *$y_\pi : R_\pi \to \mathrm{End}^0_L(\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i])$ is a homomorphism which commutes with L and preserves $*$, for every $\pi \in \Pi$.*

We fix such a quadruple $y = (Y, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi})$. Let $m_i \in (\mathbb{A}^{\infty,p} \otimes L^+)^\times$ be a preimage of l_i: A full level structure η is a L^Λ-linear similitude

$$\sum_{i\in\Lambda} \eta_i = \eta : \bigoplus_{i\in\Lambda} \mathbb{H}(m_i) \otimes_L V_i \overset{\cong}{\to} H_1^{\acute{e}t}(Y, \mathbb{A}^{\infty,p}),$$

such that the diagrams

$$\begin{array}{ccc} \mathbb{A}^{\infty,p} \otimes_{\mathbb{Q}} \bigotimes_{i\in\pi} \mathrm{End}_L(V_i) & \longleftarrow & R_\pi \\ {\scriptstyle \bigotimes_{i\in\pi}\eta_i}\downarrow & & {\scriptstyle y_\pi}\downarrow \\ \mathrm{End}(\bigotimes_{i\in\pi} H_1^{\acute{e}t}(Y[\mathfrak{e}_i], \mathbb{A}^{\infty,p})(\frac{1-\mathrm{Card}(\pi)}{2})) & \overset{H_1^{\acute{e}t}}{\longleftarrow} & \mathrm{End}^0_L(\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i]) \end{array}$$

are commutative for all $\pi \in \Pi$. We write $\mathrm{Iso}(y)$ for the right $G(K(k))$-set of L^Λ-linear similitudes

$$\sum_{i\in\Lambda} \zeta_i = \zeta : K(k) \otimes_{\mathbb{Q}} V \overset{\cong}{\to} \mathbb{D}^0(Y),$$

such that the diagrams

$$\begin{array}{ccc} K(k)\otimes_{\mathbb{Q}}\bigotimes_{i\in\pi}\mathrm{End}_L(V_i) & \longleftarrow & R_\pi \\ {\scriptstyle\bigotimes_{i\in\pi}\zeta_i}\Big\downarrow & & {\scriptstyle y_\pi}\Big\downarrow \\ \mathrm{End}_L(\bigotimes_{i\in\pi}\mathbb{D}^0(Y[\mathfrak{e}_i])(\frac{1-\mathrm{Card}(\pi)}{2})) & \xleftarrow{\mathbb{D}^0} & \mathrm{End}^0_L(\dot{\bigotimes}_{i\in\pi}Y[\mathfrak{e}_i]) \end{array}$$

are commutative for every $\pi\in\Pi$. *Note that* $\mathrm{Iso}(y)$ *has a canonical action of the Frobenius, so that it is (empty or) an isocrystal with* G*-structure. Recall our integral structure* $\{\mathfrak{V}_{i,p}\}_{i\in\Lambda}$ *for* $\mathbf{T}$ *amounting to a reductive* $\mathbb{Z}_p$*-model* $\mathfrak{G}_p:=(\mathbb{G}_{m,\mathbb{Z}_p}\times_{\mathbb{Z}_p}\prod_{\mathfrak{q}\in S_p}\mathrm{Res}_{W(\mathbb{F}_{p^{r_\mathfrak{q}}})/\mathbb{Z}_p}\mathcal{G}^1_\mathfrak{q})/\{\pm1\}$ *and let us fix a* $\mathfrak{G}_p(W(k))$*-orbit* $\overline{\zeta}\subset\mathrm{Iso}(y)$.

- *The canonical isomorphism class of the* $\mathbf{B}_{W(k),F}(\mathfrak{G}_p)$*-object (please see to definition 3.3) defined by the element:*

$$^{F^{-1}}\zeta^{-1}\circ\zeta=:b_{y,\zeta}\in\mathtt{Ob}_{\mathbf{B}_{W(k),F}(\mathfrak{G}_p)},$$

  does not depend on the choice of the representative $\zeta\in\overline{\zeta}$ *and is called the crystalline realization of* $(y,\overline{\zeta})$.
- *We say that* $\overline{\zeta}$ *is a* μ_p*-fake integral structure of* y, *if and only if for some (hence for any) representative* $\zeta\in\overline{\zeta}$ *there exist elements* $g_1,g_2\in\mathfrak{G}_p(W(k))$, *such that the diagram*

$$\begin{array}{ccc} K(k)\otimes_{\mathbb{Q}}V & \xrightarrow{\varrho(g_1{}^F\mu_p(p)g_2)} & K(k)\otimes_{\mathbb{Q}}V \\ {\scriptstyle\zeta}\Big\downarrow & & {\scriptstyle 1_{K(k)}\otimes\zeta}\Big\downarrow \\ \mathbb{D}^0(Y) & \xrightarrow{V} & K(k)\otimes_{F,K(k)}\mathbb{D}^0(Y) \end{array}$$

  commutes, where V *is the Verschiebung (i.e.* $\mathbb{D}^0(F_Y)$*), or equivalently if the essential image of* $\mathbf{h}_{\mu_p}$ *contains the crystalline realization (its preimage being* $^{F^{-1}}g_1^{-1}g_2^{-1}=:O\in\mathtt{Ob}_{\mathbf{B}_{W(k),F}(\mathfrak{G}_p,\mu_p)}$*). We write* $\mathrm{Iso}^{\mu_p}(y)=\{\overline{\zeta}\in\mathrm{Iso}(y)/\mathfrak{G}(W(k))|^F(\overline{\zeta}\circ\mu_p(p))\cap\overline{\zeta}\neq\emptyset\}$ *for the set of all* μ_p*-fake integral structure on* y.

We will frequently have to study the banal F-crystal (cf. subsection 6.2) with $\mathfrak{G}_p$-structure $M_{b_{y,\zeta}}$ over $W(k)$, provided that $\zeta\in\overline{\zeta}\in\mathrm{Iso}^{\mu_p}(y)$. Notice the specific quasi-isogeny

$$M_{b_{y,\zeta}}(\varrho)\overset{^{F^{-1}}\zeta}{\dashrightarrow}\mathbb{D}(Y\times_{k,F^{-1}}k),\tag{68}$$

preserving the F-actions, the L^Λ-operations, and the canonical sesquilinear perfect pairings up to a multiple in $\mathbb{Q}_p$. Our next aim is the study of the set $\tilde{M}_{\mathbf{T},\mathfrak{p}}(k)$:

Theorem 8.16. *If $\mathbf{T}$ is normalised, then there is a $G(\mathbb{A}^{\infty,p})$-equivariant bijection between the $G(\mathbb{A}^{\infty,p})$-set of sextuples $(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi},\eta,\overline{\zeta})$ comprising of*

- *a μ_p-fake integral structure $\overline{\zeta}$ on*
- *a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties $(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi})$ of type $\mathbf{T}$ with*
- *a full level structure η,*

and the $G(\mathbb{A}^{\infty,p})$-set $\tilde{M}_{\mathbf{T},\mathfrak{p}}(k)$. Moreover, the bijection is functorial in k.

Proof. Let $\mathbf{Mot}(k)$ be the set of sextuples in question. Defining the sought for bijection amounts to giving a 2-cartesian diagram

$$\begin{array}{ccc}\prod_{\mathfrak{q}\in S_p} B_k(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}}) & \xleftarrow{\prod_{\mathfrak{q}\in S_p}\tilde{\mathcal{P}}_\mathfrak{q}} & \mathbf{Mot}(k) \\ {\scriptstyle \prod_{\mathfrak{q}\in S_p}\mathrm{Flex}_k^{\mathbf{T}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}}\downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\mathfrak{B}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}_{\mathbb{F}_{p^f}}(k) & \longleftarrow & \mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r}}(k)\end{array}.$$

Let $(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi},\eta,\overline{\zeta})$ be an element of $\mathbf{Mot}(k)$. Let $O\in\mathtt{Ob}_{\mathbf{B}_{W(k),F}(\mathfrak{G}_p,\mu_p)}$ be a preimage under $\mathbf{h}_{\mu_p}$ for the crystalline realization of $(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi},\overline{\zeta})$. Let us write $O_\mathfrak{q}$ for its canonical image in $\mathbf{B}_{W(k),F}(\mathrm{Res}_{W(\mathbb{F}_{p^{r_\mathfrak{q}}})/\mathbb{Z}_p}\mathcal{G}_\mathfrak{q},\frac{\mu_\mathfrak{q}}{\alpha_\mathfrak{q}})$ and let $\tilde{\mathcal{P}}_\mathfrak{q}$ be the image of $O_\mathfrak{q}$ under the functor $\mathrm{Flex}^{\mathbf{d}_\mathfrak{q}^+}$, which was described in (39). Subsequently, we obtain a k-valued point of $\mathfrak{B}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}$, namely

$$\mathrm{Flex}^{\mathbf{T}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}(\tilde{\mathcal{P}}_\mathfrak{q})=:(\{\mathcal{S}_{i,\mathfrak{q}}\}_{i\in\Lambda},\{s_{\pi,\mathfrak{q}}\}_{\pi\in\Pi}).$$

Observe that $\hat{\mathbf{h}}_{\tilde{\upsilon}_{i,\mathfrak{q}}}(\mathcal{S}_{i,\mathfrak{q}})$ is canonically isogenous to $\mathbf{h}_{\mu_\mathfrak{q}}(O_\mathfrak{q})$, in view of one of the two results of subsubsection 4.1.5, namely lemma 4.5 or corollary 4.7. Composition with (68) yields a specific L-linear isogeny

$$K(k)\otimes_{F^{-1},W(k)}\mathcal{S}_{i,\mathfrak{q}}\to\mathbb{D}^0(Y[\mathfrak{e}_i]\times_{k,F^{-1}}k),$$

which preserves the skew-Hermitian structure in case $\mathfrak{q}^*=\mathfrak{q}$. This sets up a canonical $\mathbb{Z}_{(p)}$-isogeny class in the $\mathbb{Q}$-isogeny class (Y,λ,ι), and thus a k-valued point of $\mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r}}$, by using the full level structure η. This completes the definition of the aforementioned 2-commutative diagram, and the 2-cartesianism thereof follows from the same property of:

$$\begin{array}{ccc}\prod_{\mathfrak{q}\in S_p} B_k(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}}) & \xleftarrow{\prod_{\mathfrak{q}\in S_p}\tilde{\mathcal{P}}_\mathfrak{q}} & \mathbf{B}_{W(k),F}(\mathfrak{G}_p,\mu_p) \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p} B_k(\mathbb{G}_{m,W(\mathbb{F}_{r_\mathfrak{q}})},\{C\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}}) & \longleftarrow & \mathbf{B}_{W(k),F}(\mathbb{G}_{m,\mathbb{Z}_p},C)\end{array}.$$

□

Remark 8.17. On the one hand subsubsection 8.2.2 constructs our formally smooth scheme ${}_{U_p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ in a very specific manner, on the other hand a scheme enjoying the properties of theorem 8.16 is certainly not unique, for example the reduced induced subscheme $\tilde{M}_{\mathbf{T},\mathfrak{p},red}$ or its perfection would clearly work equally well (non-reduced formally smooth algebras do exist, e.g. $\mathbb{F}_p[t_1,t_2,\dots]/(t_1^p,t_1-t_2^p,\dots)$). This raises a plethora of questions: Can one prove that $\tilde{M}_{\mathbf{T},\mathfrak{p}}$ is of finite type, so that it is smooth, and in case one cannot, could it still be reduced, and in case it is not, is at least its reduction of finite type? We hope to come back to this in another paper, for the time being we note that in the special case of minuscule cocharacters μ_p we will indeed deduce the smoothness of $\tilde{M}_{\mathbf{T},\mathfrak{p},red}$, from proposition 6.18.
In the non-minuscule case one might perhaps speculate on the existence of compactifications of our stacks $\overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q},\{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}})$ so that one can argue as in the proof of proposition 6.18. Astonishingly the same phenomenon shows up in the minuscule but non-reductive (i.e. "ramified") situation: Again one would like to extend the method in the proof of proposition 6.18 to a compactification.

Let $y=(Y,\iota,\lambda,\{y_\pi\}_{\pi\in\Pi})$ a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties of type $\mathbf{T}$. Let us write I_y for the $\mathbb{Q}$-algebraic group $(\mathbb{G}_m\times\mathrm{Res}_{L^+/\mathbb{Q}}I_y^1)/\{\pm1\}$ where I_y^1 represents the $\operatorname{Spec}L^+$-functor:

$$\begin{aligned} R\mapsto\qquad &\{(\dots,f_i,\dots)\in R^\times\times\prod_{i\in\Lambda}R\otimes_{L^+}\mathrm{End}^0_L(Y[\mathfrak{e}_i])^\times|\\ &\forall\pi\in\Pi:\ \dot{\bigotimes}_{i\in\pi}f_i\in R\otimes_{L^+}\mathrm{End}^0_{R_\pi}(\bigotimes_{i\in\pi}Y[\mathfrak{e}_i])^\times\\ &\forall i\in\Lambda:\ f_i^*f_i=1\}\end{aligned}$$

Notice that every choice of full level structure η as in definition 8.15 furnishes $I^1_{y,\mathbb{A}^{\infty,p}\otimes L^+}$ with a $\mathbb{A}^{\infty,p}\otimes L^+$-rational group homomorphism, say $i_{y,\eta}$, to $G^1_{\mathbb{A}^{\infty,p}\otimes L^+}$ and that one has $i_{y,\eta\circ\gamma^p}=\mathrm{Int}^{G^1}(\gamma^p/\mathbb{A}^{\infty,p}\otimes L^+)^{-1}\circ i_{y,\eta}$ for all $\gamma^p\in G(\mathbb{A}^{\infty,p})$. Note also that $I_y(\mathbb{Q}_p)$ acts on $\mathrm{Iso}(y)$ from the left, in fact a "restriction to $\mathfrak{T}$"-argument shows more specifically that every choice of $\zeta\in\mathrm{Iso}(y)$ furnishes $I_{y,\mathbb{Q}_p}$ with a $\mathbb{Q}_p$-rational group homomorphism, say $j_{y,\zeta}$, to the twisted centralizer $J_{b_{y,\zeta}}$ in the sense of (31), and any $\gamma_p\in G(K(k))$ satisfies $j_{y,\zeta\circ\gamma_p}=\mathrm{Int}^G(\gamma_p/K(k))^{-1}\circ j_{y,\zeta}$, here observe that $J_{b_{y,\zeta},K(k)}$ is a subgroup of $G_{K(k)}$. Quite interestingly, the stabilizer $\mathfrak{I}_{y,\bar{\zeta}}$ in $I_y(\mathbb{Q})$ of some $\bar{\zeta}\in\mathrm{Iso}(y)/\mathfrak{G}(W(k))$ is a congruence subgroup therein, because it arises from intersecting with the p-adically bounded open group $j_{y,\zeta}^{-1}(\mathfrak{G}_p(W(k))) \subset I_y(K(k))$, note also, that $J_{b_{y,\zeta}}(\mathbb{Q}_p) = \{h \in G(K(k)) | h^{-1} b_{y,\zeta}{}^F h = b_{y,\zeta}\}$ is canonically isomorphic to the group of $\otimes$-automorphisms of the $\otimes$-functor $\varrho \mapsto \mathbb{Q} \otimes M_{b_{y,\zeta}}(\varrho)$ while $J_{b_{y,\zeta}}(\mathbb{Q}_p) \cap \mathfrak{G}_p(W(k))$ is canonically isomorphic to the automorphism group of the crystalline realization of $(y, \overline{\zeta})$ (when regarded as $\mathbf{B}_{W(k),F}(\mathfrak{G}_p)$-object), and we denote this compact open subgroup by $\mathfrak{J}_{y,\bar{\zeta}}$. We have $\mathfrak{I}_{y,\bar{\zeta}} = I_y(\mathbb{Q}) \cap j_{y,\zeta}^{-1}(\mathfrak{J}_{y,\bar{\zeta}})$ too. What we found is:

Corollary 8.18. *If $\xi \in \tilde{M}_{\mathbf{T},\mathfrak{p}}(k)$ corresponds to $(y, \eta, \overline{\zeta})$ by means of the bijection of theorem 8.16, then the group $i_{y,\eta}(\mathfrak{I}_{y,\bar{\zeta}})$ agrees with the stabilizer of ξ in $G(\mathbb{A}^{\infty,p})$.*

For any $\mathfrak{q} \in S_p$ we write

$$\sum_{i \in \Lambda} \zeta_{i,\mathfrak{q}} = \zeta_{\mathfrak{q}} : K(k) \otimes_{\iota_{\mathfrak{q}},L} V \xrightarrow{\cong} \mathbb{D}^0(Y[\mathfrak{q}^\infty])[\mathfrak{e}_{\iota_{\mathfrak{q}}}],$$

for the restriction of some $\zeta \in \operatorname{Iso}(y)$ to the $\iota_{\mathfrak{q}}$-eigenspace, and we write $j_{y,\zeta,\mathfrak{q}} : I^1_{y,K(k)} \to G^1_{\mathfrak{q},K(k)}$ for the restriction of the previously introduced $j_{y,\zeta,K(k)}$ to the $\iota_{\mathfrak{q}}$-eigenspace, i.e. the map induced by $\zeta_{\mathfrak{q}}$.

Proposition 8.19. *Let $y = (Y, \lambda, \iota, \{y_\pi\}_{\pi \in \Pi})$ be a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties of type $\mathbf{T}$, and assume that there exists at least one full level structure η.*

(i) *The image in the abelianization G^{1ab}, of the restriction of $i_{y,\eta}$, to the neutral component $I_y^{1\circ}$ is L^+-rational and independent of η (and thus gives rise to a canonical homomorphism $i_y^{ab} : I_y^{1\circ} \to G^{1ab}$).*

(ii) *For each $i \in \Lambda$, the composition of the character χ_{ρ_i} with $i_{y,\eta}$ is L-rational and independent of η (and thus gives rise to a canonical class function χ_i on $I^1_{y,L}$).*

(iii) *For any $\mathfrak{q} \in S_p$ and $\zeta \in \operatorname{Iso}(y)$ we have $\chi_{\rho_i} \circ j_{y,\zeta,\mathfrak{q}} = \chi_i$.*

Proof. Parts (ii) and (iii) are folklore, cf. [14, chapter V]. Towards proving (i) we let Z be the center of $\prod_{i \in \Lambda} R_{\{i\}}$, and we let $\omega : G^1 \to \operatorname{Res}_{Z/L^+} \mathbb{G}_{m,Z}$ be the homomorphism induced by the natural action of G^1 on the determinant $\det_Z(V) \in \operatorname{Pic}(\operatorname{Spec} Z)$ of V (as a $Vec(\operatorname{Spec} Z)$-object). It is easy to see that G^{1der} is the neutral component of $\ker(\omega)$, so that it suffices to check (i) for the composition $\omega \circ i_{y,\zeta}|_{I_y^{1\circ}}$, which follows again from the aforementioned methods (of Mumford). □

In short: The isotropy group of $(y, \eta, \overline{\zeta})$ (be it in the adelic product $\prod_{\ell \neq p} G(\mathbb{Q}_\ell)$ or in any of its factors $G(\mathbb{Q}_\ell)$) acts faithfully on the crystalline realization of $(y,\overline{\zeta})$. In general, neither full level structures nor any ones of type $(\mathfrak{G}_p,\mu_p)$ may exist for a given quadruple $y=(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi})$ satisfying both of (T1) and (T2), but one may speculate whether the existence of η implies $\mathrm{Iso}(y)\neq\emptyset$. We need another concept:

Definition 8.20. *Let $y=(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi})$ be a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian k-varieties of type $\mathbf{T}$, and let $\overline{\zeta}$ be a μ_p-fake integral structure for it. We call $(y,\overline{\zeta})$ toric if there exists a $\mathbb{Q}_p$-unramified maximal $\mathbb{Q}$-torus $T\subset I_y$, an element $b\in T(K(k))$, and a level structure $\zeta\in\overline{\zeta}$, such that the following statements hold:*

- *By restriction to $I_{y,K(k)}$, the map $j_{y,\zeta}$ induces a $\mathbb{Q}_p$-rational homomorphism from $T_{\mathbb{Q}_p}$ to a maximal torus of $G_{\mathbb{Q}_p}$ which maps the maximal compact subgroup of $T(\mathbb{Q}_p)$ into $\mathfrak{G}_p(\mathbb{Z}_p)$.*
- *$j_{y,\zeta}(b)=b_{y,\zeta}$*

We call $(y,\overline{\zeta})$ elliptic if the above $\mathbb{Q}_p$-unramified $\mathbb{Q}$-torus can be chosen to be $\mathbb{Q}_p$-elliptic.

Remark 8.21. When working with one elliptic element $(y,\overline{\zeta})$ at a time, it is harmless and convenient to assume in addition that μ_p factors through the map $j_{y,\zeta}|_{\mathfrak{T}_p}$ from the unique reductive $\mathbb{Z}_p$-model $\mathfrak{T}_p$ of $T_{\mathbb{Q}_p}$ to $\mathfrak{G}_p$, and that the essential image of $\mathbf{h}_{\mu_p}:\mathbf{B}_{W(k),F}(\mathfrak{T}_p,\mu_p)\to\mathbf{B}_{W(k),F}(\mathfrak{T}_p)$ contains the object b. This is due to subsubsection 3.3.2.

8.4. **Existence of elliptic points.** Let K be a finite Galois extension of $\mathbb{Q}$, and let A be a finitely generated torsionfree abelian group with a left $\mathrm{Gal}(K/\mathbb{Q})$-action. Recall the global Tate-Nakayama isomorphism:

$$\hat{H}^{-1}(\mathrm{Gal}(K/\mathbb{Q}),A)\to H^1(\mathrm{Gal}(K/\mathbb{Q}),A\otimes C_K),$$

where C_K is the idéle group of K. Now, let r be a place of $\mathbb{Q}$, and let $v|r$ be a place of K. Then we also have a local Tate-Nakayama isomorphism:

$$\hat{H}^{-1}(\mathrm{Gal}(K_v/\mathbb{Q}_r),A)\to H^1(\mathrm{Gal}(K_v/\mathbb{Q}_r),A\otimes K_v^\times),$$

and in either case one easily computes the left hand side Tate cohomology groups as the torsion subgroups of the groups of Galois coinvariants of A, because A is torsionfree. In order to describe the relation between the global and local isomorphisms we must choose one single place of

K over each of the places of $\mathbb{Q}$, and we write S for the set of places obtained in that manner: Then there is a natural commutative diagram:

$$\begin{array}{ccc} \bigoplus_{v\in S}(A_{\mathrm{Gal}(K_v/\mathbb{Q}_r)})_{tors} & \xrightarrow{TN} & \bigoplus_{v\in S} H^1(\mathrm{Gal}(K_v/\mathbb{Q}_r), A\otimes K_v^\times) \\ \downarrow & & \downarrow \\ (A_{\mathrm{Gal}(K/\mathbb{Q})})_{tors} & \xrightarrow{TN} & H^1(\mathrm{Gal}(K/\mathbb{Q}), A\otimes C_K) \end{array},$$

where the vertical arrow on the left is defined by the summation over S, while the v-component of the vertical arrow on the right is defined by the Shapiro isomorphism between $H^1(\mathrm{Gal}(K_v/\mathbb{Q}_r), A\otimes K_v^\times)$ and $H^1(\mathrm{Gal}(K/\mathbb{Q}), A\otimes(K\otimes\mathbb{Q}_r)^\times)$, followed by a map induced from the inclusions $(K\otimes\mathbb{Q}_r)^\times\subset C_K$. Furthermore the long exact cohomology sequence to

$$1\to A\otimes K^\times\to A\otimes(K\otimes\mathbb{A})^\times\to A\otimes C_K\to 1,$$

implies that every element in the kernel of the left hand-side vertical arrow maps to an element coming from $H^1(\mathrm{Gal}(K/\mathbb{Q}), A\otimes K^\times)$. We are going to use this in the following way: Fix a finite prime p, and suppose $\mu\in A$ is a cocharacter for which $\mathbb{N}_{K_v/\mathbb{Q}_p}(\mu)$ and $\mathbb{N}_{K_v/\mathbb{R}}(\mu)$ are both trivial (where v indicates the divisor in S of p or ∞). Now consider the element $\overline{\mu}\in\bigoplus_{v\in S}(A_{\mathrm{Gal}(K_v/\mathbb{Q}_r)})_{tors}$ of which the v-component is given by:

$$\overline{\mu}_v=\begin{cases}[\mu]_\infty & v|\infty\\ -[\mu]_p & v|p\\ 0 & \text{otherwise}\end{cases},$$

where $[\mu]_r$ denotes the class of μ in the group of coinvariants $(A_{\mathrm{Gal}(K_v/\mathbb{Q}_r)})_{tors}$. Let us say that some $c\in H^1(\mathrm{Gal}(K/\mathbb{Q}), A\otimes K^\times)$ is a Tate-Nakayama class for μ if the product of the local Tate-Nakayama isomorphisms send $\overline{\mu}$ to the image of c in $\bigoplus_{v\in S} H^1(\mathrm{Gal}(K_v/\mathbb{Q}_r), A\otimes K_v^\times)$. Finally, let us point out two consequences from Galois descent: First giving a $\mathrm{Gal}(K/\mathbb{Q})$-module A is equivalent to giving a $\mathbb{Q}$-torus T splitting over K, second $H^1(\mathrm{Gal}(K/\mathbb{Q}), A\otimes K^\times)$ classifies isomorphism classes of principal homogeneous spaces for T over $\operatorname{Spec}\mathbb{Q}$ becoming trivial over $\operatorname{Spec} K$. More specifically, when fixing a K-valued point η on some $P\in \mathit{Tors}(T)(\mathbb{Q})$, then $d_\eta(s)=\eta^{-1s}\eta$ is a derivation on $\mathrm{Gal}(K/\mathbb{Q})$ with coefficients in $T(K)=A\otimes K^\times$, and vice versa.

We say that $P\in \mathit{Tors}(T)(\mathbb{Q})$ is a principal homogeneous space of type μ if its cohomology class with respect to some choice of K is a Tate-Nakayama class for μ. We are now going to describe a certain class of tori which satisfy the Hasse-principle:

Definition 8.22. *Let L be a quadratic extension of a number field L^+, and let H be a commutative semisimple L-algebra equipped with an involution that preserves L and restricts to the non-trivial element of* $\mathrm{Gal}(L/L^+)$ *on L. We will say that H is locally odd over L, provided that for every inert place r of L the localization $H \otimes_L L_r$ possesses at least one $*$-invariant simple factor H' of odd degree over L_r.*

If H is a semisimple involutive $\mathbb{Q}$-algebra of finite degree, then we write $H^1/\mathbb{Q}$ for the algebraic group whose group of K-valued points are given by $\{\gamma \in H \otimes K | \gamma\gamma^* = 1\}$:

Exercise 1. Let H be commutative and not locally odd over L. Then $Ш(H^1/L^1)$ is trivial.

8.4.1. *Isocrystals of CM-type.* In this subsubsection we retain the notation of subsection 3.7. Let us introduce F-isocrystals in the generality we need: A F-isocrystal is a pair (M, F) where $M \in \mathit{Vec}(\mathrm{Spec}\, K(\mathbb{F}_p^{ac}))$, and $F : M \to M$ is an additive bijection with $F(ax) = F(a)F(x)$ for $a \in K(\mathbb{F}_p^{ac})$ and $x \in M$. The class of F-isocrystals forms a $\mathbb{Q}_p$-linear tannakian category which we denote by *Isoc*. It possesses a natural $K(\mathbb{F}_p^{ac})$-valued fiber functor $\omega^{K(\mathbb{F}_p^{ac})} : \mathit{Isoc} \to \mathit{Vec}(\mathrm{Spec}\, K(\mathbb{F}_p^{ac})); (M, F) \mapsto M$, and the theory of slopes implies that the proalgebraic automorphism group of $\omega^{K(\mathbb{F}_p^{ac})}$ coincides with the group $\mathbf{D}$ that we already introduced subsection 3.7. Notice the fully faithful $\otimes$-functor:

$$\mathit{Vec}(\mathrm{Spec}\, \mathbb{Q}_p) \to \mathit{Isoc}; V \mapsto K(\mathbb{F}_p^{ac}) \otimes_{\mathbb{Q}_p} V \tag{69}$$

of which the essential image consists of the F-isocrystals of slope 0. As a consequence of Steinberg's theorem Rapoport and Richartz identify $B(G)$ with the set of isomorphism classes of F-isocrystal with G-structure, i.e. faithful exact $\mathbb{Q}_p$-linear $\otimes$-functors from the representation category $\mathit{Rep}_0(G)$ to *Isoc* (cf. [45, definition 3.3/remark 3.4(i)]). In particular the twisted fiber functor ω_P of any locally trivial principal homogeneous space P under G over $\mathbb{Q}_p$ determines an element of $B(G)$, moreover there is a commutative diagram (cf. [45, Theorem 1.15]):

$$\begin{array}{ccc} \pi_1(G)_{\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)} & \xleftarrow{\gamma_G} & B(G) \\ \uparrow & & \uparrow \\ (\pi_1(G)_{\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)})_{tors} & \longrightarrow & H^1(\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p), G(\mathbb{Q}_p^{ac})) \end{array}$$

(here we think of $H^1(\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p), G(\mathbb{Q}_p^{ac}))$ as the set of isomorphism classes of the groupoid $\mathit{Tors}(G)(\mathbb{Q})$). We only use this result in the special case of a $\mathbb{Q}_p$-torus T, in which case the above algebraic fundamental group $\pi_1(T)$ is simply the group of cocharacters while the lower horizontal arrow is simply the local Tate-Nakayama isomorphism, furthermore: In the toric case the map $\gamma_T : B(T) \to X_*(T)_{\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)}$ is a bijection.

8.4.2. *Endomorphism algebras of* mod p *reductions.* Let H be a CM-algebra acting faithfully on an isogeny class of complex abelian $\frac{[H:\mathbb{Q}]}{2}$-folds, and let $\lambda : Y \to \check{Y}$ be a polarization whose Rosati involution stabilizes H. One associates a canonical cocharacter $\mu \in X_*((\mathbb{G}_m \times H^1)/\{\pm 1\})$ to this situation. Let us write ν for the projection of μ onto the space of $\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)$-invariants of $\mathbb{Q} \otimes X_*((\mathbb{G}_m \times H^1)/\{\pm 1\})$. We are interested in the mod p-reduction of Y together with these additional structures, and for that purpose we fix an embedding $K(\mathbb{F}_p^{ac}) \hookrightarrow \mathbb{C}$. It is known that Y possesses good models over sufficiently large rings of algebraic integers, and hence its reduction $\overline{Y}/\mathbb{F}_p^{ac}$ is a well-defined isogeny class of abelian $\frac{[H:\mathbb{Q}]}{2}$-folds over $\mathbb{F}_p^{ac}$. As $\mathrm{End}^0(Y) \subset \mathrm{End}^0(\overline{Y})$, the reduction is of CM-type as well, and it canonically inherits a polarization $\lambda : \overline{Y} \to \check{\overline{Y}}$ too. Furthermore, the choice of embedding gives birth to a specialization map:

$$\mathrm{sp}_Y : H_1(Y(\mathbb{C}), \mathbb{A}^{\infty,p}) \to H_1^{\acute{e}t}(\overline{Y}, \mathbb{A}^{\infty,p}),$$

where $Y(\mathbb{C})$ denotes the underlying real Lie-group of a complex abelian variety Y.

From now on we assume that there exists a CM-field $L \subset H$, such that:

(i) ν is contained in $\mathbb{Q} \otimes X_*((\mathbb{G}_m \times L^1)/\{\pm 1\})$, and
(ii) H is not locally odd over L, in the sense of definition 8.22.

The central theme of this paragraph is the study of the involutive $*$-algebra $\mathrm{End}_L^0(\overline{Y})$, under the two assumptions above. Let μ' be the image of μ in the quotient H^1/L^1. To prepare the setting of the result we need to fix more choices:

- Let $P \in \mathit{Tors}(H^1/L^1)(\mathbb{Q})$ be of type μ'.
- Let $M : \mathit{Rep}_0((\mathbb{G}_m \times H^1)/\{\pm 1\}) \to \mathit{Isoc}$ be a F-isocrystal with $(\mathbb{G}_m \times H^1)/\{\pm 1\}$-structure whose invariant $\gamma(M)$ agrees with μ^{-1} as an element of $X_*((\mathbb{G}_m \times H^1)/\{\pm 1\})_{\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)}$.
- Let $\sigma_p : M|_{\mathit{Rep}_0(H^1/L^1)} \to K(\mathbb{F}_p^{ac}) \otimes \omega_P$ be a $\mathbb{Q}_p$-linear $\otimes$-isomorphism, where the target on the right is the functor defined by composing $\mathbb{Q}_p \otimes \omega_P$ with (69).

The existence of σ_p is due to the assumption (i). Our Hasse-principle for the $\mathbb{Q}$-torus H^1/L^1 implies the uniqueness of the isomorphism class of P.

Remark 8.23. Observe that any two choices of σ_p differ by an element in the group $H^1/L^1(\mathbb{Q}_p)$. Since H^1/L^1 is a $\mathbb{Q}_p$-unramified $\mathbb{Q}$-torus it satisfies weak approximation for the prime p. In particular every $\mathbb{Q}_p$-rational element of H^1/L^1 is the product of a $\mathbb{Q}$-rational element of H^1/L^1 and an element in the image of the map $((\mathbb{G}_m \times H^1)/\{\pm 1\})(\mathbb{Q}_p) \to H^1/L^1(\mathbb{Q}_p)$. Since $((\mathbb{G}_m \times H^1)/\{\pm 1\})(\mathbb{Q}_p)$ and $H^1/L^1(\mathbb{Q})$ are the respective $\otimes$-automorphism groups of M and ω_P, it follows that the triple (P, M, σ_p) is still unique up to a non-unique automorphism (being an element in the inverse image of $H^1/L^1(\mathbb{Q})$ in $((\mathbb{G}_m \times H^1)/\{\pm 1\})(\mathbb{Q}_p)$).

Lemma 8.24. *Let $L \subset H \subset \mathrm{End}^0(Y)$ satisfy the above properties (i) and (ii), where Y is an isogeny class of complex abelian $\frac{[H:\mathbb{Q}]}{2}$-folds, equipped with a polarization λ, and let (P, M, σ_p) be as above. Then there exists a triple of isomorphisms:*

$$\eta^{(0)} : \omega_P(\mathrm{End}^0_L(Y(\mathbb{C}))) \to \mathrm{End}^0_L(\overline{Y})$$
$$\eta^{(1)} : M(H_1(Y(\mathbb{C}), \mathbb{Q})) \to \mathbb{D}^0(\overline{Y})$$
$$\eta^{(2)} : M(\mathbb{Q}(1)) \to \mathbb{Q}_p(1)$$

such that the following properties are fullfilled:

(i) *The map $\eta^{(1)}$ preserves the H-action, and the diagram*

$$\begin{array}{ccc} \mathbb{D}^0(\overline{Y}) \otimes \mathbb{D}^0(\overline{Y}) & \longrightarrow & \mathbb{Q}_p(1) \\ {\scriptstyle \eta^{(1)}\otimes\eta^{(1)}}\uparrow & & {\scriptstyle \eta^{(2)}}\uparrow \\ M(H_1(Y(\mathbb{C}), \mathbb{Q})) \otimes M(H_1(Y(\mathbb{C}), \mathbb{Q}))) & \longrightarrow & M(\mathbb{Q}(1)) \end{array}$$

is commutative, where the horizontal arrows are the Weil-pairings.

(ii) *The diagram*

$$\begin{array}{ccc} \omega_P(\mathrm{End}^0_L(Y(\mathbb{C}))) & \xrightarrow{\eta^{(0)}} & \mathrm{End}^0_L(\overline{Y}) \\ \uparrow & & \uparrow \\ H & \longrightarrow & \mathrm{End}^0_L(Y) \end{array}$$

is commutative, and $\eta^{(0)}$ preserves $*$.

(iii) *The diagram*

$$\begin{array}{ccc} \mathrm{End}^0_L(\overline{Y}) \otimes \mathbb{D}^0(\overline{Y}) & \longrightarrow & \mathbb{D}^0(\overline{Y}) \\ {\scriptstyle \eta^{(0)}\otimes\eta^{(1)}}\uparrow & & {\scriptstyle \eta^{(1)}}\uparrow \\ \omega_P(\mathrm{End}^0_L(Y(\mathbb{C}))) \otimes M(H_1(Y(\mathbb{C}), \mathbb{Q})) & \longrightarrow & M(H_1(Y(\mathbb{C}), \mathbb{Q})) \end{array}$$

is commutative, where the lower horizontal arrow is defined by using the canonical isomorphism $K(\mathbb{F}_p^{ac}) \otimes \omega_P(\mathrm{End}^0_L(Y(\mathbb{C}))) \cong$

$M(\operatorname{End}_L^0(Y(\mathbb{C})))$ *arising from the evaluation of* σ_p *on the* $Rep_0(H^1/L^1)$-*object* $\operatorname{End}_L^0(Y(\mathbb{C}))$.

Proof. The existence of a pair $(\eta^{(1)}, \eta^{(2)})$ with property (i) follows readily from the result [46, Satz(1.6)], which describes $\mathbb{D}^0(\overline{Y})$ as an element of $B((\mathbb{G}_m \times H^1)/\{\pm 1\})$. The remark 8.23 tells us how to adjust such a pair $(\eta^{(1)}, \eta^{(2)})$ to an isomorphism $\eta^{(0)}$ satisfying property (ii) alone, in order to achieve all three properties (i), (ii) and (iii). It remains to demonstrate the existence of $\eta^{(0)}$: Let us denote $\operatorname{End}_L^0(Y(\mathbb{C}))$ by $\tilde{A}$ and $\omega_P(\tilde{A})$ by $\overline{A}$. Notice that $\overline{A}$ is an involutive L-algebra in a natural way, because ω_P is a $\otimes$-functor, and both the involution and the L-algebra structure of $\tilde{A}$ can be expressed by certain morphisms in $Rep_0(H^1/L^1)$ between $\tilde{A}$ and its tensor powers, moreover $H = \omega_P(H)$ is naturally contained in $\overline{A}$, and it is preserved under $*$. Let K be a $\mathbb{Q}$-algebra. In the category of K-algebras with K-linear involution we want to consider the set $C(K)$ of commutative diagrams:

$$\begin{array}{ccc} R \otimes \overline{A} & \xrightarrow{\cong} & R \otimes \operatorname{End}_L^0(\overline{Y}) \\ \uparrow & & \uparrow \\ R \otimes H & \longrightarrow & R \otimes \operatorname{End}_L^0(Y) \end{array}$$

where all arrows, except the upper horizontal one, are the canonical ones. Notice that the natural H^1/L^1-action on $\overline{A}$ fixes H elementwise, and hence it is clear that C is a formal principal homogeneous space, and H^1/L^1 satisfies the Hasse principle. The set of p-adic points on C are the diagrams:

$$\begin{array}{ccc} \mathbb{Q}_p \otimes \overline{A} & \xrightarrow{\cong} & \operatorname{End}_L(\mathbb{D}^0(\overline{Y})) \\ \uparrow & & \uparrow \\ \mathbb{Q}_p \otimes H & \longrightarrow & \mathbb{Q}_p \otimes \operatorname{End}_L^0(Y) \end{array},$$

where we have used the Tate-conjecture to rewrite the upper right entry. From the existence of $(\eta^{(1)}, \eta^{(2)})$ we can now deduce $P(\mathbb{Q}_p) \neq \emptyset$. If we use the ℓ-adic Tate-conjectures, we easily get points for the completions at all other finite primes. The triviality of C follows once $*$ acts positively on $\mathbb{R} \otimes \overline{A}$. At the infinite place the result of the inner twist by our Tate-Nakayama class is simply the endomorphism ring of $H_1(Y(\mathbb{C}), \mathbb{R})$, together with the involution that accommpanies the positive definite pairing given by $\psi(h(\sqrt{-1})x, y)$. □

Remark 8.25. The choice of $\eta^{(0)}$ can be regarded as a choice of isomorphism from P to the locally trivial principal homogeneous space for H^1/L^1, of which the K-valued points are given by the following diagrams of K-linear $*$-preserving maps:

$$\begin{array}{ccc} K\otimes \mathrm{End}^0_L(Y(\mathbb{C})) & \xrightarrow{\cong} & K\otimes \mathrm{End}^0_L(\overline{Y}) \\ \uparrow & & \uparrow \\ K\otimes H & \longrightarrow & K\otimes \mathrm{End}^0_L(Y) \end{array}.$$

By transport of structure, this yields a canonical $\mathbb{A}^{\infty,p}$-valued element $\sigma^{\infty,p}$ of P, in view of $\mathbb{A}^{\infty,p}\otimes \mathrm{End}^0_L(Y(\mathbb{C}))\cong \mathbb{A}^{\infty,p}\otimes \mathrm{End}^0_L(\overline{Y})$. Stated differently, we have a diagram:

$$\begin{array}{ccc} \mathrm{End}^0_L(\overline{Y})\otimes H_1(\overline{Y},\mathbb{A}^{\infty,p}) & \longrightarrow & H_1(\overline{Y},\mathbb{A}^{\infty,p}) \\ {\scriptstyle \eta^{(0)}\otimes \mathrm{sp}_Y}\uparrow & & {\scriptstyle \mathrm{sp}_Y}\uparrow \\ \omega_P(\mathrm{End}^0_L(Y(\mathbb{C})))\otimes H_1(Y(\mathbb{C}),\mathbb{A}^{\infty,p}) & \longrightarrow & H_1(Y(\mathbb{C}),\mathbb{A}^{\infty,p}) \end{array},$$

where the lower horizontal arrow is defined by using the canonical isomorphism $\mathbb{A}^{\infty,p}\otimes\omega_P(\mathrm{End}^0_L(Y(\mathbb{C})))\cong \mathbb{A}^{\infty,p}\otimes \mathrm{End}^0_L(Y(\mathbb{C}))$ arising from the trivialization $\sigma^{\infty,p}\in P(\mathbb{A}^{\infty,p})$.

8.4.3. *Enriched poly-unitary Shimura data.* Assume that (G^1,h) is a Hodge datum and that h factors through a $\mathbb{Q}_p$&$\mathbb{R}$-elliptic, $\mathbb{Q}_p$-unramified maximal $\mathbb{Q}$-torus of the form

$$\mathbb{S}\xrightarrow{h} T=(\mathbb{G}_m\times \mathrm{Res}_{L^+/\mathbb{Q}}T^1)/\{\pm 1\}\subset G.$$

We fix an embedding $\iota_p: K(\mathbb{F}_p^{ac})\to\mathbb{C}$, and we let $\nu\in\mathbb{Q}\otimes X_*(\mathrm{Z}^G)$ be the projection of the cocharacter $\mu_h\in X_*(T)$ onto the space of $\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)$-invariants of $\mathbb{Q}\otimes X_*(T)$ (N.B.: $T_{\mathbb{Q}_p}/\,\mathrm{Z}^G_{\mathbb{Q}_p}$ contains no copy of $\mathbb{G}_{m,\mathbb{Q}_p}$). Let μ' be the image of μ_h in $X_*(T/\,\mathrm{Z}^G)$. Eventually, we want to make the following choices:

(i) Let $P\in \mathit{Tors}(T/\,\mathrm{Z}^G)(\mathbb{Q})$ be of type μ'.
(ii) Let $M:\mathit{Rep}_0(T)\to \mathit{Isoc}$ be the F-isocrystal whose invariant $\gamma_T(M)$ agrees with μ^{-1} as an element of $X_*(T)_{\mathrm{Gal}(\mathbb{Q}_p^{ac}/\mathbb{Q}_p)}$.
(iii) Let $\sigma_p: M|_{\mathit{Rep}_0(T/\,\mathrm{Z}^G)}\to K(\mathbb{F}_p^{ac})\otimes\omega_P$ be a $\mathbb{Q}_p$-linear $\otimes$-isomorphism.

The existence of P is due to the above assumptions on T. We have to introduce the important $\mathbb{Q}$-group $I:=P\times^{T/\,\mathrm{Z}^G}G$. The isomorphism σ_p induces a further $I(\mathbb{Q}_p)\cong \mathrm{Aut}^{\otimes}(M|_{\mathit{Rep}_0(G)})$ one.
Let us come back to a normalised, L-metaunitary Shimura datum,

$$\mathbf{T}=(\mathbf{d}^+,\{(V_i,\Psi_i,\rho_i,\mathbf{j}_i,l_i)\}_{i\in\Lambda},\{(R_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$$

for (G^1,h). By a T-enrichment we mean further choices of maximal commutative $*$-invariant L-algebras H_i in each $\mathrm{End}_L(V_i)$ such that:

- The image of T under ϱ_i is contained in $(\mathbb{G}_m\times H_i^1)/\{\pm 1\}$.

- No H_i is locally odd over L, in the sense of definition 8.22.
- Each H_i is unramified at p.

At the presence of a T-enrichment we show that (i),(ii) and (iii) lead to a (homogeneously) polarized $\mathbb{Q}$-isogeny class of abelian L^Λ-varieties of type $\mathbf{T}$. In fact we will be able to describe its $\mathbb{Q}$ group of self-isogenies, the isogeny class of its crystalline realization, and the canonical action of the former on the latter. From now on we fix $\{H_i\}_{i\in\Lambda}$ satisfying (i), (ii) and (iii), we let $\overline{R}\subset\mathbb{C}$ be the splitting field of the CM-algebra $H=\bigoplus_{i\in\Lambda}H_i$, so that the set of ring homomorphisms $H_{an}=\dot{\bigcup}_{i\in\Lambda}\operatorname{Hom}(H_i,\mathbb{C})$ carries a natural left $\operatorname{Gal}(\overline{R}/\mathbb{Q})$-action. We write $\overline{\vartheta}\in\operatorname{Gal}(\overline{R}/\mathbb{Q})$ for the restriction of the absolute Frobenius on $K(\mathbb{F}_p^{ac})$ to $\overline{R}$, notice $\vartheta=\overline{\vartheta}|_R$. Finally, let $\overline{\mathbf{d}}^+$ (resp. $\bar{\mathbf{j}}_i$) denote the composition of the function $\mathbf{d}^+$ (resp. $\mathbf{j}_i$) with the natural from H_{an} (resp. $H_{i,an}$) to L_{an}.

Writing the L-valued skew-Hermitian pairing on V_i in the usual form $\Psi_i=\operatorname{tr}_{H_i/L}\Psi_i'$ gives rise to skew-Hermitian H_i-modules (V_i,Ψ_i') and thus to H_i-unitary representations $\rho_i':T^1_{H_i^+}\to\mathrm{U}(V_i/H_i,\Psi_i')$, and it is easy to see that $(V_i,\Psi_i',\rho_i',\bar{\mathbf{j}}_i)$ is a normalised $\overline{\vartheta}$-gauged H_i-unitary representation (in the sense of remark 8.4). Thus, we are allowed to apply the lemma 8.2 to the skew-Hermitian Hodge H_i-modules $(V_i,\Psi_i',\varrho_{i,\mathbb{R}}\circ h)$, the outcome being some family of skew-Hermitian Hodge structures $(\tilde{V}_i,\tilde{\Psi}_i',\tilde{h}_i)$, again with coefficients in H_i. For each i we fix a complex abelian variety $Y_i/\mathbb{C}$ with polarization λ_i and Rosati-invariant H_i-action ι_i, such that $H_1(Y_i(\mathbb{C}),\mathbb{Q})$ is isometric to $(\tilde{V}_i,\operatorname{tr}_{H_i/\mathbb{Q}}\tilde{\Psi}_i',\tilde{h}_i)$. Let $\tilde{\upsilon}_i\in X_*((\mathbb{G}_m\times H^1)/\{\pm1\})$ be the associated cocharacters, as in subsubsection 8.4.2. Let Q_i be the $Tors(H_i^1/L^1)(\mathbb{Q})$-object with $\omega_{Q_i}(\operatorname{End}_L(V_i))=\operatorname{End}_L(\tilde{V}_i)$, and notice that:

$$\mathsf{Ob}_{Tors(H_i^1/L^1)(\mathbb{Q})}\ni\tilde{P}_i:=Hom_{T/\mathrm{Z}^G}(Q_i,P)\tag{70}$$

is of type $\tilde{\upsilon}_i'\in X_*(H_i^1/L^1)$. Consider compatible isomorphisms,

$$\tilde{\eta}_i^{(0)}:\omega_{\tilde{P}_i}(\operatorname{End}_L(\tilde{V}_i))\xrightarrow{\cong}\operatorname{End}^0_L(\overline{Y}_i)$$
$$\tilde{\eta}_i^{(1)}:M(\tilde{V}_i)\xrightarrow{\cong}\mathbb{D}^0(\overline{Y}_i)$$
$$\eta^{(2)}:M(\mathbb{Q}(1))\to K(\mathbb{F}_p^{ac})(1)$$

of which the existence is granted by applying lemma 8.24 to the orthogonal direct sum of the (Y_i,λ_i,ι_i)'s, which we will denote by (Y,λ,ι). We proceed by writing $\eta_i^{(0)}$ for the composition of $\tilde{\eta}_i^{(0)}$ with the canonical isomorphism from $\omega_P(\operatorname{End}_L(V_i))$ to $\omega_{\tilde{P}_i}(\operatorname{End}_L(\tilde{V}_i))$ while letting $\eta_i^{(1)}$ be the composition of $\tilde{\eta}_i^{(1)}$ with $M(\varepsilon_{i,p})$, where $\varepsilon_{i,p}:\mathbb{Q}_p\otimes V_i\to$

$\mathbb{Q}_p \otimes \tilde{V}_i$ is an arbitrary $\mathbb{Q}_p \otimes H_i$-linear isometry. Choose a trivialization $\sigma^{\infty,p} \in P(\mathbb{A}^{\infty,p})$, and recall that the choice of $\tilde{\eta}_i^{(0)}$ induces further trivializations $\tilde{\sigma}_i^{\infty,p} \in \tilde{P}_i(\mathbb{A}^{\infty,p})$, as discussed in remark 8.25. Let $\varsigma_i^{\infty,p} : \mathbb{A}^{\infty,p} \otimes \mathrm{End}_L(V_i) \to \mathbb{A}^{\infty,p} \otimes \mathrm{End}(\tilde{V}_i)$ be the unique trivialization of $Q_{i,\mathbb{A}^{\infty,p}}$ coming from $\sigma^{\infty,p}$ and the formula (70) together with the above canonical trivialization $\tilde{\sigma}_i^{\infty,p}$. Furthermore choose a L-linear similarity $\varepsilon_i^{\infty,p} : \mathbb{A}^{\infty,p} \otimes V_i \to \mathbb{A}^{\infty,p} \otimes \tilde{V}_i$ inducing $\varsigma_i^{\infty,p}$, and let $n_i \in (\mathbb{A}^{\infty,p} \otimes L^+)^\times$ be the multiplier of $\varepsilon_i^{\infty,p}$, and let us put

$$\mathbf{S} := (\mathbf{d}^+, \{(V_i, \Psi_i, \rho_i, \mathbf{j}_i, n_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi}).$$

In view of the theorem 8.10 it is now clear that one obtains a (homogeneously) polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian $\mathbb{F}_p^{ac}$-varieties of type $\mathbf{S}$ from $(\overline{Y}, \lambda, \iota)$ if one puts

$$y_\pi := (\bigotimes_{i\in\pi} \eta_i^{(0)}) \circ \omega_P(\iota_\pi),$$

and that $\varepsilon_i^{\infty,p}$ is a full level structure for it.

Theorem 8.26. *Let* $\mathbf{T} = (\mathbf{d}^+, \{(V_i, \Psi_i, \rho_i, \mathbf{j}_i, l_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$ *be a normalised, L-metaunitary Shimura datum for a Hodge datum* (G^1, h) *(and the prime* $\mathfrak{r} \mid p$*) with T-enrichment* $\{H_i\}_{i\in\Lambda}$*, and let* (P, M, σ_p) *be a triple as above. Then there exists a fully levelled and homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian $\mathbb{F}_p^{ac}$-varieties*

$$(\overline{Y}, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi}, \varepsilon^{\infty,p})$$

of type $\mathbf{S} \approx \mathbf{T}$ *with the following properties:*

- *There exists an isomorphism* $\zeta^{(0)}$ *from the twisted $\mathbb{Q}$-group I to the group of self-isogenies* $I_{\overline{Y},\lambda,\iota,\{y_\pi\}_{\pi\in\Pi}}$
- *There exists a L^Λ-linear similitude of isocrystals with L^Λ-operation:*

$$\sum_{i\in\Lambda} \zeta_i^{(1)} = \zeta^{(1)} : M(V) \overset{\cong}{\to} \mathbb{D}^0(Y),$$

  such that

$$\begin{array}{ccc} \mathrm{End}_L(M(V^\pi)) & \longleftarrow & R_\pi \\ {\scriptstyle \bigotimes_{i\in\pi}\zeta_i^{(1)}}\downarrow & & \downarrow{\scriptstyle y_\pi} \\ \mathrm{End}_L(\bigotimes_{i\in\pi} \mathbb{D}^0(Y[\mathfrak{e}_i])(\frac{1-\mathrm{Card}(\pi)}{2})) & \overset{\mathbb{D}^0}{\longleftarrow} & \mathrm{End}^0_L(\dot{\bigotimes}_{i\in\pi} Y[\mathfrak{e}_i]) \end{array}$$

  is commutative for every $\pi \in \Pi$.
- $\zeta^{(1)}$ *carries the canonical I-action on M, which results from* σ_p*, to the I-action on* $(\overline{Y}, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi})$ *that results from* $\zeta^{(0)}$.

Assume in addition, that $\mathbf{T}$ *is unramified, and that there exists a choice of integral structure* $\mathfrak{V}_{i,p}$*, such that the corresponding hyperspecial group* U_p^1 *contains a hyperspecial subgroup of* $T^1(\mathbb{Q}_p \otimes L^+)$*, and let us write* $\mathfrak{T}_p \subset \mathfrak{G}_p$ *for the reductive model of* T *that results from it. Then the* $\mathfrak{G}_p(W(\mathbb{F}_p^{ac}))$*-orbit* $\overline{\zeta}$ *of* $\zeta^{(1)}$ *is a* μ_h*-fake integral structure on* $(\overline{Y}, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi})$*, thus making*

$$\xi = (\overline{Y}, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi}, \varepsilon^{\infty,p}, \overline{\zeta})$$

into a $\mathbb{F}_p^{ac}$*-valued elliptic point of* $\tilde{M}_{\mathbf{S},\mathfrak{p}}$ *(cf. definition 8.20). The unique preimage of* M *(under the functor* $\mathbf{h}^0_{\mu_h}$ *from* $\mathbf{B}_{W(\mathbb{F}_p^{ac}),F}(\mathfrak{T}_p, \mu_h)$ *to* $\mathbf{B}_{K(\mathbb{F}_p^{ac}),F}(\mathfrak{T}_p)$ *of definition 3.3) maps to the crystalline realization of* $(\overline{Y}, \lambda, \iota, \{y_\pi\}_{\pi\in\Pi}, \overline{\zeta})$ *(under the forgetful functor from* $\mathbf{B}_{W(\mathbb{F}_p^{ac}),F}(\mathfrak{T}_p, \mu_h)$ *to* $\mathbf{B}_{W(\mathbb{F}_p^{ac}),F}(\mathfrak{G}_p, \mu_h)$*).*

8.5. **Construction of** ${}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}$**.** In this subsection we assume that (G^1, h) is a Shimura datum with coefficients in L^+, so that the quotient of $G^1(\mathbb{R} \otimes L^+)$ by the centralizer of h is a bounded symmetric domain X^+. This has several consequences:

- Proposition 6.18 implies, that the reduced induced subscheme structure on ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ is finite over ${}_{U^p}\mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r}}$, in particular the product of the line bundles (65) remains ample, when pulled back to ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$.
- For every closed point $x \in {}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ the complete local ring $\hat{\mathcal{O}}_{{}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}},x}$ prorepresents the universal equicharacteristic deformation of the fiber of $\prod_{\mathfrak{q}\in S_p} \tilde{\mathcal{P}}_\mathfrak{q}$ over x, according to remark 6.19. In particular ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p},red}$ is smooth over $\mathbb{F}_{p^f}$ (N.B.: ${}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p}}$ might be non-noetherian).
- $(\mathcal{G}_\mathfrak{q}, \{\mu_{\mathfrak{q},\sigma}\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}})$ is automatically a Φ-datum, so that $\mathcal{B}(\mathcal{G}_\mathfrak{q}, \{\mu_{\mathfrak{q},\sigma}\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}})$ is a lift of $\overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q}, \{\frac{\mu_{\mathfrak{q},\sigma}}{\alpha_{\mathfrak{q},\sigma}}\}_{\sigma\in\Sigma_\mathfrak{q}})$.

Part (iii) of theorem 5.7 yields the 2-commutative diagram:

$$\begin{array}{ccc}
\prod_{\mathfrak{q}\in S_p} \overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q}, \{\frac{\mu_{\mathfrak{q},\sigma}}{\alpha_{\mathfrak{q},\sigma}}\}_{\sigma\in\Sigma_\mathfrak{q}}) & \xleftarrow{\prod_{\mathfrak{q}\in S_p} \overline{\mathcal{P}}_\mathfrak{q}} & {}_{U^p}\overline{\mathcal{M}}_{\mathbf{T},\mathfrak{p}} \\
{\scriptstyle \prod_{\mathfrak{q}\in S_p} \mathrm{Flex}^{\mathbf{d}_\mathfrak{q}^+}} \downarrow & & \downarrow \\
\prod_{\mathfrak{q}\in S_p} \overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q}, \{\upsilon_{\mathfrak{q},\sigma}\}_{\sigma\in\tilde{\Sigma}_\mathfrak{q}}) & \xleftarrow{\prod_{\mathfrak{q}\in S_p} \tilde{\mathcal{P}}_\mathfrak{q}} & {}_{U^p}\tilde{M}_{\mathbf{T},\mathfrak{p},red}
\end{array},$$

which becomes 2-cartesian upon evaluation on algebraically closed fields, as does the diagram:

$$\begin{array}{ccc} \overline{\mathcal{B}}(\mathfrak{G}_p,\mu_p) & \xleftarrow{\overline{\mathcal{P}}} & {}_{U^p}\overline{\mathcal{M}}_{\mathbf{T},\mathfrak{p}} \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\mathfrak{B}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}_{\mathbb{F}_{p^f}} & \longleftarrow & {}_{U^p}\mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r},\mathbb{F}_{p^f}} \end{array},$$

which is obtained from the canonical morphism ${}_{U^p}\mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r},\mathbb{F}_{p^f}}\to\overline{\mathcal{B}}(\mathbb{G}_{m,\mathbb{Z}_p},C)$ together with the 2-cartesianism of:

$$\begin{array}{ccc} \prod_{\mathfrak{q}\in S_p}\overline{\mathcal{B}}(\mathcal{G}_\mathfrak{q},\{\frac{\mu_{\mathfrak{q},\sigma}}{\alpha_{\mathfrak{q},\sigma}}\}_{\sigma\in\Sigma_\mathfrak{q}}) & \longleftarrow & \overline{\mathcal{B}}(\mathfrak{G}_p,\mu_p) \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\overline{\mathcal{B}}(\mathbb{G}_{m,W(\mathbb{F}_{r_\mathfrak{q}})},\{C\}_{\sigma\in\mathbb{Z}/r_\mathfrak{q}\mathbb{Z}}) & \longleftarrow & \overline{\mathcal{B}}(\mathbb{G}_{m,\mathbb{Z}_p},C) \end{array}.$$

In view of corollary 7.6 to describe a lift of ${}_{U^p}\overline{\mathcal{M}}_{\mathbf{T},\mathfrak{p}}$ going with it, all we have to do is define certain characters: $\chi_{i,\mathfrak{q},\sigma}:\mathcal{I}_0^{\mu_{\mathfrak{q},\sigma}}\to\mathbb{G}_{m,W(\mathbb{F}_{p^f})}$ for $i\in\Lambda$, $\mathfrak{q}\in S_p$, and $\sigma\in[0,r_\mathfrak{q}-1]$, such that $\omega_{q_0(\overline{\mathcal{P}}_\mathfrak{q})}(\chi_{i,\mathfrak{q},\sigma})$ agrees with the line bundle $\mathcal{L}_{i,\mathfrak{q},\sigma}$, which is given by the formula (65). We let $\chi_{i,\mathfrak{q},\sigma}$ be the $\mathcal{I}_0^{\mu_{\mathfrak{q},\sigma}}$-character defined by $\prod_{l\in\mathbb{Z}}\chi_{\mu_{\mathfrak{q},\sigma}}(l,{}^{F^{-\sigma}}\rho_{i,\mathfrak{q}})^{d_{i,\mathfrak{q},\sigma,l}}$ where

$$d_{i,\mathfrak{q},\sigma,l}:=-\sum_{\mathbf{d}_\mathfrak{q}(\omega)=\sigma,\mathbf{j}_{i,\mathfrak{q}}(\omega)<l}p^{\mathbf{d}_\mathfrak{q}^+(\omega)}.$$

At last we apply the said corollary to the $\mathcal{I}_0^{\mu_p}$-character defined by

$$\prod_{i\in\Lambda}\prod_{\mathfrak{q}\in S_p}\prod_{\sigma=0}^{r_\mathfrak{q}-1}\chi_{i,\mathfrak{q},\sigma}=\chi:\mathcal{I}_0^{\mu_p}\to\mathbb{G}_{m,W(\mathbb{F}_{p^f})},$$

and we are done because of the subsubsection 4.1.2, here notice, that we have a canonical inclusion

$$\mathcal{I}_0^{\mu_p}\hookrightarrow\prod_{\mathfrak{q}\in S_p}\prod_{\sigma=0}^{r_\mathfrak{q}-1}\mathcal{I}_0^{\mu_{\mathfrak{q},\sigma}}.$$

Observe the 2-commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{B}(\mathfrak{G}_p,\mu_p) & \xleftarrow{\mathcal{P}^{(\nu)}} & {}_{U^p}\mathcal{M}^{(\nu)}_{\mathbf{T},\mathfrak{p}} \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{q}\in S_p}\mathfrak{B}^{\mathbf{R}_\mathfrak{q}(\{\mathfrak{f}_\pi\}_{\pi\in\Pi})}_{\mathbb{F}_{p^f}} & \longleftarrow & {}_{U^p}\mathfrak{M}^{\{\mathfrak{f}_\pi\}_{\pi\in\Pi}}_{\mathbf{P},\mathfrak{r},\mathbb{F}_{p^f}} \end{array}$$

For each ν there is a canonical display with $(\mathfrak{G}_p, \mu_p)$-structure over ${}_{U^p}\mathcal{M}^{(\nu)}_{\mathbf{T},\mathfrak{p}}$, of which ${}_{U^p}\mathcal{F}^{(\nu)}_{\mathbf{T},\mathfrak{p}} \in \mathit{Tors}(\mathcal{I}_0^{\mu_p})({}_{U^p}\mathcal{M}^{(\nu)}_{\mathbf{T},\mathfrak{p}})$ is the level-0 truncation. An analogous truncation of Witt-connections yields canonical elements

$$\nabla^{(\nu)}_{\mathbf{T},\mathfrak{p}} \in \mathit{Conn}_{W_\nu(\mathbb{F}_{p^f})}({}_{U^p}\mathcal{F}^{(\nu)}_{\mathbf{T},\mathfrak{p}} \times^{\mathcal{I}_0^{\mu_p}} \mathfrak{G}_p / {}_{U^p}\mathcal{M}^{(\nu)}_{\mathbf{T},\mathfrak{p}}),$$

whose curvatures vanish. Let $({}_{U^p}\mathcal{F}_{\mathbf{T},\mathfrak{p}}, \nabla_{\mathbf{T},\mathfrak{p}})$ be the limit of the projective system $({}_{U^p}\mathcal{F}^{(\nu)}_{\mathbf{T},\mathfrak{p}}, \nabla^{(\nu)}_{\mathbf{T},\mathfrak{p}})$ as $\nu \to \infty$. In all of this the omission of level structures indicates passage to the limit, i.e. $\mathcal{M}_{\mathbf{T},\mathfrak{p}} := \lim_{U^p \to 1} {}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}$, note that $\mathcal{F}_{\mathbf{T},\mathfrak{p}}$ is a $G(\mathbb{A}^{\infty,p})$-equivariant locally trivial principal homogeneous space for $\mathcal{I}_0^{\mu_p}$ over $\mathcal{M}_{\mathbf{T},\mathfrak{p}}$ and $G(\mathbb{A}^{\infty,p})$ fixes $\nabla_{\mathbf{T},\mathfrak{p}}$.

Remark 8.27. Notice that the $W(\mathbb{F}_{p^f})$-schemes ${}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}$ are smooth of relative dimension equal to $\dim X^+$. It would be interesting to know whether

$$\omega_{{}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}/W(\mathbb{F}_{p^f})} = \Omega^{\dim X^+}_{{}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}/W(\mathbb{F}_{p^f})}$$

is an ample line bundle.

8.5.1. *Automorphisms of lifts.* Recall that

$$\mathbf{T} = (\mathbf{d}^+, \{(V_i, \rho_i, \Psi_i, \mathbf{j}_i, l_i)\}_{i \in \Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi \in \Pi})$$

is a L-metaunitary Shimura datum for our Shimura datum with L^+-coefficients (G^1, h). Throughout this and the next subsubsection we fix a complete discretely valued $K(\mathbb{F}_{p^f})$-extension N with algebraically closed residue field l.

Definition 8.28. *Let $y = (Y, \lambda, \iota, \{y_\pi\}_{\pi \in \Pi})$ be a homogeneously polarized $\mathbb{Q}$-isogeny class of L^Λ-abelian l-varieties of type $\mathbf{T}$. By a μ_p-fake $\mathcal{O}_N$-lift of y we mean a pair $(\mathcal{P}, \epsilon)$ where $\mathcal{P}$ is a display with $(\mathfrak{G}_p, \mu_p)$-structure over $\mathcal{O}_N$, and ϵ is an isomorphism from $\mathbf{h}^0_{\mu_p}(\mathcal{P}_l)$ to the isocrystal with G-structure* $\mathrm{Iso}(y)$. *By a μ_p-fake N-motive with $\mathfrak{G}_p$-structure we mean a sextuple $\tilde{y} = (Y, \lambda, \iota, \{y_\pi\}_{\pi \in \Pi}, \mathcal{P}, \epsilon)$ with these properties.*

A μ_p-fake lift determines a μ_p-fake integral structure $\overline{\zeta}$, moreover it is convenient to choose a base point $\zeta \in \overline{\zeta}$ such that one has $g_1 = 1$ i.e. ${}^F b_{y,\zeta} = O^F \mu_p(\frac{1}{p})$ in definition 8.15. Notice that $\mathbf{B}_{W(l),F}(\mathfrak{G}_p, \mu_p)$ and $B_l(\mathfrak{G}_p, \mu_p)$ are absolutely the same thing, and that every display $\mathcal{P}$ as above is automatically banal, and since $\hat{B}_{W(\mathcal{O}_N), I(\mathcal{O}_N)}(\mathfrak{G}_p, \mu_p)$ and $B_{\mathcal{O}_N}(\mathfrak{G}_p, \mu_p)$ are absolutely the same thing too, the above can be rephrased in more banal language: Look at pairs $(U, \zeta) \in \mathfrak{G}_p(W(\mathcal{O}_N)) \times$

$\mathrm{Iso}(y)$ rendering the diagram

$$\begin{array}{ccc} K(k)\otimes V & \xleftarrow{\varrho(O^F\mu_p(\frac{1}{p}))\circ(F\otimes \mathrm{id}_V)} & K(k)\otimes V \\ \zeta\downarrow & & \zeta\downarrow \\ \mathbb{D}^0(Y) & \xleftarrow{F} & \mathbb{D}^0(Y) \end{array}$$

commutative, where $O \in \mathfrak{G}_p(W(l))$ is the $\mod \mathfrak{m}_N$-reduction of U, and look at the equivalence relation $(U',\zeta') \sim (U,\zeta)$ defined by the conditions

$$U' = k^{-1}U\Phi^{\mu_p}(k) \tag{71}$$

$$\zeta' = \zeta \circ \varrho(k_0) \tag{72}$$

where k is an element of $\mathcal{I}^{\mu_p}(\mathcal{O}_N)$ and k_0 is its $\mod \mathfrak{m}_N$-reduction. Then a μ_p-fake lift is an equivalence class of such pairs (U,ζ). We associated groups I_y over $\mathbb{Q}$ and $\hat{\Gamma}_U \subset J_{^F b_{y,\zeta}}$ over $\mathbb{Q}_p$ and homomorphisms $j_{y,\zeta} : I_{y,\mathbb{Q}_p} \to J_{b_{y,\zeta}}$ (and $j_U : \hat{\Gamma}_{U,N} \to \mathcal{I}^{\mu_p}_{0,N}$) to y, ζ and U. Let us define $K_{y,U,\zeta}$ to be the largest $\mathbb{Q}$-algebraic subgroup of I_y which is contained in $j_{y,\zeta}^{-1}(\mathrm{Int}^G(b_{y,\zeta}/K(l))(\hat{\Gamma}_U))$ (N.B. $\mathrm{Int}^G(b_{y,\zeta}/K(l))$ is the canonical isomorphism from $J_{^F b_{y,\zeta}}$ to $J_{b_{y,\zeta}}$). We define the semilocal congruence subgroup $\mathfrak{K}_{y,U,\zeta} \subset K_{y,U,\zeta}(\mathbb{Q})$ to be the inverse image of $\mathfrak{G}(W(\mathcal{O}_N))$ under $\mathrm{Int}^G(b_{y,\zeta}^{-1}/K(l)) \circ j_{y,\zeta}$. Finally notice that $K_{y,U,\zeta}$ is anisotropic over $\mathbb{R}$, as I_y^1 is totally compact.

Remark 8.29. Let us assume that $\mathbf{T}$ is normalised. An easy modification of theorem 8.16 yields a $G(\mathbb{A}^{\infty,p})$-equivariant bijection between the $G(\mathbb{A}^{\infty,p})$-sets of μ_p-fake N-motives with $\mathfrak{G}_p$-structure and the N-valued points of $\mathcal{M}_{\mathbf{T},\mathfrak{p}}$ (i.e. $\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathcal{O}_N)$). By the same token the $\mathcal{I}_0^{\mu_p}(\mathcal{O}_N) \times G(\mathbb{A}^{\infty,p})$-set $\mathcal{F}_{\mathbf{T},\mathfrak{p}}(\mathcal{O}_N)$ classifies data comprising of a fully levelled, homogeneously polarized $\mathbb{Q}$-isogeny classes of L^Λ-abelian l-varieties (y,η) of type $\mathbf{T}$ together with a $\sim$-class of pairs $(U,\zeta) \in \mathfrak{G}_p(W(\mathcal{O}_N)) \times \mathrm{Iso}(y)$ with respect to the slightly smaller equivalence relation that $(U',\zeta') \sim (U,\zeta)$ if (71) and (72) hold for some $k \in \mathfrak{G}_p(I(\mathcal{O}_N))$. These bijections are functorial in N. At last, a straightforward modification of corollary 8.18 shows, that the group homomorphism $\mathrm{Int}^G(b_{y,\zeta}^{-1}/K(l)) \circ j_{y,\zeta} \times i_{y,\eta}$ maps $\mathfrak{K}_{y,U,\zeta}$ onto the stabilizer in $\mathcal{I}_0^{\mu_p}(\mathcal{O}_N) \times G(\mathbb{A}^{\infty,p})$ of the element $\kappa \in \mathcal{F}_{\mathbf{T},\mathfrak{p}}(\mathcal{O}_N)$ corresponding to the data $(Y,\lambda,\iota,\{y_\pi\}_{\pi\in\Pi},\eta,U,\zeta)$ by means of the said bijection.

8.5.2. *Canonical lifts.* Typical examples of μ_p-fake lifts arise in the following way: Assume that $(y,\overline{\zeta})$ is toric with respect to some $T \subset I_y$ and $b \in T(K(\mathbb{F}_p^{ac}))$, and that $\zeta \in \overline{\zeta}$ is chosen according to definition

8.20. We take the convention in remark 8.21 for granted and fix a $\mathbf{B}_{W(\mathbb{F}_p^{ac}),F}(\mathfrak{T}_p)$-isomorphism from $\mathbf{h}_{\mu_p}(1_{\mathfrak{T}_p})$ to b (N.B.: all $B_{\mathbb{F}_p^{ac}}(\mathfrak{T}_p,\mu_p)$-objects are represented by the neutral element $1_{\mathfrak{T}_p}$). Applying $j_{y,\zeta}$ to it gives us the sought for isomorphism ϵ from $\mathcal{P}_{\mathbb{F}_p^{ac}}$ to the crystalline realization of $(y,\overline{\zeta})$, where $\mathcal{P}$ is the banal $W(\mathbb{F}_{p^f})$-display with $(\mathfrak{T}_p,\mu_p)$-structure over $\operatorname{Spec} W(\mathbb{F}_{p^f})$ represented by the neutral element $1_{\mathfrak{T}_p}$ (N.B.: The mod p-reduction $B_{W(\mathbb{F}_p^{ac})}(\mathfrak{T}_p,\mu_p)\to B_{\mathbb{F}_p^{ac}}(\mathfrak{T}_p,\mu_p)$ is an equivalence of categories). The significance of this example is a very geometric one: Let η be a full level structure for y, so that we obtain points $(y,\eta,\overline{\zeta})=\xi\in\overline{\mathcal{M}}_{\mathbf{T},\mathfrak{p}}(\mathbb{F}_p^{ac})$ and $(y,\eta,\mathcal{P},\epsilon)=\kappa\in\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{Z}_p^{sh})$, where $\mathbb{Z}_p^{sh}$ denotes the subring $\bigcup_{s=1}^{\infty}W(\mathbb{F}_{p^s})$ of $W(\mathbb{F}_p^{ac})$. We have a 2-commutative diagram

$$\begin{array}{ccc}
\mathcal{B}(\mathfrak{T}_p,\mu_p) & \longleftarrow & \operatorname{Spf} W(\mathbb{F}_p^{ac}) \\
{\scriptstyle j_{y,\zeta}|_{\mathfrak{T}_p}}\downarrow & & {\scriptstyle\kappa}\downarrow \\
\mathcal{B}(\mathfrak{G}_p,\mu_p) & \xleftarrow{\mathcal{P}^{(\infty)}} & \mathcal{M}_{\mathbf{T},\mathfrak{p}}^{(\infty)}
\end{array},$$

where $\mathcal{M}_{\mathbf{T},\mathfrak{p}}^{(\infty)}$ is the projective limit of the formal schemes ${}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}^{(\infty)}=\lim_{\nu\to\infty}{}_{U^p}\mathcal{M}_{\mathbf{T},\mathfrak{p}}^{(\nu)}$, as U^p tends to 1. The following properties are fulfilled:

- The bottom row is $G(\mathbb{A}^{\infty,p})$-equivariant.
- The top row is $\mathfrak{T}_p(\mathbb{Z}_p)$-equivariant with respect to the trivial action on $\operatorname{Spf} W(\mathbb{F}_p^{ac})$ and the natural action of $\mathfrak{T}_p(\mathbb{Z}_p)$ on any $\mathcal{B}(\mathfrak{T}_p,\mu_p)$-object.
- The vertical map κ preserves the $T(\mathbb{Q})\cap\mathfrak{T}_p(\mathbb{Z}_p)$-equivariance, with respect to the map to $G(\mathbb{A}^{\infty,p})$ given by $i_{y,\eta}$.

In this scenario we will call κ the T-canonical lift of the toric point ξ.

9. Conclusions

Let us choose a $\mathcal{O}_{ER}$-linear embedding $\iota_p: W(\mathbb{F}_p^{ac})\hookrightarrow\mathbb{C}$ making $\mathbb{C}$ into a $W(\mathbb{F}_p^{ac})$-algebra. We would like to describe the $G(\mathbb{A}^{\infty,p})$-space $\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$ in the analytic category, ideally together with the complexification of the pair $(\mathcal{F}_{\mathbf{T},\mathfrak{p}},\nabla_{\mathbf{T},\mathfrak{p}})$, again taking into account its $G(\mathbb{A}^{\infty,p})$-equivariance. Let us choose a base point $s_0\in\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$, and let M_{s_0} be the set of pairs (s_1,H) where H is a homotopy class of paths from s_0 to some $s_1\in\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$. Let Δ_{s_0} be the set of pairs (g,H) where H is a homotopy class of paths from s_0g to s_0 for some $g\in G(\mathbb{A}^{\infty,p})$. It is clear that M_{s_0} is a simply connected complex manifold, while

$$(g_0,H_0)*(g_1,H_1):=(g_0g_1,(H_0g_1)*H_1)$$

defines a group law on Δ_{s_0}, as the homotopy classes $H_0 g_1$ of paths from $s_0(g_0 g_1)$ to $s_0 g_1$ and H_1 of paths from $s_0 g_1$ to s_0 can be concatenated. The isotropy group Γ_{s_0} of s_0 in $G(\mathbb{A}^{\infty,p})$ can be identified with the subgroup of elements $(g, H) \in \Delta_{s_0}$ for which H is constant. The group Δ_{s_0} acts on M_{s_0} from the left by means of

$$(g_0, H_0) * (s_1, H) := (s_1 g_0^{-1}, (H_0 * H) g_1^{-1}),$$

and $\Delta_{s_0} \backslash (M_{s_0} \times G(\mathbb{A}^{\infty,p}))$ can be identified with an open $G(\mathbb{A}^{\infty,p})$-invariant union of path-component of $\mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$. Upon having chosen a horizontal section $\tilde{s}$ of $\mathcal{F}_{\mathbf{T},\mathfrak{p}} \times^{\mathcal{I}_0^{\mu_p}} \mathfrak{G}_p$ over M_{s_0}, we obtain a canonical monodromy homomorphism $\varphi_{\tilde{s}} : \Delta_{s_0} \to G(\mathbb{C})$ and the locally biholomorphic Δ_{s_0}-equivariant period map $I_{\tilde{s}} : M_{s_0} \to (G/P)(\mathbb{C})$, where $P = U^0_{\iota_p(\mu_p^{-1})}$ is the complexification of $\mathcal{I}_0^{\mu_p}$. Before we proceed we give the provisional outcome of these methods, U_p^{ab} denotes the maximal compact subgroup of $G^{ab}(\mathbb{Q}_p)$:

Theorem 9.1. *Let*

$$\mathbf{T} = (\{(V_i, \Psi_i, \rho_i, \mathbf{j}_i, l_i)\}_{i \in \Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi \in \Pi})$$

be a normalised, L-metaunitary Shimura datum, for a Shimura datum (G^1, h)*, fix a integral structure* $\{\mathfrak{V}_{i,p}\}_{i \in \Lambda}$*. Suppose that* $\kappa \in \mathcal{M}_{\mathbf{T},\mathfrak{p}}(\mathbb{Z}_p^{sh})$ *is a canonical lift of an elliptic point* $\xi \in \overline{\mathcal{M}}_{\mathbf{T},\mathfrak{p}}(\mathbb{F}_p^{ac})$*, and let* $\mathcal{M}^*_{\mathbf{T},\mathfrak{p}}$ *be the smallest open and closed* $G(\mathbb{A}^{\infty,p})$*-invariant subscheme of* $\mathcal{M}_{\mathbf{T},\mathfrak{p}}$ *which contains these.*

- *The complexification* $\mathcal{M}^*_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$ *is* $G(\mathbb{A}^{\infty,p})$*-equivariantly biholomorphically equivalent to the space*

$$\Delta^* \backslash (X^+ \times G(\mathbb{A}^{\infty,p})),$$

for a discrete and cocompact subgroup $\Delta^* \subset \mathrm{Aut}(X^+)^+ \times G(\mathbb{A}^{\infty,p})$.

- *The image of* Δ^* *in* $G^{ab}(\mathbb{A}^{\infty,p})$ *is contained in*

$$G^{ab}(\mathbb{Z}_{(p)})^+ := G^{ab}(\mathbb{Q}) \cap ((\mathbb{R}^\times \times G^{1ab}(\mathbb{R} \otimes L^+))/\{\pm 1\} \times U_p^{ab} \times G^{ab}(\mathbb{A}^{\infty,p})),$$

the intersection taking place in $G^{ab}(\mathbb{A})$*, and it has finite index in that group.*

Proof. Let s_0 be the generic fiber of κ and let $\tilde{s}$ be as above. Let G' be the quotient of $G_\mathbb{R}$ by the product of its maximal normal compact subgroup with$w(\mathbb{G}_{m,\mathbb{R}})$, let P' be the image of P in $G'_\mathbb{C}$, and let $I'_{\tilde{s}}$ be the precomposition of $I_{\tilde{s}}$ with the canonical projection $(G/P)(\mathbb{C}) \to (G'_\mathbb{C}/P')(\mathbb{C})$. We show that one obtains a bounded period morphism $(\Delta_{s_0}, M_{s_0}, G'_\mathbb{C}, P', \varphi'_{\tilde{s}}, I'_{\tilde{s}})$ in the sense that all properties (M1), (M2), (M3), and (M4) of the part E of the appendix are valid, (M3) follows immediately from subsubsection 8.5.2. Property (M2) is a mediate consequence of corollary 3.36 (cf. subsubsection 8.5.1). Property (M1) holds because the monodromy group is already maximal in the formal category by lemma 7.2, and property (M4) holds because $\mathcal{L}$ is ample. Now recall our prefered choice of $h \in X^+$, and that part E of the appendix with [9, Proposition 1.2.2] enable us to choose $\tilde{s}$ in such a way that

$$\varphi'_{\tilde{s}}(\Delta_{s_0}) \subset G'(\mathbb{R})^+ \tag{73}$$

$$I_{\tilde{s}}(s_0) = h \tag{74}$$

hold, giving the requested bijection from the complexification of $\mathcal{M}^*_{\mathbf{T},\mathfrak{p}}$ to $\Delta_{s_0}\backslash(X^+ \times G(\mathbb{A}^{\infty,p}))$, which sends the base point s_0 to h. From now on we will treat Δ_{s_0} as a subgroup of $G(\mathbb{A}^{\infty,p})$. Notice that every element of Δ_{s_0} can be written as $\gamma = \gamma_1\gamma_2$ with regular semisimple elements $\varphi_{\tilde{s}}(\gamma_1)$ and $\varphi_{\tilde{s}}(\gamma_2)$. Following the method of Ihara we let s_1 and s_2 be their fixed points on M_{s_0}, and consider corresponding maximal tori $T_1 \subset K_{s_1}$ and $T_2 \subset K_{s_2}$ containing γ_1 and γ_2. Now notice that we can apply part (i) of proposition 8.19, because tori are connected. □

We choose an auxiliary compact open subgroup of $G(\mathbb{A}^{\infty,p})$, we prefer it neat and of the form $U^p = \prod_{\ell\neq p} U_\ell$. Spurred by the above result we continue with the study of the groups $\Delta_{s_0} \cap G^{der}(\mathbb{A}^{\infty,p}) =: \Delta_{s_0}^{der}$ and $\Delta_{s_0}^{der} \cap \prod_{p\neq\ell\leq n} G(\mathbb{Q}_\ell) \times \prod_{p\neq\ell>n} U_\ell =: \Delta_{s_0,n}^{der}$. We consider the maps $\phi : \Delta_{s_0} \to G'(\mathbb{R}) \times G(\mathbb{A}^{\infty,p})$ (resp. $\phi_n : \Delta_{s_0} \to G'(\mathbb{R}) \times \prod_{p\neq\ell\leq n} G(\mathbb{Q}_\ell)$) defined by the cartesian product of $\varphi'_{\tilde{s}_0}$ with the natural inclusion into $G(\mathbb{A}^{\infty,p})$ (resp. projection to $\prod_{p\neq\ell\leq n} G(\mathbb{Q}_\ell)$). For any place $v \in L^+_{an}\dot{\cup}(\operatorname{Spec}\mathcal{O}_{L^+}\backslash\{0\})$ we let G^1_v be the algebraic group $G^1_{L^+_v}$ where $L^+_\iota = \mathbb{R}_{[\iota]}$ for any $\iota \in L^+_{an}$ while $L^+_{\mathfrak{m}}$ is the $\mathfrak{m}$-adic completion for any maximal ideal $\mathfrak{m} \subset \mathcal{O}_{L^+}$. We are particularly interested in the sets of places

$$S_n^{\infty,p} = \{\mathfrak{m} \in \operatorname{Spec}\mathcal{O}_{L^+} | p \notin \mathfrak{m} \ni n!\}$$
$$S^{\infty,p} = \{\mathfrak{m} \in \operatorname{Spec}\mathcal{O}_{L^+} | p \notin \mathfrak{m} \neq 0\}.$$

For the discussion in the remainder of this section we need the following additional assumptions on (G^1, h) and $\mathbf{T}$:

- p is inert in L^+ and G^{1der} is a simply connected and absolutely almost simple of type B_l, C_l or E_7.
- There exist $i_1, i_2 \in \Lambda$ such that at least one of the two equivalences $\rho_{i_1} \otimes (L \oplus \operatorname{Ad}^{G_L^{1ad}}) \simeq \rho_{i_2}(\text{or } \overline{\rho}_{i_2})$ hold.

We will be interested in the irreducible lattices

$$\phi_n(\Delta_{s_0,n}^{der}) \subset G'(\mathbb{R}) \times \prod_{p\neq\ell\leq n} G^{der}(\mathbb{Q}_\ell).$$

Here notice, that part (ii) of our proposition 8.19 together with the above assumption (iii) implies the "property f_1" of [53, remark 3.8]. Let Σ be the set of real embeddings of L^+ such that $G^1(\mathbb{R}_{[\iota]})$ is not compact. Following [53, step 2V] there exists a simply connected algebraic group D over a number field K together with

- embeddings $\iota_v : K \to L_v^+$ and
- isomorphisms $\psi_v : D \times_{K,\iota_v} L_v^+ \stackrel{\cong}{\to} G_v^{1der}$

for any $v \in \Sigma\dot{\cup}S^{\infty,p}$, such that the corresponding topological map

$$\psi_n : \prod_{v\in\Sigma\dot{\cup}S_n^{\infty,p}} D(L^+_{v\,[\iota_v]}) \to \prod_{v\in\Sigma\dot{\cup}S_n^{\infty,p}} G_v^{1der}(L_v^+)$$

sends some (hence any) congruence subgroup of the $\mathcal{O}_K[\prod_{p\neq\ell\leq n}\ell^{-1}]$-valued points of some $\mathcal{O}_K[\prod_{p\neq\ell\leq n}\ell^{-1}]$-form of D to a group commensurable with $\phi_n(\Delta_{s_0,n})$ (N.B.: Since A_l is excluded, the map of places $v \mapsto \iota_v^{-1}(v)$ is an injection, but this is not essential, cf. [53, step 2IV]). According to the method described in [53, step 2VI], we can use the property f_1 again to show that:

- $K = L^+$
- ι_v is equal to the canonical embedding of L^+ into L_v^+ for each $v \in \Sigma\dot{\cup}S^{\infty,p}$

We need one more property of the ψ_v's, namely, that their product gives rise to a map

$$\psi : \prod_{\iota\in\Sigma} D^{ad}(\mathbb{R}_{[\iota]}) \times D(\mathbb{A}^{\infty,p}\otimes L^+) \stackrel{\cong}{\to} G'(\mathbb{R}) \times G^{1der}(\mathbb{A}^{\infty,p}\otimes L^+),$$

which is not obvious, however the proof of an analog, that is given in [53, step 2VII], can be carried over to the case at hand, ψ turns out to be continuous, and it sends some (hence any) $S^{\infty,p}$-congruence subgroup of $D(L^+)$ to a group commensurable with $\phi(\Delta_{s_0}^{der})$. Again following [53, step 2VIII] in part, we observe that $\psi(D(L^+))$ contains $\phi(\Delta_{s_0}^{der})$: Since the subgroups $\{\gamma \in \Delta_{s_0} \cap \Gamma_{s_1} | \phi(\gamma) \in \psi(D(L^+))\}$ are of finite index in $\Delta_{s_0} \cap \Gamma_{s_1}$, for each s_1, their Zariski closures must contain the neutral component of the K_{s_1}'s, which suffices according to the method of Ihara (N.B.: our subsubsection 8.5.1 serves as analog to the "property g)" of [52, Proposition 3.6]). Under the above assumption, that p is inert in L^+, one can still use the Hasse principle to deduce $D \cong G^{der}$. Let

$\mathcal{M}^0_{\mathbf{T},\mathfrak{p}}$ the connected component of $\mathcal{M}_{\mathbf{T},\mathfrak{p},\mathbb{Z}_p^{sh}}$ that contains κ, let us sum up what we have shown:

- $\mathcal{M}^0_{\mathbf{T},\mathfrak{p}}$ is a separated, quasicompact, universally closed and formally smooth $\mathbb{Z}_p^{sh}$-scheme, which inherits a $G^{der}(\mathbb{A}^{\infty,p})$-action from $\mathcal{M}_{\mathbf{T},\mathfrak{p}}$.
- the complexification $\mathcal{M}^0_{\mathbf{T},\mathfrak{p}}(\mathbb{C}_{[\iota_p]})$ is biholomorphically equivalent to the space
$$\Delta^{(1)}\backslash(X^+ \times G^{der}(\mathbb{A}^{\infty,p})),$$
for a $S^{\infty,p}$-arithmetic subgroup of $\Delta^{(1)} \subset G^{der}(L)$.

The proof of theorem 1.1 is completed by the usual methods of [10, paragraph 6], together with the known local cases of the congruence subgroup property, cf. [49], [43, Theorem 9.1]. At last notice, that
$$\coprod_{\mathbf{T}\approx\mathbf{S}/\simeq} \overline{\mathcal{M}}_{\mathbf{S},\mathfrak{p}}$$
does contain elliptic points (and there are $2^{\mathrm{Card}(\Lambda)}$ many $\simeq$-classes of such $\mathbf{S}$).

Appendix A. Some facts on group schemes

Recall, that every element g in an abstract group G gives rise to an inner automorphism, which is defined by sending any element $x \in G$ to its conjugate gxg^{-1}, and denoted by $\mathrm{int}^G(g) \in \mathrm{Aut}(G)$. Now let X be a scheme and let $\mathcal{G}$ be an X-functor in groups. We write $\mathrm{End}(\mathcal{G})$ for the class of functorial transformations $\mathcal{G} \to \mathcal{G}$ which preserve the group structure, and the subclass of invertible elements therein is denoted by $\mathrm{Aut}(\mathcal{G})$. Again, the elements $g \in \mathcal{G}(R)$ give rise to natural elements $\mathrm{Int}^{\mathcal{G}}(g/R) \in \mathrm{Aut}(\mathcal{G}_R)$, because for varying X-morphisms $\phi : R \to S$ the families of inner automorphisms $\mathrm{int}^{\mathcal{G}(S)}(\mathcal{G}(\phi)(g))$ form invertible transformations from the group functor $\mathcal{G}_R$ to itself. By $End(\mathcal{G})$ we mean the monoidal X-functor $R \mapsto \mathrm{End}(\mathcal{G}_R)$, whenever it exists i.e. provided that each of the classes on the right hand-side is a set, and similarly for the group functor $Aut(\mathcal{G})$. Moreover, in case $Aut(\mathcal{G})$ is well-defined, there is an associated group structure-preserving functorial transformation $\mathrm{Int}^{\mathcal{G}} : \mathcal{G} \to Aut(\mathcal{G})$, which is given by $\mathrm{Int}^{\mathcal{G}}_R : \mathcal{G}(R) \to \mathrm{Aut}(\mathcal{G}_R); g \mapsto \mathrm{Int}^{\mathcal{G}}(g/R)$ on the set of points over $R \in \mathtt{Ob}_{Alg_X}$. We let $\mathrm{Z}^{\mathcal{G}}$ be the kernel of $\mathrm{Int}^{\mathcal{G}}$. Finally, let P be a functor over a X-scheme Y which is endowed with a right $\mathcal{G} \times_X Y$-action. We will say that P is a formal principal homogeneous space for $\mathcal{G}$ over Y if each non-empty $P(R)$ becomes a principal homogeneous $\mathcal{G}(R)$-set.

If $\mathcal{G}$ is a fpqc-sheaf of groups on X, then a formal principal homogeneous space P for $\mathcal{G}$ over Y is called locally trivial if the following holds:

- P is a fpqc-sheaf on Y
- For every $R \in \mathtt{Ob}_{Alg_Y}$ there exists a faithfully flat R-algebra S such that $P(S)$ is non-empty.

If one requires that $\mathcal{G}$ be representable by a flat and relatively affine group scheme over X, then a locally trivial principal homogeneous space for $\mathcal{G}$ over Y is also representable by a flat and relatively affine Y-scheme, by descent theory (and [19, Proposition(2.7.1.xiii)]). Let us write $\mathit{Tors}(\mathcal{G})$ for the X-stack of locally trivial principal homogeneous spaces for $\mathcal{G}$, so that the groupoid of X-morphisms from some X-scheme Y to $\mathit{Tors}(\mathcal{G})$ is equivalent to the category of locally trivial principal homogeneous spaces for $\mathcal{G}$ over Y. Notice that both, the diagonal 1-morphism

$$\Delta_{\mathit{Tors}(\mathcal{G})} : \mathit{Tors}(\mathcal{G}) \to \mathit{Tors}(\mathcal{G}) \times_X \mathit{Tors}(\mathcal{G}),$$

as well as the classifying morphism of the globally trivial principal homogeneous space for $\mathcal{G}$ over X, for which we write

$$\mathbf{b}(\mathcal{G}) : X \to \mathit{Tors}(\mathcal{G}),$$

are affine and flat. For the remainder of this section $\mathcal{G}$ will denote a fixed flat and affine group scheme over a fixed base ring B, which we assume to be a Dedekind ring or a field. The following conventions on B-algebras R will be in force: By an augmentation of R, one means a B-linear homomorphism p from the B-algebra R to the B-algebra B, so that $R = B \oplus \mathfrak{a}$, where $\mathfrak{a} = \ker p$ is the so-called augmentation ideal. It is customary to denote the kernel of the natural map $\mathcal{G}(R) \to \mathcal{G}(B)$ by $\mathcal{G}(\mathfrak{a})$. By an ideal of $\mathfrak{a}$ one means a B-submodule $\mathfrak{b} \subset \mathfrak{a}$ with $\mathfrak{b}\mathfrak{a} \subset \mathfrak{b}$ so that both $B \oplus \mathfrak{b}$ and $B \oplus \mathfrak{a}/\mathfrak{b}$ are augmented B-algebras. In this situation there is a canonical exact sequence

$$1 \to \mathcal{G}(\mathfrak{b}) \to \mathcal{G}(\mathfrak{a}) \to \mathcal{G}(\mathfrak{a}/\mathfrak{b}),$$

simply because the diagram

$$\begin{array}{ccc} \mathcal{G}(B \oplus \mathfrak{b}) & \longrightarrow & \mathcal{G}(B \oplus \mathfrak{a}) \\ \downarrow & & \downarrow \\ \mathcal{G}(B) & \longrightarrow & \mathcal{G}(B \oplus \mathfrak{a}/\mathfrak{b}) \end{array}$$

is (commutative and) cartesian, just think of $B \oplus \mathfrak{b}$ being the fiber product $(B \oplus \mathfrak{a}) \times_{B \oplus \mathfrak{a}/\mathfrak{b}} B$.

Appendix B. Some facts on Connections

Let Z be a B-algebra, let A be a Z-algebra, let $\mathrm{pr}_1, \mathrm{pr}_2 : A \to A \otimes_Z A$ and $\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : A \otimes_Z A \to A \otimes_Z A \otimes_Z A$ be the Z-algebra homomorphisms which are defined by:

$$\begin{aligned}\mathrm{pr}_1(x) &= x \otimes 1\\ \mathrm{pr}_2(x) &= 1 \otimes x\\ \mathrm{pr}_{12}(x \otimes y) &= x \otimes y \otimes 1\\ \mathrm{pr}_{23}(x \otimes y) &= 1 \otimes x \otimes y\\ \mathrm{pr}_{13}(x \otimes y) &= x \otimes 1 \otimes y.\end{aligned}$$

The set $\{(\mathrm{pr}_2(x) - \mathrm{pr}_1(x))(\mathrm{pr}_2(y) - \mathrm{pr}_1(y)) | x, y \in A\}$ generates an ideal, which we denote by $I \subset A^{\otimes_Z 2}$, and for each $i \in \{1, 2\}$ we denote the composition of pr_i with the natural map from $A^{\otimes_Z 2}$ to the quotient Z-algebra $A' := A^{\otimes_Z 2}/I$ by:

$$\overline{pr}_i : A \to A'.$$

Now consider the Z-algebra $A'' := A^{\otimes_Z 3}/\,\mathrm{pr}_{12}(I) + \mathrm{pr}_{23}(I) + \mathrm{pr}_{13}(I)$, and notice that for any indices $1 \leq i < j \leq 3$, the composition of pr_{ij} with the natural map from $A^{\otimes_Z 3}$ to the quotient A'' factors through further canonical Z-algebra homomorphisms:

$$\overline{pr}_{ij} : A' \to A''.$$

In the following result $\mathrm{alt}^2_A M$ denotes the cokernel of the endomorphism $M^{\otimes_A 2} \to M^{\otimes_A 2}; x \otimes y \mapsto x \otimes y - y \otimes x$, for any A-module M.

Proposition B.1. *The above Z-algebra A' is canonically isomorphic to $A \oplus \Omega^1_{A/Z}$, on which the multiplication law is given by*

$$(x, \phi)(x', \phi') = (xx', x\phi' + x'\phi).$$

Moreover, by transport of structure the above Z-linear homomorphism $\overline{pr}_1$ (resp. $\overline{pr}_2$) gives rise to the Z-linear homomorphism from A to $A \oplus \Omega^1_{A/Z}$ which is given by $x \mapsto x$ (resp. $x \mapsto x + d_{A/Z}(x)$). The above Z-algebra A'' is canonically isomorphic to $A \oplus \Omega^1_{A/Z} \oplus \Omega^1_{A/Z} \oplus \mathrm{alt}^2_A \Omega^1_{A/Z}$ on which the multiplication law is given by

$$(x, \epsilon, \delta, \eta)(x', \epsilon', \delta', \eta') = (xx', x\epsilon' + x'\epsilon, x\delta' + x'\delta, x\eta' + x'\eta + \epsilon \otimes \delta' + \epsilon' \otimes \delta).$$

Moreover, by transport of structure the above homomorphism $\overline{pr}_{12}$ (resp. $\overline{pr}_{23}$ or $\overline{pr}_{13}$) gives rise to the Z-linear homomorphism from $A \oplus \Omega^1_{A/Z}$ to $A \oplus \Omega^1_{A/Z} \oplus \Omega^1_{A/Z} \oplus \mathrm{alt}^2_A \Omega^1_{A/Z}$ which is given by $(x, \phi) \mapsto (x, \phi, 0, 0)$ (resp. $(x, \phi) \mapsto (x, d_{A/Z}(x), \phi, d_{A/Z}(\phi))$ or $(x, \phi) \mapsto (x, \phi, \phi, 0)$).

In this optic the diagonal $x \otimes y \mapsto xy$ (resp. the two degeneracy homomorphisms $x \otimes y \otimes z \mapsto xy \otimes z$ or $x \otimes yz$) give rise to the homomorphisms $(x, \phi) \mapsto x$ (resp. $(x, \epsilon, \delta, \eta) \mapsto (x, \epsilon \text{ or } \delta)$). Observe that the intersection of the kernels of the two degeneracy maps is the ideal $\mathrm{alt}_A^2 \Omega^1_{A/Z}$. Now, let P be a locally trivial principal homogeneous space for $\mathcal{G}$ over A. Let $\tilde{\mathcal{G}} = Aut_{\mathcal{G}}(P)$ be the twist of $\mathcal{G}$, that is determined by P. Notice that $\tilde{\mathcal{G}}$ is an affine and flat group scheme over A, simply because it is fpqc locally isomorphic to $\mathcal{G}$.

Definition B.2. (i) *Write P_i for the locally trivial principal homogeneous spaces for $\mathcal{G}$ over $A \oplus \Omega^1_{A/Z}$ which are obtained by pull-back along $\overline{pr}_i$, for $1 \leq i \leq 2$. By a connection (relative to Z) one means an isomorphism*

$$\alpha : P_2 \to P_1$$

whose pull-back along the diagonal $A \oplus \Omega^1_{A/Z} \to A$ agrees with the identity section. We denote the set of all connections on P by $Conn_Z(P/A)$. If P is equipped with a descent to Z, then we call the natural isomorphism from P_2 to P_1 the trivial connection.

(ii) *To any two connections α and α' one may consider the map:*

$$\alpha' \circ \alpha^{-1} : P_1 \to P_1,$$

which is seen to be a $A \oplus \Omega^1_{A/Z}$-valued automorphism of our locally trivial principal homogeneous spaces P, whose pull-back along the diagonal map $(x, \phi) \mapsto x$ agrees with identity section. The canonical element $\beta \in \tilde{\mathcal{G}}(\Omega^1_{A/Z})$, which is thus obtained is called the difference of α' and α and it will be denoted by $\alpha' - \alpha$, and one also writes $\beta + \alpha := \alpha'$.

(iii) *Write α_{ij} for the isomorphism between locally trivial principal homogeneous spaces for $\mathcal{G}$ over $A \oplus \Omega^1_{A/Z} \oplus \Omega^1_{A/Z} \oplus \mathrm{alt}_A^2 \Omega^1_{A/Z}$ which are obtained by pull-back of some $\alpha \in Conn_Z(P/A)$ along the maps $\overline{pr}_{ij}$ for $1 \leq i < j \leq 3$. The identities $\overline{pr}_{13} \circ \overline{pr}_2 = \overline{pr}_{23} \circ \overline{pr}_2$ and $\overline{pr}_{23} \circ \overline{pr}_1 = \overline{pr}_{12} \circ \overline{pr}_2$ allow us to consider the composition*

$$\alpha_{12} \circ \alpha_{23} \circ \alpha_{13}^{-1},$$

which we are allowed to regard as a $A \oplus \Omega^1_{A/Z} \oplus \Omega^1_{A/Z} \oplus \mathrm{alt}_A^2 \Omega^1_{A/Z}$-valued automorphism of our locally trivial principal homogeneous space P, given that $\overline{pr}_{13} \circ \overline{pr}_1 = \overline{pr}_{12} \circ \overline{pr}_1$, furthermore, since its pull-backs under the two degeneracy maps $(x, \epsilon, \delta, \eta) \mapsto (x, \epsilon \text{ or } \delta)$ agree with identity sections, we obtain a canonical element $\mathrm{curv}_Z(\alpha) \in \tilde{\mathcal{G}}(\mathrm{alt}_A^2 \Omega^1_{A/Z})$, which is called the curvature of α.

Remark B.3. The curvature map from $Conn_Z(P/A)$ to $\tilde{\mathcal{G}}(\mathrm{alt}^2_A \Omega^1_{A/Z})$ as well as the summation from $\tilde{\mathcal{G}}(\Omega^1_{A/Z}) \times Conn_Z(P/A)$ to $Conn_Z(P/A)$ are $\tilde{\mathcal{G}}(A)$-equivariant with respect to the natural left actions defined on each of $\tilde{\mathcal{G}}(\Omega^1_{A/Z})$, $\tilde{\mathcal{G}}(\mathrm{alt}^2_A \Omega^1_{A/Z})$ and $Conn_Z(P/A)$.

Let S be a B-scheme, let X be a S-scheme, and let P be a locally trivial principal homogeneous space for $\mathcal{G}$ over X. It is completely clear that the concepts of B.2 generalize immediately to the scheme theoretic setting: The set $Conn_S(P/X)$ of S-connections still forms a $\tilde{\mathcal{G}}(X)$-set, which is empty or a $\tilde{\mathcal{G}}(X)$-equivariant principal homogeneous space under $\tilde{\mathcal{G}}(\Omega^1_{X/S})$, and the curvature map reads $\mathrm{curv}_S : Conn_S(P/X) \to \tilde{\mathcal{G}}(\mathrm{alt}^2_{\mathcal{O}_X} \Omega^1_{X/S})$, and it is $\tilde{\mathcal{G}}(X)$-equivariant too.

B.1. **A remark on the Cartan-Maurer form.** From now on, we assume that X is smooth over S, that $\mathcal{G}$ is smooth over B, and that 2 is invertible in B, and we clarify the relation of the above notions to more classical ones. Recall that the $\mathcal{O}_X$-module of m-forms is denoted by $\Omega^m_{X/S} = \bigwedge^m_{\mathcal{O}_X} \Omega^1_{X/S}$, where $\Omega^1_{X/S}$ is the locally free $\mathcal{O}_X$-module of relative Kähler differentials, and fix another locally free $\mathcal{O}_X$-module $\mathcal{F}$. The sections in the $\mathcal{O}_X$-module $\Omega^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}$ are called $\mathcal{F}$-valued homogeneous differential forms of degree m, while an $\mathcal{F}$-valued differential form is just a formal sum of homogeneous ones. The sheaf $\bigoplus_m \Omega^m_{X/S}$ of differential forms has a interesting $\mathcal{O}_S$-linear endomorphism $d_{X/S}$, obeying the following rules:

(i) The endomorphism $d_{X/S}$ is of degree 1, and the Leibniz rule

$$d_{X/S}(\eta_1 \wedge \eta_2) = d_{X/S}(\eta_1) \wedge \eta_2 + (-1)^{\deg(\eta_1)} \eta_1 \wedge d_{X/S}(\eta_2) \tag{75}$$

holds for homogeneous sections η_1 and η_2.

(ii) $d_{X/S} \circ d_{X/S} = 0$.

(iii) The restriction of $d_{X/S}$ to $\mathcal{O}_X$ (i.e. to 0-forms) agrees with the universal $\mathcal{O}_S$-linear derivation from the $\mathcal{O}_S$-algebra $\mathcal{O}_X$ to the $\mathcal{O}_X$-module of relative Kähler differentials.

A $\mathcal{O}_S$-linear endomorphism ∇, that is defined on the sheaf of $\mathcal{F}$-valued differential forms, is called a connection (relative to S) if it is homogeneous of degree 1 and satisfies the equation (75) (in which η_1 denotes still a homogeneous differential form, while η_2 must now be an $\mathcal{F}$-valued one, [9]). Sections killed by ∇ are called horizontal, and the map $\nabla \circ \nabla$ can be written uniquely as $\eta \mapsto R \wedge \eta$, where $R =: \mathrm{curv}(\nabla)$ is called the curvature of ∇, it is a global section in the $\mathcal{F} \otimes_{\mathcal{O}_X} \check{\mathcal{F}}$-valued 2-forms on X (in fact a horizontal one). If $\mathrm{curv}(\nabla)$ vanishes, then one calls ∇ integrable. Now suppose that $\mathcal{F}$ is of the form $\mathcal{F}_1 \oplus \mathcal{F}_2$ (resp. $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$), where both bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ are equipped with connections ∇_1 and ∇_2. Consequently there exists one and only one connection ∇ on $\mathcal{F}$, which satisfies in addition $\nabla(\eta_1 + \eta_2) = \nabla_1(\eta_1) + \nabla_2(\eta_2)$ (resp. $\nabla(\eta_1 \otimes \eta_2) = \nabla_1(\eta_1) \otimes \eta_2 + \eta_1 \otimes \nabla_2(\eta_2)$) for any two sections η_i of the $\mathcal{O}_X$-modules $\mathcal{F}_i$ ($i \in \{1, 2\}$). If this holds we write $\nabla = \nabla_1 \oplus \nabla_2$ (resp. $\nabla = \nabla_1 \otimes \nabla_2$). Therefore, the class of connections (relative to S) on vector bundles on X, together with the class of horizontal $\mathcal{O}_X$-module homomomorphisms forms an additive rigid $\otimes$-category, for which we write: $Sys_{X/S}$. It is easy to see that $\operatorname{curv}(\nabla_1 \oplus \nabla_2) = \operatorname{curv}(\nabla_1) + \operatorname{curv}(\nabla_2)$ (resp. $\operatorname{curv}(\nabla_1 \otimes \nabla_2) = \operatorname{curv}(\nabla_1) \otimes \operatorname{id}_{\mathcal{F}_2} + \operatorname{id}_{\mathcal{F}_1} \otimes \operatorname{curv}(\nabla_2)$) holds. In particular, direct sums and tensor product of integrable connections are again integrable (and similarly for duals). Therefore, the class of integrable connections (relative to S) on vector bundles on X, together with the class of horizontal $\mathcal{O}_X$-module homomomorphisms forms an additive rigid $\otimes$-category, for which we write: $MiC_{X/S}$.

Let P be a locally trivial principal homogeneous space for $\mathcal{G}$ over X and recall that it gives rise to a twisted fiber functor ω_P, furthermore applying ω_P to the adjoint representation $\operatorname{Ad}^{\mathcal{G}}$ (together with the Lie-bracket from $\operatorname{Ad}^{\mathcal{G}} \otimes_B \operatorname{Ad}^{\mathcal{G}}$ to $\operatorname{Ad}^{\mathcal{G}}$) yields canonically the Lie-$\mathcal{O}_X$-algebra $\tilde{\mathfrak{g}} = \operatorname{Lie} \tilde{\mathcal{G}}$. In the same vein every $\alpha \in Conn_S(P/X)$ gives rise to S-connections $\nabla_\alpha(\rho)$ on each $\omega_P(\rho) \in \mathtt{Ob}_{Vec(X)}$, which are compatible in the sense that

$$Rep_0(\mathcal{G}) \to Sys_{X/S}; \rho \mapsto (\omega_\alpha(\rho), \nabla_\alpha(\rho))$$

sets up a $\otimes$-functor (see e.g. [48, VI.1.2.3.1]). Moreover, applying ω_P to the $Rep_0(\mathcal{G})$-morphism $\rho^{der} : \mathfrak{g} \to \operatorname{End}_B(\omega^{\mathcal{G}}(\rho))$ yields a map, say $\omega_P(\rho^{der})$ from $\tilde{\mathfrak{g}}$ to $\operatorname{End}_B(\omega_P(\rho))$ and the formula $\nabla_{E+\alpha}(\rho) = \omega_P(\rho^{der})E + \nabla_\alpha(\rho)$ holds for all $E \in \Gamma(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \tilde{\mathfrak{g}})$, recall that the set of all connections for $\mathcal{F}$ forms a principal homogeneous space for the group of global sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \check{\mathcal{F}}$-valued 1-forms, unless it is empty. The following formulae are well-known:

- $\nabla_\alpha(\operatorname{Ad}^{\mathcal{G}}) \operatorname{curv}_S(\alpha) = 0$
- $\operatorname{curv}_S(E + \alpha) = \frac{[E,E]}{2} + \nabla_\alpha(\operatorname{Ad}^{\mathcal{G}})E + \operatorname{curv}_S(\alpha)$

Here notice, that the global sections in the sheaf of $\tilde{\mathfrak{g}}$-valued differential forms possesses a Lie-superalgebra structure. At last we want to make these things more explicit for the case of the globally trivial principal homogeneous space $P = \mathcal{G} \times_B X$, in which case $\tilde{\mathcal{G}}$ and $\mathcal{G} \times_B X$ agree canonically, so that $Conn_S(P/X) = \Omega^1_{X/S} \otimes_B \mathfrak{g}$ (by decreeing the neutral element to be the trivial connection), and the $\mathcal{G}(X)$-equivariant map $\operatorname{curv}_S : \Omega^1_{X/S} \otimes_B \mathfrak{g} \to \Omega^2_{X/S} \otimes_B \mathfrak{g}$ can be described by $E \mapsto \frac{[E,E]}{2} + d_{X/S}(E)$. However, the left actions of some $g \in \mathcal{G}(X)$

on $\Omega^2_{X/S} \otimes_B \mathfrak{g}$ and $\Omega^1_{X/S} \otimes_B \mathfrak{g}$ are a little subtle: While the g-action on $\Omega^2_{X/S} \otimes_B \mathfrak{g}$ is simply given by $\mathrm{id}_{\Omega^2_{X/S}} \otimes \mathrm{Ad}^{\mathcal{G}}(g)$, it acts by means of $E \mapsto (\mathrm{id}_{\Omega^1_{X/S}} \otimes \mathrm{Ad}^{\mathcal{G}}(g))E - \eta \circ g$ on $\Omega^1_{X/S} \otimes_B \mathfrak{g}$ where $\eta \in \Omega^1_{\mathcal{G}/B} \otimes_B \mathfrak{g}$ is the canonical right-invariant Cartan-Maurer form, and $\eta \circ g$ denotes the pull-back of the 1-form η by means of $g : X \to \mathcal{G}$. We remark that the Cartan-Maurer form of $\mathrm{GL}(n)$ is $(dA)A^{-1}$.

Appendix C. Gradations and Filtrations

By a gradation of type T of some torsion-free finitely generated B-module $\mathcal{V}$, we mean a direct sum $\mathcal{V} = \bigoplus_{l \in T} \mathcal{V}_l$ (N.B. we do not require that $\mathcal{V}_l \neq 0$ for $l \in T$). Now let us write $\mathrm{GL}(\mathcal{V}/B)$ (resp. $\underline{\mathcal{V}}$) for the multiplicative (resp. additive) group functor, of which the R-valued points are the units in $\mathrm{End}_B(\mathcal{V}) \otimes_B R$ (resp. $\mathcal{V} \otimes_B R$). It is immediate that the group homomorphisms from the multiplicative group $\mathbb{G}_{m,B}$ to $\mathrm{GL}(\mathcal{V}/B)$, i.e. the cocharacters, are precisely given by their respective weight-gradations of $\mathcal{V}$, which are of type $\mathbb{Z}$, as the group $\mathbb{Z}$-scheme $\mathbb{G}_m$ is given by the spectrum of $\mathbb{Z}[x^{\pm 1}] = \bigoplus_{l \in \mathbb{Z}} \mathbb{Z}x^l$. By slight abuse of language we will say that some cocharacter ν was of type $T \subset \mathbb{Z}$ if and only if its weight-gradation would have that property, and we will say that ν is a homothety if and only if it is a cocharacter of type $\{l\}$ for some l. The homothety of type $\{1\}$ will be denoted by $C_{\mathcal{V}}$. Let us write $\mathbb{A}^1 := \operatorname{Spec} \mathbb{Z}[x]$ for the affine line, and notice in particular, that a cocharacter is of type $\mathbb{N}_0$ if and only if it extends to a homomorphism

$$\mathbb{A}^1_B \to \underline{\mathcal{V} \otimes_B \check{\mathcal{V}}},$$

respecting the natural monoidal structures on either side. A cocharacter thus extended to $\mathbb{A}^1$ will be referred to as an effective one. Now we need to introduce a couple of operations for a cocharacter ν of the group $\mathrm{GL}(\mathcal{V}/B)$, say defined by a weight gradation $\mathcal{V} = \bigoplus_{l \in \mathbb{Z}} \mathcal{V}_l$:
Foremost, there exists a canonical isomorphism

$$\mathrm{GL}(\mathcal{V}/B) \to \mathrm{GL}(\check{\mathcal{V}}/B);\ g \mapsto \check{g},$$

where $\check{\mathcal{V}} = \mathrm{Hom}_B(\mathcal{V}, B)$. It follows that there is a canonically determined dual cocharacter $\check{\nu} : \mathbb{G}_{m,B} \to \mathrm{GL}(\check{\mathcal{V}}/B)$, and indeed one finds easily that the $\check{\nu}$-weight of $\check{\mathcal{V}}_l$ (regarded as a subspace of $\check{\mathcal{V}}$) is equal to $-l$. Let $H_0 : \mathbb{Z} \to \mathbb{Z}$ be the Heaviside step function

$$H_0(l) := \begin{cases} 0 & l \leq 0 \\ 1 & l \geq 1 \end{cases},$$

consider the B-modules $\tilde{\mathcal{V}}_l := \bigoplus_{H_0(m)=l} \mathcal{V}_m$, and let $H_0(\nu)$ be the cocharacter (of type $\{0,1\}$) which is defined by the weight-gradation

$\mathcal{V} = \tilde{\mathcal{V}}_0 \oplus \tilde{\mathcal{V}}_1$, i.e. $H_0(\nu)$ acts with weight $H_0(l)$ on each direct summand $\mathcal{V}_l$. Observe that the dual of $H_0(\nu C_{\mathcal{V}})$ is equal to $\frac{H_0(\check{\nu})}{C_{\check{\mathcal{V}}}}$, because $H_0(1-l) = 1 - H_0(l)$ holds for every integer l, the dual of $C_{\mathcal{V}}$ is $C_{\check{\mathcal{V}}}^{-1}$. By a filtration on a B-algebra R one means a family of ideals $\{\mathfrak{a}_n\}_{n\in\mathbb{Z}}$ with

- $\mathfrak{a}_n\mathfrak{a}_m \subset \mathfrak{a}_{n+m}$
- $R = \mathfrak{a}_0 = \mathfrak{a}_{-1}$

Equivalently, a filtration is given by a function $v : R \to \mathbb{N}_0$ with

- $v(a+b) \geq \min\{v(a), v(b)\}$
- $v(ab) \geq v(a) + v(b)$

by means of $\mathfrak{a}_n = \{x \in R | v(x) \geq n\}$. We call v the characteristic valuation of the filtration. For an ideal $\mathfrak{a} \subset R$ we let $v_{\mathfrak{a}}(x) = \begin{cases} \infty & x \in \mathfrak{a} \\ 0 & x \notin \mathfrak{a} \end{cases}$ be the characteristic function of the ideal (corresponding to the filtration $\dots \supset R \supset \mathfrak{a} \supset \mathfrak{a}\dots$). Now, if m is a fixed integer and $\mathcal{V} = \bigoplus_{l\in\mathbb{Z}} \mathcal{V}_l$ is the gradation defined by a cocharacter $\nu : \mathbb{G}_{m,B} \to \mathrm{GL}(\mathcal{V}/B)$, then we will denote the R-module $\bigoplus_{l\in\mathbb{Z}} \mathfrak{a}_{m-l} \otimes_B \mathcal{V}_l$ by: $\hat{\mathrm{Fil}}_\nu^m(\mathcal{V}, R, v)$ (while $\mathrm{Fil}_\nu^m(\mathcal{V})$ just means $\bigoplus_{l\geq m} \mathcal{V}_l$).

C.1. **Fiber Functors.** If X is a scheme, we write $\mathit{Vec}(X)$ for the additive rigid $\otimes$-category of vector bundles on X, i.e. locally free $\mathcal{O}_X$-modules which are locally of finite type. We will also utilize the B-linear, additive, rigid $\otimes$-category $\mathit{Rep}_0(\mathcal{G})$ of 'representations', i.e. $\mathcal{G} - B$-modules which are finitely generated and projective over B, this has been introduced and studied in [48, II.4.1.2.1], along with various functors defined thereon: There is a natural forgetful faithful fiber functor

$$\omega^{\mathcal{G}} : \mathit{Rep}_0(\mathcal{G}) \to \mathit{Vec}(\mathrm{Spec}\, B),$$

which takes a $\mathcal{G} - B$-module to its underlying B-module. Furthermore, for every locally trivial principal homogeneous space P for $\mathcal{G}$ over some B-scheme X, there also exists a certain twisted fiber functor

$$\omega_P : \mathit{Rep}_0(\mathcal{G}) \to \mathit{Vec}(X).$$

This takes a representation $(\mathcal{V}, \rho)$ to the locally free $\mathcal{O}_X$-module $P \times^{\mathcal{G}} \mathcal{V}$, which is obtained by the usual extension of structure group (the map ρ takes P naturally to a locally trivial principal homogeneous space for $\mathrm{GL}(\mathcal{V}/B)$ over X, and in turn this defines a vector bundle of constant rank equal to $\mathrm{rank}_B\, \mathcal{V}$, cf. [48, II.3.2.3.4]). Notice that, both of these functors are exact, and respect the rigid $\otimes$-structure defined on their source and target. Moreover, the group $\mathcal{G}/B$ (resp. the locally trivial principal homogeneous space P/X) can be recovered from $\omega^{\mathcal{G}}$ (resp. ω_P) by means of the natural isomorphisms $\mathcal{G}(R) \cong \mathrm{Aut}^{\otimes}(R \otimes_B \omega^{\mathcal{G}})$ (resp. $P(R) \cong \mathrm{Iso}^{\otimes}(R \otimes_B \omega^{\mathcal{G}}, R \otimes_{\mathcal{O}_X} \omega_P)$), where $\mathrm{Aut}^{\otimes}$ (resp. $\mathrm{Iso}^{\otimes}$) denotes the $\otimes$-preserving subset of functorial isomorphisms from the $\mathit{Vec}(\mathrm{Spec}\, R)$-valued $\otimes$-functor $R \otimes_B \omega^{\mathcal{G}}$ to itself (resp. to $R \otimes_{\mathcal{O}_X} \omega_P$). Now let us assume that $\mathcal{G}$ is smooth and that M is a B-module. Specializing the above to the B-algebra $B \oplus M$ leads to an interesting characterization of the Lie-algebra $\mathfrak{g}$: Let us say that a B-linear transformation d from $\omega^{\mathcal{G}}$ to $M \otimes_B \omega^{\mathcal{G}}$ is a M-valued derivation of $\omega^{\mathcal{G}}$ if the identity

$$d_{\rho_1 \otimes \rho_2} = d_{\rho_1} \otimes_B \mathrm{id}_{\omega^{\mathcal{G}}(\rho_2)} + \mathrm{id}_{\omega^{\mathcal{G}}(\rho_1)} \otimes_B d_{\rho_2}$$

is valid for any two objects ρ_1 and ρ_2 of $\mathit{Rep}_0(\mathcal{G})$ (N.B. both sides are maps from $\omega^{\mathcal{G}}(\rho_1 \otimes \rho_2)$ to $M \otimes_B \omega^{\mathcal{G}}(\rho_1 \otimes \rho_2)$, because $\omega^{\mathcal{G}}$ is a fiber functor). If $\mathrm{Der}_M^{\otimes}(\omega^{\mathcal{G}})$ denotes the B-module of M-valued derivations of $\omega^{\mathcal{G}}$, then we have a natural isomorphism $M \otimes_B \mathfrak{g} \cong \mathrm{Der}_M^{\otimes}(\omega^{\mathcal{G}})$.
Notice that every cocharacter $\mu : \mathbb{G}_{m,B} \to \mathcal{G}$ defines a gradation of type $\mathbb{Z}$ on $\omega^{\mathcal{G}}$, [48, Corollaire IV.1.2.2.2], i.e. compatible gradations $\omega^{\mathcal{G}}(\rho) = \bigoplus_{l \in \mathbb{Z}} \omega_\mu^{\mathcal{G}}(l, \rho)$ as ρ runs through $\mathit{Rep}_0(\mathcal{G})$. Following [48, IV.2.1.3], we consider the associated closed subgroup functors

$$U_\mu^m(R) = \{g \in \mathcal{G}(R) | \forall \rho, l_0 :$$
$$(\rho(g) - 1)(\omega_\mu^{\mathcal{G}}(l_0, \rho)) \subset R \otimes_{W(\mathbb{F}_{p^f})} \bigoplus_{l=m+l_0}^{\infty} \omega_\mu^{\mathcal{G}}(l, \rho)\}.$$

These are representable by (and will be identified with) smooth subschemes, by [48, Proposition IV.2.1.4.1]. Since U_μ^0 preserves the filtration $\{\bigoplus_{l=l_0}^{\infty} \omega_\mu^{\mathcal{G}}(l, \rho) | l_0 \in \mathbb{Z}\}$ it acts on each $\omega_\mu^{\mathcal{G}}(l_0, \rho)$, and it is useful to have notations for

$$d_\mu(l_0, \rho) := \mathrm{rank}_B\, \omega_\mu^{\mathcal{G}}(l_0, \rho)$$

and for the character

$$\chi_\mu(l_0, \rho) : U_\mu^0 \to \mathbb{G}_{m,B}$$

corresponding to the representation of U_μ^0 on $\bigwedge_B^{d_\mu(l_0,\rho)} \omega_\mu^{\mathcal{G}}(l_0, \rho)$, note that $\chi_\mu(l_0, \rho)|_{U_\mu^1}$ is trivial. It also follows that there is a canonical homomorphism

$$L_\mu : \mathbb{A}^1_{W(\mathbb{F}_{p^f})} \to \mathit{End}(U_\mu^0),$$

of multiplicative monoids which extends the interior $\mathbb{G}_{m,B}$-action $\mathrm{Int}^{U_\mu^0} \circ \mu$ on U_μ^0. Now, if m is a fixed integer, and if the family $\mathfrak{a}_n \subset R$ is the filtration corresponding to a valuation $v : R \to \mathbb{N}_0$, then we need to introduce a certain subgroup of $\mathcal{G}(R)$ by means of:

$$\hat{U}_\mu^m(R, v) = \{g \in \mathcal{G}(R) | \forall \rho, l_0 :$$
$$(\rho(g) - 1)(\omega_\mu^{\mathcal{G}}(l_0, \rho)) \subset \bigoplus_{l \in \mathbb{Z}} \mathfrak{a}_{m+l_0-l} \otimes_{W(\mathbb{F}_{p^f})} \omega_\mu^{\mathcal{G}}(l, \rho)\},$$

which is going to serve as a "topological" analog to the groups U_μ^m, notice that $R = \mathfrak{a}_0 = \mathfrak{a}_{-1} = \dots$, so that $U_\mu^m(R)$ is indeed contained in $\hat{U}_\mu^m(R, v)$. Note that $\mathcal{G}(Q) \cap \hat{U}_\mu^m(R, v) = \hat{U}_\mu^m(Q, v|_Q)$ holds for B-subalgebras $Q \subset R$, and that $\hat{U}_\mu^m(R, v_\mathfrak{a})$ is the inverse image of $U_\mu^m(R/\mathfrak{a})$ in $\mathcal{G}(R)$.

Lemma C.1. *Let $i : \mathcal{G} \to \mathcal{H}$ be a closed immersion of affine and flat B-groups. Then $\hat{U}_\mu^m(R, v) = \{g \in \mathcal{G}(R) | i(g) \in \hat{U}_{i \circ \mu}^m(R, v)\}$ holds.*

Definition C.2. *Let $\mu : \mathbb{G}_m \to \mathcal{G}$ be a cocharacter, and consider the weight decomposition $\bigoplus_{h \in \mathbb{Z}} \mathfrak{g}_h = \mathfrak{g} = \operatorname{Lie} \mathcal{G}$.*

- *Let h be a positive integer. By an additive 1-parameter subgroup of μ-weight h, we mean a homomorphism $\epsilon : \mathbb{G}_a \to \mathcal{G}$, of which the derived map $\operatorname{Lie} \epsilon$ sends $\mathbb{G}_a = \operatorname{Lie} \mathbb{G}_a$ into $\mathfrak{g}_h$.*
- *A d-tuple of additive 1-parameter subgroups $\epsilon_1, \dots, \epsilon_d : \mathbb{G}_a \to \mathcal{G}$ of μ-weights $0 < h_1 \leq \dots \leq h_{d-1} \leq h_d$ is called a μ-basis if the map $\mathbb{G}_a^d \to \operatorname{Lie} U_\mu^1; (x_1, \dots, x_d) \mapsto \sum_{i=1}^d (\operatorname{Lie} \epsilon_i)(x_i)$ is an isomorphism.*
- *If there exist a μ-basis, then μ will be called to be of triangular type.*

Theorem C.3. *Assume that the d-tuple of additive 1-parameter subgroups $\epsilon_1, \dots, \epsilon_d : \mathbb{G}_a \to \mathcal{G}$ forms a μ-basis for a B-rational cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$, in the sense of definition C.2. Then, the map*

$$\mathbb{G}_a^d \to U_\mu^1; (x_1, \dots, x_d) \mapsto \epsilon_1(x_1) \cdot \dots \cdot \epsilon_d(x_d)$$

is an isomorphism.

Proof. Let $0 < h_1 \leq \dots \leq h_{d-1} \leq h_d$ be the μ-weights of $\epsilon_1, \dots, \epsilon_d$. It is clear that ϵ_i factors through $U_\mu^{h_i}$, and that it is enough to show that each of the maps:

$$U_\mu^{m+1} \times \prod_{h_i = m} \mathbb{G}_a \to U_\mu^m; (x_0, \dots, x_i, \dots) \mapsto x_0 \prod_{h_i = m} \epsilon_i(x_i)$$

is an isomorphism (where the order of multiplication is according to increasing i). However, this follows from the commutativity of the diagram

$$\begin{array}{ccc} \prod_{h_i=m} \mathbb{G}_a & \xrightarrow{\prod_{h_i=m} \mathrm{Lie}(\epsilon_i)} & \mathfrak{g}_m \\ {\scriptstyle \prod_{h_i=m} \epsilon_i}\downarrow & & \uparrow{\scriptstyle \log} \\ U_\mu^m & \longrightarrow & U_\mu^{m+1} \backslash U_\mu^m \end{array}$$

and the fact, that its upward arrow is a monomorphism. □

Notice that the group multiplication identifies $U^0_{\mu^{-1}} \times_B U^1_\mu$ with an open subscheme of $\mathcal{G}$ (use the implication "b)⇒a)" in [19, Théorème (17.9.1)] followed by the "d)⇒b)"-one in the result [19, Théorème (17.11.1)]), we have the following consequence:

Theorem C.4. *Let v be the characteristic valuation of the filtration $\mathfrak{a}_n$ on a B-algebra R, where $\mathfrak{a}_1 \subset \mathrm{rad}(R)$. Let $\mu : \mathbb{G}_{m,B} \to \mathcal{G}$ be a cocharacter of triangular type, and choose a μ-basis of additive* 1-*parameter subgroups $\epsilon_1, \dots, \epsilon_d : \mathbb{G}_{a,B} \to \mathcal{G}$, as in definition C.2. Then*

$$\hat{U}^0_{\mu^{-1}}(R, v) = U^0_{\mu^{-1}}(R) \prod_{i=1}^{d} \epsilon_i(\mathfrak{a}_{h_i}).$$

Proof. By the previous observation we know that $\hat{U}^0_{\mu^{-1}}(R, v)$ is contained in (the R-valued points of) the image of $U^0_{\mu^{-1}} \times_B U^1_\mu$ in $\mathcal{G}$. In view of theorem C.3 it is enough to show the inclusions:

$$\hat{U}^0_{\mu^{-1}}(R, v) \cap \prod_{i=j}^{d} \epsilon_i(R) \subset \prod_{i=j}^{d} \epsilon_i(\mathfrak{a}_{h_i}),$$

which you can handle by downward induction on j. □

In this paper we usually consider groups $\mathcal{G}$ which are defined over $B = W(\mathbb{F}_{p^f})$ (or subrings thereof). In this case there exist μ-bases for a $W(\mathbb{F}_{p^f})$-rational cocharacter μ, provided that one of the following two properties holds:

- $\mathcal{G}$ is reductive
- $\dots = \mathfrak{g}_{p+1} = \mathfrak{g}_p = 0$

Appendix D. Existence of poly-unitary Shimura data

Let K be an algebraically closed field of characteristic 0, and let $\mathfrak{g}$ be a semi-simple Lie-algebra over K. We call a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}_K(U)$ asymmetrical if for every non-inner automorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ one can find some sub-representation σ of ρ which is nonequivalent to $\sigma \circ \alpha$. The raw material for sample poly-unitary Shimura data is supplied by:

Proposition D.1. *Let $\rho_0 : G_0 \to \mathrm{GL}(U_0/K)$ be a linear representation of the semi-simple K-group G_0 and write ρ_0' for the direct sum of ρ_0 and* Ad *on the representation space $U_0' = U_0 \oplus \mathfrak{g}_0$. Assume that ρ_0' is faithful and has an asymmetrical derived representation ${\rho_0'}^{der} : \mathfrak{g}_0 \to \mathrm{End}_K(U_0')$. Write $U' := U \oplus \mathfrak{g}_0$, where U is the trivial one-dimensional representation of G_0, and fix an element $e \in U$. Consider the sub-K-algebra $K[a] \subset \mathrm{End}_{G_0}(U \otimes_K U' \otimes_K U_0')$ that is generated by the single endomorphism*

$$x_1 \otimes (x + x') \otimes (x_0 + x_0') \overset{a}{\mapsto} x_1 \otimes (x_0' \otimes x' + e \otimes {\rho_0'}^{der}(x')(x_0 + x_0')),$$

where $x_0 \in U_0$, $x, x_1 \in U$, and $x_0', x' \in \mathfrak{g}_0$. Write G^0 for the stabilizer in $\mathrm{GL}(U/K) \times_K \mathrm{GL}(U'/K) \times_K \mathrm{GL}(U_0'/K)$ of $\mathrm{End}_{G_0}(U')$, $\mathrm{End}_{G_0}(U_0')$, and $K[a]$. Then one has:

$$G^0 = G_0\,\mathrm{Z}^{G^0}$$

More specifically, write Z_0' for the center of $\mathrm{End}_{G_0}(U_0')^\times$, and write $Z' \subset Z_0'$ for the sub-torus consisting of elements whose action on $\mathfrak{g}_0$ is a scalar. Then Z' is naturally contained in Z^{G^0}, and indeed one has $\mathbb{G}_{m,K}^2 \times_K Z' = \mathrm{Z}^{G^0}$, where the two copies of $\mathbb{G}_m$ act as scalars on U and U'. The rank of Z^{G^0} is equal to

$$3 + \mathrm{Card}\{\textit{isotypic components of } U_0'\} - \mathrm{Card}\{\textit{simple factors of } \mathfrak{g}_0\},$$

and it is connected.

Proof. The proposition and its proof are both similar to [5, Lemma 7.3]. Fix an element of G^0, according to the presence of $\mathrm{End}_{G_0}(U_0')$ and $\mathrm{End}_{G_0}(U')$, we can write $\begin{pmatrix} g_0 & 0 \\ 0 & g_0' \end{pmatrix}$, $\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}$, and g_1, for the induced maps on U_0', U', and U. Notice that g_0' and g' are proportional, and that:

$$\frac{g_0}{g}\rho^{der}(x')x_0 = \rho^{der}(\frac{g'}{g}x')(\frac{g_0}{g}x_0)$$
$$\frac{g_0'}{g}[x', x_0'] = [\frac{g'}{g}x', \frac{g_0'}{g}x_0']$$

according to the presence of a. This means that $\alpha := \frac{g'}{g}$ is an automorphism of the Lie algebra $\mathfrak{g}_0$, and that $\beta := \frac{1}{g}\begin{pmatrix} g_0 & 0 \\ 0 & g_0' \end{pmatrix}$ intertwines ${\rho_0'}^{der}$ and ${\rho_0'}^{der} \circ \alpha$. Using the asymmetry of ${\rho_0'}^{der}$, and again the presence of $\mathrm{End}_{G_0}(U_0')$, we see that α must be an inner automorphism, so that it is induced from an element in $G_0(K)$. Upon an adjustment we are allowed to assume $\alpha = 1$, so that g' is equal to the multiplication by the scalar g. Consequently β lies in the center of $\mathrm{End}_{G_0}(U_0')^\times$. Upon a further adjustment we are allowed to assume that each of g_0, g_0' and g' is equal to the multiplication by the scalar g. The remaining degrees of freedom are g, $g_1 \in K^\times$. □

In the special case of a simply connected algebraic group G_0, the condition on the asymmetry of the derived representation ${\rho_0'}^{der}$ may be removed from the assumptions of the previous proposition, because this is an automatic consequence of the faithfulness of ρ_0'. In fact, if G_0 is simply connected and semisimple without simple factors of type E_8, F_4 or G_2, then the faithfulness of ρ_0 implies already the asymmetry of ρ_0^{der} (Sketch: every automorphism of $\mathfrak{g}_0$ comes from an automorphism of G_0, which is inner if and only if it restricts to the identity on the center).
In the special case of an algebraic group of adjoint type over an algebraically closed field, one can easily give a specific example of a representation ρ_0 such that ${\rho_0'}^{der}$ is faithful and asymmetrical: Let $G_0 = \prod_{i=1}^d G_i$ be the decomposition into simple factors. Let us write $\rho_i : G_i \to \mathrm{GL}(U_i/K)$ for the following asymmetrical representations:

- If G_i is of type B_l, C_l, E_8, E_7, F_4, G_2 or A_1 then $U_i := 0$,
- If $G_i \cong \mathrm{PGL}(n)$, where $n \geq 3$, then $U_i := (\mathrm{sym}_K^2 \mathrm{std}) \otimes_K \bigwedge_K^2 \check{\mathrm{std}}$,
- If $G_i \cong \mathrm{SO}(2n)/\{\pm 1\}$, then $U_i := \begin{cases} \bigwedge_K^n \mathrm{std} & n \equiv 0 \pmod 2 \\ \mathrm{sym}_K^2(\bigwedge_K^n \mathrm{std}) & n \equiv 1 \pmod 2 \end{cases}$,
- If G_i is of type E_6, then $U_i := \mathrm{sym}_K^3(\mathbb{J})$,

and let us write $\rho_0 : G_0 \to \mathrm{GL}(U_0/K)$ for the natural representation on the (exterior) direct sum $U_0 := \bigoplus_{i=1}^d U_i$ (here std means standard representation and $\mathbb{J}$ is the 27-dimensional exceptional Jordan algebra). In general U_0 need not be faithful nor asymmetrical, but it is easy to see that $U_0' = U_0 \oplus \mathfrak{g}_0$ does indeed possess both of these properties. In the sequel this is our prime example. We also need the following $\mathbb{N}_0$-valued function on the set of isomorphism classes of connected groups of adjoint type, which we call the radius:

- $r(G_1 \times_K \cdots \times_K G_d) := \max\{r(G_1), \ldots, r(G_d)\}$ if $G_1, \ldots, G_d$ are simple.
- $r(\mathrm{PGL}(n)) := 2$ for all $n \geq 3$
- $r(\mathrm{SO}(2n)/\{\pm 1\}) := \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 2 \\ n & n \equiv 1 \pmod 2 \end{cases}$ for all $n \geq 4$
- If G_0 is of type E_6, then $r(G_0) = 4$.
- If G_0 is of type B_l, C_l, E_7, or A_1 then $r(G_0) := 1$.
- If G_0 is of type E_8, F_4, or G_2 then $r(G_0) := 0$.

The number $r(G_0)$ has the following significance: If U_0' is the representation above, then no minuscule cocharacter of G_0 possesses a weight on U_0' which is strictly greater than $r(G_0)$ or strictly smaller than $-r(G_0)$.

D.1. **Preliminary reduction steps.** Just as in subsection 8.2 we let (G_0, h) be a Hodge datum with coefficients in a totally real field L^+, we let L be totally imaginary extension of L^+, we let $\mathfrak{r}$ be an unramified prime of L lying over some rational odd prime p. We write $R \subset \mathbb{C}$ for the splitting field of L and we write $\vartheta \in \mathrm{Gal}(R/\mathbb{Q})$ for the geometric Frobenius induced by $\mathfrak{r}$. The next two subsections address the existence of L-metaunitary Shimura data:

Lemma D.2. *If $(\mathbf{d}^+, \{(V_i, \Psi_i, \rho_i, \mathbf{j}_i, l_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$ is a L-metaunitary Shimura datum for (G_0, h), then there exist*

- *elements $n_i \in (\mathbb{Z}_{(p)} \otimes \mathcal{O}_L)^\times$ for each $i \in \Lambda$, and*
- *a central cocharacter $\gamma : \mathbb{S}^1 \to (\mathrm{Res}_{L^+/\mathbb{Q}} G_0)_\mathbb{R}$*
- *a family of functions $c_i : L_{an} \to \mathbb{Z}$*

such that $(\mathbf{d}^+, \{(V_i, n_i\Psi_i, \rho_i, \mathbf{j}_i - c_i, l_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$ is a normalized, L-metaunitary Shimura datum for $(G_0, \frac{h}{\gamma\circ\mathbb{N}})$.

Proof. For each $i \in \Lambda$ we write $a_{i,\iota} \leq b_{i,\iota} = 1 - a_{i,\iota\circ*}$ for the family of integers corresponding to the ϑ-gauged L-unitary representation $(V_i, \rho_i, \Psi_i, \mathbf{j}_i)$ by means of remark 8.4. Our proof begins with a choice of a disjoint union $L_{an} = S \cup \{\iota | \iota \circ * \in S\}$ where we may require that each split cycle is entirely contained in one of S or $\{\iota|\iota \circ * \in S\}$. Observe that families satisfying (N1) are already given by their values on S, since one can rewrite this as $a_{i,\iota} + a_{i,\iota\circ*} = 1 - \mathrm{Card}(\mathbf{d}^{-1}(\{\iota\}))$. So let us define another family by decreeing

$$a'_\iota := \begin{cases} 0 & \mathbf{d}(\iota) \in S \\ 1 & \mathbf{d}(\iota) \notin S \end{cases}$$

for all $\iota \in S$, which happens to satisfy not only (N1) but also (N2) and (N3). Let c_i be the function $\iota \mapsto c_{i,\iota} := a_{i,\iota} - a'_\iota$, and let $n_i \in (\mathbb{Z}_{(p)} \otimes \mathcal{O}_L)^\times$ satisfy $(-1)^{c_{i,\iota}}\iota(n_i) > 0$. Since $0 = c_{i,\iota} + c_{i,\iota\circ*}$ holds, it is easy to see that there exists a cocharacter γ such that the weights of $\rho_i \circ \gamma$ are given by the family c_i (N.B.: $\underbrace{L^1 \times \cdots \times L^1}_{\Lambda}$ is canonically contained in the center of $\mathrm{Res}_{L^+/\mathbb{Q}} G_0$ and the norm maps $\mathbb{S}$ canonically onto $\mathbb{S}^1$). ☐

In the sequel we say that $(\{(V_i, \Psi_i, \rho_i)\}_{i\in\Lambda}, \{(R_\pi, *, \iota_\pi)\}_{\pi\in\Pi})$ is a L-metaunitary collection for the Hodge datum (G_0, h) if and only if:

- Each triple (V_i, Ψ_i, ρ_i) satisfies the two conditions (U1) and (U2), and

- $(\{(V_i,\Psi_i,\rho_i)\}_{i\in\Lambda},\{(R_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$ is a L-metaunitary collection for the L^+-group $(\mathbb{G}_{m,L^+}^\Lambda\times_{L^+}G_0)/\{\pm1\}^\Lambda$,

and without further notice we write $h_i:=\varrho_{i,\mathbb{R}}\circ h$ for the skew-Hermitian Hodge structure of weight -1 on (V_i,Ψ_i) which is induced by the first of the two conditions. We have the following further criterion:

Lemma D.3. *Fix (G_0,h) together with an L-metaunitary collection $(\{(V_i,\Psi_i,\rho_i)\}_{i\in\Lambda},\{(R_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$. Furthermore, we assume $1\equiv\mathrm{Card}(\Lambda)\pmod 2$ and $\Pi=\{\Lambda\}\cup\{\{i\}|i\in\Lambda\}$. We let $[a_{i,\iota},b_{i,\iota}]$ be the smallest intervals, such that the Hodge decomposition of (V_i,Ψ_i,h_i) is of type*

$$\{(-b_{i,\iota},b_{i,\iota}-1),\dots,(-a_{i,\iota},a_{i,\iota}-1)\},$$

and we let $w_{i,\iota}:=b_{i,\iota}-a_{i,\iota}$ be their lengths. Then there exists a L-metaunitary Shimura datum for (G_0,h) of the form

$$(\mathbf{d}^+,\{(V_i,\Psi_i,\rho_i,\mathbf{j}_i,l_i)\}_{i\in\Lambda},\{(R_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$$

if and only if

$$\sum_{i\in\Lambda}\lim_{N\to\infty}\frac{\sum_{k=1}^N w_{i,\vartheta^k\circ\iota}}{N}<1,\tag{76}$$

holds for every $\iota\in L_{an}$.

Proof. Observe that the above convention on the families $a_{i,\iota}\le b_{i,\iota}=1-a_{i,\iota\circ*}$ follows remark 8.3. This is not compatible and almost opposite to the convention of remark 8.4, but it is more useful in the non-normalised situation. In view of the lemma 8.2 (with the weakened version of (i) therein) all we have to do is find $(\mathbf{d}^+,\dots,\mathbf{j}_i,\dots)$ such that $\mathbf{d}(\iota)=\vartheta^{-\mathbf{d}^+(\iota)}\circ\iota$ satisfies the following:

(i) For each $i\in\Lambda$ and $l\in[a_{i,\iota},b_{i,\iota}-1]$, there exists a unique $\kappa\in L_{an}$, with $\mathbf{d}(\kappa)=\iota$ and $\mathbf{j}_i(\kappa)=l$.
(ii) For every $\kappa\in L_{an}$, the cardinality of both sets $\{i\in\Lambda|\mathbf{j}_i(\kappa)<a_{i,\mathbf{d}(\kappa)}\}$, and $\{i\in\pi|\mathbf{j}_i(\kappa)\ge b_{i,\mathbf{d}(\kappa)}\}$ is at least $\frac{\mathrm{Card}(\Lambda)-1}{2}$.
(iii) Each cycle of L_{an} contains at least one element κ which satisfies $\mathbf{j}_i(\kappa)\notin[a_{i,\mathbf{d}(\kappa)},b_{i,\mathbf{d}(\kappa)}-1]$ for all $i\in\Lambda$.

Note that (i), (ii), and (iii) might not automatically imply the condition (N1), but that we do have $b_\iota-a_\iota\ge\mathrm{Card}(\mathbf{d}^{-1}(\{\iota\}))$ in any case, and in fact the condition $(N1)$ can be enforced by a slight adjustment of the function $\mathbf{j}$ and the intervals $[a_{i,\iota},b_{i,\iota}]$ in the sense of remark 8.3. It does no harm to assume $\Lambda=\{1,\dots,n\}$. It is very easy to see that the existence of a single function $\mathbf{j}_i(\kappa)=-\mathbf{j}_i(\kappa\circ*)$, satisfying (i) alone is equivalent to $\lim_{N\to\infty}\frac{\sum_{k=1}^N w_{i,\vartheta^k\circ\iota}}{N}\le1$ for each ι, in particular the inequality (76) implies the existence of ϑ-gauge $\mathbf{j}$ for the type $\{(-b_\iota,b_\iota-$

$1),\dots,(-a_\iota,a_\iota-1)\}$ where $a_\iota := \frac{1-n}{2}+\sum_{i=1}^n a_{i,\iota}$ and $b_\iota := \frac{1-n}{2}+\sum_{i=1}^n b_{i,\iota}$. Now we have to construct a multicompact family of gauges $\{\mathbf{j}_i\}_{1\le i\le n}$ as in (i),(ii), and (iii). We begin with a choice of disjoint union:

$$L_{an} = S\cup\{\iota|\iota\circ *\in S\}.$$

Consider some κ with $\iota := \mathbf{d}(\kappa)\in S$. Thinking of $[a_\iota,b_\iota-1]$ as the concatenation of suitable translates of the intervals $[a_{1,\iota},b_{1,\iota}-1],\dots,[a_{n,\iota},b_{n,\iota}-1]$ leads to unique integers l and m with the properties:

$$\begin{aligned}&\frac{n-1}{2}+\mathbf{j}(\kappa)=\sum_{i<m} b_{i,\iota}+l+\sum_{i>m} a_{i,\iota}\\ &m\in[0,n+1]\\ &l\in\begin{cases}]-\infty,-1] & m=0\\ [a_{m,\iota},b_{m,\iota}-1] & m\in[1,n]\\ [0,\infty[& m=n+1\end{cases}.\end{aligned}$$

So that we may define, for every $i\in\Lambda$:

$$-\mathbf{j}_i(\kappa\circ *)=\mathbf{j}_i(\kappa):=\begin{cases} a_{i,\iota}-1 & (-1)^i(i-m)<0\\ l & i=m\\ b_{i,\iota} & (-1)^i(i-m)>0\end{cases}.$$

The further details are left to the reader, as is the other implication, which we do not need. □

We write $Q_n := \mathrm{Mat}(n\times 1,\mathbb{Q})$ for the standard n-dimensional $\mathbb{Q}$-vector space endowed with the standard euclidean pairing $E_n(x,y) := x^t y$.

Lemma D.4. *Let $(\{(V_i,\Psi_i,\rho_i)\}_{i\in\Lambda},\{(R_\pi,*,\iota_\pi)\}_{\pi\in\Pi})$ be a L-metaunitary collection for (G_0,h). Assume that G_0 and L are unramified at all divisors of p.*

(i) *The L-metaunitary collection*

$$(\{(Q_2\otimes V_i,E_2\otimes\Psi_i,\rho_i^{\oplus 2})\}_{i\in\Lambda},\{(\mathrm{Mat}(2^{\mathrm{Card}(\pi)},R_\pi),*,\iota_\pi)\}_{\pi\in\Pi})$$

has a T-enrichment for every $\mathbb{Q}_p\&\mathbb{R}$-elliptic, $\mathbb{Q}_p$-unramified maximal $\mathbb{Q}$-torus T, in the sense of subsubsection 8.4.3.

(ii) *The L-metaunitary collection*

$$(\{(Q_8\otimes V_i,E_8\otimes\Psi_i,\rho_i^{\oplus 8})\}_{i\in\Lambda},\{(\mathrm{Mat}(8^{\mathrm{Card}(\pi)},R_\pi),*,\iota_\pi)\}_{\pi\in\Pi})$$

is unramified in the sense of subsection 8.2.

Proof. Part (i) follows because the unramifiedness of G_0 implies that $\mathbb{Q}_p \otimes \mathrm{End}(\rho_i)$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$, while the Rosati-involution is positive on $\mathbb{R} \otimes \mathrm{End}(\rho_i)$. To prove (ii) we pick a hyperspecial subgroup $U_p \subset \mathrm{Res}_{L^+/\mathbb{Q}} G_0(\mathbb{Q}_p)$. Sufficiently small U_p-invariant $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L$-lattices $\mathfrak{V}_{i,p} \subset V_i$ for $i \in \Lambda$, and a sufficiently large integer m would satisfy: $\mathfrak{V}_{i,p} \subset \mathfrak{V}_{i,p}^\perp \subset p^{-m}\mathfrak{V}_{i,p}$. Now recall Zarhin's trick: Choose an even self-dual lattice $Z^\perp = Z \subset Q_8$ containing a direct factor M of rank 4, such that $\{x^t y | x, y \in M\} \subset p^m\mathbb{Z}$. Replacing $\mathfrak{V}_{i,p}$ by the $\mathbb{Z}_{(p)} \otimes \mathcal{O}_L$-lattices:

$$Z \otimes \mathfrak{V}_{i,p} + M \otimes \mathfrak{V}_{i,p}^\perp \subset Q_8 \otimes V_i$$

proves the result. □

D.2. **Gauged metaunitary collections for Shimura data.** In the next two propositions we fix a reductive group G_0 of adjoint type over a totally real field L^+, we let K be the smallest L^+-extension over which G_0 is an inner form of a split form, and we let (G_0, h_0) be a Shimura datum with coefficients in L^+. For a totally imaginary quadratic extension L of a totally real field L^+ we have to introduce the following algebraic torus:

$$C_{L/L^+} := \ker(\mathrm{Res}_{L/L^+} \mathbb{G}_{m,L} \overset{\mathbb{N}_{L/L^+}}{\to} \mathbb{G}_{m,L^+})$$

Note that $\mathrm{Res}_{L^+/\mathbb{Q}} C_{L/L^+} \cong L^1$. Every CM-type for L endows $(\mathbb{G}_{m,\mathbb{Q}} \times L^1)/\{\pm 1\}$ with the structure of a Shimura datum. We fix a prime $p \neq 2$ such that G_0 and L^+ are unramified at all divisors of p. Now we turn to our poly-unitary examples:

Proposition D.5. *If*

$$\lim_{N\to\infty} \frac{\mathrm{Card}\{k \in \{1, \ldots, N\} | G_0(\mathbb{R}_{[\vartheta^k \circ \iota]}) \textit{ is compact }\}}{N} > \frac{2r(G_{0,\mathbb{C}}) + 1}{2r(G_{0,\mathbb{C}}) + 2}$$

holds for all ι. Then there exists a L^+-torus Z^0 which splits over the composite of L with K, together with a normalised L-metaunitary Shimura datum for the pair

$$(Z^0, \{c\}) \times (G_0, h_0),$$

where c factors through C_{L/L^+}.

Proof. We begin with any representation $\rho_0 : G_{0,L} \to \mathrm{GL}(W_0/L)$ whose scalar extension $L^{ac} \otimes_L W_0$ is a direct sum of any number of copies of $\mathrm{Gal}(L^{ac}/L)$-conjugates of the previously described U_0. Let $\rho_1 : G_0 \to \mathrm{GL}(W_1/L)$ be the trivial one-dimensional representation. For $i \in \{0, 1\}$ we pick G_0-invariant pairings $\Psi_i : W_i \to \check{\overline{W}}_i$ such that:

- $\overline{\Psi_i(x,y)} = \Psi_i(y,x)$ holds for all $x, y \in W_i$, and
- the symmetric form $(\operatorname{tr}_{L/\mathbb{Q}} \Psi_i)(\rho_i(h_0(\sqrt{-1}))x,y)$ is positive definite on $\mathbb{R} \otimes W_i$.

(N.B.: This is meaningful only because $h_0(-1) = 1$). Consider the prolongation of the negative of the Killing form on $\mathfrak{g}_0$ to a sesquilinear form Ψ' on $W' := L \otimes_{L^+} \mathfrak{g}_0$. The triple of polarized $G_{0,L}$-representations, that we wish to work with is: W_1, $W_2 := W_1 \oplus W'$, and $W_3 := W_0 \oplus W'$. The G_0-invariant sub-algebras $R_\pi \subset \operatorname{End}_L(W^\pi)$ are constructed along the lines of proposition D.1 and are easily seen to be $*$-invariant, here $\Lambda := \{1,2,3\}$ and $\Pi := \{\{1\},\{2\},\{3\},\{1,2,3\}\}$. The same proposition exhibits a connected L^+-torus Z^0 such that $Z^0 \times G_0$ is the stabilizer in $\mathrm{U}(W_1/L,\Psi_1) \times_{L^+} \mathrm{U}(W_2/L,\Psi_2) \times_{L^+} \mathrm{U}(W_3/L,\Psi_3)$ of $\bigcup_{\pi\in\Pi} R_\pi$, moreover the proof thereof shows that Z^0_L splits over the splitting field of $G_{0,L}$. Let us note in passing that $(\mathbb{G}_{m,\mathbb{Q}} \times L^1)/\{\pm 1\}$ is embedded diagonally into $(\mathbb{G}_{m,\mathbb{Q}} \times \operatorname{Res}_{L^+/\mathbb{Q}} Z^0)/\{\pm 1\}$, as L^1, the complement in $(\mathbb{G}_{m,\mathbb{Q}} \times L^1)/\{\pm 1\}$ to $\mathbb{G}_{m,\mathbb{Q}}$, is embedded diagonally into $\operatorname{Res}_{L^+/\mathbb{Q}} Z^0$. Let c be an arbitrary CM type for L and notice that $w_{\{c\}}$ maps -1 to -1, and thus we obtain a specific L-metaunitary collection for the datum $(Z^0,\{c\}) \times (G_0,h_0)$, if we only divide out each of the pairings Ψ_i by a suitable purely imaginary element of L. The normalised L-metaunitary Shimura datum is found by means of lemma D.3 and lemma D.2. □

For the rest of this part of the appendix we assume that G_0 is an absolutely simple group over L^+. We round off the discussion with another family of examples, where the field L is adapted to G_0, in the sense that:

$$L = \begin{cases} L^+[\sqrt{[\sqrt{p}]^2 - p}] & E_7, C_l, B_l, A_1, D_4, D_6, \dots \\ K & \text{otherwise} \end{cases},$$

for in the last cases only, K is totally imaginary quadratic extension because the Weyl-opposition is a non-trivial outer automorphism and there are no other ones. We have to introduce a class of provisional central extensions $(\overline{G}_0,\overline{h}_0) \twoheadrightarrow (G_0,h_0)$, defined by a specific push-out diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{C} & \longrightarrow & \overline{G}_0 & \longrightarrow & \overline{G}_0^{ad} & \longrightarrow & 1 \\ \cong\uparrow & & \zeta\uparrow & & \uparrow & & \cong\uparrow & & \cong\uparrow \\ 1 & \longrightarrow & \mathrm{Z}^{G_1} & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 1 \end{array},$$

where G_1 is the semisimple and simply-connected covering group of G_0. In the D_l-case with $l \equiv 0 \pmod 2$, we start with the description of $\overline{C}$

being the qudratic twist:

$$\overline{C} := \ker(\operatorname{Res}_{L/L^+} T_L \overset{\mathbb{N}_{L/L^+}}{\to} T),$$

where T is the L^+-torus whose group of cocharacters agrees with the lattice

$$X^*(T) = \{\chi : \mathrm{Z}^{G_1}(\mathbb{C}) \to \mathbb{Z} | 0 = \chi(1) = \sum_{g \neq 1} \chi(g)\},$$

which is of rank 2. Identifying the Klein four-group scheme Z^{G_1} with $T[2] \cong \overline{C}[2]$ by decreeing the cocharacter χ to attain the value $(-1)^{\chi(g)}$ on an element $g \in \mathrm{Z}^{G_1}(\mathbb{C})$ defines an inclusion ζ of Z^{G_1} into $\overline{C}$.
In the remaining cases we define $\overline{C}$ to be C_{L/L^+}: If G_0 is of type E_7, C_l, B_l or A_1 we have $\mathrm{Z}^{G_1} = \{\pm 1\}$, and we let ζ be the inclusion of Z^{G_1} into C_{L/L^+}, and otherwise we introduce ζ by observing that the theory of root data produces an isomorphism:

$$Z^{G_1} \cong \begin{cases} C_{L/L^+}[3] & E_6 \\ C_{L/L^+}[4] & D_{odd} \\ C_{L/L^+}[l+1] & A_l \end{cases},$$

which is canonical up to composition with the reciprocal function. Now, given that $\overline{C}(\mathbb{R})$ is connected, and that h_0 factors through $\mathbb{S}^1$, there exists a diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_0} & (\operatorname{Res}_{L^+/\mathbb{Q}} G_0^{ad})_{\mathbb{R}} \\ {\scriptstyle \mathbb{N}}\downarrow & & \uparrow \\ \mathbb{S}^1 & \xrightarrow{\overline{h}^0} & (\operatorname{Res}_{L^+/\mathbb{Q}} \overline{G}_0)_{\mathbb{R}} \end{array}.$$

Recall that (G_0, h_0) is said to be of type $D_l^{\mathbb{R}}$ if all simple factors of $G_{0,\mathbb{R}}$ are of the form $\mathrm{SO}(2l-2,2)/\{\pm 1\}$ or $\mathrm{SO}(2l)/\{\pm 1\}$. In the following proposition the type $D_l^{\mathbb{R}\&\mathbb{H}}$ stands for data (G_0, h_0) with $G_{0,\mathbb{C}}$ of type D_l, but (G_0, h_0) not of type $D_l^{\mathbb{R}}$ (N.B.: $D_4^{\mathbb{R}\&\mathbb{H}}$ does not exist, because $\mathrm{SO}^*(8)/\{\pm 1\} \cong \mathrm{SO}(6,2)/\{\pm 1\}$).

Proposition D.6. *Let* (G_0, h_0) *and* L *be as above. If*

$$\lim_{N\to\infty} \frac{\operatorname{Card}\{k \in \{1, \dots, N\} | G_0(\mathbb{R}_{[\vartheta^k \circ \iota]}) \text{ is compact }\}}{N} > \begin{cases} 0 & A_l \\ \frac{3}{4} & E_6, B_l, C_l, D_l^{\mathbb{R}} \\ \frac{4}{5} & E_7 \\ \frac{l+1}{l+3} & D_l^{\mathbb{R}\&\mathbb{H}},\ 2 \nmid l \\ \frac{l+2}{l+4} & D_l^{\mathbb{R}\&\mathbb{H}},\ 2 \mid l \end{cases}$$

holds for all embeddings $\iota : L^+ \to \mathbb{R}$*, then there exists a normalised* L*-metaunitary Shimura datum for some central extension* $(G^0, h^0) \twoheadrightarrow (G_0, h_0)$ *where* G^{0der} *is simply connected, and* Z^{G^0} *is connected, contains* $\overline{C}$ *and is unramified at all divisors of* p*.*

Proof. In the A_l-case there is nothing to prove. Excluding the case D_{even} for a while we let $\rho_0 : \overline{G}_{0,L} \to \mathrm{GL}(W_0/L)$ be a isotypic L-rational representation which is a direct sum of copies of minuscule irreducible representations each of whose restrictions to the center $\overline{C}_L = \mathbb{G}_{m,L}$ agree with scalar multiplication. The sets of $\mu_{\overline{h}_0}$-weights of each eigenspace (to the embeddings $L \hookrightarrow \mathbb{C}$) can be read off from [11, Table 1.3.9], so these are translates of

$$\begin{aligned} &\{-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\} && E_6 \\ &\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\} && E_7 \\ &\{-\frac{1}{2}, \frac{1}{2}\} && B_l, C_l, D_l^{\mathbb{R}} \\ &\{\frac{2-l}{4}, \dots, \frac{l}{4}\} && D_l^{\mathbb{R}\&\mathbb{H}} \end{aligned}$$

Notice that $\overline{h}^0$ could be multiplied with an arbitrary homomorphism $\mathbb{S}^1 \to L^1_{\mathbb{R}}$ without changing h_0, so that we can adjust the $\mu_{\overline{h}_0}$-weights to lie in $\{0, \pm 1\}$. Again we let ρ_1 be the trivial representation, and we pick $\overline{G}_0$-invariant pairings $\Psi' : W' \to \overline{\check{W}}'$ and $\Psi_i : W_i \to \overline{\check{W}}_i$ with properties as in the proof of the previous proposition, for any $i \in \{0, 1\}$. The proof is completed by the arguments in the proof of proposition D.5, notice that the L^+-group G^0 (or equivalently of Z^{G^0}) rests on the choice of the polarizations which in turn assumes a choice of $\overline{h}_0$.
Finally, both $D_l^{\mathbb{R}}$-cases are very similar to the B_l-one, and left to the reader, but it remains to do the $D_l^{\mathbb{R}\&\mathbb{H}}$-case with $l \equiv 0 \pmod 2$, so let K be the smallest L^+-extension over which G_0 is an inner form of a split form. If $K = L^+$, the theory of root data produces an isomorphism

$$\overline{C} \cong C^2_{L/L^+},$$

which is canonical up to a swap of the factors. So let us set ρ_0 to be the direct sum of (possibly several copies of) the two minuscule representations ρ_0^+ and ρ_0^-, with central characters being the two projections $\overline{C}_L \to \mathbb{G}_{m,L}$. Again, the weight sets of each eigenspace (to the embeddings $L \hookrightarrow \mathbb{C}$) can be read off from [11, Table 1.3.9], so depending on

$\mu_{\overline{h}_0}$ these are translates of one of the sets:

$$\{-\frac{1}{2},\frac{1}{2}\}$$
$$\{\frac{2-l}{4},\dots,\frac{l-2}{4}\}$$
$$\{-\frac{l}{4},\dots,\frac{l}{4}\}$$

In general K is a totally real quadratic extension of L^+, $\overline{C}$ is canonically isomorphic to $\operatorname{Res}_{K/L^+} C_{L/L^+}$ and the argument is analogous. $\square$

Appendix E. Varshavsky's characterization method

The paper [53] deals with characterizations of Shimura varieties. In this work we want to be slighty more basic and confine to characterizations of symmetric Hermitian spaces, here are the objects under consideration:

- Let Δ be a group and let M be a separated, non-empty, connected, complex manifold with a holomorphic left Δ-action.
- Let $G_1,\dots,G_z$ be connected, simple, non-abelian algebraic groups over $\mathbb{C}$, and let P_i be a proper minuscule parabolic subgroup of G_i. Denote the associated irreducible symmetric Hermitian domains of compact type by $X_i := G_i(\mathbb{C})/P_i(\mathbb{C})$, and write $G := \prod_{i=1}^z G_i$ and $P := \prod_{i=1}^z P_i$ and $X := \prod_{i=1}^z X_i$.
- Let $\varphi : \Delta \to G(\mathbb{C})$ be a group homomorphism, and let $I : M \to X$ be a locally biholomorphic Δ-equivariant map (use φ to define a left action of Δ on X)

A sextuple $(\Delta, M, G, P, \varphi, I)$ as above is called a period map if:

(M1) The image of Δ in $G(\mathbb{C})$ is Zariski-dense.
(M2) The image in $G(\mathbb{C})$ of the stabilizer in Δ of any element in M possesses a compact closure.
(M3) There exists some $y_0 \in M$ whose stabilizer in Δ contains some subgroup Δ_0 the closure of whose image in $G(\mathbb{C})$ is equal to $\prod_{i=1}^z \mathbf{T}_i(\mathbb{R})$, where each $\mathbf{T}_i$ is a maximal compact torus in $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} G_i$ (i.e. the $\mathbb{R}$-rank of $\mathbf{T}_i$ is equal to the $\mathbb{C}$-rank of G_i).

Without any attempt of originality we wish to give a slight reformulation of [52, p.89,p.92-94] in the above axiomatic setup:

Theorem E.1 (Varshavsky). *Suppose that $(\Delta, M, G_i, P_i, \varphi, I)$ is a period map. Then there exist real forms $\mathbf{J}_i$ of G_i for every $i \in \{1,\dots,z\}$, such that $\overline{\varphi(\Delta)} = \mathbf{J}(\mathbb{R})^\circ$, where $\mathbf{J} := \prod_{i=1}^z \mathbf{J}_i$. The locally biholomorphic map I is actually an injection and the Δ-action on M can be*

extended to a continuous and transitive $\mathbf{J}(\mathbb{R})^\circ$*-action thereon.*
Pick a base point $\tilde{y} \in M$*, and assume for notational convenience that* $I(\tilde{y})$ *is the canonical base point of* $\prod_{i=1}^{z} X_i$*, i.e. equal to* $(1, \dots, 1)$*. Write* $\mathbf{U}_i := P_i \cap \overline{P}_i$*, where* $\overline{P}_i$ *denotes the complex conjugate of* P_i *with respect to the real form* $\mathbf{J}_i$*. Consider the homogeneous spaces* $M_i := \mathbf{J}_i(\mathbb{R})^\circ / \mathbf{U}_i(\mathbb{R})$*, so that* $M = \prod_{i=1}^{z} M_i$*. Then for each* $i \in \{1, \dots, z\}$ *one and only one of the following alternatives hold:*

- $\mathbf{J}_i$ *is compact, and* $\mathbf{U}_i$ *is a maximal proper connected subgroup,*
- *or* $\mathbf{J}_i$ *is not compact, and* $\mathbf{U}_i$ *is a maximal compact subgroup,*

and in any case $\mathbf{U}_i$ *has indiscrete center so that* M_i *is a symmetric Hermitian domain of compact or of non-compact type.*

We give a synopsis of the proof: The solution to the 5th problem of Hilbert implies that $J := \overline{\varphi(\Delta)}$ is a Lie-group. Note that $\mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie} J$ is semisimple, because $\operatorname{Lie} G$ is semisimple and both of $\operatorname{Lie} J \cap \sqrt{-1} \operatorname{Lie} J$ and $\operatorname{Lie} J + \sqrt{-1} \operatorname{Lie} J$, being G-invariant $\mathbb{C}$-subspaces of $\operatorname{Lie} G$ in view of (M1), are semisimple too. Moreover, there exists a semi-simple real algebraic group $\mathbf{J} \subset \operatorname{Res}_{\mathbb{C}/\mathbb{R}} G$ such that $\operatorname{Lie} \mathbf{J} = \operatorname{Lie} J$, as semisimple Lie-algebras are algebraic. Finally the existence of $\mathbf{T}$ tells us that $\mathbf{J} = \prod_{i=1}^{r} \mathbf{J}_i$ where each single $\mathbf{J}_i$ contains $\mathbf{T}_i$ and it is either a real form of G_i or it is equal to $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} G_i$. Let us write $\operatorname{Aut}(M)$ for the the homeomorphism group of M, and let us endow it with the compact-open topology. Let $\tilde{J}$ (resp. $\tilde{T}$) denote the closure of the image of Δ (resp. Δ_0) in $\operatorname{Aut}(M)$. The group $\tilde{T}$ is compact because it fixes a point and preserves a suitable Riemannian metric, [25, II, Theorem 1.2]. It is straightforward to see that φ extends to a continuous group homomorphism, say $\tilde{\varphi} : \tilde{J} \to J$, the kernel of which is a discrete subgroup of $\tilde{J}$, cf. [52, Lemma 3.1]. The following argument shows that $\tilde{\varphi}$ is surjective: We clearly have $\overline{\tilde{\varphi}(\tilde{J})} = J$ and note also that $\tilde{\varphi}(\tilde{T}) = \mathbf{T}(\mathbb{R})$ because $\tilde{T}$ is compact. Now there exist elements $\gamma_1, \dots, \gamma_n \in \Delta$ such that $\operatorname{Ad}(\gamma_1) \operatorname{Lie} \mathbf{T} + \cdots + \operatorname{Ad}(\gamma_n) \operatorname{Lie} \mathbf{T} = \operatorname{Lie} J$. It follows that the product $\tilde{\varphi}(\gamma_1 \tilde{T} \gamma_1^{-1}) \cdot \dots \cdot \tilde{\varphi}(\gamma_n \tilde{T} \gamma_n^{-1})$ contains an open neighborhood of the identity in J and whence it follows that $\tilde{\varphi}(\tilde{J}) = J$. One shows the openness of $\tilde{\varphi}$ along the same lines.
If a Cartan subgroup $C \subset J$ fixes some point in the image of M in X, then it must be compact. This can be shown as in [52, Lemma 3.6] using the property (M2) only, together with the fact that $\tilde{J}$ is a Lie-group, which is implied by the openness of $\tilde{\varphi}$. Finally two more facts follow easily from that observation: First, no $\mathbf{J}_i$ is equal to $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} G_i$ so that $\mathbf{J}$ is actually a real form of G and second, the stabilizer in J of any point on X has to contain some Cartan subgroup C of J, this is

because it is equal to the intersection of two complex mutually conjugate parabolics, namely the stabilizer in G of that very point and its complex conjugate.
We note in passing that $(\tilde{J}, M, G, P, \tilde{\varphi}, I)$ is a period map too, by [52, Proposition 3.5]. One next establishes the transitivity of the $\tilde{J}$-action, which is accomplished by a simple dimension count (look at the stabilizers and notice that all of the maximal compact subgroups of G act transitively on G/P).

Remark E.2. Let us say that a period map $(\Delta, M, G, P, \varphi, I)$ is bounded if

(M4) there exist positive integers p_j such that the pull-back of some line bundle of the form $\bigotimes_j \omega_j^{\otimes p_j}$ by means of I is generated by its holomorphic global sections, where ω_j denotes the canonical line bundle on X_i.

It is clear that bounded period maps give rise to bounded symmetric Hermitian domains M.

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