COHOMOLOGICAL INVARIANTS: EXCEPTIONAL GROUPS AND SPIN GROUPS

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Preface

These notes are divided into three parts.

The first part is based on material developed for inclusion in Serre's lecture notes in [GMS03], but was finally omitted. I learned most of that material from Serre. This part culminates with the determination of the invariants of PGL_p mod p (for p prime) and the invariants of Albert algebras (equivalently, groups of type F_4) mod 3.

The second part describes a general recipe for finding a subgroup N of a given semisimple group G such that the natural map $H^1_{\text{fppf}}(*, N) \to H^1(*, G)$ is surjective. It is a combination of two ideas: that parabolic subgroups lead to representations with open orbits, and that such representations lead to surjective maps in Galois cohomology. I learned the second idea from Rost [Ros99b], but both ideas seem to have been discovered and re-discovered many times. We bring the two ideas together here, apparently for the first time. Representation theorists will note that our computations of stabilizers N for various G and V — summarized in Table 21a — are somewhat more precise than the published tables, in that we compute full stabilizers and not just identity components. The surjectivities in cohomology are used to describe the mod 3 invariants of the simply connected split E_6 and split E_7's.

The last two sections of this part describe a construction of groups of type E_8 that is "surjective at 5", see Prop. 13.7. We use it to determine the mod 5 invariants of E_8 and to give new examples of anisotropic groups of that type. These examples have already been applied in [PSZ06].

The third part describes the mod 2 invariants of the groups Spin_n for $n \leq 12$ and $n = 14$. It may be viewed as a fleshed-out version of Markus Rost's unpublished notes [Ros99b] and [Ros99c]. A highlight of this part is Rost's Theorem 19.3 on 14-dimensional quadratic forms in I^3.

There are also two appendices. The first uses cohomological invariants to give new examples of anisotropic groups of types E_7, answering a question posed by Kirill Zainoulline. The second appendix—written by Detlev W. Hoffmann—proves a generalization of the "common slot theorem" for 2-Pfister quadratic forms. This result is used to construct invariants of Spin_{12} in §18.

These are notes for a series of talks I gave in a "mini-cours" at the Université d'Artois in Lens, France, in June 2006. Consequently, some material has been included in the form of exercises. Although this is a convenient device to avoid going into tangential details, no substantial difficulties are hidden in this way. The exercises are typically of the "warm up" variety. On the other end of the spectrum, I have included several open problems. "Questions" lie somewhere in between.

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1. Definitions and notations

1.1. Definition of cohomological invariant. We assume some familiarity with the notes from Serre's lectures from [GMS03], which we refer to hereafter as S.[a] A reader seeking a more leisurely introduction to the notion of invariants should see pages 7–11 of those notes.

We fix a base field k_0 and consider functors

$$A\colon \mathsf{Fields}_{/k_0} \to \mathsf{Sets}$$

and

$$H\colon \mathsf{Fields}_{/k_0} \to \mathsf{Abelian\ Groups},$$

where $\mathsf{Fields}_{/k_0}$ denotes the category of field extensions of k_0. In practice, $A(k)$ will be the Galois cohomology set $H^1(k, G)$ for G a linear algebraic group[b] over k_0. In S, various functors H were considered (e.g., the Witt group), but here we only consider abelian Galois cohomology.

An *invariant of A (with values in H)* is a morphism of functors $a\colon A \to H$, where we view H as a functor with values in Sets. Unwinding the definition, an invariant of A is a collection of functions $a_k\colon A(k) \to H(k)$, one for each $k \in \mathsf{Fields}_{/k_0}$, such that for each morphism $\phi\colon k \to k'$ in $\mathsf{Fields}_{/k_0}$, the diagram

$$\begin{array}{ccc} A(k) & \xrightarrow{a_k} & H(k) \\ {\scriptstyle A(\phi)}\downarrow & & \downarrow{\scriptstyle H(\phi)} \\ A(k') & \xrightarrow{a_{k'}} & H(k') \end{array}$$

commutes.

1.2. **Examples.** (1) Fix a natural number n and write $\mathscr{S}_n$ for the symmetric group on n letters. The set $H^1(k, \mathscr{S}_n)$ classifies étale k-algebras of degree n up to isomorphism. The sign map $\mathrm{sgn}\colon \mathscr{S}_n \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism of algebraic groups and so defines a morphism of functors—an invariant—$\underline{\mathrm{sgn}}\colon H^1(*, \mathscr{S}_n) \to H^1(*, \mathbb{Z}/2\mathbb{Z})$. The set $H^1(k, \mathbb{Z}/2\mathbb{Z})$ classifies quadratic étale k-algebras, i.e., separable quadratic field extensions together with the trivial class corresponding to $k \times k$, and $\underline{\mathrm{sgn}}$ sends a degree n algebra to its discriminant algebra.

This example is familiar in the case where the characteristic of k_0 is not 2. Given a separable polynomial $f \in k[x]$, one can consider the étale k-algebra $K := k[x]/(f)$. The discriminant algebra of K—here, $\underline{\mathrm{sgn}}(K)$—is $k[x]/(x^2 - d)$, where d is the usual elementary notion of discriminant of f, i.e., the product of squares of differences of roots of f.

(For a discussion in characteristic 2, see [Wat87].)

One of the main results of S is that $\mathscr{S}_n$ only has "mod 2" invariants, see S24.12.

(2) Let G be a semisimple algebraic group over k_0. It fits into an exact sequence

$$1 \longrightarrow C \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1,$$

[a] We systematically refer to specific contents of S by S followed by a reference number. For example, Proposition 16.2 on page 39 will be referred to as S16.2.

[b] Below, we only consider algebraic groups that are linear.

where $\widetilde{G}$ is simply connected and C is finite and central in $\widetilde{G}$. This gives a connecting homomorphism in Galois cohomology

$$H^1(k, G) \xrightarrow{\ \delta\ } H^2(k, C)$$

that defines an invariant $\delta\colon H^1(*, G) \to H^2(*, C)$.

(3) The map e_n that sends the n-Pfister quadratic form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ (over a field k of characteristic $\neq 2$) to the class $(a_1)\cdot(a_2)\cdot \cdots \cdot (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ depends only on the isomorphism class of the quadratic form. (Compare §18 of S.)

The Milnor Conjecture (now a theorem, see [Voe03, 7.5] and [OVV, 4.1]) states that e_n extends to a well-defined additive map

$$e_n\colon I^n \to H^n(k, \mathbb{Z}/2\mathbb{Z})$$

that is zero on I^{n+1} and induces an isomorphism $I^n/I^{n+1} \xrightarrow{\sim} H^n(k, \mathbb{Z}/2\mathbb{Z})$. (Here I^n denotes the n-th power of the ideal I of even-dimensional forms in the Witt ring of k.)

(4) For G a quasi-simple simply connected algebraic group, there is an invariant $r_G : H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$ called the *Rost invariant*. It is the main subject of [Mer03]. When G is Spin_n, i.e., a split simply connected group of type B or D, the Rost invariant amounts to the invariant e_3 in (3) above, cf. [Mer03, 2.3].

The Rost invariant has the following useful property: If G' is also a quasi-simple simply connected algebraic group and $\rho\colon G' \to G$ is a homomorphism, then the composition

$$H^1(*, G') \xrightarrow{\ \rho\ } H^1(*, G) \xrightarrow{\ r_G\ } H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

equals $n_\rho\, r_{G'}$ for some natural number n_ρ, called the *Rost multiplier* of ρ, see [Mer03, p. 122].

(5) Suppose that k_0 contains a primitive 4-th root of unity. The trace quadratic form on a central simple algebra A of dimension 4^2 is Witt-equivalent to a direct sum $q_2 \oplus q_4$ where q_i is an i-Pfister form, see [RST06]. The maps $f_i : A \mapsto e_i(q_i)$ define invariants $H^1(*, PGL_4) \to H^i(*, \mathbb{Z}/2\mathbb{Z})$ for $i = 2$ and 4. Rost-Serre-Tignol prove that $f_2(A)$ is zero if and only if $A \otimes A$ is a matrix algebra and $f_4(A)$ is zero if and only if A is cyclic.[c]

(For the case where k_0 has characteristic 2, see [Tig06].)

1.3. Let C be a finite $\mathrm{Gal}(k_0)$-module of exponent not divisible by the characteristic of k_0. We define a functor M by setting

$$M^d(k, C) := H^d(k, C(d-1))$$

where $C(d-1)$ denotes the $(d-1)$-st Tate twist of C as in S7.8 and

$$M(k, C) := \bigoplus_{d \geq 0} M^d(k, C).$$

We are mainly interested in

$$M(k, \mathbb{Z}/n\mathbb{Z}) = H^0(k, \mathrm{Hom}(\boldsymbol{\mu}_n, \mathbb{Z}/n\mathbb{Z})) \oplus \bigoplus_{d\geq 1} H^d(k, \boldsymbol{\mu}_n^{\otimes(d-1)}).$$

Many invariants take values in $M(*, \mathbb{Z}/n\mathbb{Z})$, for example:

[c]The term "cyclic" is defined in 5.4 below.

(2bis) For $G = PGL_n$, the invariant δ in Example 1.2.2 is

$$\delta \colon H^1(*, PGL_n) \to H^2(*, \boldsymbol{\mu}_n) \subset M(*, \mathbb{Z}/n\mathbb{Z}).$$

We remark that $H^2(k, \boldsymbol{\mu}_n)$ can be identified with the n-torsion in the Brauer group of k via Kummer theory.

(4bis) Let G be a group as in 1.2.4. Write i for the Dynkin index of G as in [Mer03, p. 130] and put $n := i$ if $\operatorname{char} k_0 = 0$ or $i = p^\ell n$ for n not divisible by p if $\operatorname{char} k_0$ is a prime p. The Rost invariant maps

$$H^1(*, G) \to H^3(*, \boldsymbol{\mu}_n^{\otimes 2}) \subset M(*, \mathbb{Z}/n\mathbb{Z}).$$

(6) If the characteristic of k_0 is different from 2, then $\mathbb{Z}/2\mathbb{Z}(d)$ equals $\mathbb{Z}/2\mathbb{Z}$ for every d, and $M(k, \mathbb{Z}/2\mathbb{Z})$ is the mod 2 cohomology ring $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$. So most of the cohomological invariants considered in S take values in the functor $k \mapsto M(k, \mathbb{Z}/2\mathbb{Z})$.

We remark that for n dividing 24, $\boldsymbol{\mu}_n^{\otimes 2}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ [KMRT98, p. 444, Ex. 11]. In that case,

$$M^d(k, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} H^0(k, \operatorname{Hom}(\boldsymbol{\mu}_n, \mathbb{Z}/n\mathbb{Z})) & \text{if } d = 0 \\ H^d(k, \mathbb{Z}/n\mathbb{Z}) & \text{if } d \text{ is odd} \\ H^d(k, \boldsymbol{\mu}_n) & \text{if } d \text{ is even and } d \neq 0 \end{cases}$$

The other main target for invariants is $M(*, \boldsymbol{\mu}_n)$, characterized by

$$M(k, \boldsymbol{\mu}_n) = \mathbb{Z}/n\mathbb{Z} \oplus \bigoplus_{d \geq 1} H^d(k, \boldsymbol{\mu}_n^{\otimes d}).$$

This is naturally a ring, and we write $R_n(k)$ for $M(k, \boldsymbol{\mu}_n)$ when we wish to view it as such. (This ring is a familiar one: the Bloch-Kato Conjecture asserts that it is isomorphic to the quotient $K_\bullet^M(k)/n$ of the Milnor K-theory ring $K_\bullet^M(k)$.) When C is n-torsion, the abelian group $M(k, C)$ is naturally an $R_n(k)$-module.

For various algebraic groups G and Galois-modules C, we will determine the invariants $H^1(*, G) \to M(*, C)$. We abuse language by calling these "invariants of G with values in C", "C-invariants of G", etc. We write $\operatorname{Inv}(G, C)$ or $\operatorname{Inv}_{k_0}(G, C)$ for the collection of such invariants.[d] For example, the invariants in (2bis) and (4bis) above belong to $\operatorname{Inv}(PGL_n, \mathbb{Z}/n\mathbb{Z})$ and $\operatorname{Inv}(G, \mathbb{Z}/n\mathbb{Z})$ respectively. Note that $\operatorname{Inv}(G, C)$ is an abelian group for every algebraic group G, and, when G is n-torsion, $\operatorname{Inv}_{k_0}(G, C)$ is an $R_n(k_0)$-module.

1.4. Constant and normalized. Fix an element $m \in M(k_0, C)$. For every group G, the collection of maps that sends every element of $H^1(k, G)$ to the image of m in $M(k, C)$ for every extension k/k_0 is an invariant in $\operatorname{Inv}(G, C)$. Such invariants are called *constant*.

An invariant $a \in \operatorname{Inv}(G, C)$ is *normalized* if a sends the neutral class in $H^1(k, G)$ to zero in $M(k, C)$ for every extension k/k_0. We write $\operatorname{Inv}^{\text{norm}}(G, C)$ for the normalized invariants in $\operatorname{Inv}(G, C)$.

The reader can find a typical application of cohomological invariants in Appendix A.

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[d] Strictly speaking, this notation disagrees with the notation defined on page 11 of S. But there is no essential difference, because in S the target C is nearly always taken to be $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}(d)$ is canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all d.

2. Invariants of $\boldsymbol{\mu}_n$

Fix a natural number n not divisible by the characteristic of the field k_0. In this section, we determine the invariants of $\boldsymbol{\mu}_n$ with values in $\boldsymbol{\mu}_n$, along with a small variation.

There are two obvious invariants of $\boldsymbol{\mu}_n$:

(1) The constant invariant (as in 1.4) given by the element $1 \in \mathbb{Z}/n\mathbb{Z} \subset M(k_0, \boldsymbol{\mu}_n)$.
(2) The invariant $\underline{\mathrm{id}}$ that is the identity map
$$H^1(k, \boldsymbol{\mu}_n) \to H^1(k, \boldsymbol{\mu}_n) \subset M(k, \boldsymbol{\mu}_n)$$
for every k/k_0.

2.1. **Proposition.** *$\mathrm{Inv}_{k_0}(\boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$ is a free $R_n(k_0)$-module with basis $1, \underline{\mathrm{id}}$.*

This proposition can easily be proved by adapting the proof of S16.2. Alternatively, it is [MPT03, Cor. 1.2]. In the interest of exposition, we give an elementary proof in the case where k_0 is algebraically closed.

We need the following lemma, which is a special case of S12.3.

2.2. **Lemma.** *If invariants $a, a' \in \mathrm{Inv}_{k_0}(\boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$ agree on $(t) \in H^1(k_0(t), \boldsymbol{\mu}_n)$, then a and a' are equal.*

Proof. Replacing a, a' with $a - a', 0$ respectively, we may assume that a' is identically zero.

Fix an extension E of k_0 and an element $y \in E^\times$. Write M for the functor $M(*, \boldsymbol{\mu}_n)$ as in 1.3 and consider the commutative diagram

$$\begin{array}{ccccc} H^1(E, \boldsymbol{\mu}_n) & \longrightarrow & H^1(E((t-y)), \boldsymbol{\mu}_n) & \longleftarrow & H^1(k_0(t), \boldsymbol{\mu}_n) \\ \downarrow a_E & & \downarrow a_{E((t-y))} & & \downarrow a_{k_0(t)} \\ M(E) & \longrightarrow & M(E((t-y))) & \longleftarrow & M(k_0(t)) \end{array} \tag{2.3}$$

The polynomial $x^n - y/t$ in $E((t-y))[x]$ has residue $x^n - 1$ in $E[x]$, which has a simple root, namely $x = 1$. Therefore $x^n - y/t$ has a root over $E((t-y))$ by Hensel's Lemma, and the images of $(y) \in H^1(E, \boldsymbol{\mu}_n)$ and $(t) \in H^1(k_0(t), \boldsymbol{\mu}_n)$ in $H^1(E((t-y)), \boldsymbol{\mu}_n)$ agree. The commutativity of the diagram implies that the image of (y) in $M(E((t-y)))$ is the same as the image of $a_{k_0(t)}(t)$, i.e., zero. But the map $M(E) \to M(E((t-y)))$ is an injection by S7.7, so $a_E(y)$ is zero. This proves the lemma. □

Proof of Prop. 2.1. We assume that k_0 is algebraically closed. Fix an invariant $a \in \mathrm{Inv}_{k_0}(\boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$, and consider the torsor class $(t) \in H^1(k_0(t), \boldsymbol{\mu}_n)$. We claim that $a(t)$ is unramified away from $\{0, \infty\}$. Indeed, any other point on the affine line over k_0 is an ideal $(t-y)$ for some $y \in k_0^\times$ because k_0 is algebraically closed. Consider the diagram (2.3) with the E's replaced with k_0's. As in the proof of Lemma 2.2, the images of $(y) \in H^1(k_0, \boldsymbol{\mu}_n)$ and $(t) \in H^1(k_0(t), \boldsymbol{\mu}_n))$ agree in $H^1(k_0((t-y)), \boldsymbol{\mu}_n)$ by Hensel's Lemma, hence the image of (t) in $M(k_0((t-y)))$ comes from $M(k_0)$. That is, $a(t)$ is unramified at $(t-y)$. This proves the claim, and by S9.4 we have:

$$a(t) = \lambda_0 + \lambda_1 \cdot (t)$$

for uniquely determined elements $\lambda_0, \lambda_1 \in M(k_0)$.

Put $a' := \lambda_0 \cdot 1 + \lambda_1 \cdot \underline{\mathrm{id}}$. Since the invariants a, a' agree on (t), the two invariants are the same by Lemma 2.2. This proves that $1, \underline{\mathrm{id}}$ span $\mathrm{Inv}_{k_0}(\boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$.

As for linear independence, suppose that the invariant $\lambda_0 \cdot 1 + \lambda_1 \cdot \underline{\mathrm{id}}$ is zero. Then λ_0—the value of a on the trivial class—is zero. The other coefficient, λ_1, is the residue at $t = 0$ of $a(t)$ in $M(k_0)$. □

Recall from S4.5 that every invariant can be written uniquely as (constant) + (normalized). Clearly, the proposition proves that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$ is a free $R_n(k_0)$-module with basis $\underline{\mathrm{id}}$.

Really, the proof of Prop. 2.1 given above is the same as the proof of S16.2 in the case where k_0 is algebraically closed, except that we have unpacked the references to S11.7 and S12.3 (which are both elaborations of the Rost Compatibility Theorem) with the core of the Rost Compatibility Theorem that is sufficient in this special case.

2.4. *Remark.* The argument using Hensel's Lemma in the proof of Lemma 2.2 has real problems when the characteristic of k_0 divides n. For example, when the characteristic of k_0 (and hence E) is a prime p, the element t/y has no p-th root in $E((t-y))$ for every $y \in E^\times$. Speaking very roughly, this is the reason for the global assumption that the characteristic of k_0 does not divide the exponent of C.

2.5. $\boldsymbol{\mu}_n$ INVARIANTS OF $\boldsymbol{\mu}_{sn}$. Let s be a positive integer not divisible by the characteristic of k_0. The s-th power map (the natural surjection) $s\colon \boldsymbol{\mu}_{sn} \to \boldsymbol{\mu}_n$ fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_{sn} & \longrightarrow & \mathbb{G}_m & \xrightarrow{sn} & \mathbb{G}_m & \longrightarrow & 1 \\ & & {\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow & & \| & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{n} & \mathbb{G}_m & \longrightarrow & 1 \end{array} \tag{2.6}$$

It induces an invariant $\underline{s}\colon H^1(*, \boldsymbol{\mu}_{sn}) \to H^1(*, \boldsymbol{\mu}_n)$. A diagram chase on (2.6) shows that for each k, $\underline{s}$ is the surjection

$$k^\times / k^{\times sn} \to k^\times / k^{\times n} \quad \text{given by } x k^{\times sn} \mapsto x k^{\times n}.$$

The proof of Prop. 2.1 with obvious modifications gives:

Proposition. $\mathrm{Inv}_{k_0}(\boldsymbol{\mu}_{sn}, \boldsymbol{\mu}_n)$ *is a free* $R_n(k_0)$*-module with basis* $1, \underline{s}$. □

We will apply this in 15.8 and §18 below in the case $s = n = 2$. In that case, we will continue to write $\underline{s}$ instead of the more logical $\underline{2}$.

2.7. EXERCISE. Let C be a finite $\mathrm{Gal}(k_0)$-module whose order is a power of n, and suppose that k_0 contains a primitive n-th root of unity. For $x \in H(k_0, C(-1))$ such that $nx = 0$, define a cup product

$$- \bullet x\colon H^1(k, \boldsymbol{\mu}_n) \to H^{(}k, C)$$

mimicking §23 of S. Prove that every normalized invariant $H^1(*, \boldsymbol{\mu}_n) \to H(*, C)$ can be written uniquely as $\underline{\mathrm{id}} \bullet x$ for such an x.

2.8. EXERCISE ($\mathrm{char}\, k_0 \neq 2$). Consider the group $SL(Q)$ whose k-points are the norm 1 elements of $Q \otimes_{k_0} k$ for some quaternion algebra Q over k_0. The center of this group is $\boldsymbol{\mu}_2$ and the natural map

$$H^1(k, \boldsymbol{\mu}_2) \to H^1(k, SL(Q)) \tag{2.9}$$

is surjective for every k/k_0. The Rost invariant of $SL(Q)$ (as defined in Example 1.2.4) takes values in $H^3(k, \boldsymbol{\mu}_2^{\otimes 2})$, and its composition with (2.9) is the cup product $x \mapsto x \cdot [Q]$. Prove that the Rost invariant generates $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(SL(Q), \mathbb{Z}/2\mathbb{Z})$ as an $R_2(k_0)$-module.

3. Quasi-Galois extensions and invariants of $\mathbb{Z}/p\mathbb{Z}$

3.1. Let $p_1, p_2, \ldots, p_r$ be the distinct primes dividing the exponent of C. There is a canonical identification $C = \prod_{i=1}^r {}_{p_i}C$, where ${}_{p_i}C$ denotes the submodule of C consisting of elements of order a power of p_i. This gives an identification

$$\mathrm{Inv}_{k_0}(G, C) = \prod_{i=1}^{r} \mathrm{Inv}_{k_0}(G, {}_{p_i}C)$$

that is functorial with respect to changes in the field k_0 and the group G.

3.2. **Lemma.** *If k_1 is a finite extension of k_0 of dimension relatively prime to the exponent of C, then the natural map*

$$\mathrm{Inv}_{k_0}(G, C) \to \mathrm{Inv}_{k_1}(G, C)$$

is an injection.

Proof. By 3.1, we may assume that the exponent of C is a power of a prime p.

Let a be an invariant in the kernel of the displayed map. Fix an extension E/k_0 and an element $x \in H^1(E, G)$; we show that $a(x)$ is zero in $M(E, C)$, hence a is the zero invariant.

First suppose that k_1/k_0 is separable. The tensor product $E \otimes_{k_0} k_1$ is a direct product of fields $E_1 \times E_2 \times \cdots \times E_r$ (since k_1 is separable over k_0), and at least one of them—say, E_i—has dimension over E not divisible by p (because p does not divide $[k_1 : k_0]$). We have

$$\mathrm{res}_{E_i/E}\, a(x) = a(\mathrm{res}_{E_i/E}\, x) = 0$$

because k_1 injects into E_i. But the dimension $[E_i : E]$ is not divisible by p, so $a(x)$ is zero in $M(E, C)$.

If k_1/k_0 is purely inseparable, then there is a compositum E_1 of E and k_1 such that the dimension of E_1/E is a power of the characteristic, which (by global hypothesis) is not p. As in the previous paragraph, $a(x)$ is zero in $M(E, C)$.

In the general case, let k_s be the separable closure of k_0 in k_1. The map displayed in the lemma is the composition

$$\mathrm{Inv}_{k_0}(G, C) \to \mathrm{Inv}_{k_s}(G, C) \to \mathrm{Inv}_{k_1}(G, C),$$

and both arrows are injective by the preceding two paragraphs. Hence the composition is injective. □

3.3. Suppose that k_1/k_0 is finite of dimension relatively prime to the exponent of C as in 3.2, and suppose further that k_1/k_0 is quasi-Galois (= normal), i.e., k_1 is the splitting field for a collection of polynomials in $k_0[x]$. The separable closure k_s of k_0 in k_1 is a Galois extension of k_0. (See [Bou Alg, §V.11, Prop. 13] for the general structure of k_1/k_0.) We write $\mathrm{Gal}(k_1/k_0)$ for the group of k_0-automorphisms of k_1.

The group $\mathrm{Gal}(k_1/k_0)$ acts on $H^1(k_1, G)$ as follows. An element $g \in \mathrm{Gal}(k_1/k_0)$ sends a 1-cocycle b to a 1-cocycle $g * b$ defined by

$$(g * b)_s = {}^g b_{g^{-1}s}.$$

The Galois group acts similarly acts on $M(k_1, C)$, see e.g. [Wei69, Cor. 2-3-3].

Lemma. *If k_1/k_0 is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of C, then the restriction map*

$$M(k_0, C) \to M(k_1, C)$$

identifies $M(k_0, C)$ with the subgroup of $M(k_1, C)$ consisting of elements fixed by $\mathrm{Gal}(k_1/k_0)$. □

Proof. Write k_i for the maximal purely inseparable subextension of k_1/k_0; the extension k_1/k_i is Galois. It is standard that the restriction map $M(k_i, C) \to M(k_1, C)$ identifies $M(k_i, C)$ with the $\mathrm{Gal}(k_1/k_i)$-fixed elements of $M(k_1, C)$. To complete the proof, it suffices to note that restriction identifies $\mathrm{Gal}(k_1/k_i)$ with $\mathrm{Gal}(k_1/k_0)$ and $M(k_0, C)$ with $M(k_i, C)$, because k_i/k_0 is purely inseparable. □

3.4. Invariants under quasi-Galois extensions. Continue the assumption that k_1 is a finite quasi-Galois extension of k_0. For every extension E of k_0, there is—up to k_0-isomorphism—a unique compositum E_1 of E and k_1; the field E_1 is quasi-Galois over E and $\mathrm{Gal}(E_1/E)$ is identified with a subgroup of $\mathrm{Gal}(k_1/k_0)$. We say that an invariant $a \in \mathrm{Inv}_{k_1}(G, C)$ is *Galois-fixed* if for every E/k_0, $x \in H^1(E_1, G)$, and $g \in \mathrm{Gal}(E_1/E)$, we have

$$g * a(g^{-1} * x) = a(x) \quad \in M(E_1, C).$$

Proposition. *If k_1/k_0 is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of C, then the restriction map*

$$\mathrm{Inv}_{k_0}(G, C) \to \mathrm{Inv}_{k_1}(G, C)$$

identifies $\mathrm{Inv}_{k_0}(G, C)$ with the subgroup of Galois-fixed invariants in $\mathrm{Inv}_{k_1}(G, C)$.

Proof. The restriction map is an injection by Lemma 3.2.

Fix an invariant $a_1 \in \mathrm{Inv}_{k_1}(G, C)$. If a_1 is the restriction of an invariant defined over k_0, then a commutes with every morphism in $\mathrm{Aut}_{\mathsf{Fields}_{/E}}(E_1)$ for every extension E/k_0, i.e., a_1 is Galois-fixed.

To prove the converse, suppose that a_1 is Galois-fixed. For $x \in H^1(E, G)$ and $g \in \mathrm{Gal}(E_1/E)$, we have

$$g * a_1(\mathrm{res}_{E_1/E}\, x) = a_1(g * \mathrm{res}_{E_1/E}\, x) = a_1(\mathrm{res}_{E_1/E}\, x) \quad \in M(E_1, C)$$

since a_1 is Galois-fixed. Lemma 3.3 gives that $a_1(\mathrm{res}_{E_1/E}\, x)$ is the restriction of a unique element $a_0(x)$ in $M(E, C)$. In this way, we obtain a function $H^1(E, G) \to M(E, C)$. It is an exercise to verify that this defines an invariant $a_0 \colon H^1(*, G) \to M(*, C)$. Clearly, the restriction of a_0 to k_1 is a_1. □

3.5. Continue the assumption that k_1/k_0 is finite quasi-Galois and $[k_1 : k_0]$ is relatively prime to the exponent of C.

We fix a natural number n not divisible by the characteristic of k_0 such that $nC = 0$, and we suppose that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C)$ contains $a_1, a_2, \ldots, a_r$ whose restrictions form an $R_n(k_1)$-basis of $\mathrm{Inv}^{\mathrm{norm}}_{k_1}(G, C)$. We find:

Corollary. *$a_1, a_2, \ldots, a_r$ is an $R_n(k_0)$-basis of* $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C)$.

[Clearly, the corollary also holds if one can replaces $\mathrm{Inv}^{\mathrm{norm}}$ with Inv throughout.]

Proof. Since k_1 is finite quasi-Galois over k_0, restriction identifies $R_n(k_0)$ with the $\mathrm{Gal}(k_1/k_0)$-fixed elements in $R_n(k_1)$ (by Lemma 3.3 with $C = \boldsymbol{\mu}_n$) and the natural map

$$\mathrm{Inv}_{k_0}(G, C) \to \mathrm{Inv}_{k_1}(G, C) \tag{3.6}$$

is an injection by Prop. 3.4.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in R_n(k_0)$ be such that $\sum \lambda_i a_i$ is zero in $\mathrm{Inv}_{k_0}(G, C)$. Every λ_i is killed by k_1, hence λ_i is zero in $R_n(k_0)$ for all i. This proves that the a_i are linearly independent over k_0.

As for spanning, let a be in $\mathrm{Inv}_{k_0}(G, C)$. The restriction of a to k_1 equals $\sum \lambda_i a_i$ for some $\lambda_i \in R_n(k_1)$. But a is fixed by $\mathrm{Gal}(k_1/k_0)$, hence so are the λ_i, i.e., λ_i is the restriction of an element of $R_n(k_0)$ which we may as well denote also by λ_i. Since $a - \sum \lambda_i a_i$ is zero over k_1, it is zero over k_0. This proves that the a_i span over k_0. □

3.7. **Proposition.** *If p is a prime not equal to the characteristic of k, then* $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ *is a free $R_p(k_0)$-module with basis* $\underline{\mathrm{id}}$.

[The reader may wonder why we have switched to describing the normalized invariants, whereas in the proposition above and in S, the full module of invariants was described. The difficulty is that here the invariants are taking values in

$$H^0(*, \mathrm{Hom}(\boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z})) \oplus H^1(*, \mathbb{Z}/p\mathbb{Z}) \oplus H^2(*, \boldsymbol{\mu}_p) \oplus \cdots,$$

and it is not clear how to specify a basis for the constant invariants.]

Proof. If k_0 contains a primitive p-th root of unity, then we may use it to identify $\mathbb{Z}/p\mathbb{Z}$ with $\boldsymbol{\mu}_p$ and apply Prop. 2.1.

For the general case, take k_1 to be the extension obtained by adjoining a primitive p-th root of unity; it is a Galois extension of degree not divisible by p, and the proposition holds for k_1 by the previous paragraph. Cor. 3.5 finishes the proof. □

3.8. Exercise. Extend Prop. 3.7 by describing $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, where n is square-free and not divisible by the characteristic.

3.9. Exercise (mod p Bockstein). Let p be a prime not equal to the characteristic of k_0. The natural exact sequence

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1$$

leads to a connecting homomorphism

$$\delta_k \colon H^1(k, \mathbb{Z}/p\mathbb{Z}) \to H^2(k, \mathbb{Z}/p\mathbb{Z})$$

for each extension k/k_0. This is a normalized invariant of $\mathbb{Z}/p\mathbb{Z}$, and arguments similar to those above show that it is of the form

$$\delta_k(x) = c \cdot x$$

for a uniquely determined $c \in H^1(k_0, \mathbb{Z}/p\mathbb{Z})$. Compute c.

[In case $p = 2$, the answer is well-known to be the class of $-1 \in k_0^\times / k_0^{\times 2}$. In general, c can be expressed in terms of the cyclotomic character $\mathrm{Gal}(k_0) \to \mathbb{Z}_p^\times$ and a homomorphism $\mathbb{Z}_p^\times \to \mathbb{Z}/p\mathbb{Z}$.]

3.10. Exercise. Let k_0 be a field of characteristic zero. What are the mod 2 invariants of the dihedral group G of order 8? That is, what is $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, \mathbb{Z}/2\mathbb{Z})$?

[Note that G is the Weyl group of a root system of type B_2, so one may apply S25.15: an invariant of G is determined by its restriction to the elementary abelian 2-subgroups of G.]

4. Restricting invariants

4.1. Let A and A' be functors $\mathsf{Fields}_{/k_0} \to \mathsf{Sets}$, and fix a morphism $\phi\colon A' \to A$. (For example, a homomorphism of algebraic groups $G' \to G$ induces such a morphism of functors $H^1(*, G') \to H^1(*, G)$.) We are interested in the following condition:

(4.2) For every extension k_1/k_0 and every $x \in A(k_1)$ there is a finite extension k_2 of k_1 such that

1. $\mathrm{res}_{k_2/k_1}(x) \in A(k_2)$ is $\phi(x')$ for some $x' \in A'(k_2)$ and
2. the dimension $[k_2 : k_1]$ is relatively prime to the exponent of C.

(In the case where $[k_2 : k_1]$ can always be chosen to be not divisible by a prime p, we say that ϕ is *surjective at* p.)

Lemma. *If* (4.2) *holds, then the restriction map*

$$\phi^*\colon \mathrm{Inv}_{k_0}(A, C) \to \mathrm{Inv}_{k_0}(A', C)$$

induced by ϕ *is an injection.*

We will strengthen this result in Section 6.

Proof. ϕ^* is a group homomorphism, so it suffices to prove that the kernel of ϕ^* is zero; let a be in the kernel of ϕ^*. Fix an extension k_1 of k_0 and a class $x \in A(k_1)$, and let k_2 be as in (4.2). By the assumption on a, the class $a(x) \in M(k_1, C)$ is killed by k_2. But the map $M(k_1, C) \to M(k_2, C)$ is injective by 4.2.2, so $a(x)$ is zero in $M(k_1, C)$. That is, a is the zero invariant. □

4.3. Killable classes. Suppose that there is a natural number e such that every element of $H^1(k, G)$ is killed by an extension of k of degree dividing e for every extension k of k_0. This happens, for example, when:

1. $G = PGL_e$, a standard result from the theory of central simple algebras
2. G is a finite constant group and $e = |G|$, because every 1-cocycle is a homomorphism $\varphi\colon \mathrm{Gal}(k_1) \to G$ and φ is killed by the extension k_2 of k_1 fixed by $\ker\varphi$. The dimension of k_2 over k_1 equals the size of the image of φ, which divides the order of G. (Compare S15.4.)

Applying Lemma 4.1 with G' the group with one element gives: *If the exponent of* C *is relatively prime to* e*, then* $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C)$ *is zero.*

4.4. **Example.** Suppose that k_0 is algebraically closed of characteristic zero, G is a connected algebraic group, and the exponent of C is relatively prime to the order of the Weyl group of a Levi subgroup of G. *Then* $\mathrm{Inv}(G, C)$ *is zero.* Indeed, the paper [CGR06] gives a finite constant subgroup S of G such that the exponent of C is relatively prime to $|S|$ and the map $H^1(k, S) \to H^1(k, G)$ is surjective for every extension k of k_0. (We remark that the existence of such a subgroup S answers the question implicit in the final paragraph of S22.10.) As a consequence of the surjectivity, the restriction map $\mathrm{Inv}(G, C) \to \mathrm{Inv}(S, C)$ is an injection. Hence $\mathrm{Inv}(G, C)$ is zero by 4.3.

4.5. The previous example gives a "coarse bound" in the case where G is simple. For G simple of type E_8, the order of the Weyl group is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. So—roughly speaking—the previous example shows there are no nonconstant cohomological invariants mod p for $p \neq 2, 3, 5, 7$. Tits [Tit92] showed that every E_8-torsor is split by an extension of degree dividing $2^9 \cdot 3^3 \cdot 5$, hence by 4.3 there are no nonconstant invariants mod p also for $p = 7$.

In Table 4, for each type of exceptional group G and prime p, we give a reference for classification results regarding the invariants $\mathrm{Inv}^{\mathrm{norm}}(G, C)$ where the exponent of C is a power of p. The preceding argument shows that $\mathrm{Inv}^{\mathrm{norm}}(G, C)$ is zero for all exceptional G and $p \neq 2, 3, 5$.

type of G	$p = 2$	$p = 3$	$p = 5$
G_2	S18.4	X	X
F_4	S22.5	Th. 7.4	X
inner type E_6	Exercise 22.9 in S	Th. 10.9	X
outer type E_6	?	Exercise 10.10	X
E_7	?	12.2, Exercise 12.3	X
E_8	?	?	Th. 14.1

Table 4. References for results on $\mathrm{Inv}^{\mathrm{norm}}(G, C)$ where G is exceptional and the exponent of C is a power of a prime p

For entries marked with an X, $\mathrm{Inv}^{\mathrm{norm}}(G, C)$ is zero. For conjectures regarding the question marks, see Problems 12.4 and 14.3.

4.6. **Example** (Groups of square-free order). Suppose now that k_0 is algebraically closed, G is a finite constant group, and $|G|$ is square-free and not divisible by the characteristic of k_0. Then *every normalized invariant* $H^1(*, G) \to H^d(*, C)$ *is zero for* $d > 1$. Indeed, by 3.1 and 4.3, we may assume that C is a power of prime p dividing $|G|$. For G' a p-Sylow subgroup of G, S15.4 (a more powerful version of Lemma 4.1 that is specifically for finite groups) says the restriction map

$$\mathrm{Inv}^{\mathrm{norm}}(G, C) \to \mathrm{Inv}^{\mathrm{norm}}(G', C)$$

is injective. As G' is isomorphic to $\boldsymbol{\mu}_p$, Exercise 2.7 says that every normalized invariant $H^1(*, \boldsymbol{\mu}_p) \to H^d(*, C)$ can be written uniquely as $\underline{\mathrm{id}} \bullet x$ for some p-torsion element $x \in H^{d-1}(k_0, C(-1))$. But this last set is zero because k_0 is algebraically closed.

4.7. Ineffective bounds for essential dimension. Recall from S5.7 that the *essential dimension* of an algebraic group G over k_0—written $\mathrm{ed}(G)$—is the minimal transcendence degree of K/k_0, where K is the field of definition of a versal G-torsor. Cohomological invariants can be used to prove lower bounds on $\mathrm{ed}(G)$: *If* k_0 *is algebraically closed and there is a nonzero invariant* $H^1(*, G) \to H^d(*, C)$, *then* $\mathrm{ed}(G) \geq d$, see S12.4.

But this bound need not be sharp, as Example 4.6 shows. Indeed, in that example we find the lower bound $\mathrm{ed}(G) \geq 1$ for G not the trivial group. But when k_0 has characteristic zero and G is neither cyclic nor dihedral of order $2 \cdot (\text{odd})$, $\mathrm{ed}(G) \geq 2$ by [BR97, Th. 6.2].

Another example is furnished by the alternating group A_6. The bound provided by cohomological invariants is $\mathrm{ed}(A_6) \geq 2$ but the essential dimension cannot be 2 (Serre, unpublished).

5. Mod p invariants of PGL_p

In this section, we fix a prime p not equal to the characteristic of k_0. Our goal is to determine the invariants of PGL_p with values in $\mathbb{Z}/p\mathbb{Z}$.

The short exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_n \longrightarrow SL_n \longrightarrow PGL_n \longrightarrow 1$$

gives a connecting homomorphism $\delta : H^1(k, PGL_n) \to H^2(k, \boldsymbol{\mu}_n)$, cf. [Ser02, Ch. III], [KMRT98, p. 386], or Example 1.2.2. We remark that δ has kernel zero because $H^1(k, SL_n)$ is zero.

5.1. **Proposition.** $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(PGL_p, \mathbb{Z}/p\mathbb{Z})$ *is a free* $R_p(k_0)$*-module with basis* δ*.*

The proposition will be proved at the end of this section. The reader is invited to compare this result with the examples of invariants of PGL_4 given in Example 1.2.5.

5.2. Cyclic algebras of degree n. Let n be a natural number not divisible by $\operatorname{char} k_0$ and fix a primitive n-th root of unity ζ in some separable closure of k_0. Let e_i denote the i-th standard basis vector of k_0^n. Define $u, v \in GL_n$ to be the matrices such that

$$u(e_i) = \zeta^i e_i \text{ and } v(e_i) = \begin{cases} e_{i+1} & \text{for } 1 \le i < n \\ e_1 & \text{for } i = n. \end{cases}$$

The maps $\mathbb{Z}/n\mathbb{Z} \to GL_n$ and $\boldsymbol{\mu}_n \to GL_n$ given by $i \mapsto v^i$ and $j \mapsto u^j$ are defined over k_0. Since $uv = \zeta vu$, there is a map

$$c\colon \mathbb{Z}/n\mathbb{Z} \times \boldsymbol{\mu}_n \to PGL_n$$

defined over k_0 given by $(i, \zeta^j) \mapsto \overline{v}^i \overline{u}^j$ for $\overline{u}$, $\overline{v}$ the images of u, v in PGL_n.

The sets $H^1(k, \mathbb{Z}/n\mathbb{Z})$ and $H^1(k, PGL_n)$ classify cyclic extensions k' of k and central simple k-algebras of degree n respectively. Recall that $H^1(k, \boldsymbol{\mu}_n) = k^\times / k^{\times n}$. *The map*

$$c_*\colon H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \boldsymbol{\mu}_n) \to H^1(k, PGL_n) \tag{5.3}$$

sends the cyclic extension k' *and* $\alpha \in k^\times / k^{\times n}$ *to the class of the cyclic algebra* (k', α), see Exercise 5.4 below.

5.4. Exercise. The cyclic algebra (k', α) is defined to be the k-algebra generated by k' and an element z such that $z\ell = \rho(\ell)z$ for all $\ell \in k'$ and ρ a fixed generator of $\mathrm{Gal}(k'/k)$. Justify the italicized claim in 5.2.

[One possible solution: Fix a separable closure k_{sep} of k. The image of k' and α under c define a 1-cocycle in $H^1(k, PGL_n)$, which defines a twisted Galois action on $M_n(k_{\mathrm{sep}})$. A 1-cocycle determining k' also determines a preferred generator of $\mathrm{Gal}(k'/k)$; fix an element $\rho \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$ which restricts to this preferred generator. Prove that the map $f : k' \to M_n(k_{\mathrm{sep}})$ given by $f(\beta)e_i = \rho^i(\beta)e_i$ is defined over k. Fix an n-th root a of α. Prove that the element $z = av^{-1}$ in $M_n(k_{\mathrm{sep}})$ is F-defined. Conclude that the fixed subalgebra of $M_n(k_{\mathrm{sep}})$ is isomorphic to (k', α).]

5.5. *Remark.* The composition δc_* is a map

$$H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \boldsymbol{\mu}_n) \to H^2(k, \boldsymbol{\mu}_n).$$

There is another such map given by the cup product; they are related by

$$\delta c_*(k', \alpha) = -(k') \cdot (\alpha),$$

see [KMRT98, pp. 397, 415].

5.6. **Lemma.** *If A is a central simple algebra over k of dimension p^2, there is a finite extension k'/k, of degree prime to p, over which A becomes cyclic.*

That is, the map (5.3) is surjective at p.

Proof. This is well known. Recall the proof. We may assume that A is a division algebra, in which case it contains a field L that is a separable extension of k of degree p. Let E be the smallest Galois extension of k containing L (in some algebraic closure of k); the Galois group Γ of E/k is a transitive subgroup of the symmetric group S_p; a p-Sylow subgroup S of Γ is thus cyclic of order p. Take for k' the subfield of E fixed by S. We have $E = Lk'$. Hence E is a cyclic extension of k' of degree p which splits A over k'. □

5.7. Invariants of a product. Suppose we have algebraic groups G and G' such that

(5.8) There is a set $\{a_i\} \subset \mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C)$ that is an $R_n(k)$-basis of $\mathrm{Inv}^{\mathrm{norm}}_{k}(G, C)$ for every extension k/k_0, and

(5.9) There is an $R_n(k_0)$-basis $\{b_j\}$ of $\mathrm{Inv}_{k_0}(G', \boldsymbol{\mu}_n)$,

where n is the exponent of C. The cup product

$$H^{d_1}(*, C(d_1 - 1)) \times H^{d_2}(*, \mathbb{Z}/n\mathbb{Z}(d_2)) \to H^{d_1+d_2}(*, C(d_1 + d_2 - 1))$$

induces an $R_n(k_0)$-module homomorphism

$$\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C) \otimes_{R_n(k_0)} \mathrm{Inv}_{k_0}(G', \boldsymbol{\mu}_n) \to \mathrm{Inv}^{\mathrm{norm}}_{k_0}(G \times G', C). \tag{5.10}$$

Lemma. *The map* (5.10) *is injective. Its image I is the set of normalized invariants whose restriction to $H^1(*, G')$ is zero. The images of the $a_i \otimes b_j$ form a basis for I as an $R_n(k_0)$-module.*

This is a slight variation of Exercise 16.5 in S. We give a proof because we will use this result repeatedly later.

Proof. Let c be a normalized invariant of $G \times G'$ with values in C that vanishes on $H^1(*, G')$. For a given k-G'-torsor T', the map

$$c_{T'} : T \mapsto c(T \times T')$$

is an invariant of G with values in C. As c vanishes on $H^1(*, G')$, $c_{T'}$ is normalized. By (5.8), $c_{T'}$ is the map $T \mapsto \sum_i \lambda_{i,T'} a_i(T)$ for uniquely determined $\lambda_{i,T'} \in R_n(k)$. The maps $T' \mapsto \lambda_{i,T'}$ are invariants of G' and belong to $\mathrm{Inv}_{k_0}(G', \boldsymbol{\mu}_n)$, which by (5.9) can be written uniquely as $\sum \lambda_{i,j} b_j$ for $\lambda_{i,j} \in R_n(k_0)$. This proves that c is the image of $\sum \lambda_{i,j} a_i \otimes b_j$, hence that the image of (5.10) includes every normalized invariant whose restriction to $H^1(*, G')$ is zero. As the reverse inclusion is trivial, we have proved the second sentence in the lemma.

The proof of the first sentence is similar. Suppose that the invariant

$$T \times T' \mapsto \sum_{i,j} \lambda_{i,j} \cdot b_j(T') \cdot a_i(T)$$

of $G \times G'$ is zero, where the $\lambda_{i,j}$ are in $R_n(k_0)$. For each k-G'-torsor T', we find that $\sum_j \lambda_{i,j} \cdot b_j(T')$ is zero by (5.8), hence the invariant $\sum_j \lambda_{i,j} \cdot b_j$ is zero. By (5.9), $\lambda_{i,j}$ is zero for all i, j.

Because $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G, C)$ and $\mathrm{Inv}_{k_0}(G', \boldsymbol{\mu}_n)$ are free $R_n(k_0)$-modules, the third sentence in the lemma follows from the first two. ☐

In the examples below, the set $\{b_j\}$ is a basis of $\mathrm{Inv}_k(G', \boldsymbol{\mu}_n)$ for every extension k/k_0. This implies that the lemma holds when k_0 is replaced with k.

We can now prove Prop. 5.1.

Proof of Prop. 5.1. Combining Lemmas 5.6 and 4.1, we find that the map

$$c^* \colon \mathrm{Inv}^{\mathrm{norm}}_{k_0}(PGL_p, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathbb{Z}/p\mathbb{Z} \times \boldsymbol{\mu}_p, \mathbb{Z}/p\mathbb{Z}) \tag{5.11}$$

induced by c is an injection.

It follows from Remark 5.5 or [KMRT98, 30.6] that $c_*(x, 1)$ and $c_*(1, y)$ are the neutral class in $H^1(k, PGL_p)$ for every extension k/k_0, every $x \in H^1(k, \mathbb{Z}/p\mathbb{Z})$, and every $y \in H^1(k, \boldsymbol{\mu}_p)$. In particular the image of (5.11) is contained in the submodule I of invariants that are zero on $H^1(*, \boldsymbol{\mu}_p)$.

Fix a normalized invariant a in $\mathrm{Inv}_{k_0}(PGL_p, \mathbb{Z}/p\mathbb{Z})$. By Lemma 5.7, Prop. 3.7, and Prop. 2.1, its image c^*a under (5.11) is of the form

$$(x, y) \mapsto \lambda_1 \cdot x + \lambda_2 \cdot y \cdot x \qquad (x \in H^1(k, \mathbb{Z}/p\mathbb{Z}), y \in H^1(k, \boldsymbol{\mu}_p))$$

for uniquely determined $\lambda_1, \lambda_2 \in R_p(k_0)$. But

$$(c^*a)(x, 1) = a(c_*(x, 1)) = a(M_p(k)) = 0$$

for every $x \in H^1(k, \mathbb{Z}/p\mathbb{Z})$ and every extension k, so λ_1 is zero. Therefore,

$$(c^*a)(x, y) = \lambda_2 \cdot y \cdot x.$$

Since c^*a is a $R_p(k_0)$-multiple of $c^*\delta$, we conclude that δ spans $\mathrm{Inv}^{\mathrm{norm}}(PGL_p, \mathbb{Z}/p\mathbb{Z})$. ☐

5.12. *Remark.* A versal torsor for $\mathbb{Z}/p\mathbb{Z} \times \boldsymbol{\mu}_p$ gives a PGL_p-torsor T. The injectivity of (5.11) combined with S12.3 shows that invariants a, a' of PGL_p that agree on T are the same. One may view T as a "p-versal torsor" (appropriately defined) for PGL_p.

5.13. **Open problem.** (Reichstein-Youssin [RY00, p. 1047]) Let k_0 be an algebraically closed field of characteristic zero. Is there a nonzero invariant $H^1(*, PGL_{p^r}) \to H^{2r}(*, \mathbb{Z}/p\mathbb{Z})$?

[For $p = r = 2$, one has the Rost-Serre-Tignol invariant described in Example 1.2.5.]

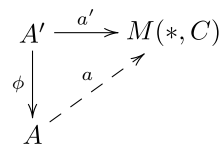
5.14. **Question.** Let k_0 be an algebraically closed field of characteristic zero. What are the mod 2 invariants of PGL_4? That is, what is $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(PGL_4, \mathbb{Z}/2\mathbb{Z})$?

[This is a "question" and not an "exercise" because there are central simple algebras of dimension 4^2 that are neither cyclic nor tensor products of two quaternion algebras [Alb33].]

6. Extending invariants

6.1. Fix functors A and A' mapping $\mathsf{Fields}_{/k_0} \to \mathsf{Sets}$ and a morphism $\phi \colon A' \to A$. *When can an invariant $a' : A' \to M(*, C)$ be extended to an invariant $a : A \to$*

$M(*, C)$? That is, when is there an invariant a that makes the diagram



commute?

Clearly, we must have

(6.2) $$\text{For every extension } k/k_0 \text{ and every } x, y \in A'(k): \quad \phi(x) = \phi(y) \implies a'(x) = a'(y)$$

Proposition. *If ϕ satisfies* (4.2)*, then the restriction*

$$\phi^* : \mathrm{Inv}_{k_0}(A, C) \to \mathrm{Inv}_{k_0}(A', C)$$

defines an isomorphism of $\mathrm{Inv}_{k_0}(A, C)$ with the invariants a' of A' satisfying (6.2)*.*

That is, assuming (4.2), condition (6.2) is sufficient as well as necessary.

Note that the proposition gives a solution to Exercise 22.9 in S as a corollary. That is, if ϕ satisfies (4.2) and ϕ is injective, then *the restriction map*

$$\phi^* : \mathrm{Inv}_{k_0}(A, C) \to \mathrm{Inv}_{k_0}(A', C)$$

is an isomorphism.

The rest of this section is a proof of the proposition. The homomorphism ϕ^* is injective by Lemma 4.1, so it suffices to prove that every invariant a' of A' satisfying (6.2) is in the image. As in 3.1, we may assume that the exponent of C is the power of a prime p.

6.3. For each *perfect* field k/k_0 and each $x \in A(k)$, we define an element $a(x) \in M(k, C)$ as follows. Fix an extension k_2 of k as in (4.2), i.e., such that there is an $x' \in A'(k_2)$ such that $\phi(x')$ is the restriction of x.

Lemma A. *$a'(x')$ is the restriction of a unique element of $M(k, C)$.*

We define $a(x)$ to be the unique element of $M(k, C)$ such that $\mathrm{res}_{k_2/k} a(x)$ is $a'(x')$. For the proof of this lemma and Lemma B below, we fix a separable closure k_{sep} of k_2 (hence also of k).

Proof. Uniqueness is easy, so we prove that $a'(x')$ is defined over k.

For each finite extension k_3 of k_2 in k_{sep} and every $\sigma \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$ such that $\sigma(k_3) \supseteq k_2$, we claim that

(6.4) $$\sigma_* \, \mathrm{res}_{k_3/k_2} \, a'(x') = \mathrm{res}_{\sigma(k_3)/k_2}(a'(x'))$$

in $M(\sigma(k_3), C)$, i.e., that $a'(x')$ is "stable" in $M(k_2, C)$. The invariant a' commutes with σ_* and res. By (6.2), Equation (6.4) is equivalent to

$$\phi(\sigma_* \, \mathrm{res}_{k_3/k_2} \, x') = \phi(\mathrm{res}_{\sigma(k_3)/k_2} \, x').$$

The morphism ϕ also commutes with σ_* and res, so this equation is equivalent to

$$\sigma_* \, \mathrm{res}_{k_3/k_2} \, x = \mathrm{res}_{\sigma(k_3)/k_2} \, x,$$

which holds because x is defined over k. This proves (6.4).

Combining (6.4) with the double coset formula for the composition $\mathrm{res} \circ \mathrm{cor}$ as in [AM04, Th. II.6.6] shows that $a'(x')$ is the restriction of an element of $M(k, C)$. □

Lemma B. *The element $a(x) \in M(k, C)$ depends only on x* (and not on the choice of k_2 and x').

Proof. Let ℓ_2 be a finite extension of k in k_{sep} such that $\mathrm{res}_{\ell_2/k}\, x$ is the image of some $y' \in A'(\ell_2)$ and the prime p does not divide $[\ell_2 : k]$. (I.e., ℓ_2 is an extension as provided by (4.2), and it is separable because k is perfect.) We prove that $a'(x') \in M(k_2, C)$ and $a'(y') \in M(\ell_2, C)$ are restrictions of the same element in $M(k, C)$.

Case 1: ℓ_2 is a conjugate of k_2. Suppose that there is a $\sigma \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$ such that $\sigma(\ell_2)$ equals k_2. One quickly checks that $\phi(\sigma_* y')$ equals $\phi(x')$ in $A(k_2)$, hence $a'(x')$ equals $a'(\sigma_* y')$ by (6.2), i.e., $a'(x')$ is $\sigma_* a'(y')$. The lemma follows in this special case.

Case 2. Suppose that the compositum K of k_2 and ℓ_2 in k_{sep} has degree $[K : k]$ not divisible by the prime p. Since $\phi(\mathrm{res}_{K/k_2}\, x')$ and $\phi(\mathrm{res}_{K/\ell_2}\, y')$ equal $\mathrm{res}_{K/k}(x)$, the restriction of $a'(x')$ and $a'(y')$ in K agree. By the hypothesis on the degree $[K : k]$, the lemma holds in this special case.

Case 3: general case. Let S be a p-Sylow in $\mathrm{Gal}(k_{\mathrm{sep}}/k)$ fixing k_2 elementwise. There is a $\sigma \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$ such that $\sigma(\ell_2)$ is also fixed elementwise by S. It follows that the compositum of $\sigma(\ell_2)$ and k_2 has degree over K not divisible by p. A combination of cases 1 and 2 gives the lemma in the general case. □

6.5. For an arbitrary extension k of k_0, write k_p for the "perfect closure" of k. Since $M(k, C)$ is canonically isomorphic to $M(k_p, C)$, we define $a(x)$ to be the element $a(\mathrm{res}_{k_p/k}\, x) \in M(k_p, C)$ defined in 6.3 above.

For every extension k of k_0, we have defined a function a_k making the diagram

$$\begin{array}{ccc} A'(k) & \xrightarrow{a'_k} & M(k,C) \\ {\scriptstyle \phi_k}\downarrow & \nearrow{\scriptstyle a_k} & \\ A(k) & & \end{array}$$

commute. We leave the proof that this defines a morphism of functors $A \to M(*, C)$ to the reader.

7. Mod 3 invariants of Albert algebras

In this section, we assume that k_0 has characteristic $\neq 2, 3$ and classify the normalized mod 3 invariants of Albert algebras. Recall that Albert k-algebras are 27-dimensional exceptional Jordan algebras—see [SV00, Ch. 5], [PR94a], or [KMRT98, Ch. IX]—and we write Alb for the functor such that $\mathrm{Alb}(K)$ is the isomorphism classes of Albert K-algebras. We compute $\mathrm{Inv}^{\mathrm{norm}}(\mathrm{Alb}, \mathbb{Z}/3\mathbb{Z})$.

The automorphism group of the "split" Albert algebra is a split algebraic group of type F_4, and by Galois descent we have an isomorphism of functors $H^1(*, F_4) \cong \mathrm{Alb}(*)$, see [KMRT98, p. 517]. This isomorphism identifies $\mathrm{Inv}^{\mathrm{norm}}(\mathrm{Alb}, \mathbb{Z}/3\mathbb{Z})$ with $\mathrm{Inv}^{\mathrm{norm}}(F_4, \mathbb{Z}/3\mathbb{Z})$.

7.1. **Example.** Let $M = M_3(k)$ be the algebra of 3-by-3 matrices over k. On the 27-dimensional space $J = M \times M \times M$, define a cubic form N by

$$N(a, b, c) = \det(a) + \det(b) + \det(c) - \mathrm{tr}(abc).$$

Write 1 for the element $(1,0,0)$ in J. The "Springer construction" endows J with the structure of an Albert k-algebra induced by N and the choice of the element 1, see [McC69, §5]. It is the split Albert algebra and its automorphism group F_4 is the subgroup of $GL(J)$ consisting of elements that fix 1 and N [Jac59, Th. 4].

If (g,z) is a point of $PGL_3 \times \boldsymbol{\mu}_3$, let $t(g,z)$ be the element of $GL(J)$ defined by

$$(a,b,c) \mapsto (i_g(a), z\cdot i_g(b), z^2\cdot i_g(c)),$$

where i_g is the inner autormorphism of M defined by g. Since (g,z) fixes both 1 and N, it belongs to the group F_4. This gives an inclusion $t\colon PGL_3 \times \boldsymbol{\mu}_3 \to F_4$ and a corresponding map

$$t_*\colon H^1(*, PGL_3) \times H^1(*, \boldsymbol{\mu}_3) \to H^1(*, F_4) \cong \mathrm{Alb}(*). \tag{7.2}$$

The image of a pair (A,α) is often denoted by $J(A,\alpha)$; such algebras are known as *first Tits constructions*, cf. [KMRT98, §39.A].

Every Albert k-algebra is a first Tits construction or becomes one over a quadratic extension of k—see, e.g., [KMRT98, 39.19]—so the map in (7.2) satisfies (4.2) when C has odd exponent. In particular, it is surjective at 3, hence the restriction map

$$t^*\colon \mathrm{Inv}^{\mathrm{norm}}(F_4, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Inv}^{\mathrm{norm}}(PGL_3 \times \boldsymbol{\mu}_3, \mathbb{Z}/3\mathbb{Z}) \tag{7.3}$$

is injective.

7.4. Invariants of F_4 mod 3. Consider the invariant

$$g_3\colon H^1(*, PGL_3) \times H^1(*, \boldsymbol{\mu}_3) \to H^3(*, \boldsymbol{\mu}_3^{\otimes 2})$$

defined by $g_3(A,\alpha) = \delta(A)\cdot(\alpha)$ for δ as defined in §5. We now give two arguments that g_3 is the restriction of an invariant of F_4.

Proof #1. The meat of [PR96] is their Lemma 4.1, which says that g_3 "factors through" the image of (7.2) in $H^1(*, F_4)$. That is, if the first Tits constructions $J(A,\alpha)$ and $J(A',\alpha')$ are isomorphic, then $g_3(A,\alpha)$ equals $g_3(A',\alpha')$. Prop. 6.1 gives that g_3 extends to an invariant of F_4. □

Proof #2. The Dynkin index of F_4 is 6 [Mer03, 16.9], so the mod 3 portion of the Rost invariant gives a nonzero invariant

$$g_3'\colon H^1(*, F_4) \to H^3(*, \boldsymbol{\mu}_3^{\otimes 2}).$$

Applying Lemma 5.7, we conclude that $t^* g_3'$ equals λg_3 for some fixed $\lambda \in R_3(k_0)$. Since the image of g_3' under $\mathrm{Inv}_{k_0}(G, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Inv}_K(G, \mathbb{Z}/3\mathbb{Z})$ is nonzero for every extension K/k_0, $t^* g_3'$ is nonzero over every K, and we conclude that $\lambda = \pm 1$, i.e., $t^* g_3'$ is $\pm g_3$. □

We abuse notation by writing g_3 also for the invariant $(t^*)^{-1}(g_3)$ of F_4. This invariant was originally constructed in [Ros91].

Proposition. $\mathrm{Inv}_{k_0}^{\mathrm{norm}}(F_4, \mathbb{Z}/3\mathbb{Z})$ *is a free* $R_3(k_0)$*-module with basis* g_3*.*

Proof. We imitate the proof of Prop. 5.1, with the role of $\mathbb{Z}/p\mathbb{Z} \times \boldsymbol{\mu}_p$ played by $PGL_3 \times \boldsymbol{\mu}_3$. For every central simple algebra A over every extension k/k_0 and every $\alpha \in k^\times$, the algebra $J(A,\alpha)$ is "split", i.e., $t_*(A,\alpha)$ is the neutral class in $H^1(k, F_4)$, if and only if α is the reduced norm of an element of $A^\times$ by [Jac68, p. 416, Th. 20] or [McC69, Th. 6]. In particular, $t_*(M_3(k),\alpha)$ is the neutral class for every α, and Lemma 5.7 gives that the restriction of a normalized invariant in $\mathrm{Inv}_{k_0}(F_4, \mathbb{Z}/3\mathbb{Z})$ to $PGL_3 \times \boldsymbol{\mu}_3$ can be written as

$$(A, \alpha) \mapsto \lambda_1 \cdot [A] + \lambda_2 \cdot (\alpha) \cdot [A]$$

for uniquely determined $\lambda_1, \lambda_2 \in R_3(k_0)$. But the algebra $J(A, 1)$ is also split for every A. It follows that λ_1 is zero. This proves that g_3 spans $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(F_4, \mathbb{Z}/3\mathbb{Z})$. □

Combining the proposition with the classification of the invariants mod 2 in S22.5, we have found just three interesting invariants of F_4, namely g_3, f_3, and f_5. Perhaps the outstanding open problem in the theory of Albert algebras is:

7.5. **Open problem.** (Serre [Ser95, §9.4], [PR94a, Q. 1, p. 205]) Is the map

$$g_3 \times f_3 \times f_5 \colon H^1(*, F_4) \to H^3(*, \mathbb{Z}/3\mathbb{Z}) \times H^3(*, \mathbb{Z}/2\mathbb{Z}) \times H^5(*, \mathbb{Z}/2\mathbb{Z})$$

injective? That is, is an Albert algebra J determined up to isomorphism by its invariants $g_3(J)$, $f_3(J)$, and $f_5(J)$?

[The map is injective on the kernel of g_3 [SV00, 5.8.1], i.e., "reduced Albert algebras are classified by their trace form". Also, Rost has an unpublished result on this problem, see [Ros02]. Note that it is still unknown if the map is injective on the kernel of $f_3 \times f_5$, i.e., for first Tits constructions.]

7.6. Symbols. We now drop the assumption that the characteristic of k_0 is $\neq 2, 3$, and instead assume that it does not divide some fixed natural number n. We call an element $x \in H^d(k, \boldsymbol{\mu}_n^{\otimes(d-1)}) = M^d(k, \mathbb{Z}/n\mathbb{Z})$ (for $d \geq 2$) a *symbol* if it is in the image of the cup product map

$$H^1(k, \mathbb{Z}/n\mathbb{Z}) \times \underbrace{H^1(k, \boldsymbol{\mu}_n) \times \cdots \times H^1(k, \boldsymbol{\mu}_n)}_{d-1 \text{ copies}} \to H^d(k, \boldsymbol{\mu}_n^{\otimes(d-1)}).$$

In particular, the zero class is always a symbol. In the usual identification of $H^2(k, \boldsymbol{\mu}_n)$ with the n-torsion in the Brauer group of k, symbols are identified with cyclic algebras of dimension n^2 as defined in §5.

7.7. **Example.** In the case $n = 2$, $M^d(k, \mathbb{Z}/2\mathbb{Z})$ is just $H^d(k, \mathbb{Z}/2\mathbb{Z})$, and it is isomorphic to I^d/I^{d-1} as in 1.2.3. Symbols in $M^d(k, \mathbb{Z}/2\mathbb{Z})$ correspond to the (equivalence classes of) d-Pfister quadratic forms. Further, one has the following nice property: If there is an odd-degree extension K/k such that $\mathrm{res}_{K/k}(x)$ is a symbol in $H^d(k, \mathbb{Z}/2\mathbb{Z})$, then x is itself a symbol by [Ros99a, Prop. 2].

In the case $n = 3$ (and $\mathrm{char}\, k_0 \neq 3$), we have the following weaker property, mentioned in [Ros99a]:

7.8. **Lemma.** *Fix $x \in H^2(k, \boldsymbol{\mu}_3)$. If there is an extension K/k such that 3 does not divide the dimension $[K : k]$ and $\mathrm{res}_{K/k}(x)$ is a symbol in $H^2(K, \boldsymbol{\mu}_3)$, then x is itself a symbol.*

Proof. We identify $H^2(k, \boldsymbol{\mu}_3)$ and $H^2(K, \boldsymbol{\mu}_3)$ with the 3-torsion in the Brauer group of k and K respectively. We assume that x is nonzero, hence that it corresponds to a central division k-algebra A of dimension $(3^r)^2$ for some positive r. By hypothesis, $A \otimes K$ is isomorphic to $M_r(B)$ for a cyclic K-algebra B of dimension 3^2. But as 3 does not divide $[K : k]$, the index of A and $A \otimes K$ agree [Dra83, §9, Th. 12]. It follows that A is a division algebra of dimension 3^2 over k, hence by Wedderburn's Theorem [KMRT98, 19.2] A is cyclic, i.e., x is a symbol. □

Returning to groups of type F_4, the image of the invariant

$$g_3 \colon H^1(k, F_4) \to H^3(k, \boldsymbol{\mu}_3^{\otimes 2})$$

consists of symbols by [Tha99, p. 303]. For an alternative proof, combine [KMRT98, 40.9] with Lemma 7.8.

Part II. Surjectivities and invariants of E_6, E_7, and E_8

8. Surjectivities: internal Chevalley modules

Consider the following well-known example.

8.1. **Example.** Let q be a nondegenerate quadratic form on a vector space V over a field k of characteristic $\neq 2$. Fix an anisotropic vector $v \in V$. Over a separable closure k_{sep} of k, the orthogonal group $\mathrm{O}(q)(k_{\mathrm{sep}})$ acts transitively on the open subset of $\mathbb{P}(V)$ consisting of anisotropic vectors by Witt's Extension Theorem. The stabilizer of an anisotropic line $[v]$ in $\mathrm{O}(q)$ is isomorphic to $\boldsymbol{\mu}_2 \times \mathrm{O}(v^\perp)$. It follows from [Ser02, §I.5.5, Prop. 37] that the natural map

$$H^1(k, \boldsymbol{\mu}_2 \times \mathrm{O}(v^\perp)) \to H^1(k, \mathrm{O}(q)) \tag{8.2}$$

is surjective. Repeating this procedure, we find a surjection

$$\bigoplus^{\dim V} H^1(k, \boldsymbol{\mu}_2) \to H^1(k, \mathrm{O}(q)).$$

Since $H^1(k, \boldsymbol{\mu}_2)$ is the same as $k^\times / k^{\times 2}$, this surjection can be viewed as reflecting the fact that quadratic forms can be diagonalized.

8.3. Let G be an algebraic group over k. Roughly speaking, we now abstract the example by finding a subgroup N of G such that the natural map $H^1(*, N) \to H^1(*, G)$ is surjective. We suppose that k is infinite[e] and that G has a representation V such that there is an open G-orbit in $\mathbb{P}(V)$ over an algebraic closure of k. As k is infinite, there is a k-point $[v]$ in the open orbit.

Theorem. *The natural map*

$$H^1_{\mathrm{fppf}}(k, N) \to H^1(k, G)$$

is surjective, where N is the scheme-theoretic stabilizer of $[v]$ in G.

We write $H^1_{\mathrm{fppf}}(k, N)$ for the pointed set of k-N-torsors relative to the fppf topology as in [DG70]. When N is smooth, this group agrees with the usual Galois cohomology set $H^1(k, N)$ [DG70, p. 406, III.5.3.6], so the reader who wishes to avoid flat cohomology may simply add hypotheses that various groups are smooth or—more restrictively—only consider fields of characteristic zero.

In the case where N is smooth, a concrete proof of the theorem can be found in [Gar01a, 3.1] or by applying [Ser02, §III.2.1, Exercise 2] with B, C, D replaced by $G, N, GL(V)$.

Proof. Write $\mathcal{O}$ for the G-orbit of $[v]$ in $\mathbb{P}(V)$ (equivalently, G/N). For $z \in H^1(k, G)$, there is an inclusion of twisted objects $\mathcal{O}_z \to \mathbb{P}(V)_z$. As G acts on $\mathbb{P}(V)$ through $GL(V)$, the twisted variety $\mathbb{P}(V)_z$ is isomorphic to $\mathbb{P}(V)$ and the k-points are dense in $\mathbb{P}(V)_z$ (because k is infinite). Moreover, $\mathcal{O}_z$ is open in $\mathbb{P}(V)_z$ because $\mathcal{O}$ is open in $\mathbb{P}(V)$. Hence $\mathcal{O}_z$ has a k-point and the map $H^1_{\mathrm{fppf}}(k, N) \to H^1(k, G)$ is surjective [DG70, p. 373, Prop. III.4.4.6b]. □

[e] This hypothesis is harmless. In the examples, G will be connected, so $H^1(k, G)$ will be zero when k is finite.

8.4. **Example** (char $k = 0$). Let G be a semisimple group. *The adjoint representation V of G has an open orbit in $\mathbb{P}(V)$ if and only if G has absolute rank 1*, i.e., G is of type A_1. Indeed, if G is of rank 1, then the regular semisimple elements in V form an open orbit in $\mathbb{P}(V)$, because G acts transitively on the collection of maximal toral subalgebras of V [Hum80, 16.4]. Conversely, if there is an open G-orbit in $\mathbb{P}(V)$, it contains a regular semisimple element v. The stabilizer of $[v]$ normalizes the centralizer of v in the Lie algebra, i.e., normalizes a maximal toral subalgebra $\mathfrak{t}$ of V containing v. Hence N normalizes the maximal torus T of G with Lie algebra $\mathfrak{t}$. As T fixes v, we have:

$$\operatorname{rank} G = \dim N = \dim G - \dim \mathbb{P}(V) = 1.$$

8.5. A non-example is furnished by a representation V of G on which G acts trivially. If $\dim V = 1$, then $\mathbb{P}(V)$ is a point, N equals G, and the conclusion of Th. 8.3 is uninteresting. If $\dim V$ is at least 2, then $\mathbb{P}(V)$ does not have an open orbit (exercise).

8.6. **Example** (Reducible representations). Let V be a representation of G as in 8.3, and suppose that there is a proper G-invariant subspace W of V. The quotient map $V \to V/W$ gives a G-equivariant rational surjection $f\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/W)$. If $[v]$ is in the open G-orbit in $\mathbb{P}(V)$, then f is defined at $[v]$ and the orbit of $f([v])$ is dense in $\mathbb{P}(V/W)$, hence open.

8.7. **Example** (char $k = 0$). A group G of type E_8 has no nontrivial representations V with an open G-orbit in $\mathbb{P}(V)$. Indeed, by Example 8.6, it suffices to prove that no faithful *irreducible* representation V of G has an open orbit in $\mathbb{P}(V)$. By the following exercise, the constraint $\dim G \geq \mathbb{P}(V)$ leaves the adjoint representation as the only possibility, and there is no open G-orbit in that case by Example 8.4.

8.8. Exercise (char $k = 0$). Check that the split group of type E_8 has unique irreducible representations of dimensions 1, 248 (adjoint), 3875, 27000, and 30380, and no others of dimension $< 10^5$.

[Compare [Gar].]

8.9. Internal Chevalley modules. How to find groups and representations that satisfy the hypotheses of Theorem 8.3? We now give a mechanism from representation theory that produces such.

Let $\widetilde{G}$ be a semisimple algebraic group that is defined and isotropic over k. We fix a maximal k-torus $\widetilde{T}$ in $\widetilde{G}$ that contains a maximal k-split torus $\widetilde{T}_d$. Fix also a set $\widetilde{\Delta}$ of simple roots of $\widetilde{G}$ with respect to $\widetilde{T}$. We suppose that there is some $\pi \in \widetilde{\Delta}$ that is fixed by the Galois group (under the $*$-action, which permutes $\widetilde{\Delta}$) and is not constant on $\widetilde{T}_d$. (In the notation of Tits's classification paper [Tit66], the vertex π in the Dynkin diagram is circled and the circle does not include any other vertices.) Finally, we assume that k has characteristic $\neq 2$ if $\widetilde{G}$ is of type B, C, or F_4 and $\neq 2, 3$ if $\widetilde{G}$ has type G_2. This concludes our list of assumptions.

We define G to be the semisimple subgroup of $\widetilde{G}$ that is generated over a separable closure k_{sep} of k by the 1-dimensional unipotent subgroups U_α of $\widetilde{G}$ as α varies over the roots of $\widetilde{G}$ with π-coordinate zero. The Dynkin diagram of G is the diagram of $\widetilde{G}$ with the vertex π deleted. If $\widetilde{G}$ is simply connected, then so is G by [SS70, 5.4b]. The reader can find a list of Dynkin diagrams in Table 8 below and

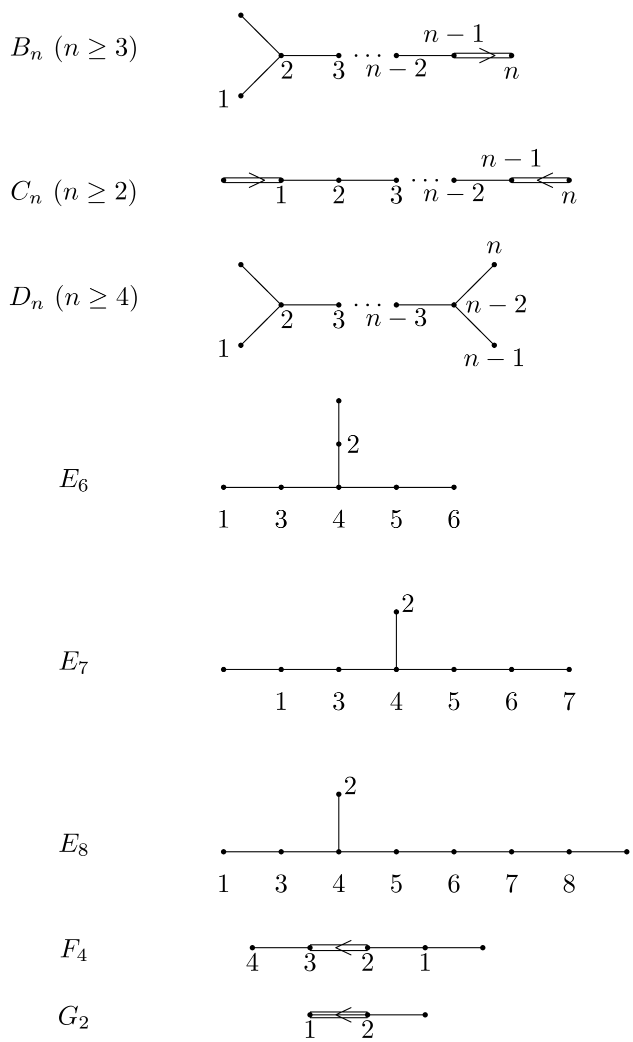


TABLE 8. Extended Dynkin diagrams.

Vertices are numbered as in [Bou Lie]. The unlabeled vertex corresponds to the negative $-\widetilde{\alpha}$ of the highest root, and omitting this vertex leaves the usual Dynkin diagram. Type A is omitted entirely.

a list of concrete examples of groups G that we will consider by looking ahead at Tables 21a or 11 below.

Over k_{sep}, there is a cocharacter $\lambda\colon \mathbb{G}_m \to \widetilde{T}$ such that $\pi(\lambda)$ is negative and $\alpha(\lambda)$ is zero for $\alpha \in \widetilde{\Delta} \setminus \{\pi\}$. The cocharacter λ is even defined over k by [BT65, 6.7, 6.9]; its image is in $\widetilde{T}_d$. We take $\widetilde{P}$ (respectively, L) to be the parabolic subgroup of $\widetilde{G}$ (resp., Levi subgroup of $\widetilde{P}$) picked out by λ in the sense of [Spr98, 13.4.1], i.e., the subgroup generated over k_{sep} by $\widetilde{T}$ and the U_α where α varies over the roots of $\widetilde{G}$ with non-positive π-coordinate (resp., π-coordinate zero). Note that G is the derived subgroup of L.

The Levi subgroup L acts on the unipotent radical Q of $\widetilde{P}$. We fix a positive integer i and write $Q(i)$ for the subgroup of Q spanned by the U_α where the π-coordinate of α is $\leq -i$. We put $V := Q(1)/Q(2)$; it is a representation of L and there is an open L-orbit in V over an algebraic closure of k [ABS90, Th. 2]. The representation V is called an *internal Chevalley module*. It is irreducible with highest weight $-\pi$ [ABS90, Th. 2].

8.10. *Remarks.* (1) The addition on V comes from the multiplication in $\widetilde{G}$. What is the scalar multiplication that turns V into a k-vector space? Suppose that $\widetilde{T}$ is split. Number the roots of $\widetilde{G}$ with π-coordinate -1 arbitrarily as $\rho_1, \rho_2, \ldots, \rho_s$. The product map

$$U_{\rho_1} \times U_{\rho_2} \times \cdots \times U_{\rho_s} \xrightarrow{m} V$$

is an isomorphism by [Bor91, Prop. 14.4(2)]. The group U_α is the image of a homomorphism $x_\alpha \colon \mathbb{G}_a \to \widetilde{G}$ and the scalar multiplication is the naive one: For $\lambda \in k^\times$ and $u_i \in k$, we have

$$\lambda \cdot m\left(\prod x_{\rho_i}(u_i)\right) = m\left(\prod x_{\rho_i}(\lambda u_i)\right),$$

see [ABS90, p. 554].

(2) If, instead of the parabolic $\widetilde{P}$, we chose the "opposite" parabolic, then everything would work out the same except that the highest weight of V would be the highest positive root with π-coordinate 1—something that is more difficult to read off of the Dynkin diagrams. The resulting V would be the L-module that is dual to the one we consider here.

(3) In most of the examples considered below, the vertex π of the Dynkin diagram is adjacent to only one other vertex—call it δ—and the two vertices are joined by a single bond, so the highest weight of V is the fundamental weight corresponding to δ.

(4) Although one could consider the modules $Q(i)/Q(i+1)$ for various i, no real generality is gained, see [Röh93c, 1.8].

(5) Although the root π is fixed by the Galois group under the $*$-action (and the cocharacter λ is k-defined), π need not be fixed by the usual Galois action. Indeed, [Gar98] gives a concrete construction of groups of type ${}^3D_4$ with k-rank 1 where the root $\pi := \alpha_2$ is fixed by the $*$-action (and is non-constant on the split torus), but the usual Galois action interchanges α_2 and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

(6) In case $\widetilde{G}$ is split, there is an open $\widetilde{P}$-orbit in the unipotent radical Q whose elements are known as "Richardson elements". Clearly, any Richardson element with π-coordinate -1 maps to an element of the open L-orbit in V. For $\widetilde{G}$ of classical type, the reader can find concrete examples of Richardson elements in [Bau06].

8.11. We maintain the assumptions from 8.9, and we further assume—as in 8.3—that the field k is infinite. We fix a k-point $[v]$ in the open L-orbit in $\mathbb{P}(V)$.

Theorem. *The natural map*

$$H^1_{\text{fppf}}(k, N) \to H^1(k, G)$$

is surjective, where N is the scheme-theoretic stabilizer of $[v] \in \mathbb{P}(V)$ in G.

Proof. Write $\mathcal{O}$ for the L-orbit of $[v]$ in $\mathbb{P}(V)$. Note that since the L-orbit of v is dense in V, $\mathcal{O}$ is dense in $\mathbb{P}(V)$, hence open in $\mathbb{P}(V)$ because orbits are locally closed.

As V is an irreducible representation of L, the torus S in the center of L acts on V by scalar multiplication. But G and S generate L, so the G- and L-orbits in $\mathbb{P}(V)$ coincide. That is, the G-orbit of $[v] \in \mathbb{P}(V)$ is open. Theorem 8.3 completes the proof. $\square$

Note that Theorems 8.3 and 8.11 give surjections $H^1_{\mathrm{fppf}}(K, N) \to H^1(K, G)$ for every extension K/k, where N is the scheme-theoretic stabilizer of a k-point in the open orbit. We summarize this by saying that the morphism of functors $H^1_{\mathrm{fppf}}(*, N) \to H^1(*, G)$ *is surjective* or that the inclusion $N \subset G$ *induces a surjection on H^1's.*

8.12. **Example** ($F_4 \times \boldsymbol{\mu}_3 \subset E_6$)**.** The natural inclusion of root systems leads to an inclusion of split simply connected groups $E_6 \subset E_7$. We take these groups as G and $\widetilde{G}$ respectively in the notation of 8.11, so that π is the root α_7 of E_7. (We number the simple roots as in Table 8.) The representation V of G is irreducible and 27-dimensional with highest weight ω_6.

There is a split group of type F_4 inside of E_6, and we denote it also by F_4. Writing $x_\alpha \colon \mathbb{G}_a \to E_6$ for the generators of E_6 as in [Ste68], F_4 is generated by the maps

$$x_{\alpha_2}, \quad x_{\alpha_4}, \quad u \mapsto x_{\alpha_3}(u)\, x_{\alpha_5}(u), \quad u \mapsto x_{\alpha_1}(u)\, x_{\alpha_6}(u), \tag{8.13}$$

etc., where the displayed maps correspond to the roots α_1, α_2, α_3, and α_4 respectively in F_4, cf. [Spr98, §10.3]. We claim that N is the direct product of F_4 with the center Z of E_6, which is isomorphic to $\boldsymbol{\mu}_3$.

Restricting the representation V of E_6 to F_4, we find that V is a direct sum of an F_4-invariant line $[v]$ (for some v) and an indecomposable 26-dimensional representation W (which is even irreducible if the characteristic of k_0 is not 3 [GS88, p. 412]). We take v to be a generator of the line.

Note that the maximal proper parabolics of L have Levi subgroups of type

$$D_5, \quad A_1 \times A_4, \quad A_1 \times A_2 \times A_2, \quad A_5,$$

and these semisimple parts all have dimension strictly less than 52, the dimension of F_4. Therefore F_4 is not contained in a proper parabolic subgroup of L. By [Röh93a, Prop. 3.5], it follows that v belongs to the open L-orbit in V.

Clearly, F_4 is contained in the stabilizer N of $[v]$ in E_6, and by dimension count it is the identity component of N. Since Z is also contained in N, it suffices to prove that F_4 and Z generate the normalizer of F_4 in E_6. But every automorphism of F_4 is inner and F_4 has trivial center, so the normalizer of F_4 in E_6 is the product $F_4 \times C$, where C is the centralizer of F_4 in E_6. Therefore it suffices to prove that the center Z is all of C.

Write T_4 for the maximal torus $(F_4 \cap T)^\circ$ of F_4. The centralizer of T_4 in E_6 contains T, is reductive [Bor91, 13.17, Cor. 2a], and is generated by T and the images of the x_γ's, where γ varies over the roots of E_6 whose inner product with $\alpha_2, \alpha_4, \alpha_3 + \alpha_5$, and $\alpha_1 + \alpha_6$ is zero. Such a root γ is a $\mathbb{Q}$-linear combination of the weights

$$\omega_3 - \omega_5 = \frac{1}{3}(\alpha_1 + 2\alpha_3 - 2\alpha_5 - \alpha_6) \quad \text{and} \quad \omega_1 - \omega_6 = \frac{1}{3}(2\alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_6).$$

But such a γ would have disconnected support,[f] which is impossible by [Bou Lie, §VI.1.6, Cor. 3a to Prop. 19]. So the centralizer of T_4 in E_6 is T, and in particular C is contained in T. But C commutes with the images of the maps in (8.13), hence with the image of x_{α_i} for $1 \le i \le 6$. That is, C is contained in the center Z of E_6. This completes the proof that N equals $F_4 \times Z$.

Combining this example with Th. 8.11 gives that every k-E_6-torsor can be written (not necessarily uniquely) as a pair (J, β), where J is an Albert k-algebra and β belongs to $k^\times/k^{\times 3}$. For a classical proof of this in characteristic $\ne 2, 3$, see [Spr62]. For an application, see [GH06, §5] or 10.9 below.

Context. The representations appearing in 8.3 are nearly the same as the prehomogeneous vector spaces appearing in [SK77]. Recall that a *prehomogeneous vector space* is a representation V of an algebraic group G such that there is a G-orbit in V. These too lead to surjections in cohomology, by the same proof as in 8.3. However, we are interested in the case where G is semisimple (and not merely reductive), for which there are not enough prehomogeneous vector spaces.

Continuing the comparison of G-orbits in V and $\mathbb{P}(V)$, we note that in the examples of 8.11 considered below (listed in Table 21a), the G-orbit of v in the affine space V is a hypersurface, more specifically a level set of a homogeneous G-invariant polynomial on V. However, this need not be true, as considering $G = \mathrm{Spin}_{10}$ shows: Viewing G as a subgroup of E_6, the recipe of 8.11 gives that V is a half-spin representation, and the G-orbit of v in that case is dense in V [Igu70, Prop. 2]. (In Example 15.8, we view G as a subgroup of Spin_{12}, the resulting V is the 10-dimensional vector representation, and the G-invariant polynomial on V is the quadratic form.)

In the setup for Th. 8.11, we cited [ABS90] because it is a convenient reference, but the core idea can certainly be found in other, earlier references, e.g., [Vin76].

9. New invariants from homogeneous forms

A *(homogeneous) form of degree* d on a k_0-vector space V is a nonzero element of the d-th symmeric power $S^d(V^*)$. Equivalently, fixing a basis $x_1, x_2, \ldots, x_n$ for the dual space V^*, it is a homogeneous polynomial of degree d in $k_0[x_1, x_2, \ldots, x_n]$. In this section, we give a mechanism for constructing new invariants of a group G from G-invariant forms.

Suppose that V is a representative of an algebraic group G and that V supports a G-invariant form f. Each $y \in H^1(k, G)$ defines a twisted form f_y on $V \otimes k$.

We are concerned with the case where f is a form of degree d such that $dC = 0$. For an invariant $a \in \mathrm{Inv}_{k_0}(G, C)$ and $v \in V \otimes k$ such that $f_y(v)$ is not zero, we consider the element

$$a(y) \cdot (f_y(v)) \quad \in M(k, C). \tag{9.1}$$

(We view $(f_y(v))$ as an element of $H^1(k, \boldsymbol{\mu}_d)$.) The following proposition is adapted from [Ros99c, Prop. 5.2].

---

[f]Recall that every root γ can be written uniquely as an integral linear combination of the simple roots. The *support* of γ is the set of those simple roots whose coefficient is nonzero in this expression.

9.2. **Proposition.** *If $a(y)$ is zero whenever f_y has a nontrivial zero, then the element* (9.1) *depends only on y (and not on the choice of v) and the map $y \mapsto a(y) \cdot (f_y(v))$ defines an invariant of G over k_0.*

Recall that f_y is said to *have a nontrivial zero* if there is some nonzero $v \in V \otimes k$ such that $f_y(v) = 0$. (Obviously, f_y always has the "trivial" zero $f_y(0) = 0$.)

9.3. **Example.** In the "smallest" case, when V is 1-dimensional, we can see the proposition directly. If we fix a dual basis x for V, then f is αx^d for some $\alpha \in k_0^\times$. The action of G on V is given by a homomorphism $\chi \colon G \to \boldsymbol{\mu}_d$ and this defines an invariant $\underline{\chi} \colon H^1(*, G) \to H^1(*, \boldsymbol{\mu}_d)$. For $y \in H^1(k, G)$, f_y is the form $\alpha \chi(y) x^d$, and for nonzero $v \in V \otimes k$, we have $(f_y(v)) = (\alpha) + (\chi(y))$. In particular, (9.1) is the value of the invariant $(\alpha) \cdot a + \underline{\chi} \cdot a$ at y.

Proof of the proposition. By the example, we may assume the dimension of V is at least 2. Fix a basis for V^* as above. Writing f_y (viewed as an element of $k[V]$) in terms of this basis is equivalent to evaluating it at the generic point of V. Put

$$\omega := a(y) \cdot (f_y) \quad \in M(k(V), C).$$

We claim that ω is the restriction of some $\omega_0 \in M(k, C)$. By S10.1, it suffices to check that ω is unramified at every discrete valuation of $k(V)/k$ that corresponds to an irreducible hypersurface in V. Such a hypersurface is defined by some irreducible $\pi \in k[V]$. If π does not divide f_y (i..e, the hypersurface is not a component of the variety $\{f_y = 0\}$), then ω is unramified on the hypersurface by definition. If π does divide f_y in $k[V]$, we write $f_y = \pi^n \varepsilon$ for some ε not divisible by π, so

$$\omega = a(y) \cdot (\varepsilon) + n\, a(y) \cdot (\pi).$$

The residue of the first term is zero and the residue of the second is a multiple of $\operatorname{res}_{k(\pi)/k} a(y)$. The form f_y is zero on the sum of the vectors in the dual basis in $V \otimes k(\pi)$, and this is a nontrivial zero because the dimension of V is not 1. It follows that ω has residue zero. This proves the claim.

Specializing the generic point to $v \in V \otimes k$ maps $f_y \mapsto f_y(v)$ and $\omega \mapsto (a(y)) \cdot (f_y(v))$, but does not change ω_0. This proves that $a(y) \cdot (f_y(v))$ does not depend on the choice of v. The remainder of the proposition is clear. □

9.4. **Example.** The invariants produced by the lemma need not be interesting. In the following examples, we consider the case where f is a quadratic form.

(1) Suppose that a is an invariant as in the lemma. Applying the proposition once produces an invariant a', and this new invariant also satisfies the hypothesis of the proposition. Applying the proposition again, we obtain an invariant

$$a'' \colon y \mapsto a(y) \cdot (f_y(v)) \cdot (f_y(v)) = a'(y) \cdot (-1).$$

That is, $a'' = (-1) \cdot a'$.

(2) Suppose that—in the situation of the proposition—the form f_y is Witt-equivalent to an n-Pfister form for every $y \in H^1(k, G)$ and every k/k_0, and a is the invariant $y \mapsto e_n(f_y)$. (For example, take G to be the split group of type G_2 and a to be the invariant from S18.4, i.e., the Rost invariant.) Then for each y, f_y is Witt-equivalent to some $\otimes_{i=1}^n \langle 1, -\alpha_i \rangle$ and the invariant a' given by the proposition satisfies

$$a'(y) = [(\alpha_1) \cdot (\alpha_2) \cdots (\alpha_n)] \cdot (-\alpha_n) = 0.$$

That is, a' is the zero invariant.

10. Mod 3 invariants of simply connected E_6

In this section we assume that the characteristic of k_0 is different from 2 and 3.

10.1. Invariants of the split E_6. We compute the invariants of the simply connected split group of type E_6, which we denote simply by E_6. The mod 2 invariants were computed in Exercise 22.9 in S. We note that E_6 has no invariants modulo primes $\neq 2, 3$ by the remarks in 4.5.

As in Example 8.12, we have an inclusion

$$i \colon F_4 \times \boldsymbol{\mu}_3 \hookrightarrow E_6$$

that identifies $\boldsymbol{\mu}_3$ with the center of E_6 such that the induced map

$$i_* \colon H^1(*, F_4 \times \boldsymbol{\mu}_3) \to H^1(*, E_6) \tag{10.2}$$

is a surjection. Two classes (J, β) and (J', β') have the same image in $H^1(k, E_6)$ if and only if there is a vector space isomorphism $f \colon J \to J'$ such that $\beta N_J = \beta' N_{J'} f$, where N_J and $N_{J'}$ denote the cubic norms on J and J', see [Gar01b, 2.8(2)].

10.3. Exercise. Albert algebras J, J' are isotopic (see [Jac68] for a definition) if and only if their norm forms are similar, i.e., $i_*(J, \beta) = i_*(J', \beta')$ for some $\beta, \beta' \in k^\times$, see pages 242–244 of [Jac68]. Prove that J and J' are isotopic if and only if their norms are *isomorphic*, i.e., $i_*(J, 1) = i_*(J', 1)$. Prove also that $i_*(J, 1) = i_*(J, \beta)$ if and only if β is the norm of an element of J.

Composing (10.2) and (7.2) gives a functor

$$H^1(*, (PGL_3 \times \boldsymbol{\mu}_3) \times \boldsymbol{\mu}_3) \to H^1(*, E_6)$$

where the $PGL_3 \times \boldsymbol{\mu}_3$ in parentheses is the subgroup of F_4 from §7. This functor is surjective at 3 because every Albert algebra is in the image of (7.2) after an extension of the base field of degree 1 or 2. Therefore the restriction map

$$\mathrm{Inv}^{\mathrm{norm}}(E_6, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Inv}^{\mathrm{norm}}(PGL_3 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3, \mathbb{Z}/3\mathbb{Z}) \tag{10.4}$$

is injective.

10.5. An invariant of degree 3. Consider the invariant g_3 of $PGL_3 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$ defined by

$$g_3 \colon (A, \alpha, \beta) \mapsto [A] \cdot (\alpha) \quad \in H^3(k, \boldsymbol{\mu}_3^{\otimes 2}) \tag{10.6}$$

for (A, α, β) defined over k. We now give two arguments to show that it is in the image of (10.4).

Proof #1. If (A, α, β) and (A', α', β') have the same image in $H^1(k, E_6)$, then the Albert algebras $J(A, \alpha), J(A', \alpha')$ have similar norms. But as they are first Tits constructions, this implies that the algebras are isomorphic [PR84, 4.9], hence $[A] \cdot (\alpha)$ equals $[A'] \cdot (\alpha')$ as in 7.4. That is, g_3 satisfies (6.2) and so extends uniquely to an invariant of E_6. □

Proof #2. The Dynkin index of E_6 is 6 [Mer03, 16.6], so the mod 3 portion of the Rost invariant defines a nonzero invariant

$$g'_3 \colon H^1(*, E_6) \to H^3(*, \boldsymbol{\mu}_3^{\otimes 2}).$$

As the inclusion $F_4 \hookrightarrow E_6$ has Rost multiplier one [Gar01a, 2.4], the restriction of g_3' to $H^1(*, F_4)$ is εg_3 for $\varepsilon = \pm 1$ and g_3 the invariant from 7.4. The composition

$$H^1(k, F_4) \times H^1(k, \boldsymbol{\mu}_3) \to H^1(k, E_6) \xrightarrow{g_3'} H^3(k, \boldsymbol{\mu}_3^{\otimes 2})$$

sends an Albert k-algebra J and a $\beta \in k^\times / k^{\times 3}$ to the element $\varepsilon g_3(J)$. (When β is 1, this is clear. In general, one uses a twisting argument as in [GQ06, Remark 2.5(i)].) The invariant $\varepsilon g_3'$ restricts to the map g_3 from (10.6). □

We abuse notation and write also g_3 for the invariant of E_6 that restricts to the g_3 from (10.6). Note that the image of this invariant of E_6 consists of symbols in $H^3(k, \boldsymbol{\mu}_3^{\otimes 2})$, because the same is true for the invariant g_3 of F_4.

10.7. An invariant of degree 4. Define an invariant g_4 of $PGL_3 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$ by putting

$$g_4 \colon (A, \alpha, \beta) \mapsto [A] \cdot (\alpha) \cdot (\beta) \quad \in H^4(k, \boldsymbol{\mu}_3^{\otimes 3}). \tag{10.8}$$

We give two proofs of the fact that g_4 extends to an invariant $H^1(*, E_6) \to H^4(*, \boldsymbol{\mu}_3^{\otimes 3})$.

Proof #1. We check (6.2). Suppose that (A, α, β) and (A', α', β') have the same image in $H^1(k, E_6)$. As in 10.5, $J(A, \alpha)$ and $J(A', \alpha')$ are isomorphic and $[A] \cdot (\alpha)$ equals $[A'] \cdot (\alpha')$. Further, β/β' is a similarity of the norm of $J(A, \alpha)$. By Exercise 10.3, β/β' is a norm from $J(A, \alpha)$, hence $J(A, \alpha)$ is isomorphic to $J(A'', \alpha'')$ for some central simple algebra A'' such that β/β' is reduced norm from A'' [PR84, 4.2]. We conclude that

$$[A] \cdot (\alpha) \cdot (\beta) - [A'] \cdot (\alpha') \cdot (\beta') = [A] \cdot (\alpha) \cdot (\beta/\beta') = [A''] \cdot (\alpha'') \cdot (\beta/\beta') = 0.$$

This verifies (6.2), hence g_4 extends to an invariant of $H^1(*, E_6)$. □

Proof #2 (sketch). Observe that $H^1(k, E_6)$ classifies cubic forms that become isomorphic to the norm of an Albert algebra over a separable closure of k. The statement "i_* is surjective" says that such a cubic form is a scalar multiple—say, $\beta.N_J$—of the norm on an Albert k-algebra J. Moreover, $g_3(J)$ is zero whenever the norm N_J has a nontrivial zero (i.e., whenever J is reduced), so Prop. 9.2 gives that the map

$$g_4 \colon \beta.N_J \mapsto g_3(J) \cdot (\beta)$$

is a well-defined invariant of E_6. □

As usual, we write also g_4 for the invariant of E_7 that restricts to give the g_4 defined in (10.8).

10.9. **Proposition.** $\operatorname{Inv}_{k_0}^{\mathrm{norm}}(E_6, \mathbb{Z}/3\mathbb{Z})$ *is a free* $R_3(k_0)$*-module with basis* g_3, g_4.

Proof. We imitate the proofs of Propositions 5.1 and 7.4. The restriction map

$$i^* \colon \operatorname{Inv}_{k_0}^{\mathrm{norm}}(E_6, \mathbb{Z}/3\mathbb{Z}) \to \operatorname{Inv}_{k_0}^{\mathrm{norm}}(F_4 \times \boldsymbol{\mu}_3, \mathbb{Z}/3\mathbb{Z})$$

is an injection.

The center of E_6 is contained in a maximal split torus, hence the image of the map $H^1(*, \boldsymbol{\mu}_3) \to H^1(*, E_6)$ is zero. Applying 5.7 and Propositions 2.1 and 7.4, we find that g_3 and g_4 span $\operatorname{Inv}_{k_0}^{\mathrm{norm}}(E_6, \mathbb{Z}/3\mathbb{Z})$. □

10.10. Exercise (Mod 3 invariants of the quasi-split E_6). For K a quadratic field extension of k_0, write E_6^K for the simply connected quasi-split group of type E_6 associated with the extension K/k. Describe the "mod 3" invariants of E_6^K.

10.11. **Open problem.** [PR94a, p. 205, Q. 4] Let J, J' be Albert k-algebras. If J and J' have similar norms, then their images in $H^1(k, E_6)$ are the same, hence they have the same Rost invariant. In the notation §7 of this note and §22 of S, $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Does the converse hold? That is, if $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$, are the norms of J and J' necessarily similar?

[If J and J' are reduced, the answer is "yes", see [Jac68, p. 369, Th. 2].]

11. Surjectivities: the highest root

We now describe a general situation where — in the setting of 8.11 — we can describe the identity component N° of the stabilizer. We will use this to apply Th. 8.11 to the simply connected group of type E_7.

11.1. Let $\widetilde{G}$ be a simply connected split algebraic group *not* of type A. The highest root $\widetilde{\alpha}$ is connected to a unique simple root—see Table 8—which we take to be π in the notation of 8.11. This situation was studied by Röhrle in [Röh93b], and for convenience of reference, we adopt the hypotheses of his main theorem. Namely, we assume that π is long (equivalently, $\widetilde{G}$ is not of type C), the rank of G is at least 4, and the characteristic is $\neq 2$.

As $-\widetilde{\alpha}$ is joined to π by a single bond, $\widetilde{\alpha}$ is the fundamental weight corresponding to the simple root π, i.e., for every root β, the integer $\langle \widetilde{\alpha}, \beta \rangle$ is the coordinate of π in β. For example, the π-coordinate of $\widetilde{\alpha}$ is $\langle \widetilde{\alpha}, \widetilde{\alpha} \rangle = 2$. For w_0 the longest element of the Weyl group of $\widetilde{G}$, clearly $w_0(\widetilde{\alpha}) = -\widetilde{\alpha}$, hence $w_0(\pi) = -\pi$.

We take V to be $Q(1)/Q(2)$, where Q is the unipotent radical of the parabolic subgroup opposite to the one chosen in 8.11, so that Q is generated over k_{sep} by the U_α where α has positive π-coordinate. We do this both to agree with Röhrle's notation and for the convenience of working with positive roots. As mentioned in Remark 8.10.1, this V is the dual of the irreducible L-module with highest weight $\widetilde{\alpha}$, meaning it has highest weight $-w_0(\widetilde{\alpha}) = \widetilde{\alpha}$ also. Changing from the parabolic in 8.11 to its opposite has not changed the isomorphism class of V.

$\widetilde{G}$	G	V	$\dim V$	$(N^\circ)^{\mathrm{ss}}$	$\dim Z(N^\circ)$
D_4	$(SL_2)^{\times 3}$	$k^2 \otimes k^2 \otimes k^2$	8	1	2
F_4	Sp_6		14	SL_3	0
E_6	SL_6	$\wedge^3 k^6$	20	$SL_3 \times SL_3$	0
E_7	Spin_{12}	half-spin	32	SL_6	0
E_8	E_7	minuscule	56	E_6	0
Spin_d ($d \geq 9$)	$SL_2 \times \mathrm{Spin}_{d-4}$	$k^2 \otimes$ vector	$2d-8$	Spin_{d-6}	1

Table 11. Internal Chevalley modules corresponding to the highest root

The last line of the table combines the cases where $\widetilde{G}$ is of type B_n ($n \geq 4$) or D_n ($n \geq 5$).

Table 11 describes the possibilities we consider. The last two columns will be explained below. Readers who know some nonassociative algebra will immediately recognize that V must be a Freudenthal triple system. Some convenient comparisons are [Mey68, (8.4)] and [Kru, Table 1].

The top five rows of the table are "sisters": The groups $\widetilde{G}$ from these rows form the bottom row of Freudenthal's "magic square", resp. the G's form the next-to-the-bottom row. The representations V are the "preferred representations" from the bottom row of the magic triangle in [DG02, Table 2]. These rows of the table are related to triple systems coming from cubic Jordan algebras of dimension 3, 6, 9, 15, and 27 respectively. The little one (with $\dim V = 8$) has appeared in Bhargava's work on higher reciprocity laws [Bha04, p. 220] and in Gopal Prasad's solution [Pra05] to the Kneser-Tits Problem for rank 1 groups of type ${}^3D_4$ and ${}^6D_4$. Note that in Prasad's case, the group G is not split, but π is circled in the index of G. In that case, the representation V is defined and there is an open G-orbit in $\mathbb{P}(V)$ as in 8.11. The stabilizer of a k-point in the open orbit is a "twisted form" of the N we now compute.

Write $\widetilde{\Phi}$ for the set of roots of $\widetilde{G}$. We view $\widetilde{G}$ as defined by generators and relation as in [Ste68]. In particular, for each root $\alpha \in \widetilde{\Phi}$, the unipotent subgroup U_α is the image of a homomorphism $x_\alpha \colon \mathbb{G}_a \to G$.

We put

$$v := x_\pi(r) x_{\widetilde{\alpha}-\pi}(s) U_{\widetilde{\alpha}} \quad \in V$$

for some $r, s \in k^\times$. It belongs to the open L-orbit in V [Röh93b, 4.4], and we define N to be the scheme-theoretic stabilizer of $[v]$ in G.

11.2. **Lemma** ($\operatorname{char} k_0 \neq 2$)**.** *The identity component N° of N is reductive. Its semisimple part is simply connected and generated by the subgroups U_β as β varies over the roots in $\widetilde{\Phi}$ whose support contains neither π nor any root adjacent to π. The rank of its central torus equals* $\deg \pi - 1$.

The notation $\deg \pi$ denotes the degree of the vertex π of the Dynkin diagram, i.e., the number of simple roots that are distinct from and not orthogonal to π. The lemma says that the Dynkin diagram of N is obtained from the Dynkin diagram of $\widetilde{G}$ by deleting π and every vertex adjacent to π.

Proof. Let $\beta \in \widetilde{\Phi}$ be as in the statement of the lemma. The support of $\pi \pm \beta$ has two connected components—the support of π and β—so $\pi \pm \beta$ is not a root of $\widetilde{G}$. For sake of contradiction, suppose that $\widetilde{\alpha} - \pi \pm \beta$ is a root of $\widetilde{G}$. It has π-coordinate 1, hence

$$\langle \widetilde{\alpha}, \widetilde{\alpha} - \pi \pm \beta \rangle = 1 \quad \text{and} \quad s_{\widetilde{\alpha}-\pi\pm\beta}(\widetilde{\alpha}) = \pi \mp \beta.$$

(Here and below we write s_β for the reflection defined by a root β.) That is, $\pi \mp \beta$ is a root of $\widetilde{G}$, a contradiction.

The previous paragraph is summarized by saying: β is strongly orthogonal to π and to $\widetilde{\alpha} - \pi$. It follows that the subgroup H of G generated by the U_β's fixes v and so is a subgroup of N. The type of H is listed in the next-to-the-last column of Table 11, and H is simply connected by [SS70, 5.4b]. We note that, line-by-line in the table, H has dimension

$$0, 8, 16, 35, 78, \frac{d^2 - 13d}{2} + 21.$$

Next consider the largest subtorus T_Z of $\widetilde{T}$ on which π, $\widetilde{\alpha}$, and the simple roots belonging to H vanish. This torus belongs to N, commutes with H, and has dimension

$$\operatorname{rank}\widetilde{G} - \operatorname{rank} H - 2 = \deg \pi - 1.$$

This number is listed in the last column of Table 11.

The subgroup $H.T_Z$ of G is connected and reductive with derived subgroup H. To complete the proof of the lemma, it suffices to check that $H.T_Z$ and N have the same dimension, i.e., to to check the equation

$$\dim H + \dim T_Z = \dim G - \dim V + 1. \tag{11.3}$$

The dimension of G, line-by-line in the table, is

$$9, 21, 35, 66, 133, \frac{d^2 - 9d}{2} + 13,$$

so equation (11.3) holds in each case. □

11.4. Orthogonal long roots in $\widetilde{\Phi}_1$. We put $\widetilde{\Phi}_j$ for the roots whose π-coordinate is j. For a positive root $\beta \in \widetilde{\Phi}$, the π-coordinate of β is 0, 1, or 2, and it is 2 if and only if β equals $\widetilde{\alpha}$, see [Bou Lie, §VI.1.8, Prop. 25(iv)]. That is, $\widetilde{\Phi}_j$ is nonempty only for $j = 0, \pm 1, \pm 2$ and $\widetilde{\Phi}_2$ is the singleton $\{\widetilde{\alpha}\}$.

As in [Röh93b, p. 145], there is a sequence $\mu_1, \mu_2, \mu_3, \mu_4$ of pairwise orthogonal long roots in $\widetilde{\Phi}_1$.

Lemma. *The roots $\mu_1, \mu_2, \mu_3, \mu_4$ are pairwise strongly orthogonal.*

Proof. If $\mu_i + \mu_j$ is a root, then it has π-coordinate 2, hence it equals $\widetilde{\alpha}$. But

$$0 = \langle \mu_i, \mu_j \rangle = \langle \widetilde{\alpha} - \mu_j, \mu_j \rangle = -1,$$

a contradiction. Further, μ_i and μ_j are orthogonal, so since $\mu_i + \mu_j$ is not a root, neither is $\mu_i - \mu_j$. □

11.5. Strongly orthogonal roots in G. The Weyl group of G acts transitively on the roots in $\widetilde{\Phi}_1$ of the same length [ABS90, §2, Lemma 1], so we may assume that μ_1 equals π. For $j = 2, 3, 4$, we set:

$$\gamma_j := \widetilde{\alpha} - \pi - \mu_j.$$

Lemma. *$\gamma_2, \gamma_3, \gamma_4$ are pairwise strongly orthogonal long roots of G. For various x, y, the value of $\langle x, y \rangle$ is given by the table:*

		y: $\widetilde{\alpha}$	π	μ_j	γ_j
	$\widetilde{\alpha}$	2	1	1	0
x	π	1	2	0	−1
	μ_j	1	0	2	−1
	γ_j	0	−1	−1	2

Proof. The top row of the table is the π-coordinate of y, and we know these already. We calculate that γ_j equals $s_\pi s_{\mu_j}(\widetilde{\alpha})$, so in particular, γ_j has the same length as $\widetilde{\alpha}$: long. As all the roots in table have the same length, the table is symmetric. As for π and μ_j, they are orthogonal by construction. The entries for $\langle \gamma_j, \pi \rangle$ and $\langle \gamma_j, \mu_j \rangle$ are straightforward computations.

Similarly, for $i \neq j$ we have $\langle \gamma_j, \mu_i \rangle = 1$, hence $\langle \gamma_i, \gamma_j \rangle = 0$. Further, $\gamma_i - \gamma_j = \mu_i - \mu_j$ is not a root. As in the proof of Lemma 11.4, $\gamma_i + \gamma_j$ is not a root. □

We note that $\mu_2 + \mu_3 + \mu_4 = 2\widetilde{\alpha} - \pi$ [Röh93b, 1.4], so

$$\gamma_2 + \gamma_3 + \gamma_4 = \widetilde{\alpha} - 2\pi. \tag{11.6}$$

11.7. A COPY OF SL_2. Define 1-parameter subgroups $x, y\colon \mathbb{G}_a \to G$ via

$$x(u) := \prod_{j=2}^{4} x_{\gamma_j}(u) \quad \text{and} \quad y(u) := \prod_{j=2}^{4} x_{-\gamma_j(u)}.$$

Since the γ_j's are strongly orthogonal, the images of the x_{γ_j} commute [Ste68, p. 30, (R2)], i.e., it does not matter in what order the displayed products are written. The images of x and y generate a copy of SL_2 in G which we denote simply by SL_2. For $t \in \mathbb{G}_m$, we set

$$w(t) := x(t)y(-t^{-1})x(t) \quad \text{and} \quad h(t) := w(t)w(-1).$$

The map h is a homomorphism and its image is a maximal torus in SL_2. How does SL_2 act on V? Identity (R8) from [Ste68, p. 30] says that for roots β, δ, we have:

$$h_\beta(t)x_\delta(u) = x_\delta(t^{\langle \delta, \beta \rangle} u)h_\beta(t) \tag{R8}$$

where $h_\beta\colon \mathbb{G}_m \to \widetilde{T}$ is the cocharacter corresponding to the coroot $\check{\beta}$. In particular,

$$h(t)x_\pi(u) = x_\pi(t^{-3}u)h(t) \quad \text{and} \quad h(t)x_{\widetilde{\alpha}-\pi}(u) = x_{\widetilde{\alpha}-\pi}(t^3u)h(t) \tag{11.8}$$

Moreover, we have:

Lemma. *There exists a $c \in \{\pm 1\}$ such that*

$$w(t)x_\pi(u) = x_{\widetilde{\alpha}-\pi}(ct^3u)w(t) \quad \textit{and} \quad w(t)x_{\widetilde{\alpha}-\pi}(u) = x_\pi(-ct^{-3}u)w(t)$$

for all $t \in \mathbb{G}_m$ and $u \in \mathbb{G}_a$.

Proof. Steinberg gives the formula [Ste68, p. 67, Lemma 37a]:

$$w_\beta(t)x_\delta(u) = x_{s_\beta \delta}(c(\beta, \delta)t^{-\langle \delta, \beta \rangle}u)w_\beta(t),$$

where $w_\beta(t)$ is defined to be $x_\beta(t)x_{-\beta}(-t^{-1})x_\beta(t)$ and $c(\beta, \delta) = \pm 1$ depends only on β and δ. Applying this with $\delta = \pi$ and successively with $\beta = \gamma_2, \gamma_3, \gamma_4$, we find $c \in \{\pm 1\}$ such that

$$w(t)x_\pi(u) = x_{\widetilde{\alpha}-\pi}(ct^3u)w(t).$$

(For the exponent of t, note e.g. that $\langle s_{\gamma_2}\pi, \gamma_3 \rangle = \langle \pi, s_{\gamma_2}\gamma_3 \rangle = \langle \pi, \gamma_3 \rangle = -1$.) Similarly, we obtain

$$w(t)x_{\widetilde{\alpha}-\pi}(u) = x_\pi(c't^{-3}u)w(t)$$

for some $c' \in \{\pm 1\}$.

The equations (11.8) give $h(-1)x_\pi(u) = x_\pi(-u)h(-1)$ and since $h(-1) = w(-1)^2$, we have:

$$x_\pi(-u)h(-1) = w(-1)^2 x_\pi(u) = x_\pi(cc'u)h(-1).$$

So $c' = -c$. □

11.9. *Remark.* We can describe this copy of SL_2 concretely in the notation of Dynkin [Dyn57b, Ch. III]. For simplicity, we consider the cases where $\widetilde{G}$ is simply laced, so we may identify roots and coroots by defining all roots to have length 2 with respect to the Weyl-invariant inner product (,). By (11.6), the intersection of the maximal torus $\widetilde{T}$ of $\widetilde{G}$ with SL_2 is the image of the cocharacter $h_{\widetilde{\alpha}-2\pi}$. For δ a simple root of G, the inner product $(\widetilde{\alpha} - 2\pi, \delta)$ is 2 if δ is adjacent to π and 0 otherwise. (Recall that π is not a root of G.) That is, Dynkin would denote the corresponding copy of $\mathfrak{sl}_2$ in the Lie algebra of G by attaching a 2 to the vertices of the Dynkin diagram of G that are adjacent to π.

11.10. We take N to be the scheme-theoretic stabilizer of

$$v := x_\pi(1)\, x_{\widetilde{\alpha}-\pi}(-c)\, U_{\widetilde{\alpha}} \quad \in V$$

for c as in Lemma 11.7. For a primitive 4-th root of unity i, we have $w(i)v = iv$. (See Remark 8.10 for the vector space structure on V.) The map $i \mapsto w(i)$ defines an injection $\boldsymbol{\mu}_4 \hookrightarrow N$, and we abuse notation by writing also $\boldsymbol{\mu}_4$ for the image in N.

So far, what we have written holds for the general setting of 11.1. We now specialize to the case where G is E_7.

11.11. **Lemma.** *In E_7, the centralizer C of E_6 is the rank 1 torus from 11.9 and the normalizer of E_6 is the group generated by C, E_6, and the copy of $\boldsymbol{\mu}_4$ from 11.10.*

Proof. Write T_6 and T_7 for the maximal tori in E_6 and E_7 respectively, obtained by intersecting with the maximal torus $\widetilde{T}$ of E_8. We argue along the lines of Example 8.12. First note that the centralizer of T_6 in E_7 contains T_7, is reductive, and is generated by root subgroups U_γ of E_7 for roots γ of E_7 whose inner product with the simple root α_i is zero for $1 \le i \le 6$. Such a γ is a multiple of the fundamental weight ω_7 with integer coefficients, i.e., an integer multiple of $2\omega_7$. However, $2\omega_7$ has height 27 and the highest root of E_7 has height 17, so no such γ exists. Therefore the centralizer of T_6 in E_7 is T_7.

It follows that the centralizer of E_6 in E_7 is the subgroup of T_7 formed by intersecting the kernels of the roots of E_6. This is a computation in terms of root systems: the character group of this centralizer is the quotient of the E_7 weight lattice by the sublattice generated by the α_i for $1 \le i \le 6$; this quotient is free of rank 1. Therefore the centralizer is a rank 1 torus in T_7. To prove the first claim in the lemma, it suffices to observe that the inner product $(\widetilde{\alpha} - 2\pi, \delta)$ is zero for every root δ of E_6, which is clear because $\widetilde{\alpha} - 2\pi$ equals $2\omega_7$.

The quotient group of "outer automorphisms" (automorphisms modulo inner automorphisms) of E_6 is $\mathbb{Z}/2\mathbb{Z}$, so to prove the claim about the normalizer it suffices to show that conjugation by a generator $w(i)$ of $\boldsymbol{\mu}_4(k_{\mathrm{sep}})$ gives an outer automorphism of E_6. As $\boldsymbol{\mu}_4$ belongs to N, it normalizes the identity component E_6 of N. Further, conjugation by $w(i)$ inverts elements of the maximal torus C of SL_2. But C contains the center of E_6 by the previous paragraph, so conjugation by $w(i)$ is an outer automorphism of E_6. ∎

11.12. *Remark.* The torus C appearing above is the image of the cocharacter $h_{2\omega_7} : \mathbb{G}_m \to \widetilde{T}$, which maps

$$t \mapsto h_{\alpha_1}(t^2) h_{\alpha_2}(t^3) h_{\alpha_3}(t^4) h_{\alpha_4}(t^6) h_{\alpha_5}(t^5) h_{\alpha_6}(t^4) h_{\alpha_7}(t^3).$$

Restricting this homomorphism to $\boldsymbol{\mu}_3$ and $\boldsymbol{\mu}_2$ respectively, we find

$$\zeta \mapsto h_{\alpha_1}(\zeta^2)h_{\alpha_3}(\zeta)h_{\alpha_5}(\zeta^2)h_{\alpha_6}(\zeta) \quad \text{and} \quad \varepsilon \mapsto h_{\alpha_2}(\varepsilon)h_{\alpha_5}(\varepsilon)h_{\alpha_7}(\varepsilon).$$

The images of these maps are the centers of E_6 and E_7 respectively, see [GQ06, 8.2, 8.1].

11.13. **Example** ($E_6 \rtimes \boldsymbol{\mu}_4 \subset E_7$)**.** We now show that the inclusion $E_6 \rtimes \boldsymbol{\mu}_4 \subset E_7$ from 11.10 induces a surjection

$$H^1(k, E_6 \rtimes \boldsymbol{\mu}_4) \to H^1(k, E_7) \tag{11.14}$$

for every extension k/k_0. By Th. 8.11, it suffices to show that $E_6 \rtimes \boldsymbol{\mu}_4$ is the stabilizer N of $[v] \in \mathbb{P}(V)$ for v as in 11.10. The subgroup of the torus C stabilizing $[v]$ is the image of $\boldsymbol{\mu}_6$ by (11.8), which is the subgroup of C generated by the center of E_6 and the copy of $\boldsymbol{\mu}_2$ in $\boldsymbol{\mu}_4$. Combining Lemma 11.11 and the fact that E_6 and $\boldsymbol{\mu}_4$ belong to N, we conclude that N equals $E_6 \rtimes \boldsymbol{\mu}_4$.

The surjectivity of (11.14) can be interpreted as a statement about Freudenthal triple systems; see [Gar01b, 4.15] for a precise statement and an algebraic proof.

12. Mod 3 invariants of E_7

The goal of this section is to compute the invariants of a split group of type E_7 (simply connected or adjoint) with values in $\mathbb{Z}/3\mathbb{Z}$. We write E_7 for the simply connected split group of that type, and we assume throughout this section that the characteristic of k_0 is $\neq 2, 3$. (Roughly speaking, we avoid characteristic 2 in order to use the results of the previous section, and we avoid characteristic 3 because we wish to describe the invariants mod 3, cf. Remark 2.4.) The "heavy lifting" was already done in the previous section.

Recall that the split group F_4 of that type can be viewed as a subgroup of E_6 as in Example 8.12.

12.1. **Lemma.** *The inclusion $F_4 \subset E_7$ gives a morphism*

$$H^1(*, F_4) \to H^1(*, E_7)$$

that is surjective at 3.

Proof. We have inclusions

$$F_4 \times \boldsymbol{\mu}_3 \subset E_6 \subset E_6 \rtimes \boldsymbol{\mu}_4 \subset E_7.$$

The first and third of these induce surjections on H^1 by Example 8.12 and 11.13. The second inclusion gives a morphism that is surjective at 3.

The image of $\boldsymbol{\mu}_3$ in E_7 is the center of E_6 as in Remark 11.12, so the inclusion $F_4 \times \boldsymbol{\mu}_3 \subset E_7$ factors through the subgroup $F_4 \times C$. As $H^1(k, C)$ is zero for every k/k_0, the images of $H^1(k, F_4)$ and $H^1(k, F_4 \times \boldsymbol{\mu}_3)$ in $H^1(k, E_7)$ agree. The claim follows. □

12.2. Mod 3 invariants of E_7. We now give two proofs that the invariant g_3 of F_4 defined in 7.4 extends to an invariant of E_7, which we will also denote by g_3.

Proof #1. Let $J, J' \in H^1(k, F_4)$ be Albert algebras whose images in $H^1(k, E_7)$ agree. Then J and J' have similar norms by [Fer72, 6.8] and $g_3(J)$ equals $g_3(J')$ by 10.5. Lemma 6.1 gives that g_3 extends to an invariant of E_7. □

Proof #2. The Rost invariant of E_7 has order 12 [Mer03, 16.7], so the mod 3 portion defines a nonzero invariant of E_7 with values in $\mathbb{Z}/3\mathbb{Z}$. Moreover, the inclusion $F_4 \subset E_7$ has Rost multiplier 1, so the restriction ofthis invarinat of E_7 to F_4 is — up to sign — the g_3 from 7.4. □

Combining Lemmas 4.1 and 12.1, we conclude that the restriction

$$\mathrm{Inv}^{\mathrm{norm}}_{k_0}(E_7, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Inv}^{\mathrm{norm}}_{k_0}(F_4, \mathbb{Z}/3\mathbb{Z})$$

is injective; by the above and Th. 7.4, it is an isomorphism. We conclude:

Theorem. $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(E_7, \mathbb{Z}/3\mathbb{Z})$ *is a free* $R_3(k_0)$*-module with basis* g_3. □

12.3. Exercise (Mod 3 invariants of adjoint E_7). Write E_7^{adj} for the split adjoint group of type E_7. Prove that the invariant g_3 of E_7 induces an invariant $g_3^{\mathrm{adj}} : H^1(*, E_7^{\mathrm{adj}}) \to H^3(*, \boldsymbol{\mu}_3^{\otimes 2})$ and that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(E_7^{\mathrm{adj}}, \mathbb{Z}/3\mathbb{Z})$ is a free $R_3(k_0)$-module with basis g_3^{adj}.

For the mod 2 invariants of E_7, the situation is much less clear.

12.4**. Open problem.** (Reichstein-Youssin [RY00, p. 1047]) Let k_0 be an algebraically closed field of characteristic zero. Is there a nonzero invariant

$$H^1(*, E_7^{\mathrm{adj}}) \to H^8(*, \mathbb{Z}/2\mathbb{Z})?$$

[Some readers have expressed skepticism about the precise degree — 8 — suggested above. Nonetheless, the core of the question remains: What are the invariants of degree > 3?]

13. Construction of groups of type E_8

Write E_8 for the split algebraic group of that type over k_0. Since it is adjoint and every automorphism is inner, the set $H^1(k, E_8)$ is identified with the group of isomorphism classes of groups of type E_8 over k. (This same phenomenon occurs with groups of type G_2 and F_4.) Here we describe a construction of groups of type E_8 that is analogous to the first Tits construction of groups of type F_4 (equivalently, Albert algebras) from 7.1. The fruit of this construction will appear in the next section. As the results of §8 do not apply to E_8 by Example 8.7, we take a new approach here. We assume throughout this section that the characteristic of k_0 is $\neq 5$.

13.1. A subgroup H of E_8. Let G be split of type E_8. We write H for the subgroup of G generated by the root subgroups $U_{\pm\widetilde{\alpha}}$ and $U_{\pm\alpha_i}$ for $i \neq 5$. This subgroup is of type $A_4 \times A_4$. We identify the first component of H — generated by $U_{\pm\alpha_i}$ for $i = 1, 2, 3, 4$ — with SL_5 via an irreducible representation whose highest weight is 1 on α_1 and 0 on α_2, α_3, and α_4. We identify the second component of H with SL_5 via an irreducible representation whose highest weight is 1 on α_6 and 0 on α_7, α_8, and $-\widetilde{\alpha}$.

Write ϱ for the homomorphism $\boldsymbol{\mu}_5 \to E_8$ defined by

$$\varrho\colon \zeta \mapsto h_1(\zeta)h_2(\zeta^4)h_3(\zeta^2)h_4(\zeta^3). \tag{13.2}$$

Applying the method described in [GQ06, §8], one finds that the image of ϱ is the center of both copies of SL_5 in E_8. More precisely, the canonical identification of the center of SL_5 with $\boldsymbol{\mu}_5$ is the map ϱ for the first component of H and ϱ^3 for the second component of H. That is, we have identified H with the quotient of $SL_5 \times SL_5$ by the subgroup generated by (ζ, ζ^2) for $\zeta \in \boldsymbol{\mu}_5$.

For $i = 1, 2$, write $\pi_i \colon H \to PGL_5$ for the projection on the i-th factor.

13.3. **Lemma.** *For $\eta \in H^1(k, H)$, write A_i for the central simple k-algebra of degree 5 defined by $\pi_i(\eta)$. Then $2[A_1] = [A_2]$ in the Brauer group of k.*

The twisted group H_η is isomorphic to $(SL(A_1) \times SL(A_2))/\boldsymbol{\mu}_5$.

Proof. Consider the diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_5 \times \boldsymbol{\mu}_5 & \longrightarrow & SL_5 \times SL_5 & \longrightarrow & PGL_5 \times PGL_5 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle q} & & \downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_5 & \xrightarrow{\varrho} & H & \xrightarrow{\pi_1 \times \pi_2} & PGL_5 \times PGL_5 & \longrightarrow & 1
\end{array}$$

where q is given by $(x, y) \mapsto y/x^2$. The diagram commutes because $y/x^2 = x(y/x^3)$ and $(y/x^2)^3 = y(y/x^3)^2$. We obtain a commutative diagram with exact rows:

$$\begin{array}{ccccc}
1 & \longrightarrow & H^1(k, PGL_5 \times PGL_5) & \longrightarrow & \mathrm{Br}_5 \times \mathrm{Br}_5 \\
\downarrow & & \| & & \downarrow{\scriptstyle q} \\
H^1(k, H) & \xrightarrow{\pi_1 \times \pi_2} & H^1(k, PGL_5 \times PGL_5) & \xrightarrow{\delta} & \mathrm{Br}_5\,.
\end{array}$$

In the Brauer group, we find the equation:

$$0 = \delta(\pi_1 \times \pi_2)(\eta) = q(\pi_1(\eta), \pi_2(\eta)) = -2[A_1] + [A_2]. \qquad \square$$

13.4. The subgroup C of H. Write C for the group $\mathbb{Z}/5\mathbb{Z} \times \boldsymbol{\mu}_5$. We define a homomorphism $t \colon C \times \boldsymbol{\mu}_5 \to H$ such that t restricted to $\boldsymbol{\mu}_5$ is the map ϱ from (13.2) and the restriction of t to C is given by

$$t|_C(i, \zeta^j) = (v^i u^j, v^i u^{2j})$$

in the notation of 5.2, where ζ is a fixed primitive 5-th root of unity. The formula $uv = \zeta vu$ shows that t is indeed a group homomorphism.

For each extension k/k_0, there is an induced function

$$t_* \colon H^1(k, C \times \boldsymbol{\mu}_5) \to H^1(k, E_8). \tag{13.5}$$

Because $H^1(k, E_8)$ classifies groups of type E_8 over k, we view t_* as a *construction* of groups of type E_8 via Galois descent.

13.6. **Example.** We now compute $t_*(\gamma, z)$ for some $\gamma \in H^1(k, C)$ and $z \in H^1(k, \boldsymbol{\mu}_5)$. Put $\eta := t_*(\gamma, 1)$ and write A_1 for $\pi_1(\eta)$ as in Lemma 13.3. Twisting SL_5 and E_8 by η, we find a subgroup $SL(A_1)$ of $(E_8)_\eta$.

The diagram

$$\begin{array}{ccc}
C & \xrightarrow{t} & H \\
 & \searrow^{c} & \downarrow{\scriptstyle \pi_1} \\
 & & PGL_5
\end{array}$$

commutes for c as in 5.2, so $\pi_1(\eta) = c_*(\gamma)$.

(1) Suppose that $c_*(\gamma)$ is zero, i.e., A_1 is split. Then H_η is split. Since η and $t_*(\gamma, z)$ — viewed as elements of $H^1(k, H)$ — differ by a central cocycle $t_*(1, z)$, the twisted group $H_{t_*(\gamma,z)}$ is also split. Hence E_8 twisted by $t_*(\gamma, z)$ is split. We conclude that $t_*(\gamma, z)$ is zero in $H^1(k, E_8)$.

(2) Suppose now that $z_1, z_2 \in H^1(k, \boldsymbol{\mu}_5)$ differ by a reduced norm from A. The inclusion ϱ of $\boldsymbol{\mu}_5$ into $H \subset E_8$ is unaffected by twisting by η, and we obtain a map $H^1(k, \boldsymbol{\mu}_5) \to H^1(k, (E_8)_\eta)$. The composition

$$H^1(k, \boldsymbol{\mu}_5) \xrightarrow{\varrho} H^1(k, (E_8)_\eta) \xrightarrow[\tau_\eta]{\cong} H^1(k, E_8),$$

where τ_η is the twisting isomorphism, sends z_i to $t_*(\gamma, z_i)$. However, the first arrow factors through $H^1(k, SL(A_1))$, hence z_1 and z_2 have the same image in $H^1(k, E_8)$.

13.7. **Proposition.** *The morphism*

$$t_* \colon H^1(*, C \times \boldsymbol{\mu}_5) \to H^1(*, E_8)$$

is surjective at 5.

Proof. Step 1. We first show that — for H the subgroup of E_8 defined in 13.1 — the morphism $H^1(*, H) \to H^1(*, E_8)$ is surjective at 5.

Let ξ be in $Z^1(k, E_8)$. Fix a maximal and k-split torus T of E_8 and a k-defined maximal torus T' in the twisted group $(E_8)_\xi$. There is some $g \in E_8(k_{\mathrm{sep}})$ such that $g^{-1}T'g = T$, and replacing ξ with $\sigma \mapsto g^{-1}\xi_\sigma\, {}^\sigma g$, we may assume that $\xi_\sigma \sigma(T) = T$ for every $\sigma \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$, i.e., that ξ takes values in $N_{E_8}(T)$.

The Galois group acts trivially on the Weyl group $N_{E_8}(T)/T$, so the image $\overline{\xi} \in Z^1(k, N_{E_8}(T)/T)$ is a continuous homomorphism $\overline{\xi} \colon \mathrm{Gal}(k_{\mathrm{sep}}/k) \to N_{E_8}(T)/T$. Fix a 5-Sylow subgroup S of $N_{E_8}(T)/T$. Take K to be the subfield of k_{sep} fixed by $\overline{\xi}^{-1}(S)$; it is an extension of K of dimension not divisible by 5.

Because S and a 5-Sylow in $N_H(T)/T$ both have order 5^2, there is a $\overline{w} \in (N_{E_8}(T)/T)(K)$ such that the image of the map $\sigma \mapsto \overline{w}^{-1}\overline{\xi}_\sigma\overline{w}$ is contained in $N_H(T)/T$. Further, T is K-split, so there is some $w \in N_{E_8}(T)(K)$ such that w maps to $\overline{w}$. Replacing ξ with $\sigma \mapsto w^{-1}\xi_\sigma\, {}^\sigma w$, we may assume that $\mathrm{res}_{K/k}(\xi) \in H^1(K, E_8)$ is in the image of $H^1(K, H)$.

Step 2. We now show that the morphism $H^1(*, C \times \boldsymbol{\mu}_5) \to H^1(*, H)$ is surjective at 5. Fix $\eta \in Z^1(K, H)$ and let A be the central simple algebra of degree 5 representing $\pi_1(\eta) \in H^1(K, PGL_5)$. By Lemma 5.6, there is an extension L/K of dimension not divisible by 5 such that $A \otimes L$ is cyclic, i.e., equals $c_*(\gamma)$ for some $\gamma \in H^1(L, C)$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_5 & \longrightarrow & C \times \boldsymbol{\mu}_5 & \longrightarrow & C & \longrightarrow & 1 \\
 & & \| & & \downarrow t & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_5 & \xrightarrow{\varrho} & H & \xrightarrow{\pi_1 \times \pi_2} & PGL_5 \times PGL_5 & \longrightarrow & 1.
\end{array}$$

By Lemma 13.3, γ and η have the same image in $H^1(k, PGL_5 \times PGL_5)$, namely the class of $(\pi_1(\eta), \pi_2(\eta))$. It follows that η and $t_*(\gamma)$ are in the same $H^1(k, \boldsymbol{\mu}_5)$-orbit. Fixing a $\lambda \in H^1(k, \boldsymbol{\mu}_5)$ such that $\eta = \lambda \cdot t_*(\gamma)$, we have:

$$t_*(\lambda \cdot \gamma) = \lambda \cdot t_*(\gamma) = \eta,$$

as desired. □

13.8. The Rost invariant. We now compute the composition

$$H^1(k, C \times \boldsymbol{\mu}_5) \xrightarrow{t_*} H^1(k, E_8) \xrightarrow{r_{E_8}} H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \tag{13.9}$$

for every extension k/k_0. As the Dynkin index of E_8 is $60 = 5 \cdot 12$ [Mer03, 16.8], 4.3 says that the image of the composition is 5-torsion, hence lies in $H^3(k, \boldsymbol{\mu}_5^{\otimes 2})$. Recall that C is $\mathbb{Z}/5\mathbb{Z} \times \boldsymbol{\mu}_5$. We have:

13.10**. Lemma.** *There is a uniquely determined* $\lambda \in H^1(k_0, \boldsymbol{\mu}_5)$ *and a natural number* m *not divisible by* 5 *such that the composition* (13.9) *is given by*

$$((x, y), z) \mapsto \lambda \cdot x \cdot y + m\, x \cdot y \cdot z$$

for every $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ *and* $y, z \in H^1(k, \boldsymbol{\mu}_5)$ *and every* k/k_0.

Proof. We first prove the claim in the case where z is zero and k_0 contains a primitive 5-th root of unity, which we use to identify $\boldsymbol{\mu}_5$ with $\mathbb{Z}/5\mathbb{Z}$. If x or y is zero, the class $t_*(x, y, 1)$ is zero in $H^1(k, E_8)$ by Example 13.6.1. Applying Lemma 5.7, we conclude that the composition (13.9) is $(x, y, 1) \mapsto \lambda \cdot x \cdot y$ for a unique $\lambda \in R_5(k_0)$. That is, the claim holds in this case.

We now consider the case where z is zero, but the extension k_1 obtained by adjoining a primitive 5-th root of unity to k_0 may be proper. By the previous paragraph, the restriction of (13.9) to $H^1(*, C)$ and viewed as an invariant $\mathsf{Fields}_{/k_1} \to \mathsf{Abelian\ Groups}$ is given by

$$(x, y) \mapsto \lambda_1 \cdot x \cdot y$$

for a uniquely determined $\lambda_1 \in H^1(k_1, \boldsymbol{\mu}_5)$. Write λ_0 for the unique class in $H^1(k_0, \boldsymbol{\mu}_5)$ whose restriction to k_1 is λ_1. Since the invariants (13.9) and $(x, y) \mapsto \lambda_0 \cdot x \cdot y$ agree over every extension k/k_1, Lemma 3.2 proves the claim.

Finally, we consider the general case. Put $\eta := c_*(x, y)$ and consider the diagram

$$\begin{array}{ccccc} H^1(k, \boldsymbol{\mu}_5) & \longrightarrow & H^1(k, (E_8)_\eta) & \xrightarrow{r_{(E_8)_\eta}} & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \\ & \searrow^{t_*(x,y,?)} & \cong \downarrow \tau_\eta & & \downarrow ?+r_{E_8}(\eta) \\ & & H^1(k, E_8) & \xrightarrow{r_{E_8}} & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

where τ_η is the twisting isomorphism. The triangle obviously commutes and the square commutes by [Gil00, p. 76, Lemma 7]. The image of $z \in H^1(k, \boldsymbol{\mu}_5)$ in the lower right corner going counterclockwise is $r_{E_8}(t_*(x, y, z))$, i.e., the image of (x, y, z) under (13.9). The arrow in the upper left factors as $H^1(k, \boldsymbol{\mu}_5) \xrightarrow{\varrho} H^1(k, SL(A_1)) \to H^1(k, (E_8)_\eta)$. Since the inclusion of $SL(A_1)$ in $(E_8)_\eta$ has Rost multiplier 1, the composition on the top row is $z \mapsto m\, x \cdot y \cdot z$ for some natural number m not divisible by 5. This proves the claim. □

13.11. A TWISTED MORPHISM. Fix a 1-cocycle $\mu \in Z^1(k, \boldsymbol{\mu}_5)$ such that $\mu = -m^* \lambda$ in $H^1(k, \boldsymbol{\mu}_5)$, where m^* denotes a natural number such that mm^* is congruent to 1 mod 5, and λ is as in Lemma 13.10. Define i to be the composition

$$i\colon H^1(*, C \times \boldsymbol{\mu}_5) \xrightarrow{\mu+?} H^1(*, C \times \boldsymbol{\mu}_5) \xrightarrow{t_*} H^1(*, E_8).$$

Since $t_*(x, y, z) = i(x, y, z - \mu)$, Prop. 13.7 holds with t_* replaced by i. Furthermore, by Lemma 13.10 we have:

$$r_{E_8} i(x, y, z) = r_{E_8} t_*(x, y, \mu + z) = m\, x \cdot y \cdot z. \tag{13.12}$$

13.13. **Theorem.** *Suppose that k is perfect. For $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ and $y, z \in H^1(k, \boldsymbol{\mu}_5)$, we have: $i(x,y,z)$ is zero in $H^1(k, E_8)$ if and only if $r_{E_8} i(x,y,z)$ is zero.*

Proof. The "only if" direction is a basic property of the Rost invariant, so we suppose that $r_{E_8} i(x,y,z)$ is zero, i.e., that $x \cdot y \cdot z$ is zero in $H^3(k, \boldsymbol{\mu}_5^{\otimes 2})$. By the Merkurjev-Suslin Theorem, z is a reduced norm from the cyclic algebra $c_*(x,y)$, so by Example 13.6.2 we have:

$$i(x,y,z) = t_*(x,y,z+\mu) = t_*(x,y,\mu) = i(x,y,1).$$

Now consider the class $i(x,u,1)$ in $H^1(k(u), E_8)$ for u an indeterminate. Note that this class is split by the cyclic extension of degree 5 defined by x and it has $r_{E_8} i(x,u,1) = 0$ by (13.12). The proof of [Gil02a, 1.4] shows that — for every completion K of $k(u)$ with respect to a discrete valuation trivial on k — the image of $i(x,u,1)$ in $H^1(K, E_8)$ is the image of some element of $H^1(k, E_8)$, i.e., $i(x,u,1)$ is unramified on $\mathbb{A}^1_k$. (This argument uses Bruhat-Tits theory, in particular the hypothesis that k is perfect.) We conclude that $i(x,u,1) \in H^1(k(u), E_8)$ is also the image of a class in $H^1(k, E_8)$. By specialization, the value of $i(x,y,1)$ does not depend on y. In particular, we have:

$$i(x,y,1) = i(x,1,1) = t_*(x,1,\mu) \quad \text{for } y \in H^1(k, \boldsymbol{\mu}_5).$$

But $c_*(x,1)$ is the the matrix algebra $M_5(k)$, so $t_*(x,1,\mu)$ is zero by Example 13.6.1. □

14. Mod 5 invariants of E_8

We now derive consequences of the construction in the previous section. We classify the invariants mod 5 of the split group E_8 of that type, recover Chernousov's result on the kernel of these invariants, and give new examples of anisotropic groups of type E_8 over a broad class of fields. We continue the assumption that the characteristic of k_0 is $\neq 5$.

As in 13.4, the 5-torsion in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ is identified with $H^3(k, \boldsymbol{\mu}_5^{\otimes 2})$. Composing the Rost invariant r_{E_8} with the projection on 5-torsion, we find a normalized invariant

$$h_3 \colon H^1(*, E_8) \to H^3(*, \boldsymbol{\mu}_5^{\otimes 2}).$$

14.1. **Theorem.** *$\mathrm{Inv}^{\mathrm{norm}}_{k_0}(E_8, \mathbb{Z}/5\mathbb{Z})$ is a free $R_5(k_0)$-module with basis h_3.*

Proof. By Cor. 3.5, we may assume that k_0 is perfect and that it contains a primitive 5-th root of unity, which we use to identify $\boldsymbol{\mu}_5$ with $\mathbb{Z}/5\mathbb{Z}$. Because i is surjective at 5, the restriction map

$$i^* \colon \mathrm{Inv}^{\mathrm{norm}}(E_8, \mathbb{Z}/5\mathbb{Z}) \to \mathrm{Inv}^{\mathrm{norm}}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}) \tag{14.2}$$

is an injection. By the same proof as for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in S16.4, we see that every normalized invariant of $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is of the form

$$(x,y,z) \mapsto \lambda_x \cdot x + \lambda_y \cdot y + \lambda_z \cdot z + \lambda_{xy} \cdot x \cdot y + \lambda_{xz} \cdot x \cdot z + \lambda_{yz} \cdot y \cdot z + \lambda_{xyz} \cdot x \cdot y \cdot z$$

for uniquely determined λ's in $R_5(k_0)$. However, if x, y, or z is zero, then $x \cdot y \cdot z$ is zero, hence $i(x,y,z)$ is zero in $H^1(k, E_8)$ by Th. 13.13. It follows that the image of (14.2) is contained in the span of the invariant

$$(x,y,z) \mapsto \lambda_{xyz} \cdot x \cdot y \cdot z.$$

But this $R_5(k_0)$-submodule of $\mathrm{Inv}^{\mathrm{norm}}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z})$ is also the submodule spanned by the restriction of h_3 by (13.12), so the theorem is proved. □

14.3. **Open problem.** (Reichstein-Youssin [RY00, p. 1047]) Let k_0 be an algebraically closed field of characteristic zero. Do there exist nonzero invariants mapping $H^1(*, E_8)$ into $H^9(*, \mathbb{Z}/2\mathbb{Z})$ and $H^5(*, \mathbb{Z}/3\mathbb{Z})$?

14.4. Comparison with groups of type F_4. There are tantalizing similarities between the behavior of groups of type F_4 relative to the prime 3 and groups of type E_8 relative to the prime 5, see e.g. [Gil02a, 3.2] or compare Theorems 7.4 and 14.1. We now investigate these similarities. For E_8, the morphism i from 13.11 plays the role of the first Tits construction of Albert algebras.

As groups of type F_4 and E_8 have trivial centers and only inner automorphisms, the groups $H^1(k, F_4)$ and $H^1(k, E_8)$ are in bijection with isomorphism classes of split groups of type F_4 and E_8 respectively. Using this bijection, it makes sense to write $g_3(G)$ when G is of type F_4 and g_3 denotes the invariant from 7.4, as well as to write $h_3(G)$ for G of type E_8 and h_3 as defined above.

14.5. Splitting by extensions prime to p. Every group G of type F_4 over k is of the form $\mathrm{Aut}(J)$ for some Albert k-algebra J. If $g_3(G) \in H^3(k, \boldsymbol{\mu}_3^{\otimes 2})$ is nonzero, then clearly G cannot be split by an extension of degree not divisible by 3. Conversely, if $g_3(G)$ is zero, then J is reduced, i.e., constructed from an octonion algebra O and a 2-Pfister form. In that case, every quadratic extension of k that splits O also splits J and G [Jac68, p. 369, Th. 2].

For groups of type E_8, the analogous result is the following. It is due to Chernousov, see [Che95].

Proposition. *An algebraic group G of type E_8 over k is split by an extension of k of dimension not divisible by 5 if and only if $h_3(G) = 0$.*

Proof. As h_3 is normalized, the "only if" direction is clear. So assume that $h_3(G)$ is zero. After replacing k by an extension of dimension not divisible by 5, we may assume that k is perfect and that G equals $i(x, y, z)$ for i the map defined in 13.11 and some x, y, z. Since $r_{E_8} i(x, y, z)$ equals $h_3(G)$, Th. 13.13 gives the claim. □

14.6. Anisotropy. For a group $G = \mathrm{Aut}(J)$ of type F_4, one knows that G is isotropic if and only if J has nonzero nilpotents [CG06, 9.1]. If $g_3(G)$ is nonzero, then J has no zero divisors (i.e., is not reduced), see [Ros91] or [PR96], and in particular G is anisotropic.

We now prove the corresponding result for E_8.

Proposition. *If a group G of type E_8 has $h_3(G) \neq 0$, then G is anisotropic.*

Proof. If G is split, then clearly $h_3(G) = 0$. So suppose that G is isotropic but not split. According to the list of possible indexes in [Tit66, p. 60], the semisimple anisotropic kernel A of G is a strongly inner group of type D_4, D_6, D_7, E_6, or E_7. That is, A is obtained by twisting a split simply connected group S of one of these types by a 1-cocycle $\eta \in Z^1(k, S)$. Tits's Witt-type Theorem [Spr98, 16.4.2] implies that G is isomorphic to E_8 twisted by η.

The inclusion of S in E_8 comes from the obvious inclusion of Dynkin diagrams, so has Rost multiplier one. That is, the diagram

$$\begin{array}{ccc} H^1(k,S) & \xrightarrow{r_S} & H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \| \\ H^1(k,E_8) & \xrightarrow{r_{E_8}} & H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

commutes. However, for each of the possibilities for S, the Dynkin index is 2, 2, 2, 6, or 12 respectively by [Mer03, 15.4, 16.6, 16.7], so the mod 5 portion of $r_{E_8}(\eta)$, namely $h_3(G)$, is zero. □

14.7. Anisotropic groups split by extensions of degree p. If $G = \mathrm{Aut}(J)$ is a group of type F_4 over k where J is a first Tits construction that is not split, then G is anisotropic over k but split by a cubic extension of k. That is, nonzero symbols in $H^3(k, \boldsymbol{\mu}_3^{\otimes 2})$ give anisotropic groups of type F_4 that are split by a cubic extension.

The analogous statement for E_8 is the following:

Corollary (of Prop. 14.6). *If $H^3(k, \boldsymbol{\mu}_5^{\otimes 2})$ contains a nonzero symbol, then k supports an anisotropic group of type E_8 that is split by a cyclic extension of dimension 5.*

Proof. Fix $x \in H^1(k, \mathbb{Z}/5\mathbb{Z})$ and $y, z \in H^1(k, \boldsymbol{\mu}_5)$ such that $x \cdot y \cdot z$ is not zero in $H^3(k, \boldsymbol{\mu}_5^{\otimes 2})$. Then $h_3 i(x,y,z)$ is not zero by (13.12), so the group obtained by twisting E_8 by $i(x,y,z)$ is anisotropic by Prop. 14.6. It is split by the cyclic extension of k of dimension 5 determined by x by Example 13.6. □

The interesting part of the corollary is that the groups are split by an extension of degree 5, and not merely that the groups are anisotropic. Indeed, anisotropic groups of type E_8 abound. For example, a number field k supports an anisotropic group G of type E_8 if and only if k has a real embedding by the Hasse Principle [PR94b, p. 286, Th. 6.6]. But the Hasse Principle also implies that G cannot be split by an odd-degree extension of k.

As a concrete illustration of the corollary, fix a number field k. It supports a cyclic division algebra A of dimension 5^2. (One can specify A by local data, see [Rei75, §32].) For t an indeterminate, the symbol $[A] \cdot (t)$ is nonzero in $H^3(k(t), \boldsymbol{\mu}_5^{\otimes 2})$.

14.8. Failure of the analogy. If $G = \mathrm{Aut}(J)$ is a group of type F_4 such that the Rost invariant $r_{F_4}(G)$ is 3-torsion (i.e., $r_{F_4}(G)$ equals $g_3(G)$), then it is a result of Petersson and Racine that J is a first Tits construction [KMRT98, 40.5]. In this case, the analogy between first Tits constructions and the map i fails. Gille [Gil02b, App.] has given an example of a group G of type E_8 over a particular field k such that $r_{E_8}(G)$ is zero but G is not split. By Th. 13.13, such a G cannot be in the image of i.

14.9. Exercise (prime-to-5-closed fields). Suppose that k is a field such that every finite separable extension of k has degree a power of 5. Prove that every group of type E_8 over k is split or anisotropic.

[The assumption on k is stronger than necessary; it suffices to assume that the group $H^3(k, \mathbb{Z}/6\mathbb{Z}(2))$ defined in [Mer03, App. A] is zero.]

Part III. Spin groups

15. Surjectivities: Spin_n for $n \le 12$

We continue the examples of internal Chevalley modules as defined in 8.11, focusing on the case where G is Spin_n for $n \le 12$. We assume throughout this section that the characteristic of k_0 is different from 2.

15.1. **Example** ($\mathrm{Spin}_{2n-1} \cdot Z \subset \mathrm{Spin}_{2n}$)**.** Taking $\widetilde{G}$ to be the split simply connected group of type D_{n+1} and π to be α_1, we find that G is the split simply connected group Spin_{2n} of type D_n, and V is the vector representation.

There is a G-invariant quadratic form on V and we fix an anisotropic vector v. The stabilizer of v in $\mathrm{SO}(V)$ is easily seen to be $\boldsymbol{\mu}_2 . \mathrm{SO}(v^\perp)$ (using that V is even-dimensional), hence the stabilizer of $[v]$ in G is the compositum $Z . \mathrm{Spin}_{2n-1}$, where the center Z of G meets Spin_{2n-1} in a copy of $\boldsymbol{\mu}_2$ that is the kernel of the vector representation. By dimension count, the orbit of $[v]$ is the open orbit in $\mathbb{P}(V)$.

Theorem 8.11 gives that the induced map

$$H^1(k, \mathrm{Spin}_{2n-1} \cdot Z) \to H^1(k, \mathrm{Spin}_{2n}) \tag{15.2}$$

is surjective for every k/k_0. But we can say a little more. Since Z is central, the multiplication map $\mathrm{Spin}_{2n-1} \times Z \to \mathrm{Spin}_{2n-1} \cdot Z$ is a group homomorphism, and composing this with (15.2) gives a map

$$H^1(k, \mathrm{Spin}_{2n-1}) \times H^1(k, Z) \to H^1(k, \mathrm{Spin}_{2n}) \tag{15.3}$$

and *this map is also surjective*. Indeed, the intersection $\mathrm{Spin}_{2n-1} \cap Z$ is the center of Spin_{2n-1}, i.e., $\boldsymbol{\mu}_2$, and there is an exact sequence

$$1 \longrightarrow \mathrm{Spin}_{2n-1} \longrightarrow \mathrm{Spin}_{2n-1} \cdot Z \xrightarrow{q} \boldsymbol{\mu}_2 \longrightarrow 1. \tag{15.4}$$

The center Z of Spin_{2n} satisfies

$$Z \cong \begin{cases} \boldsymbol{\mu}_4 & \text{if } n \text{ is odd,} \\ \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 & \text{if } n \text{ is even} \end{cases}$$

and in either case the restriction of q to Z yields a surjection $H^1(k, Z) \to H^1(k, \boldsymbol{\mu}_2)$. (For surjectivity in the n odd case, see 2.5.) A twisting argument combined with the exactness of (15.4) now gives that the map

$$H^1(k, \mathrm{Spin}_{2n-1}) \times H^1(k, Z) \to H^1(k, \mathrm{Spin}_{2n-1} \cdot Z)$$

is surjective, hence that (15.3) is surjective, as claimed.

Attempting to do the same for groups of type B (equivalently, odd-dimensional quadratic forms) gives a stabilizer that is less attractive.

15.5. **Example** ($G_2 \times \boldsymbol{\mu}_2 \subset \mathrm{Spin}_7$)**.** Take $\widetilde{G}$ to be the split group of type F_4 and $\pi := \alpha_4$. The subgroup G is the split simply connected group Spin_7 of type B_3 and V is its spin representation.

Write G_2 for the split group of that type. The irreducible representation W with highest weight ω_1 is 7-dimensional (in characteristic $\ne 2$ [GS88, p. 413]) and supports a G_2-invariant nonsingular quadratic form q. It gives an embedding of G_2 in Spin_7. We claim that N may be taken to be the direct product of G_2 with the center $\boldsymbol{\mu}_2$ of Spin_7.

As a representation of G_2, V is a direct sum of W and a 1-dimensional representation, say $k_0 v$. As in Example 8.12, dimension considerations imply that $[v]$ belongs to the open L-orbit in $\mathbb{P}(V)$ and G_2 is the identity component of the stabilizer N of $[v]$.

As the kernel $\boldsymbol{\mu}_2$ of the map $\mathrm{Spin}_7 \to \mathrm{SO}(W)$ clearly belongs to N, we may compute N by determining its image in $GL(W)$. Since W is an irreducible representation of G_2 and every automorphism of G_2 is inner, the normalizer of G_2 in $GL(W)$ consists of scalar matrices. It follows that N is contained in $G_2.\boldsymbol{\mu}_2$, hence N equals $G_2 \times \boldsymbol{\mu}_2$.

For a version of this example over the reals, see [Var01, Th. 3].

Combining this example with Th. 8.11 gives that every 8-dimensional form in I^3 that represents 1 is the norm quadratic form of an octonion algebra, hence every 8-dimensional form in I^3 is similar to a 3-Pfister form. This is a special case of the general theorem: a 2^n-dimensional form in I^n is similar to an n-Pfister form [Lam05, X.5.6].

15.6. Exercise. Prove: If q is an 8-dimensional quadratic form over k such that $C_0(q)$ is isomorphic to $M_8(K)$ for some quadratic étale k-algebra K, then q is similar to $\langle 1 \rangle \oplus \langle \alpha \rangle q_0$ for $\alpha \in k^\times$ such that $K = k[x]/(x^2 - \alpha)$ and a uniquely determined 7-dimensional form q_0 such that $\langle 1 \rangle \oplus q_0$ is a 3-Pfister form.

[This can be proved using standard quadratic form theory, or by combining Examples 15.1 and 15.5.]

In the examples above, we have used internal Chevalley modules as in 8.11 to produce representations with open orbits. For the cases where G is Spin_9 or Spin_{11}, such arguments are somewhat more complicated than the naive setup in 8.11. (See [Rub04, 4.3(3), 5.1] for details.) Instead, we refer to Igusa's paper [Igu70]; he proves the existence of an open orbit using concrete computations in the Clifford algebra.

15.7. **Example** ($\mathrm{Spin}_7 \times \boldsymbol{\mu}_2 \subset \mathrm{Spin}_9$)**.** As in [Igu70, p. 1017], there are inclusions

$$\mathrm{Spin}_7 \to \mathrm{Spin}_8 \to \mathrm{Spin}_9$$

such that Spin_9 has an open orbit in $\mathbb{P}(V)$ for V its (16-dimensional) spin representation, and Spin_7 is the stabilizer of a $v \in V$ whose image in $\mathbb{P}(V)$ is in the open orbit. Recall that there are three non-conjugate embeddings of Spin_7 in Spin_8, distinguished by which copy of $\boldsymbol{\mu}_2$ in the center of Spin_8 they contain, cf. [Dyn57a, Th. 6.3.1] or [Var01, Th. 5]. The $\boldsymbol{\mu}_2$ in this Spin_7 is *not* in the kernel of the map $\mathrm{Spin}_9 \to \mathrm{SO}_9$, i.e., is not the center of Spin_9.

Write Z for the copy of $\boldsymbol{\mu}_2$ that is the center of Spin_9; the element $-1 \in Z$ sends v to $-v$. But v is an anisotropic vector for the Spin_9-invariant quadratic form on V, hence $Z \times \mathrm{Spin}_7$ is the stabilizer of the line $[v]$ in Spin_9.

15.8. **Example** ($G_2 \times \boldsymbol{\mu}_4 \subset \mathrm{Spin}_{10}$)**.** Example 15.1 gives a surjection

$$H^1(k, \mathrm{Spin}_9 \cdot \boldsymbol{\mu}_4) \to H^1(k, \mathrm{Spin}_{10}).$$

Example 15.7 gives an inclusion

$$\mathrm{Spin}_7 \times \boldsymbol{\mu}_2 \subset \mathrm{Spin}_9$$

that induces a surjection on H^1's, i.e., the map

$$H^1(k, \mathrm{Spin}_7 \times \boldsymbol{\mu}_2) \times H^1(k, \boldsymbol{\mu}_4) \to H^1(k, \mathrm{Spin}_{10})$$

is surjective. The copy of $\boldsymbol{\mu}_2$ here is the center of Spin_9, which is contained in $\boldsymbol{\mu}_4$. So combining all the previous statements we obtain an inclusion

$$\mathrm{Spin}_7 \times \boldsymbol{\mu}_4 \subset \mathrm{Spin}_{10}$$

that gives a surjection on H^1's.

In terms of quadratic forms, we view Spin_{10} as the spin group of the quadratic form $q := \langle 1, -1\rangle \oplus 4\langle 1, -1\rangle$, where Spin_7 acts on the second summand. Therefore, we have proved that the image of the map

$$H^1(k, \mathrm{Spin}(q)) \to H^1(k, \mathrm{SO}(q)) \tag{15.9}$$

consists of isotropic quadratic forms. On the other hand, the image of (15.9) is precisely the collection of 10-dimensional forms in I^3, so we have recovered Pfister's result—see [Pfi66, p. 123] or [Lam05, XII.2.8]—that such forms are isotropic. (Pfister's proof used quadratic form theory. Tits gave a characteristic-free proof using algebraic groups in [Tit90, 4.4.1(ii)]. We remark that this theorem has been generalized by Hoffmann, Vishik, and Karpenko: There are no anisotropic forms in I^n of dimension d such that $2^n < d < 2^n + 2^{n-1}$ for $n \geq 2$, see e.g. [EKM].)

We can find a subgroup of Spin_{10} that is smaller than $\mathrm{Spin}_7 \times \boldsymbol{\mu}_4$ and yet still gives a surjection on H^1's. As in the remarks at the end of Example 15.5, everything in the image of (15.9) is in the image of

$$H^1(k, G_2 \times \boldsymbol{\mu}_4) \to H^1(k, \mathrm{Spin}(q)) \to H^1(k, \mathrm{SO}(q)), \tag{15.10}$$

where G_2 is a subgroup of Spin_7 in the natural way. Said differently, everything in $H^1(k, \mathrm{Spin}(q))$ is in the $H^1(k, \boldsymbol{\mu}_4)$-orbit of something in the image of $H^1(k, G_2)$, i.e., the first map in (15.10) is surjective.

Instead of starting with Example 15.1, we could have viewed Spin_{10} as a subgroup of E_6, in which case the representation V given by 8.11 is a half-spin representation. However, this gives an ugly stabilizer, see [Igu70, Prop. 2].

The following exercise gives an example of a useful surjection on cohomology.

15.11. Exercise. Recall that for every quadratic étale k_0-algebra k_1, there is a surjective functor $\mathrm{Quad}_n \to \mathrm{Herm}_{k_1/k_0,n}$ that sends a quadratic form q to a k_1/k_0-hermitian form q_H ("hermitian forms can be diagonalized"). That is, in the commutative diagram

$$\begin{array}{ccc} H^1(*, \mathrm{O}(q)) & \longrightarrow & H^1(*, \mathrm{U}(q_H)) \\ \uparrow & & \uparrow \\ H^1(*, \mathrm{SO}(q)) & \longrightarrow & H^1(*, \mathrm{SU}(q_H)), \end{array}$$

the top arrow is a surjection. Prove that the bottom arrow is also a surjection.

15.12. **Example** ($\mathrm{SO}(6) \times \boldsymbol{\mu}_4 \subset \mathrm{Spin}_{12}$)**.** Take $\widetilde{G}$ to be the split simply connected group of type E_7 and $\pi := \alpha_1$. The subgroup G is the split simply connected group Spin_{12} of type D_6 and V is a half-spin representation. Speaking concretely, we view Spin_{12} as the spin group of the symmetric bilinear form b on the space with basis $e_1, e_2, \ldots, e_{12}$ such that

$$b(e_i, e_j) = b(e_{6+i}, e_{6+j}) = 0 \quad \text{and} \quad b(e_i, e_{6+j}) = \delta_{ij} \quad (1 \leq i, j \leq 6)$$

Our b is the same bilinear form used by Igusa in [Igu70]; as he did, we write $e_L := e_1 e_2 \cdots e_6$. The element $v := 1 + e_L \in V$ belongs to the open orbit in $\mathbb{P}(V)$, and the stabilizer of v in Spin_{12} is isomorphic to SL_6 by [Igu70, Prop. 3] such that the copy of $\boldsymbol{\mu}_2$ in the center of SL_6 is the kernel of the (half-spin) representation on V.

We identify SL_6 with its image in Spin_{12}. As SL_6 does not meet the kernel of the vector representation, it is identified with its image in $\mathrm{SO}(b)$. With respect to our fixed basis, b is the form

$$b(x,y) = x^t \left(\begin{smallmatrix} 0 & I_6 \\ I_6 & 0 \end{smallmatrix}\right) y$$

and SL_6 sits inside GL_{12} as the matrices of the form

$$\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-t} \end{smallmatrix}\right) \quad (a \in SL_6)$$

We claim that the stabilizer N of $[v]$ is isomorphic to $SL_6 \rtimes \boldsymbol{\mu}_4$. Fix a primitive 4-th root of unity ζ (in some algebraic closure of k_0) and put

$$s := \zeta\,(e_1 + \zeta e_7)(e_2 + \zeta e_8) \cdots (e_6 + \zeta e_{12}).$$

This element belongs to Spin_{12} and satisfies $s \cdot v = \zeta v$, and s^2 is the element -1 in the Clifford algebra, i.e., the nontrivial element in the kernel of the vector representation of Spin_{12}. As V supports a Spin_{12}-invariant quartic form [Igu70, Prop. 3], it follows that N is generated by SL_6 and s, hence N is isomorphic to $SL_6 \rtimes \boldsymbol{\mu}_4$.

We now compute the action of $\boldsymbol{\mu}_4$ on SL_6. Write $\chi\colon \mathrm{Spin}(b) \to \mathrm{SO}(b)$ for the vector representation of Spin_{12}. Then

$$\chi(s) = \begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix} \quad \text{and} \quad \chi(sas^{-1}) = \chi(a^{-t})$$

for $a \in SL_6$. Hence sas^{-1} equals a^{-t} in Spin_{12}.

As in Th. 8.11 (or Th. 11.10) we have a surjection

$$H^1(*, SL_6 \rtimes \boldsymbol{\mu}_4) \to H^1(*, \mathrm{Spin}_{12}).$$

Write $\mathrm{SO}(6)$ for the special orthogonal group of the dot product on k_0^6. It is a subgroup of SL_6 and is fixed elementwise by the map $g \mapsto g^{-t}$, so there is a natural inclusion

$$\mathrm{SO}(6) \times \boldsymbol{\mu}_4 \hookrightarrow SL_6 \rtimes \boldsymbol{\mu}_4$$

that is the identity on $\boldsymbol{\mu}_4$. The induced map

$$H^1(k, \mathrm{SO}(6) \times \boldsymbol{\mu}_4) \to H^1(k, SL_6 \rtimes \boldsymbol{\mu}_4)$$

is a surjection for every extension k/k_0. (To see that a given class in $\eta \in H^1(k, SL_6 \rtimes \boldsymbol{\mu}_4)$ is in the image, twist by the image of η in $H^1(k, \boldsymbol{\mu}_4)$ and then apply Exercise 15.11.) It follows that the inclusion $\mathrm{SO}(6) \times \boldsymbol{\mu}_4 \to \mathrm{Spin}_{12}$ induces a surjection on H^1's.

Concretely, this says that every 12-dimensional quadratic form in I^3 is isomorphic to $\langle 1, -a\rangle q$ for some $a \in k^\times$ and some 6-dimensional quadratic form q with determinant 1, a result due to Pfister [Pfi66, pp. 123, 124]. Hoffmann has conjectured [Hof98, Conj. 2] a generalization of this statement for forms of dimension $2^n + 2^{n-1}$ in I^n with $n \geq 4$.

15.13. **Example** ($\mathrm{SO}(5) \times \boldsymbol{\mu}_4 \subset \mathrm{Spin}_{11}$)**.** We view Spin_{12} as the spin group of the bilinear form b from Example 15.12. In this way, we see Spin_{11} as a subgroup of Spin_{12} consisting of elements that fix the vector

$$\varepsilon_1 := e_6 - e_{12}$$

in the space underlying b. The image of Spin_{11} under the vector representation $\chi\colon \mathrm{Spin}_{12} \to \mathrm{SO}(b)$ is the special orthogonal group of b restricted to the subspace with basis

$$\varepsilon_0 := e_6 + e_{12}, e_1, e_2, \ldots, e_5, e_7, e_8, e_9, \ldots, e_{11}.$$

(This is the description of $\mathrm{Spin}_{2n-1} \subset \mathrm{Spin}_{2n}$ used on pages 1000, 1001 of [Igu70].)

The spin representation of Spin_{11} is the restriction of the half-spin representation V of Spin_{12} from the previous example, and $v := 1 + e_L$ is again a representation of an open orbit in $\mathbb{P}(V)$ [Igu70, Prop. 6]. The stabilizer N of $[v]$ in Spin_{11} is the intersection of Spin_{11} with $SL_6 \rtimes \boldsymbol{\mu}_4$.

We first compute the intersection of $\chi(\mathrm{Spin}_{11})$ with $\chi(SL_6 \rtimes \boldsymbol{\mu}_4)$ in $\mathrm{SO}(b)$, i.e., we find the subgroup of $\chi(SL_6 \rtimes \boldsymbol{\mu}_4)$ that fixes ε_1. The intersection is $SL_5 \rtimes \boldsymbol{\mu}_2$, where SL_5 is viewed as the subgroup of matrices

$$\left(\begin{array}{cc|cc} a & & & \\ & I_1 & & \\ \hline & & a^{-t} & \\ & & & I_1 \end{array}\right) \qquad (a \in SL_5)$$

of GL_{12} and the nontrivial element of $\boldsymbol{\mu}_2$ is

$$\chi\left(\begin{pmatrix} \zeta^{-1} I_5 & \\ & \zeta \cdot I_1 \end{pmatrix} s\right) = \left(\begin{array}{cc|cc} & & I_5 & \\ & & & -I_1 \\ \hline I_5 & & & \\ & -I_1 & & \end{array}\right).$$

It follows that the stabilizer N of $[v]$ in Spin_{11} is $SL_5 \rtimes \boldsymbol{\mu}_4$. As in the previous example, the composition

$$\mathrm{SO}(5) \times \boldsymbol{\mu}_4 \to SL_5 \rtimes \boldsymbol{\mu}_4 \to \mathrm{Spin}_{11}$$

induces a surjection on H^1's.

16. Invariants of Spin_n for $n \leq 10$

We now determine the invariants of Spin_n (for $n \leq 10$) with values in $\mathbb{Z}/2\mathbb{Z}$. We assume throughout this section that the field k_0 has characteristic $\neq 2$.

16.1. Invariants of Spin_8. Combining Examples 15.5 and 15.1, we find an inclusion

$$i\colon G_2 \times Z \to \mathrm{Spin}_8$$

such that the induced map i_* on H^1's is surjective, where Z is the center of Spin_8 and is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. As Spin_8 is split, the image of $H^1(k, Z)$ in $H^1(k, \mathrm{Spin}_8)$ is zero. Applying Lemma 5.7, i^* identifies $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_8, \mathbb{Z}/2\mathbb{Z})$ with an $R_2(k_0)$-submodule of the free module I with basis the invariants

$$e_3 \cdot 1 \cdot 1, \quad e_3 \cdot \underline{\mathrm{id}} \cdot 1, \quad e_3 \cdot 1 \cdot \underline{\mathrm{id}}, \quad e_3 \cdot \underline{\mathrm{id}} \cdot \underline{\mathrm{id}}$$

of $G_2 \times Z$, where 1 and $\underline{\mathrm{id}}$ are as defined in §2. Fix inequivalent 8-dimensional representations $\chi_1, \chi_2 : \mathrm{Spin}_8 \to \mathrm{SO}_8$. They restrict to characters $\chi_j : Z \to \boldsymbol{\mu}_2$ which induce invariants $\underline{\chi}_j : H^1(*, Z) \to H^1(*, \boldsymbol{\mu}_2)$. Clearly, the invariants

$$e_3, \quad e_3 \cdot \underline{\chi}_1, \quad e_3 \cdot \underline{\chi}_2, \quad e_3 \cdot \underline{\chi}_1 \cdot \underline{\chi}_2 \tag{16.2}$$

are also an $R_2(k_0)$-basis for the module I. We prove that each of the invariants in (16.2) is the restriction of an invariant of Spin_8.

Let (C,ζ) be a class in $H^1(k, G_2 \times Z)$, where C is an octonion algebra. Abusing notation, we write $\chi_j(\zeta)$ for the corresponding element of $k^\times/k^{\times 2}$. The composition

$$H^1(k, G_2 \times Z) \longrightarrow H^1(k, \mathrm{Spin}_8) \xrightarrow{\chi_j} H^1(k, \mathrm{SO}_8) \tag{16.3}$$

sends (C,ζ) to the quadratic form $\langle \chi_j(\zeta)\rangle N_C$. Composing (16.3) with the Arason invariant e_3 defined in Example 1.2.3 sends (C,ζ) to $e_3(C)$. That is, the invariant e_3 from (16.2) is the restriction of an invariant of Spin_8.

Of course, $e_3(C)$ is zero whenever $\langle \chi_j(\zeta)\rangle N_C$ is isotropic. Applying Prop. 9.2 to the representations χ_1 and χ_2, we find that $e_3 \cdot \underline{\chi}_1$ and $e_3 \cdot \underline{\chi}_2$ are also restrictions of invariants of Spin_8. Finally, $e_3 \cdot \underline{\chi}_1$ is zero whenever $\langle \chi_2(\zeta)\rangle N_C$ is isotropic, and applying Prop. 9.2 again gives that $e_3 \cdot \underline{\chi}_1 \cdot \underline{\chi}_2$ is the restriction of an invariant of Spin_8.

We have proved that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_8, \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module of rank 4 with generators of degree 3, 4, 4, 5.

16.4. Invariants of Spin_7. By Example 15.5, there is a subgroup $G_2 \times \boldsymbol{\mu}_2$ of Spin_7 such that the induced map

$$i_* \colon H^1(*, G_2 \times \boldsymbol{\mu}_2) \to H^1(*, \mathrm{Spin}_8)$$

is surjective. Combined with the inclusion $\mathrm{Spin}_7 \hookrightarrow \mathrm{Spin}_8$ obtained by viewing Spin_7 as the identity component of the stabilizer of a vector of length 1, we have maps

$$\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G_2 \times \boldsymbol{\mu}_2, \mathbb{Z}/2\mathbb{Z}) \hookleftarrow \mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_7, \mathbb{Z}/2\mathbb{Z}) \leftarrow \mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_8, \mathbb{Z}/2\mathbb{Z}).$$

As in 16.1, the image in $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G_2 \times \boldsymbol{\mu}_2, \mathbb{Z}/2\mathbb{Z})$ is contained in the free $R_2(k_0)$-module I with basis $e_3, e_3 \cdot \underline{\mathrm{id}}$ and the invariant e_3 is the restriction of the Arason invariant on SO_8. Similarly, the copy of $\boldsymbol{\mu}_2$ in Spin_7 is the kernel of a representation $\mathrm{Spin}_8 \to \mathrm{SO}_8$, say χ_2. The invariant $e_3 \cdot \underline{\chi}_1$ of Spin_8 restricts to the invariant $e_3 \cdot \underline{\mathrm{id}}$ of $G_2 \times \boldsymbol{\mu}_2$. This proves that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_7, \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module of rank 2 with basis elements of degrees 3 and 4.

16.5. Invariants of Spin_{10}. From Example 15.8 we have an inclusion $i \colon G_2 \times \boldsymbol{\mu}_4 \to \mathrm{Spin}_{10}$ such that the induced map i_* on H^1's is surjective. For a Cayley k-algebra C and $\alpha \in k^\times/k^{\times 4}$, define

$$a_3(C,\alpha) = e_3(C) \quad \text{and} \quad a_4(C,\alpha) = e_3(C) \cdot \underline{s}(\alpha)$$

in $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G_2 \times \boldsymbol{\mu}_4, \mathbb{Z}/2\mathbb{Z})$. (The invariant $\underline{s}$ is defined in 2.5.) As for Spin_8, i^* identifies $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_{10}, \mathbb{Z}/2\mathbb{Z})$ with a submodule of the free $R_2(k_0)$-module with basis a_3, a_4. The image of a pair (C,α) in $H^1(k, \mathrm{SO}_{10})$ corresponds to the quadratic form $\langle 1,-1\rangle \oplus \langle\alpha\rangle N_C$, so a_3 and a_4 are obviously restrictions of invariants of Spin_{10}.

16.6. Invariants of Spin_9. We view $\mathrm{Spin}_8 \subset \mathrm{Spin}_9 \subset \mathrm{Spin}_{10}$ as the spin groups of the quadratic forms

$$4\langle 1,-1\rangle, \quad \langle -1\rangle \oplus 4\langle 1,-1\rangle, \quad \text{and} \quad \langle 1,-1\rangle \oplus 4\langle 1,-1\rangle$$

in the obvious manner. Combining Examples 15.5 and 15.7 and putting $Z = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, we find an inclusion of $G_2 \times Z$ in Spin_9 that gives a surjection

$$H^1(*, G_2 \times Z) \to H^1(*, \mathrm{Spin}_9)$$

and identifies $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_9, \mathbb{Z}/2\mathbb{Z})$ with a submodule of $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(G_2 \times Z, \mathbb{Z}/2\mathbb{Z})$, contained in the free $R_2(k_0)$-module with basis (16.2).

For the sake of fixing notation, suppose that the restriction of the vector representation of Spin_9 to Spin_8 is the direct sum of χ_1 (as opposed to χ_2 or χ_3) and a 1-dimensional trivial representation. The image of a pair $(C, \zeta) \in H^1(k, G_2 \times Z)$ under the maps

$$H^1(k, G_2 \times Z) \to H^1(k, \mathrm{Spin}_8) \to H^1(k, \mathrm{Spin}_{10}) \to H^1(k, \mathrm{SO}_{10})$$

is $\langle 1, -1\rangle \oplus \langle \chi_1(\zeta)\rangle N_C$. Thus the invariants a_3 and a_4 of Example 16.5 restrict to invariants e_3 and $e_3 \cdot \underline{\chi}_1$ of $G_2 \times Z$ from (16.2).

We can also view Spin_9 as the subgroup of the automorphism group of the split Albert algebra J consisting of the algebra automorphisms that fix a primitive idempotent in J [Jac68, §IX.3]. Restricting J to a representation of Spin_8, we find a direct sum of a 3-dimensional trivial representation and the three inequivalent irreps $\chi_1, \chi_2, \chi_3 \colon \mathrm{Spin}_8 \to \mathrm{SO}_8$. The invariant

$$f_5 \colon H^1(*, \mathrm{Aut}(J)) \to H^5(*, \mathbb{Z}/2\mathbb{Z})$$

defined in §22 of S restricts to be nonzero on $G_2 \times Z$. If -1 is a square in k_0, then its restriction is the invariant $e_3 \cdot \underline{\chi}_1 \cdot \underline{\chi}_2$ on $G_2 \times Z$ from (16.2). (The assumption on -1 is here only for the convenience of ignoring various factors of -1.)

Finally we claim that the invariant $\lambda \cdot e_3 \cdot \underline{\chi}_2$ of $G_2 \times Z$, for every nonzero $\lambda \in R_2(k_0)$, is *not* the restriction of an invariant of Spin_9. Let k be the extension of k_0 obtained by adjoining indeterminates x, y, z, w, and write C for the Cayley k-algebra with $e_3(C)$ equal to $(x)\cdot(y)\cdot(z)$. Fix a $\zeta \in H^1(k, Z)$ such that $\chi_1(\zeta) = (1)$ and $\chi_2(\zeta) = (w)$. The invariant $e_3 \cdot \underline{\chi}_2$ takes different values on $(C, 1)$ and (C, ζ), namely 0 and $\lambda \cdot (x) \cdot (y) \cdot (z) \cdot (w)$. However, the two classes have the same image in $H^1(k, \mathrm{SO}_9)$, the form $\langle -1\rangle \oplus N_C$. As this form is isotropic, its spinor norm map is onto and the fiber of

$$H^1(k, \mathrm{Spin}_9) \to H^1(k, \mathrm{SO}_9)$$

over $\langle -1\rangle \oplus N_C$ is a singleton. That is, $(C, 1)$ and (C, w) have the same image in $H^1(k, \mathrm{Spin}_9)$, proving the claim.

In the case where -1 is a square in k_0, this determines $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_9, \mathbb{Z}/2\mathbb{Z})$: it is free of rank 3 with basis elements of degree 3, 4, 5.

17. Divided squares in the Grothendieck-Witt ring

In this section, we define a function $P_n \colon I^n \to I^{2n}$ in the Witt ring that will be used to construct invariants of Spin_n for $n = 11, 12, 14$. It can also be used to give bounds on the symbol length of a class in $H^d(k, \mathbb{Z}/2\mathbb{Z})$, cf. Example A.3.

Recall the Grothendieck-Witt ring $\widehat{W}$ (denoted WGr in S) over a field k of characteristic $\neq 2$: it is the ring of formal differences of (nondegenerate) quadratic forms over k. It is a λ-ring in the sense of Grothendieck, see e.g. S27.1. For a quadratic form $q = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ and $0 < p \leq n$, we have

$$\lambda^p q = \oplus_{i_1 < i_2 < \cdots < i_p} \langle \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_p}\rangle.$$

In particular, $\lambda^0 q = \langle 1\rangle$ and $\lambda^1 q = q$.

17.1. **Example.** Writing $\mathcal{H}$ for a hyperbolic plane, we have:

$$\lambda^2(n\mathcal{H}) \cong (n^2 - n)\mathcal{H} \oplus n\langle -1\rangle.$$

17.2. Exercise. Prove: The Killing form on the Lie algebra $\mathfrak{so}(q)$ is $\langle -2\rangle\langle \dim q - 2\rangle \lambda^2 q$.

We will only make use of λ^2. Here are a few useful identities in $\widehat{W}$, where x and y denote quadratic forms:

$$\lambda^2(x+y) = \lambda^2 x + xy + \lambda^2 y \tag{17.3}$$

$$\lambda^2(\langle c \rangle x) = \lambda^2 x \tag{17.4}$$

$$\lambda^2(x-y) = \lambda^2 x - y(x-y) - \lambda^2 y = \lambda^2 x - xy + \dim y + \lambda^2 y \tag{17.5}$$

$$\lambda^2(xy) = x^2\lambda^2 y + y^2 \lambda^2 x - 2(\lambda^2 x)(\lambda^2 y) \tag{17.6}$$

17.7. **Example.** For a quadratic form z and a natural number n, the image of $\lambda^2(z - n\mathcal{H})$ in the Witt ring is $n + \lambda^2 z$, as can be seen by combining (17.5) and Example 17.1.

17.8. **Lemma.** *For every n-Pfister form ϕ with $n \geq 1$, we have: $\lambda^2\phi \cong 2^{n-1}\phi'$.*

Proof. By induction on n. As $\lambda^2\langle 1, -\alpha\rangle$ is isomorphic to $\langle -\alpha \rangle$, the case $n = 1$ holds. For ϕ an n-Pfister form with $n > 1$, we may write[g] $\phi = \langle\langle \alpha \rangle\rangle \psi$ for some $\alpha \in k^\times$ and $(n-1)$-Pfister ψ. In $\widehat{W}$, we have

$$\lambda^2\phi = \langle\langle \alpha \rangle\rangle^2 \lambda^2\psi + \langle -\alpha \rangle \psi^2 - 2\langle -\alpha\rangle \lambda^2 \psi$$

by (17.6). In the Witt ring, $\langle\langle \alpha \rangle\rangle^2 - 2\langle -\alpha \rangle$ equals 2, and

$$\lambda^2 \phi = 2\lambda^2\psi + \langle -\alpha \rangle \psi^2,$$

which by the induction hypothesis is

$$2^{n-1}\psi' + \langle -\alpha \rangle \psi^2 = 2^{n-1}(\psi' + \langle -\alpha \rangle \psi) = 2^{n-1}(\langle\alpha\rangle\psi)'.$$

Since $\lambda^2\phi$ equals $2^{n-1}\phi'$ in the Witt ring and both have dimension $2^{n-1}(2^n - 1)$, the conclusion follows. □

For q an even-dimensional quadratic form, there is a canonical lift $\hat{q}$ to the Grothendieck-Witt ring $\widehat{W}$, namely

$$\hat{q} := q - r\mathcal{H}, \quad \text{where } \dim q = 2r. \tag{17.9}$$

Note that $\hat{q} \in \widehat{W}$ only depends on q up to Witt-equivalence. (This is just a restatement of the fact that the quotient map $\widehat{W} \to W$ restricts to an isomorphism $\widehat{I} \xrightarrow{\sim} I$, where $\widehat{I}$ is ideal of zero-dimensional virtual forms; $\hat{q}$ is the inverse image of q under this isomorphism.) For $n \geq 1$, we define

$$P_n \colon I \to W \quad \text{via} \quad P_n(x) := \lambda^2\hat{x} - 2^{n-1}x,$$

where we conflate $\lambda^2\hat{x}$ with its image in the Witt ring. We remark that the device of replacing x with $\hat{x}$ is necessary, as λ^2 is not well-behaved with respect to Witt-equivalence. (For example, the dimensions of $\lambda^2\mathcal{H}$ and $\lambda^2(2\mathcal{H})$ are not even congruent mod 2.)

Using Example 17.7, it is easy to check that

$$P_n(x+y) = P_n(x) + xy + P_n(y) \tag{17.10}$$

and

$$P_n(\langle c \rangle x) = P_n(x) + 2^{n-1}\langle\langle c \rangle\rangle x \tag{17.11}$$

hold, for $x, y \in I$ and $c \in k^\times$.

---

[g]Here and below we write $\langle\langle \alpha_1, \ldots, \alpha_n \rangle\rangle$ for the n-Pfister form $\otimes_{i=1}^n \langle 1, -\alpha_i \rangle$.

17.12. **Proposition.** *For $n \geq 1$:*

(1) *P_n is zero on n-Pfister forms.*
(2) *P_n restricts to a map $I^n \to I^{2n}$.*

If $n \geq 2$ and -1 is a square in k:

(3) *P_n induces a map $I^n/I^{n+1} \to I^{2n}/I^{2n+1}$.*
(4) *For $c_i \in k^\times$ and n-Pfister forms ϕ_i, we have:*

$$P_n\left(\sum_i \langle c_i\rangle\phi_i\right) = \sum_{i<j} \langle c_i c_j\rangle \phi_i\phi_j$$

Proof. Combining Example 17.7 and Lemma 17.8 gives (1). For (2), we use that every element of I^n is a sum of elements of the form $\langle c\rangle\phi$, where ϕ is an n-Pfister form and c is in $k^\times$. By (17.10) and (17.11), it suffices to prove that $P_n(\phi)$ belongs to I^{2n}, which is true by (1).

Both (3) and (4) rest on the fact that $2^{n-1} = 0$ in the Witt ring because n is at least 2 and -1 is a square. We prove (3). Let $x, y \in I^n$ be such that $z := x - y$ belongs to I^{n+1}. Then $\hat{z} = \hat{x} - \hat{y}$ in $\widehat{W}$, and we have

$$P_{n+1}(z) = \lambda^2\hat{x} - \lambda^2\hat{y} - yz - 2^n z$$

by (17.5). So:

$$P_n(x) - P_n(y) = P_{n+1}(z) + yz + 2^{n-1}z.$$

All three summands on the right belong to I^{2n+1}. For the first term, this is (2). For the last term, it is because $2^{n-1} = 0$.

As for (4), under our special hypotheses, Equation (17.11) takes the nice form:

$$P_n(\langle c\rangle x) = P_n(x).$$

Applying (17.10) and (1) gives (4). □

For the remainder of this section, we maintain the hypotheses that -1 is a square in k_0 and n is at least 2. Applying the map e_{2n} from Example 1.2.3 to Prop. 17.12.4 gives:

$$e_{2n}\left(P_n\left(\sum_i \langle c_i\rangle\phi_i\right)\right) = \sum_{i<j} e_n(\phi_i)e_n(\phi_j). \tag{17.13}$$

17.14. **Example** (Invariants of SO(6))**.** We write the invariants of SO(6) (the special orthogonal group of the dot product) in terms of the maps e_n and P_n. By S20.6, the normalized invariants of SO(6) with values in $\mathbb{Z}/2\mathbb{Z}$ form a free $R_2(k_0)$-module with basis w_2, w_4, b, where b satisfies

$$b(\langle\alpha_1, \alpha_2, \ldots, \alpha_5, \alpha_6\rangle) = (\alpha_1)\cdot(\alpha_2)\cdots(\alpha_5).$$

(In S20.1, this b was denoted "b_1", where 1 is the nonzero element of $H^0(k_0, \mathbb{Z}/2\mathbb{Z})$, i.e., the identity element of $R_2(k_0)$.)

An element of $H^1(k, \mathrm{SO}(6))$ corresponds to a 6-dimensional form q in I^2. Such a form is isomorphic to $\langle\beta\rangle(\phi_1' \oplus \langle -1\rangle\phi_2')$ for some $\beta \in k^\times$ and 2-Pfister forms ϕ_1, ϕ_2. Direct computation gives

$$w_2(q) = e_2(q),$$
$$w_4(q) = w_2(\langle\beta\rangle\phi_1')\cdot w_2(\langle\beta\rangle\phi_2') = e_2(\phi_1)\cdot e_2(\phi_2) = e_4(P_2(q))$$

by (17.13), and

$$b(q) = (\beta)\cdot w_4(q).$$

17.15. Prop. 17.12.4 makes P_n look like a "divided square", meaning a squaring operation from a divided power structure. We remark that—still assuming that -1 is a square—there are also divided square operations on Milnor K-theory $P_n^M : K_n^M/2 \to K_{2n}^M/2$, see [Kah00, App. A]. For $n \geq 2$, the diagram

$$\begin{array}{ccc} K_n^M/2 & \xrightarrow{P_n^M} & K_{2n}^M/2 \\ \downarrow & & \downarrow \\ I^n/I^{n+1} & \xrightarrow{P_n} & I^{2n}/I^{2n+1} \end{array}$$

commutes, where the vertical arrows are the natural surjections that send the symbol $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ to the class of the Pfister form $\langle\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle\rangle$.

18. Invariants of Spin_{11} and Spin_{12}

We now determine the invariants of Spin_{12} and Spin_{11} with values in $\mathbb{Z}/2\mathbb{Z}$. We assume throughout this section that the field k_0 has characteristic $\neq 2$. We begin with some results on quadratic forms.

18.1. **Lemma.** *Let x, y be quadratic forms of the same dimension and fix $c \in k^\times$. if $\langle\langle c \rangle\rangle(x - y)$ is zero in the Witt ring, then $\langle\langle c \rangle\rangle \lambda^2(x - y) \in \widehat{W}$ maps to zero in the Witt ring.*

Proof. Replacing x, y with $x \oplus \langle -1 \rangle y$, $(\dim y)\mathcal{H}$ respectively does not change $\langle\langle c \rangle\rangle(x - y)$ nor the image of $\lambda^2(x - y)$ in the Witt ring. Therefore, we may assume that x has even dimension $2r$ and $y = r\mathcal{H}$.

The hypothesis on $\langle\langle c \rangle\rangle(x - y)$ says that the quadratic form $\langle\langle c \rangle\rangle x$ is hyperbolic, so by [EL73, 2.2], x is isomorphic to a sum $\oplus_{i=1}^r \langle c_i \rangle \langle\langle n_i \rangle\rangle$ such that $c_i \in k^\times$ and n_i is a norm from $k(\sqrt{c})$. Then by Example (17.7) and (17.3), $\langle\langle c \rangle\rangle \lambda^2(x - r\mathcal{H})$ maps to

$$r\langle\langle c \rangle\rangle + \langle\langle c \rangle\rangle \sum_{i=1}^r \langle -n_i \rangle + \sum_{1 \leq i < j \leq r} \langle\langle c, n_i, n_j \rangle\rangle \quad \text{in } W. \tag{18.2}$$

Because the n_i's are norms, the middle term equals $-r\langle\langle c \rangle\rangle$ and each of the forms $\langle\langle c, n_i, n_j \rangle\rangle$ is hyperbolic. That is, (18.2) is zero. □

18.3. **Proposition.** *For $x \in I^2$, the class of $\langle\langle c \rangle\rangle \lambda^2 \hat{x}$ in the Witt ring depends only on the isomorphism class of $\langle\langle c \rangle\rangle x$ (and not on c or x).*

(See (17.9) for a definition of $\hat{x}$.)

Proof. Write $\dim x = 2r$ and suppose that $\langle\langle c \rangle\rangle x$ is isomorphic to $\langle\langle d \rangle\rangle y$ for some $d \in k^\times$ and $2r$-dimensional form y. We must show that $\langle\langle c \rangle\rangle \lambda^2(x - r\mathcal{H})$ and $\langle\langle d \rangle\rangle \lambda^2(y - r\mathcal{H})$ have the same image in the Witt ring. Let τ be the $2r$-dimensional form provided by Cor. B.5 such that

$$\langle\langle c \rangle\rangle x = \langle\langle c \rangle\rangle \tau = \langle\langle d \rangle\rangle \tau = \langle\langle d \rangle\rangle y.$$

We first prove

$$\langle\langle c \rangle\rangle \lambda^2(x - r\mathcal{H}) = \langle\langle c \rangle\rangle \lambda^2(\tau - r\mathcal{H}) \quad \text{in } W. \tag{18.4}$$

In view of Example 17.7, it suffices to prove that $\langle\langle c\rangle\rangle(\lambda^2 x - \lambda^2\tau)$ is zero in the Witt ring. Applying (17.5), we find:

$$\langle\langle c\rangle\rangle(\lambda^2 x - \lambda^2\tau) = \langle\langle c\rangle\rangle(\lambda^2(x-\tau) + (x-\tau)\tau) \quad \text{in } \widehat{W}. \tag{18.5}$$

Since $\langle\langle c\rangle\rangle x$ and $\langle\langle c\rangle\rangle\tau$ are isomorphic, $\langle\langle c\rangle\rangle(x-\tau)$ is zero in the Witt ring, and Lemma 18.1 gives that $\langle\langle c\rangle\rangle\lambda^2(x-\tau)$ is hyperbolic. We conclude that (18.5) is zero and hence that (18.4) holds. By symmetry, we also have (18.4) where c and x are replaced by d and y.

It remains to prove

$$\langle\langle c\rangle\rangle\lambda^2(\tau - r\mathcal{H}) = \langle\langle d\rangle\rangle\lambda^2(\tau - r\mathcal{H}) \quad \text{in } W. \tag{18.6}$$

Since

$$\langle\langle c\rangle\rangle - \langle\langle d\rangle\rangle = \langle -c, d\rangle = \langle d\rangle\langle\langle cd\rangle\rangle \quad \text{in } W$$

and $\langle\langle c\rangle\rangle\tau$ is isomorphic to $\langle\langle d\rangle\rangle\tau$, the form $\langle\langle cd\rangle\rangle\tau$ is hyperbolic. Applying Lemma 18.1, we find that $\langle\langle cd\rangle\rangle\lambda^2(\tau - r\mathcal{H})$ is zero in the Witt ring. That is, (18.6) holds. □

From here until the end of this section, *we assume that -1 is a square in k_0*. We now construct invariants a_5 and a_6 of Spin_{12} as in Rost's paper [Ros99c].

18.7. Definition of a_5. For $\eta \in H^1(k, \mathrm{Spin}_{12})$, Example 15.12 says that the corresponding quadratic form $q_\eta \in H^1(k, \mathrm{SO}_{12})$ is isomorphic to $\langle\langle c\rangle\rangle x$ for some $c \in k^\times$ and 6-dimensional form x of determinant 1. As -1 is a square in k, the form x belongs to I^2 and $\langle\langle c\rangle\rangle P_2(x)$ is in I^5. We define $a_5(\eta) \in H^5(k, \mathbb{Z}/2\mathbb{Z})$ to be $e_5(\langle\langle c\rangle\rangle P_2(x))$, equivalently, $(c)\cdot e_4(P_2(x))$. Prop. 18.3 shows that $a_6(\eta)$ is well defined: As -1 is a square in k, we have $2 = 0$ in the Witt ring, so $P_2(x) = \lambda^2(x - 3\mathcal{H})$.

18.8. **Example.** The invariant a_5 is not zero. Indeed, let k be the field obtained by adjoining indeterminates u, w and v_1, v_2, v_3, v_4 to k_0. The 12-dimensional form $q := \langle w\rangle\langle\langle u\rangle\rangle(\langle\langle v_1, v_2\rangle\rangle - \langle\langle v_3, v_4\rangle\rangle)$ belongs to I^3, hence it is of the form q_η for some class $\eta \in H^1(k, \mathrm{Spin}_{12})$. By (17.13), we find:

$$a_5(\eta) = (u)\cdot(v_1)\cdot(v_2)\cdot(v_3)\cdot(v_4).$$

18.9. **Lemma.** *If q_η is isotropic, then $a_5(\eta)$ is zero.*

Proof. If q_η is isotropic, then it is Witt-equivalent to a 10-dimensional form in I^3, hence by Example 15.8 it is isomorphic to $\langle d\rangle\langle\langle c\rangle\rangle\phi + 2\mathcal{H}$ for some $c, d \in k^\times$ and 2-Pfister form ϕ, equivalently, is isomorphic to $\langle\langle c\rangle\rangle(\langle d\rangle\phi + \mathcal{H})$. As -1 is a square in k_0, we have:

$$a_5(\eta) = (c)\cdot e_4(P_2(\langle d\rangle\phi)) = 0.$$ □

18.10. Definition of a_6. Prop. 9.2 applied to a_5 gives an invariant a_6 of Spin_{12} defined by setting

$$a_6(\eta) = a_5(\eta)\cdot(\alpha),$$

where α is a nonzero element of $k^\times$ represented by q_η.

In Example 18.8, the form q_η represents wv_3, so

$$a_6(\eta) = a_5(\eta)\cdot(wv_3) = (u)\cdot(v_1)\cdot(v_2)\cdot(v_3)\cdot(v_4)\cdot(w).$$

In particular, a_6 is not the zero invariant.

18.11. **Proposition.** ($\sqrt{-1} \in k_0$) *$\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_{12}, \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module with basis e_3 (the Rost invariant), a_5, a_6.*

Proof. Recall from S20.6 or Example 17.14 that $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{SO}(6), \mathbb{Z}/2\mathbb{Z})$ is a free $R_2(k_0)$-module with basis w_2, w_4, b. By Example 15.12, the inclusion $\mathrm{SO}(6) \times \boldsymbol{\mu}_4 \to \mathrm{Spin}_{12}$ induces a surjection on H^1's. The image of SO(6) in Spin_{12} sits in a copy of SL_6, so $H^1(*, \mathrm{SO}(6)) \to H^1(*, \mathrm{Spin}_{12})$ is the zero map. Applying Lemma 5.7 with $G = \boldsymbol{\mu}_4$ and $G' = \mathrm{SO}(6)$, we find that restricting invariants of Spin_{12} to $\mathrm{SO}(6) \times \boldsymbol{\mu}_4$ identifies $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_{12}, \mathbb{Z}/2\mathbb{Z})$ with a submodule of the free $R_2(k_0)$-module with basis

$$1 \cdot \underline{s}, \quad w_2 \cdot \underline{s}, \quad w_4 \cdot \underline{s}, \quad b \cdot \underline{s}.$$

The last three are invariants of Spin_{12} by Example 17.14, e.g., $b \cdot \underline{s}$ is a restriction of a_6. However, $\lambda \cdot 1 \cdot \underline{s}$ is not such a restriction for any nonzero $\lambda \in R_2(k_0)$. To see this, one argues as in 16.6, comparing the images of the trivial class and an indeterminate $(t) \in H^1(k_0(t), \boldsymbol{\mu}_4)$ in $H^1(k_0(t), \mathrm{Spin}_9)$. □

18.12. Invariants of Spin_{11}. There are two invariants of Spin_{11} with values in $\mathbb{Z}/2\mathbb{Z}$ that we can find without doing any work. As always, one has the Rost/Arason invariant $e_3 \colon H^1(*, \mathrm{Spin}_{11}) \to H^3(*, \mathbb{Z}/2\mathbb{Z})$. On the other hand, the inclusion of Spin_{11} in Spin_{12} from Example 15.13 leads to an invariant of Spin_{11} of degree 5 via the composition

$$H^1(*, \mathrm{Spin}_{11}) \to H^1(*, \mathrm{Spin}_{12}) \xrightarrow{a_5} H^5(*, \mathbb{Z}/2\mathbb{Z}).$$

We denote this invariant also by a_5. (Note that restricting a_6 to Spin_{11} gives the zero invariant. Indeed, the image of $H^1(*, \mathrm{Spin}_{11})$ in $H^1(*, \mathrm{SO}_{12})$ consists of those forms that represent 1.)

Proposition. ($\sqrt{-1} \in k_0$) $\mathrm{Inv}^{\mathrm{norm}}_{k_0}(\mathrm{Spin}_{12}, \mathbb{Z}/2\mathbb{Z})$ *is a free* $R_2(k_0)$*-module with basis* e_3, a_5*.*

Proof. As in the proof of Prop. 18.11, we restrict the invariants of Spin_{11} to the subgroup $\mathrm{SO}(5) \times \boldsymbol{\mu}_4$. Recall from S19.1 that $\mathrm{Inv}_{k_0}(\mathrm{SO}(5), \mathbb{Z}/2\mathbb{Z})$ is a free $\mathbb{Z}/2\mathbb{Z}$-module with basis $1, w_2, w_4$. Therefore, the set of normalized invariants of Spin_{11} with values in $\mathbb{Z}/2\mathbb{Z}$ is identified with a subspace of the free $R_2(k_0)$-module with basis

$$1 \cdot \underline{s}, \quad w_2 \cdot \underline{s}, \quad w_4 \cdot \underline{s}.$$

We have a commutative diagram

$$\begin{array}{ccc} H^1(k, \mathrm{SO}(5)) \times H^1(k, \boldsymbol{\mu}_4) & \longrightarrow & H^1(k, \mathrm{SO}(6)) \times H^1(k, \boldsymbol{\mu}_4) \\ \downarrow & & \downarrow \\ H^1(k, \mathrm{Spin}_{11}) & \longrightarrow & H^1(k, \mathrm{Spin}_{12}) \end{array}$$

The inclusion $\mathrm{SO}(5) \to \mathrm{SO}(6)$ is given by $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so the arrow $H^1(k, \mathrm{SO}(5)) \to H^1(k, \mathrm{SO}(6))$ sends a 5-dimensional quadratic form q to $q \oplus \langle 1 \rangle$. The restriction of $w_j \colon H^1(k, \mathrm{SO}(6)) \to H^j(k, \mathbb{Z}/2\mathbb{Z})$ to SO(5) is

$$w_j(q \oplus \langle 1 \rangle) = (1) \cdot w_{j-1}(q) + w_j(q) = w_j(q),$$

so the invariants e_3 and a_5 of Spin_{12} restrict to $w_2 \cdot \underline{s}$ and $w_4 \cdot \underline{s}$ on $H^1(k, \mathrm{SO}(5) \times \boldsymbol{\mu}_4)$.

As in the proof of Prop. 18.11, one checks that $\lambda \cdot 1 \cdot \underline{s}$ is not the restriction of an invariant of Spin_{11} for any nonzero $\lambda \in R_2(k_0)$. □

19. Surjectivities: Spin_{14}

Throughout this section, we assume that the field k_0 has characteristic different from 2. We fix a primitive 4-th root of unity i in a separable closure of k_0.

19.1. **Example** $((G_2 \times G_2) \rtimes \boldsymbol{\mu}_8 \subset \mathrm{Spin}_{14})$**.** Returning to the methods of 8.11, we take $\widetilde{G}$ to be the split group of type E_8 and we omit the root $\pi := \alpha_1$. The semisimple subgroup G is simply connected of type D_7—i.e., it is isomorphic to Spin_{14}—and the representation V is a half-spin representation.

Fix a 7-dimensional quadratic form q such that $\langle 1 \rangle \oplus q$ is hyperbolic. We view Spin_{14} as the spin group of the quadratic form $q \oplus -q$, which gives a homomorphism $\mathrm{Spin}(q) \times \mathrm{Spin}(-q) \to \mathrm{Spin}_{14}$. We may identify the vector spaces underlying the form q and underlying the 7-dimensional fundamental representation of G_2 (which we call the standard representation of G_2) so that q is G_2-invariant. (Note that the standard representation of G_2 is irreducible since the characteristic is different from 2 [GS88, p. 413].) This gives an embedding of G_2 in $\mathrm{Spin}(q)$, hence of $G_2 \times G_2$ in Spin_{14}.

We now argue as in Example 8.12. The restriction of the representation V to $\mathrm{Spin}(q) \times \mathrm{Spin}(-q)$ is the tensor product of the (8-dimensional) spin representations of $\mathrm{Spin}(q)$ and $\mathrm{Spin}(-q)$. As in Example 15.5, each of these restricts to be a direct sum of the 7-dimensional irreducible representation of G_2 and a 1-dimensional trivial representation. We take v to be a tensor product of nonzero vectors that are fixed by the two G_2's.

To see that $G_2 \times G_2$ is not contained in a proper parabolic subgroup of L, we note that $G_2 \times G_2$ has no faithful representations of dimension < 14 [KL90, 5.4.13], so it cannot be contained in a group of type D_n for $n < 7$ or A_n for $n < 13$. (Popov [Pop80, p. 225, Prop. 11] gives a proof that $G_2 \times G_2$ is the identity component of N using concrete computations in the half-spin representation in the style of Igusa's paper [Igu70].)

We conclude that $G_2 \times G_2$ is the identity component of the stabilizer N of $[v]$ in Spin_{14}. Rather than computing the full stabilizer N, we compute instead the normalizer of $G_2 \times G_2$ in Spin_{14}, which contains N.

Write W for the 14-dimensional vector space underlying $q \oplus -q$. The image of $G_2 \times G_2$ in $GL(W)$ has normalizer

$$((G_2.\mathbb{G}_m) \times (G_2.\mathbb{G}_m)) \rtimes \mathbb{Z}/2\mathbb{Z}, \tag{19.2}$$

where the nonidentity element in $\mathbb{Z}/2\mathbb{Z}$ is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The normalizer of $G_2 \times G_2$ in $\mathrm{SO}(W)$ is the intersection of (19.2) with $\mathrm{SO}(W)$, namely $(G_2 \times G_2) \rtimes \boldsymbol{\mu}_4$, where a primitive 4-th root of unity i in $\boldsymbol{\mu}_4$ is identified with the matrix

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathrm{SO}(W).$$

Fix orthogonal bases $\{x_j\}$ and $\{y_j\}$ of the two standard representation of G_2 in W such that $q(x_j) = -q(y_j) = \pm 1$ for all j. The element

$$s := \prod_{j=1}^{7} \frac{1 + i x_j y_j}{\sqrt{2}}$$

in the even Clifford algebra belongs to Spin_{14}, has order 8 since $s^2 = \prod i x_j y_j$, and maps to $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ in $\mathrm{SO}(W)$. Therefore, the normalizer of $G_2 \times G_2$ in Spin_{14} is $(G_2 \times G_2) \rtimes \boldsymbol{\mu}_8$, where the copy of $\boldsymbol{\mu}_8$ is generated by s.

Th. 8.11 says that the inclusion

$$(G_2 \times G_2) \rtimes \boldsymbol{\mu}_8 \to \mathrm{Spin}_{14}$$

induces a surjection on H^1's.

We now interpret this result in terms of quadratic forms. Fix a quadratic extension $K := k(\sqrt{d})$ of k. The *trace* $\mathrm{tr}_*(q)$ of a quadratic form q over K is a quadratic form over k of dimension $2\dim q$. It is defined by viewing the K-vector space V underlying q as a vector space over k and taking the bilinear form

$$V \times V \xrightarrow{\text{bilinearization of } q} K \xrightarrow{\mathrm{tr}_{K/k}} k.$$

In other words, $\mathrm{tr}_*(q)$ is the Scharlau transfer of q via the linear map $\mathrm{tr}_{K/k}$, see e.g. [Lam05, §VII.1].

The goal of this section is to prove:

19.3**. Theorem.** (Rost [Ros99b]) *Every* 14*-dimensional form in* I^3k *is of (at least) one of the following two types:*

(1) $\langle a\rangle(\phi_1' - \phi_2')$ *for some* $a \in k^\times$ *and* ϕ_1, ϕ_2 3*-Pfister forms over* k.
(2) $\mathrm{tr}_*(\sqrt{d}\phi')$ *for some nonsquare* $d \in k^\times$ *and* ϕ *a* 3*-Pfister form over* $k(\sqrt{d})$.

[Here we have written $'$ for the pure part of a Pfister form, so for example ϕ equals $\langle 1\rangle \oplus \phi'$ in (2).]

We remark that a 14-dimensional form in I^3k is as in (1) if and only if it contains a subform similar to a 2-Pfister form, see [HT98, 2.3] or [IK00, 17.2]. The two papers just cited give concrete examples of 14-dimensional forms that cannot be written as in (1), see [HT98, p. 211] and [IK00, 17.3]. Izhboldin and Karpenko applied Th. 19.3 to give a concrete description of 8-dimensional forms in I^2k whose Clifford algebra has index 4, see [IK00, 16.10].

19.4. First, we compute that trace of a 1-dimensional form. Directly from the definition, we find:

(19.5) *For* $\ell \in K^\times$*, the 2-dimensional quadratic form* $\mathrm{tr}_*(\langle\sqrt{d}\ell\rangle)$ *represents* $\mathrm{tr}_{K/k}(\sqrt{d}\ell) \in k$ *and has determinant* $-N_{K/k}(\ell) \in k^\times/k^{\times 2}$.

That is,

$$\mathrm{tr}_*(\langle\sqrt{d}\ell\rangle) \cong \begin{cases} \text{hyperbolic plane} & \text{if } \ell \in k^\times, \text{ i.e., if } \mathrm{tr}_{K/k}(\sqrt{d}\ell) = 0; \\ \langle \mathrm{tr}_{K/k}(\sqrt{d}\ell)\rangle\langle 1, -N_{K/k}(\ell)\rangle & \text{otherwise.} \end{cases}$$

To see that this isomorphism holds, it suffices by [Lam05, I.5.1] to observe that the forms on either side of the isomorphism have the same determinant and represent $\mathrm{tr}_{K/k}(\sqrt{d}\ell)$, which follows from (19.5).

Next we compute a toy example.

19.6**. Example.** Write V for the vector space k^2 endowed with the quadratic form $q\colon \binom{x}{y} \mapsto x^2 - y^2$. Map the group $(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rtimes \boldsymbol{\mu}_4$ into the orthogonal group $\mathrm{O}(q)$ of q by sending

$$(\varepsilon_1, \varepsilon_2, i^r) \mapsto \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^r$$

for $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $r \in \mathbb{Z}$. The set $H^1(k, \mathrm{O}(q))$ classifies 2-dimensional quadratic forms over k and we ask: Given a class $\eta \in H^1(k, (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rtimes \boldsymbol{\mu}_4)$, what is the 2-dimensional quadratic form q_η deduced from it?

The quotient map $(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rtimes \boldsymbol{\mu}_4 \to \boldsymbol{\mu}_4$ sends η to an element $\overline{\eta} \in H^1(k, \boldsymbol{\mu}_4)$, i.e., some $dk^{\times 4} \in k^\times/k^{\times 4}$.

If d is a square in k, then η comes from $H^1(k, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$, i.e., η corresponds to a triple $(\alpha, \beta, \gamma) \in (k^\times/k^{\times 2})^{\times 3}$. The 2-dimensional k-subspace of $V \otimes_k k_{\mathrm{sep}}$ fixed by $\eta_\sigma \sigma$ for all σ in the Galois group of k is spanned by

$$\begin{pmatrix} \sqrt{\alpha\gamma} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \sqrt{\beta\gamma} \end{pmatrix}.$$

The quadratic form q_η is the restriction of q to this subspace, i.e., q_η is isomorphic to $\langle\gamma\rangle\langle\alpha, -\beta\rangle$.

Suppose now that d is not a square in k. Fix a 4-th root δ of d such that $\overline{\eta}_\sigma \sigma(\delta) = \delta$. Note that

$$\overline{\eta}_\sigma \sigma(\delta^3) = \begin{cases} \delta^3 & \text{if } \sigma \text{ is the identity on } K \\ -\delta^3 & \text{otherwise.} \end{cases}$$

If we twist $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ by $\overline{\eta}$, we find the transfer $R_{K/k}(\boldsymbol{\mu}_2)$ for $K := k(\sqrt{d})$. Moreover, η is in the image of the map

$$K^\times/K^{\times 2} = H^1(k, R_{K/k}(\boldsymbol{\mu}_2)) \xrightarrow{\sim} H^1(k, (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)_{\overline{\eta}}) \to H^1(k, (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rtimes \boldsymbol{\mu}_4),$$

i.e., η is the image of a class $\ell K^{\times 2} \in K^\times/K^{\times 2}$. Write $\overline{\ell}$ for the image of ℓ under the nonidentity k-automorphism of K and fix square roots $\sqrt{\ell}, \sqrt{\overline{\ell}} \in k_{\mathrm{sep}}$. Then η is the image of the 1-cocycle

$$\sigma \mapsto \begin{cases} (\sigma(\sqrt{\ell})^{-1}\sqrt{\ell}, \sigma(\sqrt{\overline{\ell}})^{-1}\sqrt{\overline{\ell}}) & \text{if } \sigma \text{ is the identity on } K \\ (\sigma(\sqrt{\overline{\ell}})^{-1}\sqrt{\ell}, \sigma(\sqrt{\ell})^{-1}\sqrt{\overline{\ell}}) & \text{otherwise} \end{cases}$$

with values in $(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)_{\overline{\eta}}$. By considering separately the cases where σ is and is not the identity on K, it is easy to check that $\eta_\sigma \sigma$ fixes the vectors

$$\begin{pmatrix} \delta\sqrt{\ell} \\ \delta\sqrt{\overline{\ell}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta^3\sqrt{\ell} \\ -\delta^3\sqrt{\overline{\ell}} \end{pmatrix}$$

in $V \otimes k_{\mathrm{sep}}$; the quadratic form q_η is the restriction of q to the subspace they span. The value of q on the first vector is

$$\delta^2(\ell - \overline{\ell}) = \mathrm{tr}_{K/k}(\sqrt{d}\ell).$$

The determinant of the restriction q_η of q to this subspace is

$$\det \begin{pmatrix} \delta^2(\ell - \overline{\ell}) & \delta^4(\ell + \overline{\ell}) \\ \delta^4(\ell + \overline{\ell}) & \delta^6(\ell - \overline{\ell}) \end{pmatrix} = -4d^2 N_{K/k}(\ell).$$

As in 19.4, q_η is $\mathrm{tr}_*(\langle\sqrt{d}\ell\rangle)$.

Sketch of proof of Th. 19.3. By Example 19.1, it suffices to describe the quadratic form deduced from a $(G_2 \times G_2) \rtimes \boldsymbol{\mu}_4$-torsor, as that is the image of $(G_2 \times G_2) \rtimes \boldsymbol{\mu}_8$ in $\mathrm{SO}(W)$. Reasoning as in Example 8.1, one can reduce the descent computation to the case of a 2-dimensional quadratic form. This computation was done in Example 19.6. □

20. Invariants of Spin_{14}

In this section, we exhibit some invariants of Spin_{14} with $\mathbb{Z}/2\mathbb{Z}$ coefficients using results from §19. The results here are all derived from [Ros99c]. We write k for a fixed base field of characteristic $\neq 2$.

20.1. We define an invariant a_6 of Spin_{14} to be the composition

$$a_6 \colon H^1(k, \mathrm{Spin}_{14}) \to I^3 \xrightarrow{P_3} I^6 \xrightarrow{e_6} H^6(k, \mathbb{Z}/2\mathbb{Z}),$$

where P_3 and e_6 are the maps from §17 and Example 1.2.3 respectively.

We argue that a_6 is not the zero invariant. For a given base field k, define k_1 to be the field obtained by adjoining 6 indeterminates t_{rs} for $r = 1, 2$ and $s = 1, 2, 3$, and—if it is not already in k—a square root of -1. Put $\phi_r := \langle\langle t_{r1}, t_{r2}, t_{r3}\rangle\rangle$ and take $\eta \in H^1(k_1, \mathrm{Spin}_{14})$ to have corresponding quadratic form $q_\eta = \phi_1' - \phi_2'$. By (17.13), we have:

$$a_6(\eta) = e_3(\phi_1) \cdot e_3(\phi_2) = \prod_{r,s} (t_{rs}) \neq 0.$$

20.2. **Proposition.** *Fix $\eta \in H^1(k, \mathrm{Spin}_{14})$ and write q_η for the quadratic form deduced from it. Suppose that -1 is a square in k.*

1. *If q_η is isotropic, then $a_6(\eta)$ is zero.*
2. *If 3 is a square in k and k has characteristic $\neq 3$ (and $\neq 2$), then $a_6(\eta)$ is a symbol.*

Proof. (1): If q_η is isotropic, then it is Witt-equivalent to a 12-dimensional form in I^3. By Example 15.12, q_η is isomorphic to $\langle\langle c\rangle\rangle x + \mathcal{H}$ for some $c \in k^\times$ and some 6-dimensional form x of determinant 1. As -1 is a square in k, x is an Albert form, i.e., $x = \langle d\rangle(\psi_1' - \psi_2')$ for 2-Pfister forms ψ_1, ψ_2 and some $d \in k^\times$ [Lam05, XII.2.13]. In the Witt ring,

$$q_\eta = \langle d\rangle(\langle\langle c\rangle\rangle\psi_1 - \langle\langle c\rangle\rangle\psi_2).$$

Equation (17.13) gives:

$$a_6(\eta) = (c) \cdot (c) \cdot e_2(\psi_1) \cdot e_2(\psi_2).$$

As -1 is a square in k, $(c) \cdot (c)$ is zero, proving (1).

We now prove (2). Computing in the Witt ring, $P_3(q_\eta)$ is $7 + \lambda^2 q_\eta$ by Example 17.7, which equals $\lambda^2 q_\eta + \langle 1\rangle$.

Now the Lie algebra $\mathfrak{so}(q_\eta)$ contains a subalgebra of type $G_2 \times G_2$ or the transfer of a G_2 from a quadratic extension. The Killing form on a Lie algebra of type G_2 associated with a 3-Pfister form ψ is $\langle -1, -3\rangle\psi'$—see e.g. S27.21—so it is is hyperbolic, and contains a 7-dimensional totally isotropic subspace. Hence the Killing form $\langle -24\rangle\lambda^2 q$ (see Exercise 17.2) contains a totally isotropic subspace of dimension at least 14. By the previous paragraph, the class of $P_3(q_\eta)$ in the Witt ring is represented by an anisotropic quadratic form of dimension at most

$$\dim \lambda^2 q_\eta + 1 - 28 = 64.$$

But $P_3(q_\eta)$ belongs to I^6, so it is similar to a 6-Pfister form [Lam05, X.5.6]. □

In the proof of (2) above, we assumed that the characteristic was not 3 so that the Killing form of $\mathfrak{so}(q_\eta)$ was not identically zero.

20.3. Suppose now that -1 is a square in k. We define an invariant

$$a_7 \colon H^1(*, \mathrm{Spin}_{14}) \to H^7(*, \mathbb{Z}/2\mathbb{Z}) \quad \text{via } a_7(\eta) := a_6(\eta)\cdot(\alpha)$$

where α is any nonzero element of k represented by q_η. By Propositions 20.2.1 and 9.2, this is a well-defined invariant of Spin_{14}.

20.4. **Example.** (Assuming $\sqrt{-1} \in k$.) Let $\eta \in H^1(k, \mathrm{Spin}_{14})$ be such that q_η equals $\langle c\rangle(\phi_1' - \phi_2')$ for some $c \in k^\times$ and ϕ_1, ϕ_2 3-Pfister forms. Write ϕ_1 as $\langle\langle \alpha_1, \alpha_2, \alpha_3\rangle\rangle$. we have

$$a_7(\eta) = (-c\alpha_1)\cdot e_3(\phi_1)\cdot e_3(\phi_2)$$

by (17.13). But $(-\alpha_1)\cdot e_3(\phi_1)$ is zero as in Example 9.4.2, hence

$$a_7(\eta) = (c)\cdot e_3(\phi_1)\cdot e_3(\phi_2).$$

As in 20.1, it is easy to see that a_7 is not the zero invariant.

21. Partial summary of results

Surjectivities. Table 21a summarizes the examples of surjectivites given above. The restrictions on the characteristic listed in the table should not be taken seriously. They only reflect the availability of easy-to-cite results in the literature.

N	$\subset$	G	char k_0	Ref.
$\mathrm{Spin}_{2n-1} \times \boldsymbol{\mu}_2$	$\subset$	Spin_{2n}	$\neq 2$	15.1
$G_2 \times \boldsymbol{\mu}_2$	$\subset$	Spin_7	$\neq 2$	15.5
$G_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	$\subset$	Spin_8	$\neq 2$	15.1 and 15.5
$\mathrm{Spin}_7 \times \boldsymbol{\mu}_2$	$\subset$	Spin_9	$\neq 2$	15.7
$G_2 \times \boldsymbol{\mu}_4$	$\subset$	Spin_{10}	$\neq 2$	15.8
$\mathrm{SO}(5) \times \boldsymbol{\mu}_4$	$\subset$	Spin_{11}	$\neq 2$	15.13
$\mathrm{SO}(6) \times \boldsymbol{\mu}_4$	$\subset$	Spin_{12}	$\neq 2$	15.12
$(G_2 \times G_2) \rtimes \boldsymbol{\mu}_8$	$\subset$	Spin_{14}	$\neq 2$	19.1
$F_4 \times \boldsymbol{\mu}_3$	$\subset$	E_6	any	8.12
$E_6 \rtimes \boldsymbol{\mu}_4$	$\subset$	E_7	$\neq 2$	11.13

Table 21a. Examples of inclusions for which $H^1_{\mathrm{fppf}}(*, N) \to H^1(*, G)$ is surjective

This table is obviously not exhaustive. We have only considered a short list of internal Chevalley modules; the recipe in 8.11 gives others. For example, taking $\widetilde{G}$ to be E_6, E_7, E_8 and $\pi = \alpha_2$, one finds that there is an open GL_n-orbit in $\wedge^3 k^n$ ("alternating trilinear forms") for $n = 6, 7, 8$, hence an open SL_n-orbit in $\mathbb{P}(\wedge^3 k^n)$. Alternatively, other examples where there is an open G-orbit in $\mathbb{P}(V)$ can be found by consulting the table at the end of [PV94] or the lists of prehomogeneous vector spaces in [SK77].

n	ed(Spin_n)	$\mathrm{Inv}_{k_0}^{\mathrm{norm}}(\mathrm{Spin}_n, \mathbb{Z}/2\mathbb{Z})$ has basis with elements of degree	Restrictions on k_0?	Ref.
≤ 6	0	$\varnothing$		
7	4	3, 4		16.4
8	5	3, 4, 4, 5		16.1
9	5	3, 4, 5	$\sqrt{-1} \in k_0$	16.6
10	4	3, 4		16.5
11	5	3, 5	$\sqrt{-1} \in k_0$	18.12
12	6	3, 5, 6	$\sqrt{-1} \in k_0$	18.11
13	6	?		[Ros99c, §10]
14	7	?	$\sqrt{-1} \in k_0$	20.3

TABLE 21B. Invariants and essential dimension of Spin_n for $n \leq 14$

All statements are under the global hypothesis that the characteristic of k_0 is $\neq 2$.

Invariants and essential dimensions of Spin groups. Table 21b summarizes the results on invariants of Spin_n for $n \leq 14$. We remark that in the examples considered in S (O_n, SO_n, the symmetric group on n letters, ...), the description of the invariants depended in a regular way on n; clearly, that is not the case here.

The values for the essential dimension given in the table are easily deduced from various results in Part III. For example, the table claims that the essential dimension of Spin_7 is 4. Since Spin_7 has a nontrivial cohomological invariant of degree 4, the essential dimension is ≥ 4, cf. 4.7. (All lower bounds on essential dimension here are proved by constructing nonzero cohomological invariants. These bounds can also be obtained by less ad hoc means, see [CS].) On the other hand, the essential dimension of $G_2 \times \boldsymbol{\mu}_2$ is 4, so the surjectivity from Example 15.5 shows that the essential dimension is ≤ 4.

What of Spin_{13}, which we have not yet discussed? One knows that the essential dimension is at least 6 by [CS] or because the invariant a_6 of Spin_{14} restricts to be nonzero on Spin_{13}. One cannot get an upper bound by imitating the methods of §15 to get a surjectivity in Galois cohomology because the spin representation V does not have an open orbit in $\mathbb{P}(V)$. See [Ros99c] for a proof that the essential dimension is at most 6.

21.1**. Open problem.** (Reichstein-Youssin [RY00, p. 1047]) Let k_0 be an algebraically closed field of characteristic zero. Is there a nonzero invariant $H^1(*, \mathrm{Spin}_n) \to H^{\lfloor n/2 \rfloor + 1}(*, \mathbb{Z}/2\mathbb{Z})$ when $n \equiv 0, \pm 1 \mod 8$?

[For $n = 7, 8, 9$, one has the invariants described in Examples 16.4, 16.1, and 16.6 above.]

Appendixes

Appendix A. Examples of anisotropic groups of types E_7

We use cohomological invariants to give examples of algebraic groups of type E_7 that are anisotropic over "prime-to-2 closed" fields or are anisotropic but split by an extension of degree 2.

A.1. Groups of type E_7. Write E_7 for the split simply connected group of that type over a field k The Rost invariant r_{E_7} recalled in Example 1.2.4 maps

$$r_{E_7}\colon H^1(*, E_7) \to H^3(*, \mathbb{Z}/12\mathbb{Z}(2)),$$

see [Mer03, pp. 150, 154]. (In this appendix, the group $H^3(k, \mathbb{Z}/n\mathbb{Z}(2))$ is as defined in [Mer03]. If the characteristic of k does not divide n, then $H^3(k, \mathbb{Z}/n\mathbb{Z}(2))$ is $H^3(k, \boldsymbol{\mu}_n^{\otimes 2})$, as in the main body of the notes. In any case, it is n-torsion.) The group $H^3(k, \mathbb{Z}/12\mathbb{Z}(2))$ is 12-torsion, and its 4- and 3-torsion are identified with $H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$ and $H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$ respectively. We write r' for the composition of r_{E_7} with the projection of $H^3(k, \mathbb{Z}/12\mathbb{Z}(2))$ onto its 4-torsion, i.e.:

$$r'\colon H^1(k, E_7) \xrightarrow{r_{E_7}} H^3(k, \mathbb{Z}/12\mathbb{Z}(2)) \to H^3(k, \mathbb{Z}/4\mathbb{Z}(2)).$$

Proposition. *Suppose that, for $\eta \in H^1(k, E_7)$, the twisted group $(E_7)_\eta$ is isotropic. Then $2r'(\eta) = 0$. If furthermore k contains a primitive 4-th root of unity, then $r'(\eta)$ has symbol length ≤ 2 in $H^3(k, \mathbb{Z}/2\mathbb{Z})$.*

The last sentence of the proposition warrants some comments. Note that the hypothesis implies that k has characteristic $\neq 2$, so $H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$ and $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ are simply the Galois cohomology groups $H^3(k, \mathbb{Z}/4\mathbb{Z})$ and $H^3(k, \mathbb{Z}/2\mathbb{Z})$ respectively. It makes sense to speak of $r'(\eta)$ as belonging to $H^3(k, \mathbb{Z}/2\mathbb{Z})$, because the natural map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ identifies $H^3(k, \mathbb{Z}/2\mathbb{Z})$ with the 2-torsion in $H^3(k, \mathbb{Z}/4\mathbb{Z})$. As for the symbol length, recall that every element of $H^3(k, \mathbb{Z}/2\mathbb{Z})$ can be written as a sum of symbols. The *symbol length* of a class in $z \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ is the smallest natural number n such that z can be written as a sum of n symbols.

Proof. We consult the list of the possible Tits indexes of groups of type E_7 from [Tit66, p. 59]. (See 2.3 in that paper for the definition of the Tits index.) In three of these indexes ($E_{7,1}^{48}$, $E_{7,2}^{31}$, and $E_{7,4}^{9}$), one of the summands of the semisimple anisotropic kernel is of the from $SL(Q)$ for some quaternion division algebra Q. However, $(E_7)_\eta$ has trivial Tits algebras, so by [Tit71, p. 211] the semisimple anisotropic kernel cannot have such a summand. In the remaining cases, the vertex 1 or 7 is circled, where the vertices are numbered as in Table 8. We refer to these possibilities as cases 1 and 7 respectively. If both vertices are circled, we arbitrarily say we are in case 1.

Fix a maximal split torus T in the split group E_7. As E_7 is simply connected and all roots have the same length, the cocharacter group T_* is identified with the root lattice. In case c, write S for the image of the cocharacter corresponding to twice the fundamental weight ω_c. Write G for the derived subgroup of the centralizer of S in E_7; it is simply connected and split; it has type D_6 in case 1 and type E_6 in case 7. For precision, we write i for the inclusion $G \hookrightarrow E_7$ and i_* for the induced map on H^1's.

By Tits's Witt-type theorem, $(E_7)_\eta$ is isomorphic to $(E_7)_{i_*\tau}$ for some class τ in $H^1(k, G)$. It follows that $i_*\tau = \zeta \cdot \eta$ where ζ is a 1-cocycle taking values in the center Z of E_7. As the Rost invariant is compatible with twisting [Gil00, p. 76, Lem. 7], we have

$$r'(i_*\tau) = r'(\zeta \cdot \eta) = r'(\zeta) + r'(\eta),$$

cf. [Gar01a, 7.1]. However, E_7 is split, so the image of $H^1(k, Z)$ in $H^1(k, E_7)$ is zero. In particular, $r'(\zeta)$ is zero and $r'(i_*\tau) = r'(\eta)$. Replacing η with $i_*\tau$, we may assume that η is the image of τ. Since the inclusion i of G in E_7 has Rost multiplier one, $r_{E_7}(\eta)$ equals $r_G(\tau)$.

To prove the first claim in the proposition, it suffices to observe that the order of r_G is 2 in case 1 and 6 in case 7 by [Mer03, 15.4, 16.6]. In both cases, the 2-primary part is 2 and not 4.

We now prove the last claim. In case 7, the "mod 2" portion of the Rost invariant is a symbol over an odd-degree extension of k by S22.9, hence it is a symbol over k by [Ros99a], see Example 7.7. In case 1, G is the split group Spin_{12} and as in 18.7 and 18.8 the quadratic form $q_\tau \in H^1(k, \mathrm{SO}_{12})$ deduced from τ is of the form $q_\tau = \langle d \rangle \langle\langle c \rangle\rangle (\phi'_1 - \phi'_2)$ for some $c, d \in k^\times$ and 2-Pfister forms ϕ_1, ϕ_2. The Rost invariant of τ is

$$r_G(\tau) = e_3(q_\tau) = (c) \cdot e_2(\phi_1) + (c) \cdot e_2(\phi_2),$$

a sum of two symbols. □

Suppose that k is a prime-to-2 closed field, i.e., every finite extension of k has degree a power of 2. Every group of inner type E_6 is isotropic. In case $k = \mathbb{R}$, the unique anisotropic simply connected group of type E_7 is not a strongly inner form of E_7 (i.e., it has nontrivial Tits algebras). We now give an example of a prime-to-2 closed field k that supports an anisotropic strongly inner form of E_7.

A.2. **Example.** Rost [Gil00, p. 91, Prop. 8] gives an extension k_0 of $\mathbb{Q}$ and a class $\eta \in H^1(k_0, E_7)$ such that $2r'(\eta)$ is not zero. If we take k to be the extension of k_0 fixed by a 2-Sylow subgroup of the absolute Galois group of k_0, then every finite extension of k has degree a power of 2, yet k supports the strongly inner form of E_7 obtained by twisting by $\mathrm{res}_{k/k_0}(\eta)$, and this group is anisotropic by the proposition.

In the preceding example, $2r'(\eta)$ is not zero over k, so a restriction/corestriction argument shows that the twisted group $(E_7)_\eta$ is not split by a quadratic extension of k. We can use the second criterion in the proposition to give an example of a strongly inner form of E_7 that is anisotropic but is split by a quadratic extension.

A.3. **Example.** Let F be a field of characteristic zero containing a primitive 4-th root of unity. Let k_0 be the field obtained by adjoining the indeterminates $t_1, t_2, \ldots, t_6$ and put k for the field $k_0(d)$, for d an indeterminate. We construct a strongly inner form G of E_7 that is anisotropic over k and split over the quadratic extension $K := k(\sqrt{d})$. Let H denote the quasi-split simply connected group of type ${}^2E_6$ associated with the quadratic extension K/k; it is a subgroup of the split simply connected group E_7 and the inclusion has Rost multiplier one. Chernousov [Che03, p. 321] gives a 1-cocycle $\eta \in H^1(K/k, H)$ whose image under r' is

$$(d) \cdot [(t_1) \cdot (t_3) + (t_2 t_3 t_5) \cdot (t_4) + (t_5) \cdot (t_6)] \quad \in H^3(k, \mathbb{Z}/2\mathbb{Z}). \tag{A.4}$$

We take G to be E_7 twisted by η. As η is killed by K, G is K-split. For sake of contradiction, suppose that G is isotropic over k. Applying Prop. A.1, we note that $r'(\eta)$ can be written as a sum of ≤ 2 symbols in $H^3(k, \mathbb{Z}/2\mathbb{Z})$. It follows that the residue with respect to d, namely

$$(t_1)\cdot(t_3) + (t_2t_3t_5)\cdot(t_4) + (t_5)\cdot(t_6) \quad \in H^2(k_0, \mathbb{Z}/2\mathbb{Z}) \tag{A.5}$$

can be written as a sum of ≤ 2 symbols in $H^2(k_0, \mathbb{Z}/2\mathbb{Z})$. For P_n as in Prop. 17.12.3, the image of (A.5) under the composition

$$H^2(k_0, \mathbb{Z}/2\mathbb{Z}) \xrightarrow[e_2^{-1}]{\sim} I^2/I^3 \xrightarrow{P_2} I^4/I^5 \xrightarrow[e_4]{\sim} H^4(k_0, \mathbb{Z}/2\mathbb{Z})$$

is

$$(t_1)\cdot(t_3)\cdot(t_2t_5)\cdot(t_4) + (t_1)\cdot(t_3)\cdot(t_5)\cdot(t_6) + (t_2t_3)\cdot(t_4)\cdot(t_5)\cdot(t_6).$$

By parts 1 and 4 of Prop. 17.12, this is a (possibly zero) symbol in $H^4(k_0, \mathbb{Z}/2\mathbb{Z})$. Taking residues with respect to t_2 and then t_4, we find

$$(t_1)\cdot(t_3) + (t_5)\cdot(t_6) \quad \in H^3(F(t_1, t_3, t_5, t_6), \mathbb{Z}/2\mathbb{Z}).$$

Our assumption implies that this is a symbol, which is impossible as the t_i's are indeterminates. We conclude that G is anisotropic over k.

Appendix B. A generalization of the Common Slot Theorem

By Detlev W. Hoffmann

The purpose of this appendix is to prove Cor. B.5, which is used in the construction of the degree 5 invariant of Spin_{12} in §18. The corollary as such is due to Rost, but his original argument had a small flaw. The version we present here is actually more general and can be considered as a generalization of the well known Common Slot Theorem, see, e.g., [Lam05, III.4.13]. Recall that the Common Slot Theorem says that if $A = \left(\frac{a,x}{k}\right)$ and $B = \left(\frac{b,y}{k}\right)$ are quaternion algebras over a field k with $\mathrm{char}(k) \neq 2$ such that $A \cong B$, then there exists $z \in k^*$ with $A \cong \left(\frac{a,z}{k}\right)$ and $B \cong \left(\frac{b,z}{k}\right)$. Translated into Pfister forms, it means that if $\langle\langle a, x\rangle\rangle \cong \langle\langle b, y\rangle\rangle$ then

$$\langle\langle a, x\rangle\rangle \cong \langle\langle a, z\rangle\rangle \cong \langle\langle b, z\rangle\rangle \cong \langle\langle b, y\rangle\rangle$$

for some $z \in k^*$. Furthermore, $z \in D_k(\langle\langle ab\rangle\rangle)$, i.e., z is represented by the form $\langle\langle ab\rangle\rangle$ and hence is a norm of the extension $k(\sqrt{ab})/k$. Indeed, $\langle 1, -a, -z, az\rangle \cong \langle\langle a, z\rangle\rangle \cong \langle\langle b, z\rangle\rangle \cong \langle 1, -b, -z, bz\rangle$ implies after Witt cancellation and scaling that $\langle\langle ab\rangle\rangle \cong \langle z\rangle\langle\langle ab\rangle\rangle$.

In the sequel, all fields are assumed to be of characteristic different from 2. To state our version, we first recall the notion of *linkage* of Pfister forms introdued by Elman and Lam [EL72]. Let α and β be Pfister forms over k of folds m and n, respectively. Then α and β are called *r-linked* form some nonnegative integer $r \leq \min(n, m)$ if there exist Pfister forms ρ, σ, τ of folds $m - r$, $n - r$ and r, respectively, such that $\alpha \cong \rho\tau$ and $\beta \cong \sigma\tau$. In other words, α and β are r-linked if they can be written with r slots in common. It can be shown that α and β are r-linked if and only if the Witt index of $\alpha \oplus \langle -1\rangle\beta$ is $\geq 2^r$ (see [EL72, 4.4]).

If $m \geq n$, we call α and β *linked* if they are $(n-1)$-linked in the above sense, and we say that they are *strictly linked* if they are $(n-1)$-linked but not n-linked (i.e., α is not similar to a subform of β). So if $n = m$, being (strictly) linked means that there exist an $(n-1)$-fold Pfister form π and $a, b \in k^*$ such that $\alpha \cong \langle\langle a\rangle\rangle\pi$ and $\beta \cong \langle\langle b\rangle\rangle\pi$ (and $\alpha \not\cong \beta$). Note that in this situation, we have in the Witt ring Wk that $\alpha - \beta = \langle b\rangle\langle\langle ab\rangle\rangle\pi$.

Recall also that a form ϕ is called *round* if $\phi \cong \langle a \rangle \phi$ if and only if $a \in D_k(\phi)$, i.e., the group of similarity factors $G_k(\phi)$ coincides with the set $D_k(\phi)$ of nonzero elements represented by ϕ. It is well known that Pfister forms are round. The following facts about round forms are also well known, see [WS77, Theorem 2] for a proof (or [EL72, 1.4] in case of Pfister forms).

B.1. **Lemma.** *Let α and q be forms over k and assume that α is round.*

(1) *If $x \in D_k(\alpha q)$, then there exists a form q_1 such that $\phi q \cong \alpha(\langle x \rangle \oplus q_1)$.*
(2) *If ϕ is anisotropic and αq isotropic, then there exists a form q_2 such that $\alpha q \cong \alpha(\mathcal{H} \oplus q_2)$.*

The crucial ingredient in the proof of our result is the following theorem by Wadsworth and Shapiro [WS77, Theorem 3].

B.2. **Theorem.** *Let α and β be strictly linked Pfister forms over k of folds m and n, respectively, with $m \geq n \geq 1$. Let q be an anisotropic form over k and suppose that there exist forms ϕ and ψ over k with $q \cong \alpha\phi \cong \beta\psi$. Then there exist forms q_i, ϕ_i, ψ_i, $1 \leq i \leq r$, such that*

- *$q \cong q_1 \oplus q_2 \oplus \cdots \oplus q_r$, and*
- *$\dim \phi_i = 2$, $\dim \psi_i = 2^{m-n+1}$ for each i, and*
- *$q_i \cong \alpha\phi_i \cong \beta\psi_i$ for each i.*

Our result now reads as follows.

B.3. **Proposition.** *Let α and β be n-fold Pfister forms over k that are strictly linked. Let π be an $(n-1)$-fold Pfister form and $a, b \in k^*$ such that $\alpha \cong \pi\langle\langle a \rangle\rangle$ and $\beta \cong \pi\langle\langle b \rangle\rangle$, and let $\gamma \cong \pi\langle\langle ab \rangle\rangle$.*

If ϕ, ψ are forms over k such that $\alpha\phi = \beta\psi$ in Wk, then there exists a form τ over k, a nonnegative integer r, $c_i \in k^*$ and $d_i \in D_k(\gamma)$ $(1 \leq i \leq r)$ such that $\tau \cong \bigoplus_{i=1}^r \langle c_i \rangle \langle\langle d_i \rangle\rangle$, $\alpha\tau$ anisotropic and

$$\alpha\phi = \alpha\tau = \beta\tau = \beta\psi \in Wk \ .$$

Proof. Note that the assumption on α and β being strictly linked implies that γ is anisotropic.

By Lemma B.1(2), we may assume that $\alpha\phi$ and $\beta\psi$ are anisotropic and hence $\alpha\phi \cong \beta\psi$. We denote this anisotropic form by q and apply Theorem B.2 to deduce that there exist forms q_i, ϕ_i, ψ_i, $1 \leq i \leq r$ such that

- $q \cong q_1 \oplus q_2 \oplus \cdots \oplus q_r$, and
- $\dim \phi_i = \dim \psi_i = 2$ for each i, and
- $q_i \cong \alpha\phi_i \cong \beta\psi_i$ for each i.

But then, by Lemma B.1(1), there exist $c_i, x_i, y_i \in k^*$ such that $c_i \in D_k(q_i)$ and

$$q_i \cong \langle c_i \rangle \alpha \langle\langle x_i \rangle\rangle \cong \langle c_i \rangle \beta \langle\langle y_i \rangle\rangle \ .$$

Hence, $\alpha\langle\langle x_i \rangle\rangle \cong \beta\langle\langle y_i \rangle\rangle$, and with $\alpha \cong \pi\langle\langle a \rangle\rangle$, $\beta \cong \pi\langle\langle b \rangle\rangle$, $\gamma \cong \pi\langle\langle ab \rangle\rangle$, we get in Wk that

$$0 = \alpha\langle\langle x_i \rangle\rangle - \beta\langle\langle y_i \rangle\rangle = \alpha - \beta + \langle y_i \rangle \beta - \langle x_i \rangle \alpha$$

and therefore

$$\langle b \rangle \gamma = \langle x_i \rangle \alpha - \langle y_i \rangle \beta \ .$$

Comparing dimensions shows that $\langle x_i\rangle\alpha\oplus\langle -y_i\rangle\beta$ is isotropic. Thus, there exists $d_i\in D_k(\langle x_i\rangle\alpha)\cap D_k(\langle y_i\rangle\beta)$, and by Lemma B.1(1), we have $\langle x_i\rangle\alpha\cong\langle d_i\rangle\alpha$ and $\langle y_i\rangle\beta\cong\langle d_i\rangle\beta$. We conclude that $\alpha\langle\langle x_i\rangle\rangle\cong\alpha\langle\langle d_i\rangle\rangle$ and $\beta\langle\langle y_i\rangle\rangle\cong\beta\langle\langle d_i\rangle\rangle$, hence

$$q_i\cong\langle c_i\rangle\langle\langle d_i\rangle\rangle\alpha\cong\langle c_i\rangle\langle\langle d_i\rangle\rangle\beta\ .$$

The proof is now finished by putting $\tau\cong\bigoplus_{i=1}^r\langle c_i\rangle\langle\langle d_i\rangle\rangle$. □

B.4. *Remarks.* (i) One could relax the condition on being strictly linked by linked, and the above statement would still hold provided $\dim\phi$ is even. But this doesn't really yield anything new of interest. Indeed, if α and β are linked but not strictly so, then this just means that $\alpha\cong\beta$ which in turn implies that γ is hyperbolic. Hence, $D_k(\gamma)=k^*$. By Lemma B.1, there exists a (necessarily even-dimensional) form τ such that $(\alpha\phi)_{\rm an}\cong\alpha\tau$, and one can simply take *any* orthogonal decomposition $\tau\cong\bigoplus_{i=1}^r\langle c_i\rangle\langle\langle d_i\rangle\rangle$.

(ii) Recall that a field k is called *linked* if any two Pfister forms over k are linked. In fact, it is not difficult to check that k is linked iff any two 2-fold Pfister forms over k are linked. This notion of a linked field has been coined in [EL73]. Well known examples of linked fields are finite, local and global fields, fields of transcendence degree ≤ 1 over a real closed field or of transcendence degree ≤ 2 over an algebraically closed field.

Hence, if we assume the field k in Proposition B.3 to be linked, then the condition of the two Pfister forms being strictly linked can be replaced by the two Pfister forms being nonisometric.

Let us state the case $n=1$ for the above proposition explicitly.

B.5**. Corollary.** *Let $a,b\in k^*$ represent different nontrivial square classes. Let $\ell=k(\sqrt{ab})$. If ϕ, ψ are forms over k such that $\langle\langle a\rangle\rangle\phi=\langle\langle b\rangle\rangle\psi$ in Wk, then there exists a form τ over k, a nonnegative integer r, $c_i\in k^*$ and $d_i\in D_k(\langle\langle ab\rangle\rangle)=N_{\ell/k}(\ell^*)$ $(1\leq i\leq r)$ such that $\tau\cong\bigoplus_{i=1}^r\langle c_i\rangle\langle\langle d_i\rangle\rangle$, $\alpha\tau$ anisotropic and*

$$\alpha\phi=\alpha\tau=\beta\tau=\beta\psi\in Wk\ .$$

Suppose that, as in the corollary, a and b represent different nontrivial square classes in k^*. Let ϕ be an anisotropic form over k. If $\langle\langle a\rangle\rangle\phi$ is isotropic, it is well known and not difficult to see that there exists a 2-dimensional subform ϕ' of ϕ such that already $\langle\langle a\rangle\rangle\phi'$ is isotropic (and hence hyperbolic as it is similar to a 2-fold Pfister form), cf. [EL73, 2.2].

Indeed, $\langle\langle a\rangle\rangle\phi\cong\phi\oplus\langle -a\rangle\phi$ being isotropic clearly implies that there are nonzero vectors x,y in an underlying vector space V of ϕ such that $\phi(x)=a\phi(y)$. Since a is not a square, x and y span a 2-dimensional subspace W of V. Then just take ϕ' to be the restriction of ϕ to W.

Now let $K=k(\sqrt{b})$ and suppose that $\langle\langle a\rangle\rangle\phi_K$ is isotropic (or possibly even hyperbolic, in which case $\langle\langle a\rangle\rangle\phi\cong\langle\langle b\rangle\rangle\psi$ for some form ψ, see [Lam05, VII.3.2]). By the above, we see that there exists over K (!) a 2-dimensional subform ϕ' of ϕ_K such that $\langle\langle a\rangle\rangle\phi'$ is isotropic over K. If, in this situation, one could always find a 2-dimensional subform ϕ' of ϕ already over k (!) such that $\langle\langle a\rangle\rangle\phi'_K$ is isotropic (and hence hyperbolic) over K, then one could use the Common Slot Theorem plus a staightforward induction on $\dim\phi$ to easily deduce the above corollary. In fact, for $\dim\phi=2$, the above corollary is essentially nothing else but the Common Slot Theorem.

However, such a 2-dimensional subform ϕ' of ϕ over k doesn't exist in general as the following counterexamples will show for forms ϕ of dimension n for any given $n > 2$.

B.6. **Example.** Recall that the Pythagoras number $p(k)$ of a field k is defined to be the least positive integer p (provided such an integer exists) such that each sum of squares in k can be written as a sum of $\leq p$ squares. If no such integer exists, then we put $p(k) = \infty$.

Let k be a formally real field with $p(k) = \infty$ (e.g., the rational function field over the reals in infinitely many variables, cf. [Lam05, IX.2.4]). Let $n \geq 3$ and s be such that $2^s < n \leq 2^{s+1}$. Pick an element $-b$ that is a sum of $2^{s+1} + 2$ squares but not fewer. Note that this is always possible since $p(k) > 2^{s+1} + 1$. Now let $a = -1$, so $\langle\langle a \rangle\rangle \cong \langle 1, 1\rangle$, $\phi \cong \langle 1, \ldots, 1\rangle$ (sum of n squares), and let $K = k(\sqrt{b})$.

Then $\langle\langle a \rangle\rangle \phi$ is a Pfister neighbor of $P \cong \langle\langle -1, -1, \ldots, -1 \rangle\rangle$, a sum of 2^{s+2} squares. Now $P \cong \langle 1 \rangle \oplus P'$ with P' a sum of $2^{s+2} - 1$ squares. In particular, P' represents $-b$, and P has therefore a subform $\langle 1, -b\rangle$ which becomes isotropic over K. Hence P_K is hyperbolic and the Pfister neighbor $\langle\langle a \rangle\rangle \phi_K$ is isotropic. Note that if $n = 2^{s+1}$ then in fact $\langle\langle a \rangle\rangle \phi_K \cong P_K$ is hyperbolic.

Suppose now that ϕ contains a subform $\langle u, v\rangle$ over k with $\langle\langle a \rangle\rangle \langle u, v\rangle \cong \langle 1, 1\rangle \langle u, v\rangle$ isotropic over K. Note that both u and v are necessarily sums of $n \leq 2^{s+1}$ squares in k as both are represented by ϕ.

Let $w = uv$. Then $\langle 1, 1, w, w\rangle \cong \langle\langle -1, -w \rangle\rangle$ is similar to $\langle 1, 1\rangle\langle u, v\rangle$ and thus isotropic (and hence hyperbolic) over K. But then b can be chosen as a slot of the Pfister form $\langle\langle -1, -w \rangle\rangle$: $\langle\langle -1, -w \rangle\rangle \cong \langle\langle b, c \rangle\rangle$ for some $c \in k^*$ (cf. [Lam05, III.4.1]). By Witt cancellation, $\langle 1, w, w\rangle \cong \langle -b, -c, bc\rangle$ and thus $-b$ is represented by $\langle 1, w, w\rangle$. In particular, there exist $x, y, z \in k^*$ with $-b = x^2 + w(y^2 + z^2)$.

Now $w(y^2 + z^2) = uv(y^2 + z^2)$ is the product of three factors, each of which being a sum of at most 2^{s+1} squares. A famous result by Pfister states that, for each nonnegative integer m, the nonzero sums of 2^m squares in a field form a multiplicative group (see, e.g., [Lam05, X.1.9]). Hence, we have that $w(y^2 + z^2)$ can be expressed itself as a sum of at most 2^{s+1} squares. But then, $-b$ can be written as a sum of at most $2^{s+1} + 1$ squares, a contradiction!

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