WILD PFISTER FORMS OVER HENSELIAN FIELDS, K-THEORY, AND CONIC DIVISION ALGEBRAS

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Abstract. The epicenter of this paper concerns Pfister quadratic forms over a field F with a Henselian discrete valuation. All characteristics are considered but we focus on the most complicated case where the residue field has characteristic 2 but F does not. We also prove results about round quadratic forms, composition algebras, generalizations of composition algebras we call conic algebras, and central simple associative symbol algebras. Finally we give relationships between these objects and Kato's filtration on the Milnor K-groups of F.

Introduction

The theory of quadratic forms over a field F with a Henselian discrete valuation is well understood in case the residue field $\overline{F}$ has characteristic different from 2 thanks to Springer [45]. But when $\overline{F}$ is imperfect of characteristic 2, the theorems are much more complicated—see, e.g., [49] and [21]—reflecting perhaps the well-known fact that quadratic forms are not determined by valuation-theoretic data, as illustrated below in Example 11.14. However, more can be said when one focuses on Pfister quadratic forms over F and more generally round forms, see Part II below.

In Part III we change our focus to Kato's filtration on the mod-p Milnor K-theory of a Henselian discretely valued field of characteristic zero where the residue field has characteristic p. (Again, if $\overline{F}$ has characteristic different from p, the mod-p Galois cohomology and Milnor K-theory of F are easily described in terms of $\overline{F}$, see [17, pp. 17–19] and [18, 7.1.10].) We give translations between valuation-theoretic properties of Pfister forms, octonion algebras, central simple associative algebras of prime degree (really, symbol algebras), and cyclic field extensions of prime degree over F on the one hand and properties of the corresponding symbols in Milnor K-theory on the other.

Along the way, we prove some results that are of independent interest, which we now highlight. Part I treats quadratic forms and composition algebras over arbitrary fields. It includes a Skolem-Noether Theorem for purely inseparable subfields of composition algebras (Th. 5.7) and a result on factoring quadratic forms (Prop. 3.12). We also give a new family of examples of what we call conic division algebras, which are roughly speaking division algebras where every element satisfies a polynomial of degree 2, see Example 6.6. More precisely, we show that—contrary to what is known, e.g., over the reals—a Pfister quadratic form of characteristic 2 is anisotropic if and only if it is the norm of such a division algebra (Cor. 6.5).

Part II focuses on round quadratic forms and composition algebras over a field F with a 2-Henselian discrete valuation with residue field of characteristic 2. From this part, our Local Norm Theorem 8.10 has already been applied in [55]. We also relate Tignol's height ω from [50] with Saltman's level $\mathrm{hgt}_{\mathrm{com}}$ from [44], with a nonassociative version of Saltman's level that we denote by $\mathrm{hgt}_{\mathrm{ass}}$, and with valuation-theoretic properties of composition algebras, see Th. 12.11 and Cor. 19.3. We show that composition division algebras over F having pre-assigned valuation data, subject to a few obvious constraints,

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always exist (Cor. 11.13), though they are far from unique up to isomorphism (Example 11.14).

Part III gives a K-theoretic proof of the Local Norm Theorem (Th. 15.2); it has an easy proof but stronger hypotheses than the version in Part II. Finally, our Gathering Lemma 16.1 is independent of the rest of the paper and says that one may rewrite symbols in a convenient form.

Let us now discuss what happens "under the hood". The basic technical result in Part I is a non-orthogonal analogue of the classical Cayley-Dickson construction for algebras of degree 2. It is used to prove the Skolem-Noether Theorem mentioned above as well as to construct the examples of conic division algebras.

In Part II, we proceed to a more arithmetic set-up by considering a base field F, discretely valued by a normalized discrete valuation $\lambda\colon F \to \mathbb{Z} \cup \{\infty\}$, which is 2-Henselian in the sense that it satisfies Hensel's lemma for quadratic polynomials. Our main goal is to understand composition algebras and Pfister quadratic forms over F. For this purpose, the results of [40]—where the base field was assumed to be *complete* rather than Henselian—carry over to this more general (and also more natural) setting virtually unchanged; we use them here here without further ado. Moreover, we will be mostly concerned with "wild" composition algebras over F (see 7.15 below for the precise definition of this term in a more general context) since a complete description of the "tame" ones in terms of data living over the residue field of F has been given in [40]. The approach adopted here owes much to the work of Kato [22, § 1], Saltman [44] and particularly Tignol [50] on wild associative division algebras of degree the residual characteristic $p > 0$ of their (possibly non-discrete) Henselian base field. Moreover, our approach is not confined to composition algebras but, at least to a certain extent, works more generally for (non-singular) pointed quadratic spaces that are round (e.g., Pfister) and anisotropic. We attach valuation data to these spaces, among which not so much the usual ones (ramification index (7.9 (b)) and pointed quadratic residue space (7.9 (c))), but wildness-detecting invariants like the trace exponent (8.1) play a significant role. After imitating the quadratic defect [34, 63A] for round and anisotropic pointed quadratic spaces (8.8) and extending the local square theorem [34, 63:1] to this more general setting (Thm. 8.10), we proceed to investigate the behavior of our valuation data when passing from a wild, round and anisotropic pointed quadratic space P having ramification index 1 as input to the output $Q := \langle\langle\mu\rangle\rangle \otimes P$, for any non-zero scalar $\mu \in F$ (Section 9). In all cases except one, the output, assuming it is anisotropic, will again be a wild pointed quadratic space. Remarkably, the description of the exceptional case (Thm. 9.9), where the input is assumed to be Pfister and the output turns out to be tame, when specialized to composition algebras, relies critically on the non-orthogonal Cayley-Dickson construction encountered in the first part of the paper (Cor. 10.18). This connection is due to the fact that our approach also lends itself to the study of what we call λ-normed and λ-valued conic algebras (Section 10), the latter forming a class of conic division algebras over F that generalize ordinary composition algebras and turn out to exist in all dimensions 2^n, $n = 0, 1, 2, \ldots$, once F has been chosen appropriately (Examples 10.7,10.15). There is yet another unusual feature of the exceptional case: though exclusively belonging to the theory of quadratic forms (albeit in an arithmetic setting), it can be resolved here only by appealing to elementary properties of flexible conic algebras (Thm. 10.17). The second part of the paper concludes with extending Tignol's notion of height [50], which agrees with Saltman's notion of level [44], to composition division algebras over F and relating them to the valuation data introduced before (Thm. 12.11).

In the third part of the paper, we consider the case where F has characteristic zero and a primitive p-th root of unity and has a Henselian discrete valuation with residue field of characteristic p. In that setting, Kato, Bloch, and Gabber gave a description of the mod-p Milnor K-groups $k_q(F)$ in [22, 23, 24, 4]; we use [10] as a convenient reference. We relate properties of a symbol in $k_q(F)$ with valuation-theoretic properties of the corresponding algebra, see Prop. 19.1 and Th. 19.2.

Contents

Part I. Base fields of characteristic 2

1. Standard properties of conic algebras

Although composition algebras are our main concern in this paper, quite a few of our results remain valid under far less restrictive conditions. The appropriate framework for some of these conditions is provided by the category of conic algebras. They are the subject of the present section.

We begin by fixing some terminological and notational conventions about non-associative algebras in general and about quadratic forms. For the time being, we let k be a field of arbitrary characteristic. Only later on (Sections 3−6) will we confine ourselves to base fields of characteristic 2.

1.1. **Algebras.** Non-associative (= not necessarily associative) algebras play a dominant role in the present investigation. For brevity, they will often be referred to simply as algebras (over k) or as k-algebras. A good reference for the standard vocabulary is [56].

Left and right multiplication of a k-algebra A will be denoted by $x \mapsto L_x$ and $x \mapsto R_x$, respectively. A is called unital if it has an identity (or unit) element, denoted by 1_A. A subalgebra of A is called unital if it contains the identity element of A. Algebra homomorphisms are called unital if they preserve identity elements. Commutator and associator of A will be denoted by $[x, y] = xy - yx$ and $[x, y, z] = (xy)z - x(yz)$, respectively. If A is unital, then

$$\mathrm{Nuc}(A) := \big\{x \in A \mid [A, A, x] = [A, x, A] = [x, A, A] = \{0\}\big\}$$

is a unital associative subalgebra of A, called its nucleus, and

$$\mathrm{Cent}(A) := \big\{x \in \mathrm{Nuc}(A) \mid [A, x] = \{0\}\big\}$$

is a unital commutative associative subalgebra of A, called its centre. We say A is central (resp. has trivial nucleus) if $\mathrm{Cent}(A) = k1_A$ (resp. $\mathrm{Nuc}(A) = k1_A$).

A k-algebra A is called flexible if it satisfies the flexible law

$$xyx := (xy)x = x(yx). \tag{1}$$

A is said to be alternative if the associator is an alternating (trilinear) function of its arguments. This means that A is flexible and satisfies the left and right alternative laws

$$x(xy) = x^2y, \quad (yx)x = yx^2. \tag{2}$$

Furthermore, the left, middle and right Moufang identities

$$x\big(y(xz)\big) = (xyx)z, \quad x(yz)x = (xy)(zx), \quad \big((zx)y\big)x = z(xyx) \tag{3}$$

hold.

1.2. **Quadratic forms.** Our main reference in this paper for the algebraic theory of quadratic forms is [15], although our notation will occasionally be different and we sometimes work in infinite dimensions. Let V be vector space over k, possibly infinite-dimensional. Deviating from the notation used in [15], we write the polar form of a quadratic form $q\colon V \to k$, also called the bilinear form associated with q or its bilinearization, as ∂q, so $\partial q\colon V \times V \to k$ is the symmetric bilinear form given by

$$\partial q(x, y) := q(x + y) - q(x) - q(y) \qquad (x, y \in V).$$

Most of the time we simplify notation and write $q(x, y) := \partial q(x, y)$ if there is no danger of confusion. The quadratic form q is said to be *non-singular* if it has finite dimension and its polar form is non-degenerate in the usual sense, i.e., for any $x \in V$, the relations $\partial q(x, y) = 0$ for all $y \in V$ imply $x = 0$. Recall that non-singular quadratic forms have even dimension if the characteristic is 2 [15, Chap. 2, Remark 7.22].

1.3. **Conic algebras.** We consider a class of non-associative algebras that most authors refer to as quadratic [56] or algebras of degree 2 [32]. In order to avoid confusion with Bourbaki's notion of a quadratic algebra [5], we adopt a different terminology. A k-algebra C is said to be *conic* if it has an identity element $1_C \neq 0$ and there exists a quadratic form $n\colon C \to k$ with $x^2 - t(x)x + n(x)1_C = 0$ for all $x \in C$, where t is defined by $t := \partial n(1_C, -)\colon C \to k$ and hence is a linear form. The quadratic form n is uniquely determined by these conditions and is called the *norm* of C, written as n_C. We call $t_C := t = \partial n_C(1_C, -)$ the *trace* of C and have

$$x^2 - t_C(x)x + n_C(x)1_C = 0, \quad n_C(1_C) = 1, \quad t_C(1_C) = 2 \qquad (x \in C). \tag{1}$$

Finally, the linear map

$$\iota_C\colon C \longrightarrow C, \quad x \longmapsto \iota_C(x) := x^* := t_C(x)1_C - x, \tag{2}$$

called the *conjugation* of C, has period 2 and is characterized by the condition

$$1_C^* = 1_C, \quad xx^* = n_C(x)1_C \qquad (x \in C). \tag{3}$$

The property of an algebra to be conic is inherited by unital subalgebras. Injective unital homomorphisms of conic algebras are automatically norm preserving. A conic algebra C over k is said to be *non-degenerate* if the polar form ∂n_C has this property. Thus finite-dimensional conic algebras are non-degenerate iff their norms are non-singular as quadratic forms. Orthogonal complementation in C always refers to ∂n_C. We say C *is simple as an algebra with conjugation* if only the trivial (two-sided) ideals $I \subseteq C$ satisfy $I^* = I$.

1.4. **Invertibility in conic algebras.** Let C be a conic algebra over k. By (1.3.1), the unital subalgebra of C generated by an element $a \in C$, written as $k[a]$, is commutative associative and spanned by $1_C, a$ as a vector space over k; in particular it has dimension at most 2. We say a is *invertible* in C if this is so in $k[a]$, i.e., if there exists an element $a^{-1} \in k[a]$ (necessarily unique and called the *inverse* of a in C) such that $aa^{-1} = 1_C$. For a to be invertible in C it is necessary and sufficient that $n_C(a) \neq 0$, in which case $a^{-1} = n_C(a)^{-1}a^*$. The set of invertible elements in C will always be denoted by $C^\times$.

As usual, a non-associative k-algebra A is called a *division algebra* if for all $a, b \in A$, $a \neq 0$, the equations $ax = b$, $ya = b$ can be solved uniquely in A. The quest for conic division algebras is an important topic in the present investigation. The following necessary criterion, though trivial, turns out to be useful.

1.5. **Proposition.** *The norm of a conic division algebra is anisotropic.* □

The converse of this proposition does not hold (cf. 6.1). For $\mathrm{char}(k) \neq 2$, conditions that are necessary and sufficient for a conic algebra to be division have been given by Osborn [35, Thm. 3].

1.6. **Inseparable field extensions.** Exotic examples of conic algebras arise in connection with inseparability. Suppose k has characteristic 2 and let K/k be a purely inseparable field extension of exponent at most 1, so $K^2 \subseteq k$. Then K is a conic k-algebra with $n_K(u) = u^2$ for all $u \in K$, $\partial n_K = 0$, $t_K = 0$ and $\iota_K = \mathbf{1}_K$. In particular, inseparable field extensions of exponent at most 1 over k are degenerate, hence singular, conic division algebras.

1.7. **Composition algebras.** Composition algebras form the most important class of conic algebras. Convenient references, including base fields of characteristic 2, are [28, 46], although [28] introduces a slightly more general notion. An algebra C over k is said to be a *composition algebra* if it is non-zero, contains a unit element and carries a non-singular quadratic form $n\colon C \to k$ that *permits composition*: $n(xy) = n(x)n(y)$ for all $x, y \in C$. Composition algebras are automatically conic. In fact, the only quadratic form on C permitting composition is the norm of C in its capacity as a conic algebra.

1.8. **Basic properties of composition algebras.** Composition algebras exist only in dimensions $1, 2, 4, 8$ and are alternative. They are associative iff their dimension is at most 4, and commutative iff their dimension is at most 2. The base field is a composition algebra if and only if it has characteristic different from 2. Composition algebras of dimension 2 are the same as quadratic étale algebras. Composition algebras of dimension 4 (resp. 8) are called quaternion (resp. octonion or Cayley) algebras. Two composition algebras are isomorphic if and only if their norms are isometric (as quadratic forms). The conjugation of a composition algebra C over k is an algebra involution.

What is denied to arbitrary conic algebras holds true for composition algebras:

1.9. **Norm criterion for division algebras.** *A composition algebra C over k is a division algebra if and only if its norm is anisotropic.* Otherwise its norm is hyperbolic, in which case we say C is split. Up to isomorphism, split composition algebras are uniquely determined by their dimension, and their structure is explicitly known.

1.10. **The Cayley-Dickson construction.** The main tool for dealing with conic algebras in general and composition algebras in particular is the Cayley-Dickson construction. Its inputs are a conic algebra B and a non-zero scalar $\mu \in k$. Its output is a conic algebra $C := \mathrm{Cay}(B, \mu)$ that is given on the vector space direct sum $C = B \oplus Bj$ of two copies of B by the multiplication

$$(1) \quad (u_1 + v_1 j)(u_2 + v_2 j) := (u_1 u_2 + \mu v_2^* v_1) + (v_2 u_1 + v_1 u_2^*)j \quad (u_i, v_i \in B, i = 1, 2).$$

Norm, polarized norm, trace and conjugation of C are related to the corresponding data of B by the formulas

$$n_C(u + vj) = n_B(u) - \mu n_B(v), \tag{2}$$

$$n_C(u_1 + v_1 j, u_2 + v_2 j) = n_B(u_1, u_2) - \mu n_B(v_1, v_2), \tag{3}$$

$$t_C(u + vj) = t_B(u), \tag{4}$$

$$(u + vj)^* = u^* - vj \tag{5}$$

for all $u, v, u_i, v_i \in B$, $i = 1, 2$. Note that B embeds into C as a unital conic subalgebra through the first summand; we always identify $B \subseteq C$ accordingly. The Cayley-Dickson construction $\mathrm{Cay}(B, \mu)$ is clearly functorial in B, under injective unital homomorphisms.

It is a basic fact that C is a composition algebra iff B is an associative composition algebra. Conversely, we have the following embedding property, which fails for arbitrary conic algebras, cf. Example 10.8 below.

1.11. **Embedding property.** Any proper composition subalgebra B of a composition algebra C over k is associative and *admits a scalar* $\mu \in k^\times$ *such that the inclusion* $B \hookrightarrow C$ *extends to an embedding* $\mathrm{Cay}(B, \mu) \to C$ *of conic algebras.* More precisely, $\mu \in k^\times$ satisfies this condition iff $\mu = -n_C(y)$ for some $y \in B^\perp \cap C^\times$.

1.12. **The Cayley-Dickson process.** Let B be a conic k-algebra. Using non-zero scalars $\mu_1, \dots, \mu_n \in k^\times$ ($n \geq 1$), we write inductively

$$C := \mathrm{Cay}(B; \mu_1, \dots, \mu_n) := \mathrm{Cay}\big(\mathrm{Cay}(B; \mu_1, \dots, \mu_{n-1}), \mu_n\big)$$

for the corresponding iterated Cayley-Dickson construction starting from B. It is a conic k-algebra of dimension $2^n \dim_k(B)$. We say C arises from B and the $\mu_1, \dots, \mu_n$ by means of the *Cayley-Dickson process.* The norm of C is given by

$$n_C = \langle\langle \mu_1, \dots, \mu_n \rangle\rangle \otimes n_B. \tag{1}$$

Here are the most important special cases of the Cayley-Dickson process.

Case 1. $B = k$, $\mathrm{char}(k) \neq 2$.
Then $n_C = \langle\langle \mu_1, \dots, \mu_n \rangle\rangle$ is an n-Pfister quadratic form. C is a composition algebra iff $n \leq 3$.

Case 2. $B = k$, $\mathrm{char}(k) = 2$.
Then $n_C = \langle\langle \mu_1, \dots, \mu_n \rangle\rangle_q$ is a quasi-Pfister (quadratic) form [15, § 10, p. 56]. Moreover, n_C is anisotropic iff $C = k(\sqrt{\mu_1}, \dots, \sqrt{\mu_n})$ is an extension field of k, necessarily purely inseparable of exponent 1, hence never a composition algebra.

Case 3. B is a quadratic étale k-algebra.
Then $n_B = \langle\langle \mu]\!]$ for some $\mu \in k$ [15, Example 9.4] and (1) shows that

$$n_C = \langle\langle \mu_1, \dots, \mu_n, \mu]\!]$$

is an $(n + 1)$-Pfister quadratic form over k. Moreover, C is a composition algebra iff $n \leq 2$.

Composition algebras other than the base field itself always contain quadratic étale subalgebras. Hence, by Cases 1, 3 above and by the embedding property 1.11, they may all be obtained from each one of these, even from the base field itself if the characteristic is not 2, by the Cayley-Dickson process. The preceding discussion also shows that all Pfister and all quasi-Pfister quadratic forms are the norms of appropriate conic algebras.

1.13. **Inseparable subfields.** Let C be a composition *division* algebra over k. A unital subalgebra of C is either a composition (division) algebra itself or an inseparable extension field of k; in the latter case, k has characteristic 2 and the extension is purely inseparable of exponent at most 1 [52]. The extent to which this case actually occurs may be described somewhat more generally as follows.

Suppose k has characteristic 2, B is a conic k-algebra and $\mu_1, \dots, \mu_n \in k^\times$ are such that the norm of

$$C := \mathrm{Cay}(B; \mu_1, \dots, \mu_n) \tag{1}$$

is anisotropic, so all non-zero elements of C are invertible (1.4). Then, by Case 2 of 1.12,

$$K := \mathrm{Cay}(k; \mu_1, \dots, \mu_n) \subseteq C$$

is a purely inseparable subfield of degree 2^n and exponent 1.

Specializing this observation to $n = 2$ and B quadratic étale over k, we conclude *that every octonion division algebra over a field of characteristic* 2 *contains an inseparable subfield of degree* 4.

2. Flexible and alternative conic algebras.

This section is devoted to some elementary properties of flexible and alternative conic algebras. In particular, we derive expansion formulas for the norm of commutators and associators that turn out to be especially useful in subsequent applications.

Phrased with appropriate care, most of the results obtained here remain valid over any commutative associative ring of scalars. For simplicity, however, we continue to work over a field k of arbitrary characteristic. We fix a conic algebra C over k and occasionally adopt the abbreviations $1 = 1_C$, $n = n_C$, $t = t_C$.

2.1. **Identities in arbitrary conic algebras.** The following identities, some of which have been recorded before, are assembled here for the convenience of the reader and either hold by definition or are straightforward to check.

$$n_C(1_C) = 1_C, \tag{1}$$

$$t_C(1_C) = 2, \tag{2}$$

$$t_C(x) = n_C(1_C, x), \tag{3}$$

$$x^2 = t_C(x)x - n_C(x)1_C, \tag{4}$$

$$x \circ y := xy + yx = t_C(x)y + t_C(y)x - n_C(x, y)1_C, \tag{5}$$

$$x^* = t_C(x)1_C - x, \tag{6}$$

$$xx^* = n_C(x)1_C, \tag{7}$$

$$n_C(x^*) = n_C(x). \tag{8}$$

2.2. **Identities in flexible conic algebras.** We now assume that C is flexible. By McCrimmon [32, 3.4,Thm. 3.5], this implies the following relations:

$$n_C(xy, x) = n_C(x)t_C(y) = n_C(yx, x), \tag{1}$$

$$n_C(x, zy^*) = n_C(xy, z) = n_C(y, x^*z), \tag{2}$$

$$n_C(x, y) = t_C(xy^*) = t_C(x)t_C(y) - t_C(xy), \tag{3}$$

$$t_C(xy) = t_C(yx), \quad t_C(xyz) := t_C\big((xy)z\big) = t_C\big(x(yz)\big). \tag{4}$$

Moreover, the conjugation is an algebra involution of C, so we have $(xy)^* = y^*x^*$ for all $x, y \in C$. Dealing with flexible conic algebras has the additional advantage that this property is preserved under the Cayley-Dickson construction [32, Thm. 6.8].

2.3. **Remark.** By [32, 3.4], each one of the four(!) identities in (2.2.1),(2.2.2) is actually *equivalent* to C being flexible.

The norm of a flexible conic algebra will in general not permit composition. But we have at least the following result.

2.4. **Proposition.** *Let C be a flexible conic algebra over k. Then*

(1) $$n_C(xy) = n_C(yx),$$

(2) $$n_C([x,y]) = 4n_C(xy) - t_C(x)^2 n_C(y) - t_C(y)^2 n_C(x) + t_C(xy)t_C(xy^*)$$

for all $x, y \in C$.

Proof. Expanding the expression $n(x \circ y)$ by means of (2.1.5),(2.1.3) yields

$$n(x \circ y) = t(x)^2 n(y) + t(y)^2 n(x) + n(x,y)^2 - t(x)t(y)n(x,y),$$

where flexibility allows us to invoke (2.2.3); we obtain

(3) $$n(x \circ y) = t(x)^2 n(y) + t(y)^2 n(x) - t(xy)t(xy^*).$$

Now let $\varepsilon = \pm 1$. Then

$$n(xy + \varepsilon yx) = n(xy) + \varepsilon n(xy, yx) + n(yx) = n(xy) + (1 - 2\varepsilon)n(yx) + \varepsilon n(x \circ y, yx),$$

and combining (2.1.5) with (2.2.1)(2.2.3),(2.2.4), we conclude

(4) $$n(xy + \varepsilon yx) = n(xy) + (1 - 2\varepsilon)n(yx) + \varepsilon\big(t(x)^2 n(y) + t(y)^2 n(x) - t(xy)t(xy^*)\big).$$

Comparing (3) and (4) for $\varepsilon = 1$ yields (1), while (1) and (4) for $\varepsilon = -1$ yield (2). □

2.5. **Proposition.** *Let C be a non-degenerate conic algebra over k.*

(a) *C is simple as an algebra with conjugation* (cf. 1.3).

(b) *If ι_C is an algebra involution of C, in particular, if C is flexible, then C is either simple or split quadratic étale.*

Proof. (a) Let $I \subseteq C$ be an ideal with $I^* = I$. For $x \in I$, $y \in C$ we linearize (2.1.7) and obtain $n_C(x,y)1_C = xy^* + yx^* \in I$. Then either $I = C$ or $n_C(x,y) = 0$ for all $x \in I$, $y \in C$, forcing $I = \{0\}$ by non-degeneracy.

(b) Assuming C is not simple, we must show it has dimension 2. Since (C, ι_C) is simple as an algebra with involution by (a), there exists a k-algebra A such that $(C, \iota_C) \cong (A^{\mathrm{op}} \oplus A, \varepsilon)$ as algebras with involution, ε being the exchange involution of $A^{\mathrm{op}} \oplus A$. Then A, embedded diagonally into $A^{\mathrm{op}} \oplus A$, and

$$H := H(C, \iota_C) := \{x \in C \mid x = x^*\} \subseteq C$$

identify canonically as vector spaces over k. In particular, not only the dimension of H but also its codimension in C agree with the dimension of A. Applying (2.1.6), we conclude that H has dimension 1 for $\mathrm{char}(k) \neq 2$ and codimension 1 for $\mathrm{char}(k) = 2$. In both cases, C must be 2-dimensional. □

2.6. **Proposition.** *Let C be a flexible conic algebra over k whose norm is anisotropic. Then either C is non-degenerate or $\partial n_C = 0$.*

Proof. Since, by 1.4, all non-zero elements of C are invertible, C is a simple algebra. On the other hand, (2.2.2) shows that $I := C^\perp \subseteq C$ is an ideal. If $I = \{0\}$, then C is non-degenerate. If $I = C$, then $\partial n_C = 0$. □

2.7. **Identities in conic alternative algebras.** If C is a conic alternative algebra, then by [32, p. 97] its norm permits composition:

(1) $$n_C(xy) = n_C(x)_C n(y).$$

Linearizing (1), we obtain

(2) $$n_C(x_1 y, x_2 y) = n_C(x_1, x_2)n_C(y),$$

(3) $$n_C(xy_1, xy_2) = n_C(x)n_C(y_1, y_2),$$

(4) $$n_C(x_1 y_1, x_2 y_2) + n_C(x_1 y_2, x_2 y_1) = n_C(x_1, x_2)n_C(y_1, y_2).$$

Moreover, by [32, Prop. 3.9] and (2.2.3),

(5) $$xyx = n_C(x, y^*)x - n_C(x)y^* = t_C(xy)x - n_C(x)y^*.$$

2.8. **Theorem.** *Let C be a conic alternative algebra over k. Then*

$$n_C([x_1,x_2,x_3]) = 4n_C(x_1)n_C(x_2)n_C(x_3) - \sum t_C(x_i)^2 n_C(x_j)n_C(x_l) + \sum t_C(x_ix_j)t_C(x_ix_j^*)n_C(x_l) - t_C(x_1x_2)t_C(x_2x_3)t_C(x_3x_1) + t_C(x_1x_2x_3)t_C(x_2x_1x_3) \tag{1}$$

for all $x_1,x_2,x_3 \in C$, where both summations on the right of (1) *are taken over the cyclic permutations* (ijl) *of* (123).

Proof. Expanding

$$n([x_1,x_2,x_3]) = n((x_1x_2)x_3) - n((x_1x_2)x_3, x_1(x_2x_3)) + n(x_1(x_2x_3))$$

and applying (2.7.1), we conclude

$$n([x_1,x_2,x_3]) = 2n(x_1)n(x_2)n(x_3) - n\big((x_1x_2)x_3, x_1(x_2x_3)\big). \tag{2}$$

Turning to the second summand on the right of (2), we obtain, by (2.7.4),

$$n\big((x_1x_2)x_3, x_1(x_2x_3)\big) = n(x_1x_2,x_1)n(x_3,x_2x_3) - n\big((x_1x_2)(x_2x_3), x_1x_3\big),$$

where applying (2.2.1) to the first summand on the right yields

$$n\big((x_1x_2)x_3, x_1(x_2x_3)\big) = t(x_2)^2 n(x_3)n(x_1) - n\big((x_1x_2)(x_2x_3), x_1x_3\big). \tag{3}$$

Manipulating the expression $(x_1x_2)(x_2x_3)$ by means of (2.1.5) and the Moufang identities (1.1.3), we obtain

$$\begin{aligned}(x_1x_2)(x_2x_3) &= (x_1x_2)\circ(x_2x_3) - (x_2x_3)(x_1x_2)\\ &= t(x_1x_2)x_2x_3 + t(x_2x_3)x_1x_2 - n(x_1x_2,x_2x_3)1 - x_2(x_3x_1)x_2,\end{aligned}$$

where (2.7.5),(2.2.4) yield

$$(x_1x_2)(x_2x_3) = t(x_1x_2)x_2x_3 + t(x_2x_3)x_1x_2 - n(x_1x_2,x_2x_3)1 - t(x_1x_2x_3)x_2 + n(x_2)(x_3x_1)^*. \tag{4}$$

Here we use (2.2.2),(2.1.6) to compute

$$n(x_1x_2,x_2x_3) = n(x_1,x_2x_3x_2^*) = t(x_2)n(x_1,x_2x_3) - n(x_1,x_2x_3x_2)$$

and (2.2.3),(2.7.5) give

$$\begin{aligned}n(x_1x_2,x_2x_3) &= t(x_2)n(x_1,x_2x_3) - t(x_2x_3)n(x_1,x_2) + t(x_3x_1)n(x_2)\\ &= t(x_1)t(x_2)t(x_2x_3) - t(x_2)t(x_1x_2x_3) - t(x_1)t(x_2)t(x_2x_3) +\\ &\quad t(x_1x_2)t(x_2x_3) + t(x_3x_1)n(x_2),\end{aligned}$$

hence

$$n(x_1x_2,x_2x_3) = t(x_1x_2)t(x_2x_3) - t(x_2)t(x_1x_2x_3) + t(x_3x_1)n(x_2).$$

Inserting this into (4), and (4) into the second term on the right of (3), we conclude

$$\begin{aligned}n\big((x_1x_2)(x_2x_3), x_1x_3\big) &= t(x_1x_2)n(x_2x_3,x_1x_3) + t(x_2x_3)n(x_1x_2,x_1x_3) -\\ &\quad t(x_1x_2)t(x_2x_3)t(x_3x_1) + t(x_2)t(x_3x_1)t(x_1x_2x_3) -\\ &\quad t(x_3x_1)^2 n(x_2) - t(x_1x_2x_3)n(x_2,x_1x_3) +\\ &\quad n\big((x_3x_1)^*, x_1x_3\big)n(x_2)\\ &= t(x_1x_2)t(x_1x_2^*)n(x_3) + t(x_2x_3)t(x_2x_3^*)n(x_1) -\\ &\quad t(x_1x_2)t(x_2x_3)t(x_3x_1) + t(x_2)t(x_3x_1)t(x_1x_2x_3) -\\ &\quad t(x_3x_1)^2 n(x_2) - t(x_2)t(x_3x_1)t(x_1x_2x_3) +\\ &\quad t(x_1x_2x_3)t(x_2x_1x_3) + t(x_3x_1^2x_3)n(x_2),\end{aligned}$$

where we may use (2.1.4),(2.2.3),(2.2.2) to expand

$$\begin{aligned} t(x_3x_1^2x_3)n(x_2) - t(x_3x_1)^2n(x_2) &= t(x_3^2x_1^2)n(x_2) - t(x_3x_1)^2n(x_2) \\ &= t\big([t(x_3)x_3 - n(x_3)1][t(x_1)x_1 - n(x_1)1]\big)n(x_2) - \\ &\quad t(x_3x_1)^2n(x_2) \\ &= t(x_3)t(x_1)t(x_3x_1)n(x_2) - t(x_1)^2n(x_2)n(x_3) - \\ &\quad t(x_3)^2n(x_1)n(x_2) + 2n(x_1)n(x_2)n(x_3) - \\ &\quad t(x_3x_1)^2n(x_2) \\ &= t(x_3x_1)t(x_3x_1^*)n(x_2) - t(x_1)^2n(x_2)n(x_3) - \\ &\quad t(x_3)^2n(x_1)n(x_2) + 2n(x_1)n(x_2)n(x_3). \end{aligned}$$

Inserting the resulting expression

$$\begin{aligned} n\big((x_1x_2)(x_2x_3), x_1x_3\big) = \sum t(x_ix_j)t(x_ix_j^*)n(x_l) - t(x_1x_2)t(x_2x_3)t(x_3x_1) + \\ t(x_1x_2x_3)t(x_2x_1x_3) - t(x_1)^2n(x_2)n(x_3) - \\ t(x_3)^2n(x_1)n(x_2) + 2n(x_1)n(x_2)n(x_3) \end{aligned}$$

into (3) and (3) into (2), the theorem follows. □

Remark. The associator of a conic alternative algebra being alternating, its norm must be totally symmetric in all three variables. Since the expression $t_C(xy^*)$ is symmetric in $x, y \in C$ by (2.2.3), this fact is in agreement with the right-hand side of (2.8.1).

3. Interlude: Pfister bilinear and quadratic forms in characteristic 2

Working over an arbitrary field k of characteristic 2, the main purpose of this section is to collect a few results on Pfister bilinear and Pfister quadratic forms that are hardly new but, at least in their present form, apparently not in the literature. The final result, Prop. 3.12, concerns quadratic forms in arbitrary characteristic and may be amusing even for experts.

Recall the following from, for example, [20, 8.5(iv)]. If one is given an anisotropic n-Pfister bilinear form $b \cong \langle\langle \alpha_1, \ldots, \alpha_n \rangle\rangle$, then $K := k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$ is a field of dimension 2^n over k such that $K^2 \subseteq k$. Conversely, given such an extension K/k, we can view K as a k-vector space endowed with a k-valued quadratic form $q \colon x \mapsto x^2$, and one checks that q is isomorphic to the quadratic form $v \mapsto b(v, v)$. This defines a bijection between isomorphism classes of extensions K/k of dimension 2^n with $K^2 \subseteq k$ and anisotropic n-quasi-Pfister quadratic forms [15, 10.4].

3.1. **Unital linear forms.** We now refine this bijection. Fix an extension K/k as in the previous paragraph, and a k-linear form $s \colon K \to k$ such that $s(1_K) = 1$, i.e., s is a retraction of the inclusion $k \hookrightarrow K$. (We say that s is *unital*.) Define a symmetric bilinear

$$b_{K,s} \colon K \times K \longrightarrow k \quad \text{via} \quad b_{K,s}(u, v) := s(uv)$$

for $u, v \in K$; it is the transfer $s_*\langle 1 \rangle$ in the notation of [15, §20.A]. Moreover, it is anisotropic (hence non-degenerate) since $b_{K,s}(u, u) = u^2$ for $u \in K$.

We remark that if both s and t are unital linear forms on K, then (by the nondegeneracy of $b_{K,s}$) there is some $u \in K^\times$ such that $b_{K,s}(u, –) = t$, i.e., $s(uv) = t(v)$ for all $v \in K$.

3.2. **Pfister bilinear forms.** Fix a finite extension K/k such that $K^2 \subseteq k$ as assumed above. We compute $b_{K,s}$ for a particular s. Fix a 2-basis $\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}$ of K. The monomials $\sqrt{\alpha_1}^{i_1} \cdots \sqrt{\alpha_n}^{i_n}$ with $i_j \in \{0, 1\}$ are a k-basis for K and we define s to be 1 on 1_k and 0 on the other monomials. One sees immediately that $b_{K,s}$ is isomorphic to the Pfister bilinear form $\langle\langle \alpha_1, \ldots, \alpha_n \rangle\rangle$.

In fact, the s constructed above is the general case. Suppose we are given a unital linear form $s : K \to k$; we will construct a 2-basis so that s is as in the previous paragraph. Start with $A = \emptyset$ and repeat the following loop: If $k(A) = K$, then we are done. Otherwise, $[K : k(A)]$ is at least 2, hence there is some $a \neq 0$ in the intersection of the k-subspaces $\mathrm{Ker}(s)$ and $k(A)^{\perp}$, orthogonal complementation relative to $b_{K,s}$. As a is not in $k(A)$, $A \cup \{a\}$ is p-free by [6, §V.13.1, Prop. 3]. We replace A by $A \cup \{a\}$ and repeat.

In this way, we have proved: If $[K : k] = 2^n$, then $b_{K,s}$ is an anisotropic n-Pfister bilinear form. Conversely, if we are given an anisotropic symmetric bilinear form $\langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle$ over k, then $\sqrt{\alpha_1}, \dots, \sqrt{\alpha_n}$ is a 2-basis for a purely inseparable extension K/k as in the coarser correspondence recalled at the beginning of the section.

The preceding considerations can be made more precise by looking at the category of pairs (K, s), where K/k is a finite purely inseparable field extension of exponent at most 1, $s\colon K \to k$ is a unital linear form, and morphisms are (automatically injective) k-homomorphisms of field extensions preserving unital linear forms. The map $(K, s) \mapsto b_{K,s}$ defines a functor from this category into the category of anisotropic Pfister bilinear forms over k where morphisms are (automatically injective) isometries of bilinear forms.

3.3. **Proposition.** *The functor just defined is an equivalence of categories.*

Proof. Given two pairs (K, s), (K', s') of finite purely inseparable field extensions of exponent at most 1 over k with unital linear forms, we need only show that any isometry $\varphi\colon b_{K,s} \to b_{K',s'}$ is, in fact, a field homomorphism preserving unital linear forms. We have $s'(\varphi(u)\varphi(v)) = s(uv)$ for all $u, v \in K$, hence $\varphi(u)^2 = s'(\varphi(u)^2) = s(u^2) = u^2$ by unitality of s, s', which implies $\varphi(uv)^2 = (uv)^2 = u^2v^2 = (\varphi(u)\varphi(v))^2$ and therefore $\varphi(uv) = \varphi(u)\varphi(v)$ since we are in characteristic 2. Thus φ is a k-homomorphism of fields preserving unital linear forms in view of $s'(\varphi(u)) = s'(\varphi(1_K)\varphi(u)) = s(1_K u) = s(u)$. □

3.4. **The passage to Pfister quadratic forms.** Let $\alpha \in k$ be a scalar and $s\colon K \to k$ a unital linear form. We define the quadratic form $q_{K;\alpha,s}$ to be the transfer $s_*(\langle\langle \alpha]\!] \otimes K)$, where, as usual, $\langle\langle \alpha]\!]$ stands for the binary quadratic form given on $k \oplus kj$ by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & \alpha \end{smallmatrix}\right)$, so

$$\langle\langle \alpha]\!](\beta + \gamma j) = \beta^2 + \beta\gamma + \alpha\gamma^2$$

for $\beta, \gamma \in k$. By the projection formula [15, p. 84], $q_{K;\alpha,s}$ is isomorphic to $b_{K,s} \otimes \langle\langle \alpha]\!]$. More concretely, on the vector space direct sum $K \oplus Kj$ of two copies of K over k, we can define $q_{K;\alpha,s}$ by the formula

(1) $$q_{K;\alpha,s}(u + vj) := u^2 + s(uv) + \alpha v^2 \qquad (u, v \in K).$$

3.5. **Example.** For K and s as in the first paragraph of 3.2 and $\alpha_{n+1} \in k$, the form $q_{K;\alpha_{n+1},s}$ is isomorphic to $\langle\langle \alpha_1, \dots, \alpha_n, \alpha_{n+1}]\!]$.

3.6. **Theorem.** *For $\alpha \in k$ and $s\colon K \to k$ a unital linear form, $q_{K;\alpha,s}$ is an $(n+1)$-Pfister quadratic form over k. Conversely, every* anisotropic *$(n+1)$-Pfister quadratic form over k is isomorphic to $q_{K;\alpha,s}$ for some purely inseparable field extension K/k of exponent at most 1 and degree 2^n, some $\alpha \in k$ and some unital linear form $s\colon K \to k$.*

Proof. Combine 3.2 and Example 3.5. □

The Pfister quadratic forms $q_{K;\alpha,s}$ of 3.4 need not be anisotropic, nor can isometries between two of them be described as easily as in the case of Pfister bilinear forms (cf. Prop. 3.3).

3.7. **Proposition.** *Given scalars $\alpha, \beta \in k$ and unital linear forms $s, t\colon K \to k$, the following conditions are equivalent.*

(i) *The Pfister quadratic forms $q_{K;\alpha,s}$ and $q_{K;\beta,t}$ are isometric.*
(ii) *There exist elements $u_0, v_0 \in K$ such that*

(1) $$\beta = u_0^2 + s(u_0v_0) + \alpha v_0^2, \quad t(u) = s(uv_0) \qquad (u \in K).$$

Proof. We identify $K \subseteq K \oplus Kj$ canonically through the first summand.

(i) $\Longrightarrow$ (ii). If $q := q_{K;\alpha,s}$ and $q' := q_{K;\beta,t}$ are isometric, Witt's theorem [15, Thm. 8.3] yields a bijective k-linear isometry $\varphi\colon K \oplus Kj \xrightarrow{\sim} K \oplus Kj$ from q' to q (so $q \circ \varphi = q'$) that is the identity on K. Then there are elements $u_0, v_0 \in K$ such that

$$\varphi(j) = u_0 + v_0 j, \tag{2}$$

and there are k-linear maps $f, g\colon K \to K$ such that

$$\varphi(u + vj) = \big(u + f(v)\big) + g(v)j \qquad (u, v \in K). \tag{3}$$

Combining (2), (3), we conclude

$$f(1_K) = u_0, \quad g(1_K) = v_0, \tag{4}$$

while evaluating $q \circ \varphi = q'$ at $u + vj \in K \oplus Kj$ with the aid of (3) and (1) yields

$$t(uv) + \beta v^2 = f(v)^2 + s\big(ug(v)\big) + s\big(f(v)g(v)\big) + \alpha g(v)^2$$

for all $u, v \in K$. Setting $u = 0$, $v = 1_K$ and observing (4), we obtain the first equation of (1). The second one now follows by setting $v = 1_K$ again but keeping $u \in K$ arbitrary.

(ii) $\Longrightarrow$ (i). $s(v_0) = t(1_K) = 1$ implies $v_0 \in K^\times$, and a straightforward verification using (1) shows that the map

$$K \oplus Kj \xrightarrow{\sim} K \oplus Kj, \quad u + vj \longmapsto (u + u_0 v) + (v_0 v)j,$$

is a bijective isometry from q' to q. □

3.8. **Corollary.** *Let $s\colon K \to k$ be a unital linear form. Given $\beta \in k$ and a unital linear form $t\colon K \to k$, there exists an element $\alpha \in k$ such that*

$$q_{K;\beta,t} \cong q_{K;\alpha,s}.$$

Proof. By 3.1, some $v_0 \in K$ satisfies $t(u) = s(uv_0)$ for all $u \in K$. Now Prop. 3.7 applies. □

3.9. **The Artin-Schreier map.** Let $s\colon K \to k$ be a unital linear form. Then

$$\wp_{K,s}\colon K \longrightarrow k, \quad u \longmapsto \wp_{K,s}(u) := u^2 + s(u)$$

is called the *Artin-Schreier map* of K/k relative to s. It is obviously additive and becomes the usual Artin-Schreier map (in characteristic $p = 2$), simply written as $\wp$, for $K = k$. Given another unital linear form $t\colon K \to k$, we conclude from 3.1 that there is a unique element $v_0 \in K^\times$ satisfying $s(v_0) = 1$ and $t(u) = s(uv_0)$ for all $u \in K$. Hence

$$\wp_{K,t}(u) = v_0^2 \wp_{K,s}(uv_0^{-1}) \qquad (u \in K)$$

since the left-hand side is equal to $u^2 + s(uv_0) = v_0^2[(uv_0^{-1})^2 + s(uv_0^{-1})]$.

With $K' := \mathrm{Ker}(s)$, we obtain the orthogonal splitting $K = k1_K \perp K'$ relative to $b_{K,s}$, so the symmetric bilinear form $b'_{K,s} := b_{K,s}|_{K' \times K'}$ up to isometry is uniquely determined by $b_{K,s} \cong \langle 1 \rangle \perp b'_{K,s}$. For $\alpha \in k$, $u' \in K'$ we obtain

$$\wp_{K,s}(\alpha 1_K + u') = \wp(\alpha) + b'_{K,s}(u', u') = \wp(\alpha) + u'^2. \tag{1}$$

3.10. **Corollary.** *Let $s\colon K \to k$ be a unital linear form and $\alpha, \beta \in k$.*

(a) *$q_{K;\alpha,s}$ is isotropic if and only if $\alpha \in \mathrm{Im}(\wp_{K,s})$.*

(b) *$q_{K;\alpha,s}$ and $q_{K;\beta,s}$ are isometric if and only if $\alpha \equiv \beta \bmod \mathrm{Im}(\wp_{K,s})$.*

Proof. (a) If $q := q_{K;\alpha,s}$ is isotropic, (3.4.1) shows $u^2 + s(uv) + \alpha v^2 = 0$ for some $u, v \in K$ not both zero, which implies $v \neq 0$ and then $\alpha = (uv^{-1})^2 + s(uv^{-1}) \in \mathrm{Im}(\wp_{K,s})$. Conversely, $\alpha \in \mathrm{Im}(\wp_{K,s})$ implies $\alpha = u^2 + s(u)$ for some $u \in K$, forcing $q(u + j) = 0$ by (3.4.1).

(b) By Prop. 3.7, the two quadratic forms are isometric iff (3.7.1) holds with $t = s$. But this forces $v_0 = 1_K$, and (3.7.1) becomes equivalent to $\alpha \equiv \beta \bmod \mathrm{Im}(\wp_{K,s})$. □

3.11. **Remark.** (a) For a symmetric bilinear form b on a vector space V over k, we adopt the usual notation from the theory of quadratic forms by writing $\tilde{D}(b) := \{b(v,v) | v \in V\}$ [15, Def. 1.12] as an additive subgroup of k (recall $\mathrm{char}(k) = 2$). With this notation, and observing (3.9.1), Cor. 3.10 (a) amounts to saying that the quadratic form $q_{K;\alpha,s}$ is isotropic if and only if α belongs to $\mathrm{Im}(\wp) + \tilde{D}(b'_{K,s})$; written in this way, Cor. 3.10 (a) agrees with [15, Lemma 9.11].

(b) The definition of the quadratic forms $q_{K;\alpha,s}$ in 3.4 as well as of the Artin-Schreier map in 3.9 make sense also if K has infinite degree over k, and the isomorphism $q_{K;\alpha,s} \cong b_{K,s} \otimes \langle\langle\alpha]]$ as well as Cor. 3.10 (a) continue to hold under these more general circumstances.

We close this section with an observation that will play a useful role in the discussion of types for composition algebras over 2-Henselian fields, cf. in particular 12.7 below. For the remainder of this section, we dispense ourselves from the overall restriction that k have characteristic 2.

3.12. **Proposition.** *Let q be a Pfister quadratic form over a field k of arbitrary characteristic and suppose q_1, q_2 are Pfister quadratic subforms of q with $\dim(q_1) \leq \dim(q_2)$. Then there are Pfister bilinear forms b_1, b_2 over k such that*

$$b_2 \otimes b_1 \otimes q_1 \cong q \cong b_2 \otimes q_2.$$

In particular, if $\dim(q_1) = \dim(q_2)$, then $b \otimes q_1 \cong q \cong b \otimes q_2$ for some Pfister bilinear form b over k.

Proof. We may assume that q is anisotropic. Writing $q\colon V \to k$, $q_i = q|_{V_i}\colon V_i \to k$ for $i = 1, 2$, with a vector space V of dimension n over k and subspaces $V_i \subseteq V$ of dimension n_i $(0 \leq n_1 \leq n_2 \leq n)$, we then argue by induction on $r := n - n_1$. If $r = 0$, we put $b_1 := b_2 := \langle 1 \rangle$. Now suppose $r > 0$. Since the q_i represent 1, there are $e_i \in V_i$ with $q(e_i) = q_i(e_i) = 1$. Now Witt's theorem [15, Thm. 8.3] yields an orthogonal transformation $f \in O(V, q)$ with $f(e_1) = e_2$, and replacing q_1 by $q_1 \circ f^{-1}|_{f(V_1)}\colon f(V_1) \to k$ if necessary, we may assume $e_1 = e_2 \in V_1 \cap V_2$. Assuming $n_2 < n$, and taking orthogonal complements relative to ∂q, we obtain

$$\begin{aligned}\dim_k(V_1^\perp \cap V_2^\perp) &= \dim_k\big((V_1 + V_2)^\perp\big) = 2^n - \dim_k(V_1 + V_2) \\ &= 2^n - \dim_k(V_1) - \dim_k(V_2) + \dim_k(V_1 \cap V_2) > 2^n - 2^{n_1} - 2^{n_2} \\ &\geq 2^n - 2^{n_2+1} \geq 0.\end{aligned}$$

Hence we can find a non-zero vector $j \in V_1^\perp$ that also belongs to $V_2^\perp$ if $n_2 < n$. Setting $\mu := -q(j) \in k^\times$, we conclude from [15, Lemma 23.1] that $q'_1 := \langle\langle\mu\rangle\rangle \otimes q_1$ is a subform of q, and the same holds true for $q'_2 := \langle\langle\mu\rangle\rangle \otimes q_2$ unless $n_2 = n$, in which case we put $q'_2 := q_2 = q$. In any event, the induction hypothesis yields Pfister bilinear forms b'_1, b'_2 over k such that b'_2 is a subform of b'_1, $b'_2 = \langle 1 \rangle$ for $n_2 = n$ and $b'_1 \otimes q'_1 \cong q \cong b'_2 \otimes q'_2$. Hence $b_1 := b'_1 \otimes \langle\langle\mu\rangle\rangle$,

$$b_2 := \begin{cases} \langle 1 \rangle & \text{for } n_2 = n, \\ b'_2 \otimes \langle\langle\mu\rangle\rangle & \text{for } n_2 < n \end{cases}$$

are Pfister bilinear forms that satisfy $b_1 \otimes q_1 \cong b'_1 \otimes \langle\langle\mu\rangle\rangle \otimes q_1 \cong b'_1 \otimes q'_1 \cong q \cong b_2 \otimes q_2$, completing the induction step since b_2 obviously is a subform of b_1. □

4. A non-orthogonal Cayley-Dickson construction

By the embedding property 1.11, the Cayley-Dickson construction may be regarded as a tool to recover the structure of a composition algebra C having dimension 2^n, $n = 1, 2, 3$, from a composition subalgebra $B \subseteq C$ of dimension 2^{n-1}: there exists a scalar $\mu \in k^\times$ with $C \cong \mathrm{Cay}(B, \mu)$. In the present section, a similar construction will be developed achieving the same objective when B is replaced by an inseparable subfield of degree 2^{n-1}; concerning the existence of such subfields, see 1.13. The construction we are going to present is less general but more intrinsic than the one recently investigated by Pumplün [42]. Throughout this section, we work over a field k *of characteristic* 2.

4.1. **Unital linear forms and conic algebras.** Let K/k be a purely inseparable field extension of exponent at most 1, C a conic alternative k-algebra containing K as a unital subalgebra and let $l \in C$. Since K has trivial conjugation by 1.6 and C satisfies (2.2.3), the relation

$$s_l(u) := n_C(u, l) = t_C(ul) \tag{1}$$

holds for all $u \in K$ and defines a linear form $s_l\colon K \to k$, which is unital in the sense of 3.1 if $t_C(l) = 1$. In this case, s_l is called the unital linear form on K *associated with* l.

The following proposition paves the way for the non-orthogonal Cayley-Dickson construction we have in mind.

4.2. **Proposition.** *Let C be a conic alternative algebra over k and $K \subseteq C$ a purely inseparable subfield of exponent at most* 1*. Suppose $l \in C$ satisfies $t_C(l) = 1$, put $\mu := n_C(l) \in k$ and write $s := s_l$ for the unital linear form on K associated with l. Then $B := K + Kl \subseteq C$ is the subalgebra of C generated by K and l. More precisely, the relations*

$$(vl)u = s(u)v + uv + (uv)l, \tag{1}$$

$$u(vl) = s(uv)1_C + s(u)v + s(v)u + uv + (uv)l, \tag{2}$$

$$\begin{aligned}(v_1l)(v_2l) = {} & s(v_1v_2)1_C + s(v_1)v_2 + s(v_2)v_1 + (1+\mu)v_1v_2 + \\ & \big(s(v_1v_2)1_C + s(v_1)v_2 + v_1v_2\big)l\end{aligned} \tag{3}$$

hold for all $u, v, v_1, v_2 \in K$.

Proof. The assertion about B will follow once we have established the relations (1)−(3). To do so, we first prove

$$u(vl) + (vl)u = s(uv)1_C + s(v)u. \tag{4}$$

Since K has trace zero, we combine the definition of s with (2.1.5),(2.2.2) and obtain

$$\begin{aligned}u(vl) + (vl)u &= t_C(u)vl + t_C(vl)u - n_C(u, vl)1_C \\ &= s(v)u - n_C(v^*u, l)1_C = s(v)u + n_C(uv, l)1_C \\ &= s(uv)1_C + s(v)u,\end{aligned}$$

giving (4). In order to establish (1), it suffices to show

$$(vx)u = t_C(ux)v + t_C(x)uv + (uv)x \qquad (u, v \in K, x \in C) \tag{5}$$

since this implies (1) for $x = l$. The assertion being obvious for $u = v = 1$, we may assume $K \neq k$, forcing k to be infinite. By Zariski density, we may therefore assume that x is invertible. Then the Moufang identities (1.1.3) combine with (2.7.5), (2.1.4), the right alternative law (1.1.2) and (4.1.1) to imply

$$\begin{aligned}\big((vx)u\big)x = v(xux) &= v\big(t_C(xu)x - n_C(x)u^*\big) = t_C(ux)vx + n_C(x)uv \\ &= t_C(ux)vx + (uv)\big(t_C(x)x + x^2\big) = \big(t_C(ux)v + t_C(x)uv + (uv)x\big)x,\end{aligned}$$

hence (5). Combining (4) and (1), we now obtain (2). Finally, making use of the Moufang identities again and of (1),(4),(4.1.1), we conclude

$$\begin{aligned}(v_1l)(v_2l) &= (v_1l + lv_1)(v_2l) + (lv_1)(v_2l) = \big(s(v_1)1_C + s(1_C)v_1\big)(v_2l) + l(v_1v_2)l \\ &= s(v_1)v_2l + v_1(v_2l) + t_C\big((v_1v_2)l\big)l - n_C(l)(v_1v_2)^* \\ &= s(v_1)v_2l + v_1(v_2l) + s(v_1v_2)l + \mu v_1v_2.\end{aligned}$$

Applying (2) to the second summand on the right gives (3). ∎

4.3. **Embedding inseparable field extensions into conic algebras.** We now look at the converse of the situation described in Prop. 4.2. Let K/k be a purely inseparable field extension of exponent at most 1. Suppose we are given a unital linear form $s : K \to k$ and a scalar $\mu \in k$. Inspired by the relations (4.2.1)−(4.2.3), we now observe the obvious fact that the vector space direct sum $K \oplus Kj$ of two copies of K carries a unique unital non-associative k-algebra structure

$$C := \mathrm{Cay}(K; \mu, s)$$

into which K embeds as a unital subalgebra through the first summand such that the relations

$$(vj)u = \big(s(u)v + uv\big) + (uv)j, \tag{1}$$

$$u(vj) = \big(s(uv)1_K + s(u)v + s(v)u + uv\big) + (uv)j, \tag{2}$$

$$\begin{aligned}(v_1j)(v_2j) = &\big(s(v_1v_2)1_K + s(v_1)v_2 + s(v_2)v_1 + (1+\mu)v_1v_2\big)+ \\ &\big(s(v_1v_2)1_K + s(v_1)v_2 + v_1v_2\big)j\end{aligned} \tag{3}$$

hold for all $u, v, v_1, v_2 \in K$. Adding (1) to (2), we obtain

$$u \circ (vj) = u(vj) + (vj)u = s(uv)1_K + s(v)u. \tag{4}$$

4.4. **Proposition.** *With the notations and assumptions of* 4.3, $C = \mathrm{Cay}(K; \mu, s)$ *is a non-degenerate flexible conic k-algebra with norm, polarized norm, trace, conjugation respectively given by the formulas*

$$n_C(u + vj) = u^2 + s(uv) + \mu v^2, \tag{1}$$

$$n_C(u_1 + v_1j, u_2 + v_2j) = s(u_1v_2) + s(u_2v_1), \tag{2}$$

$$t_C(u + vj) = s(v), \tag{3}$$

$$(u + vj)^* = s(v)1_K + u + vj \tag{4}$$

for all $u, u_1, u_2, v, v_1, v_2 \in K$. Moreover, $n_C = q_{K;\mu,s}$ (cf. 3.4*), and this is an $(n+1)$-Pfister quadratic form if K has finite degree 2^n over k.*

Proof. The right-hand side of (1) defines a quadratic form $n : C \to k$ whose polarization is given by the right-hand side of (2). In particular, setting $t := \partial n(1_C, -)$, we obtain $t(u + vj) = s(v)$ and then, using (4.3.3),(4.3.4),

$$\begin{aligned}(u + vj)^2 &= u^2 + u \circ (vj) + (vj)^2 = u^2 + s(uv)1_C + s(v)u + v^2 + (1+\mu)v^2 + s(v)vj \\ &= s(v)(u + vj) + \big(u^2 + s(uv) + \mu v^2\big)1_C \\ &= t(u + vj)(u + vj) - n(u + vj)1_C.\end{aligned}$$

Thus C is indeed a conic k-algebra, and (1)−(4) hold. Since K is a field and s is unital, forcing $b_{K,s}$ to be non-degenerate by 3.1, we conclude from (2) that ∂n_C is non-degenerate as well. The final statement of the proposition follows from comparing (1) with (3.4.1) and applying Thm. 3.6. It therefore remains to show that C is flexible. We do so by invoking Remark 2.3 and verifying the first relation of (2.2.1). Letting $u, v, w \in K$ be arbitrary and setting $x = u + vj$, we may assume $y = w$ or $y = wj$. Leaving the former case to the reader, we apply (1),(3),(4.3.1−3) and compute

$$\begin{aligned}
n_C(xy,x) - n_C(x)t_C(y) &= n_C\big(u(wj) + (vj)(wj), u + vj\big) - n_C(u+vj)t_C(wj) \\
&= n_C\Big(s(uw)1_K + s(u)w + s(w)u + uw + s(vw)1_K + s(v)w + s(w)v \\
&\quad + (1+\mu)vw + \big(uw + s(vw)1_K + s(v)w + vw\big)j, u + vj\Big) + \big(u^2 + s(uv) + \mu v^2\big)s(w) \\
&= u^2 s(w) + s(u)s(vw) + s(v)s(uw) + s(uvw) + s(v)s(uw) + s(u)s(vw) \\
&\quad + s(w)s(uv) + s(uvw) + s(v)s(vw) + s(v)s(vw) + v^2 s(w) + v^2 s(w) \\
&\quad + \mu v^2 s(w) + u^2 s(w) + s(w)s(uv) + \mu v^2 s(w) \\
&= 0,
\end{aligned}$$

which completes the proof. □

Prop. 4.2 not only serves to illuminate the intuitive background of the non-orthogonal Cayley-Dickson construction presented in 4.3 and Prop. 4.4. It also allows for the following application.

4.5. **Proposition.** *Let C be a conic alternative algebra over k and $K \subseteq C$ a purely inseparable subfield of exponent at most 1. Suppose $l \in C$ satisfies $t_C(l) = 1$, put $\mu := n_C(l) \in k$ and write $s := s_l$ for the unital linear form on K associated with l. Then there is a unique homomorphism*

$$\varphi : \mathrm{Cay}(K;\mu,s) \longrightarrow C$$

extending the identity of K and satisfying $\varphi(j) = l$. Moreover, φ is injective, and its image is the subalgebra of C generated by K and l.

Proof. The uniqueness assertion being obvious, define $\varphi : \mathrm{Cay}(K;\mu,\delta) \to C$ by $\varphi(u + vj) := u + vl$ for $u, v \in K$. Then φ is a k-linear map whose image by Prop. 4.2 agrees with the subalgebra of C generated by K and l. But φ is also a homomorphism of algebras, which follows by comparing (4.3.1)−(4.3.3) with the corresponding relations (4.2.1)−(4.2.3). It remains to show that φ is injective, i.e., that $u + vl = 0$ for $u, v \in K$ implies $u = v = 0$. Otherwise, we would have $v \neq 0$, which implies $0 = v^{-1}(u + vl) = v^{-1}u + l$ and applying t_C yields a contradiction. □

It is a natural question to ask for conditions that are necessary and sufficient for a non-orthogonal Cayley-Dickson construction as in 4.3 to be a composition algebra. The answer is given by the following result.

4.6. **Theorem.** *Let K/k be a purely inseparable field extension of exponent at most 1, $s\colon K \to k$ a unital linear form and $\mu \in k$ a scalar. We put $C := \mathrm{Cay}(K;\mu,s)$.*

(a) *The map $f_C\colon K^4 \to k$ defined by*

$$f_C(u_1,u_2,u_3,u_4) := s(u_1u_2u_3u_4) + \sum s(u_i)s(u_ju_lu_m) + \sum s(u_iu_j)s(u_lu_4)$$

for $u_1,u_2,u_3,u_4 \in K$, where the first (resp. second) sum on the right is taken over all cyclic permutations $ijlm$ (resp. ijl) of 1234 (resp. 123), is an alternating quadri-linear map. Setting $\dot K := K/k1_K$, f_C induces canonically an alternating quadri-linear map $\dot f_C\colon \dot K^4 \to k$. Moreover, the relation

$$n_C(xy) = n_C(x)n_C(y) + f_C(u_1,u_2,v_1,v_2) \tag{1}$$

holds for all $x = u_1 + v_1 j, y = u_2 + v_2 j \in C$ with $u_i, v_i \in K$, $i = 1,2$.

(b) *The following conditions are equivalent.*

(i) *C is a composition algebra.*
(ii) *$f_C = 0$.*
(iii) *$[K : k] \leq 4$.*

Proof. (a) Setting $f := f_C$, which is obviously quadri-linear, it is straightforward to check that it is alternating as well (hence symmetric since we are in characteristic 2) and satisfies $f(1_K, u_2, u_3, u_4) = 0$ for all $u_i \in K$, $i = 1, 2, 3$. This proves existence of $\dot{f}$ with the desired properties. It remains to establish (1). Subtracting the first summand on the right from the left, and expanding the resulting expression in the obvious way, we conclude that it decomposes into the sum of terms

$$\begin{aligned}
n_C\big((vj)u\big) &- n_C(vj)n_C(u),\\
n_C\big(u(vj)\big) &- n_C(u)n_C(vj),\\
n_C\big((v_1j)(v_2j)\big) &- n_C(v_1j)n_C(v_2j),\\
n_C\big(u_1u_2, u_1(v_2j)\big) &- n_C(u_1)n_C(u_2, v_2j),\\
n_C(u_1u_2, (v_1j)u_2)\big) &- n_C(u_1, v_1j)n_C(u_2),\\
n_C\big(u_1(v_2j), (v_1j)(v_2j)\big) &- n_C(u_1, v_1j)n_C(v_2j),\\
n_C\big((v_1j)u_2, (v_1j)(v_2j)\big) &- n_C(v_1j)n_C(u_2, v_2j),\\
n_C\big(u_1u_2, (v_1j)(v_2j)\big) + n_C\big(u_1(v_2j), (v_1j)u_2\big) &- n_C(u_1, v_1j)n_C(u_2, v_2j).
\end{aligned}$$

A tedious but routine computation, involving (4.2.1)−(4.2.3) and (4.4.1),(4.4.2) shows that all but the very last one of these expressions are equal to zero. Hence (1) follows from

$$\begin{aligned}
&n_C\big(u_1u_2, (v_1j)(v_2j)\big) + n_C\big(u_1(v_2j), (v_1j)u_2\big)\\
&= n_C\Big(u_1u_2, \big(s(v_1v_2)1_K + s(v_1)v_2 + s(v_2)v_1 + (1+\mu)v_1v_2\big) +\\
&\quad \big(s(v_1v_2)1_K + s(v_1)v_2 + v_1v_2\big)j\Big) +\\
&\quad n_C\Big(\big(s(u_1v_2)1_K + s(u_1)v_2 + s(v_2)u_1 + u_1v_2\big) + (u_1v_2)j, \big(s(u_2)v_1 + u_2v_1\big) + (u_2v_1)j\Big)\\
&= s(u_1u_2)s(v_1v_2) + s(v_1)s(u_1u_2v_2) + s(u_1u_2v_1v_2) + s(u_1v_2)s(u_2v_1) +\\
&\quad s(u_1)s(u_2v_1v_2) + s(v_2)s(u_1u_2v_1) + s(u_1u_2v_1v_2) + s(u_2)s(u_1v_1v_2) + s(u_1u_2v_1v_2)\\
&= s(u_1u_2v_1v_2) + s(u_1)s(u_2v_1v_2) + s(u_2)s(v_1v_2u_1) + s(v_1)s(v_2u_1u_2) + s(v_2)s(u_1u_2v_1) +\\
&\quad s(u_1u_2)s(v_1v_2) + s(u_2v_1)s(u_1v_2) + s(v_1u_1)s(u_2v_2) + s(u_1v_1)s(u_2v_2)\\
&= f(u_1, u_2, v_1, v_2) + n_C(u_1, v_1j)n_C(u_2, v_2j).
\end{aligned}$$

(b) (i) $\Longrightarrow$ (iii) follows from the fact that composition algebras have dimension at most 8 and that the dimension of C is twice the degree of K/k.
(iii) $\Longrightarrow$ (ii) follows from the fact that $\dot{K}$ has dimension at most 3, so *any* alternating quadri-linear linear map on $\dot{K}$ must be zero.
(ii) $\Longrightarrow$ (i) follows immediately from (1) since ∂n_C by Prop. 4.4 is non-degenerate. □

Remark. Thm. 4.6 (b) can be proved without recourse to any a priori knowledge of composition algebras. To do so, it suffices to show directly that $[K : k] > 4$ implies $f_C \neq 0$, which can be done quite easily. We omit the details.

5. The Skolem-Noether theorem for inseparable subfields

The non-orthogonal Cayley-Dickson construction introduced in the preceding section will be applied in two ways. Recalling from [46] that every isomorphism between composition subalgebras of a composition algebra C can be extended to an automorphism of C, the aim of our first application will be to derive an analogous result where the composition subalgebras are replaced by inseparable subfields. We begin by exploiting more fully our description of Pfister quadratic forms presented in Section 3 within the framework of composition algebras.

Throughout we continue to work over an arbitrary field k of characteristic 2.

5.1. **Comparing non-orthogonal Cayley-Dickson constructions.** Let K/k be a purely inseparable field extension of exponent at most 1 and degree 2^{n-1}, $1 \le n \le 3$. Suppose we are given scalars $\mu, \mu' \in k$ and unital linear forms $s, s'\colon K \to k$. By Thm. 4.6, $C := \mathrm{Cay}(K;\mu,s)$ and $C' := \mathrm{Cay}(K;\mu',s')$ are composition algebras over k; the notational conventions of 4.3 will remain in force for C and will be extended to $C' = K \oplus Kj'$ in the obvious manner. By a *K-isomorphism* from C to C' we mean an isomorphism that induces the identity on K; we say C and C' are *K-isomorphic* if a K-isomorphism from C to C' exists.

5.2. **Proposition.** *With the notations and assumptions of* 5.1, *we have:*

(a) *C is split if and only if $\mu \in \mathrm{Im}(\wp_{K,s})$.*

(b) *C and C' are K-isomorphic if and only if there exist $u_0, v_0 \in K$ such that*

$$\mu' = u_0^2 + s(u_0v_0) + \mu v_0^2, \quad s'(u) = s(uv_0) \qquad (u \in K). \tag{1}$$

Proof. (a) C is split iff n_C is isotropic iff $\mu \in \mathrm{Im}(\wp_{K,s})$ by Cor. 3.10 (a) and Prop. 4.4.

(b) If C and C' are K-isomorphic, their norms are isometric, so by Prop. 3.7, some $u_0, v_0 \in K$ satisfy (1). Conversely, let this be so. Setting $l := u_0 + v_0j \in C$ and combining (4.4.1)$-$(4.4.3)with (1), we conclude $t_C(l) = s(v_0) = s'(1_K) = 1$, $n_C(l) = u_0^2 + s(u_0v_0) + \mu v_0^2 = \mu'$ and $n_C(u,l) = s(uv_0) = s'(u)$ for all $u \in K$, so Prop. 4.5 yields a unique K-isomorphism $C' \xrightarrow{\sim} C$ sending j' to l. □

5.3. **Remark.** While the set-up described in 5.1 above extends to the case of allowing purely inseparable extensions of exponent 1 and arbitrary degree 2^{n-1}, $n \ge 1$, in the obvious manner (replacing composition algebras by flexible conic ones in the process), our methods of proof become unsustainable in this generality. A typical example is provided by the proof of Prop. 5.2 (b), where a K-isomorphism from C' to C sending j' to l in general does not exist unless $n \le 3$. Indeed, assuming that every element $l \in C$ of trace 1 allows a K-isomorphism

$$\varphi\colon \mathrm{Cay}\big(K; n_C(l), \partial n_C(-,l)\big) \xrightarrow{\sim} C$$

with $\varphi(j') = l$, one computes the expression $\varphi((vj')u) - \varphi(vj')\varphi(u) = 0$ and arrives at the conclusion that the trilinear map defined by

$$H(u_1,u_2,u_3) := s(u_1u_2u_3)1_K + \sum \Big(s(u_iu_j)u_l + s(u_i)u_ju_l\Big) + u_1u_2u_3 \tag{1}$$

for $u_1, u_2, u_3 \in K$ is zero, the sum on the right being extended over all cyclic permutations ijl of 123. Since H is obviously alternating and satisfies the relations $H(1_K, u_2, u_3) = H(u_1, u_2, u_1u_2) = 0$ for all $u_1, u_2, u_3 \in K$, it is easy to check (see the corresponding argument in the proof of Thm. 4.6) that H vanishes for $n \le 3$ (as it should). But for $n > 3$, we may choose $u_1, u_2, u_3 \in K$ to be 2-independent over k [6, V §13.2, Thm. 2], forcing the right-hand side of (1) to be a k-linear combination of linear independent vectors over k [6, V §13.2, Prop. 1], whence $H(u_1,u_2,u_3) \ne 0$.

5.4. **Proposition.** *Let K/k be a purely inseparable field extension of exponent at most 1 and degree 2^{n-1}, $1 \le n \le 3$. Furthermore, let $s\colon K \to k$ be a unital linear form.*

(a) *If C is a composition algebra of dimension 2^n over k containing K as a unital subalgebra, then there exists a scalar $\mu \in k$ such that C and $\mathrm{Cay}(K;\mu,s)$ are K-isomorphic.*

(b) *For $\mu, \mu' \in k$, the following conditions are equivalent.*

(i) $\mathrm{Cay}(K;\mu,s)$ *and* $\mathrm{Cay}(K;\mu',s)$ *are K-isomorphic.*
(ii) $\mathrm{Cay}(K;\mu,s) \cong \mathrm{Cay}(K;\mu',s)$.
(iii) $\mu \equiv \mu' \bmod \mathrm{Im}(\wp_{K,s})$.

Proof. (a) Pick any $l \in C$ of trace 1. Then $C = K \oplus Kl$, and Prop. 4.5 yields a K-isomorphism $C \xrightarrow{\sim} \mathrm{Cay}(K;\mu',s')$, for some $\mu' \in k$ and some unital linear form $s'\colon K \to k$. Hence there is a unique $v_0 \in K^\times$ such that $s'(u) = s(uv_0)$ for all $u \in K$. Setting

$\mu := v_0^{-2}\mu' + \wp_{K,s}(u_0 v_0^{-1})$ and consulting Prop. 5.2 (b), we obtain a K-isomorphism $\mathrm{Cay}(K;\mu',s') \xrightarrow{\sim} \mathrm{Cay}(K;\mu,s)$.

(b) While (i) $\Rightarrow$ (ii) is obvious, (ii) $\Rightarrow$ (iii) follows from Cor. 3.10 (b). It remains to show (iii) $\Rightarrow$ (i). But (iii) implies $\mu' = \mu + \wp_{K,s}(u_0)$ for some $u_0 \in K$, so (5.2.1) holds with $v_0 = 1_K$. This gives (i). □

Every element of an octonion algebra over a field is contained in a suitable quaternion subalgebra [46, Prop. 1.6.4]. However, it doesn't seem entirely obvious that, if the octonion algebra is split, the quaternion subalgebra can be chosen to be split as well. But, in fact, it can:

5.5. **Proposition.** *Let C be a split octonion algebra over an arbitrary field F. Then every element of C is contained in a split quaternion subalgebra of C.*

Proof. Let $x \in C$. We may assume that $R := k[x]$ has dimension 2 over k. There are three cases [5, III §2 Prop. 3].

Case 1. R is the algebra of dual numbers. In particular, it contains zero divisors. Hence so does every quaternion subalgebra of C containing x, which therefore must be split.

Case 2. R is (quadratic) étale. Since C up to isomorphism is uniquely determined by splitness, it may be obtained from R by the Cayley-Dickson process as $C \cong \mathrm{Cay}(R;1,1)$, which implies that x is contained in the split quaternion subalgebra $\mathrm{Cay}(R,1)$ of C.

Case 3. We are left with the most delicate possibility that $F = k$ has characteristic 2 and $K := R$ is a purely inseparable field extension of k having exponent at most 1. Let $c \in C$ be an idempotent different from $0, 1_C$, which exists since C is split, and write B for the subalgebra of C generated by K and c. Since c has trace 1, Prop. 4.5 yields a scalar $\mu \in k$, a unital linear form $s\colon K \to k$ and an isomorphism from $B' := \mathrm{Cay}(K;\mu,s)$ onto B. But B' is a quaternion algebra by Thm. 4.6 whereas B, containing c, has zero divisors. Hence B is a split quaternion subalgebra of C containing x. □

And finally, we need a purely technical result.

5.6. **Lemma.** *Let C be an octonion division algebra over k and suppose $\varphi\colon K_1 \xrightarrow{\sim} K_2$ is an isomorphism of inseparable quadratic subfields $K_1, K_2 \subseteq C$. Then there exist inseparable subfields $L_1, L_2 \subseteq C$ of degree 4 over k such $K_i \subseteq L_i$ for $i = 1,2$ and φ extends to an isomorphism $\psi\colon L_1 \xrightarrow{\sim} L_2$.*

Proof. Given any elements $y \in C$, $x_i \in K_i \setminus k1_C$ for $i = 1,2$, denote by L_i the subalgebra of C generated by K_i and y. Since C has no zero divisors, L_i is either a composition algebra or an inseparable field extension, the latter possibility being equivalent to the trace of C vanishing identically on L_i. In any event, the dimension of L_i is either 2 or 4. Moreover, L_i is spanned by $1_C, x_i, y, x_i y$ as a vector space over k. Summing up, L_i/k is therefore an inseparable field extension of degree 4 if and only if $y \notin K_i$ satisfies the condition $t_C(y) = t_C(x_i y) = 0$. To choose y appropriately, we now write C^0 for the space of trace zero elements in C and consider the hyperplane intersection $V := C^0 \cap x_1 C^0 \cap x_2 C^0 \subseteq C$, which is a subspace of dimension at least 5. Since a group cannot be the union of two proper subgroups, we conclude $V \nsubseteq K_1 \cup K_2$ and, accordingly, pick a $y \in V$ that neither belongs to K_1 nor to K_2. Then y has trace zero and $y = x_i z_i$ for some $z_i \in C^0$, so $x_i y = x_i^2 z_i = n_C(x_i) z_i \in k z_i$ has trace zero as well, and by the above, L_i/k is an inseparable field extension of degree 4 containing K_i. Moreover, $K_1 \cong K_2$ implies $K_1^2 = K_2^2$, hence

$$L_1^2 = K_1^2 + K_1^2 y^2 = K_2^2 + K_2^2 y^2 = L_2^2,$$

so there is an isomorphism $\psi : L_1 \xrightarrow{\sim} L_2$, which necessarily extends φ since the only k-embedding $K_1 \to L_2$ is the one induced by φ. □

After these preparations, we can now establish the Skolem-Noether theorem for inseparable subfields of composition algebras.

5.7. **Theorem.** (Skolem-Noether) *Let C be a composition algebra over k. Then every isomorphism between inseparable subfields of C can be extended to an automorphism of C.*

Proof. Write $\dim_k(C) = 2^n$, $0 \leq n \leq 3$ and let $\varphi\colon K_1 \xrightarrow{\sim} K_2$ be an isomorphism of inseparable subfields $K_1, K_2 \subseteq C$ having degree $2^{n'}$, $0 \leq n' < n$, over k. We may assume $n' > 0$ and first reduce to the case $n' = n - 1$. To do so, suppose the theorem holds for $n' = n - 1$ and let $n' < n - 1$. Then $n' = 1$, $n = 3$, so C is an octonion algebra containing K_1, K_2 as inseparable quadratic subfields. If C is split, there are split quaternion subalgebras $B_i \subseteq C$ containing K_i for $i = 1, 2$ (Prop. 5.5). But B_1, B_2 are isomorphic, hence conjugate under the automorphism group of C, by the classical Skolem-Noether theorem for composition algebras. Hence up to conjugation by automorphisms of C, we may assume $B_1 = B_2 =: B$. But then φ extends to an automorphism of B, which in turn extends to an automorphism of C. We are left with the case that C is a division algebra. By Lemma 5.6, there are inseparable subfields $K_i \subseteq L_i \subseteq C$ of degree 4 ($i = 1, 2$) such that φ extends to an isomorphism $\psi\colon L_1 \xrightarrow{\sim} L_2$. But ψ in turn extends to an automorphism of C, completing the reduction to the case $n' = n - 1$. From now on we assume $n' = n - 1$ and fix a unital linear form $s_2 : K_2 \to k$. Then $s_1 := s_2 \circ \varphi\colon K_1 \to k$ is a unital linear form as well. For $i = 1, 2$, Prop. 5.4 (a) yields scalars $\mu_i \in k$ and K_i-isomorphisms $\psi_i\colon \mathrm{Cay}(K_i; \mu_i, s_i) \xrightarrow{\sim} C$. Now observe that the non-orthogonal Cayley-Dickson construction of 4.3 is functorial in the parameters involved. Hence φ determines canonically an isomorphism

$$\psi := \mathrm{Cay}(\varphi)\colon \mathrm{Cay}(K_1; \mu_1, s_1) \xrightarrow{\sim} \mathrm{Cay}(K_2; \mu_1, s_1 \circ \varphi^{-1}) = \mathrm{Cay}(K_2; \mu_1, s_2).$$

Putting things together, we thus obtain an isomorphism

$$\psi_2^{-1} \circ \psi_1 \circ \psi^{-1} : \mathrm{Cay}(K_2; \mu_1, s_2) \xrightarrow{\sim} \mathrm{Cay}(K_2; \mu_2, s_2).$$

Applying Prop. 5.4 (b), we see that there exists a K_2-isomorphism

$$\chi\colon \mathrm{Cay}(K_2; \mu_2, s_2) \xrightarrow{\sim} \mathrm{Cay}(K_2; \mu_1, s_2),$$

giving rise to the automorphism $\phi := \psi_2 \circ \chi^{-1} \circ \psi \circ \psi_1^{-1}$ of C, which extends φ since $\psi_2 = \chi^{-1} = \mathbf{1}$ on K_2 and $\psi = \varphi$, $\psi_1^{-1} = \mathbf{1}$ on K_1. □

Remark. Let C be an octonion algebra over k and $K \subseteq C$ an inseparable subfield of degree 4. Changing scalars to the algebraic closure, $\bar{k}$, of k, $K \otimes_k \bar{k}$ becomes a unital $\bar{k}$-subalgebra of $\bar{C} := C \otimes_k \bar{k}$ containing a 3-dimensional subalgebra N that consists entirely of nilpotent elements. Hence $N \subseteq \bar{C}$ is a Borel subalgebra in the sense of [37]. The fact that all Borel subalgebras of $\bar{C}$ are conjugate under its automorphism group [37, § 2, 1.] corresponds nicely with the Skolem-Noether theorem.

It is a natural question to ask how our non-orthogonal Cayley-Dickson construction can be converted into the classical orthogonal one. When dealing within the framework of composition (and not of arbitrary conic) algebras, here is a simple answer.

5.8. **Orthogonalizing the non-orthogonal Cayley-Dickson construction.** Let K/k be a purely inseparable field extension of exponent at most 1 and degree 2^n, $n = 1, 2$. Suppose we are given an intermediate subfield $k \subseteq K' \subseteq K$ of degree 2^{n-1}, a scalar $\mu \in k$ and a unital linear form $s\colon K \to k$. Then $s' := s|_{K'}\colon K' \to k$ is a unital linear form on K' and $B := \mathrm{Cay}(K'; \mu, s')$ is a composition subalgebra of $C := \mathrm{Cay}(K; \mu, s)$. Moreover, (4.4.2) implies

(1) $$B^{\perp} = K'^{\perp} \oplus K'^{\perp} j,$$

orthogonal complementation in K (resp. C) being taken relative to $b_{K,s}$ (resp. ∂n_C). From (1) we conclude

$$C = \mathrm{Cay}(B, -n_C(u)) = \mathrm{Cay}(B; u^2) \tag{2}$$

for any non-zero element $u \in K'^{\perp}$.

To be more specific, let C be an octonion algebra over k and $K \subseteq C$ an inseparable subfield of degree 4. Pick a 2-basis $a = (a_1, a_2)$ of K/k. By Prop. 5.4 (a), there exists a scalar $\mu \in k$ with $C \cong \mathrm{Cay}(K; \mu, s_a)$. On the other hand,

$$L = \mathrm{Cay}(k; \mu, \mathbf{1}_k) = k[\mathbf{t}]/(\mathbf{t}^2 + \mathbf{t} + \mu)$$

is a quadratic étale k-algebra, and (2) yields

$$C = \mathrm{Cay}(L; a_1^2, a_2^2)$$

as an ordinary Cayley-Dickson process starting from L.

6. Conic division algebras in characteristic 2.

We now turn to a second application of the non-orthogonal Cayley-Dickson construction which consists in finding new examples of conic division algebras. A few comments on the historical context seem to be in order.

6.1. **Conic division algebras over arbitrary fields.** Among all conic division algebras, it is the composition division algebras that are particularly well understood and particularly easy to construct: by 1.9, it suffices to ensure that their norms be anisotropic. Of course, composition division algebras exist only in dimensions $1, 2, 4, 8$, as do all non-associative division algebras over the reals, by the Bott-Kervaire-Milnor theorem [13, Kap. 10,§ 2]; in particular, the Cayley-Dickson process 1.12 leads to conic algebras

$$\mathrm{Cay}(\mathbb{R}; \mu_1, \dots, \mu_n), \quad \mu_1 = \cdots = \mu_n = -1 \qquad (n \in \mathbb{Z},\ n \geq 1)$$

over the reals whose norms are positive definite (hence anisotropic) but which fail to be division algebras unless $n \leq 3$. Hence it is natural to ask for examples of conic division algebras in dimensions other than $1, 2, 4, 8$, over fields other than the reals.

From the point of view of non-associative algebras, conic division algebras that are not central, like purely inseparable field extensions of characteristic 2 and exponent 1 as in 1.6, are not particularly interesting. Over appropriate fields of characteristic not 2, the first examples of central conic division algebras in all dimensions 2^n, $n = 0, 1, 2 \dots$ are apparently due to Brown [7, pp. 421-422]. They all arise from the base field, an iterated Laurent series field in finitely many variables, by the Cayley-Dickson process; generalizations of these examples will be discussed in Example 10.7 below. Other examples of central conic division algebras in dimension 16, using a refinement of the Cayley-Dickson construction, have been exhibited by Becker [3, Satz 16]. Examples of central *commutative* conic division algebras in characteristic 2 and all dimensions 2^n, $n \geq 0$, have been constructed by Albert [2, Thm. 2]. These algebras are closely related to purely inseparable field extensions of exponent 1 since their norms bilinearize to zero and hence degenerate when extending scalars to the algebraic closure.

In view of the preceding results one may ask whether the dimension of a finite-dimensional conic division algebra is always a power of 2. Though a feeble result along these lines has been obtained by Petersson [38], the answer to this question doesn't seem to be known.

In this paper, two classes of conic division algebras in all dimensions 2^n, $n = 0, 1, 2 \dots$, will be constructed. The examples of the first class, to be discussed in the present section, depend strongly on the non-orthogonal Cayley-Dickson construction, hence exist only in characteristic 2 but differ from Albert's by being central and highly non-commutative and by allowing *arbitrary* anisotropic Pfister quadratic forms as their norms, which in particular remain non-singular under all scalar extensions. The second class of examples will be discussed in 10.7 and 10.15 below.

6.2. **Notations and conventions.** For the remainder of this section, we fix a base field k of characteristic 2, a purely inseparable field extension K/k of exponent at most 1, a scalar $\mu \in k$ and a unital linear form $s\colon K \to k$ to consider the non-orthogonal Cayley-Dickson construction $C := \mathrm{Cay}(K; \mu, s)$ as in 4.3. We put $[K : k] = 2^n$, $n = 0, 1, 2, \ldots$ and explicitly allow the possibility $n = \infty$, i.e., that K has infinite degree over k. The following proposition paves the way for the application we have in mind.

6.3. **Proposition.** *With the notations and conventions of* 6.2, *the following assertions hold.*

(a) *C is locally finite-dimensional.*

(b) *For $n \geq 2$, every element of C belongs to an octonion subalgebra of C.*

(c) *C is central simple for $n \geq 1$ and has trivial nucleus for $n \geq 2$.*

Proof. (a) It suffices to note that finitely many elements $x_i = u_i + v_i j \in C$ with $u_i, v_i \in K$ $(1 \leq i \leq m)$ are contained in $\mathrm{Cay}(K'; \mu, s|_{K'})$ $(K' := k(u_1, v_1, \ldots, u_m, v_m))$, which is a subalgebra of C having dimension at most 2^{2m+1}.

(b) Let $x = u + vj \in C$, $u, v \in K$. Then there is a subfield $K' \subseteq K$ containing u, v and having degree 4 over k, so $C' := \mathrm{Cay}(K'; \mu, s|_{K'}) \subseteq C$ by Thm. 4.6 is an octonion subalgebra containing x.

(c) Standard properties of composition algebras allow us to assume $n > 2$. Combining Props. 2.5, 4.4 we see that C is simple. Let $x \in \mathrm{Nuc}(C)$ and apply (b) to pick an octonion subalgebra $C' \subseteq C$ containing x. Since C' has trivial nucleus [46, Prop. 1.9.2], we conclude $x \in k1_C$, so C has trivial nucleus as well; in particular, C is central. ☐

Referring the reader to our version of the Artin-Schreier map (3.9, Remark 3.11), we can now state the main result of this section.

6.4. **Theorem.** *With the notations and conventions of* 6.2, *the following conditions are equivalent.*

(i) *C is a division algebra.*
(ii) *n_C is anisotropic.*
(iii) *$\mu \notin \mathrm{Im}(\wp_{K,s})$.*

Proof. The implication (i) ⇒ (ii) follows from 1.5, while (ii) and (iii) are equivalent by Cor. 3.10 (a) and Remark 3.11 (b). It remains to prove
(ii) ⟹ (i). Since C is locally finite-dimensional by Prop. 6.3 (a), it suffices to show that there are no zero divisors, so suppose $x_1, x_2 \in C$ satisfy $x_1 x_2 = 0$. By (ii) and (2.4.1), this implies $x_2 x_1 = 0$, and from (2.1.5) we conclude $0 = x_1 x_2 + x_2 x_1 = t_C(x_1)x_2 + t_C(x_2)x_1 - n_C(x_1, x_2)1_C$. If $t_C(x_1) \neq 0$, this yields $x_2 \in k[x_1]$, hence $x_2 = 0$ since n_C being anisotropic implies that $k[x_1]$ is a field. By symmetry, we may therefore assume $t_C(x_1) = t_C(x_2) = n_C(x_1, x_2) = 0$. Write $x_i = u_i + v_i j$ with $u_i, v_i \in K$ for $i = 1, 2$. Then $s(v_i) = 0$ by (4.4.3), $s(u_1 v_2) = s(u_2 v_1)$ by (4.4.2), and if $v_1 = 0$ or $v_2 = 0$, Thm. 4.6 (a) yields $n_C(x_1)n_C(x_2) = n_C(x_1 x_2) = 0$, hence $x_1 = 0$ or $x_2 = 0$. We are thus reduced to the case

$$v_1 \neq 0 \neq v_2, \quad s(v_1) = s(v_2) = 0, \quad s(u_1 v_2) = s(u_2 v_1). \tag{1}$$

Next we use (4.3.1–3) and (1) to expand $(u_1 + v_1 j)(u_2 + v_2 j) = 0$. A short computation gives

$$u_1 u_2 + s(u_1 v_2)1_K + s(u_1)v_2 + u_1 v_2 + s(u_2)v_1 + u_2 v_1 + s(v_1 v_2)1_K + (1 + \mu)v_1 v_2 = 0,$$

$$u_1 v_2 + u_2 v_1 + s(v_1 v_2)1_K + v_1 v_2 = 0. \tag{2}$$

Adding these two relations, we obtain

$$u_1 u_2 + s(u_1 v_2)1_K + s(u_1)v_2 + s(u_2)v_1 + \mu v_1 v_2 = 0 \tag{3}$$

and note that (2),(3) are symmetric in the indices $1, 2$ by (1). We now claim:

(∗) The subfield K' of K/k generated by u_1, u_2, v_1, v_2 is spanned by

$$1_K, u_1, u_2, v_1, v_2, u_1v_2, u_2v_1 \tag{4}$$

as a vector space over k

Suppose for the time being that this claim has been proved. Then the field extension K'/k has degree at most 7. Being purely inseparable at the same time, it has, in fact, degree at most 4. By Thm. 4.6 (b), $C' := \mathrm{Cay}(K'; \mu, s|_{K'}) \subseteq C$ is therefore a composition subalgebra containing x_1, x_2 and inheriting its anisotropic norm from C. Thus C' is a division algebra, and $x_1x_2 = 0$ implies $x_1 = 0$ or $x_2 = 0$, as desired.

We are thus reduced to showing (∗). Writing V for the linear span of the vectors in (4), it suffices to show that $V \subseteq K$ is a k-subalgebra, i.e., that the product of any two distinct elements in (4) belongs to V. Since $v_1v_2 \in V$ by (2), hence $u_1u_2 \in V$ by (3), this will follow once we have shown that

$$v_p^2u_qv_q, \quad u_pu_qv_p, \quad u_pv_pv_q \qquad\qquad (\{p, q\} = \{1, 2\})$$

all belong to V. By symmetry, we may assume $p = 1$, $q = 2$. Multiplying (2) by u_2v_1, we obtain

$$u_1u_2v_1v_2 + u_2^2v_1^2 + s(v_1v_2)u_2v_1 + v_1^2u_2v_2 = 0,$$

forcing $v_1^2u_2v_2 \equiv u_1u_2v_1v_2 \bmod V$. But multiplying (3) by v_1v_2 implies $u_1u_2v_1v_2 \in V$, so we have $v_1^2u_2v_2 \in V$ as well. Moreover, multiplying (2) first by u_1, then by v_1, yields

$$u_1u_2v_1 = u_1^2v_2 + s(v_1v_2)u_1 + u_1v_1v_2 \equiv v_1^2u_2 + s(v_1v_2)v_1 + v_1^2v_2 \equiv 0 \bmod V.$$

Hence also $u_1v_1v_2 = u_1^2v_2 + u_1u_2v_1 + s(v_1v_2)u_1 \in V$, which completes the proof. □

6.5. **Corollary.** *For a Pfister quadratic form q over a field of characteristic 2 to be the norm of a conic division algebra it is necessary and sufficient that q be anisotropic.* □

6.6. **Examples.** Letting k be the field of rational functions in countably many variables $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \dots$ over any field of characteristic 2, e.g., over $\mathbb{F}_2$, the $(n+1)$-Pfister quadratic forms $\langle\langle \mathbf{t}_1, \dots, \mathbf{t}_n, \mathbf{t}_{n+1}]]$, $n \geq 0$, by standard arguments are easily seen to be anisotropic, and we obtain central flexible conic division algebras over k in all dimensions 2^n, $n = 0, 1, 2, \dots$.

Part II. 2-Henselian base fields

7. Pointed quadratic spaces over 2-Henselian fields.

In this section, we recast the conceptual foundations for the study of quadratic forms over Henselian fields in the setting of pointed quadratic spaces. Our subsequent considerations also fit naturally into the valuation theory of Jordan division rings [39] when specialized to the Jordan algebras of pointed quadratic spaces over Henselian fields.

7.1. **Round quadratic forms.** For the time being, we work over a field k that is completely arbitrary. We recall from [15, § 9A, p. 52] that a finite-dimensional quadratic form q over k is said to be *round* if all its non-zero values are precisely its similarity factors; in particular, they form a subgroup of $k^\times$. The most important examples of round quadratic forms are Pfister forms and quasi-Pfister forms [15, Cor. 9.9, Cor. 10.13].

7.2. **Pointed quadratic spaces over arbitrary fields.** Adopting the terminology of Weiss [54, Def. 1.1], by a *pointed quadratic space* over k we mean a triple $Q = (V, q, e)$ consisting of a finite-dimensional vector space V over k, a quadratic form $q\colon V \to k$ and a vector $e \in V$ which is a *base point* for q in the sense that $q(e) = 1$. Morphisms of pointed quadratic spaces are isometries preserving base points. Q is said to be *non-singular* (resp. *anisotropic*, *round*, *Pfister*, ...) if q is. Given a conic algebra C over k, we obtain in $Q_C := (C, n_C, 1_C)$ a pointed quadratic space and every pointed quadratic space arises in this manner (Loos [30], see also Rosemeier [43]). Notationally, we do not always distinguish carefully between C and Q_C. If $Q = (V, q, e)$ is any pointed quadratic space over k, we therefore find it convenient to put $V_Q = V$ as a vector space over k and to call $n_Q := q$ the *norm*, $1_Q := e$ the *unit element*, $t_Q := \partial n_Q(1_Q, -)$ the *trace*, $\iota_Q \colon Q \to Q$, $x \mapsto x^* := t_Q(x)1_Q - x$ the *conjugation* of Q. We also put $V_Q^\times := \{x \in Q \mid n_Q(x) \neq 0\}$.

When dealing with quadratic forms representing 1, insisting on pointedness is not so much a matter of necessity but one of convenience, making the language of non-associative algebras the natural mode of communication. Just as for composition algebras, Witt's theorem [15, Thm. 8.3] implies that pointed quadratic spaces are classified by their norms:

7.3. **Proposition.** *For non-singular pointed quadratic spaces over k to be isomorphic it is necessary and sufficient that their underlying quadratic forms be isometric.* □

7.4. **Enlargements.** Let P be a pointed quadratic space over k and $\mu \in k^\times$. Writing $V_P \oplus V_P j$ for the direct sum of two copies of the vector space V_P over k as in 1.10, and identifying $V_P \subseteq V_P \oplus V_P j$ as a subspace through the first summand,

$$Q := \langle\langle\mu\rangle\rangle \otimes P := (V_Q, n_Q, 1_Q), \quad V_Q := V_P \oplus V_P j, \quad n_Q := \langle\langle\mu\rangle\rangle \otimes n_P, \quad 1_Q := 1_P$$

with

$$\big(\langle\langle\mu\rangle\rangle \otimes n_P\big)(u + vj) = n_P(u) - \mu n_P(v) \qquad (u, v \in V_P) \tag{1}$$

is again a pointed quadratic space over k whose trace and conjugation are given by the formulas

$$t_Q(u + vj) = t_P(u), \tag{2}$$

$$(u + vj)^* = u^* - vj \tag{3}$$

for all $u, v \in V_P$. Moreover, if P is round (resp. Pfister), so is Q. Finally, a comparison with the Cayley-Dickson construction 1.10 shows $Q_{\mathrm{Cay}(B,\mu)} = \langle\langle\mu\rangle\rangle \otimes Q_B$ for any conic k-algebra B and any $\mu \in k^\times$. The following two statements are standard facts about round quadratic forms, translated into the setting of pointed quadratic spaces; we refer to [15, Prop. 9.8, Lemma 23.1] for details.

7.5. **Proposition.** *Let P be a non-singular round pointed quadratic space over k.*
(a) *For $\mu \in k^\times$, the following conditions are equivalent.*
  (i) *$\langle\langle\mu\rangle\rangle \otimes P$ is isotropic.*
  (ii) *$\mu \in n_P(V_P^\times)$.*
  (iii) *$\langle\langle\mu\rangle\rangle \otimes P$ is hyperbolic.*

(b) *Let $\mu_1, \mu_2 \in k^\times$. Then $\langle\langle\mu_1\rangle\rangle \otimes P \cong \langle\langle\mu_2\rangle\rangle \otimes P$ if and only if $\mu_1 = \mu_2 n_P(u)$ for some $u \in P^\times$.* □

7.6. **Proposition.** (Embedding property) *Let Q be a pointed Pfister quadratic space over k and $P \subset Q$ a proper pointed Pfister quadratic subspace. Then the inclusion $P \hookrightarrow Q$ extends to an embedding from $\langle\langle\mu\rangle\rangle \otimes P$ to Q, for some $\mu \in k^\times$.* □

7.7. **2-Henselian fields.** Let F be a field of arbitrary characteristic that is endowed with a normalized discrete valuation λ, so $\lambda \colon F \to \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$ is a surjective map satisfying the following conditions, for all $\alpha, \beta \in F$.

$$\begin{aligned} \lambda \text{ is definite:} &\quad \lambda(\alpha) = \infty \Longleftrightarrow \alpha = 0. \\ \lambda \text{ is sub-additive:} &\quad \lambda(\alpha + \beta) \geq \min\{\lambda(\alpha), \lambda(\beta)\}. \\ \lambda \text{ is multiplicative:} &\quad \lambda(\alpha\beta) = \lambda(\alpha) + \lambda(\beta). \end{aligned}$$

As convenient references for the theory of valuations we mention [14, 16], and particularly [31] for the discrete case. We write $\mathfrak{o} \subseteq F$ for the valuation ring of F relative to λ, $\mathfrak{p} \subseteq \mathfrak{o}$ for its valuation ideal and $\bar{F} := \mathfrak{o}/\mathfrak{p}$ for the residue field of F. The natural map from $\mathfrak{o}$ to $\bar{F}$ will always be indicated by $\alpha \mapsto \bar{\alpha}$. Throughout the remainder of this paper, we fix a prime element $\pi \in \mathfrak{o}$. The quantity

$$e_F := \lambda(2 \cdot 1_F), \tag{1}$$

which is either a non-negative integer or ∞, will play an important role in the sequel. If $\bar{F}$ has characteristic 2, then $e_F > 0$ agrees with what is usually called the *absolute ramification index* of F.

Due to the quadratic character of the gadgets we are interested in (composition and conic algebras, pointed quadratic spaces), requiring F to be Henselian (with respect to λ) is too strong a condition. It actually suffices to assume that F be 2-*Henselian* in the sense of Dress [12] or [16, §4.2], i.e., that F satisfies the following two equivalent conditions [12, Satz 1].

(i) For any quadratic field extension K/F, there is a unique extension of λ to a discrete valuation λ_K of K taking values in $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$.

(ii) For all $\alpha_0, \alpha_1, \alpha_2 \in \mathfrak{o}$ with $\alpha_0 \in \mathfrak{p}$, $\alpha_1 \notin \mathfrak{p}$, the polynomial

$$\alpha_0 \mathbf{t}^2 + \alpha_1 \mathbf{t} + \alpha_2 \in F[\mathbf{t}]$$

is reducible.

In this case, the extension λ_K of λ in (i) is given by

$$\lambda_K(u) = \frac{1}{2}\lambda\big(N_{K/F}(u)\big) \qquad (u \in K). \tag{2}$$

From now on, F is assumed to be a fixed 2-Henselian field with respect to a normalized discrete valuation λ. For simplicity, all algebras, quadratic forms etc. over F are assumed to be finite-dimensional. The characteristic of F is arbitrary.

7.8. **Quadratic forms over 2-Henselian fields.** Let $q\colon V \to F$ be a quadratic form over F and suppose q is anisotropic. Following Springer [45] (in the case of a complete rather than Henselian valuation),

$$\lambda\big(q(x+y)\big) \geq \min\{\lambda\big(q(x)\big), \lambda\big(q(y)\big)\} \qquad (x, y \in V). \tag{1}$$

For convenience, we give the easy proof of this inequality. It evidently suffices to show

$$\lambda\big(q(x,y)\big) \geq \min\{\lambda\big(q(x)\big), \lambda\big(q(y)\big)\} \tag{2}$$

for all $x, y \in V$. Suppose there are $x, y \in V$ such that (2) does not hold. Then $q(x,y) \neq 0$, and the polynomial

$$\frac{1}{q(x,y)} q(\mathbf{t}x + y) = \frac{q(x)}{q(x,y)}\mathbf{t}^2 + \mathbf{t} + \frac{q(y)}{q(x,y)} \in F[\mathbf{t}]$$

has no zero in F since q is anisotropic, but is reducible by 7.7 (ii), a contradiction. □

Relation (2) may be strengthened to

$$2\lambda\big(q(x,y)\big) \geq \lambda\big(q(x)\big) + \lambda\big(q(y)\big) \qquad (x, y \in V), \tag{3}$$

either by appealing to a general result of Bruhat-Tits [8, Thm. 10.1.15], or by using an ad-hoc argument from the valuation theory of Jordan rings [39] in disguise: for $x, y \in V$, $x \neq 0$, we obtain

$$Q_x y := q(x,y)x - q(x)y = -q(x)\tau_x(y),$$

where $\tau_x\colon V \to V$ is the reflection in the hyperplane perpendicular to x. In particular, τ_x leaves q invariant, which implies $\lambda(q(Q_x y)) = 2\lambda(q(x)) + \lambda(q(y))$, and since $q(x,y)x = Q_x y + q(x)y$, we obtain (3).

7.9. **Valuation data for pointed quadratic spaces.** For the rest of this section, we fix a pointed quadratic space Q over F which is round and anisotropic.

(a) The map

$$\lambda_Q\colon V_Q \longrightarrow \mathbb{Q}_\infty, \quad x \longmapsto \lambda_Q(x) := \frac{1}{2}\lambda\big(n_Q(x)\big), \tag{1}$$

is a norm of V_Q as an F-vector space in the sense of Bruhat-Tits [9, 1.1], that is, the following relations hold for all $\alpha \in F$, $x, y \in V_Q$.

$$\lambda_Q \text{ is definite:} \quad \lambda_Q(x) = \infty \Longleftrightarrow x = 0, \tag{2}$$

$$\lambda_Q \text{ is sub-additive:} \quad \lambda_Q(x+y) \geq \min\{\lambda_Q(x), \lambda_Q(y)\}, \tag{3}$$

$$\lambda_Q \text{ is scalar-compatible:} \quad \lambda_Q(\alpha x) = \lambda(\alpha) + \lambda_Q(x), \tag{4}$$

where (3) is a consequence of (7.8.1). Moreover,

$$\lambda_Q(x^*) = \lambda_Q(x), \tag{5}$$

$$\lambda\big(n_Q(x,y)\big) \geq \lambda_Q(x) + \lambda_Q(y), \tag{6}$$

$$\lambda\big(t_Q(x)\big) \geq \lambda_Q(x), \tag{7}$$

for all $x, y \in V_Q$, where (5) follows from conjugation invariance of n_Q, (6) from (7.8.3), and (7) from (6) for $y = 1_Q$.

(b) n_Q being round, $\Gamma_Q := \lambda_Q(V_Q^\times)$ is an additive subgroup of $\mathbb{Q}$ for which (1) implies

$$\mathbb{Z} \subseteq \Gamma_Q \subseteq \frac{1}{2}\mathbb{Z}, \quad e_{Q/F} := [\Gamma_Q : \mathbb{Z}] \in \{1, 2\}, \quad \Gamma_Q = \frac{1}{e_{Q/F}}\mathbb{Z}. \tag{8}$$

We call $e_{Q/F}$ the *ramification index* of Q.

(c) We put

$$\mathfrak{o}_Q := \{x \in V_Q \mid \lambda_Q(x) \geq 0\}, \tag{9}$$

$$\mathfrak{p}_Q := \{x \in V_Q \mid \lambda_Q(x) > 0\} = \{x \in V_Q \mid \lambda_Q(x) \geq \frac{1}{e_{Q/F}}\} \subseteq \mathfrak{o}_Q, \tag{10}$$

which are both full $\mathfrak{o}$-lattices in V_Q, and

$$\mathfrak{o}_Q^\times := \mathfrak{o}_Q \setminus \mathfrak{p}_Q = \{x \in V_Q \mid \lambda_Q(x) = 0\}, \tag{11}$$

which is just a subset of $\mathfrak{o}_Q$ containing 1_Q. By abuse of language, the elements of $\mathfrak{o}_Q^\times$ are called *units* of $\mathfrak{o}_Q$. An element $\Pi \in V_Q^\times$ such that $\lambda_Q(\Pi) > 0$ generates the infinite cyclic group Γ_Q belongs to $\mathfrak{p}_Q$ and is called a *prime element* of $\mathfrak{o}_Q$. Writing $x \mapsto \bar{x}$ for the natural map from $\mathfrak{o}_Q$ to $\mathfrak{o}_Q/\mathfrak{p}_Q$ and setting

$$\bar{Q} := (V_{\bar{Q}}, n_{\bar{Q}}, 1_{\bar{Q}}), \quad V_{\bar{Q}} := \mathfrak{o}_Q/\mathfrak{p}_Q, \quad 1_{\bar{Q}} := \overline{1_Q},$$

where $n_{\bar{Q}} \colon \bar{Q} \longrightarrow \bar{F}, \quad \bar{x} \longmapsto n_{\bar{Q}}(\bar{x}) := \overline{n_Q(x)}$ is the first residue form of n_Q, we obtain a pointed quadratic space over $\bar{F}$, called the *pointed quadratic residue space* of Q, which is round and anisotropic. Here only roundness of $n_{\bar{Q}}$ demands a proof, so let $u \in \mathfrak{o}_Q^\times$. Since n_Q is round, there exists a linear bijection $f \colon V_Q \to V_Q$ with $n_Q(f(x)) = n_Q(u)n_Q(x)$ for all $x \in V_Q$. Hence f stabilizes $\mathfrak{o}_Q$ as well as $\mathfrak{p}_Q$ and thus canonically induces a similarity transformation $V_{\bar{Q}} \to V_{\bar{Q}}$ relative to $n_{\bar{Q}}$ with multiplier $n_{\bar{Q}}(\bar{u})$.

We call

$$f_{Q/F} := \dim_{\bar{F}}(V_{\bar{Q}}) \tag{12}$$

the *residue degree* of Q. For convenience, we collect some of the above properties of $\bar{Q}$ in the following proposition, which is really stating the obvious.

7.10. **Proposition.** *Norm, trace and conjugation of the pointed quadratic residue space $\bar{Q}$ of Q are given by the formulas*

$$n_{\bar{Q}}(\bar{x}) = \overline{n_Q(x)}, \tag{1}$$

$$t_{\bar{Q}}(\bar{x}) = \overline{t_Q(x)}, \tag{2}$$

$$\bar{x}^* = \overline{x^*} \tag{3}$$

for all $x \in \mathfrak{o}_Q$. Moreover, $n_{\bar{Q}}$ is round and anisotropic. □

Here is the easiest example that is not totally trivial.

7.11. **Example.** Let $Q = (V, q, e)$ be such that V has basis e, j and $q(ue+vj) = u^2 - \mu v^2$ for some $\mu \in \mathfrak{o}^\times$ and all $u, v \in F$. (This is the case $P = \langle 1 \rangle$ of 7.4; $q \cong \langle\langle \mu \rangle\rangle$.) If $\bar{\mu}$ is not a square in $\overline{F}$, then $\mathfrak{o}_Q$ and $\mathfrak{p}_Q$ are the $\mathfrak{o}$- and $\mathfrak{p}$-spans of e, j respectively and the pointed quadratic residue space of Q is naturally identified with the quadratic extension $\overline{F}(\sqrt{\bar{\mu}})$ with quadratic form the squaring map $z \mapsto z^2$ and base point $1 \in \overline{F}$.

7.12. **Lemma.** *The map*

$$\mathbb{Z} \times \left\{ y \in V_Q^\times \,\middle|\, 0 \le \lambda_Q(y) \le 1 - \frac{1}{e_{Q/F}} \right\} \longrightarrow V_Q^\times, \quad (m, y) \longmapsto \pi^m y,$$

is surjective.

Proof. For $x \in V_Q^\times$, some $n \in \mathbb{Z}$ has $\lambda_Q(x) = \frac{n}{e_{Q/F}}$ by (7.9.8), and writing $n = me_{Q/F} + r$, $m, r \in \mathbb{Z}$, $0 \le r \le e_{Q/F} - 1$, the element $y := \pi^{-m} x \in V_Q^\times$ satisfies $0 \le \lambda_Q(y) \le 1 - \frac{1}{e_{Q/F}}$. □

7.13. **Proposition.** $e_{Q/F} f_{Q/F} = \dim_F(V_Q)$.

Proof. If $e_{Q/F} = 1$, then $\mathfrak{p}_Q = \pi \mathfrak{o}_Q$, and the dimension of $V_{\bar{Q}} = \mathfrak{o}_Q \otimes_{\mathfrak{o}} \bar{F}$ over $\bar{F}$ agrees with the dimension of V_Q over F. We may therefore assume $e_{Q/F} = 2$. Let Π be a prime element of $\mathfrak{o}_Q$ and $f\colon V_Q \to V_Q$ a norm similarity with multiplier $n_Q(\Pi)$. Then $\lambda_Q(f(x)) = \frac{1}{2} + \lambda_Q(x)$ for all $x \in V_Q^\times$, which implies $f(\mathfrak{o}_Q) = \mathfrak{p}_Q$, $f(\mathfrak{p}_Q) = \pi \mathfrak{o}_Q$, and f induces canonically an $\bar{F}$-linear bijection from $V_{\bar{Q}}$ onto $\mathfrak{p}_Q / \pi \mathfrak{o}_Q$. Combined with the filtration $\mathfrak{o}_Q \otimes_{\mathfrak{o}} \bar{F} = \mathfrak{o}_Q / \pi \mathfrak{o}_Q \supset \mathfrak{p}_Q / \pi \mathfrak{o}_Q \supset \{0\}$ and the isomorphism $(\mathfrak{o}_Q / \pi \mathfrak{o}_Q) / (\mathfrak{p}_Q / \pi \mathfrak{o}_Q) \cong V_{\bar{Q}}$ as vector spaces over $\bar{F}$, this implies $\dim_F(V_Q) = 2 \dim_{\bar{F}}(V_{\bar{Q}})$, as desired. □

Remark. Prop. 7.13 becomes *false* if Q is not assumed to be round, for example if $e_{Q/F} = 2$ and $\dim_F(V_Q)$ is odd, see [39, Satz 6.3] for generalization.

7.14. **Connecting with conic algebras.** Let C be a finite-dimensional conic algebra over F and suppose its norm is round and anisotropic; for example, C could itself be a composition algebra, or it could arise from a composition algebra by means of the Cayley-Dickson process. Applying as we may the preceding considerations to Q_C, the pointed quadratic space associated with C via 7.2, we systematically adhere to the following convention: all notation and terminology developed up to now and later on for pointed quadratic spaces will be applied without further comment to C in place of Q_C, modifying subscripts accordingly whenever possible. For example,

$$\text{(1)} \qquad \lambda_C\colon\, C \longrightarrow \mathbb{Q}_\infty, \quad x \longmapsto \lambda_C(x) := \frac{1}{2} \lambda\big(n_C(x)\big),$$

is a norm of C as an F-vector space, $e_{C/F} := e_{Q_C/F}$ is the *ramification index*, $f_{C/F} := f_{Q_C/F}$ is the *residue degree* and $\bar{C} := \overline{Q_C}$ is the *pointed quadratic residue space* of C. But note that $\bar{C}$ in general is *not* a conic algebra over $\bar{F}$ in a natural way unless C is a composition algebra; see 7.15 and Section 10 below for further discussion.

7.15. **Tame and wild pointed quadratic spaces.** If $t_{\bar{Q}}$ is non-zero, then Q is said to be *tame*. Otherwise, i.e., if $t_{\bar{Q}} = 0$, then Q is said to be *wild*. For Q to be wild it is clearly necessary that $\bar{F}$ have characteristic 2. Applying [40, Prop. 1], and bearing in mind the conventions of 7.14, a composition division algebra C over F is tame (resp. wild) iff $\bar{C}$ is a composition algebra (resp. a purely inseparable field extension of exponent at most 1) over $\bar{F}$. Extending the terminology of [40] to the present more general set-up, we call Q *unramified* (resp. *ramified*) if Q is tame with $e_{Q/F} = 1$ (resp. $e_{Q/F} = 2$). For Q to be unramified it is necessary and sufficient that the quadratic form n_Q have good reduction with respect to λ in the sense of Knebusch [26, 27]. This will follow from Prop. 8.2 (c) below.

The preceding definitions seem to assign a distinguished role to the base point of a pointed quadratic space. But this is not so as will be seen in Prop. 8.2 below.

8. Trace and norm exponent.

This section serves a double purpose. Working with a fixed non-singular, round and anisotropic pointed quadratic space Q over a 2-Henselian field F, we attach wildness-detecting invariants to Q. Moreover, we present a device measuring how far a scalar in $\mathfrak{o}^\times$ is removed from being the norm of an appropriate element in V_Q.

8.1. **Trace ideal and trace exponent.** Since $\mathfrak{o}_Q \subseteq V_Q$ is a full $\mathfrak{o}$-lattice, its image under the trace of Q by non-singularity is a non-zero ideal in $\mathfrak{o}$, called the *trace ideal* of Q. But $\mathfrak{o}$ is a discrete valuation ring, so there is a unique integer $\mathrm{texp}(Q) \geq 0$ such that

$$t_Q(\mathfrak{o}_Q) = \mathfrak{p}^{\mathrm{texp}(Q)}. \tag{1}$$

We call $\mathrm{texp}(Q)$ the *trace exponent* of Q and have

$$\mathrm{texp}(Q) = \min\{\lambda\big(t_Q(x)\big) \mid x \in \mathfrak{o}_Q\}. \tag{2}$$

The image of $\mathfrak{o}_Q \otimes_{\mathfrak{o}} \mathfrak{o}_Q$ under the linear map $x \otimes y \mapsto n_Q(x, y)$ is an ideal in $\mathfrak{o}$ denoted by $\partial n_Q(\mathfrak{o}_Q \otimes_{\mathfrak{o}} \mathfrak{o}_Q)$. The following result relates the ideals just defined to one another but also to wild and tame pointed quadratic spaces.

8.2. **Proposition.** (a) *Q is wild if and only if* $\mathrm{texp}(Q) > 0$.

(b) *If $P \subseteq Q$ is pointed quadratic subspace that is round and non-singular, then* $\mathrm{texp}(P) \geq \mathrm{texp}(Q)$.

(c) $t_Q(\mathfrak{o}_Q) = \partial n_Q(\mathfrak{o}_Q \otimes_{\mathfrak{o}} \mathfrak{o}_Q)$.

(d) *Q is tame if and only if $\bar{Q}$ is non-singular.*

Proof. (a) and (b) are obvious. Before proving (c),(d), we let $x \in Q^\times$ and, by using a Jordan isotopy argument in disguise, pass to $Q^x := (V_Q, n_Q(x)^{-1}n_Q, x)$, which is a non-singular quadratic space over F. Since n_Q is round, the norms of Q and Q^x are isometric, forcing Q and Q^x to be isomorphic as pointed quadratic spaces by Prop. 7.3. This implies

$$\begin{aligned} t_Q(\mathfrak{o}_Q) = t_{Q^x}(\mathfrak{o}_{Q^x}) &= \big\{n_Q\big(x, n_Q(x)^{-1}y\big) \mid y \in V_Q,\ \lambda_Q(y) \geq \lambda_Q(x)\big\} \\ &= \{n_Q(x, y) \mid y \in V_Q,\ \lambda_Q(y) \geq -\lambda_Q(x)\}. \end{aligned} \tag{1}$$

(c) The left-hand side is clearly contained in the right, so it suffices to show $n_Q(x, y) \in t_Q(\mathfrak{o}_Q)$ for all $x, y \in \mathfrak{o}_Q$, $x \neq 0$. But this follows from (1) and $\lambda_Q(x) \geq 0$.
(d) Non-singularity of $\bar{Q}$ is clearly sufficient for Q to be tame. Conversely, suppose Q is tame and let $x \in \mathfrak{o}_Q^\times$. Then (1) produces an element $y \in \mathfrak{o}_Q$ with $n_Q(x, y) = 1 \in t_Q(\mathfrak{o}_Q)$, hence $n_{\bar{Q}}(\bar{x}, \bar{y}) \neq 0$. □

8.3. **Trace generators.** An element $w_0 \in \mathfrak{o}_Q$ where the minimum in (8.1.2) is attained, i.e., with $\lambda(t_Q(w_0)) = \mathrm{texp}(Q)$ is called a *trace generator* of Q. If even $t_Q(w_0) = \pi^{\mathrm{texp}(Q)}$, we speak of a *normalized* trace generator, dependence on π being understood. Trace generators always exist (as do normalized ones) and their traces generate the trace ideal of C. Moreover, they satisfy the inequalities

$$0 \leq \lambda_Q(w_0) \leq 1 - \frac{1}{e_{Q/F}}. \tag{1}$$

Indeed, assuming $\lambda_Q(w_0) > 1 - \frac{1}{e_{Q/F}}$ implies

$$\lambda_Q(\pi^{-1}w_0) = \lambda_Q(w_0) - 1 > -\frac{1}{e_{Q/F}},$$

and since $\lambda_Q(\pi^{-1}w_0)$ belongs to $\frac{1}{e_{Q/F}}\mathbb{Z}$, we conclude $\pi^{-1}w_0 \in \mathfrak{o}_Q$. But now $t_Q(\pi^{-1}w_0) = \pi^{\mathrm{texp}(Q)-1} \notin \mathfrak{p}^{\mathrm{texp}(Q)}$ leads to a contradiction.

8.4. **Tignol's invariant** $\omega(C)$**.** Another invariant that fits into the present set-up is due to Tignol [50, pp. 9,17] for separable field extensions and central associative division algebras of degree p over Henselian fields (not necessarily discrete) having residual characteristic $p > 0$. As a straightforward adaptation of Tignol's definition to the situation we are interested in, we define

(1) $$\omega(Q) := \min\{\lambda\big(t_Q(x)\big) - \lambda_Q(x) \mid x \in V_Q^\times\};$$

thanks to (7.9.7), it is a non-negative rational number. Moreover, it is closely related to the trace exponent, as the following proposition shows.

8.5. **Proposition.** (a) $\omega(Q) = \mathrm{texp}(Q)$ *or* $\omega(Q) = \mathrm{texp}(Q) - \frac{1}{2}$.

(b) *There exists a trace generator* w_0 *of* Q *with*

(1) $$\omega(Q) = \mathrm{texp}(Q) - \lambda_Q(w_0).$$

(c) *If* $e_{Q/F} = 1$, *then* $\omega(Q) = \mathrm{texp}(Q)$.

Proof. Since the map

$$\varphi\colon V_Q^\times \longrightarrow \Gamma_Q = \frac{1}{e_{Q/F}}\mathbb{Z}, \quad x \longmapsto \varphi(x) := \lambda\big(t_Q(x)\big) - \lambda_Q(x),$$

is homogeneous of degree zero, so $\varphi(\alpha x) = \varphi(x)$ for all $\alpha \in F^\times$, $x \in V_Q^\times$, Lemma 7.12 implies $\omega(Q) = \min\{\varphi(x) \mid x \in S\}$, where

$$S := \{x \in V_Q^\times \mid 0 \le \lambda_Q(x) \le 1 - \frac{1}{e_{Q/F}}\}.$$

Accordingly, let $w_0 \in S$ satisfy

(2) $$\varphi(w_0) = \omega(Q).$$

Given any trace generator w_0' of Q, the chain of inequalities

(3) $$\begin{aligned} 0 \le \lambda_Q(w_0') = \lambda\big(t_Q(w_0')\big) - \varphi(w_0') \le \mathrm{texp}(Q) - \omega(Q) \le \lambda\big(t_Q(w_0)\big) - \varphi(w_0) \\ = \lambda_Q(w_0) \le 1 - \frac{1}{e_{Q/F}} \end{aligned}$$

implies (a) and (c), while in (b) we may assume $e_{Q/F} = 2$ since otherwise (1) holds for w_0', i.e., for any trace generator of Q. Since all quantities in (3) belong to $\frac{1}{2}\mathbb{Z}$, we either have $\lambda_Q(w_0') = \mathrm{texp}(Q) - \omega(Q)$ or

$$\mathrm{texp}(Q) - \omega(Q) = \lambda\big(t_Q(w_0)\big) - \varphi(w_0) = \lambda_Q(w_0) = \frac{1}{2}.$$

In the latter case, (2) shows that w_0 is a trace generator of Q, and the proof of (b) is complete. □

8.6. **Regular trace generators.** Trace generators of Q satisfying (8.5.1) are called *regular*. If Q has ramification index 1, then every trace generator by (8.3.1) and Prop. 8.5 (c) is regular but, in general, this need not be so, cf. Cor. 11.7 and Thm. 12.3 below.

8.7. **Example.** If F has characteristic not 2, it is a composition division algebra over itself, with norm and trace given by $n_F(\alpha) = \alpha^2$, $t_F(\alpha) = 2\alpha$ for all $\alpha \in F$. Hence (7.7.1), (8.1.1) and (8.4.1) imply

(1) $$\mathrm{texp}(F) = \omega(F) = e_F,$$

and $w_0 = \frac{\pi^{e_F}}{2} \in \mathfrak{o}^\times$ is the unique normalized trace generator of F; it is obviously regular. From (1) and Props. 8.2 (c), 8.5 we now conclude

(2) $$0 \le \omega(Q) \le \mathrm{texp}(Q) \le e_F,$$

which for trivial reasons also holds in characteristic 2.

8.8. **The norm exponent.** We wish to measure how far a given unit in the valuation ring of F is removed from being the norm of an element in V_Q or, equivalently, in $\mathfrak{o}_Q^\times$. To this end, we observe that, given $\alpha \in \mathfrak{o}^\times$ and an integer $d \geq 0$, the following conditions are equivalent.

(i) α is a norm of $\mathfrak{o}_Q^\times$ mod $\mathfrak{p}^d$, i.e., there exists an element $v \in \mathfrak{o}_Q^\times$ with

$$\alpha - n_Q(v) \in \mathfrak{p}^d.$$

(ii) There exist elements $\beta \in \mathfrak{o}$, $v \in \mathfrak{o}_Q^\times$ with

$$\alpha = (1 - \pi^d \beta) n_Q(v).$$

Thus the set $N_Q(\alpha)$ of non-negative integers d satisfying (i)/(ii) above contains 0, and $d \in N_Q(\alpha)$ implies $d' \in N_Q(\alpha)$ for all integers d', $0 \leq d' \leq d$. We therefore put

$$\operatorname{nexp}_Q(\alpha) := \sup N_Q(\alpha) \tag{1}$$

and call this the *norm exponent* of α relative to Q; it is either a non-negative integer or ∞. Roughly speaking, the bigger the norm exponent becomes, the closer α gets to being a norm of Q. *More precisely,* $2^{-\operatorname{nexp}_Q(\alpha)}$ *is the (minimum) distance of* α *from the subset* $n_Q(\mathfrak{o}_Q^\times) \subseteq F^\times$ *relative to the metric induced by the absolute value* $\xi \mapsto |\xi| := 2^{-\lambda(\xi)}$.

A number of useful elementary properties of the norm exponent are collected in the following proposition, whose straightforward proof is left to the reader.

8.9. **Proposition.** *With the notations and assumptions of* 8.8, *let* $\alpha, \alpha' \in \mathfrak{o}^\times$.

(a) *If* $\alpha \in n_Q(\mathfrak{o}_Q^\times)$, *then* $\operatorname{nexp}_Q(\alpha) = \infty$.

(b) *If* $d = \operatorname{nexp}_Q(\alpha)$ *is finite, then* α *can be written in the form*

$$\alpha = (1 - \pi^d \beta) n_Q(v), \quad \beta \in \mathfrak{o}, \quad v \in \mathfrak{o}_Q^\times,$$

and every such representation of α *satisfies* $\beta \in \mathfrak{o}^\times$.

(c) $\operatorname{nexp}_Q(\alpha) = 0$ *if and only if* $\bar{\alpha} \notin n_{\bar{Q}}(V_{\bar{Q}}^\times)$.

(d) $\alpha \equiv \alpha' \bmod n_Q(V_Q^\times)$ *implies* $\operatorname{nexp}_Q(\alpha) = \operatorname{nexp}_Q(\alpha')$.

(e) $\operatorname{nexp}_P(\alpha) \leq \operatorname{nexp}_Q(\alpha)$ *for any round non-singular pointed quadratic subspace* $P \subseteq Q$.

□

By Prop. 8.9 (a), the elements of $n_Q(\mathfrak{o}_Q^\times)$ have infinite norm exponent. While the converse is also true, we can do better than that by showing that the norm exponents of elements in $\mathfrak{o}^\times \setminus n_Q(\mathfrak{o}_Q^\times)$ are uniformly bounded from above. Indeed, we have the following result.

8.10. **Local Norm Theorem.** *Let* $\alpha \in \mathfrak{o}^\times \setminus n_Q(\mathfrak{o}_Q^\times) = \mathfrak{o}^\times \setminus n_Q(V_Q^\times)$. *Then*

$$\operatorname{nexp}_Q(\alpha) \leq 2\,\omega(Q).$$

More precisely, letting $w_0 \in \mathfrak{o}_Q$ *be a normalized regular trace generator of* Q, *every* $\beta \in \mathfrak{o}$ *admits a* $\gamma \in \mathfrak{o}$ *with*

$$1 - \pi^{2\,\omega(Q)+1}\beta = n_Q\big(1_Q + \pi^{\operatorname{texp}(Q)}\gamma w_0\big). \tag{1}$$

Proof. Arguing indirectly, and using roundness of n_Q, the first part of the theorem follows from the second. To establish the second part, we apply Prop. 8.5 (a),(b) and obtain

$$0 \leq d := \lambda\big(n_Q(w_0)\big) = 2\lambda_Q(w_0) = 2\big(\operatorname{texp}(Q) - \omega(Q)\big) \leq 1.$$

Since F is 2-Henselian, the polynomial

$$g := \pi^{1-d} n_Q(w_0)\mathbf{t}^2 + \mathbf{t} + \beta \in \mathfrak{o}[\mathbf{t}] \subseteq F[\mathbf{t}]$$

by 7.7 (ii) is reducible, and the two roots $\delta, \delta' \in F$ of $\pi^{d-1} n_Q(w_0)^{-1} g$ satisfy the relation $\delta\delta' = \pi^{d-1} n_Q(w_0)^{-1}\beta \in \mathfrak{p}^{-1}$. Thus we may assume $\delta \in \mathfrak{o}$ and have $\pi^{1-d} n_Q(w_0)\delta^2 + \delta + \beta = 0$. Setting $\gamma := \pi^{1-d}\delta \in \mathfrak{o}$, we therefore obtain

$$\begin{aligned} n_Q\big(1 + \pi^{\mathrm{texp}(Q)}\gamma w_0\big) &= 1 + \pi^{\mathrm{texp}(Q)+1-d}\delta t_Q(w_0) + \pi^{2\,\mathrm{texp}(Q)}\pi^{2(1-d)}\delta^2 n_Q(w_0) \\ &= 1 + \pi^{2\,\mathrm{texp}(Q)-d+1}\big(\delta + \pi^{1-d} n_Q(w_0)\delta^2\big) \\ &= 1 - \pi^{2\,\omega(Q)+1}\beta. \end{aligned}$$

□

For a version of this result addressed to central associative division algebras of degree $p = \mathrm{char}(\bar{F}) > 0$, see Kato [22, Prop. 2 (iii)].

8.11. **Corollary.** *Let $\mu, \mu' \in \mathfrak{o}^\times$. Then*

$$\mu \equiv \mu' \bmod n_Q(Q^\times) \iff \exists\, x \in \mathfrak{o}_Q^\times : \mu \equiv \mu' n_Q(x) \bmod \mathfrak{p}^{2\,\omega(Q)+1}.$$

□

8.12. **The connection with the quadratic defect.** We assume $\mathrm{char}(F) \neq 2$ and return to the composition division algebra $C = F$ of 8.7. Comparing 8.8 with [34, §63A], we see that $\mathfrak{p}^{\mathrm{nexp}_F(\alpha)}$ is the quadratic defect of $\alpha \in \mathfrak{o}^\times$. Moreover, $\mathrm{texp}(F) = \omega(F) = e_F$, and $w_0 := \frac{\pi^{e_F}}{2} \in \mathfrak{o}^\times$ is the unique normalized regular trace generator of F. In particular, $\frac{4}{\pi^{2e_F}} \in \mathfrak{o}^\times$, and the change of variables $\beta = -\frac{4}{\pi^{2e_F}}\beta'$, $\gamma = \frac{4}{\pi^{2e_F}}\gamma'$ converts (8.10.1) into the relation

$$1 + 4\pi\beta' = (1 + 2\pi\gamma')^2.$$

Hence the local norm theorem becomes the local square theorem of [34, 63:1] or [29, VI.2.19] in the special case $Q = Q_F$. See also [10, Prop. 4.1.2] for an extension of this result to residual characteristics other than 2.

The local norm theorem has numerous applications. One of these can already be given in this section; a useful technical lemma prepares the way.

8.13. **Lemma.** *Suppose Q is wild and has ramification index $e_{Q/F} = 1$. For $d \in \mathbb{Z}$, $0 \le d \le 2\,\mathrm{texp}(Q) = 2\,\omega(Q)$ and $\beta \in \mathfrak{o}^\times$, the following conditions are equivalent.*

(i) $1 - \pi^d\beta \in n_Q(\mathfrak{o}_Q^\times)$.

(ii) $1 - \pi^d\beta \in \mathfrak{o}^\times$ *and there are elements $m \in \mathbb{Z}$, $w \in \mathfrak{o}_Q^\times$ with*

(1) $$d = 2m, \quad \beta = -n_Q(w) + \pi^{-m} t_Q(w).$$

Proof. (i) $\Longrightarrow$ (ii). There exists an element $v \in \mathfrak{o}_Q^\times$ with $1 - \pi^d\beta = n_Q(v) \in \mathfrak{o}^\times$. For $d = 0$ we put $w = 1_Q - v \in \mathfrak{o}_Q$ and obtain $1 - \beta = n_Q(1_Q - w) = 1 - t_Q(w) + n_Q(w)$, hence $\beta = -n_Q(w) + t_Q(w)$. But $t_Q(w) \in \mathfrak{p}^{\mathrm{texp}(Q)} \subseteq \mathfrak{p}$ since Q is wild, forcing $w \in \mathfrak{o}_Q^\times$. Thus (ii) holds with $m = 0$. We may therefore assume $d > 0$. Then $1 = \overline{n_Q(v)} = n_{\bar{Q}}(\bar{v})$, forcing $\bar{v} = 1$ since $n_{\bar{Q}}$ is anisotropic and $n_{\bar{Q}}(1_{\bar{Q}} - \bar{v}) = 0$ by wildness of Q. Combining with $e_{Q/F} = 1$, we find an integer $m > 0$ and a unit $w \in \mathfrak{o}_Q^\times$ with $v = 1_Q - \pi^m w$. Expanding the right-hand side of $1 - \pi^d\beta = n_Q(1_Q - \pi^m w)$, we conclude

(2) $$\pi^d\beta = -\pi^{2m} n_Q(w) + \pi^m t_Q(w),$$

which in turn yields the estimate

$$2\ \mathrm{texp}(Q) \ge d = \lambda(\pi^d\beta) = \lambda\big(-\pi^{2m} n_Q(w) + \pi^m t_Q(w)\big) \ge \min\{2m, m + \mathrm{texp}(Q)\}.$$

This implies

(3) $$m \le \mathrm{texp}(Q), \quad d \ge 2m,$$

and (2) attains the form

(4) $$\pi^d\beta = -\pi^{2m} n_Q(w)(1 - \gamma), \quad \gamma := \pi^{-m} n_Q(w)^{-1} t_Q(w) \in \mathfrak{p}^{\mathrm{texp}(Q)-m}.$$

If $m < \mathrm{texp}(Q)$, then (1) follows from (2),(4). On the other hand, if $m = \mathrm{texp}(Q)$, then (3) implies $d = 2\,\mathrm{texp}(Q) = 2m$, and (1) follows from (2).

(ii) $\Longrightarrow$ (i). Setting $v := 1_Q - \pi^m w \in \mathfrak{o}_Q$ and applying (1), we conclude $n_Q(v) = 1 - \pi^d\beta \in \mathfrak{o}^\times$, hence $v \in \mathfrak{o}_Q^\times$, and (i) holds. □

Remark. For $d > 0$, the condition $1 - \pi^d\beta \in \mathfrak{o}^\times$ in (ii) is of course automatic.

8.14. **Proposition.** *Let P be a pointed quadratic space over F that is non-singular, round and anisotropic with $e_{P/F} = 1$, and let d be an odd integer with $0 \le d < 2\,\mathrm{texp}(P)$.*
(a) *If $\beta \in \mathfrak{o}$, then $\mu := 1 - \pi^d\beta \in \mathfrak{o}^\times$ and*

$$d = \mathrm{nexp}_P(\mu) \iff \beta \in \mathfrak{o}^\times.$$

(b) *If $\beta_i \in \mathfrak{o}^\times$ and $\mu_i := 1 - \pi^d\beta_i$ for $i = 1, 2$, then*

$$\langle\!\langle \mu_1 \rangle\!\rangle \otimes P \cong \langle\!\langle \mu_2 \rangle\!\rangle \otimes P \Longrightarrow \overline{\beta_1} = \overline{\beta_2}.$$

Proof. (a) $d = \mathrm{nexp}_P(\mu)$ implies $\beta \in \mathfrak{o}^\times$ by Prop. 8.9 (b). Conversely, suppose $\beta \in \mathfrak{o}^\times$. Then $\mu \notin n_P(V_P^\times)$ by Lemma 8.13. Thus the local norm theorem 8.10 implies $d \le d' := \mathrm{nexp}_P(\mu) \le 2\,\mathrm{texp}(P)$, and from Prop. 8.9 (b) we obtain a representation $\mu = \mu' n_P(v')$, $\mu' = 1 - \pi^{d'}\beta'$ for some $\beta' \in \mathfrak{o}^\times$, $v' \in \mathfrak{o}_P^\times$. Assuming $d < d'$ would imply that $\mu\mu' = 1 - \pi^d\gamma$, $\gamma := \beta + \pi^{d'-d}\beta' - \pi^{d'}\beta\beta' \in \mathfrak{o}^\times$, does not belong to $n_P(V_P^\times)$, again by Lemma 8.13, in contradiction to $\mu\mu' = n_P(\mu' v')$.

(b) Arguing indirectly, let us assume $\overline{\beta_1} \ne \overline{\beta_2}$. Then $\mu_1\mu_2 = 1 - \pi^d\beta$, where $\beta = \beta_1 + \beta_2 - \pi^d\beta_1\beta_2 \in \mathfrak{o}$ satisfies $\bar\beta = \overline{\beta_1} - \overline{\beta_2} \ne 0$, hence $\beta \in \mathfrak{o}^\times$. By (a), the norm exponent of $\mu_1\mu_2$ relative to P is d, forcing $\mu_1\mu_2 \notin n_P(V_P^\times)$ (this also follows from Lemma 8.13). Hence $\langle\!\langle \mu_1 \rangle\!\rangle \otimes P$ and $\langle\!\langle \mu_2 \rangle\!\rangle \otimes P$ are not isomorphic by Prop. 7.5 (b). □

We will see in Example 9.11 (b) below that the converse of Prop. 8.14 (b) does not hold.

8.15. **Lemma.** *Let $\beta \in \mathfrak{o}$, $w \in \mathfrak{o}_Q$ and suppose d, m are non-negative integers. Determine $\alpha \in \mathfrak{o}$ by $t_Q(w) = \pi^{\mathrm{texp}(Q)}\alpha$. Then*

$$(1 - \pi^d\beta) n_Q(1_Q - \pi^m w) = 1 - \pi^d\beta + \pi^{2m} n_Q(w) - \pi^{\mathrm{texp}(Q)+m}\alpha + \pi^{\mathrm{texp}(Q)+d+m}\alpha\beta - \pi^{d+2m}\beta n_Q(w).$$

Proof. Expand the left-hand side in the obvious way. □

9. Valuation data under enlargements.

In this section we will be concerned with the question of what happens to the valuation data ramification index (7.9 (b)), pointed quadratic residue space (7.9 (c)) and trace exponent (8.1) when passing from a pointed quadratic space P to $\langle\!\langle \mu \rangle\!\rangle \otimes P$, $\mu \in F^\times$. We will answer this question not in full generality but only under the additional hypothesis that P have ramification index 1. This hypothesis derives its justification from the fact that, if F has characteristic zero, every anisotropic pointed $(n + 1)$-Pfister quadratic space over F contains a pointed n-Pfister quadratic subspace of ramification index 1. We will prove this in Prop. 17.2 and Thm. 19.2(i) below by using methods from algebraic K-theory. It would be interesting to know whether the result in question also holds for F having characteristic 2.

9.1. **The general set-up.** (a) We fix a 2-Henselian field F and a pointed quadratic space P over F which is non-singular, round and anisotropic. We also assume throughout that P has ramification index $e_{P/F} = 1$, which implies

$$\omega(P) = \mathrm{texp}(P) \tag{1}$$

by Prop. 8.5 (c) and

$$\Gamma_P = \lambda_P(P^\times) = \mathbb{Z}, \quad \lambda\big(n_P(P^\times)\big) = 2\mathbb{Z} \tag{2}$$

by 7.9 (b) and (7.9.1).

(b) We are interested in pointed quadratic spaces $Q = \langle\!\langle \mu \rangle\!\rangle \otimes P$, $\mu \in F^\times$, as in 7.4; in particular, we recall $V_Q = V_P + V_P j$ as vector spaces over F. By Prop. 7.5 (b), we may and always will assume $\lambda(\mu) \in \{0,1\}$, so μ is either a unit or a prime element in $\mathfrak{o}$.

There are two harmless cases which we treat first. One of them arises when μ is a prime, the other when μ is a unit and P is tame.

9.2. **Proposition.** *If μ is a prime element in $\mathfrak{o}$, then $Q := \langle\langle\mu\rangle\rangle \otimes P$ is a non-singular, round and anisotropic pointed quadratic space over F with*

(1) $$\lambda_Q(u + vj) = \min\{\lambda_P(u), \lambda_P(v) + \frac{1}{2}\} \qquad (u, v \in V_P),$$

(2) $$\mathfrak{o}_Q = \mathfrak{o}_P \oplus \mathfrak{o}_P j, \quad \mathfrak{p}_Q = \mathfrak{p}_P \oplus \mathfrak{o}_P j,$$

(3) $$e_{Q/F} = 2, \quad \bar{Q} = \bar{P}, \quad \mathrm{texp}(Q) = \mathrm{texp}(P).$$

Proof. From (9.1.2) we conclude $\mu \notin n_P(P^\times)$. Thus Q is not only round and non-singular but also anisotropic, so λ_Q satisfying (7.9.1)$-$(7.9.4) exists. Since $\lambda_Q(vj) = \lambda_P(v) + \frac{1}{2} \neq \lambda_P(u)$ for all $u, v \in V_P^\times$, by (7.4.1),(7.9.1) and again by (9.1.2), we obtain (1), hence (2) and the first two relations of (3), while the last one is immediately implied by (7.4.2). □

9.3. **Proposition.** *If P as in* 9.1 (a) *is tame and $\mu \in \mathfrak{o}^\times$, then $Q = \langle\langle\mu\rangle\rangle \otimes P$ is a non-singular and round pointed quadratic space over F. Moreover the following conditions are equivalent.*

(i) *Q is anisotropic.*
(ii) *$\mu \notin n_P(V_P^\times)$.*
(iii) *$\bar{\mu} \notin n_{\bar{P}}(V_{\bar{P}}^\times)$.*

In this case,

(1) $$\lambda_Q(u + vj) = \min\{\lambda_P(u), \lambda_P(v)\} \qquad (u, v \in P),$$

(2) $$\mathfrak{o}_Q = \mathfrak{o}_P \oplus \mathfrak{o}_P j, \quad \mathfrak{p}_Q = \mathfrak{p}_P \oplus \mathfrak{p}_P j,$$

and Q is tame of ramification index $e_{Q/F} = 1$ with $\bar{Q} \cong \langle\langle\bar{\mu}\rangle\rangle \otimes \bar{P}$.

Proof. While the first statement is obvious, the equivalence of (i),(ii),(iii) follows from Prop. 7.5 (a) and Cor. 8.11 since $\omega(P) = \mathrm{texp}(P) = 0$ by (9.1.1) and tameness of P. If (i),(ii),(iii) hold, then λ_Q exists and it suffices to show that (1) holds. Since $\lambda_Q(vj) = \lambda_P(v)$ by (7.4.1),(7.9.1), we certainly have $\lambda_Q(u + vj) \geq \min\{\lambda_P(u), \lambda_P(v)\}$. To prove equality, Lemma 7.12 allows us to assume $\lambda_P(u) = \lambda_P(v) = 0$. Here $\lambda_Q(u + vj) > 0$ would imply $n_{\bar{P}}(\bar{u}) = \bar{\mu} n_{\bar{P}}(\bar{v})$, hence $\bar{\mu} \in n_{\bar{P}}(\bar{P}^\times)$ since $n_{\bar{P}}$ is round, and we obtain a contradiction to (iii). □

The remaining cases where P is wild and $\mu \in \mathfrak{o}$ is a unit are much more troublesome.

9.4. **Some easy reductions.** For the rest of this section, we assume that P as given in 9.1 (a) is wild, so $\omega(P) = \mathrm{texp}(P) > 0$ by Prop. 8.2 (a). We are interested in the pointed quadratic spaces $\langle\langle\mu\rangle\rangle \otimes P$, $\mu \in \mathfrak{o}^\times$, only when they are anisotropic. By Prop. 7.5 (a) and the local norm theorem 8.10, this is equivalent to μ having norm exponent $\leq 2\,\mathrm{texp}(P)$, so by Prop. 8.9 (b) it will be enough to consider units in $\mathfrak{o}$ having the form $\mu = (1 - \pi^d\beta)n_P(v)$, where $d \in \mathbb{Z}$ satisfies $0 \leq d \leq 2\,\mathrm{texp}(P)$, $\beta \in \mathfrak{o}^\times$ and $v \in \mathfrak{o}_P^\times$. Here Prop. 7.5 (b) allows us to assume $v = 1_P$. We are thus reduced to working with scalars μ that may be written as

(1) $$\mu = 1 - \pi^d\beta, \quad d \in \mathbb{Z}, \quad 0 \leq d \leq 2\,\mathrm{texp}(P), \quad \beta \in \mathfrak{o}^\times.$$

Setting

(2) $$0 \leq m := \left\lfloor \frac{d}{2} \right\rfloor \leq \mathrm{texp}(P), \Theta_d := \pi^{-m}(1_P + j) \in V_Q = V_P + V_P j,$$

we define, inspired by the Cayley-Dickson construction of conic algebras (cf. (1.10.1)),

(3) $$v\Theta_d := \pi^{-m}(v + vj) \quad (v \in V_P), \quad V_P\Theta_d := \{v\Theta_d \mid v \in V_P\}$$

and obtain after a straightforward computation, involving (7.4.1),(7.4.2),

(4) $$V_Q = V_P \oplus V_P\Theta_d,$$

(5) $$n_Q(u + v\Theta_d) = n_P(u) + \pi^{-m} n_P(u, v) + \pi^{d-2m}\beta n_P(v),$$

(6) $$t_Q(u + v\Theta_d) = t_P(u) + \pi^{-m} t_P(v)$$

for all $v \in V_P$.

9.5. **Pointed quadratic residue spaces and inseparable extensions.** (cf. [15, Remark 10.4]) For P as in 9.1,9.4, we claim *that $V_{\bar{P}}$ carries a unique structure of a purely inseparable extension field over $\bar{F}$ having exponent at most 1 such that $n_{\bar{P}}(u') = u'^2$ for all $u' \in V_{\bar{P}}$.* To see this, it suffices to note that $n_{\bar{P}}$ is round and anisotropic with $\partial n_{\bar{P}} = 0$, making $n_{\bar{P}}(V_{\bar{P}})$ a subfield of $\bar{F}$, so an $\bar{F}$-bilinear multiplication $V_{\bar{P}} \times V_{\bar{P}} \to V_{\bar{P}}$, $(u', v') \mapsto u'v'$, gives a purely inseparable extension field structure as indicated iff $n_{\bar{P}}(u'v') = n_{\bar{P}}(u')n_{\bar{P}}(v')$ for all $u', v' \in V_{\bar{P}}$. The purely inseparable extension field thus constructed will again be denoted by $V_{\bar{P}}$ if there is no danger of confusion.

9.6. **Proposition.** *Let d be an* odd *integer with $0 \le d \le 2\,\mathrm{texp}(P)$ and $\beta \in \mathfrak{o}^\times$. Then*

$$\mu := 1 - \pi^d\beta \in \mathfrak{o}^\times$$

and $Q := \langle\langle\mu\rangle\rangle \otimes P$ is a non-singular, round and anisotropic pointed quadratic space over F. Moreover, Q is wild and

(1) $$\Pi := \Theta_d = \pi^{-\frac{d-1}{2}}(1_B + j) \in V_Q$$

is a prime element of $\mathfrak{o}_Q$ with

(2) $$n_Q(\Pi) = \pi\beta, \quad t_Q(\Pi) = 2\pi^{-\frac{d-1}{2}}, \quad V_Q = V_P \oplus V_P\Pi,$$

(3) $$\lambda_Q(u + v\Pi) = \min\{\lambda_P(u), \lambda_P(v) + \frac{1}{2}\} \quad (u, v \in V_P),$$

(4) $$\mathfrak{o}_Q = \mathfrak{o}_P \oplus \mathfrak{o}_P\Pi, \quad \mathfrak{p}_Q = \mathfrak{p}_P \oplus \mathfrak{o}_P\Pi,$$

(5) $$e_{Q/F} = 2, \quad \bar{Q} = \bar{P}, \quad \mathrm{texp}(Q) = \mathrm{texp}(P) - \frac{d-1}{2}.$$

Proof. By Prop. 8.14 (a), $\mathrm{nexp}_P(\mu) = d$ is finite, forcing Q to be anisotropic. Applying (9.4.4)−(9.4.6), we obtain (2); in particular, $\lambda_Q(\Pi) = \frac{1}{2}$, so Π is a prime element of $\mathfrak{o}_Q$ and the first formula of (5) holds. We proceed to establish (3). If $u, v \in V_P^\times$, then $\lambda_Q(u) = \lambda_P(u)$ is an integer by (9.1.2), while $\lambda_Q(v\Pi) = \lambda_P(v) + \frac{1}{2}$ is not. This not only proves (3) but also (4) and the second formula of (5). The last one follows from the fact that (9.4.6) establishes $\mathfrak{p}^{\mathrm{texp}(P)} + \mathfrak{p}^{\mathrm{texp}(P)-\frac{d-1}{2}} = \mathfrak{p}^{\mathrm{texp}(P)-\frac{d-1}{2}}$ as the trace ideal of Q. □

9.7. **Proposition.** *Let d be an* even *integer with $0 \le d < 2\,\mathrm{texp}(P)$ and suppose $\beta \in \mathfrak{o}$ satisfies the condition $\bar{\beta} \notin V_{\bar{P}}^2$ (cf. 9.5). Then*

(1) $$\mu = 1 - \pi^d\beta \in \mathfrak{o}^\times$$

and $Q := \langle\langle\mu\rangle\rangle \otimes P$ is a non-singular, round and anisotropic pointed quadratic space over F. Moreover, Q is wild and

(2) $$\Xi := \Theta_d = \pi^{-\frac{d}{2}}(1_P + j) \in V_Q$$

is a unit of $\mathfrak{o}_Q$ with

(3) $$n_Q(\Xi) = \beta, \quad t_Q(\Xi) = 2\pi^{-\frac{d}{2}}, \quad V_Q = V_P \oplus V_P\Xi,$$

(4) $$\lambda_Q(u + v\Xi) = \min\{\lambda_P(u), \lambda_P(v)\} \qquad (u, v \in V_P),$$

(5) $$\mathfrak{o}_Q = \mathfrak{o}_P \oplus \mathfrak{o}_P\Xi, \quad \mathfrak{p}_Q = \mathfrak{p}_P \oplus \mathfrak{p}_P\Xi,$$

(6) $$e_{Q/F} = 1, \quad \bar{Q} = \langle\langle\bar{\beta}\rangle\rangle \otimes \bar{P}, \quad \mathrm{texp}(Q) = \mathrm{texp}(P) - \frac{d}{2}.$$

Proof. The assertion $\mu \in \mathfrak{o}^\times$ is trivial for $d > 0$ but holds also for $d = 0$ since in this case $\bar{\beta} \notin V_{\bar{P}}^2$ implies $\bar{\mu} = 1_{\bar{F}} - \bar{\beta} \notin V_{\bar{P}}^2$ by wildness of P. Next we show that Q is anisotropic. Otherwise, $\mu \in n_P(V_P)$ by Prop. 7.5 (a), and Lemma 8.13 yields an element $w \in \mathfrak{o}_P^\times$ with $\beta = -n_P(w) + \pi^{-m} t_P(w)$, $m = \frac{d}{2}$, where the second summand belongs to $\mathfrak{p}^{\mathrm{texp}(P)-m} \subseteq \mathfrak{p}$ by the hypothesis on d. Thus $\bar{\beta} = n_{\bar{P}}(\bar{w}) = \bar{w}^2$, a contradiction, and we have proved that Q is indeed anisotropic. Consulting (9.4.4−6) for $u = 0$, $v = 1_P$, we end up with (3). Turning to (4), it suffices to show, by Lemma 7.12, that $u, v \in \mathfrak{o}_P^\times$ implies $u + v\Xi \in \mathfrak{o}_Q^\times$. Otherwise, observing (9.4.5),

$$\lambda_Q(u + v\Xi) = \frac{1}{2}\lambda\big(n_P(u) + \pi^{-m} n_P(u, v) + \beta n_P(v)\big)$$

were strictly positive, and since $\pi^{-m} n_P(u, v) \in \mathfrak{p}$ by Prop. 8.2 (c), we would again arrive at the contradiction $\bar{\beta} \in V_{\bar{P}}^2$. Thus (4) holds, which directly implies (5), while (6) follows from (5) and (9.4.5,6). □

In 9.4, particularly (9.4.1), we are left with the case $d = 2\,\mathrm{texp}(P)$, which turns out to be the most delicate. In order to get started, we require the following elementary but crucial observations.

9.8. **Setting the stage for the case** $d = 2\,\mathrm{texp}(P)$**.** (a) Let K/k be a purely inseparable field extension of characteristic 2, exponent at most 1 and finite degree. Consider a scalar $\alpha \in k$ and a unital linear form $s\colon K \to k$. We denote by by $Q_{K;\alpha,s}$ the pointed quadratic space over k with norm the Pfister quadratic form $q_{K;\alpha,s}$ of 3.4 and with base point $1_K \in K \subseteq K \oplus Kj$. Recall from Prop. 4.4 that $Q_{K;\alpha,s} = Q_{\mathrm{Cay}(K;\alpha,s)}$ is the pointed quadratic space corresponding to the flexible conic algebra $\mathrm{Cay}(K; \alpha, s)$ that arises form K, α, s by means of the non-orthogonal Cayley-Dickson construction.

(b) Put $m := \mathrm{texp}(P)$ and let w_0 be a normalized trace generator of P. Then $w_0 \in \mathfrak{o}_P^\times$ by (8.3.1) and the map $s_{w_0}\colon V_P \to F$, $u \mapsto \pi^{-m} n_P(u, w_0)$ is a unital linear form with $s_{w_0}(\mathfrak{o}_P) \subseteq \mathfrak{o}$, $s_{w_0}(\mathfrak{p}_P) \subseteq \mathfrak{p}$ by Prop. 8.2 (c) and since $\mathfrak{p}_P = \mathfrak{p}\mathfrak{o}_P$. Thus we obtain an induced unital linear form

(1) $$\bar{s}_{w_0}\colon V_{\bar{P}} \longrightarrow \bar{F}, \quad \bar{u} \longmapsto \bar{s}_{w_0}(\bar{u}) = \overline{\pi^{-m} n_P(u, w_0)},$$

and given any $\alpha' \in \bar{F}$, the notational conventions of (a) apply when $V_{\bar{P}}/\bar{F}$ is viewed via 9.5 as a purely inseparable field extension of exponent at most 1.

9.9. **Theorem.** *With the notations of* 9.8, *let*

(1) $$\beta \in \mathfrak{o}, \quad \mu := 1 - \pi^{2\,\mathrm{texp}(P)}\beta \in \mathfrak{o}^\times, \quad \beta_0 := n_P(w_0)\beta \in \mathfrak{o}, \quad Q := \langle\!\langle \mu \rangle\!\rangle \otimes P.$$

Then the following conditions are equivalent.

(i) *Q is anisotropic and unramified.*
(ii) *Q is anisotropic and tame.*
(iii) *Q is anisotropic.*
(iv) $\mathrm{nexp}_P(\mu) = 2\,\mathrm{texp}(P)$.

Moreover, if P is a pointed Pfister quadratic space, then these conditions are also equivalent to

(v) $\overline{\beta_0} \notin \mathrm{Im}(\wp_{V_{\bar{P}}, \bar{s}_{w_0}})$.

Proof. In this section, we will not be able to give the proof in full but must restrict ourselves to showing the equivalence of (i)−(iv), relegating the rest to the next section. Since the implications (i) ⇒ (ii) ⇒ (iii) are obvious, it suffices to show (iv) ⇔ (iii) ⇒ (i).

(iv) $\Longleftrightarrow$ (iii). From 8.8 we deduce $\mathrm{nexp}_P(\mu) \geq 2\,\mathrm{texp}(P)$. Hence (iv) holds iff $\mathrm{nexp}_P(\mu) \leq 2\,\mathrm{texp}(P)$ iff Q is anisotropic by the local norm theorem 8.10 and Prop. 8.9 (a).

(iii) $\Longrightarrow$ (i). Setting $d := 2\,\mathrm{texp}(P)$, we apply (9.4.5,6) for $u = 0$, $v = w_0$ and obtain $\Xi_0 := w_0\Theta_d \in \mathfrak{o}_Q^\times$ and $t_Q(\Xi_0) = 1$. Thus Q is tame, and since P is wild, we conclude that $\bar{P}$ is a *proper* pointed quadratic subspace of $\bar{Q}$. Hence $e_{Q/F} = 1$, forcing Q to be unramified. □

Given $\beta \in \mathfrak{o}$, $d \in \mathbb{Z}$ with $0 \le d \le 2\,\mathrm{texp}(P)$, it follows from 8.8 that $\mu := 1 - \pi^d\beta$ has norm exponent at least d, and if d is odd, Prop. 8.14 (a) yields a characterization in terms of β when equality holds. While a similar characterization for $d = 2\,\mathrm{texp}(P)$ is presented in Thm. 9.9 (v) (though as yet unproved), we are now able to provide one for d even, $d < 2\,\mathrm{texp}(P)$.

9.10. **Corollary.** *Let d be an even integer such that $0 \le d < 2\,\mathrm{texp}(P)$.*

(a) *With $\beta \in \mathfrak{o}$ and $\mu := 1 - \pi^d\beta$, the following conditions are equivalent.*
  (i) *$\mu \in \mathfrak{o}^\times$ and $\mathrm{nexp}_P(\mu) = d$.*
  (ii) *$\bar{\beta} \notin V_{\bar{P}}^2$.*

(b) *If $d > 0$, $\beta_i \in \mathfrak{o}^\times$ and $\mu_i := 1 - \pi^d\beta_i \in \mathfrak{o}^\times$ for $i = 1, 2$, then*

$$\langle\langle \mu_1 \rangle\rangle \otimes P \cong \langle\langle \mu_2 \rangle\rangle \otimes P \Longrightarrow \overline{\beta_1} \equiv \overline{\beta_2} \bmod V_{\bar{P}}^2.$$

Proof. (a) (i) $\Longrightarrow$ (ii). We have $\beta \in \mathfrak{o}^\times$ by Prop. 8.9 (b). For $d = 0$ the assertion follows from Prop. 8.9 (c). Now suppose $d \ne 0$. Arguing indirectly, we assume that there exists an element $w \in \mathfrak{o}_P^\times$ with $\bar{\beta} = \bar{w}^2 = \overline{n_P(w)}$. Then $n_P(w) = \beta + \pi\beta'$ for some $\beta' \in \mathfrak{o}$, and Lemma 8.15 with $m := \frac{d}{2}$ yields

$$\mu = 1 - \pi^d\beta = (1 - \pi^{d+1}\beta'')n_P(v'')^{-1}$$

where, setting $r := \mathrm{texp}(P)$,

$$\beta'' = -\beta' + \pi^{r-m-1}\alpha - \pi^{r+m-1}\alpha\beta + \pi^{2m-1}\beta n_P(w) \in \mathfrak{o}, \quad v'' = 1_P - \pi^m w \in \mathfrak{o}_P^\times$$

since $0 < m < r$. Now roundness of P and the definition of the norm exponent imply $\mathrm{nexp}_P(\mu) \ge d + 1$, a contradiction.

(ii) $\Longrightarrow$ (i). By (9.7.1) we have $\mu \in \mathfrak{o}^\times$ and Q is anisotropic by Prop. 9.7, forcing $\mu \notin n_P(V_P^\times)$ (Prop. 7.5 (a)) and $d \le d' := \mathrm{nexp}_P(\mu) \le 2\,\mathrm{texp}(P)$ (Thm. 8.10). Furthermore, by Prop. 8.9 (b), $\mu = \mu' n_P(v')$, $\mu' = 1 - \pi^{d'}\beta'$ for some $\beta' \in \mathfrak{o}^\times$, $v' \in \mathfrak{o}_P^\times$. In particular, $\mu' \in \mathfrak{o}^\times$ and $Q \cong Q' := \langle\langle \mu' \rangle\rangle \otimes P$. This implies $e_{Q'/F} = e_{Q/F} = 1$ by (9.7.6), so $d' < 2\,\mathrm{texp}(P)$ (Thm. 9.9) is even (Prop. 9.6), and we are allowed to apply Prop. 9.7 to Q' by the implication (i) $\Rightarrow$ (ii) already established. Thus (9.7.6) yields $d = d'$.

(b) We put $Q_i := \langle\langle \mu_i \rangle\rangle \otimes P$ for $i = 1, 2$. If $V_{\bar{P}}^2$ contains $\overline{\beta_1}$ but not $\overline{\beta_2}$, then μ_1, μ_2 have different norm exponents by (a), so Q_1, Q_2 cannot be isomorphic (Props. 8.9 (d), 7.5 (b)). We may therefore assume $\overline{\beta_1}, \overline{\beta_2} \notin V_{\bar{P}}^2$. As in the proof of Cor. 8.14 we have $\mu := \mu_1\mu_2 = 1 - \pi^d\beta$, $\beta := \beta_1 + \beta_2 - \pi^d\beta_1\beta_2 \in \mathfrak{o}$. Assuming $\bar{\beta} = \overline{\beta_1} - \overline{\beta_2} \notin V_{\bar{P}}^2$ would force $\langle\langle \mu \rangle\rangle \otimes P$ to be anisotropic by Prop. 9.7, hence μ_1, μ_2 to fall into distinct norm classes relative to P. But then Q_1, Q_2 would not be isomorphic, a contradiction. □

9.11. **Examples.** (a) Let m be an integer with $0 \le m < \mathrm{texp}(P)$ and suppose we are given an element $\gamma \in \mathfrak{o}^\times$ such that $\bar{\gamma} \in V_{\bar{P}}^2$. If $\mu := 1 - \pi^{2m}\gamma$ is a unit in $\mathfrak{o}$ (automatic unless $m = 0$), Cor. 9.10 (a) implies $\mathrm{nexp}_P(\mu) > 2m$, so it is a natural question to ask whether a more precise estimate for the norm exponent of μ can be given. Unless specific properties of γ are taken into account, the answer is no. To see this, let $d \in \mathbb{Z}$ with $d > 2m$, $\beta \in \mathfrak{o}$, and $w \in \mathfrak{o}_P^\times$ with $\bar{w} \ne 1_{\bar{P}}$. Then Lemma 8.15 implies

$$1 - \pi^{2m}\gamma \equiv 1 - \pi^d\beta \bmod n_P(P^\times), \quad \bar{\gamma} = \bar{w}^2$$

for

$$\gamma = -n_P(w) + \pi^{d-2m}\beta + \pi^{\mathrm{texp}(P)-m}\alpha - \pi^{\mathrm{texp}(P)+d-m}\alpha\beta + \pi^d\beta n_P(w) \in \mathfrak{o}^\times.$$

Hence $\mathrm{nexp}_P(1 - \pi^{2m}\gamma) = \infty$ for $d > 2\,\mathrm{texp}(P)$ by the local norm theorem 8.10, and, by Cors. 8.14,9.10, β may be so chosen that $\mathrm{nexp}_P(1 - \pi^{2m}\gamma)$ attains any finite pre-assigned value d with $2m < d < 2\,\mathrm{texp}(P)$ provided $V_{\bar{P}}^2 \ne \bar{F}$.

(b) Let $d \in \mathbb{Z}$, $0 < d \le 2\,\mathrm{texp}(P)$, $\beta, \beta' \in \mathfrak{o}^\times$ and put $\mu := 1 - \pi^d\beta$, $\mu' = 1 - \pi^d\beta' \in \mathfrak{o}^\times$. We wish to refute the converse of Cors. 8.14 (b) and 9.10 (b) by showing that $\bar\beta = \bar\beta'$ *does not* imply $\mu \equiv \mu' \bmod n_P(V_P^\times)$. Indeed, if $\beta \neq \beta'$, then $\bar\beta = \bar\beta'$ amounts to the same as $\mu' = 1 - \pi^d\beta - \pi^q\gamma$ for some integer $q > d$ and some $\gamma \in \mathfrak{o}^\times$, allowing us to conclude

$$\mu\mu' = \mu^2(1 - \pi^q\mu^{-1}\gamma) \equiv 1 - \pi^q\mu^{-1}\gamma \bmod n_P(V_P^\times).$$

Therefore,

- $\mu \equiv \mu' \bmod n_P(V_P^\times)$ for $q > 2\,\mathrm{texp}(P)$ (Thm. 8.10),

but

- $\mu \not\equiv \mu' \bmod n_P(V_P^\times)$ for $d < q \le 2\,\mathrm{texp}(P)$, provided $\bar\gamma \notin V_{\bar P}^2$ if $q < 2\,\mathrm{texp}(P)$ is even (Cors. 8.14,9.10), and $\bar w_0^2\bar\gamma \notin \mathrm{Im}(\wp_{V_{\bar P};\bar s_{w_0}})$ if $d = 2\,\mathrm{texp}(P)$ (Thm. 9.9). ☐

The preceding results on the behavior of the ramification index, the pointed quadratic residue space and the trace exponent under the passage from P to $\langle\langle\mu\rangle\rangle \otimes P$, $\mu \in F^\times$, can be stated in a particularly concise way when addressed to pointed Pfister quadratic spaces by combining them with the embedding property of Prop. 7.6. In order to do so, we introduce the following terminology.

9.12. **Scalars of standard type.** We say a scalar $\mu \in F$ has *standard type* relative (or with respect) to P if it satisfies one of the following mutually exclusive conditions.

(a) μ is a prime element of $\mathfrak{o}$ (possibly distinct from π).
(b) $\mu = 1 - \pi^d\beta$ for some *odd* integer d with $0 \le d < 2\,\mathrm{texp}(B)$ and some $\beta \in \mathfrak{o}^\times$.
(c) $\mu = 1 - \pi^d\beta$ for some *even* integer d with $0 \le d < 2\ \mathrm{texp}(B)$ and some $\beta \in \mathfrak{o}$ with $\bar\beta \notin V_{\bar P}^2$.

9.13. **Theorem.** *Let P be a pointed n-Pfister quadratic space over F that is anisotropic and wild of ramification index $e_{P/F} = 1$. For Q to be a wild anisotropic pointed $(n+1)$-Pfister quadratic space over F into which P embeds as a pointed quadratic subspace it is necessary and sufficient that Q be a pointed quadratic space isomorphic to $\langle\langle\mu\rangle\rangle \otimes P$, for some scalar $\mu \in F$ of standard type relative to P. In this case, precisely one of the following implications holds.*

(a) *If μ is a prime element in $\mathfrak{o}$, then*

$$e_{Q/F} = 2, \quad \bar Q \cong \bar P, \quad \mathrm{texp}(Q) = \mathrm{texp}(P).$$

(b) *If $\mu = 1 - \pi^d\beta$ for some* odd *integer d with $0 \le d < 2\,\mathrm{texp}(P)$ and some $\beta \in \mathfrak{o}^\times$, then*

$$e_{Q/F} = 2, \quad \bar Q \cong \bar P, \quad \mathrm{texp}(Q) = \mathrm{texp}(P) - \frac{d-1}{2}.$$

(c) *If $\mu = 1 - \pi^d\beta$ for some* even *integer d, $0 \le d < 2\,\mathrm{texp}(P)$ and some $\beta \in \mathfrak{o}$ with $\bar\beta \notin V_{\bar P}^2$, then*

$$e_{Q/F} = 1, \quad \bar Q \cong \langle\langle\bar\beta\rangle\rangle \otimes \bar P, \quad \mathrm{texp}(Q) = \mathrm{texp}(P) - \frac{d}{2}.$$

Proof. By Props. 9.2,9.6,9.7, the condition is sufficient, and (a)−(c) hold. Conversely, suppose Q is a pointed anisotropic wild $(n+1)$-Pfister quadratic space over F containing P as a pointed quadratic subspace. Up to isomorphism, $Q = \langle\langle\mu\rangle\rangle \otimes P$ for some $\mu \in F^\times$ by the embedding property (Prop. 7.6), where the reduction of 9.1 (b) allows us to assume that $\mu \in \mathfrak{o}^\times$ is a unit in $\mathfrak{o}$. Q being anisotropic implies $0 \le d := \mathrm{nexp}_P(\mu) \le 2\,\mathrm{texp}(P)$ by the local norm theorem 8.10 and without loss $\mu = 1 - \pi^d\beta$ for some $\beta \in \mathfrak{o}$. Since Q is wild, we conclude $d < 2\,\mathrm{texp}(P)$ from the part of Thm. 9.9 already established, and if d is odd, then $\beta \in \mathfrak{o}^\times$ (Prop. 8.14 (a)), while if d is even, then $\bar\beta \notin V_{\bar P}^2$ (Cor. 9.10). In any event, μ has standard type relative to P. ☐

9.14. **Corollary.** *With P as in Thm.* 9.13, *let Q be a pointed quadratic space over F.*

(a) *The following conditions are equivalent.*

(i) *Q is an anisotropic pointed $(n+1)$-Pfister quadratic space into which P embeds as a pointed quadratic subspace such that $e_{Q/F} = 2$ and* $\mathrm{texp}(Q) = \mathrm{texp}(P)$.

(ii) *$Q \cong \langle\langle \pi\beta \rangle\rangle \otimes P$ or $Q \cong \langle\langle 1 - \pi\beta \rangle\rangle \otimes P$ for some $\beta \in \mathfrak{o}^\times$.*

In this case $\bar{Q} \cong \bar{P}$.

(b) *The following conditions are equivalent.*

(i) *Q is an anisotropic pointed $(n+1)$-Pfister quadratic space into which P embeds as a pointed quadratic subspace such such that $e_{Q/F} = 1$ and* $\mathrm{texp}(Q) = \mathrm{texp}(P)$.

(ii) *$Q \cong \langle\langle \mu \rangle\rangle \otimes P$ for some $\mu \in \mathfrak{o}$ with $\bar{\mu} \notin V_{\bar{P}}^2$.*

In this case, $\bar{Q} \cong \langle\langle \bar{\mu} \rangle\rangle \otimes P$.

Proof. In (a) and (b), condition (i) implies that Q is wild (Prop. 8.2 (a)). Hence the assertions follow immediately from Thm. 9.13. □

There is an analogue of Thm. 9.13 dealing with tame rather than wild enlargements of pointed n-Pfister quadratic spaces. We omit the proof since it proceeds along the same lines as the one of Thm. 9.13, applying Thm. 9.9 in full rather than Props. 9.2,9.6,9.7.

9.15. **Theorem.** *Keeping the notations of* 9.8 (b), *let P be a pointed n-Pfister quadratic space over F that is anisotropic and wild of ramification index $e_{P/F} = 1$. For Q to be a tame and anisotropic $(n+1)$-Pfister pointed quadratic space over F into which P embeds as a pointed quadratic subspace it is necessary and sufficient that Q be a pointed quadratic F-space and there exist an element $\beta \in \mathfrak{o}$ with*

$$Q \cong \langle\langle \mu \rangle\rangle \otimes P, \quad \mu := 1 - \pi^{2\,\mathrm{texp}(B)}\beta, \quad \overline{\beta_0} \notin \mathrm{Im}(\wp_{V_{\bar{P}}, \bar{s}_{w_0}}), \quad \beta_0 := n_P(w_0)\beta.$$

□

10. λ-normed and λ-valued conic algebras.

In order to illuminate the intuitive background of the present section, we recall the notion of an absolute-valued algebra. Following Albert [1] (see also Palacios [36]), an absolute-valued algebra is a non-associative real algebra A equipped with a norm $x \mapsto \|x\|$ that permits composition: $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in A$. Since absolute valued algebras obviously have no zero divisors, the finite-dimensional ones are division algebras, hence exist only in dimensions $1, 2, 4, 8$ (Albert [1] gave an ad-hoc proof of this result, the Bott-Kervaire-Milnor thoerem not having been known at the time). By contrast, natural analogues of absolute-valued algebras over 2-Henselian fields will be discussed in the present section that exist in all dimensions 2^n, $n = 0, 1, 2 \ldots$.

Throughout we continue to work over a fixed 2-Henselian field F as in 7.7 and alert the reader to the terminological conventions of 7.14. All vector spaces, algebras, etc. over F are tacitly assumed to be finite-dimensional.

10.1. **The basic concepts.** A conic algebra C over F is said to be λ-*normed* if the following conditions hold.

(i) C is non-singular, round and anisotropic.

(ii) λ_C is sub-multiplicative: $\lambda_C(xy) \geq \lambda_C(x) + \lambda_C(y)$ for all $x, y \in C$.

We speak of a λ-*valued* conic algebra if C satisfies (i), and if instead of (ii) the following stronger condition holds:

(iii) λ_C is multiplicative: $\lambda_C(xy) = \lambda_C(x) + \lambda_C(y)$ for all $x, y \in C$.

The norm of a λ-valued conic algebra C over F will typically not permit composition (for example, if the dimension of C differs from $1, 2, 4, 8$) but, remarkably, its failure to do so is not detected by λ since λ_C being multiplicative by (7.14.1) amounts to $\lambda(n_C(xy)) = \lambda(n_C(x)n_C(y))$ for all $x, y \in C$. This looks like a pretty far-fetched phenomenon but, in fact, turns out to be quite common.

We start with a trivial but useful observation.

10.2. **Proposition.** (a) *λ-valued conic algebras over F are division algebras.*
(b) *A composition algebra over F is a λ-valued conic algebra if and only if it is a division algebra.* □

10.3. **Proposition.** *Let C be a non-singular, round and anisotropic conic algebra over F. Then C is λ-normed if and only if $\mathfrak{o}_C \subseteq C$ is an $\mathfrak{o}$-subalgebra and $\mathfrak{p}_C \subseteq \mathfrak{o}_C$ is an ideal with $\mathfrak{p}_C^2 \subseteq \mathfrak{p}\mathfrak{o}_C$. In this case, $\bar{C} := \mathfrak{o}_C/\mathfrak{p}_C$ is a conic algebra over $\bar{F}$ whose norm, trace and conjugation are given by the formulas*

$$n_{\bar{C}}(\bar{x}) = \overline{n_C(x)}, \tag{1}$$

$$t_{\bar{C}}(\bar{x}) = \overline{t_C(x)}, \tag{2}$$

$$(\bar{x})^* = \overline{(x^*)} \tag{3}$$

for all $x \in \mathfrak{o}_C$. Moreover, the norm of $\bar{C}$ is round and anisotropic.

Proof. By Lemma 7.12, sub-multiplicativity of λ_C amounts to

$$\lambda_C(xy) \geq \lambda_C(x) + \lambda_C(y) \qquad (x, y \in C, \quad 0 \leq \lambda_C(x), \lambda_C(y) \leq \frac{1}{2}).$$

The first part of the proposition follows from this at once. The second part is a restatement of Prop. 7.10. □

Remark. The conic algebra $\bar{C} = \mathfrak{o}_C/\mathfrak{p}_C$ is called the *residue algebra* of C. If C is wild, we do not know whether this residue algebra always agrees with the purely inseparable extension field of $\bar{F}$ attached to C via 9.5, though it does if C is a composition algebra [40, Prop. 1].

10.4. **Corollary.** *With the notations of Prop.* 10.3 *suppose in addition that C has ramification index $e_{C/F} = 1$. Then*:

(a) *C is λ-normed if and only if $\mathfrak{o}_C \subseteq C$ is an $\mathfrak{o}$-subalgebra.*

(b) *C is λ-valued if and only if C is λ-normed and $\bar{C}$ is a division algebra.*

Proof. (a) $\mathfrak{p}_C = \mathfrak{p}\mathfrak{o}_C$.

(b) Consulting Lemma 7.12 again, λ_C is multiplicative iff $\lambda_C(xy) = 0$ for all $x, y \in \mathfrak{o}_C^\times$ iff C is λ-normed and $\bar{C}$ is a division algebra. □

We now proceed to re-examine the main results of the preceding section within the framework of λ-normed and λ-valued conic algebras.

10.5. **Convention.** For the remainder of this section, we fix a λ-normed conic algebra B over F having ramification index $e_{B/F} = 1$.

10.6. **Proposition.** *If μ is a prime element in $\mathfrak{o}$, then $C := \mathrm{Cay}(B, \mu)$ is a λ-normed conic algebra over F with $\bar{C} = \bar{B}$ as conic $\bar{F}$-algebras. Moreover, C is λ-valued if and only if B is λ-valued.*

Proof. By Prop. 9.2, C is non-singular, round and anisotropic. Combining Prop. 10.3 with (1.10.1), we conclude that C is λ-normed. It remains to show that if B is λ-valued, so is C. By Lemma 7.12, we must show $\lambda_C(x_1x_2) = \lambda_C(x_1) + \lambda_C(x_2)$ for all $x_i = u_i + v_ij \in C$, $u_i, v_i \in B$, $0 \leq \lambda_C(x_i) \leq \frac{1}{2}$, $i = 1, 2$. There are four cases: (i) $\lambda_C(x_1) = \lambda_C(x_2) = 0$, (ii) $\lambda_C(x_1) = 0$, $\lambda_C(x_2) = \frac{1}{2}$, (iii) $\lambda_C(x_1) = \frac{1}{2}$, $\lambda_C(x_2) = 0$, (iv) $\lambda_C(x_1) = \lambda_C(x_2) = \frac{1}{2}$. We only treat (iv) and leave the other three cases to the reader. From (9.2.1) we deduce $u_i \in \mathfrak{p}_B$, $v_i \in \mathfrak{o}_B^\times$ for $i = 1, 2$ and (1.10.1) yields

$$x_1x_2 = u + vj, \quad u = u_1u_2 + \mu v_2^*v_1, \quad v = v_2u_1 + v_1u_2^*,$$

where $\lambda_B(u) = 1$, $\lambda_B(v) \geq 1$, hence $\lambda_C(x_1x_2) = 1 = \lambda_C(x_1) + \lambda_C(x_2)$ □

10.7. **Example.** Specializing Prop. 10.6 to (iterated) Laurent series fields of characteristic not 2, we recover examples of λ-valued conic algebras that originally go back to to Brown [7, pp. 421-422]. In a slightly more general vein, let k be any field, L/k a separable quadratic field extension and write

$$A = \mathrm{Cay}(L; \mu_1, \dots, \mu_{n-1}) \qquad\qquad (n \in \mathbb{Z},\ n \geq 1)$$

for the k-algebra arising from L and scalars $\mu_1, \dots, \mu_{n-1} \in k^\times$ by means of the Cayley-Dickson process as in 1.12. Then A is a flexible conic algebra with norm an n-Pfister quadratic form. We now assume that A is a division algebra, forcing A to be non-singular, round and anisotropic. Consider the field $F = k((\mathbf{t}))$ of formal Laurent series in a variable $\mathbf{t}$ with coefficients in k, which is complete and therefore Henselian under the standard discrete valuation $\lambda\colon F \to \mathbb{Z}_\infty$. Setting

$$B := A \otimes_k F = A((\mathbf{t})),$$

we obtain a flexible conic division F-algebra whose norm is an anisotropic n-Pfister quadratic form over F. Using (7.9.1), a straightforward verification shows

$$\lambda_B\Big(\sum_{r \gg -\infty}^{\infty} a_r \mathbf{t}^r\Big) = \min\{r \in \mathbb{Z} \mid a_r \neq 0\} \qquad\qquad (a_r \in A,\ r \in \mathbb{Z}),$$

which immediately implies that B is an unramified λ-valued conic algebra over F. By Prop. 10.6 we thus find in $C := \mathrm{Cay}(B, \mathbf{t})$ a λ-valued conic algebra over F having dimension 2^{n+1}. Starting from $A = L$ (i.e., from $n = 1$) and continuing in this way, we obtain λ-valued conic algebras over appropriate iterated Laurent series fields in all dimensions 2^n, $n = 0, 1, 2, \dots$.

10.8. **Example.** We specify Example 10.7 a bit further by setting $k = \mathbb{R}$, $L = \mathbb{C}$, $n = 3$, $\mu_1 = \mu_2 = -1$. Then $A = \mathbb{O}$, the real algebra of Graves-Cayley octonions, and $B = \mathbb{O}((\mathbf{t}))$ is the unique unramified octonion division algebra over $F = \mathbb{R}((\mathbf{t}))$. Moreover, the 16-dimensional conic division algebra

$$C = \mathrm{Cay}(B, \mathbf{t}) = \mathrm{Cay}(F; -1, -1, -1, \mathbf{t})$$

over F contains $B' := \mathrm{Cay}(F; -1, -1, \mathbf{t})$ as a ramified octonion subalgebra. In particular, B and B' are not isomorphic, allowing us to conclude from [7, Thm. 2] that the subalgebra $B' \subseteq C$ does *not* satisfy the embedding property 1.11.

10.9. **Proposition.** *If B as in* 10.5 *is tame and $\mu \in \mathfrak{o}^\times \setminus n_B(B^\times)$, then $C := \mathrm{Cay}(B, \mu)$ is a λ-normed conic algebra over F with $\bar{C} = \mathrm{Cay}(\bar{B}, \bar{\mu})$ as conic $\bar{F}$-algebras. Moreover, C is λ-valued if and only if $\bar{C}$ is a division algebra.*

Proof. By Prop. 9.3, C is a tame non-singular, round and anisotropic conic algebra over F with $e_{C/F} = 1$. Moreover, $\mathfrak{o}_B$ being a $\mathfrak{o}$-subalgebra of B by Prop. 10.3, we conclude from (9.3.2) that $\mathfrak{o}_C$ is an $\mathfrak{o}$-subalgebra of C. Now everything follows from Cor. 10.4. □

Dealing with the case that B as in 10.5 is wild turns out to be more troublesome. We not only need a few preparations but also have to add an extra hypothesis by requiring that the conic algebras involved be flexible.

10.10. **Proposition.** *Let C be a flexible λ-normed conic algebra over F. Then*

(1) $$\lambda\big(t_C(x)\big) \geq\ \omega(C) + \lambda_C(x),$$

(2) $$\lambda_C([x_1, x_2]) \geq\ \omega(C) + \lambda_C(x_1) + \lambda_C(x_2),$$

(3) $$\lambda_C(x - x^*) \geq\ \omega(C) + \lambda_C(x)$$

for all $x, x_1, x_2 \in C^\times$.

Proof. (1) follows immediately from (8.4.1). To establish (2), we combine (1) with (2.4.2),(7.9.5),(8.7.2) and use the fact that λ_C is sub-multiplicative. Finally, applying (1) and (2.1.6), we obtain

$$\lambda_C(x - x^*) = \lambda_C\big(2x - t_C(x)1_C\big) \geq \min\{e_F + \lambda_C(x), \omega(C) + \lambda_C(x)\},$$

and (3) follows from (8.7.2). □

10.11. **Lemma.** *Suppose B as in* 10.5 *is wild and $\mu \in \mathfrak{o}^\times$ has the form*

(1) $$\mu = 1 - \pi^d\beta, \quad d \in \mathbb{Z}, \quad 0 \leq d \leq 2\,\mathrm{texp}(B), \quad \beta \in \mathfrak{o}^\times.$$

Setting $C = \mathrm{Cay}(B, \mu)$ and

(2) $$0 \leq m := \left\lfloor \frac{d}{2} \right\rfloor < \mathrm{texp}(B), \quad \Theta_d := \pi^{-m}(1_B + j) \in C$$

as in (9.4.1), *the relations*

(3) $$u(v\Theta_d) = \pi^{-m}[u, v] + (vu)\Theta_d,$$

(4) $$(v\Theta_d)u = \pi^{-m}v(u - u^*) + (vu^*)\Theta_d,$$

(5) $$(v_1\Theta_d)(v_2\Theta_d) = \Big(\pi^{-2m}[v_1, v_2 - v_2^*] - \pi^{d-2m}\beta v_2^*v_1\Big) + \pi^{-m}\Big(t_B(v_2)v_1 - [v_1, v_2]\Big)\Theta_d$$

hold for all $u, v, v_1, v_2 \in B$. Moreover, if B is flexible, then $\mathfrak{O} := \mathfrak{o}_B \oplus \mathfrak{o}_B\Theta_d$ is an $\mathfrak{o}$-subalgebra of C.

Proof. A slightly involved but straightforward computation using the transition formulas

$$u + v\Theta_d = (u + \pi^{-m}v) + \pi^{-m}vj, \quad u + vj = (u - v) + \pi^m v\Theta_d$$

implies (3)−(5). Combining these with Prop. 10.10 leads to the final assertion of the lemma. □

10.12. **Theorem.** *Suppose B as in* 10.5 *is flexible and wild,*

$$\mu = 1 - \pi^d\beta, \quad d \in \mathbb{Z}, \quad 0 \leq d < 2\,\mathrm{texp}(B), \quad \beta \in \mathfrak{o}^\times,$$

and d is odd. *Then $C := \mathrm{Cay}(B, \mu)$ is a flexible λ-normed conic algebra over F with $\bar{C} = \bar{B}$ as conic $\bar{F}$-algebras. Moreover, C is λ-valued if and only if B is λ-valued.*

Proof. The proof is similar to, but a bit more complicated than, the one of Prop. 10.6. By Prop. 9.6, C is non-singular, round and anisotropic. Combining the final statement of Lemma 10.11 with (9.6.4), we conclude that $\mathfrak{o}_C \subseteq C$ is an $\mathfrak{o}$-subalgebra which, thanks to (10.11.3−5) and to d being odd contains $\mathfrak{p}_C$ as an ideal with $\mathfrak{p}_C^2 \subseteq \mathfrak{p}\mathfrak{o}_C$. Thus C is λ-normed (Prop. 10.3), and it remains to show that if B is λ-valued, so is C. Let $x_i = u_i + v_i\Pi \in C^\times$, $u_i, v_i \in B$, $i = 1, 2$ and $x_1x_2 = u + v\Pi$, where (10.11.3−5) imply

(1) $$u = u_1u_2 + \pi^{-m}\big([u_1, v_2] + v_1(u_2 - u_2^*)\big) + \pi^{-2m}[v_1, v_2 - v_2^*] - \pi\beta v_2^*v_1,$$

(2) $$v = v_2u_1 + v_1u_2^* + \pi^{-m}\big(t_B(v_2)v_1 - [v_1, v_2]\big).$$

We must show $\lambda_C(x_1x_2) = \lambda_C(x_1) + \lambda_C(x_2)$. To this end, invoking Lemma 7.12, we may assume $0 \leq \lambda_C(x_i) \leq \frac{1}{2}$, $i = 1, 2$. Since conjugation is an algebra involution of C leaving λ_C invariant, there are three cases: (i) $\lambda_C(x_1) = \lambda_C(x_2) = 0$, (ii) $\lambda_C(x_1) = 0$, $\lambda_C(x_2) = \frac{1}{2}$, (iii) $\lambda_C(x_1) = \lambda_C(x_2) = \frac{1}{2}$. Among these cases, we treat only (iii) since the other ones can be treated analogously. In (iii) we have $u_1, u_2 \in \mathfrak{p}_B$, $v_1, v_2 \in \mathfrak{o}_B^\times$, observe Prop. 10.10 and obtain $u \equiv -\pi\beta v_2^*v_1 \bmod \mathfrak{p}_B^2$ by (1), hence $\lambda_B(u) = 1$, while (2) yields $\lambda_B(v) \geq 1$. Therefore $\lambda_C(x_1x_2) = 1 = \lambda_C(x_1) + \lambda_C(x_2)$. □

10.13. **Theorem.** *Suppose B as in* 10.5 *is flexible and wild,*

$$(1) \qquad \mu = 1 - \pi^d\beta, \quad d \in \mathbb{Z}, \quad 0 \le d < 2\,\mathrm{texp}(B), \quad \beta \in \mathfrak{o}, \quad \bar{\beta} \notin V_{\bar{B}}^2$$

and d is even. *Then $C := \mathrm{Cay}(B, \mu)$ is a wild λ-normed conic algebra over F with $\bar{C} \cong \mathrm{Cay}(\bar{B}, \bar{\beta})$ as conic $\bar{F}$-algebras. Moreover, C is λ-valued if and only if $\bar{C}$ is a division algebra.*

Proof. By Prop. 9.7, C is a non-singular, round and anisotropic conic F-algebra. Moreover, C is wild of ramification index 1. The final statement of Lemma 10.11 combined with (9.7.5) shows that $\mathfrak{o}_C \subseteq C$ is an $\mathfrak{o}$-subalgebra, forcing C to be λ-normed (Cor. 10.4 (a)). Moreover, writing $\mathrm{Cay}(\bar{B}, \bar{\beta}) = \bar{B} \oplus \bar{B}j'$, $j'^2 = \bar{\beta}1_{\bar{B}}$ as in 1.10, consulting (10.11.3−5) and observing $d < \mathrm{texp}(B) = \omega(B)$, Prop. 10.10 shows that (in the notations of Prop. 9.7) the assignment $u + v\Xi \mapsto \bar{u} + \bar{v}j'$ determines an isomorphism $\bar{C} \xrightarrow{\sim} \mathrm{Cay}(\bar{B}, \bar{\beta})$ of $\bar{F}$-algebras. The final statement of the theorem follows immediately from Cor. 10.4 (b). □

10.14. **Corollary.** *With the notations and assumptions of Thm.* 10.13, *suppose in addition that $\bar{B}/\bar{F}$ is a purely inseparable field extension of exponent at most* 1. *Then C is a λ-valued conic algebra over F with $\bar{C} \cong \bar{B}(\sqrt{\bar{\beta}})$.*

Proof. It suffices to note that the last condition of (10.13.1) makes $\mathrm{Cay}(\bar{B}, \bar{\beta}) = \bar{B}(\sqrt{\bar{\beta}})$ (cf. 1.12, Case 2) a division algebra. □

Remark. The additional hypothesis in Cor. 10.14 is fulfilled if, e.g., B is a composition algebra (Remark to 10.3).

10.15. **Examples.** In Brown's examples of conic division algebras (cf. 10.7), one basically keeps building up ramified λ-valued conic algebras over iterated Laurent series fields of characteristic not 2. By contrast, we will now be able to construct wild λ-valued conic algebras of ramification index 1 over appropriate Henselian fields of characteristic zero. Let k be any field of characteristic 2 and write K for the field of rational functions in an infinite number of variables over k. Then $[K : K^2] = \infty$. Pick an infinite chain

$$K = K_0 \subset K_1 \subset \cdots \subset K_{n-1} \subset K_n \subset \cdots$$

of purely inseparable field extensions of K having exponent at most 1 and $[K_n : K] = 2^n$ for all integers $n \ge 0$. Following Teichmüller [48], there is an essentially unique complete field F under a discrete valuation $\lambda\colon F \to \mathbb{Z}_\infty$ such that F has characteristic zero, residue field $\bar{F} = K$ and absolute ramification index $e_F = 1$. For $n \ge 1$ choose $\beta_n \in \mathfrak{o}^\times$ such that $K_n = K_{n-1}(\sqrt{\bar{\beta}_n})$, put $\mu_n = 1 - \beta_n$, observe (8.7.1) and apply Cor. 10.14 successively for $d = 0$ and $n = 1, 2, 3, \ldots$ to conclude that the Cayley-Dickson process leads to a wild λ-valued conic algebra

$$C_n := \mathrm{Cay}(F; \mu_1, \ldots, \mu_n)$$

over F having dimension 2^n and ramification index 1 such that $\bar{C}_n \cong K_n$.

10.16. **Corollary.** *Let Q be a pointed Pfister quadratic space over F that is anisotropic and wild of ramification index $e_{Q/F} = 1$. Then there exists a flexible λ-valued conic algebra C over F such that $Q_C \cong Q$ and $\bar{C}/\bar{F}$ is a purely inseparable field extension of exponent at most* 1.

Proof. Arguing by induction, we let Q be a pointed $(n+1)$-Pfister quadratic space and pick a pointed n-Pfister quadratic subspace $P \subseteq Q$. Clearly, P is wild with $e_{P/F} = 1$. By Theorem 9.13, some scalar μ of standard type relative to P satisfies $Q = \langle\langle \mu \rangle\rangle \otimes P$ up to isomorphism, and since the implications (a),(b) of that theorem do not hold for μ, implication (c) does. On the other hand, the induction hypothesis leads to a flexible λ-valued conic algebra B over F with $Q_B \cong P$ such that $\bar{B}/\bar{F}$ is a purely inseparable field extension of exponent at most 1. By Cor. 10.14, $C := \mathrm{Cay}(B, \mu)$ is a flexible λ-valued conic algebra over F with $Q_C = \langle\langle \mu \rangle\rangle \otimes P = Q$ and $\bar{C} \cong \bar{B}(\sqrt{\bar{\beta}})$. □

We still haven't closed the gap in our proof of Thm. 9.9 but will now be able to do so by appealing to the connection with conic algebras. In view of Cors. 10.14,10.16, the missing equivalence of (v) and (i)−(iv) in Thm. 9.9 will be a consequence of the following result.

10.17. **Theorem.** *Suppose B as in 10.5 is flexible, wild and λ-normed having $\bar{B}/\bar{F}$ as a purely inseparable field extension of exponent at most 1 and, with the notations of 9.8 (b), let*

(1) $$\beta \in \mathfrak{o}, \quad \mu := 1 - \pi^{2\,\mathrm{texp}(B)}\beta \in \mathfrak{o}^\times, \quad \beta_0 := n_B(w_0)\beta.$$

Then $C := \mathrm{Cay}(B, \mu)$ is anisotropic if and only if $\overline{\beta_0} \notin \mathrm{Im}(\wp_{\bar{B},\bar{s}_{w_0}})$. In this case, setting

(2) $$\Xi_0 := \pi^{-\,\mathrm{texp}(B)}(w_0 + w_0 j) \in C,$$

we obtain the relations

(3) $$t_C(\Xi_0) = 1, \quad n_C(\Xi_0) = \beta_0, \quad C = B \oplus B\Xi_0,$$

(4) $$\lambda_C(u + v\Xi_0) = \min\{\lambda_B(u), \lambda_B(v)\} \qquad (u, v \in B),$$

(5) $$\mathfrak{o}_C = \mathfrak{o}_B \oplus \mathfrak{o}_B\Xi_0, \quad \mathfrak{p}_C = \mathfrak{p}_B \oplus \mathfrak{p}_B\Xi_0$$

and C is λ-normed with $Q_{\bar{C}} \cong Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}}$.

Proof. As in the proof of the implication (iii) ⇒ (i) in Thm. 9.9, we put $d := 2\,\mathrm{texp}(B)$, $\Xi := \Theta_d$ and have $\Xi_0 = w_0\Xi$. Thus the first two relations of (3) follow from (9.4.5,6), while the last one is a straightforward consequence of the fact that B is λ-valued by Cor. 10.4 (b), hence a division algebra (Prop. 10.2 (a)). Setting $m := \mathrm{texp}(B)$, we let $v \in B$ and compute, using (1.10.1,2),(2.4.1) and (1),

$$\begin{aligned} n_C(v\Xi_0) &= \pi^{-2m} n_C\big(v(w_0 + w_0 j)\big) = \pi^{-2m} n_C\big(vw_0 + (w_0 v)j\big) \\ &= \pi^{-2m}\big(n_B(vw_0) - \mu n_B(w_0 v)\big) = \pi^{-2m} n_B(vw_0)(1-\mu) = n_B(vw_0)\beta. \end{aligned}$$

But since $\bar{B}/\bar{F}$ is a purely inseparable field extension of exponent at most 1, we have $n_B(vw_0) \equiv n_B(v)n_B(w_0) \bmod \mathfrak{p}$ and conclude

(6) $$n_C(v\Xi_0) \equiv n_B(v)\beta_0 \bmod \mathfrak{p}.$$

Suppose first that C is anisotropic. Then $\bar{C}$ is an anisotropic conic algebra over $\bar{F}$ containing $\bar{B}$ as a subalgebra and $l := \overline{\Xi_0} \in \bar{C}$ as a distinguished element with $t_{\bar{C}}(l) = 1$, $n_{\bar{C}}(l) = \overline{\beta_0}$, $n_{\bar{C}}(\bar{u}, l) = \bar{s}_{w_0}(\bar{u})$ for all $u \in \mathfrak{o}_B$. Moreover, (6) implies $n_{\bar{C}}(\bar{v}l) = \overline{\beta_0}\bar{v}^2$ for all $v \in \mathfrak{o}_B$. Writing $\bar{B} \oplus \bar{B}j'$ for the vector space underlying the pointed quadratic space $Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}}$, the assignment $\bar{u} + \bar{v}j' \mapsto \bar{u} + \bar{v}l$ therefore and by (2.2.2) gives an embedding φ from $Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}}$ to $Q_{\bar{C}}$ of pointed quadratic spaces. Comparing dimensions (observe $e_{C/F} = 1$ by (i) of Thm. 9.9), $\varphi\colon Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}} \xrightarrow{\sim} Q_{\bar{C}}$ is, in fact, an isomorphism, and since $Q_{\bar{C}}$ is anisotropic, so is $Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}}$. Now $\overline{\beta_0} \notin \mathrm{Im}(\wp_{\bar{B},\bar{s}_{w_0}})$ follows from Cor. 3.10 (a). Moreover, we claim that (4) holds (which immediately implies (5)). As usual, we may assume $u, v \in \mathfrak{o}_B^\times$, which yields $\lambda_C(u + v\Xi_0) \geq 0$, and if this were strictly positive, we would end up with $\varphi(\bar{u} + \bar{v}j') = \bar{u} + \bar{v}l = 0$, forcing $\bar{u} = \bar{v} = 0$, a contradiction. Suppose next $\overline{\beta_0} \notin \mathrm{Im}(\wp_{\bar{B},\bar{s}_{w_0}})$ and consider the full $\mathfrak{o}$-lattice

(7) $$\mathfrak{O} := \mathfrak{o}_B \oplus \mathfrak{o}_B\Xi_0 \subseteq C,$$

on which n_C takes integral values. More precisely, (6) and (2.2.2) imply

(8) $$n_C(u + v\Xi_0) \equiv n_B(u) + \pi^{-m} n_B(v^*u, \Xi_0) + n_B(v)\beta_0 \bmod \mathfrak{p}$$

for all $u, v \in \mathfrak{o}_B$. Reducing mod $\mathfrak{p}$, we obtain a pointed quadratic space

$$\bar{\mathfrak{O}} := \mathfrak{O} \otimes_{\mathfrak{o}} \bar{F} = \bar{B} \oplus \bar{B}l', \quad l' := \Xi_0 \otimes_{\mathfrak{o}} 1_{\bar{F}},$$

over $\bar{F}$, and the assignment $\bar{u} \oplus \bar{v}j' \mapsto \bar{u} + \bar{v}l'$ by (7),(8) gives an isomorphism from $Q_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}}$ onto $\bar{\mathfrak{O}}$. The former being anisotropic by Cor. 3.10 (a), so is the latter. But then C must be anisotropic as well since every non-zero element $x \in C$ satisfies $\pi^m x \in \mathfrak{O} \setminus \mathfrak{p}\mathfrak{O}$ for some integer m.

It remains to show that C is λ-normed provided it is anisotropic. By Cor. 10.4 (a), it suffices to show that $\mathfrak{o}_C \subseteq C$ is an $\mathfrak{o}$-subalgebra. In order to do so, we use Lemma 10.11 to derive the following formulas by a straightforward computation, for all $u, v, v_1, v_2 \in B$.

$$v\Xi_0 = \pi^{-m}[v, w_0] + (w_0 v)\Xi, \tag{9}$$

$$v\Xi = \pi^{-m}[w_0, L_{w_0}^{-1}v] + (L_{w_0}^{-1}v)\Xi_0, \tag{10}$$

$$u(v\Xi_0) = \pi^{-m}\big(u[v, w_0] + [u, w_0 v]\big) + \big((w_0 v)u\big)\Xi, \tag{11}$$

$$(v\Xi_0)u = \pi^{-m}\big([v, w_0]u + (w_0 v)(u - u^*)\big) + \big((w_0 v)u^*\big)\Xi \tag{12}$$

$$\begin{aligned}(v_1\Xi_0)(v_2\Xi_0) = \pi^{-2m}\Big(&[v_1, w_0][v_2, w_0] + [[v_1, w_0], w_0 v_2] + \\ &[w_0 v_1, w_0 v_2 - (w_0 v_2)^*] + (w_0 v_1)\big([v_2, w_0] - [v_2, w_0]^*\big)\Big) - \\ &\beta(w_0 v_2^*)(w_0 v_1) + \pi^{-m}\Big((w_0 v_2)[v_1, w_0] + (w_0 v_1)[v_2, w_0]^* + \\ &t_B(w_0 v_2)w_0 v_1 - [w_0 v_1, w_0 v_2]\Big)\Xi.\end{aligned} \tag{13}$$

Since B is λ-valued, (10) implies $\mathfrak{o}_B\Xi \subseteq \mathfrak{o}_C$, and then (11)$-$(13) combine with Prop. 10.10 to establish $\mathfrak{o}_C$ as an $\mathfrak{o}$-subalgebra of C. □

10.18. **Corollary.** *Suppose in Thm.* 10.17 *that B is an associative composition division algebra and $\overline{\beta_0} \notin \mathrm{Im}(\wp_{\bar{B};\bar{\beta}_0,\bar{s}_{w_0}})$. Then C is an unramified composition division algebra over F with $\bar{C} \cong \mathrm{Cay}(\bar{B}; \overline{\beta_0}, \bar{s}_{w_0})$ as a non-orthogonal Cayley-Dickson construction.*

Proof. Composition algebras are classified by their norms. □

Remark. If C as in Thm. 10.17 is anisotropic, its pointed quadratic residue space is described explicitly by the theorem. But C is also a λ-normed conic algebra, making $\bar{C}$ canonically a conic algebra (over $\bar{F}$) in its own right. It would be interesting to obtain an equally explicit description of that algebra. Cor. 10.18 provides one if C is a composition algebra but it is not at all clear whether this description prevails in the general case, nor whether C is always a λ-*valued* conic algebra.

11. Applications to composition algebras.

There are obvious and less obvious applications of the preceding results to composition algebras. Working over a fixed 2-Henselian field F of arbitrary characteristic as before (cf. 7.7), the obvious ones may be described as follows.

11.1. **Translations.** Since composition division algebras over F are classified by their norms and are λ-valued conic F-algebras by Prop. 10.2 (b), the results of Sections 8, 9 translate immediately into this more special setting, where the ones in Section 9 in particular yield explicit descriptions of how the valuation data ramification index, residue algebra and trace exponent behave under the Cayley-Dickson construction. Rather than carrying out these translations in full detail, suffice it to point out that all one has to do is replace

- the pointed quadratic space P of 8.14,9.1,9.4 by an associative composition division algebra B over F having ramification index $e_{B/F} = 1$,
- the pointed quadratic space Q by a composition algebra C, and the condition of Q being anisotropic by the one of C being a division algebra,
- the passage from P to $\langle\langle\mu\rangle\rangle \otimes P$, $\mu \in F^\times$, by the Cayley-Dickson construction $\mathrm{Cay}(B, \mu)$.

The less obvious applications of our results to composition algebras are all related, in one way or another, to the following innocuous observation, which we have not been able to extend to pointed Pfister quadratic spaces.

11.2. **Proposition.** *Let C be a composition division algebra over F and $B' \subseteq \bar{C}$ a unital subalgebra. Assume* $\mathrm{char}(F) \neq 2$ *or* $\dim_{\bar{F}}(B') > 1$. *Then there exists a composition subalgebra $B \subseteq C$ having ramification index $e_{B/F} = 1$ and satisfying $\bar{B} = B'$.*

Proof. One adapts the proof of [40, Lemma 3] to the present more general set-up; for completeness, we include the details. Since B' is either a composition division algebra over $\bar{F}$ or a purely inseparable field extension of characteristic 2 and exponent at most 1, it has dimension 2^m, $m \in \mathbb{Z}$, $0 \leq m \leq 3$. Moreover B' is generated by m elements $\overline{x_1}, \dots, \overline{x_m}$, for some $x_1, \dots, x_m \in \mathfrak{o}_C$, where we may assume $m \geq 1$ since $m = 0$ implies $\mathrm{char}(F) \neq 2$ by hypothesis and $B := F$ does the job. The elements of non-zero trace in C form a Zariski open and dense subset, which therefore is open and dense in the valuation topology as well, so we may assume $t_C(x_1) \neq 0$. Then B, the unital subalgebra of C generated by $x_1, \dots, x_m$, is a composition division algebra with $\dim_F(B) \leq 2^m$, $B' \subseteq \bar{B}$. Now Prop. 7.13 implies $\bar{B} = B'$ and $e_{B/F} = 1$. □

The preceding result can be refined in various ways. For example, given a composition division algebra C over F, we will exhibit (chains of) proper composition subalgebras of C having ramification index 1 and the same trace exponent as C. From this we derive normal forms for octonion and quaternion algebras over F and show that quantities subject to a few obvious constraints are the valuation data of an appropriate composition division algebra. We begin by listing a few properties of wild separable quadratic field extensions which should be well known but seem to lack a convenient reference. We therefore include the details.

11.3. **Proposition.** *Let L be an F-algebra and suppose $\bar{F}$ has characteristic 2. Then the following conditions are equivalent.*

(i) *L/F is a wild separable quadratic field extension and $e_{L/F} = 1$.*
(ii) *There are a positive integer r and elements $\alpha \in \mathfrak{o}^\times$, $\beta \in \mathfrak{o}$ such that $\bar{\beta} \notin \bar{F}^2$ and*

$$L \cong F[\mathbf{t}]/(\mathbf{t}^2 - \pi^r \alpha \mathbf{t} + \beta).$$

If these conditions hold, $\bar{L} = \bar{F}(\sqrt{\bar{\beta}})$. Moreover, setting

$$\vartheta := \mathbf{t} \bmod (\mathbf{t}^2 - \pi^r \alpha \mathbf{t} + \beta) \in L \tag{1}$$

in (ii), *the following relations hold.*

$$\lambda_L(\gamma + \delta\vartheta) = \min\{\lambda(\gamma), \lambda(\delta)\} \qquad (\gamma, \delta \in F), \tag{2}$$

$$\mathfrak{o}_L = \mathfrak{o}1_L \oplus \mathfrak{o}\vartheta, \quad \mathfrak{p}_L = \mathfrak{p}1_L \oplus \mathfrak{p}\vartheta, \tag{3}$$

$$\mathrm{texp}(L) = \min\{e_F, r\}. \tag{4}$$

Proof. (i) $\Longrightarrow$ (ii). By (i) there exists an element $\beta \in \mathfrak{o}$ such that $\bar{\beta} \notin \bar{F}^2$ and $\bar{L} = \bar{F}(\sqrt{\bar{\beta}})$. Pick an element $\vartheta \in \mathfrak{o}_L$ satisfying $\bar{\vartheta} = \sqrt{\bar{\beta}}$. Then $t_L(\vartheta) \in \mathfrak{p}$, and replacing β by $n_L(\vartheta)$ if necessary, we may assume $n_L(\vartheta) = \beta$. Here $\bar{\beta} \notin \bar{F}^2$ forces $L = F(\vartheta) = F[\vartheta]$. We claim there is no harm in assuming $t_L(\vartheta) \neq 0$. Indeed, for $\mathrm{char}(F) = 2$, this is automatic while, if $\mathrm{char}(F) = 0$, we may replace ϑ by $\vartheta' := 1_L + \vartheta$. Thus, without loss, $t_L(\vartheta) \neq 0$. But this yields a unit $\alpha \in \mathfrak{o}^\times$ such that $t_L(\vartheta) = \pi^r \alpha$, $r := \lambda(t_L(\vartheta)) \in \mathbb{Z}$, $r > 0$, and L has the form described in (ii).

(ii) $\Longrightarrow$ (i). By the hypotheses on β, the monic polynomial $f := \mathbf{t}^2 - \pi^r \alpha \mathbf{t} + \beta \in \mathfrak{o}[\mathbf{t}] \subseteq F[\mathbf{t}]$ is irreducible over F, and L/F is a separable quadratic field extension. Define ϑ as in (1). We have $n_L(\vartheta) = \beta \in \mathfrak{o}^\times$, hence $\vartheta \in \mathfrak{o}_L^\times$, and $\bar{\vartheta}^2 = \bar{\beta}$, which implies $\bar{L} = \bar{F}(\sqrt{\bar{\beta}})$, so L is wild and has ramification index $e_{L/F} = 1$. Hence (i) holds.

A standard argument now yields (2), which immediately implies the remaining assertions of the proposition. □

11.4. **Corollary.** *For a separable quadratic field extension L/F to be wild and to have ramification index 1 it is necessary and sufficient that* $\mathrm{texp}(L) > 0$ *and there exist a trace generator u of L with $\overline{n_L(u)} \notin \bar{F}^2$. In this case, $L = k[u]$ and u may be so chosen as to satisfy the additional relation*

$$\lambda_L(u - u^*) = \mathrm{texp}(L). \tag{1}$$

Proof. Necessity and the final statement. If L is wild and $e_{L/F} = 1$, we obtain $\mathrm{texp}(L) > 0$ and first deal with the case $e_F > \mathrm{texp}(L)$. Then we find a positive integer r and elements $\alpha \in \mathfrak{o}^\times$, $\beta \in \mathfrak{o}$ as in Prop. 11.3 (ii) and conclude $\mathrm{texp}(L) = r$ from (11.3.4). Moreover, $u := \vartheta$ as defined in (11.3.1) is a trace generator of L satisfying $\overline{n_L(u)} \notin \bar{F}^2$ and $L = F[u]$. Finally, $\lambda_L(u - u^*) = \lambda_L(2u - \pi^r\alpha 1_L)$, where $\lambda_L(2u) = e_F > r = \lambda_L(\pi^r\alpha 1_L)$, which implies (1) as well. By (8.7.2), we are left with the case $e_F = \mathrm{texp}(L) < \infty$. Then F has characteristic 0, allowing us to apply Cor. 9.14 (b) with $B = P = F$: there exists a scalar $\mu \in \mathfrak{o}$ such that $\bar{\mu} \notin \bar{F}^2$ and $L = F(\sqrt{\mu})$, so some $y \in \mathfrak{o}_L^\times$ has

$$L = F[y], \quad t_L(y) = 0, \quad \overline{n_L(y)} \notin \bar{F}^2. \tag{2}$$

Hence $u := 1_L + y \in \mathfrak{o}_L$ is a trace generator of L satisfying $\overline{n_L(u)} \notin \bar{F}^2$. Moreover, since $y^* = -y$ by (2), $\lambda_L(u - u^*) = \lambda_L(2y) = e_F = \mathrm{texp}(L)$, and the proof is complete.

Sufficiency. We have $L = k[u]$, $t_L(u) = \pi^r\alpha$, $r := \mathrm{texp}(L)$, $\alpha \in \mathfrak{o}^\times$, and condition (ii) of Prop. 11.3 holds with $\beta = n_L(u)$. $\square$

11.5. **Proposition.** *Let L be an F-algebra and suppose $\bar{F}$ has characteristic 2. Then the following conditions are equivalent.*

(i) *L/F is a separable quadratic field extension of ramification index $e_{L/F} = 2$.*
(ii) *There are a positive integer r and elements $\alpha, \beta \in \mathfrak{o}^\times$ such that*

$$L \cong F[\mathbf{t}]/(\mathbf{t}^2 - \pi^r\alpha\mathbf{t} + \pi\beta).$$

In this case, for any prime element $\Pi \in \mathfrak{o}_L$ (e.g., for

$$\Pi := \mathbf{t} \bmod (\mathbf{t}^2 - \pi^r\alpha\mathbf{t} + \pi\beta)) \tag{1}$$

the following relations hold.

$$\lambda_L(\gamma + \delta\Pi) = \min\{\lambda(\gamma), \lambda(\delta) + \frac{1}{2}\} \qquad (\gamma, \delta \in F), \tag{2}$$

$$\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}\Pi, \quad \mathfrak{p}_L = \mathfrak{p} \oplus \mathfrak{o}\Pi, \tag{3}$$

$$\mathrm{texp}(L) = \min\{e_F, r\}. \tag{4}$$

Proof. Everything is standard once it has been shown in (i) $\Rightarrow$ (ii) that $\mathfrak{o}_L$ contains prime elements Π with $t_L(\Pi) \neq 0$. But this follows from the fact that the set of elements in L with non-zero trace, by separability being open and dense in the Zariski topology, is open and dense in the valuation topology as well. $\square$

11.6. **Theorem.** *Let C be a composition division algebra of dimension 2^n, $n = 2, 3$, over F. Then there exists a separable quadratic subfield $L \subseteq C$ having ramification index* 1 *and the same trace exponent as C: $e_{L/F} = 1$, $\mathrm{texp}(L) = \mathrm{texp}(C)$.*

Proof. Setting $r := \mathrm{texp}(C)$, we proceed in four steps.

1^0. Let us first consider the case $r = 0$. Then $\bar{C}$ is a composition division algebra of dimension at least 2 over $\bar{F}$ and hence contains a separable quadratic subfield $L' \subseteq \bar{C}$, which by Prop. 11.2 or [40, Lemma 3], lifts to a separable quadratic subfield $L \subseteq C$ with $e_{L/F} = 1$, $\mathrm{texp}(L) = 0 = r$. We may therefore assume from now on that $r > 0$, so C is wild.

2^0. Next we deal with the case $e_{C/F} = 1$. Pick a trace generator w_0 of C, which belongs to $\mathfrak{o}_C^\times$ by (8.3.1). If $w_0 \notin F1_C$, then $L := F[w_0]$ is a separable quadratic subfield of C satisfying $1 \leq e_{L/F} \leq e_{C/F} = 1$, hence $e_{L/F} = 1$. From $\mathfrak{p}^r = \pi^r\mathfrak{o} = t_L(\mathfrak{o}w_0) \subseteq t_L(\mathfrak{o}_L) =$

$\mathfrak{p}^{\mathrm{texp}(L)}$ we conclude $\mathrm{texp}(L) \le r$, which implies $\mathrm{texp}(L) = r$ by Prop. 8.2 (b). On the other hand, if $w_0 \in F1_C$, then $r = e_F$ and *any* separable quadratic subfield $L \subseteq C$ satisfies $e_{L/F} = 1$ as well as $r \le \mathrm{texp}(L) \le e_F = r$ by Prop. 8.2 (b) and (8.7.2).

3^0. We are left with the case $e_{C/F} = 2$. Then Prop. 11.2 yields a compositions subalgebra $B \subseteq C$ with $e_{B/F} = 1$ and $\dim_F(B) = 2^{n-1}$. If $\mathrm{texp}(B) = r$, we are done for $n = 2$ and my apply 2^0 to B for $n = 3$ to arrive at the desired conclusion. By Prop. 8.2 (b), we may therefore assume $\mathrm{texp}(B) > r$. But then Thm. 9.13 justifies the assumption $C = \mathrm{Cay}(B, \mu)$, $\mu = 1 - \pi^d\beta$, for some *odd* integer d, $0 \le d < 2\,\mathrm{texp}(B)$, and some unit $\beta \in \mathfrak{o}^\times$. In other words, we are in the situation of Prop. 9.6.

4^0. Applying 2^0 to B if B is a quaternion algebra, we find a separable quadratic subfield $L \subseteq B$ satisfying $e_{L/F} = 1$, $\mathrm{texp}(L) = \mathrm{texp}(B)$. From Cor. 11.4 we therefore conclude that there is an element $u \in \mathfrak{o}_L^\times$ such that $t_L(u) = \pi^{\mathrm{texp}(B)}$, $\overline{n_L(u)} \notin \bar{F}^2$. Now consider the element $w := u + u\Pi$, which is a unit of $\mathfrak{o}_C$ by (9.6.4). Moreover, setting $m := \frac{d-1}{2}$ and observing Prop. 9.6,

$$t_C(w) = \pi^r\varepsilon, \quad \varepsilon := 1 + \pi^{\mathrm{texp}(B)-r} \in \mathfrak{o}^\times,$$
$$n_C(w) = n_L(u) + 2\pi^{-m}n_L(u) + \pi\beta n_L(u),$$

hence $\overline{n_C(w)} = \overline{n_L(u)} \notin \bar{F}^2$, so w by Prop. 11.3 generates a separable quadratic subfield $L' \subseteq C$ with $e_{L'/F} = 1$, $\mathrm{texp}(L') = r$. □

11.7. **Corollary.** *Every composition division algebra C of dimension 2^n, $n = 2, 3$, over F contains a trace generator which is a unit in $\mathfrak{o}_C$.*

Proof. Picking $L \subseteq C$ as in Thm. 11.6, every trace generator of L is one of C and by (8.3.1) belongs to $\mathfrak{o}_L^\times \subseteq \mathfrak{o}_C^\times$. □

11.8. **Corollary.** *Let C be a composition division algebra over F with $f_{C/F} > 1$. Then*

$$\mathrm{texp}(C) = \min\{\lambda_C(u - u^*) \mid u \in \mathfrak{o}_C\} = \min\{\lambda_C(u - u^*) \mid u \in \mathfrak{o}_C^\times\}.$$

Proof. For $u \in \mathfrak{o}_C$ we apply (2.1.6),(7.7.1),(8.7.2) and obtain

$$\lambda_C(u - u^*) = \lambda_C\big(2u - t_C(u)1_C\big) \ge \min\{e_F + \lambda_C(u), \lambda\big(t_C(u)\big)\} \ge \mathrm{texp}(C).$$

Hence it suffices to show that there is an element $u \in \mathfrak{o}_C^\times$ with $\lambda_C(u - u^*) = \mathrm{texp}(C)$. If C is tame, the conjugation of $\bar{C}$ cannot be the identity, so $\lambda_C(u - u^*) = 0 = \mathrm{texp}(C)$ for some $u \in \mathfrak{o}_C^\times$. We may therefore assume that C is wild. Combining the hypothesis $f_{C/F} > 1$ with Thm. 11.6, we find a separable quadratic subfield $L \subseteq C$ with $e_{L/F} = 1$ and $\mathrm{texp}(L) = \mathrm{texp}(C)$. Now Cor. 11.4 yields and element $u \in \mathfrak{o}_L^\times \subseteq \mathfrak{o}_C^\times$ such that $\lambda_C(u - u^*) = \lambda_L(u - u^*) = \mathrm{texp}(L) = \mathrm{texp}(C)$. □

We will see in Example 12.6 below that the hypothesis $f_{C/F} > 1$ in the preceding corollary cannot be avoided.

11.9. **Corollary.** *Let C be a composition division algebra of dimension 2^n, $n = 2, 3$, over F. Then there exists a composition subalgebra $B \subseteq C$ with*

$$\dim_F(B) = 2^{n-1}, \quad e_{B/F} = 1, \quad \mathrm{texp}(B) = \mathrm{texp}(C).$$

Proof. For $n = 2$, this is just Thm. 11.6, so we may assume $n = 3$, i.e., that C is an octonion algebra. By [40, Lemma 3], we may also assume $\mathrm{texp}(C) > 0$ and, applying Thm. 11.6 again, we find a separable quadratic subfield $L \subseteq C$ with $e_{L/F} = 1$, $\mathrm{texp}(L) = \mathrm{texp}(C)$. Hence it suffices to prove the following lemma.

11.10. **Lemma.** *Let C be a wild octonion division algebra over F and $L \subseteq C$ a separable quadratic subfield such that $e_{L/F} = 1$, $\mathrm{texp}(L) = \mathrm{texp}(C)$. Then there exists a quaternion subalgebra $L \subseteq B \subseteq C$ with $e_{B/F} = 1$, $\mathrm{texp}(B) = \mathrm{texp}(C)$.*

Proof. Since C is wild, its residue algebra $\bar{C}$ is a purely inseparable field extension of exponent 1 and degree at least 4 over $\bar{F}$ containing $\bar{L}$ as a quadratic subfield. Pick an element $y \in \mathfrak{o}_C^\times$ satisfying $\bar{y} \in \bar{C} \setminus \bar{L}$. Then L and y generate a composition subalgebra $B \subseteq C$ of dimension at most 4 whose residue algebra contains $\bar{L}(\bar{y})$. Hence $B \subseteq C$ is a quaternion subalgebra containing L and having $e_{B/F} = 1$, $\mathrm{texp}(C) = \mathrm{texp}(L) \geq \mathrm{texp}(B) \geq \mathrm{texp}(C)$ by Prop. 8.2 (b). □

11.11. **Normal Form Theorem: octonion algebras.** *For $0 < r \leq e_F$ and an F-algebra C, the following conditions are equivalent.*

(i) *C is an octonion division algebra over F having trace exponent* $\mathrm{texp}(C) = r$.

(ii) *There exist*

(I) *a separable quadratic field extension L/F such that $e_{L/F} = 1$ and* $\mathrm{texp}(L) = r$ *(in particular, L is wild),*

(II) *a scalar $\alpha \in \mathfrak{o}$ such that $\bar{\alpha} \notin \bar{L}^2$,*

(III) *either a scalar $\beta \in \mathfrak{o}$ such that $\bar{\beta} \notin \bar{L}(\sqrt{\bar{\alpha}})^2$ and*

$$C \cong \mathrm{Cay}(L; \alpha, \beta)$$

or a scalar $\beta \in \mathfrak{o}^\times$ such that

$$C \cong \mathrm{Cay}(L; \alpha, 1 - \pi\beta) \quad \text{or} \quad C \cong \mathrm{Cay}(L; \alpha, \pi\beta).$$

If this is so, then $e_{C/F} = 1$, $\bar{C} \cong \bar{L}(\sqrt{\bar{\alpha}}, \sqrt{\bar{\beta}})$ in the first alternative of (III), *while $e_{C/F} = 2$, $\bar{C} \cong \bar{L}(\sqrt{\bar{\alpha}})$ otherwise.*

Proof. (i) $\Longrightarrow$ (ii). Thm. 11.6 yields a separable quadratic subfield $L \subseteq C$ satisfying (I), and applying Lemma 11.10, we find a quaternion subalgebra $L \subseteq B \subseteq C$ with $e_{B/F} = 1$, $\mathrm{texp}(B) = \mathrm{texp}(C) = r = \mathrm{texp}(L)$. Now Cor. 9.14 (b) yields a quantity α satisfying (II) and $B \cong \mathrm{Cay}(L, \alpha)$, $\bar{B} \cong \bar{L}(\sqrt{\bar{\alpha}})$, while (a) and (b) of the same corollary yield a quantity β satisfying (III) as well as the remaining assertions of the theorem.

(ii) $\Longrightarrow$ (i). This follows immediately from Prop. 9.2 and Cor. 9.14. □

Basically the same arguments, inserting the obvious simplifications at the appropriate places, leads to the following quaternionic version of the preceding result.

11.12. **Normal Form Theorem: quaternion algebras.** *For $0 < r \leq e_F$ and an F-algebra B, the following conditions are equivalent.*

(i) *B is a quaternion division algebra over F having trace exponent* $\mathrm{texp}(B) = r$.

(ii) *There exist*

(I) *a separable quadratic field extension L/F such that $e_{L/F} = 1$ and* $\mathrm{texp}(L) = r$ *(in particular, L is wild),*

(II) *either a scalar $\alpha \in \mathfrak{o}$ such that $\bar{\alpha} \notin \bar{L}^2$ and*

$$B \cong \mathrm{Cay}(L, \alpha),$$

or a scalar $\alpha \in \mathfrak{o}^\times$ such that

$$B \cong \mathrm{Cay}(L; 1 - \pi\alpha) \quad \text{or} \quad B \cong \mathrm{Cay}(L, \pi\alpha).$$

If this is so, then $e_{B/F} = 1$, $\bar{B} \cong \bar{L}(\sqrt{\bar{\alpha}})$ in the first alternative of (II), *while $e_{B/F} = 2$, $\bar{B} \cong \bar{L}$ otherwise.* □

On the basis of the preceding results, it can now be shown that the three valuation data we are interested in can be pre-assigned in advance pretty much arbitrarily once the obvious constraints are taken into account:

11.13. **Corollary.** *Let e, n, r be integers with $n \geq 0$ and A an $\bar{F}$-algebra of dimension 2^n. There exists a composition division algebra C over F satisfying $\bar{C} \cong A$, $e_{C/F} = e$, $\mathrm{texp}(C) = r$ if and only if the following conditions are fulfilled.*

(a) $e \in \{1, 2\}$, $0 \leq n \leq 4 - e$, $0 \leq r \leq e_F$, $r = e_F$ *(for $n = 0$, $e = 1$).*
(b) *A is a composition division algebra for $r = 0$, and $A/\bar{F}$ is a purely inseparable field extension of characteristic 2 and exponent at most 1 for $r > 0$.*

Proof. If C is a composition division algebra of dimension 2^m, $0 \leq m \leq 3$, over F having the prescribed valuation data, then $e = e_{C/F} \in \{1, 2\}$ by (7.9.8), $0 \leq r = \mathrm{texp}(C) \leq e_F$ by (8.7.2) and $n = m$ or $n = m - 1$ according as $e_{C/F}$ is 1 or 2 by Prop. 7.13; moreover, $n = 0$ and $e = 1$ imply $C \cong F$. Summing up and observing (8.7.1), we obtain (a), while (b) follows from Prop. 8.2 (d) combined with the remark to Prop. 10.3. Conversely, suppose (a) and (b) are fulfilled. If $r = 0$, the existence of a composition division algebra over F with the desired properties follows from from [40, Thms. 1,2]. We may therefore assume $r > 0$, which by (b) implies that $A/\bar{F}$ is a purely inseparable field extension of characteristic 2, exponent at most 1 and degree 2^n. We first consider the case $n = 0$. If $e = 1$, then $C = F$ has the prescribed valuation data since $r = e_F$ by (a). If $e = 2$, Prop. 11.5 yields a separable quadratic field extension L/k having $e_{L/F} = 2$, $\mathrm{texp}(L) = r$. We may therefore assume $n > 0$. By Prop. 11.3, there exists a separable quadratic field extension L/F such that $e_{L/F} = 1$, $\mathrm{texp}(L) = r$ and $\bar{L} \subseteq A$. Hence $A = \bar{L}$, or $A = \bar{L}(\sqrt{\bar{\alpha}})$ for some $\alpha \in \mathfrak{o}$, $\bar{\alpha} \notin \bar{L}^2$, or $A = \bar{L}(\sqrt{\bar{\alpha}}, \sqrt{\bar{\beta}}\,)$ for some $\alpha, \beta \in \mathfrak{o}$, $\bar{\alpha} \notin \bar{L}^2$, $\bar{\beta} \notin \bar{L}(\sqrt{\bar{\alpha}})^2$ according as $n = 1, 2, 3$. In each case, Prop. 11.3 and Thms. 11.11, 11.12 yield a composition division algebra C over F having the prescribed valuation data. □

Not surprisingly, however, the above valuation data are far away from *classifying* composition division algebras over 2-Henselian fields. This fact is underscored by the following example.

11.14. **Example.** Suppose $\bar{F}$ has characteristic 2 and let $\bar{F} \subseteq K' \subseteq L'$ be a chain of purely inseparable field extensions having exponent at most 1 over $\bar{F}$ such that $[L' : \bar{F}] = 2^n$, $[K' : \bar{F}] = 2^{n-1}$, $n = 2, 3$. Furthermore, suppose $r, s \in \mathbb{Z}$ satisfy the relations $0 < s < r \leq e_F$. Then Cor. 11.13 yields a composition division algebra B over F with

$$e_{B/F} = 1, \quad \bar{B} \cong K', \quad \mathrm{texp}(B) = r. \tag{1}$$

We claim *there are an infinite number of mutually non-isomorphic composition division algebras C over F containing B as a subalgebra and having $e_{C/F} = 1$, $\bar{C} \cong L'$*, $\mathrm{texp}(C) = s$. This is in stark contrast to the fact that unramified composition division algebras up to isomorphism are uniquely determined by their residue algebras [40, Thm. 1].

To prove our claim, we fix an element $\beta \in \mathfrak{o}$ with $L' = K'(\sqrt{\bar{\beta}})$ and have $\bar{\beta} \notin \bar{B}^2$ by (1), which implies $\overline{\alpha\beta} \notin \bar{B}^2$ for any $\alpha \in \mathfrak{o}^\times$ with $\bar{\alpha} \in \bar{B}^2$. Setting

$$d := 2(r - s) = 2\,(\mathrm{texp}(B) - s) > 0, \quad \mu_\alpha := 1 - \pi^d \alpha\beta \in \mathfrak{o}, \quad C_\alpha := \mathrm{Cay}(B, \mu_\alpha),$$

we conclude from Prop. 9.7 that $\mu_\alpha \in \mathfrak{o}^\times$ and C_α is a composition division algebra over F satisfying

$$e_{C_\alpha/F} = 1, \quad \overline{C_\alpha} \cong L', \quad \mathrm{texp}(C_\alpha) = s.$$

It therefore suffices to show, for any additional element $\alpha' \in \mathfrak{o}^\times$ with $\overline{\alpha'} \in \bar{B}^2$, that $C_\alpha \cong C_{\alpha'}$ implies $\bar{\alpha} = \overline{\alpha'}$. To see this, we write $\bar{\alpha} = \delta^2$, $\overline{\alpha'} = \delta'^2$ with $\delta, \delta' \in \bar{B}$ and invoke Cor. 9.10 (b) to derive the following chain of implications.

$$\begin{aligned} C_\alpha \cong C_{\alpha'} &\Longrightarrow \overline{\alpha\beta} \equiv \overline{\alpha'\beta} \bmod \bar{B}^2 \\ &\Longrightarrow \exists\, \gamma \in \bar{B} : \delta^2\bar{\beta} = \delta'^2\bar{\beta} + \gamma^2 \\ &\Longrightarrow \exists\, \gamma \in \bar{B} : (\delta - \delta')\sqrt{\bar{\beta}} + \gamma = 0 \\ &\Longrightarrow \delta = \delta' \Longrightarrow \bar{\alpha} = \overline{\alpha'}. \end{aligned}$$

□

12. Types of composition algebras and heights.

The trace exponent has been our principal tool so far to detect wildness in pointed quadratic spaces and related objects. Other tools of this kind may be obtained by consulting the literature. It is the purpose of the present section to recast these tools in the setting of composition algebras and to compare them with the trace exponent. We begin by describing what will turn out later (see Cor. 12.4 below) to be a dichotomy of composition division algebras over a 2-Henselian field F.

12.1. **Types of composition algebras.** A composition division algebra C over F is said to be of *unitary* (resp. of *primary*) *type* if there exist an associative composition division algebra B over F with $e_{B/F} = 1$ and a scalar μ which is a unit (resp. a prime element) in $\mathfrak{o}$ such that $C \cong \mathrm{Cay}(B, \mu)$. We record a few easy but useful observations.

(a) By Prop. 11.2, C is of unitary or of primary type, provided $f_{C/F} > 1$ or $\mathrm{char}(F) \neq 2$.

(b) For C to be of primary type it is necessary by Prop. 9.2 that C have ramification index $e_{C/F} = 2$.

(c) If C is tame, then C is of unitary (resp. of primary) type if and only if C is unramified (resp. ramified) [40].

(d) Suppose F has characteristic 0 and $\bar{F}$ has characteristic 2. A quadratic field extension of F may or may not be wild. But if it is, it is of primary (resp. of unitary) type if and only if it is tamely ramified (resp. wildly ramified or wildly unramified) in the sense of [21, p. 60].

12.2. **Remark.** At this stage we cannot rule out the possibility that a composition division algebra C over F is both of unitary and of primary type: conceivably, there could exist composition division algebras B, B' over F of ramification index 1, and a unit μ as well as a prime element μ' in $\mathfrak{o}$ such that $\mathrm{Cay}(B, \mu) \cong C \cong \mathrm{Cay}(B', \mu')$. For showing that this scenario is actually impossible, the following improvement of Prop. 8.5 will play a crucial role.

12.3. **Theorem.** *Let C be a composition division algebra over F.*

(a) *If C is of primary type, then $\omega(C) = \mathrm{texp}(C)$.*

(b) *Consider the following conditions on C.*

(i) $\omega(C) = \mathrm{texp}(C) - \frac{1}{2}$.
(ii) *There are trace generators of C belonging to $\mathfrak{p}_C$.*
(iii) *C is wild of unitary type and ramification index* 2.

Then the implications

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longleftarrow \text{(iii)}$$

hold. Moreover, if $f_{C/F} > 1$ or $\mathrm{char}(F) \neq 2$, then all three conditions are equivalent.

Proof. We begin with the first part of (b).

(i) $\Longrightarrow$ (ii). Let w_0 be a regular trace generator of C. Then (i) and (8.5.1) show $\lambda_C(w_0) > 0$, hence $w_0 \in \mathfrak{p}_C$.

(ii) $\Longrightarrow$ (i). Let $w_0 \in \mathfrak{p}_C$ be a trace generator of C. Then $\omega(C) \leq \lambda(t_C(w_0)) - \lambda_C(w_0) < \mathrm{texp}(C)$, and Prop. 8.5 (a) gives (ii).

(iii) $\Longrightarrow$ (ii). If C is wild of unitary type and ramification index 2, we have $C = \mathrm{Cay}(B, \mu)$, B an associative composition division algebra over F with $e_{B/F} = 1$, $\mu \in \mathfrak{o}^\times$, and Thm. 9.13 shows that we are in the situation of Prop. 9.6 with $Q = C$, $P = B$ (cf. the translation formalism 11.1). Picking a trace generator u of B, we apply (9.6.1),(9.6.5) to obtain

$$\lambda\big(t_C(u\Pi)\big) = \lambda\Big(t_C\big(\pi^{-\frac{d-1}{2}}(u+uj)\big)\Big) = \lambda\big(t_B(u)\big) - \frac{d-1}{2}$$
$$= \operatorname{texp}(B) - \frac{d-1}{2} = \operatorname{texp}(C),$$

so $u\Pi \in \mathfrak{p}_C$ is a trace generator of C, showing (ii). Before completing the proof of (b), we turn to

(a) By hypothesis, we are in the situation of Prop. 9.2 with $Q = C$, $P = B$ as before, and by (9.2.2), an element of $\mathfrak{p}_C$ has the form $y = u + vj$, $u \in \mathfrak{p}_B$, $v \in \mathfrak{o}_B$. Hence $\pi^{-1}u \in \mathfrak{o}_B$ since $e_{B/F} = 1$, and from (9.2.3) we conclude

$$\lambda\big(t_C(y)\big) = \lambda\big(t_B(u)\big) = \lambda\big(t_B(\pi^{-1}u)\big) + 1 \geq \operatorname{texp}(B) + 1 = \operatorname{texp}(C) + 1\operatorname{texp}(C).$$

Thus $\mathfrak{p}_C$ does not contain trace generators of C, violating (ii), hence (i), in (b). Now Prop. 8.5 (a) implies (a).

It remains to show (i) $\Rightarrow$ (iii) in (b) under the assumption $f_{C/F} > 1$ or $\operatorname{char}(F) \neq 2$. Then $e_{C/F} = 2$ by Prop. 8.5 (c), and $\operatorname{texp}(C) = \omega(C) + \frac{1}{2} > 0$ by (i) forces C to be wild. Moreover, by 12.1 (a), it is of unitary or of primary type, the latter case being excluded by (i) and (a). □

12.4. **Corollary.** *A composition division algebra C over F cannot be both of unitary and of primary type.*

Proof. If C is tame, the assertion follows immediately from 12.1 (c). If C is wild of ramification index 1, it cannot be of primary type by 12.1 (b). Hence we are left with the case that C is wild of ramification index 2. Then, by Thm. 12.3, $\omega(C) = \operatorname{texp}(C)$ is an integer if C is of primary type, while $\omega(C) = \operatorname{texp}(C) - \frac{1}{2}$ is not if C is of unitary type. □

12.5. **Corollary.** *Let C be a composition division algebra of primary type over F and suppose $B \subseteq C$ is a composition subalgebra with*

$$e_{B/F} = 1, \quad \dim_k(B) = \frac{1}{2}\dim_k(C).$$

Then $\operatorname{texp}(B) = \operatorname{texp}(C)$.

Proof. Since C is not of unitary type by Cor. 12.4, we obtain $C \cong \operatorname{Cay}(B, \mu)$ for some prime element $\mu \in \mathfrak{o}$. Hence $\operatorname{texp}(B) = \operatorname{texp}(C)$ by (9.2.3). □

Both Cor. 11.8 and Thm. 12.3 (b) fail in the exceptional cases stated therein. This is the upshot of the following example.

12.6. **Example.** Let L/F be a separable quadratic field extension of ramification index 2 and suppose $\operatorname{char}(\bar{F}) = 2$. Then we are in the situation of of Prop. 11.5, and if $r \leq e_F$, then $\operatorname{texp}(L) = r$ by (11.5.4). Moreover, (11.5.1) gives $\lambda(t_L(\Pi)) = \lambda(\pi^r\alpha) = r = \operatorname{texp}(L)$, so $\Pi \in \mathfrak{p}_L$ is a trace generator of L; in particular, for $\operatorname{char}(F) = 2$, parts (i),(ii) of Thm. 12.3 (b) hold but (iii) doesn't. On the other hand, if $r > e_F$, then F has characteristic zero and

$$L = F\big(\sqrt{\pi\gamma}\big) = \operatorname{Cay}(F, \pi\gamma), \quad \gamma := -\beta + \frac{\pi^{2r-1}}{4}\alpha^2 \in \mathfrak{o}^\times,$$

is of primary type, forcing $\omega(L) = \operatorname{texp}(L)$ by Thm. 12.3 (a).

Let $u = \gamma + \delta\Pi \in \mathfrak{o}_L$, $\gamma, \delta \in \mathfrak{o}$ (cf. (11.5.3)). Applying (11.5.2),

$$\lambda_L(u - u^*) = \lambda_L(-\pi^r\alpha\delta + 2\delta\Pi) = \min\{r + \lambda(\delta), e_F + \lambda(\delta) + \frac{1}{2}\}$$
$$\geq \min\{r, e_F + \frac{1}{2}\},$$

and this minimum is attained for, e.g., $u = \Pi$. Thus, by(11.5.4),

$$\min\{\lambda_L(u - u^*) \mid u \in \mathfrak{o}_L\} = \begin{cases} \operatorname{texp}(L) & \text{for } r \le e_F, \\ \operatorname{texp}(L) + \frac{1}{2} & \text{for } r > e_F, \end{cases}$$

so in the latter case, the conclusion of Cor. 11.8 does not hold.

We remark in closing that Cor. 11.8 also fails if L/F is *tame* of ramification index 2 since this implies $\operatorname{texp}(L) = 0$ while * induces the identity on $\bar{L}$, so $\lambda_L(u - u^*) > 0$ for all $u \in \mathfrak{o}_L$.

12.7. **Remark.** There is an alternate way of proving Cor. 12.4, by working with quadratic forms. Let C be a composition division algebra over F and suppose C is of unitary and of primary type. Then there are composition division algebras B, B' over F of ramification index 1 and a unit μ as well as a prime element μ' in $\mathfrak{o}$ such that

$$\operatorname{Cay}(B, \mu) \cong C \cong \operatorname{Cay}(B', \mu').$$

Then Prop. 3.12 yields a scalar $\gamma \in F^\times$ satisfying

$$\operatorname{Cay}(B, \mu) \cong \operatorname{Cay}(B, \gamma), \quad \operatorname{Cay}(B', \mu') \cong \operatorname{Cay}(B', \gamma) \tag{1}$$

Since $e_{B/F} = e_{B'/F} = 1$, we conclude from (9.1.2) that $\lambda(n_B(B^\times)) = \lambda(n_{B'}(B'^\times)) = 2\mathbb{Z}$. Hence Prop. 7.5 (b) and the first relation of (1) show that $\lambda(\gamma)$ is even, while Prop. 7.5 (b) and the second relation of (1) show that $\lambda(\gamma)$ is odd, a contradiction.

With the aim of generalizing Thm. 11.6, Cor. 11.9 and Lemma 11.10, we next turn to the problem of finding (chains of) subalgebras having ramification index 1 and pre-assigned trace exponents. Once the obvious constraints are taken into account (provided, e.g., by Prop. 8.2 (b) and Cor. 12.5), we will show that chains of such subalgebras always exist.

12.8. **Theorem.** *Suppose $\bar{F}$ has characteristic 2 and let C be a composition division algebra of dimension 2^n, $n = 2, 3$, over F that is not a quaternion division algebra of primary type. Given $r \in \mathbb{Z}$, $\operatorname{texp}(C) \le r \le e_F$, there exists a separable quadratic subfield $L \subseteq C$ with $e_{L/F} = 1$, $\operatorname{texp}(L) = r$.*

Proof. The case $r = \operatorname{texp}(C)$ having been settled by Thm. 11.6, we may assume $r > \operatorname{texp}(C)$. Applying Cor. 11.9 and then Thm. 11.6, we find a composition subalgebra $B \subseteq C$ and a separable quadratic subfield $K \subseteq B$ with

$$\dim_F(B) = 2^{n-1}, \quad e_{K/F} = e_{B/F} = 1, \quad \operatorname{texp}(K) = \operatorname{texp}(B) = \operatorname{texp}(C).$$

Suppose for the time being that the case $n = 2$ has been solved and let $n = 3$. The quaternion algebra B, having ramification index 1, cannot be of primary type, allowing us to apply the case $n = 2$ to B in place of C and leading us to the desired conclusion.

We are thus reduced to the case $n = 2$, which we will assume for the rest of the proof. Then $K = B$ and C has dimension 4, hence is not of primary type by hypothesis. We are therefore lead to a unit $\mu \in \mathfrak{o}^\times$ with

$$C = \operatorname{Cay}(K, \mu) = K \oplus Kj. \tag{1}$$

Let us first assume $\operatorname{texp}(K) = \operatorname{texp}(C) = 0$, so K is tame. Since C is a division algebra, we conclude $\bar{\mu} \notin n_{\bar{K}}(\bar{K}^\times)$ from Prop. 9.3. In particular, we have $\bar{\mu} \notin \bar{F}^2$. The hypothesis $\operatorname{texp}(K) = 0$ yields an element $v \in \mathfrak{o}_K$ having $t_K(v) = 1$. Observing (1), we put

$$w := \pi^r v + j \in C$$

and obtain $t_C(w) = \pi^r$, $n_C(w) = \pi^{2r} n_K(v) - \mu$, $\overline{n_C(w)} = \bar{\mu} \notin \bar{F}^2$. In particular, $w \in \mathfrak{o}_C^\times \setminus F1_C$ and we conclude from Prop. 11.3 that $L = F[w] \subseteq C$ is a separable quadratic subfield of the desired kind.

We are left with the case $\operatorname{texp}(K) = \operatorname{texp}(C) > 0$. Then K is wild, and Cor. 11.4 yields an element $u \in \mathfrak{o}_K^\times$ with

$$t_K(u) = \pi^{\operatorname{texp}(K)}, \quad \overline{n_K(u)} \notin \bar{F}^2. \tag{2}$$

By (1) and Cor. 9.14, we may assume

(3) $$\bar{\mu} \notin \bar{F}^2 \quad \text{or} \quad \mu = 1 - \pi\beta \quad \text{for some } \beta \in \mathfrak{o}^\times$$

according as C has ramification index 1 or 2. We now put

$$w := \begin{cases} \pi^{r-\mathrm{texp}(K)}u + j \in C & \text{if } e_{C/F} = 1, \\ \pi^{r-\mathrm{texp}(K)}u + uj \in C & \text{if } e_{C/F} = 2. \end{cases}$$

Then $t_C(w) = \pi^r$ by (2), and for $e_{C/F} = 1$ we obtain $\overline{n_C(w)} = \bar{\mu} \notin \bar{F}^2$ by (3). Similarly, for $e_{C/F} = 2$, we obtain $\overline{n_C(w)} = \overline{n_K(u)} \notin \bar{F}^2$ by (2),(3). In either case, $w \in \mathfrak{o}_C^\times \setminus k1_C$ generates a separable quadratic subfield $L := F[w] \subseteq C$ of ramification index $e_{L/F} = 1$ and trace exponent $\mathrm{texp}(L) = r$ (Prop. 11.3). □

12.9. **Corollary.** *Suppose $\bar{F}$ has characteristic 2 and let C be an octonion division algebra over F that is not of primary type. Given $r, s \in \mathbb{Z}$, $\mathrm{texp}(C) \leq r \leq s \leq e_F$, there exists a filtration $L \subseteq B \subseteq C$ consisting of a quaternion subalgebra $B \subseteq C$ and a separable quadratic subfield $L \subseteq B$ such that $e_{L/F} = e_{B/F} = 1$ and $\mathrm{texp}(B) = r$, $\mathrm{texp}(L) = s$.*

Proof. It suffices to construct a quaternion subalgebra $B \subseteq C$ with $e_{B/F} = 1$, $\mathrm{texp}(B) = r$ because Thm. 12.8 applies to such a B and also yields an L with the desired properties.

To construct B, we first invoke Thm. 11.6 and Cor. 11.9 to find a quaternion subalgebra $B_1 \subseteq C$ and a separable quadratic subfield $L_1 \subseteq B_1$ satisfying

(1) $$e_{L_1/F} = e_{B_1/F} = 1, \quad \mathrm{texp}(L_1) = \mathrm{texp}(B_1) = \mathrm{texp}(C).$$

By hypothesis, (1) and Cor. 9.14 yield units $\mu_1, \mu_2 \in \mathfrak{o}^\times$ with

(2) $$B_1 = \mathrm{Cay}(L_1, \mu_1) = L_1 \oplus L_1 j_1, \quad \overline{\mu_1} \notin \bar{F}^2,$$
$$C = \mathrm{Cay}(B_1, \mu_2) = B_1 \oplus B_1 j_2,$$

where

$$\overline{\mu_2} \notin \bar{F}^2 \text{ for } e_{C/F} = 1 \text{ and } \mu_2 = 1 - \pi\beta_2,\ \beta_2 \in \mathfrak{o}^\times \text{ for } e_{C/F} = 2.$$

By the same token,

$$B_2 := \mathrm{Cay}(L_1, \mu_2) = L_1 \oplus L_1 j_2 \subseteq C$$

is a quaternion subalgebra not of primary type by Cor. 12.4 with $\mathrm{texp}(B_2) = \mathrm{texp}(L_1) = \mathrm{texp}(C) \leq r \leq e_F$, $e_{B_2/F} = e_{C/F}$. Hence Thm. 12.8 leads us to a separable quadratic subfield $L \subseteq B_2$ with $e_{L/F} = 1$, $\mathrm{texp}(L) = r$ and (2) combines with Cor. 9.14 (b) to show that the quaternion subalgebra

$$B = \mathrm{Cay}(L, \mu_1) = L \oplus L j_1 \subseteq C$$

satisfies $e_{B/F} = 1$, $\mathrm{texp}(B) = \mathrm{texp}(L) = r$. □

12.10. **Heights.** Over a Henselian field having residual characteristic $p > 0$, the height as an important invariant of a central associative division algebra of degree p over F has been considered by Saltman [44, pp. 1757, 1765-6] (who uses the term "level"), Kato [22, § 1] (who calls it the "ramification number"), and Tignol [50, 3.2]. It is, in particular, Tignol's approach that suggests two immediate translations to the setting of composition algebras.

Let C be a composition division algebra over our 2-Henselian field F. We use the maps $h_{\mathrm{com}} \colon C^\times \times C^\times \to \mathbb{Q}_\infty$, $h_{\mathrm{ass}} : C^\times \times C^\times \times C^\times \to \mathbb{Q}_\infty$ given by

(1) $$h_{\mathrm{com}}(x, y) := \lambda_C([x, y]) - \lambda_C(x) - \lambda_C(y) \geq 0,$$

(2) $$h_{\mathrm{ass}}(x, y, z) := \lambda_C([x, y, z]) - \lambda_C(x) - \lambda_C(y) - \lambda_C(z) \geq 0$$

for all $x, y, z \in C^\times$ to define

(3) $$\mathrm{hgt}_{\mathrm{com}}(C) := \inf\{h_{\mathrm{com}}(x, y) \mid x, y \in C^\times\},$$

(4) $$\mathrm{hgt}_{\mathrm{ass}}(C) := \inf\{h_{\mathrm{ass}}(x, y, z) \mid x, y, z \in C^\times\}$$

and to call these numbers the *commutative height* and the *associative height* of C, respectively. If C has dimension at most 2, then $\mathrm{hgt}_{\mathrm{com}}(C) = \mathrm{hgt}_{\mathrm{ass}}(C) = \infty$. On the other hand, if C is a quaternion algebra, then $\mathrm{hgt}_{\mathrm{com}}(C) < \infty = \mathrm{hgt}_{\mathrm{ass}}(C)$ and $\mathrm{hgt}_{\mathrm{com}}(C)$ agrees with what Tignol calls its height, while if C is an octonion algebra, its commutative and its associative height are both finite.

A general theorem of Tignol [50, 3.12] implies $\mathrm{hgt}_{\mathrm{com}}(C) = \omega(C)$ for any quaternion division algebra C over F. This special observation is part of a much more general picture that will be summarized in the following theorem, whose proof in the quaternionic case is independent of [50] and, in fact, works uniformly in the octonionic case as well.

12.11. **Theorem.** *If C is a quaternion division algebra over F, then*

$$\mathrm{hgt}_{\mathrm{com}}(C) = \omega(C).$$

If C is an octonion division algebra over F, then

$$\mathrm{hgt}_{\mathrm{com}}(C) = \mathrm{hgt}_{\mathrm{ass}}(C) = \omega(C).$$

Proof. Let C be a composition division algebra of dimension 2^n, $n = 2, 3$, over F. We must show

$$\mathrm{hgt}_{\mathrm{com}}(C) = \omega(C), \tag{1}$$

$$\mathrm{hgt}_{\mathrm{com}}(C) = \mathrm{hgt}_{\mathrm{ass}}(C) = \omega(C) \qquad \text{(for } n = 3\text{)}. \tag{2}$$

To do so, we combine Thm. 2.8 with (10.10.1) to obtain

$$\lambda_C([x_1, x_2, x_3]) \geq \omega(C) + \lambda_C(x_1) + \lambda_C(x_2) + \lambda_C(x_3)$$

for all $x_1, x_2, x_3 \in C$. Combining this and (10.10.2) with (12.10.1−4), we conclude

$$\mathrm{hgt}_{\mathrm{com}}(C) \geq \omega(C), \quad \mathrm{hgt}_{\mathrm{ass}}(C) \geq \omega(C). \tag{3}$$

To complete the proof of (1),(2), it therefore suffices to show that

$$h_{\mathrm{com}}(x_1, x_2) = \omega(C) \qquad \text{(for some } x_1, x_2 \in C\text{)}, \tag{4}$$

$$h_{\mathrm{ass}}(x_1, x_2, x_3) = \omega(C) \qquad \text{(for } n = 3 \text{ and some } x_1, x_2, x_3 \in C\text{)}. \tag{5}$$

For this purpose, we require two additional formulas: suppose $C = \mathrm{Cay}(B, \mu) = B \oplus Bj$ is a Cayley-Dickson construction as in 1.10, for some associative composition algebra B over F and some scalar $\mu \in F^\times$. Then a straightforward application of (1.10.1) yields

$$[u, j] = (u - u^*)j \qquad (u \in B), \tag{6}$$

$$[u_1, u_2, j] = [u_1, u_2]j \qquad (u_1, u_2 \in B). \tag{7}$$

In order to establish (4),(5), we distinguish the following cases.

Case 1. $e_{C/F} = 1$.
Then $\omega(C) = \mathrm{texp}(C)$ by Prop. 8.5 (c). If C is tame, the $\bar{C}$ is a composition division algebra of dimension 2^n over $\bar{F}$, so there are $x_1, x_2, x_3 \in \mathfrak{o}_C^\times$ with $[x_1, x_2] \in \mathfrak{o}_C^\times$, and even $[x_1, x_2, x_3] \in \mathfrak{o}_C^\times$ for $n = 3$. By (12.10.1),(12.10.2), this implies $h_{\mathrm{com}}(x_1, x_2) = 0 = \mathrm{texp}(C)$, and even $h_{\mathrm{ass}}(x_1, x_2, x_3) = 0 = \mathrm{texp}(C)$ for $n = 3$, proving (4),(5) in the tame case.

If C is wild, it must be of unitary type since $e_{C/F} = 1$, and Thm. 9.13 combined with Prop. 11.2 implies $C = \mathrm{Cay}(B, \mu)$, B an associative composition division algebra over F with $e_{B/F} = 1$, $\mu = 1 - \pi^d \beta$, $d \in \mathbb{Z}$ even, $0 \leq d < 2\,\mathrm{texp}(B)$, $\beta \in \mathfrak{o}$, $\bar{\beta} \notin \bar{B}^2$. In particular, taking into account 11.1, we are in the situation of Prop. 9.7. Applying Cor. 11.8, we find an element $u \in \mathfrak{o}_B^\times$ such that $\lambda_B(u - u^*) = \mathrm{texp}(B)$. Hence (9.7.2),(9.7.3),(9.7.6) and (6) yield

$$\begin{aligned} h_{\mathrm{com}}(u, \Xi) &= \lambda_C([u, \Xi]) - \lambda_C(u) - \lambda_C(\Xi) = \lambda_C(\pi^{-\frac{d}{2}}[u, j]) \\ &= \lambda_B(u - u^*) - \frac{d}{2} = \mathrm{texp}(B) - \frac{d}{2} = \mathrm{texp}(C). \end{aligned}$$

Thus (4) holds for $x_1 = u$, $x_2 = \Xi$, and we have established (1) in Case 1.

If $n = 3$ (C still assumed to be wild), Case 1 applies to B, so (4) yields elements $u_1, u_2 \in B$ such that $h_{\text{com}}(u_1, u_2) = \text{texp}(B)$. Hence (7) and (9.7.6) imply

$$\begin{aligned} h_{\text{ass}}([u_1, u_2, \Xi]) &= \lambda_C(\pi^{-\frac{d}{2}}[u_1, u_2, j]) - \lambda_B(u_1) - \lambda_B(u_2) \\ &= h_{\text{com}}(u_1, u_2) - \frac{d}{2} = \text{texp}(B) - \frac{d}{2} = \text{texp}(C), \end{aligned}$$

giving (5) for $x_1 = u_1$, $x_2 = u_2$, $x_3 = \Xi$, and settling Case 1 completely.

Case 2. $e_{C/F} = 2$.
If C is of primary type, then $\omega(C) = \text{texp}(C)$ by Thm. 12.3 (a), and $C = \text{Cay}(B, \mu)$ with B as before and μ a prime element in $\mathfrak{o}$. Applying Thm. 9.13 (a) and picking $u \in \mathfrak{o}_B^\times$ with $\text{texp}(C) = \text{texp}(B) = \lambda_B(u - u^*)$, we obtain

$$\begin{aligned} h_{\text{com}}(u, j) &= \lambda_C([u, j]) - \lambda_B(u) - \lambda_C(j) \\ &= \lambda_C\big((u - u^*)j\big) - \lambda_C(j) = \lambda_B(u - u^*) = \text{texp}(C), \end{aligned}$$

and (4) holds for $x_1 = u$, $x_2 = j$. Moreover, for $n = 3$, Case 1 applies to B and yields $u_1, u_2 \in B^\times$ having $h_{\text{com}}(u_1, u_2) = \text{texp}(B) = \text{texp}(C)$, allowing us to compute

$$\begin{aligned} h_{\text{ass}}(u_1, u_2, j) &= \lambda_C([u_1, u_2]j) - \lambda_B(u_1) - \lambda_B(u_2) - \lambda_C(j) \\ &= h_{\text{com}}(u_1, u_2) = \text{texp}(C) \end{aligned}$$

and completing the proof of (5) for $x_1 = u_1$, $x_2 = u_2$, $x_3 = j$.

Now suppose C is of unitary type. Then C is wild since $e_{C/F} = 2$, and Thm. 12.3 (b) shows $\omega(C) = \text{texp}(C) - \frac{1}{2}$. This time, Thm. 9.13 implies $C = \text{Cay}(B, \mu)$ with B as before, $\mu = 1 - \pi^d\beta$, $d \in \mathbb{Z}$ odd, $0 \leq d < 2\,\text{texp}(B)$, $\beta \in \mathfrak{o}^\times$, so in view of 11.1 we are in the situation of Prop. 9.6. Picking again an element $u \in \mathfrak{o}_B^\times$ with $\text{texp}(B) = \lambda_B(u - u^*)$, we obtain

$$\begin{aligned} h_{\text{com}}(u, \Pi) &= \lambda_C([u, \Pi]) - \lambda_B(u) - \lambda_C(\Pi) = \lambda_C(\pi^{-\frac{d-1}{2}}[u, j]) - \frac{1}{2} \\ &= \lambda_C\big((u - u^*)j\big) - \frac{d-1}{2} - \frac{1}{2} = \lambda_B(u - u^*) - \frac{d-1}{2} - \frac{1}{2} \\ &= \text{texp}(B) - \frac{d-1}{2} - \frac{1}{2} = \text{texp}(C) - \frac{1}{2} = \omega(C), \end{aligned}$$

and (4) holds for $x_1 = u$, $x_2 = \Pi$. If, in addition, $n = 3$, then Case 1 applies to B, yielding $u_1, u_2 \in B^\times$ with $h_{\text{com}}(u_1, u_2) = \text{texp}(B)$. Hence

$$\begin{aligned} h_{\text{ass}}(u_1, u_2, \Pi) &= \lambda_C([u_1, u_2, \Pi]) - \lambda_B(u_1) - \lambda_B(u_2) - \lambda_C(\Pi) \\ &= \lambda_C(\pi^{-\frac{d-1}{2}}[u_1, u_2, j]) - \lambda_B(u_1) - \lambda_B(u_2) - \frac{1}{2} \\ &= \lambda_C([u_1, u_2]j) - \frac{d-1}{2} - \lambda_B(u_1) - \lambda_B(u_2) - \frac{1}{2} \\ &= h_{\text{com}}(u_1, u_2) - \frac{d-1}{2} - \frac{1}{2} = \text{texp}(B) - \frac{d-1}{2} - \frac{1}{2} \\ &= \text{texp}(C) - \frac{1}{2} = \omega(C). \end{aligned}$$

□

Part III. **Connections with** K**-theory**

13. Introduction to Part III

13.1. The goal of this part of the paper is to translate the results of Part II into the language of Kato's filtration on Milnor K-theory mod 2. Because it costs little extra, we will also give a dictionary relating traditional valuation-theoretic terms on associative division algebras of prime degree p with Milnor K-theory mod p.

For the remainder of the paper, we fix a field F of characteristic zero that has a Henselian discrete valuation λ and residue field $\overline{F}$ of prime characteristic p. We assume that F contains a primitive p-th root of unity ζ and set

$$m := p \cdot \frac{\lambda(p)}{p-1}.$$

This is an integer divisible by p because $\lambda(p)/(p-1) = \lambda(\zeta - 1)$, see, e.g., [10, 4.1.2(i)].

Recall that the Milnor K-ring of F, denoted $K_*^M(F)$, is the tensor algebra (over $\mathbb{Z}$) of the abelian group $F^\times$ modulo the "Steinberg relation" $a \otimes (1-a) = 0$ for $a \in F$, $a \neq 0, 1$.

One writes $\{a_1, \ldots, a_q\}$ for the image of $a_1 \otimes \cdots \otimes a_q$ in $K_q^M(F)$. We put $k_q(F)$ for $K_q^M(F)/p$, and we abuse notation by writing $\{a_1, \ldots, a_q\}$ also for the image of that element in $k_q(F)$; such a class in $k_q(F)$ is called a *symbol*. See, e.g., [18] for basic properties. Kato, Bloch, and Gabber proved that $k_q(F)$ is isomorphic to $H^q(F, \mu_p^{\otimes q})$ via the "Galois symbol", which sends $\{a_1, \ldots, a_q\} \mapsto (a_1) \cdot (a_2) \cdots (a_q)$; this identifies nonzero symbols in $k_q(F)$ with nonzero symbols in $H^q(F, \mu_p^{\otimes q})$. We are mainly interested in the following cases:

(1) $\underline{q = 1}$: $k_1(F)$ and $H^1(F, \mu_p)$ are naturally identified with $F^\times/F^{\times p}$. A nonzero element $xF^{\times p}$ defines a degree p extension $F(\chi)$ such that $\chi^p = x$.

(2) $\underline{q = 2}$: $H^2(F, \mu_p^{\otimes 2})$ is identified (via ζ) with $H^2(F, \mu_p)$, i.e., the p-torsion in the Brauer group of F. We fix the identification with the Brauer group so that the nonzero symbol $\{x, y\}$ in $k_2(F)$ is sent to the associative central division F-algebra of dimension p^2 generated by elements χ, ψ satisfying

$$\chi^p = x, \quad \psi^p = y, \quad \text{and} \quad \chi\psi = \zeta\psi\chi.$$

(3) $\underline{p = 2}$: In this case, there is a bijection between symbols in $k_q(F)$ and q-Pfister (quadratic) forms given by sending $\{a_1, \ldots, a_q\}$ to $\langle\langle a_1, \ldots, a_q\rangle\rangle$. For $q \leq 3$, we can of course further identify anisotropic q-Pfister forms with composition algebras of dimension 2^q.

For a nonzero symbol $\gamma \in k_q(F)$ from cases (1) or (2), write D for the corresponding division F-algebra. As the valuation λ is Henselian, it extends to a discrete valuation λ_D on D via the usual formula $\lambda_D(x) := \lambda(N_{F(x)/F}(x))/[F(x) : F]$, cf. (7.14.1). The definition of residue division algebra $\overline{D}$, ramification index $e_{D/F}$, etc., is the same as for quaternion algebras, and the fundamental relation of Prop. 7.13 holds, see [53, p. 393] for references.

13.2. Below, we will recall the filtration on $k_q(F)$ and define invariants e_γ (= 1 or p) and depth(γ) for a symbol $\gamma \in k_q(F)$ in terms of K-theory. We will prove K-theoretic analogues of the Local Norm Theorem 8.10 (§15), the Normal Form Theorems 11.11 and 11.12 (§16), and Theorem 9.9 (§18). These proofs use the background results on quadratic forms over Henselian fields from §7 but not the deeper results from the rest of Part II.

In the final sections of the paper (§§19–22) we give a dictionary between properties of symbols in $k_q(F)$ in the cases $q = 1$, $q = 2$, or $p = 2$. The proofs in the case $p = 2$ rely heavily on the full strength of the results in Part II.

14. The filtration on K-theory

We now recall the Kato filtration on Milnor K-theory over F, together with the isomorphisms of the graded components with various modules of differential forms, etc., over $\overline{F}$.

14.1. Filtration and depth. Write $\mathfrak{o}$ for the valuation ring on F and $\mathfrak{p}$ for its maximal ideal. One can filter $\mathfrak{o}$ as

$$\mathfrak{o} \setminus \{0\} = U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$$

where $U_i := 1 + \mathfrak{p}^i$ for $i \geq 1$.

For $q \geq 1$, setting $U^i k_q(F)$ to be additively generated by elements $\{u\} \cdot k_{q-1}(F)$ for $u \in U_i$ defines a filtration

$$k_q(F) = U^0 k_q(F) \supseteq U^1 k_q(F) \supseteq \cdots$$

For $i > m$, U_i consists of p-th powers [10, 4.1.2(ii)], so $U^i k_q(F)$ is zero.

The *depth* of $\gamma \in k_q(F)$ is the supremum of $\{i \mid \gamma \in U^i k_q(F)\}$. The only element of depth $> m$ is zero, which has depth ∞. The filtration is compatible with the product in the sense that $U^i k_r(F) \cdot U^j k_s(F) \subseteq U^{i+j} k_{r+s}(F)$ by [10, 4.1.1b]. Said differently, for elements $\alpha \in k_r(F)$ and $\beta \in k_s(F)$, we have

$$\text{depth}(\alpha) + \text{depth}(\beta) \leq \text{depth}(\alpha \cdot \beta); \tag{1}$$

this inequality can be strict, see Example 20.4.

14.2. Kato isomorphisms. For nonzero $\gamma \in k_q(F)$ with $q \geq 1$ of depth d, we consider the (nonzero) image of γ in $\text{gr}^d k_q(F) := U^d k_q(F)/U^{d+1} k_q(F)$; this is the *initial form* of γ. The results of Kato, et al, include specific isomorphisms:

$$\text{gr}^d k_q(F) \cong \begin{cases} k_q(\overline{F}) \oplus k_{q-1}(\overline{F}) & \text{if } d = 0; \\ \Omega^{q-1} & \text{if } 0 < d < m \text{ and } p \text{ does not divide } d; \\ \frac{\Omega^{q-1}}{Z^{q-1}} \oplus \frac{\Omega^{q-2}}{Z^{q-2}} & \text{if } 0 < d < m \text{ and } p \text{ divides } d; \\ H^1(\overline{F}, \nu(q-1)) \oplus H^1(\overline{F}, \nu(q-2)) & \text{if } d = m. \end{cases}$$

Here Ω^1 denotes the $\overline{F}$-vector space of derivations $\overline{F} \to \overline{F}$, $\Omega^q := \wedge^q \Omega^1$ for $q \geq 1$, $\Omega^0 = \overline{F}$ and $\Omega^{-1} = \{0\}$. The subspace Z^q is the kernel of the differential $\Omega^q \to \Omega^{q+1}$, i.e., Z^q is the subspace of exact forms. The groups $\nu(q)$ are defined in terms of the Cartier operator [10, pp. 4,5]; they are chosen so that $H^1(\overline{F}, \nu(q-1))$ in characteristic p plays the role of the Galois cohomology group $H^q(K, \mu_p^{\otimes(q-1)})$ for K of characteristic $\neq p$.

We refer to these isomorphisms as the "Kato isomorphisms". Fix a uniformizer π for λ and write a_i and b for elements of $\mathfrak{o}^\times$. The isomorphisms are:

d	map
$d = 0$	$\{a_1, a_2, \ldots, a_q\} \mapsto (\{\bar{a}_1, \ldots, \bar{a}_q\}, 0)$ $\{\pi, a_1, \ldots, a_{q-1}\} \mapsto (0, \{\bar{a}_1, \ldots, \bar{a}_{q-1}\})$
p does not divide d	$\{1 + b\pi^d, a_1, \ldots, a_{q-1}\} \mapsto \bar{b}\frac{\mathrm{d}\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_{q-1}}{\bar{a}_{q-1}}$
p divides d and $d \neq 0, m$	$\{1 + b\pi^d, a_1, \ldots, a_{q-1}\} \mapsto (\bar{b}\frac{\mathrm{d}\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_{q-1}}{\bar{a}_{q-1}}, 0)$ $\{\pi, 1 + b\pi^d, a_1, \ldots, a_{q-2}\} \mapsto (0, \bar{b}\frac{\mathrm{d}\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_{q-2}}{\bar{a}_{q-2}})$
$d = m$	$\{1 + b(\zeta - 1)^p, a_1, \ldots, a_{q-1}\} \mapsto (\bar{b}\frac{\mathrm{d}\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_{q-1}}{\bar{a}_{q-1}}, 0)$ $\{\pi, 1 + b(\zeta - 1)^p, a_1, \ldots, a_{q-2}\} \mapsto (0, \bar{b}\frac{\mathrm{d}\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_{q-2}}{\bar{a}_{q-2}})$

The description of $\text{gr}^0 k_q(F)$ is a result of Bass-Tate that holds without restriction on the characteristic of $\overline{F}$. For depth m, of course $(\zeta - 1)^p$ has value m, and in case $p = 2$ the expression $(\zeta - 1)^p$ is 4.

14.3. We mention for later reference a useful fact about $k_q(\overline{F})$ for $q \geq 1$. As with any field, there is a group homomorphism $K_q^M(\overline{F}) \to \Omega^q$ defined by $\{x_1, \ldots, x_q\} \mapsto \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_q}{x_q}$ (one checks the Steinberg relation). But $\overline{F}$ has characteristic p, so this homomorphism induces a homomorphism $\psi \colon k_q(\overline{F}) \to \Omega^q$. Moreover, ψ is injective by [4, 2.1] or [18, 9.7.1]. In summary, we have: *for $x_1, \ldots, x_q \in \overline{F}^\times$, the following are equivalent:*

(i) *The symbol $\{x_1, \ldots, x_q\}$ is zero in $k_q(\overline{F})$.*
(ii) $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q$ *is zero in Ω^q.*
(iii) *The elements $\sqrt[p]{x_1}, \ldots, \sqrt[p]{x_q}$ are not p-free over $\overline{F}$.*

The equivalence of (ii) and (iii) is [6, §V.13.2, Th. 1]. For further statements along these lines, see e.g. [19, 8.1].

Given $y, x_1, \ldots, x_q \in \overline{F}^\times$ with $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q \neq 0$ in Ω^q, the preceding equivalence implies that $y\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q = 0$ in Ω^q/Z^q if and only if y is a p-th power in $\overline{F}(\sqrt[p]{x_1}, \ldots, \sqrt[p]{x_q})$.

15. The Local Norm Theorem 8.10 revisited

We now prove an analogue of the Local Norm Theorem 8.10. We continue the notation of 13.1, and focus on a nonzero symbol $\gamma \in k_q(F)$ where $q = 1$, $q = 2$, or $p = 2$. The symbol γ corresponds to a Galois field extension of F of degree p, a (central) associative division algebra of dimension p^2 over F, or a q-Pfister quadratic form over F. Write V for the underlying vector space, which has dimension p^q. In each case, there is a canonical choice of homogeneous polynomial $f\colon V \to F$ of degree p and representing 1: the norm, the reduced norm, or the quadratic form itself. Further, the valuation λ extends to a valuation λ_V on V via the formula $\lambda_V(v) := \lambda(f(v))/p$, and we write $\mathfrak{o}_V^\times$ for the set of $v \in V$ with value zero.

For a given $a \in \mathfrak{o}^\times$, we ask: How close is f to representing a? Imitating the definition in 8.8, we put

$$\operatorname{nexp}_\gamma(a) := \sup\left\{\lambda(a - f(v)) \mid v \in \mathfrak{o}_V^\times\right\}.$$

It is obviously equivalent to define $\operatorname{nexp}_\gamma(a)$ to be the supremum of all $d \geq 0$ such that there exist $\beta \in \mathfrak{o}$ and $v \in \mathfrak{o}_V^\times$ such that $a = (1 - \pi^d\beta)\, f(v)$.

15.1. Example. As f represents 1,

$$\operatorname{depth}\{a\} \leq \operatorname{nexp}_\gamma(a) \quad (a \in \mathfrak{o}^\times).$$

Moreover, we have equality in case $p = 2$ and $q = 0$ (so necessarily f is the quadratic form $\langle 1\rangle$ and $\gamma = 1 \in \mathbb{Z}/2\mathbb{Z} = k_0(F)$).

15.2. Local Norm Theorem. *Suppose $p = 2$ or $1 \leq q \leq 2$. For $a \in \mathfrak{o}^\times$ and a symbol $\gamma \in k_q(F)$, we have:*

$$\operatorname{nexp}_\gamma(a) + \operatorname{depth}\gamma \leq \operatorname{depth}(\{a\}\cdot\gamma). \tag{1}$$

Furthermore, the following are equivalent:

(i) $\{a\}\cdot\gamma = 0$.
(ii) $a \in f(V)$.
(iii) $\operatorname{nexp}_\gamma(a) > m - \operatorname{depth}\gamma$.
(iv) $\operatorname{nexp}_\gamma(a) = \infty$.

The number m was defined in 13.1 to be $p\,\lambda(p)/(p-1)$. Note that the inequality (15.2.1) apparently strengthens (14.1.1) by Example 15.1.

Proof. (i) and (ii) are known to be equivalent: for $q = 1$, it is [18, 4.7.5] and for $p = 2$ it is Prop. 7.5. For $q = 2$, the implication (ii) $\Rightarrow$ (i) is elementary and the converse is due to Merkurjev-Suslin [33, 12.2].

Assume (ii), i.e., $f(v) = a$ for some $v \in V$. Then $\lambda_V(v) = \lambda(a)/p = 0$, so $v \in \mathfrak{o}_V^\times$ and (iv) is obvious, hence also (iii).

We now prove equation (1). Suppose $a = (1 - \pi^d\beta)\, f(v)$ for some $d \geq 0$, $\beta \in \mathfrak{o}$ and $v \in \mathfrak{o}_V^\times$. Then

$$\{a\}\cdot\gamma = \{f(v)\}\cdot\gamma + \{1 - \pi^d\beta\}\cdot\gamma.$$

But the first term on the right side is zero by the equivalence of (i),(ii) already established. Hence

$$\operatorname{depth}(\{a\}\cdot\gamma) = \operatorname{depth}(\{1 - \pi^d\beta\}\cdot\gamma) \geq d + \operatorname{depth}\gamma,$$

and we have proved (1).

Finally, suppose (iii). By (1), the symbol $\{a\}\cdot\gamma$ has depth $> m$, hence the symbol is zero, proving (i). ∎

In order to compare this result with the Local Norm Theorem 8.10, we must relate ω with depth; for the purposes of this discussion, let us focus on the case $p = 2$ and put $\omega(\gamma) := m - (\operatorname{depth}\gamma)/2$. (This agrees with the definition of ω for composition algebras in case additionally $q = 2$ or 3 by Cor. 19.3(ii) below.) Translating Equation 15.2.1 into this notation gives:

$$\operatorname{nexp}_\gamma(a) \le 2\,\omega(\gamma) - 2\,\omega(\{a\}\cdot\gamma).$$

That is, Theorem 15.2 sharpens Theorem 8.10.

15.3. *Remark.* If every finite extension of F has dimension a power of p ("F is p-special") for some prime p, then Theorem 15.2 holds for that prime p and all $q \ge 1$ if one adjusts slightly the statement of (ii). The adjusted statement should be in terms of a norm variety for γ as is obvious from [47, Prop. 2.4]; we leave the details to the reader. The paper [47] also provides the proof of the equivalence of (i) and the adjusted form of (ii).

16. Gathering the depth

We maintain the notation of 13.1. Recall that $U^d k_q(F)$ is generated as an abelian group by $U_d \cdot k_{q-1}(F)$. In this section, we prove that a symbol in $U^d k_q(F)$ can be written as $\{u\}\cdot\alpha$ where $u \in U_d$ and α is a symbol in $k_{q-1}(F)$ (and not just as a sum of such things). More precisely, we have:

16.1. Gathering Lemma. *For every nonzero symbol $\gamma \in k_q(F)$ with $q \ge 2$, there is a $u \in \mathfrak{o}^\times$ and a symbol $\alpha \in k_{q-1}(F)$ such that $\gamma = \{u\}\cdot\alpha$,* $\operatorname{depth}\{u\} = \operatorname{depth}\gamma$*, and* $\operatorname{depth}\alpha = 0$*. The symbol α may be chosen to be $\{a_2, \ldots, a_q\}$ with $0 \le \lambda(a_2) < p$ and $\lambda(a_i) = 0$ for $3 \le i \le q$. If* $\operatorname{depth}\gamma$ *is not divisible by p, we may further arrange that $\lambda(a_2)$ has any pre-assigned value $j = 0, \ldots, p-1$ as desired.*

One should compare this lemma in the case $p = 2$ and $q = 2, 3$ with the Normal Form Theorems 11.11 and 11.12. Heuristically speaking, here we "gather the depth in the first slot" u. The Normal Form Theorems (in view of Th. 19.2 below and with u replaced by L) do the same, except when C is of unitary type, where they take $\operatorname{depth} L = \operatorname{depth} C - 1$.

We first amass some preliminary results; the proof of the Gathering Lemma will come at the end of the section. We use only background material on K-theory including the material summarized in §14; we don't use anything else from this paper.

16.2. We write $O(\pi^i)$ for an unspecified element (possibly zero) of $\mathfrak{o}$ divisible by π^i. We have the trivial but useful observation:

$$u + O(\pi^j) = u(1 + O(\pi^j)) \qquad \text{for } u \in \mathfrak{o}^\times \text{ and } j \ge 0.$$

Indeed, for $b \in \mathfrak{o}$, we have: $u + b\pi^j = u\left(1 + (u^{-1}b)\pi^j\right)$.

16.3. Example (cf. [4, p. 122])**.** Suppose that $y \in \mathfrak{o}$ has $\bar{y} = \bar{c}^p$ for some $c \in \mathfrak{o}$ and fix $0 \le s < \lambda(p)/(p-1)$ such that $1 + y\pi^{ps} \in \mathfrak{o}^\times$. Then

$$1 + y\pi^{ps} = 1 + c^p\pi^{ps} + O(\pi^{ps+1}) = (1 + c\pi^s)^p + O(\pi^{ps+1})$$

where the second equality is because

$$(1) \qquad \lambda\left(\pi^{is}\tbinom{p}{i}\right) \ge is + \lambda(p) > is + (p-1)s \ge ps \quad \text{for } 1 \le i < p.$$

As $1 + y\pi^{ps}$ is a unit, so is $(1 + c\pi^s)^p$, and 16.2 gives that $1 + y\pi^{ps} = (1 + c\pi^s)^p\,(1 + O(\pi^{ps+1}))$, hence $\{1 + y\pi^{ps}\} = \{1 + O(\pi^{ps+1})\}$ in $k_1(F)$. Noting that the hypotheses on y obviously depend only on $\bar{y}$ (or applying 16.2 once more), we find:

$$\{1 + y\pi^{ps} + O(\pi^{ps+1})\} = \{1 + O(\pi^{ps+1})\} \quad \text{in } k_1(F).$$

16.4. Lemma. *Let $a, b \in \mathfrak{o}$, $i \ge 0$, $j \ge 1$ with $1 + a\pi^i \in \mathfrak{o}^\times$. Then in $k_q(F)$ we have:*

$$\{1 + a\pi^i, 1 + b\pi^j\} = \left\{1 + c\pi^{i+j}, d\pi^{i(p-1)}\right\}$$

for some nonzero $c, d \in \mathfrak{o}$. Further, if $a, b \in \mathfrak{o}^\times$, then also $c, d \in \mathfrak{o}^\times$.

Proof. The computations in the proof of [10, 4.1.1b] or [4, p. 122] yield:

$$\{1 + a\pi^i, 1 + b\pi^j\} = -\left\{1 + \frac{ab}{1 + a\pi^i}\pi^{i+j}, -a\pi^i(1 + b\pi^j)\right\} \quad \in k_2(F).$$

As $-1 = p - 1$ in $k_0(F)$, we have:

$$\{1 + a\pi^i, 1 + b\pi^j\} = \left\{1 + \frac{ab}{1 + a\pi^i}\pi^{i+j}, \left(-a(1 + b\pi^j)\right)^{p-1} \pi^{i(p-1)}\right\}. \qquad \square$$

16.5. Lemma. *Suppose that* $\{1+b\pi^{ps}, a_2, \ldots, a_q\}$ *satisfies* $\mathrm{d}\bar{b}\wedge\mathrm{d}\bar{a}_2\wedge\mathrm{d}\bar{a}_3\wedge\cdots\wedge\mathrm{d}\bar{a}_q = 0$ *in* Ω^q*, with* $b, 1+b\pi^{ps}, a_2, \ldots, a_q \in \mathfrak{o}^\times$ *and* $0 \le s < \lambda(p)/(p-1)$*. Then* $\{1+b\pi^{ps}, a_2, \ldots, a_q\}$ *is equal to* $\{u', a'_2, \ldots, a'_q\}$ *for some* $u', a'_i \in \mathfrak{o}^\times$ *where* $\operatorname{depth}\{u'\} > ps$*.*

Proof. We may assume that $\mathrm{d}\bar{b}$ is not zero—hence that $\bar{b}$ is not a p-th power—by Example 16.3. This settles the $q = 1$ case.

Suppose $q \ge 2$ and $\{\bar{a}_2, \ldots, \bar{a}_q\}$ is zero in $k_{q-1}(\overline{F})$, so $\mathrm{d}\bar{a}_2 \wedge \cdots \wedge \mathrm{d}\bar{a}_q = 0$ by 14.3. We apply the $q - 1$ case of the lemma with $u = a_q$ and $s = 0$ (so $u = a_q = 1 + c$ with $c = a_q - 1$, hence $\mathrm{d}\bar{c} = \mathrm{d}\bar{a}_q$) to see that $\{a_2, \ldots, a_q\} = \{a'_2, \ldots, a'_q\}$ where $\operatorname{depth}\{a'_q\}$ is positive. Then Lemma 16.4 gives the claim.

So we may assume that $\{\bar{a}_2, \ldots, \bar{a}_q\}$ is not zero. Write $[i]$ for a $(q-1)$-tuple $(i_2, \ldots, i_q)$ with $0 \le i_j < p$ and put $\bar{a}^{[i]}$ for $\bar{a}_2^{i_2}\bar{a}_3^{i_3}\cdots\bar{a}_q^{i_q} \in \overline{F}$. By 14.3 there are $c_{[i]} \in \mathfrak{o}$ such that

$$\bar{b} = \left(\sum\nolimits_{[i]} \bar{c}_{[i]} \sqrt[p]{\bar{a}^{[i]}}\right)^p = \sum\nolimits_{[i]} \bar{c}_{[i]}^p \bar{a}^{[i]}.$$

If it happens that $\bar{c}_{[i]} = 0$ for all nonzero $[i]$, then $\bar{b}$ is a p-th power in $\overline{F}$ and we are done. We now show, roughly speaking, that we can make $\bar{c}_{[i]}$ zero for all nonzero $[i]$.

More precisely, fix a nonzero $[i]$ with $\bar{c}_{[i]}$ nonzero; choose a specific j_0 such that $i_{j_0} \ne 0$. Take E/F to be the extension obtained by adjoining a p-th root α of $a_{j_0}^{i_{j_0}} \prod_{j\ne j_0}(-a_j)^{i_j}$; obviously α is integral. Take $v := c_{[i]}\alpha$. As $\bar{c}_{[i]}$ is not zero, the residue of v does not belong to $\overline{F}$, hence $v\pi^s$ is not in F and so has minimal polynomial $x^p - (c_{[i]}\alpha\pi^s)^p$ in $F[x]$. By degree count, this is also the characteristic polynomial $\operatorname{chpoly}_{v\pi^s}(x)$ of $v\pi^s$ as an element of the F-algebra E. It follows that

$$N_{E/F}(1 - v\pi^s) = \operatorname{chpoly}_{v\pi^s}(1) = 1 + (-1)^p N_{E/F}(v)\pi^{ps}. \tag{1}$$

Next observe that E kills $\gamma := \{a_2, \ldots, a_q\}$: we renumber the a_j's so that $j_0 = 2$. In $k_{q-1}(E)$, we have:

$$i_2\gamma = \{a_2^{i_2}, a_3, \ldots, a_q\} = \sum_{j=3}^{q}(p - i_j)\{-a_j, a_3, \ldots, a_q\} = 0,$$

where the middle equality is because α is in E. As i_2 is not divisible by p, we deduce that γ is zero in $k_{q-1}(E)$, as required.

Now the projection formula [18, 7.2.7] gives:

$$0 = N_{E/F}\left(\{1 - v\pi^s\} \cdot \gamma\right) = \{N_{E/F}(1 - v\pi^s)\} \cdot \gamma$$

in $k_q(F)$. Combining this with (1), we find:

$$\{1 + b\pi^{ps}\} \cdot \gamma = \{1 + (b + (-1)^p N_{E/F}(v))\pi^{ps} + O(\pi^{ps+1})\} \cdot \gamma.$$

But the residue of $N_{E/F}(v)$ is $\bar{v}^p = \bar{c}_{[i]}^p \bar{a}^{[i]}$. In this way, we have replaced b with a new one that has fewer nonzero coefficients $\bar{c}_{[i]}$. Repeating this process completes the proof. $\square$

Proof of the Gathering Lemma 16.1. First, consider a symbol $\{x, y\} \in k_2(F)$. Suppose that neither $\lambda(x)$ nor $\lambda(y)$ are divisible by p. Then there is some s such that $\lambda(x) \equiv s\lambda(y) \pmod p$ and $\{x, y\} = \{x(-y)^{-s}, y\}$ because $\{-y, y\}$ is zero in $k_2(F)$. As $\lambda(x(-y)^{-s}) = \lambda(x) - s\lambda(y)$, we may assume that x has value 0.

Second, we may shuffle the entries in a symbol $\{x_1, \ldots, x_q\}$ by a permutation σ. We have $\{x_1, \ldots, x_{q-1}, x_q\} = \{x_{\sigma(1)}, \ldots, x_{\sigma(q-1)}, x_{\sigma(q)}^{\operatorname{sgn}\sigma}\}$.

Combining the two preceding paragraphs shows that we may write $\gamma = \{u\} \cdot \alpha$ for $\alpha = \{a_2, \ldots, a_q\}$ with $u, a_i \in \mathfrak{o}^\times$ for $3 \leq i \leq q$. Amongst all such ways of writing γ, fix one with depth$\{u\}$ maximal. For sake of contradiction, suppose that $d := \operatorname{depth}\{u\} < \operatorname{depth}\gamma \leq m$. Put $r = 2$ if $\lambda(a_2) = 0$ and $r = 3$ if $0 < \lambda(a_2) < p$.

We now inspect the Kato isomorphism at depth d. By hypothesis, γ is zero in $\operatorname{gr}^d k_q(F)$, so has zero image. If $d = 0$, then $\{\bar{u}, \bar{a}_r, \ldots, \bar{a}_q\}$ is zero in $k_*(\overline{F})$, hence $\mathrm{d}(\bar{u} - 1_{\overline{F}}) \wedge \mathrm{d}\bar{a}_r \wedge \cdots \wedge \mathrm{d}\bar{a}_q$ is zero by 14.3. Lemma 16.5 gives a contradiction. The case where $d = ps$ for some $0 < s = d/p < (\operatorname{depth}\gamma)/p < \lambda(p)/(p-1)$ is similar.

Finally we suppose d is not divisible by p. We claim that $\lambda(a_2)$ may be freely chosen. Indeed, for $j \in \mathbb{Z}$, $\lambda(a_2) - j \equiv sd \pmod{p}$ for some $s \geq 0$. Writing $u = 1 + b\pi^d$ for $b \in \mathfrak{o}^\times$, we have:

$$\{1 + b\pi^d, a_2\} = \{1 + b\pi^d, a_2\} - s\{1 + b\pi^d, -b\pi^d\} = \{1 + b\pi^d, a_2(-b)^{-s}\pi^{-sd}\},$$

where $\lambda(a_2(-b)^{-s}\pi^{-sd}) \equiv j \pmod{p}$, proving the claim. The hypothesis that $\{u\} \cdot \alpha$ has depth greater than d implies that $\mathrm{d}\bar{a}_2 \wedge \cdots \wedge \mathrm{d}\bar{a}_q$ is zero, and applying Lemma 16.5 to α with $s = 0$ shows that we may assume that one of the a_i has residue 1; Lemma 16.4 gives a contradiction. □

16.6. *Remark.* We can quickly deduce a useful restatement of the Gathering Lemma. Fix some $r \geq 1$ and a permutation σ of $\{2, \ldots, q\}$ and put

$$\beta := \{u, a_{\sigma(2)}, \ldots, a_{\sigma(r)}\} \quad \text{and} \quad \delta := \{a_{\sigma(r+1)}, a_{\sigma(r+2)}, \ldots, a_{\sigma(q)}\}.$$

Because of the identity $\{x, y\} = -\{y, x\}$ in $K_2^M(F)$, we find:

$$\gamma = \pm\beta \cdot \delta, \quad \operatorname{depth}\beta = \operatorname{depth}\gamma, \quad \text{and} \quad \operatorname{depth}\delta = 0.$$

Indeed, for the second and third equalities, $\geq$ is obvious and $\leq$ follows from the first equality and equation (14.1.1). We will apply this in the case $p = 2$, so the sign in the first equation will be irrelevant.

17. Ramification index for symbols

17.1. Definition. For a class $\gamma \in k_q(F)$, we put $e_\gamma = p$ if

- depth(γ) is *not* divisible by p, or
- depth(γ) is divisible by p and its initial form has nonzero projection in the second summand of $\operatorname{gr}^d k_q(F)$. (Note that this condition does not depend on the choice of uniformizer π.)

Otherwise—or if $\gamma = 0$—we put $e_\gamma = 1$.

The "ramification index" e_γ is more subtle than in the case of good residue characteristic, see Example 20.4 below. But we do have the following positive results:

17.2. Proposition. *Let $\gamma \in k_q(F)$, $q \geq 2$, be a non-zero symbol. Then there exist a symbol $\beta \in k_{q-1}(F)$ and an element $a \in F^\times$ such that $\gamma = \beta \cdot \{a\}$, $e_\beta = 1$ and one of the following holds.*

(i) depth $\beta = 0$, depth$\{a\} = \operatorname{depth}\gamma$, $a \in \mathfrak{o}^\times$ *and*

$$e_\gamma = 1 \iff \operatorname{depth}\gamma \equiv 0 \bmod p.$$

(ii) depth $\beta = \operatorname{depth}\gamma$, $0 < \lambda(a) < p$ *and* $e_\gamma = p$.

Proof. Write γ as in the Gathering Lemma 16.1, where we may assume $a_2, \ldots, a_{q-1} \in \mathfrak{o}^\times$, $0 \leq \lambda(a_q) < p$. If $\lambda(a_q) = 0$, then the Kato isomorphism at depth zero shows $e_\alpha = 1$, so with $\beta := \alpha$ and $a = u^{\pm 1}$ we are in Case (i) since the Kato isomorphism at depth $d := \operatorname{depth}\gamma$ gives $e_\gamma = 1$ iff d is divisible by p. Now suppose $\lambda(a_q) > 0$; by the final statement of the Gathering Lemma, we may also assume $d \equiv 0 \bmod p$. Arguing as before, in particular consulting the Kato isomorphisms again for the determination of e_γ, $\beta := \{u, a_2, \ldots, a_{q-1}\}$ has depth d and $e_\beta = 1$, so we are in Case (ii). □

17.3. Proposition. *For a symbol $\gamma \in k_q(F)$ for $q \geq 2$, the following are equivalent:*

(i) $\gamma = \beta \cdot \{a\}$ *for a nonzero symbol* $\beta \in k_{q-1}(F)$ *with* $e_\beta = 1$ *and* $a \in F^\times$ *of value not divisible by* p.

(ii) $e_\gamma = p$ *and* $\operatorname{depth} \gamma$ *is divisible by* p.

If these conditions hold, then additionally $\operatorname{depth} \gamma = \operatorname{depth} \beta$.

Proof. The proof amounts to looking at the explicit formulas for the Kato isomorphisms.

(i) $\Rightarrow$ (ii): The Kato isomorphisms send β to a nonzero symbol $\bar{\beta}$ (in some cohomology group over $\overline{F}$), because $e_\beta = 1$. We have $\operatorname{depth}(\gamma) \geq \operatorname{depth}(\beta)$ by (14.1.1), and an examination of the isomorphisms show that the isomorphism at the depth of β (divisible by p) sends $\beta \cdot \{a\}$ to $\lambda(a)\bar{\beta}$ in the second component of the image, which is not zero because all of the targets of the Kato isomorphisms are abelian groups killed by p; that is, $e_\gamma = p$ and $\operatorname{depth} \gamma = \operatorname{depth} \beta$.

(ii) $\Rightarrow$ (i): By (ii), alternative (i) of Prop. 17.2 does not hold. Hence alternative (ii) does. $\square$

18. Theorem 9.9 revisited

We will now prove a version of Theorem 9.9 for K-theory mod-2 when $\overline{F}$ has characteristic 2; in the notation of §14 we restrict to the case $p = 2$. We replace the anisotropic round quadratic form P with $e_{P/F} = 1$ from Th. 9.9 with a nonzero symbol $\gamma \in k_q(F)$ with $q \geq 1$ and $e_\gamma = 1$ (in particular, $\operatorname{depth} \gamma$ is even). We replace the hypothesis on texp with the hypothesis $\operatorname{depth}\{\mu\} + \operatorname{depth} \gamma = m$ (recall that $m = 2\lambda(2)$), the largest possible depth for a nonzero element of $k_q(F)$. The extreme cases where $\operatorname{depth} \gamma$ is 0 or m are comparatively easy, so we focus on the middle case. We will use the following:

18.1. Technique. If one has a *nonzero* class $\bar{b}\frac{\mathrm{d}\bar{a}_2}{\bar{a}_2} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_q}{\bar{a}_q} \in \Omega^{q-1}/Z^{q-1}$, we may apply d and obtain the nonzero symbol $x_0 := \mathrm{d}\bar{b} \wedge \frac{\mathrm{d}\bar{a}_2}{\bar{a}_2} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_q}{\bar{a}_q}$ in Ω^q. The $\overline{F}$-span of this symbol contains $x := \bar{b}^{-1}x_0$, which lies in $\nu(q)$. Indeed, it is the unique nonzero element of $\overline{F}x_0$ with this property, because $\nu(q)$ is defined to be $\ker(\gamma - 1)$ for γ the inverse Cartier operator [10, pp. 123, 124] and for $c \in \overline{F}$ we have $(\gamma - 1)(cx) = (c^2 - c)x \in \Omega^q$. A canonical isomorphism identifies x with the class of the anisotropic symmetric bilinear form $B = \langle\langle \bar{b}, \bar{a}_2, \ldots, \bar{a}_q \rangle\rangle$ in the graded Witt ring [25], hence with B itself [15, 6.20]. The equivalence from Prop. 3.3 takes B and gives an extension $K/\overline{F}$ of degree 2^q with a unital linear form $s\colon K \to \overline{F}$.

Let $\gamma \in k_q(F)$ be a nonzero symbol of even depth $d \neq 0, m$ and suppose that $e_\gamma = 1$. Then the initial form of γ is a nonzero symbol in Ω^{q-1}/Z^{q-1} and the technique in the preceding paragraph gives a (K, s) derived from γ.

18.2. Proposition. *Let* $\gamma \in k_q(F)$ *be a nonzero symbol of even depth* $d \neq 0, m$ *with* $e_\gamma = 1$. *Write* (K, s) *as in 18.1, and write* $\gamma = \{1 + x\pi^d\} \cdot \alpha$ *with* $x \in \mathfrak{o}^\times$ *as in the Gathering Lemma 16.1. For* $\mu = 1 - b\pi^{m-d}$ *with* $b \in \mathfrak{o}^\times$, *we have:* $\{\mu\} \cdot \gamma = 0$ *in* $k_{q+1}(F)$ *if and only if the residue of* $xb\pi^m/4$ *is in the image of* $\wp_{K,s}$.

Proof. Write $\alpha = \{a_2, \ldots, a_q\}$ for some $a_2, \ldots, a_q \in \mathfrak{o}^\times$. Using Lemma 16.4, we calculate:

$$\{\mu\} \cdot \{1 + x\pi^d\} = \{1 + x\pi^d, 1 - b\pi^{m-d}\} = \left\{1 - \frac{bx}{1 + x\pi^d}\pi^m, -x(1 - b\pi^{m-d})\right\}.$$

We see from this that the initial form of $\{\mu\} \cdot \gamma$ is $\bar{x}\bar{b}\bar{\varepsilon}\frac{\mathrm{d}\bar{x}}{\bar{x}} \wedge \frac{\mathrm{d}\bar{a}_2}{\bar{a}_2} \wedge \cdots \wedge \frac{\mathrm{d}\bar{a}_q}{\bar{a}_q}$ where we have set $\varepsilon := \pi^m/4$ to simplify the notation. This determines the class of the quadratic Pfister form $\langle\langle \bar{x}, \bar{a}_2, \ldots, \bar{a}_q, \bar{x}\bar{b}\bar{\varepsilon}]]$ in the graded Witt group of quadratic forms [25]. The Arason-Pfister Hauptsatz [15, 23.7(1)] implies that this class is zero (equivalently, $\{\mu\} \cdot \gamma$ is zero) if and only if the Pfister form is isotropic, if and only if $\bar{x}\bar{b}\bar{\varepsilon}$ is in the image of $\wp_{K,s}$ by Cor. 3.10(a), proving the claim. $\square$

19. Dictionary between K-theory and algebras and quadratic forms

In the cases $q = 1$, $q = 2$, or $p = 2$, we have a close relationship between properties of symbols in $k_q(F)$ relative to Kato's filtration and valuation-theoretic properties on the corresponding algebras. Specifically:

19.1. Proposition. *In cases $q = 1$ and $q = 2$ we have:*

(i) $e_\gamma = e_{D/F}$.
(ii) $\overline{D}$ *is a separable division algebra over* $\overline{F}$ *and distinct from* $\overline{F}$ *if and only if* $\text{depth}(\gamma) = m$.

That is, you can determine $e_{D/F}$ and whether or not D is tame by examining the corresponding symbol in $k_q(F)$.

We prove the proposition in §20 and 21.2 below. The calculations appearing in the proof of this result are similar to some in sections 1.1 and 2.1 of [51]. Roughly speaking, our statements here differ from those in [51] by using the language of Kato's filtration.

In contrast with Proposition 19.1, the following theorem relies heavily on the results of Part II. It is proved in §22.

19.2. Theorem. *Let $p = 2$. Fix a nonzero symbol $\gamma \in k_q(F)$ for some $q \geq 1$ and let Q be the corresponding anisotropic q-Pfister form. Then:*

(i) $e_\gamma = e_{Q/F}$.
(ii) $\overline{Q}$ *is nonsingular (i.e., Q is tame) if and only if* $\text{depth}(\gamma) = 2\lambda(2)$.
(iii) $\text{texp}(Q) = \lambda(2) - \left\lfloor \frac{\text{depth}(\gamma)}{2} \right\rfloor$.

In the statement of the theorem, Q is a quadratic form, whereas the results in §§7–9 (including the definition of tame and texp) are in terms of pointed quadratic spaces. This is harmless in view of Prop. 7.3. The theorem leads to:

19.3. Corollary. *Fix a division quaternion or octonion algebra C over F and write γ for the corresponding symbol in $k_*(F)$. We have:*

(i) *C is of primary type if and only if $e_\gamma = 2$ and* $\text{depth}(\gamma)$ *is even. (Otherwise C is of unitary type.)*
(ii) $\omega(C) = \lambda(2) - \frac{\text{depth}(\gamma)}{2}$

That is, one can read off properties of the composition algebra C—including the invariant $\omega(C)$ studied by Saltman and Tignol—from the properties of the corresponding symbol γ in Milnor K-theory. We prove Theorem 19.2 and Cor. 19.3 in §22.

19.4. *Remark.* It is natural to wonder if on can extend Prop. 19.1 to include the case $p = q = 3$, where the corresponding algebraic objects are Albert (Jordan) algebras obtained from the first Tits construction. The answer is no, because we do not know if two such Albert algebras corresponding to the same symbol in $k_3(F)$ are necessarily isomorphic. (This is a special case of open problem #4 from [41].) If this is indeed the case, then one can easily extend Prop. 19.1 to include the case $p = q = 3$ by taking advantage of the valuation theoretic results in [39] and by imitating the proof in the cases given below.

20. Proof of Proposition 19.1: case $q = 1$

20.1. Example ($q = 1$ and nonzero depth divisible by p)**.** Fix $\{x\} \in k_1(F)$ of depth d divisible by p, so $x = 1 + u\pi^d$ for some u of value 0. As the "second summand" Ω^{-1}/Z^{-1} or $H^1(\overline{F}, \nu(-1))$ in the Kato isomorphism is zero, $e_{\{x\}} = 1$.

The algebra D corresponding to $\{x\}$ is $F(\chi)$ where $\chi^p = x$. For

$$\alpha := \pi^{-d/p}(\chi - 1) \quad \in D$$

we have

$$(\alpha + \pi^{-d/p})^p = \pi^{-d}x = u + \pi^{-d},$$

so

$$\sum_{i=1}^{p} \binom{p}{i} \alpha^i \pi^{-(p-i)d/p} - u = 0. \tag{1}$$

For $1 \le i < p$, the prime p divides $\binom{p}{i}$, so

$$\lambda\left(\tbinom{p}{i}\pi^{-(p-i)d/p}\right) \ge \lambda(p) - \frac{d}{p}(p-i) = \frac{(m-d)(p-1)+d(i-1)}{p}.$$

As $i \ge 1$ and $d \le m$, this is at least 0, hence α is integral. Further, the coefficient of α^i in (1) has residue zero for $2 \le i < p$ in all cases and also for $i = 1$ if $d < m$. (Clearly, $\bar{\alpha}$ is not zero, so $\lambda(\alpha) = 0$.)

Therefore, if $d < m$, $\overline{D}$ contains $\bar{\alpha}$ satisfying $\bar{\alpha}^p - \bar{u} = 0$. As x has depth d, the Kato isomorphism shows that $\bar{u}$ is not a p-th power in $\overline{F}$, and we conclude that $\overline{D}$ is the proper extension $\overline{F}(\sqrt[p]{\bar{u}})$ and $e_{D/F} = 1$.

In case $d = m$, we set $\eta := \zeta - 1$. As $\lambda(\eta) = m/p = \lambda(p)/(p-1)$, we have

$$x = 1 + b\eta^p \quad \text{for } b = \frac{u\pi^m}{\eta^p}.$$

We put $\beta := \eta^{-1}(\chi - 1)$ and apply the same reasoning as in the case $d < m$ with $\pi^{-d/p}$ replaced with η^{-1}. The element β satisfies

$$\sum_{i=1}^{p} \binom{p}{i} \beta^i \eta^{-(p-i)} - b = 0. \tag{2}$$

Again, the coefficients of β are integral because

$$\lambda\left(\binom{p}{i}\eta^{-(p-i)}\right) \ge \lambda(p) - (p-i)\frac{\lambda(p)}{p-1} = \lambda(p)\frac{i-1}{p-1} \ge 0 \quad \text{for } 1 \le i < p. \tag{3}$$

Taking residues of (2) kills the terms with $1 < i < p$. For the $i = 1$ term, we note that expanding the equation $(1+\eta)^p = 1$ gives $\sum_{i=1}^{p} \binom{p}{i}\eta^i = 0$, so $p\eta = -\sum_{i=2}^{p} \eta^i$ and

$$p\eta^{-(p-1)} = p\eta^{1-p} = -\sum_{i=2}^{p} \binom{p}{i} \eta^{-(p-i)}.$$

Taking residues and applying (3), only the $i = p$ term is nonzero on the right side, so $p\eta^{-(p-1)}$ has residue -1. Taking residues of (2), we find the equation $\bar{\beta}^p - \bar{\beta} - \bar{b} = 0$ in $\overline{D}$. The fact that x has depth d asserts that the element $\bar{b}$ is nonzero in $H^1(\overline{F}, \nu(0)) \cong \overline{F}/\wp(\overline{F})$, i.e., $\bar{\beta}$ generates a proper extension of $\overline{F}$ and we conclude that D/F is unramified. We have verified Proposition 19.1 for $\gamma \in k_1(F)$ of nonzero depth divisible by p.

20.2. *Remark.* It might be illuminating to compare Example 20.1 in the case $p = 2$ with the material in Part II. In case $0 < d < m = 2\lambda(2) = 2\,\mathrm{texp}(F)$, we compare the example to Prop. 9.7 with $P = F$. The Kato isomorphism at depth d sends $\{x\}$ to the image of $\bar{u}$ in $\bar{F}/\bar{F}^2$, and we conclude $\bar{u} \notin \bar{F}^2$, in agreement with Prop. 9.7. Moreover, identifying $F(\sqrt{x}) = \mathrm{Cay}(F, x) = F \oplus Fj$ and $j = -\chi$, we obtain that $-\alpha = \Xi$ in the sense of (9.7.2); it is a normalized trace generator.

The situation with $d = 2\lambda(2)$ is slightly more delicate; we compare the example to Thm. 9.9 (resp. Cor. 10.18) for P (resp. C) $= F$. First of all, we have $\zeta = -1$, $\eta = -2$, $\beta = \frac{\chi-1}{2}$, and $w_0 = \frac{\pi^{\lambda(2)}}{2} \in \mathfrak{o}^\times$ is the unique normalized trace generator of F. Moreover, $\bar{s}_{w_0} = \mathbf{1}_{\bar{F}}$, forcing $\wp_{\bar{F},\bar{s}_{w_0}} = \wp$ to be the usual Artin-Schreier map on $\bar{F}$. We have $u = -\beta$, $u_0 = n_F(w_0)u = -\beta_0$ in the sense of (9.9.1) and $b = u_0$. The Kato isomorphism at depth m sends $\{x\}$ to $\bar{u}_0 = \bar{\beta}_0 \in H^1(\bar{F}, \nu(0)) = \bar{F}/\wp(\bar{F})$, forcing $\bar{\beta}_0 \notin \mathrm{Im}(\wp)$ in agreement with the equivalence (iii) $\Leftrightarrow$ (v) in Thm. 9.9. Furthermore, by Cor. 10.18,

$$\bar{C} = \bar{F}[\mathbf{t}]/(\mathbf{t}^2 - \mathbf{t} - \bar{u}_0) = \bar{F}[\mathbf{t}]/(\mathbf{t}^2 - \mathbf{t} - \bar{b}) = \overline{F(\sqrt{x})},$$

in agreement with the second part of Example 20.1.

20.3. *Proof of Proposition 19.1 for $q = 1$*: Fix a nonzero $\{x\} \in k_1(F)$ and write $D = F(\chi)$ as in Example 20.1.

Suppose first that x has depth 0; we may assume that $0 \le \lambda(x) < p$. The Kato isomorphism

$$\mathrm{gr}^0 k_1(F) \xrightarrow{\sim} k_1(\overline{F}) \oplus k_0(\overline{F}) \cong \overline{F}^\times / \overline{F}^{\times p} \oplus \mathbb{Z}/p\mathbb{Z}$$

is the direct sum of the specialization map s_π and the tame symbol ∂, described concretely in [18], e.g. As $\partial(x) \equiv \lambda(x) \pmod p$, we have $e_{\{x\}} = p$ if and only if $\lambda(x)$ is not zero. As to the extension D/F, the element χ satisfies $\chi^p - x = 0$, hence is integral. If $\lambda(x)$ is not zero, then $\bar{\chi}^p$ is zero, hence $\bar{\chi}$ is zero, $\overline{D} = \overline{F}$, and $e_{D/F} = p$. If $\lambda(x)$ is zero, then since x has depth 0, $\bar{x}$ is not a p-th power in $\overline{F}$ and $\overline{F}(\bar{\chi})$ is a proper extension of $\overline{F}$; in this case $\overline{D} = \overline{F}(\sqrt[p]{\bar{x}})$ and $e_{D/F} = 1$.

Suppose that x has depth d not divisible by p; we may take x of the form $1 + u\pi^d$ where u has value 0. Then

$$N_{D/F}(\chi - 1) = (-1)^{p-1}(x-1) = (-1)^{p-1} u \pi^d,$$

so the value of $\chi - 1$ is not an integer and $e_{D/F} = p = e_{\{x\}}$.

Note that in both of these cases, $\overline{D}$ is not a proper separable extension of $\overline{F}$. The remaining case where d has nonzero depth divisible by p was treated in Example 20.1, which concludes the proof. □

20.4. Example. Let $\overline{F} := \mathbb{F}_2((x))$, Laurent series over the field with 2 elements. Construct a field F of characteristic zero by taking the absolutely unramified Teichmüller extension of $\overline{F}$ as in Example 10.15 and adjoining $\sqrt{2}$. Then F has residue field $\overline{F}$ and $\sqrt{2}$ is a uniformizer. Consider the elements $a = 1 + \sqrt{2}^3$ and $b = 1 + 2u$ in F, where $u \in F$ is a unit with residue x. The symbols $\{a\}$ and $\{b\}$ have depth 3 and 2. The valuation λ ramifies on $F(\sqrt{a})$ and does not ramify on $F(\sqrt{b})$.

A reader familiar with the situation of good residue characteristic might assume that the quaternion algebra (a, b) is division over F, but in fact it is split. This is easily seen because the symbol $\{a, b\}$ has depth at least 5 by (14.1.1), which is greater than $2\lambda(2)$.

21. Proof of Proposition 19.1: cases $q = 2$

We now prove Proposition 19.1 and Theorem 19.2 in the case $q = 2$. We consider a nonzero symbol $\{x, y\}$ in $k_2(F)$ and write D for the corresponding symbol algebra.

21.1. Lemma. *$e_{D/F} = p$ if and only if some subfield L of D has $e_{L/F} = p$.*

Proof. Easy. If such an L exists, then there is some $\ell \in L^\times$ such that $\lambda(N_{L/F}(\ell))$ is not divisible by p. But $N_{L/F}(\ell) = \mathrm{Nrd}_{D/F}(\ell)$ by [11, p. 150]. This proves "if". For "only if", note that a nonzero element of D generates a subfield L/F. □

21.2. *Proof of Proposition 19.1 for $q = 2$*: Suppose first that the depth d of $\{x, y\}$ is not divisible by p (so $e_{\{x,y\}} = p$).

By the Gathering Lemma 16.1 we can arrange that x is in U_d and $y \in \mathfrak{o}^\times$ has residue that is not a p-th power. As in 20.3, $e_{F(\chi)/F} = p$, hence $e_{D/F} = p$ and the dimension of $\overline{D}/\overline{F}$ is p. The residue field of $F(\psi)$ is the proper extension $\overline{F}(\sqrt[p]{\overline{y}})$ of $\overline{F}$, and by dimension count it is all of $\overline{D}$ and D is wild. So we may assume that the depth of $\{x, y\}$ is divisible by p.

Suppose now that $e_{\{x,y\}} = p$. By Prop. 17.3, we may assume that $e_{\{x\}} = 1$ and $y = u\pi^n$ for some $u \in \mathfrak{o}$ and n not divisible by p, and $\mathrm{depth}\{x, y\} = \mathrm{depth}\{x\}$. On the one hand, D contains $F(\psi)$ on which the valuation ramifies, so $e_{D/F} = p$ as claimed. On the other hand, D contains $F(\chi)$ whose residue algebra is a proper extension of $\overline{F}$ that is inseparable (if $\mathrm{depth}\{x, y\} < m$) or separable (if $\mathrm{depth}\{x, y\} = m$) by Example 20.1; this proves the claim. We are left with the case where the depth is divisible by p and $e_{\{x,y\}} = 1$.

If the depth of $\{x, y\}$ is zero, then by the Kato isomorphisms $\{\bar{x}, \bar{y}\}$ is not zero in $k_2(\overline{F})$, hence $\bar{x}, \bar{y}$ are p-free over $\overline{F}^p$. Following 20.3, $\bar{\chi}$ and $\bar{\psi}$ generate purely inseparable extensions of $\overline{F}$ in $\overline{D}$. The value of $\chi\psi - \psi\chi = (\zeta - 1)\psi\chi$ is $\lambda(\zeta - 1) = m/p > 0$, so $\bar{\chi}$ and $\bar{\psi}$ commute in $\overline{D}$. We deduce that $\overline{D}$ is $\overline{F}(\sqrt[p]{\bar{x}}, \sqrt[p]{\bar{y}})$, $e_{D/F} = 1$, and D is wild.

If $0 < \operatorname{depth}\{x, y\} < m$, then we can choose $x = 1 + a\pi^d$ where $d = \operatorname{depth}\{x, y\}$ so that the initial form of $\{x, y\}$ in Ω^1/Z^1 is $\bar{a}\frac{\mathrm{d}\bar{y}}{\bar{y}}$. As the depth is d, $\bar{a}\frac{\mathrm{d}\bar{y}}{\bar{y}}$ is not in Z^1, i.e., $\mathrm{d}\bar{a} \wedge \frac{\mathrm{d}\bar{y}}{\bar{y}}$ is not 0. It follows that $\dim_{\overline{F}} \overline{F}(\sqrt[p]{\bar{a}}, \sqrt[p]{\bar{y}}) = p^2$ by 14.3. We claim that this field is $\overline{D}$. By dimension count, it suffices to note that for $\alpha := \pi^{-d/p}(\chi - 1)$, we have

$$\bar{\alpha}^p = \bar{a}, \quad \bar{\psi}^p = \bar{y}, \quad \text{and} \quad \bar{\alpha}\bar{\psi} = \bar{\psi}\bar{\alpha}.$$

The first equation is as in Example 20.1 and the second is obvious. The third follows because

$$\alpha\psi - \psi\alpha = \pi^{-d/p}(\chi\psi - \psi\chi) = \pi^{-d/p}(\zeta - 1)\psi\chi.$$

But $\zeta - 1$ has value $m/p > d/p$, hence this commutator has positive value and $\bar{\alpha}, \bar{\psi}$ commute. This shows that $e_{D/F} = 1$ and D is wild.

Finally suppose that the depth of $\{x, y\}$ is m. Then we write $x = 1 + b\eta^p$ and $\beta = \eta^{-1}(\chi - 1)$ as in Example 20.1; again $\bar{\beta}^p - \bar{\beta} - \bar{b} = 0$. Further,

$$\beta\psi - \psi\beta = \eta^{-1}(\chi\psi - \psi\chi) = \psi\chi,$$

so

$$\bar{\beta}\bar{\psi} = \bar{\psi}(\bar{\beta} + 1).$$

The elements $\bar{\beta}$ and $\bar{\psi}$ generate a division algebra of dimension p^2 over its center $\overline{F}$ [18, p. 36], so it must be $\overline{D}$. Therefore, $e_{D/F} = 1$ and D is tame. □

22. Proofs of Theorem 19.2 and Corollary 19.3

The following proofs amount to translating results of §§8, 9 into K-theory.

Proof of Theorem 19.2. It suffices to establish (i) and (iii) since by Prop. 8.2(a) (iii) implies (ii).

Case $q = 1$: Suppose first that $q = 1$, i.e., $\gamma = \{\mu\}$ for some $\mu \in F^\times$ and Q is the 1-Pfister $\langle\langle\mu\rangle\rangle$. Put $P := \langle 1\rangle$; it is wild because $\operatorname{char}\overline{F} = 2$ and $\operatorname{texp} P = \lambda(2)$ by Example 8.7. We may assume that μ has value 0 or 1. If μ has value 1, then $\operatorname{depth}\gamma = 0$, $e_\gamma = 2$, and the theorem holds by Prop. 9.2.3. If μ has value 0 and $\bar{\mu}$ is a nonsquare in $\overline{F}$, then $\operatorname{depth}\gamma = 0$, $e_\gamma = 1$, and the theorem holds by Prop. 9.7.6.

Otherwise $\overline{\mu}$ is a square in $\overline{F}$, so multiplying μ by a square in F we may assume that $\overline{\mu} = 1$. If $\operatorname{depth}\gamma < 2\lambda(2)$, then (i) and (iii) hold by Prop. 9.6.5 or Prop. 9.7.6.

Finally suppose that $\operatorname{depth}\gamma = 2\lambda(2)$. The symbol γ corresponds to the quadratic extension $F(\sqrt{\mu})$ and the residue algebra of this extension was computed in Example 20.1; this verifies (i) and that Q is tame (hence (ii)). Then $\operatorname{texp} Q = 0$ by Prop. 8.2(d), proving (iii).

Case $q \geq 2$: We argue by induction on q and decompose γ as in Prop. 17.2 (with $p = 2$). Writing P (resp. Q) for the Pfister quadratic form corresponding to β (resp. γ), we conclude $Q \cong \langle\langle a\rangle\rangle \otimes P$ and put $d := \operatorname{depth}\gamma$. Then $e_{P/F} = 1$ and $\operatorname{texp}(P) = \lambda(2) - (\operatorname{depth}\beta)/2$ by the induction hypothesis. If alternative (ii) of Prop. 17.2 holds, then (9.2.3) shows $e_\gamma = 2 = e_{Q/F}$ and $\operatorname{texp}(Q) = \operatorname{texp}(P) = \lambda(2) - \left\lfloor \frac{\operatorname{depth}\gamma}{2} \right\rfloor$ since $\operatorname{depth}\gamma = \operatorname{depth}\beta$ is even. Now suppose alternative (i) of Prop. 17.2 holds. Since β has depth zero and ramification index 1, we can write $\beta = \{a_1, \ldots, a_{q-1}\}$, $a_i \in \mathfrak{o}^\times$, $1 \leq i < q$. If $d = 2\lambda(2)$, then $e_\gamma = 1$ and $\langle\langle a\rangle\rangle$ is a *tame* 1-Pfister quadratic subspace of $Q \cong \langle\langle a, a_1, \ldots, a_{q-1}\rangle\rangle$ with $e_{\langle\langle a\rangle\rangle/F} = 1$. Applying Prop. 9.3 $q - 1$ times yields $e_{Q/F} = 1 = e_\gamma$ and that Q is tame as well, forcing $\operatorname{texp}(Q) = 0 = \lambda(2) - \lfloor d/2\rfloor$. We are left with the case $0 \leq d < 2\lambda(2)$. Assertions (i) and (iii) follow by combining the effect of the Kato isomorphism at depth d on γ with 14.3 and (9.6.5) (for d odd) or with (9.7.6) and Cor. 9.10(a) (for d even). □

Proof of Corollary 19.3. We now prove Corollary 19.3. Claim (i) amounts to reformulating Proposition 17.3. For (ii), one combines Theorem 19.2(iii) relating depth γ with $\operatorname{texp} C$ with the relation between $\operatorname{texp} C$ and $\omega(C)$ demonstrated in the proof of Theorem 12.11. □

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