Center, Centroid, Extended Centroid and Quotients of Jordan Systems

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ABSTRACT: In this paper we prove that the extended centroid of a nondegenerate Jordan system is isomorphic to the centroid (and to the center in the case of Jordan algebras) of its maximal Martindale-like system of quotients with respect to the filter of all essential ideals.

Key words: Martindale quotients, centroid, extended centroid, center.

MSC2000: 17C10, 17B05.

Introduction

The notion of Martindale rings of quotients, jointly with the notion of extended centroid, play an important role in the study of prime rings satisfying a generalized polynomial identity, see [**11**].

This notion of quotients has been a useful tool in several branches of algebra, not only in the associative setting but also for non-associative structures such as Jordan systems or Lie algebras. For example, K. McCrimmon in 1989 [**8**] extended

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Martindale's construction to semiprime associative structures to obtain what he called the Martindale system of symmetric quotients, which played a central role in McCrimmon-Zelmanov's classification of strongly prime Jordan systems [**10**].

In a recent paper [**3**], the authors introduced this type of quotients for Jordan systems and called them Martindale-like Jordan systems of quotients ($\mathfrak{M}$-quotients, for short). In their work they showed the existence of maximal systems of $\mathfrak{M}$-quotients for nondegenerate Jordan systems when 6 was invertible in the ring of scalars, and later on, in a work jointly written with J. A. Anquela [**1**], they extended the construction of maximal algebras of $\mathfrak{M}$-quotients for strongly prime Jordan algebras when only $\frac{1}{2}$ belonged to the ring of scalars.

On the other hand, similar-like algebras of quotients for Lie algebras where defined by M. Siles in [**15**]. In [**3**] it is proved that they are related to Jordan systems of $\mathfrak{M}$-quotients through the Tits-Kantor-Koecher construction.

An importan result on Martindale associative quotients is that the extended centroid of a semiprime associative ring coincides with the center of its Martindale ring of quotients [**5**, 14.18 and 14.19]. This fact implies that the central closure of a semiprime associative ring R is the subring of its Martindale ring of quotients Q generated by R and the center of Q.

The goal of this paper is to give an analogue of this result for Jordan algebras, where Martindale rings of quotients are replaced by maximal Jordan algebras of $\mathfrak{M}$-quotients with respect to the filter $\mathcal{F}_e$ of all essential ideals of the algebra. Moreover, for nondegenerate Jordan pairs and triple systems T, where the notion of center is naturally replaced by the centroid, we obtain that the extended centroid of T is isomorphic to the centroid and to the extended centroid of the maximal system of $\mathfrak{M}$-quotients of T with respect to the filter $\mathcal{F}_e$ of all essential ideals of T. Our proof is made in two steps: firstly, we show that the extended centroid of T is isomorphic to the extended centroid of any system of $\mathfrak{M}$-quotients of T with respecto to any power filter of essential ideals, and, secondly, we prove that the centroid and the extended centroid of the maximal system of $\mathfrak{M}$-quotients of T with respect to $\mathcal{F}_e$ are isomorphic.

The paper is divided into three sections. In Section 1 we recall basic linear facts and linear notions on Jordan systems and Lie algebras over a ring of scalars Φ, $\frac{1}{2} \in \Phi$, including the notions of centroid and extended centroid. In Section 2 we study some properties of Jordan systems of $\mathfrak{M}$-quotients, showing, with a purely combinatorial proof, that the extended centroid does not change when considering

Jordan systems of $\mathfrak{M}$-quotients with respect to power filters of essential ideals. In the last section we show the coincidence of the centroid and the extended centroid for the maximal Jordan system of $\mathfrak{M}$-quotients of a nondegenerate Jordan systems (pairs or triple systems) with respect to the filter of all essential ideals. Moreover, in the case of algebras, we also obtain that the center, the centroid and the extended centroid of the maximal Jordan algebra of $\mathfrak{M}$-quotients coincide. Some of our arguments are done for Lie algebras since Jordan pairs and Lie algebras are easily connected through the Tits-Kantor-Koecher construction.

1. Preliminaries

1.1 We will deal with Jordan systems over a ring of scalars Φ, where $\frac{1}{2} \in \Phi$. The reader is referred to [**4**, **6**, **10**] for basic results, notation and terminology, though we will stress some notions and basic properties. The identities JPx listed in [**6**] will be quoted with their original numbering without explicit reference to [**6**].

—For a Jordan pair $V = (V^+, V^-)$ we will denote its linear products by $Q_{x,z}y = \{x, y, z\}$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. We will write $Q_x y = \frac{1}{2}\{x, y, x\}$.

—A Jordan linear triple system T is given by its products $P_{x,z}y = \{x, y, z\}$, $x, y, z \in T$. We also have $P_x y = \frac{1}{2}\{x, y, x\}$.

—Given a Jordan algebra J, its linear products will be denoted by $x \circ y$ and $U_{x,z}y = \{x, y, z\}$, for $x, y, z \in J$. In this case, $U_x y = \frac{1}{2}\{x, y, x\}$ and $x^2 = \frac{1}{2}(x \circ x)$.

A Jordan algebra J is said to be unital if there is an element $1 \in J$ satisfying $U_1 = \mathrm{Id}_J$ and $U_x 1 = x^2$, for any $x \in J$ (such an element can be shown to be unique and it is called the unit of J). Every Jordan algebra J imbeds in a unital Jordan algebra $\hat{J} = J \oplus \Phi 1$ called its (free) unitization [**10**, 0.6].

A Jordan system J is said to be nondegenerate if zero is the only absolute zero divisor, i.e., zero is the only $x \in J$ such that $U_x = 0$ (respectively, $Q_x = 0$ or $P_x = 0$). It is semiprime if it has no nonzero nilpotent ideals, and it is prime if it has no nonzero orthogonal ideals. Strongly prime Jordan systems are those that are nondegenerate and prime.

1.2 A Jordan algebra J gives rise to a Jordan triple system J_T by simply forgetting the squaring and letting $P = U$. Moreover, J is nondegenerate if and only if J_T is so. Conversely, if a Jordan triple system T has an element 1 with

$P_1 x = x$ for every $x \in T$, then it is really a unital Jordan algebra with product $U = P$ and $x \circ x = P_{x,x}1$ [**12**, 0.1].

By doubling any Jordan triple system T one obtains the double Jordan pair $V(T) = (T, T)$ with products $Q_{x,z}y = P_{x,z}y$, for any $x, y, z \in T$ [**6**, 1.13]. Moreover, T is a nondegenerate Jordan triple system if and only if $V(T)$ is nondegenerate. Reciprocally, each Jordan pair V gives rise to a polarized triple system $T(V) = V^+ \oplus V^-$. Niceness conditions such as nondegeneracy, primeness, strong primeness, and others are inherited by the polarized triple system of a Jordan pair. Moreover, V is a nondegenerate Jordan pair if and only if $T(V)$ is nondegenerate.

1.3 In a nondegenerate Jordan system J, the annihilator $\mathrm{Ann}_J(I)$ of an ideal I of J is an ideal of J, given by $\mathrm{Ann}_J(I) = \{x \in J \mid U_x I = 0\} = \{x \in J \mid U_I x = 0\}$ [**9**, 1.3, 1.7; **13**, 1.3].

An ideal I of J will be said sturdy if $\mathrm{Ann}_J(I) = 0$. It is easy to prove that essential ideals coincide with sturdy ideals in any semiprime Jordan system.

1.4 Given a Jordan triple system T, its centroid $\Gamma(T)$ consists of all maps $\gamma \in \mathrm{End}(T)$ such that $\gamma(\{x, y, z\}) = \{\gamma(x), y, z\} = \{x, \gamma(y), z\}$ for all $x, y, z \in T$. Similarly, when dealing with Jordan pairs $V = (V^+, V^-)$, a map $\gamma = (\gamma^+, \gamma^-) \in \mathrm{End}(V)$ belongs to its centroid $\Gamma(V)$ if $\gamma^\sigma(\{x^\sigma, y^{-\sigma}, z^\sigma\}) = \{\gamma^\sigma(x^\sigma), y^{-\sigma}, z^\sigma\} = \{x^\sigma, \gamma^{-\sigma}(y^{-\sigma}), z^\sigma\}$, for all $x^\sigma, z^\sigma \in V^\sigma$ and $y^{-\sigma} \in V^{-\sigma}$, $\sigma = \pm$. The condition for a linear map γ to belong to the centroid of a Jordan algebra J is $\gamma(x \circ y) = \gamma(x) \circ y$, for all $x, y \in J$.

The centroid of a Lie algebra L is the set of linear maps $\gamma : L \to L$ such that $\gamma([x, y]) = [\gamma(x), y]$ for all $x, y \in L$.

1.5 Let T be a Jordan triple system and let I be an ideal of T. A linear mapping $f : I \to T$ will be called a T-homomorphism if for all $y \in I$, $x, z \in T$ it satisfies:

(i) $f(\{y, x, z\}) = \{f(y), x, z\}$, and

(ii) $f(\{x, y, z\}) = \{x, f(y), z\}$.

The set of T-homomorphisms with domain I will be denoted $\mathrm{Hom}_T(I, T)$.

1.6 Similarly, for a Jordan pair $V = (V^+, V^-)$ and an ideal $I = (I^+, I^-)$, we define a V-homomorphism as a pair $f = (f^+, f^-)$ of linear mappings $f^\sigma : I^\sigma \to V^\sigma$, $\sigma = \pm$, satisfying, for all $y^\sigma \in I^\sigma$, $x^\sigma, z^\sigma \in V^\sigma$, $\sigma = \pm$,

(i) $f^\sigma(\{y^\sigma, x^{-\sigma}, z^\sigma\} = \{f^\sigma(y^\sigma), x^{-\sigma}, z^\sigma\}$, and

(ii) $f^\sigma(\{x^\sigma, y^{-\sigma}, z^\sigma\}) = \{x^\sigma, f^{-\sigma}(y^{-\sigma}), z^\sigma\}$.

We denote by $\mathrm{Hom}_V(I, V)$ the set of V-homomorphisms from I into V.

1.7 For a Jordan algebra J and an ideal I, an algebra J-homomorphism is a linear map that satisfies, for all $x \in J$ and all $y \in I$,

(iii) $f(y \circ x) = f(y) \circ x$.

Again we denote the set of J-homomorphisms defined on I by $\mathrm{Hom}_J(I, J)$.

Following [**12**], a pair (f, I), where $f \in \mathrm{Hom}_J(I, J)$, for a Jordan system (pair, triple, or algebra) will be called a permissible map if the ideal I is essential.

1.8 For permissible maps (f, I) and (g, L) of the Jordan system T, define a relation $\equiv$ by $(f, I) \equiv (g, L)$ if there is an essential ideal K of T, contained in $I \cap L$, such that $f(x) = g(x)$ for all $x \in K$. It is easy to see that this is an equivalence relation. The quotient set $C(T)$ will be called the extended centroid of T. We will write $[f, I]$ for the equivalence class of the permissible map (f, I).

For nondegenerate linear systems this set has a ring structure coming from the addition of homomorphisms and from the composition of restrictions of homomorphisms, see [**2**].

1.9 As for Jordan systems, we can define the extended centroid for Lie algebras. Given a Lie algebra L and an ideal I of L, an L-homomorphism with domain I is a linear map $f : I \to L$ such that for any $x \in L$ and any $y \in I$ we have $f([x, y]) = [x, f(y)]$. A pair (f, I), where f is an L-homomorphism with domain I, will be called a permissible map if the ideal I is essential. As before, we can define an equivalence relation $\equiv$ in the set of all permissible maps, such that the quotient $C(L)$ is called the extended centroid of L. If L is nondegenerate, $C(L)$ has also a ring structure through the addition of homomorphisms and the composition of restrictions of homomorphisms.

2. **Quotients and the extended centroid**

2.1 A *filter* $\mathcal{F}$ on a Jordan system or a Lie algebra is a nonempty family of nonzero ideals such that for any $I_1, I_2 \in \mathcal{F}$ there exists $I \in \mathcal{F}$ such that $I \subset I_1 \cap I_2$. Moreover, $\mathcal{F}$ is a *power filter* if for any $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $K \subset I^\#$, where $I^\# = [I, I]$ for Lie algebras, $I^\# = (Q_{I^+}I^-, Q_{I^-}I^+)$ for Jordan pairs, $I^\# = P_I I$ for Jordan triple systems, and $I^\# = I^2$ for Jordan algebras. We highlight the filter $\mathcal{F}_e$ of all essential ideals in V. When V is semiprime it is easy to show that $\mathcal{F}_e$ is a power filter, and that it coincides with the set $\mathcal{F}_s$ of all sturdy ideals in V.

2.2 Let T be a Jordan triple system and consider a filter $\mathcal{F}$ on T. We say that a Jordan triple system Q is a *triple system of Martindale-like quotients (triple system of $\mathfrak{M}$-quotients*, for short) of T with respect to $\mathcal{F}$ if the elements of Q are $\mathcal{F}$-absorbed into T, i.e., for each $0 \neq q \in Q$ there exists an ideal $I_q \in \mathcal{F}$ such that

$$0 \neq \{q, I_q, T\} + \{q, T, I_q\} + \{I_q, q, T\} \subset T.$$

Similarly, a *pair of Martindale-like quotients ($\mathfrak{M}$-quotients*, for short) of a Jordan pair V with respect to a filter $\mathcal{F}$ in V is Jordan pair Q whose elements are $\mathcal{F}$-absorbed into V (cf. [**3**, 2.5]), and an *algebra of Martindale-like quotients ($\mathfrak{M}$-quotients*, for short) of a Jordan algebra J with respect to a filter $\mathcal{F}$ in J is a Jordan algebra that whose elements satisfy the absorption property into J by ideals of $\mathcal{F}$ (cf. [**3**, 5.1]).

To simplify our calculations, in this section we are going to work with Jordan triple systems, though the results are also valid for Jordan algebras and pairs.

2.3 Lemma. *Let Q be a Jordan triple system of $\mathfrak{M}$-quotients of a Jordan triple system T with respect to a filter $\mathcal{F}$. Then $\{q, I, T\} + \{q, T, I\} + \{I, q, T\} \neq 0$ for any $0 \neq q \in Q$ and any sturdy ideal I in $\mathcal{F}$. Moreover, if $\mathcal{F}$ is a power filter, $\{q, I, I\} \neq 0 \neq \{I, q, I\}$.*

Proof. This lemma is mainly [**3**, 4.2]. If $\mathcal{F}$ is a power filter, we already know that $\{q, I, I\} + \{I, q, I\} \neq 0$ for any $I \in \mathcal{F}$.

Now suppose that $\{q, I, I\} = 0$. Then there exist $y_1, y_2 \in I$ with $\{y_1, q, y_2\} \neq 0$. Moreover, since I has zero annihilator,

$$0 \neq P_I\{y_1, q, y_2\} \subset \{I, I, \{q, I, I\}\} + \{P_I I, I, q\} = 0,$$

by JP13, leading to a contradiction, hence $\{q, I, I\} \neq 0$. Similarly, $\{I, q, I\}$ must be nonzero. ∎

2.4 Lemma. *Let Q be a Jordan triple system of $\mathfrak{M}$-quotients of a Jordan triple system T with respect to a filter $\mathcal{F}$. Given $u, w \in Q$ and an ideal K of T,*

(i) if $K' \in \mathcal{F}$ is an ideal of T that absorbs w, and $x \in (K \cap K')^3$, then both $\{w, x, u\}$ and $\{x, w, u\}$ belong to $\{K \cap K', T, u\} + \{T, K \cap K', u\}$,

(ii) if $K' \in \mathcal{F}$ is an ideal of T that absorbs w, and $t \in (K \cap K')^9$, then both $\{t, w, u\}$ and $\{w, t, u\}$ belong to $\{(K \cap K')^3, (K \cap K')^3, u\}$.

Proof. (i). By JP16, for $x_1, x_2, x_3 \in K \cap K'$,

$$\begin{aligned}\{w, \{x_1, x_2, x_3\}, u\} &= \{\{w, x_1, x_2\}, x_3, u\} + \{\{w, x_3, x_2\}, x_1, u\} \\ &\quad - \{x_2, \{x_1, w, x_3\}, u\} \in \{K \cap K', T, u\} + \{T, K \cap K', u\},\end{aligned}$$

$$\begin{aligned}\{\{x_1, x_2, x_3\}, w, u\} &= \{x_3, \{x_2, x_1, w\}, u\} - \{\{x_3, w, x_1\}, x_2, u\} \\ &\quad + \{x_1, \{x_2, x_3, w\}, u\} \in \{K \cap K', T, u\} + \{T, K \cap K', u\}.\end{aligned}$$

(ii). Let us consider $x_1, x_2, x_3 \in (K \cap K')^3$. By JP16 and (i),

$$\begin{aligned}\{w, \{x_1, x_2, x_3\}, u\} &= \{\{w, x_1, x_2\}, x_3, u\} + \{\{w, x_3, x_2\}, x_1, u\} \\ &\quad - \{x_2, \{x_1, w, x_3\}, u\} \in \{(K \cap K')^3, (K \cap K')^3, u\}\end{aligned}$$

since $(K \cap K')^3$ is a semi-ideal of (T, T). Similarly,

$$\begin{aligned}\{\{x_1, x_2, x_3\}, w, u\} &= \{x_3, \{x_2, x_1, w\}, u\} - \{\{x_3, w, x_1\}, x_2, u\} \\ &\quad + \{x_1, \{x_2, x_3, w\}, u\} \in \{(K \cap K')^3, (K \cap K')^3, u\}\end{aligned}$$

which completes the proof. ■

2.5 LEMMA. *Let Q be a Jordan system of $\mathfrak{M}$-quotients of a nondegenerate Jordan system T with respect to a filter of essential ideals $\mathcal{F}$. Then every essential ideal of Q hits T in an essential ideal of T.*

Proof. Let us suppose without loss of generality that T and Q are Jordan triple systems. Let $\mathcal{I}$ be an essential ideal of Q and define $I = \mathcal{I} \cap T$. Given $z \in \mathrm{Ann}_T(I)$ and $q \in \mathcal{I}$, if $P_z q \neq 0$, there exists an essential ideal $K \in \mathcal{F}$ that absorbs both q and $P_z q$ into T. Let $x \in K^3$ such that $0 \neq P_x P_z q \in T$. Now, since T is nondegenerate and K^3 is an essential ideal, $0 \neq P_{P_x P_z q} K^3 = P_x P_z P_q P_z P_x K^3 \subset P_x P_z(\mathcal{I} \cap T) = P_x P_z I = 0$, a contradiction. So $P_z q = 0$ for every $q \in \mathcal{I}$, which implies that $z = 0$, i.e., $\mathrm{Ann}_T(I) = 0$ and I is an essential ideal of T. ■

The following theorem is the main result of this section: extended centroids do not change when considering triples of $\mathfrak{M}$-quotients with respect to power filters of essential ideals.

2.6 THEOREM. *Let $T \leq Q$ be nondegenerate Jordan triple systems such that Q is a triple system of $\mathfrak{M}$-quotients of T with respect to a power filter of essential ideals. Then $C(T) \cong C(Q)$ (ring isomorphism).*

Proof. Let $[f,\mathcal{I}] \in C(Q)$, i.e., $\mathcal{I}$ is an essential ideal of Q and $f : \mathcal{I} \to Q$ is a permissible map.

It is trivial to prove that $I := f^{-1}(T) \cap T$ is an ideal of T.

Let us show that I is essential. Let J be a nonzero ideal of T. Since $\mathcal{I}\cap T$ is an essential ideal of T (2.5), we can consider $0 \neq x \in J \cap \mathcal{I} \subset T$. If $f(x) \in T$ we have finished; otherwise, since Q is a triple system of $\mathfrak{M}$-quotients of T and $f(x) \neq 0$ there exist $a, b \in T$ such that $0 \neq \{f(x), a, b\} \in T$, so $0 \neq \{x, a, b\} \in J \cap I$, because $f(\{x, a, b\}) = \{f(x), a, b\} \in T$.

Let us consider the map $\varphi : C(Q) \to C(T)$ defined by $\varphi([f,\mathcal{I}]) = [f|_I, I]$. Clearly it is well defined and it is a homomorphism of associative rings.

Let us show that φ is a monomorphism. Let $[f,\mathcal{I}] \in C(Q)$ be such that $0 = \varphi[f,\mathcal{I}] = [f|_I, I]$. Then there exists an essential ideal K of T contained in I with $f|_I(K) = 0$. Now consider the ideal $\mathrm{Id}_Q(K)$ of Q generated by K, and $\mathrm{Id}_Q(K) \cap \mathcal{I}$, which is an essential ideal of Q (since $\mathrm{Id}_Q(K)$ is an essential ideal of Q) with $f(\mathrm{Id}_Q(K) \cap \mathcal{I}) = 0$, which implies $[f,\mathcal{I}] = 0$, i. e., φ is a monomorphism of associative rings.

Now we prove that φ is an epimorphism. Let $[f, I] \in C(T)$. We want to define a permissible map on the ideal $\mathrm{Id}_Q(I)$ generated by I in Q. Given an element $\xi \in \mathrm{Id}_Q(I)$ we can express it as a sum of monomials consisting of an element of I multiplied an even number of times by elements of Q, i.e., $\xi = \sum_{i=1}^k a_i^{n_i}$, for $0 \leq n_1 \leq \ldots \leq n_i \leq \ldots \leq n_k$ and where $a_i^{n_i}$, $f(a_i)^n$, and the sets A_i^n and fA_i^n are defined inductively as follows:

$$
\begin{aligned}
a_i^0 \in A_i^0 =& \{a_i\} \subset I;\ f(a_i)^0 \in fA_i^0 = \{f(a_i^0) \mid a_i^0 \in A_i^0\};\\
a_i^{n+1} \in A_i^{n+1} =& \{\{v, w, a_i^n\} \mid v, w \in Q,\ a_i^n \in A_i^n\} \cup \\
& \{\{v, a_i^n, w\} \mid v, w \in Q,\ a_i^n \in A_i^n\};\\
f(a_i)^{n+1} \in fA_i^{n+1} =& \{\{v, w, f(a_i)^n\} \mid v, w \in Q,\ f(a_i)^n \in fA_i^n\} \cup \\
& \{\{v, f(a_i)^n, w\} \mid v, w \in Q,\ f(a_i)^n \in fA_i^n\}.
\end{aligned}
$$

Sometimes, to simplify the notation, we will forget the subscrips of the $a_i \in I$ and simply write a for elements of I, a^n for elements in A^n and $f(a)^n$ for elements in fA^n.

We define

$$
\overline{f} : \mathrm{Id}_Q(I) \to W \quad \text{by} \quad \overline{f}(\sum_{i=1}^k a_i^{n_i}) := \sum_{i=1}^k f(a_i)^{n_i}.
$$

Let us show that $\overline{f}$ is well defined, i.e., if $\xi = \sum_{i=1}^k a_i^{n_i} = 0$, then also $\overline{f}(\xi) = \sum_{i=1}^k f(a_i)^{n_i} = 0$. Then it is easy to see that $[\overline{f}, \mathrm{Id}_Q(I)] \in C(Q)$ and $\varphi([\overline{f}, \mathrm{Id}_Q(I)]) = [f, I]$, proving that φ is surjective.

We claim that it is enough to prove the following property (P): for every a^n there exists an essential ideal K of T such that for every $x, y \in K$,

$$\{x, y, a^n\} \in I \quad \text{and} \quad f(\{x, y, a^n\}) = \{x, y, f(a)^n\}. \tag{P}$$

Indeed, as soon as we have (P) then we can find a common absorbing essential ideal K of T for every summand $a_i^{n_i}$ appearing in $\xi = \sum_{i=1}^k a_i^{n_i} = 0$, i.e., for every $x, y \in K$

$$\{x, y, a_i^{n_i}\} \in I \quad \text{and} \quad f(\{x, y, a_i^{n_i}\}) = \{x, y, f(a_i)^{n_i}\}.$$

Suppose that $0 \neq f(\xi) = \sum_{i=1}^k f(a_i)^{n_i} \in Q$, so we can choose $K' \in \mathcal{F}$ such that

$$0 \neq \{K', K', \sum_{i=1}^k f(a_i)^{n_i}\} \subset T,$$

i.e., we have elements $v, w \in K'$ such that $0 \neq \eta = \{v, w, \sum_{i=1}^k f(a_i)^{n_i}\} \subset T$. But using (P), for every $x, y \in K$

$$0 = f(\{x, y, \{v, w, \xi\}\}) = f(\{x, y, \{v, w, \sum_{i=1}^k a_i^{n_i}\}\}) = \{x, y, \{v, w, \sum_{i=1}^k f(a_i)^{n_i}\}\},$$

hence $0 = \{K, K, \eta\}$, which is not possible since K is an essential ideal of T. We have proved that $0 = \sum_{i=1}^k a_i^{n_i} \in \mathrm{Id}_Q(I)$ implies $\sum_{i=1}^k f(a_i)^{n_i} = 0$, i.e., $\overline{f}$ is well defined.

Now let us prove property (P). We give a proof by induction on n:

If $n = 0$, the result follows for $K = T$ because f is a permissible map of T.

Suppose it is true for n and let K be an essential ideal of T such that for every $x, y \in K$

$$\{x, y, a^n\} \in I \quad \text{and} \quad f(\{x, y, a^n\}) = \{x, y, f(a)^n\}. \tag{P$_\mathrm{n}$}$$

Let $a^{n+1} := \{v, w, a^n\}$ for some $v, w \in Q$ and let us consider an ideal $K' \in \mathcal{F}$ that absorbs v, w and a^n into T, and let K_1 be an essential ideal of T contained in $(K' \cap K \cap I)^9$. Then for every $x, y, z \in K_1$, by JP16 we have:

$$\begin{aligned}\{x, y, \{v, w, a^n\}\} = \{x, \{y, v, w\}, a^n\} - \{x, w, \{v, y, a^n\}\} \\ + \{x, \{y, a^n, w\}, v\}.\end{aligned} \tag{I}$$

Let us study each summand separately.

(1). By (2.4)(i),(ii) and the induction hypothesis: $\{y,v,w\} \in \{(K'\cap K\cap I)^3,(K'\cap K\cap I)^3,w\} \subset (K'\cap K\cap I)^3 \subset K$, hence $\{x,\{y,v,w\},a^n\} \in \{K,K,a^n\} \subset I$. Therefore,

$$f(\{x,\{y,v,w\},a^n\}) = \{x,\{y,v,w\},f(a)^n\}. \qquad (\mathrm{S}_1)$$

(2). By (2.4)(i),(ii) and the induction hypothesis: $\{v,y,a^n\} \in \{(K'\cap K\cap I)^3,(K'\cap K\cap I)^3,a^n\} \subset (K'\cap K\cap I)^3 \subset I$, hence $\{x,w,\{v,y,a^n\}\} \in \{(K'\cap K\cap I)^3,(K'\cap K\cap I)^3,I\} \subset I$.

Now, for any $z \in (K'\cap K\cap I)^3$ and $x_1,x_2,x_3 \in (K'\cap K\cap I)$, by JP16,

$$\begin{aligned} &f(\{\{x_1,x_2,x_3\},w,z\}) = f(\{x_1,\{x_2,x_3,w\},z\}) + f(\{x_3,\{x_2,x_1,w\},z\}) \\ &- f(\{\{x_1,w,x_3\},x_2,z\}) = \{x_1,\{x_2,x_3,w\},f(z)\} + \{x_3,\{x_2,x_1,w\},f(z)\} \\ &- \{\{x_1,w,x_3\},x_2,f(z)\} = \{\{x_1,x_2,x_3\},w,f(z)\}, \end{aligned}$$

since $z \in I$ and $\{x_2,x_3,w\} \in T$, $\{x_2,x_1,w\} \in T$ and $\{x_1,w,x_3\} \in T$. Therefore,

$$f(\{x,w,z\}) = \{x,w,f(z)\}, \quad \forall\, x,z \in (K'\cap K\cap I)^3 \qquad (AS_2)$$

Now for every $y_1,y_2,y_3 \in (K'\cap K\cap I)^3$, by JP16,

$$\begin{aligned} &f(\{v,\{y_1,y_2,y_3\},a^n\}) = f(\{\{v,y_1,y_2\},y_3,a^n\}) + f(\{\{v,y_3,y_2\},y_1,a^n\}) \\ &- f(\{y_2,\{y_1,v,y_3\},a^n\}) = \{\{v,y_1,y_2\},y_3,f(a)^n\} + \{\{v,y_3,y_2\},y_1,f(a)^n\} \\ &- \{y_2,\{y_1,v,y_3\},f(a)^n\} = \{v,\{y_1,y_2,y_3\},f(a)^n\} \end{aligned}$$

because we can use the induction hypothesis in each summand (each summand belongs to I and consists of elements in K and a^n). Hence for every $y \in (K'\cap K\cap I)^9$ we obtain

$$f(\{v,y,a^n\}) = \{v,y,f(a)^n\}. \qquad (BS_2)$$

By (AS_2) we get $f(\{x,w,\{v,y,a^n\}\}) = \{x,w,f(\{v,y,a^n\})\}$ and together with (BS_2) we have

$$f(\{x,w,\{v,y,a^n\}\}) = \{x,w,\{v,y,f(a)^n\}\}. \qquad (\mathrm{S}_2)$$

(3). By (2.4)(i),(ii) and the induction hypothesis: $\{y,a^n,w\} \in \{(K'\cap K\cap I)^3,(K'\cap K\cap I)^3,w\} \subset (K'\cap K\cap I)^3$, hence $\{x,\{y,a^n,w\},v\} \in \{(K'\cap K\cap$

$I)^3, (K' \cap K \cap I)^3, v\} \subset I$. If we take any $b \in K_1$ and any $x, t \in (K' \cap K \cap I)^3$, and use JP13,

$$\begin{aligned} &P_b f(\{x,t,v\}) = f(P_b\{x,t,v\}) = f(\{b,x,\{t,v,b\}\}) - f(\{P_b x, v, t\}) \\ &= \{b,x,f(\{t,v,b\})\} - \{P_b x, v, f(t)\} = \{b,x,\{f(t),v,b\}\} - \{P_b x, v, f(t)\} \\ &= P_b\{x, f(t), v\} \end{aligned}$$

by (AS_2) (which is also true if we change w by v) applied to both terms. Since this holds for every $b \in K_1$, which is an essential ideal of T, we obtain

$$f(\{x,t,v\}) = \{x, f(t), v\} \tag{AS_3}$$

for every $x, t \in (K' \cap K \cap I)^3$.

On the other hand, for every $z_1, z_2, z_3 \in (K' \cap K \cap I)^3$ we have

$$\begin{aligned} &f(\{\{z_1,z_2,z_3\},w,a^n\}) = f(\{z_1,\{z_2,z_3,w\},a^n\}) + f(\{z_3,\{z_2,z_1,w\},a^n\}) \\ &- f(\{\{z_1,w,z_3\},z_2,a^n\}) = \{z_1,\{z_2,z_3,w\},f(a)^n\} + \{z_3,\{z_2,z_1,w\},f(a)^n\} \\ &- \{\{z_1,w,z_3\},z_2,f(a)^n\} = \{\{z_1,z_2,z_3\},w,f(a)^n\} \end{aligned}$$

because we can use the induction hypothesis in each summand (each summand belongs to I and consists of elements in K and a^n). Therefore, for $z \in K_1$ we get

$$f(\{z,w,a^n\} = \{z,w,f(a)^n\}. \tag{BS_3}$$

Now, for every $b, y \in K_1$ we have by JP13

$$\begin{aligned} P_b f(\{y,a^n,w\}) &= f(P_b\{y,a^n,w\}) = f(\{b,y,\{a^n,w,b\}\}) - f(\{P_b y, w, a^n\}) \\ &= \{b,y,f(\{a^n,w,b\})\} - \{P_b y, w, f(a)^n\} \\ &= \{b,y,\{f(a)^n,w,b\}\} - \{P_b y, w, f(a)^n\} \\ &= P_b\{y, f(a)^n, w\} \end{aligned}$$

since b and $P_b y$ belong to K_1 so we can apply (BS_3) in each summand. This formula holds for every $b \in K_1$, which is an essential ideal of T, so for every $y \in K_1$

$$f(\{y,a^n,w\}) = \{y, f(a)^n, w\}. \tag{CS_3}$$

Now putting together (AS_3) and (CS_3) we get that for every $x, y \in K_1$

$$f(\{x,\{y,a^n,w\},v\}) = \{x, f(\{y,a^n,w\}), v\} = \{x,\{y,f(a)^n,w\},v\}. \tag{S$_3$}$$

And putting together (I), (S_1), (S_2) and (S_3) we get that for every $x, y \in K_1$

$$\begin{gathered}\{x, y, \{v, w, a^n\}\} \in (K' \cap K \cap I)^3 \subset I, \text{ and} \\ f(\{x, y, \{v, w, a^n\}\}) = \{x, y, \{v, w, f(a)^n\}\},\end{gathered}$$

which is $(\mathrm{P})_{n+1}$ for $a^{n+1} := \{v, w, a^n\}$.

It remains to prove the same property $(\mathrm{P})_{n+1}$ for an element of the form $a^{n+1} = \{v, a^n, w\}$. For every $x, y \in K_1$, by JP16,

$$\begin{aligned}\{x, y, \{v, a^n, w\}\} &= \{x, \{y, v, a^n\}, w\} + \{x, \{y, w, a^n\}, v\} - \{x, a^n, \{v, y, w\}\} \\ &\in (K' \cap K \cap I)^3\end{aligned}$$

using (2.4) and the induction hypothesis as in (I). Moreover, by (AS_3), (BS_3) and the induction hypothesis,

$$\begin{aligned}&f(\{x, \{y, v, a^n\}, w\}) = \{x, f(\{y, v, a_n\}), w\} = \{x, \{y, v, f(a)^n\}, w\}, \\ &f(\{x, \{y, w, a^n\}, v\}) = \{x, f(\{y, w, a^n\}, v\} \text{ (by } (AS_3)) = \{x, \{y, w, f(a)^n\}\} \text{ and} \\ &f(\{x, a^n, \{v, y, w\}\}) = \{x, f(a)^n, \{v, y, w\}\},\end{aligned}$$

so $f(\{x, y, \{v, a^n, w\}\}) = \{x, y, \{v, f(a)^n, w\}\}$, i.e., for every $x, y \in K_1$

$$\begin{gathered}\{x, y, \{v, a^n, w\}\} \in (K' \cap K \cap I)^3 \subset I, \text{ and} \\ f(\{x, y, \{v, a^n, w\}\}) = \{x, y, \{v, f(a)^n, w\}\}.\end{gathered}$$

which completes the proof. ■

Now, it is natural to extend this theorem to Jordan pairs and algebras.

2.7 Corollary. *Let $V \leq W$ be nondegenerate Jordan pairs such that W is a pair of $\mathfrak{M}$-quotients of V with respect to a power filter of essential ideals. Then $C(V) \cong C(W)$ (ring isomorphism).*

Proof. If W is a Jordan pair of $\mathfrak{M}$-quotients of V with respect to a power filter of essential ideals $\mathcal{F}$, then the polarized triple system $T(W)$ associated to W is a triple system of $\mathfrak{M}$-quotients of the polarized triple system $T(V)$ with respect to the power filter of essential ideals $T(\mathcal{F}) = \{I^+ \oplus I^- \mid (I^+, I^-) \in \mathcal{F}\}$. Now, by (2.6), $C(T(V)) \cong C(T(W))$, while by [**13**, 2.7], $C(V) \cong C(T(V))$ and $C(W) \cong C(T(W))$, which completes the proof. ■

2.8 Corollary. *Let J be a nondegenerate Jordan algebra and let Q be an algebra of $\mathfrak{M}$-quotients of J with respect to a power filter of essential ideals. Then $C(J) \cong C(Q)$ (ring isomorphism).*

Proof. Let J_T and Q_T denote the underlying triple systems of J and Q respectively. It has been shown in [**3**, 5.2] that Q_T is a triple system of $\mathfrak{M}$-quotients of J_T with respect to the same filter of ideals, so by (2.6) $C(J_T) \cong C(Q_T)$. Moreover, by [**13**, 2.4] the extended centroid of a nondegenerate Jordan algebra is isomorphic to the extended centroid of its underlying triple system, which completes the proof. ∎

3. Centroids, extended centroids and centers

From now on, we will deal with rings of scalars Φ with $\frac{1}{2}$ and $\frac{1}{3}$ in Φ. The condition $\frac{1}{3} \in \Phi$ is needed for the existence of maximal systems of $\mathfrak{M}$-quotients of nondegenerate Jordan systems and Lie algebras (cf. [**3**, 5.4]). Moreover, in this section we will always consider quotients with respect to the filter $\mathcal{F}_e$ of all essential ideals of the Jordan system or Lie algebra.

The goal of this section is to show the coincidence of the center, the centroid and the extended centroid of the maximal algebra of $\mathfrak{M}$-quotients $Q_{max}(J)$ of a nondegenerate Jordan algebra J with respect to $\mathcal{F}_e$. With this, given such J with extended centroid $C(J)$, we will get that the central closure of J is the subalgebra of $Q_{max}(J)$ generated by J and the center of $Q_{max}(J)$.

Our proof will be done through Lie algebras by the Tits-Kantor-Koecher (TKK) construction [**14**, §1].

3.1 Recall that given a semiprime Lie algebra L, we can consider algebras of quotients Q of such L with respect to the filter of all essential ideals $\mathcal{F}_e$, in the sense that any nonzero element q of Q can be absorbed into L by an essential ideal: there exists an essential ideal I of L such that $0 \neq [q, I] \subset L$. Moreover, it was shown in [**15**, 3.4, 3.6] that the maximal algebra of quotients $Q_{max}(L)$ of L can be built, in the sense that it is an algebra of quotients of L and any other algebra of quotients of L can be imbedded in such $Q_{max}(L)$. Explicitly, $Q_{max}(L)$ consists of the direct limit of all derivations defined on essential ideals of L, i.e.,

$$Q_{max}(L) = \varinjlim \operatorname{Der}(I, L), \quad I \in \mathcal{F}_e.$$

The elements q of $Q_{max}(L)$ can be seen as equivalence classes $[d_q, I]$ where I is an essential ideal of L and $d_q : I \to L$ is a derivation. Two equivalence classes

$q = [d_q, I]$ and $p = [d_p, K]$ are equal if d_q and d_p coincide on a common domain (which must be an essential ideal of L). Notice that the Lie product in $Q_{max}(L)$ is $[p, q] = [[d_p, d_q], (I \cap K)^2]$, i.e., the derivation associated to $[p, q]$ is $d_{[p,q]} = [d_p, d_q]$.

The next result relates the centroid and the extended centroid of some ideals of $Q_{max}(L)$.

3.2 THEOREM. *Let L be a semiprime Lie algebra, let $\mathcal{Q} = Q_{max}(L)$ be its maximal algebra of quotients and $\mathcal{I}$ an essential ideal of $\mathcal{Q}$ with $L \subset \mathcal{I}$. Then*

(i) the map $\Psi_{\mathcal{I}} : \Gamma(\mathcal{I}) \to C(\mathcal{I})$ defined by $\Psi_{\mathcal{I}}(\gamma) = [\gamma, \mathcal{I}]$ is a monomorphisms of associative rings.

(ii) Let us suppose that every essential ideal of $\mathcal{I}$ hits L in an essential ideal. Then every $[f, \mathcal{I}'] \in C(\mathcal{I})$ can be extended to a map $\bar{f} : \mathcal{I} \to \mathcal{Q}$ such that for every $p, q \in \mathcal{I}$, $\bar{f}[p, q] = [\bar{f}(p), q]$. Moreover, if $\bar{f}(q) \in \mathcal{I}$ for every $q \in \mathcal{I}$ and $[f, \mathcal{I}'] \in C(\mathcal{I})$, $\Gamma(\mathcal{I})$ is isomorphic to $C(\mathcal{I})$.

Proof. (i). Let us consider the map

$$\Psi_{\mathcal{I}} : \Gamma(\mathcal{I}) \to C(\mathcal{I}) \quad \text{defined by} \quad \Psi_I(\gamma) = [\gamma, \mathcal{I}].$$

It is clear that $\Psi_{\mathcal{I}}$ is a homomorphism of associative rings. Moreover, since Q is semiprime, $\mathcal{I}$ is semiprime and $\Psi_{\mathcal{I}}$ is a monomorphism: if there exists $0 \neq \gamma \in \Gamma(\mathcal{I})$ with $\Psi_{\mathcal{I}}(\gamma) = 0$, then γ annihilates an essential ideal $\mathcal{K}$ of $\mathcal{I}$, i.e., $\gamma(\mathcal{K}) = 0$, and in this case, since $0 \neq \gamma(\mathcal{I})$ is an ideal of $\mathcal{I}$ (so $\gamma(\mathcal{I}) \cap \mathcal{K} \neq 0$), there exists $0 \neq \gamma(x) \in \mathcal{K}$, which implies that $\gamma^2(x) = 0$ and therefore the (nonzero) ideal of $\mathcal{I}$ generated by $\gamma(x)$ has zero square, a contradiction.

(ii). Given $[f, \mathcal{I}'] \in C(\mathcal{I})$ and $q = [d_q, I''] \in \mathcal{I}$, we define

$$\bar{f}(q) := [f \circ d_q, (I'' \cap f^{-1}(L))^2].$$

(1) Let us show that $\bar{f}(q)$ is an element of $\mathcal{Q}$. On the one hand, let us prove that $(I'' \cap f^{-1}(L))^2$ is an essential ideal of L. It is clear that we only have to show that $f^{-1}(L)$ is an essential ideal of L: for any ideal C of L, by hypothesis, $\mathcal{I}' \cap L$ is essential in L, so, we can take $0 \neq a \in \mathcal{I}' \cap C$. If $f(a) \in L$, $a \in f^{-1}(L) \cap C$; otherwise, $0 \neq f(a) \in \mathcal{I}$ implies that there exists $b \in L$ with $0 \neq [f(a), b] \in L$, so $0 \neq [a, b] \in f^{-1}(L) \cap C$. On the other hand, let us prove that $f \circ d_q$ is a derivation on $(I'' \cap f^{-1}(L))^2$: for any $x, y \in (I'' \cap f^{-1}(L))^2$, $(f \circ d_q)[x, y] = f([d_q(x), y]) + f([x, d_q(y)]) = [fd_q(x), y] + [x, fd_q(y)]$ because $d_q(x), d_q(y) \in \mathcal{I}'$. We have proved that $\bar{f}(q) \in Q$ for any $q \in \mathcal{I}$.

(2) Let us prove that the map $\bar{f}:\mathcal{I}\to\mathcal{Q}$ satisfies that for every $p,q\in\mathcal{I}$, $\bar{f}[p,q]=[\bar{f}(p),q]$:

(2.1) Given $p=[d_p,I'']\in\mathcal{I}$, f and d_p commute on $(I''\cap f^{-1}(L))^2$: for any $x\in(I''\cap f^{-1}(L))^2$, $(f\circ d_p)(x)=f([p,x])=[p,f(x)]=(d_p\circ f)(x)$, because $[p,x]\in[p,\mathcal{I}']\subset\mathcal{I}'$, $x\in\mathcal{I}'$, and $f(x)\in I''$.

(2.2) For any two elements of $\mathcal{I}$, $p=[d_p,I'_p]$ and $q=[d_q,I'_q]$, we want to show that $\bar{f}([p,q])=[\bar{f}(p),q]=[p,\bar{f}(q)]$, i.e., the maps $f\circ d_{[p,q]}$, $d_{[\bar{f}(p),q]}$ and $d_{[p,\bar{f}(q)]}$ coincide on an essential ideal of L. Take any $x\in(I'_p\cap I'_q\cap f^{-1}(L))^4$,

$$\begin{aligned}(f\circ d_{[p,q]})(x)&=\bar{f}([[p,q],x])=\bar{f}([[p,x],q])+\bar{f}([p,[q,x]])\\&=-fd_qd_p(x)+fd_pd_q(x)=[fd_p,d_q](x)+[d_q,f]d_p(x)\\&=d_{[\bar{f}(p),q]}(x),\end{aligned}$$

because d_q and f commute on $d_q(x)\in(I'_p\cap f^{-1}(L))^2$ by (1). Now, $(f\circ d_{[p,q]})(x)=-(f\circ d_{[q,p]})(x)=-d_{[\bar{f}(q),p]}(x)=d_{[p,\bar{f}(q)]}(x)$.

(3) Therefore, if $\bar{f}(q)\in\mathcal{I}$ for every $q\in\mathcal{I}$ and $[f,\mathcal{I}']\in C(\mathcal{I})$, given $[f,\mathcal{I}']\in C(\mathcal{I})$, we have that $[f,\mathcal{I}']=\Psi_{\mathcal{I}}(\bar{f})$ which implies that Ψ is an isomorphism of rings, i.e., $\Gamma(\mathcal{I})\cong C(\mathcal{I})$. ■

This theorem has an easy consequence for prime Lie algebras.

3.3 COROLLARY. *Let L be a prime Lie algebra and $\mathcal{Q}=Q_{max}(L)$ be its maximal algebra of quotients, then $\Gamma(\mathcal{Q})$ is isomorphic to $C(\mathcal{Q})$.* ■

Now we turn to Jordan pairs and use their connection with Lie algebras through the Tits-Kantor-Koecher construction. Recall that given a Jordan pair V, TKK(V) becomes a 3-graded Lie algebra with associated Jordan pair V. To use (3.2) for Jordan pairs, we need the following technical results.

3.4 LEMMA. *Let V be a Jordan pair and let* TKK(V) *be its Tits-Kantor-Koecher algebra. If we denote by $\Gamma(\ )$ the centroid, and by $C(\ )$ the extended centroid, then $\Gamma(V)\cong\Gamma(\mathrm{TKK}(V))$ and $C(V)\cong C(\mathrm{TKK}(V))$.*

Proof. The extension of the elements of $\Gamma(V)$ and $C(V)$ to the Lie algebra TKK(V) is done in the obvious way, taking into account that given an ideal $I=(I^+,I^-)$ of the Jordan pair V, the ideal of TKK(V) generated by I is $\tilde{I}=I^+\oplus([I^+,V^-]+[V^+,I^-])\oplus I^-$. ■

3.5 THEOREM. *Let T be a nondegenerate Jordan system and let Q be its* *maximal system of* $\mathfrak{M}$*-quotients. Then the extended centroid* $C(Q)$ *is isomorphic to the centroid* $\Gamma(Q)$ *and, in the case of Jordan algebras, also to the center of* Q*.*

Proof. Let us suppose that we are working with a Jordan pair T. Let us denote by $L = \mathrm{TKK}(T)$, the Tits-Kantor-Koecher algebra of T. The maximal Lie algebra of quotients $Q_{max}(L)$ of L with respect to the filter $\mathcal{F}_e$ is a 3-graded Lie algebra, $Q_{max}(L) = \mathcal{Q}_{-1} \oplus \mathcal{Q}_0 \oplus \mathcal{Q}_1$, see [**3**, 2.4], and $Q \cong (\mathcal{Q}_{-1}, \mathcal{Q}_1)$ is the maximal pair of $\mathfrak{M}$-quotients of T with respect to $\mathcal{F}_e$, see [**3**, 3.2].

Since $Q_{max}(L)$ is nondegenerate, the ideal $\mathcal{I}$ of $Q_{max}(L)$ generated by Q is isomorphic to $\mathrm{TKK}(Q)$ and is an essential ideal of $Q_{max}(L)$ which contains L. Let us show that $\mathcal{I} \cong \mathrm{TKK}(Q)$ satisfies the hypothesis of (3.2)(ii),

(I) $\mathcal{I} \cap L$ is an essential ideal of L: since $\mathcal{I}$ is an essential ideal of $\mathcal{Q}$, $\mathcal{I} \cap Q$ is a essential ideal of Q. Now, by (2.5), $\mathcal{I} \cap Q \cap T$ is an essential ideal of T which implies that $\mathcal{I} \cap L$ is an essential ideal of L.

(II) $\bar{f}(\mathcal{I}) \subset \mathcal{I}$: Given $q_1 = [d_{q_1}, I''] \in \mathcal{I}_1$ and $[f, \mathcal{I}'] \in C(\mathcal{I})$, let us show that $\bar{f}(q_1) := [f \circ d_{q_1}(I'' \cap f^{-1}(L))^2] \in \mathcal{I}_1$. If we put $\bar{f}(q_1) = p_1 + p_0 + p_{-1} \in \mathcal{Q}$, then

$$0 = \bar{f}([q_1, \mathcal{I}_1]) = [\bar{f}(q_1), \mathcal{I}_1] \subset [p_0, \mathcal{I}_1] + [p_{-1}, \mathcal{I}_1] \in \mathcal{I}_1 \oplus \mathcal{I}_0,$$

so $[p_0, \mathcal{I}_1] = 0$ and $[p_{-1}, \mathcal{I}_0] = 0$, and, since $\mathcal{I}$ is nondegenerate, $p_{-1} = 0$. Therefore, $\bar{f}(\mathcal{I}_1) \subset \mathcal{I}_0 \oplus \mathcal{I}_1$. Similarly, $\bar{f}(\mathcal{I}_{-1}) \subset \mathcal{I}_0 \oplus \mathcal{I}_{-1}$. We know that $[\bar{f}(q_1), q_{-1}] = [q_1, \bar{f}(q_{-1})]$, but $[\bar{f}(q_1), q_{-1}] \in \mathcal{I}_{-1} \oplus \mathcal{I}_0$ and $[q_1, \bar{f}(q_{-1})] \in \mathcal{I}_0 \oplus \mathcal{I}_1$ implies that $[\bar{f}(q_1), q_{-1}] \in \mathcal{I}_0$ and therefore $[p_0, \mathcal{I}_{-1}] = 0$. So $[p_0, \mathcal{I}_1 \cup \mathcal{I}_{-1}] = 0$, which gives $[p_0, \mathcal{I}] = 0$ and, therefore, $p_0 = 0$. Hence $\bar{f}(q_1) \in \mathcal{I}_1$. Notice that $\bar{f}(q_{-1}) \in \mathcal{I}_{-1}$ follows in a similar way, which implies that $\bar{f}(\mathcal{I}) \subset \mathcal{I}$.

Then, by (3.2)(ii), $\Gamma(\mathrm{TKK}(Q)) \cong \Gamma(\mathcal{I}) \cong C(\mathcal{I}) \cong C(\mathrm{TKK}(Q))$. Moreover, the centroid (extended centroid) of a Jordan pair is isomorphic to the centroid (extended centroid) of its Tits-Kantor-Koecher algebra (3.4), so we get

$$\Gamma(Q) \cong \Gamma(\mathrm{TKK}(Q)) \cong C(\mathrm{TKK}(Q)) \cong C(Q).$$

Since the centroid and the extended centroid of a Jordan triple system are isomorphic to the centroid and the extended centroid of its double Jordan pair, we obtain the result for Jordan triple systems, and since the centroid and the extended centroid of a Jordan algebra are isomorphic to the centroid and the extended centroid of its associated Jordan triple system, we obtain the result for Jordan algebras.

Finally, by [**3**, 5.5], the maximal Jordan algebra Q of $\mathfrak{M}$-quotients of a nondegenerate Jordan algebra is unital, and we have, by [**7**, pag. 156], that the centroid of Q is essentially the center of Q. ■

Putting together (2.6) and (3.5) we have the main result of this work.

3.6 COROLLARY. *Let T be a nondegenerate Jordan system and let Q be its maximal system of $\mathfrak{M}$-quotients with respect to $\mathcal{F}_e$. Then:*

(i) $C(T) \cong C(Q) \cong \Gamma(Q)$.

(ii) *If T is a Jordan algebra,* $C(T) \cong C(Q) \cong \Gamma(Q) \cong Z(Q) \cdot \mathrm{Id}$, *where* Id *denotes the identity map in Q.* ■

3.7 COROLLARY. *Let J be a nondegenerate Jordan algebra and let Q be its maximal algebra of $\mathfrak{M}$-quotients with respect to $\mathcal{F}_e$. Then the central closure of J is the subalgebra of Q generated by J and the center of Q.* ■

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