A COMBINATORIAL CONJECTURE RELATED WITH COMPLEX BOUNDED SYMMETRIC DOMAINS

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1. Motivations

Let Ω be a bounded irreducible symmetric domain. To each such Ω are attached numerical invariants: the *multiplicities* a and b, and the *rank* r. These three invariants characterize the domain up to isomorphism. Hereunder is the table of all possible values, corresponding to the classification of irreducible complex bounded symmetric domains:

Type	a	b	r	g	d
$I_{m,n}$ $(1 \leq m \leq n)$	2	$n-m$	m	$m+n$	mn
II_n $(n \geq 2)$	4	$\begin{cases} 0 & (n=2p) \\ 2 & (n=2p+1) \end{cases}$	$\left[\frac{n}{2}\right]$	$2(n-1)$	$\frac{n(n-1)}{2}$
III_n $(n \geq 1)$	1	0	n	$n+1$	$\frac{n(n+1)}{2}$
IV_n $(n > 2)$	$n-2$	0	2	n	n
V	6	4	2	12	16
VI	8	0	3	18	27

In the last two columns of this table

$$g = 2 + a(r-1) + b$$

is the *genus* of the bounded symmetric domain Ω and

$$d = \dim_{\mathbb{C}} \Omega = r + \frac{r(r-1)}{2}a + rb$$

its complex dimension.

The *Hua polynomial* χ of the bounded symmetric domain Ω is then defined by

$$\chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}, \tag{1}$$

where $(s)_k$ denotes the *raising factorial*

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)}.$$

The polynomial χ is related to the *Hua integral* $\int_\Omega N(z,z)^s \omega(z)$ (where N is the *generic norm* for Ω) by

$$\int_\Omega N(z,z)^s \omega(z) = \frac{\chi(0)}{\chi(s)} \int_\Omega \omega \qquad (s > -1).$$

(see [1]).

One checks easily that $\deg \chi = d$. The expression of χ for the different types of bounded symmetic domains is as follows:

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- Type $\mathrm{I}_{m,n}$ $(1 \le m \le n)$: $\chi(s) = \prod_{j=1}^{m} (s+j)_n$.
- *Type* II_{2p}: $\chi(s) = \prod_{j=1}^{p} (s+2j-1)_{2p-1}$.
- *Type* II_{2p+1}: $\chi(s) = \prod_{j=1}^{p} (s+2j-1)_{2p+1}$.
- *Type* III_n: $\chi(s) = \prod_{j=1}^{n} \left(s + \frac{j+1}{2}\right)_{1+n-j}$.
- *Type* IV_n: $\chi(s) = (s+1)_{n-1} \left(s + \frac{n}{2}\right)$.
- *Type* V: $\chi(s) = (s+1)_8 (s+4)_8$.
- *Type* VI: $\chi(s) = (s+1)_9 (s+5)_9 (s+9)_9$.

Let $\mu \in \mathbb{R}$, $\mu > 0$. The following expansion

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j} \frac{(k+1)_j}{j!}. \tag{2}$$

and the associated rational function

$$F_{\chi,\mu}(t) = \sum_{j=0}^{d} c_{\mu,j} \left(\frac{1}{1-t}\right)^j \tag{3}$$

play a key role in the computation of the Bergman kernel of some Hartogs domains built over bounded symmetric domains (see [1], [2]).

2. The conjecture

Let

$$\mu_0 = \frac{g}{d+1}.$$

Conjecture. *The coefficients $c_{\mu,j}$ in (2):*

$$\chi(\mu s) = \sum_{j=0}^{d} c_{\mu,j} (s+1)_j$$

are all strictly positive if and only if

$$\mu < \mu_0.$$

For $\mu = \mu_0$, all coefficients $c_{\mu,j}$ in (2) are strictly positive, except $c_{\mu,d-1} = 0$ and except for the rank 1 type $I_{1,n}$ (where $c_{\mu,d} = 1$ and $c_{\mu,j} = 0$ for all $j < d$).

We call μ_0 the *critical exponent* for Ω. The values of the critical exponent are

Type	$I_{m,n}$	II_n	III_n	IV_n	V	VI
μ_0	$\frac{m+n}{mn+1}$	$\frac{4}{n+\frac{2}{n-1}}$	$\frac{2}{n+\frac{1}{n+1}}$	$\frac{n}{n+1}$	$\frac{12}{17}$	$\frac{9}{14}$

We have always $\mu_0 < 1$, except in the rank 1 case $I_{1,n}$.

Remark 1. The conjecture has been checked with help of computer algebra software in many significant cases:

- for $\mu = \mu_0$ and the types $I_{3,3}$, IV_3, IV_4, IV_6, V, VI;
- for type V and various values of μ.

Remark 2. As the function $F_{\chi,\mu}$ is related to the Bergman kernel of a family of bounded (non homogeneous) domains, it is known that all derivatives $F_{\chi,\mu}^{(k)}$, $k > 0$, of this function are strictly positive on $[0,1[$ for all $\mu > 0$.

Remark 3. If the conjecture is true, it would allow to compare the Bergman metric of some Hartogs domains built over bounded symmetric domains, with the Kähler-Einstein metric of the same domains.

References

[1] Yin Weiping, Lu Keping, Roos Guy, New classes of domains with explicit Bergman kernel, *Science in China* Ser. A Mathematics, **47**(2004), 352–371.

[2] Roos, Guy, Weighted Bergman kernels and virtual Bergman kernels, *Proceedings SCV2004 Beijing*, to appear. *Jordan archive, Preprint* #169.

[3] Wang An, Yin Weiping, Zhang Liyou, Roos Guy, The Kähler–Einstein metric for some Hartogs domains over bounded symmetric domains, *in preparation.*

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