ON LOCALLY FINITE SPLIT LIE TRIPLE SYSTEMS

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Abstract

Lie triple systems appear as the natural ternary extension of Lie algebras. The classification in the finite-dimensional setup (over an algebraically closed field of characteristic zero) is well-known (see [1]). In order to suggest a possible approach to a structure theory of infinite-dimensional Lie triple systems, we introduce and study split and locally finite Lie triple systems, stating that under certain conditions the standard embedding of a split Lie triple system is a split Lie algebra and that the standard embedding of a locally finite Lie triple system is a locally finite Lie algebra. We also give a description of certain locally finite simple split Lie triple systems.

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1 INTRODUCTION

The main tool used by Lister ([1]) to study finite-dimensional Lie triple systems is the standard embedding, this one being a two-graded Lie algebra $L = L^0 \oplus$

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L^1, with L^0 the span of $\{\mathcal{L}(x,y) : x, y \in T\}$, where $\mathcal{L}(x,y)$ denotes the left multiplication operator in T, $L^1 := T$, and where the product is given by

$$[(\mathcal{L}(x,y),z),(\mathcal{L}(u,v),w)] :=$$

$$= (\mathcal{L}([u,v,y],x) - \mathcal{L}([u,v,x],y) + \mathcal{L}(z,w), [x,y,w] - [u,v,z]).$$

Later, Faulkner gives in [2] an alternative approach to the classification of Lie triple systems, also in the finite-dimensional setup, by introducing a Cartan subalgebra H^0 of the even part of the standard embedding, and a decomposition of T as the direct sum of certain root spaces relative to H^0.

In the framework of infinite-dimensional Lie algebras, K. H. Neeb, N. Stumme and other authors have successfully developed over the recent years a theory of split and locally finite Lie algebras (c. f. [3, 4, 5, 6, 7, 8, 9, 10]).

In this paper, we combine all the above ideas in order to introduce and study split Lie triple systems of arbitrary dimension in sections 2 and 3, the main result is Theorem 3.1 which states that under certain conditions the standard embedding of a split Lie triple system is a split Lie algebra. In section 4 we also state that the standard embedding of a locally finite Lie triple system is a locally finite Lie algebra (Proposition 4.1), and obtain in Theorem 4.1 a description of an important class of locally finite split simple Lie triple systems.

2 BASIC DEFINITIONS

Let $\mathbb{K}$ be a field of characteristic zero and let T be a vector space over $\mathbb{K}$. We say that T is a *triple system* if it is endowed with a trilinear map

$$< \cdot, \cdot, \cdot >: T \times T \times T \to T,$$

called the *triple product* of T.

A triple system T is called a *Lie triple system* if its triple product, denoted by $[\cdot, \cdot, \cdot]$, satisfies

1. $[x, x, y] = 0$
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity)
3. $[x, y, [a, b, c]] - [a, b, [x, y, c]] = [[x, y, a], b, c] + [a, [x, y, b], c]$

for any $x, y, z, a, b, c \in T$.

An *ideal* of a Lie triple system T is a subspace I for which $[I, T, T] \subseteq I$. Let us observe that $[I, T, T] \subseteq I$ implies that $[T, I, T] \subseteq I$ and $[T, T, I] \subseteq I$. A Lie triple system T is called *simple* if the product is nonzero and its only ideals are $\{0\}$ and T.

A *two-graded* $\mathbb{K}$*-algebra* A is a $\mathbb{K}$-algebra which splits into the direct sum $A = A^0 \oplus A^1$ of subspaces (called the even and the odd part respectively) satisfying $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for any α, β in $\mathbb{Z}_2$. A homomorphism f between two-graded algebras A and B is a homomorphism which preserves gradings i.e.,

$$f(A^\alpha) \subset B^\alpha$$

for all $\alpha \in \mathbb{Z}_2$. A *graded ideal* of a two-graded algebra $A = A^0 \oplus A^1$ is an ideal I of A such that there exist two subspaces I^0 and I^1 with $I = I^0 \oplus I^1$ and $I^\alpha \subseteq A^\alpha$ for any $\alpha \in \mathbb{Z}_2$. A two-graded algebra A is *graded-simple* if the product is nonzero and its only graded ideals are $\{0\}$ and A.

The *standard embedding* of a Lie triple system T is the two-graded Lie algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{\mathcal{L}(x,y) : x, y \in T\}$, where $\mathcal{L}(x,y)$ denotes the left multiplication operator in T, $\mathcal{L}(x,y)(z) := [x,y,z]$; $L^1 := T$ and where the product is given by

$$[(\mathcal{L}(x,y), z), (\mathcal{L}(u,v), w)] :=$$

$$= (\mathcal{L}([u,v,y],x) - \mathcal{L}([u,v,x],y) + \mathcal{L}(z,w), [x,y,w] - [u,v,z]).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a Lie algebra.

By using the same arguments as in the finite dimensional case, (see [1, Theorem 2.13]), we can prove the following propositions

Proposition 2.1 *A Lie triple system T is simple if and only if its standard embedding is graded-simple.*

Proposition 2.2 *Let $L = L^0 \oplus L^1$ be a graded-simple two-graded Lie algebra. Then either*

1. *L is simple (in ungraded sense) and the grading is given by*

   $$L = Sym(L, \xi) \oplus Skw(L, \xi)$$

   where ξ is an involutive automorphism of L, or

2. *L is isomorphic to $L' \oplus L'$ where L' is a simple Lie algebra and the product is*

   $$[(a,b),(c,d)] = ([a,c] + [b,d], [a,d] + [b,c]).$$

3 SPLIT LIE TRIPLE SYSTEMS

3.1 Definitions and Basic Properties

Given an element x of a Lie algebra L, we denote by $ad(x)$ the *adjoint mapping* defined as $ad(x)(y) = [x,y]$ for any $y \in L$.

Following [7, 10, 11, 12], we recall that a *splitting Cartan subalgebra H of a Lie algebra L* is defined as a maximal abelian subalgebra, (MASA), of

L satisfying that the adjoint mappings $ad(h)$ for $h \in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a *split Lie algebra*. This means that we have a *root decomposition* $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ where $L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}$ for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\} : L_\alpha \neq 0\}$. The subspaces L_α for $\alpha \in H^*$ are called *root spaces* of L (respect to H) and the elements $\alpha \in \Lambda \cup \{0\}$ are called *roots* of L respect to H.

Definition 3.1 *Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. Following the ideas in [2], for a linear functional $\alpha \in (H^0)^*$ we define the root space of T (respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [h, t_\alpha] = \alpha(h)t_\alpha \text{ for any } h \in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^* \backslash \{0\} : T_\alpha \neq 0\}$.*

Let us observe that $T_0 = \{t_0 \in T : [h, t_0] = 0 \text{ for any } h \in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [h, v^0_\alpha] = \alpha(h)v^0_\alpha \text{ for any } h \in H^0\} \neq 0$.

Lemma 3.1 *Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. If $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\delta \in \Lambda^0 \cup \{0\}$. Then,*

1. *If $[T_\alpha, T_\beta] \neq 0$ then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$.*
2. *If $[L^0_\delta, T_\alpha] \neq 0$ then $\delta + \alpha \in \Lambda^1 \cup \{0\}$ and $[L^0_\delta, T_\alpha] \subseteq T_{\delta+\alpha}$*
3. *If $[T_\alpha, T_\beta, T_\gamma] \neq 0$ then $\alpha + \beta + \gamma \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{\alpha+\beta+\gamma}$.*

Proof. 1. For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, we have $[h, [x, y]] = -[[h, y], x] + [[h, x], y] = (\alpha + \beta)(h)[x, y]$. Therefore $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$.

2. and 3. The proof is similar to 1.□

Let us observe that if $[T_\alpha, T_0] \neq 0$ then α is a root of L^0 relative to H^0 and if $[L^0_\alpha, T_0] \neq 0$ then α is a root of T relative to H^0.

Definition 3.2 *Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. We shall call that T is a split Lie triple system (respect to H^0) if:*

1. $T = T_0 \oplus (\bigoplus_{\alpha \in \Lambda^1} T_\alpha)$,
2. $[T_0, T_0, T] = 0$.

Proposition 3.1 *Let T be a split Lie triple system respect to H^0, then L^0 is a split Lie algebra respect to the splitting Cartan subalgebra H^0.*

Proof. We first prove that $\sum_{\beta\in\Lambda^0} L^0_\beta$ is a direct sum, (let us note that we can also apply [13, 2.1 Prop.1]). If $x \in L^0_\alpha \cap \sum_{\beta\neq\alpha} L^0_\beta$, we can write $x = \sum_{\beta\neq\alpha} x_\beta$ with $x_\beta \in L^0_\beta$, and $x = x_\alpha$ with $x_\alpha \in L^0_\alpha$. From Lemma 3.1, we deduce for any $y \in T_\gamma$ and $\gamma \in \Lambda^1 \cup \{0\}$, that

$$[x,y] \in T_{\alpha+\gamma} \cap \sum_{\beta\neq\alpha} T_{\beta+\gamma}$$

and therefore $[x,y] = 0$ for any $y \in T$. As x is a linear map on T, satisfying $x(y) = [x,y]$, we deduce that $x = 0$. Let us also check that $H^0 + (\bigoplus_{\beta\in\Lambda^0} L^0_\beta)$ is a direct sum. If $x \in H^0 \cap (\bigoplus_{\beta\in\Lambda^0} L^0_\beta)$ we can write $x = \sum_{\beta\in\Lambda^0} x_\beta$, with $x_\beta \in L^0_\beta$, and $x = h^0 \in H^0$. For any $\beta \in \Lambda^0$, we can take $h_\beta \in H^0$, such that $\beta(h_\beta) \neq 0$, and then

$$0 = [h_\beta, h^0] = [h_\beta, x] = \sum_{\beta\in\Lambda^0} [h_\beta, x_\beta] = \bigoplus_{\beta\in\Lambda^0} \beta(h_\beta) x_\beta,$$

therefore $x_\beta = 0$. We conclude $x = 0$. Thus $H^0 \oplus (\bigoplus_{\beta\in\Lambda^0} L^0_\beta) \subseteq L^0$.

Let us observe that the maximal abelian character of H^0 in L^0 implies $H^0 = L^0_0$.

Finally, as

$$L^0 = [T,T] = \sum_{\alpha,\gamma\in\Lambda^1\cup\{0\}} [T_\alpha, T_\gamma]$$
$$\subseteq \sum_{\alpha,\gamma\in\Lambda^1\cup\{0\}} L^0_{\alpha+\gamma} \subseteq H^0 \oplus (\bigoplus_{\beta\in\Lambda^0} L^0_\beta),$$

the proof is complete.□

3.2 Types of Roots. Integrable Roots

In this subsection we distinguish the various types in the root system of a split Lie triple system, taking particular notice of the integrable roots.

Lemma 3.2 *For nonzero root vectors $t_{\pm\alpha} \in T_{\pm\alpha}$ the subalgebra of the standard embedding,*

$$L(t_\alpha, t_{-\alpha}) := span_{\mathbb{K}}\{t_\alpha, t_{-\alpha}, [t_\alpha, t_{-\alpha}]\}$$

is of one of the following types:

1. *If $[t_\alpha, t_{-\alpha}] = 0$, then $L(t_\alpha, t_{-\alpha})$ is two dimensional abelian. We say that $L(t_\alpha, t_{-\alpha})$ is of abelian type.*

2. *If $[t_\alpha, t_{-\alpha}] \neq 0$ but $\alpha([t_\alpha, t_{-\alpha}]) = 0$, then $L(t_\alpha, t_{-\alpha})$ is a three dimensional algebra. We say that $L(t_\alpha, t_{-\alpha})$ is of nilpotent type.*

3. *If $\alpha([t_\alpha, t_{-\alpha}]) \neq 0$, then $L(t_\alpha, t_{-\alpha}) \approx sl(2, \mathbb{K})$. We say that $L(t_\alpha, t_{-\alpha})$ is of simple type.*

Proof. This is an easy verification (cf. [10, Lemma I.2]).□

Definition 3.3 *For a root $\alpha \in \Lambda^1$ the subalgebras $L(t_\alpha, t_{-\alpha})$ are called test algebras associated to α. We say that a root $\alpha \in \Lambda^1$ is of nilpotent type if all test algebras associated to α are of abelian or nilpotent type. Note that a root $\alpha \in \Lambda^1$ with -$\alpha \notin \Lambda^1$ is of nilpotent type. We call a root $\alpha \in \Lambda^1$ of simple type if there exists an associated test algebra of simple type. A root $\alpha \in \Lambda^1$ of simple type is called integrable if there exists an associated test algebra $L(t_\alpha, t_{-\alpha})$ of simple type such that the middle multiplication operators of T,*

$$\mathcal{M}(t_{\epsilon\alpha}) : T \to T,$$

$\mathcal{M}(t_{\epsilon\alpha})(t) := [t_{\epsilon\alpha}, t, t_{\epsilon\alpha}]$, $\epsilon \in \{+, -\}$, are locally nilpotent. We denote by Λ^1_i the set of all integrable nonzero roots of T relative to H^0.

Lemma 3.3 *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1$. Then we have:*

1. *$\mathcal{M}(t_\alpha)(t) = -ad^2(t_\alpha)(t)$, for any $t_\alpha \in T_\alpha$ and for any $t \in T$.*
2. *$ad^{3+2n}(t_\alpha)([t, t']) = (-1)^n \mathcal{M}^{n+1}(t_\alpha)([t, t', t_\alpha])$, for any $n \in \mathbb{N}$, $t_\alpha \in T_\alpha$ and $t, t' \in T$.*

Proof.

1. It is clear.
2. By induction on n:

   For $n = 0$, we have $ad^3(t_\alpha)([t, t']) = [t_\alpha, [t_\alpha, [t_\alpha, [t, t']]]] = [t_\alpha, [t, t', t_\alpha], t_\alpha] = \mathcal{M}(t_\alpha)[t, t', t_\alpha]$.

   Assume 2 holds for n, let us prove it for $n + 1$. We have

   $$ad^{3+2(n+1)}(t_\alpha)([t, t']) = ad^2(t_\alpha)(ad^{3+2n}(t_\alpha)[t, t']) =$$

   $$= ad^2(t_\alpha)((-1)^n \mathcal{M}^{n+1}(t_\alpha)([t, t', t_\alpha])),$$

   and by applying 1,

   $$ad^2(t_\alpha)((-1)^n \mathcal{M}^{n+1}(t_\alpha)([t, t', t_\alpha])) = -\mathcal{M}(t_\alpha)((-1)^n \mathcal{M}^{n+1}(t_\alpha)([t, t', t_\alpha])) =$$

   $$= (-1)^{n+1} \mathcal{M}^{n+2}(t_\alpha)([t, t', t_\alpha]).$$

   The proof is complete.□

Proposition 3.2 *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1$. Then, for any $t_\alpha \in T_\alpha$, $\mathcal{M}(t_\alpha)$ is locally nilpotent on T if and only if $ad(t_\alpha)$ is locally nilpotent on L.*

Proof. Suppose that $\mathcal{M}(t_\alpha)$ is locally nilpotent on T. Let be $x = x^0 + x^1 \in L$, with $x^0 \in L^0$ and $x^1 \in L^1$. There exists $m \in \mathbb{N}$ such that $\mathcal{M}^m(t_\alpha)(x^1) = 0$. By Lemma 3.3-1 we also have $ad^{2m}(t_\alpha)(x^1) = 0$. As $x^0 = \sum_{i=1}^q [t_i, t_i']$ with $t_i, t_i' \in T$, if we consider for each $[t_i, t_i']$ the element $[t_i, t_i', t_\alpha] \in T$, there exists $m_i \in \mathbb{N}$ such that $\mathcal{M}^{m_i}([t_i, t_i', t_\alpha]) = 0$, and by Lemma 3.3-2, $ad^{3+2(m_i-1)}(t_\alpha)([t_i, t_i']) = 0$. By denoting $r = \max\{2m, \bigcup_{i=1,\ldots,q} 3 + 2(m_i - 1)\}$, we conclude that $ad^r(t_\alpha)(x) = 0$ and therefore $ad(t_\alpha)$ is locally nilpotent on L.

By Lemma 3.3-1, it is clear that if $ad(t_\alpha)$ is locally nilpotent on L, then $\mathcal{M}(t_\alpha)$ is locally nilpotent on T.$\square$

Proposition 3.3 *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $L(t_\alpha, t_{-\alpha})$ be a test algebra associated to $\alpha \in \Lambda^1$ such that $\mathcal{M}(t_{\pm\alpha})$ are locally nilpotent. Then L is a locally finite $L(t_\alpha, t_{-\alpha})-$module with respect to the adjoint representation.*

Proof. By Proposition 3.2, the adjoint mappings $ad(t_{\pm\alpha})$ are locally nilpotent on L. By using now the same arguments as for $sl(2, \mathbb{K})$ (cf. [13, Proposition 2.4.7]) we complete the proof.$\square$

Proposition 3.4 *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1_i$. Then we have:*

1. *$dim T_\alpha = 1$.*
2. *$T_{\pm(2n-1)\alpha} = L^0_{\pm 2m\alpha} = 0$ for $n > 1$ and $m \geq 1$.*
3. *If $\alpha \in \Lambda_0$, then $-\alpha \in \Lambda_0$ and $dim L^0_\alpha = dim L^0_{-\alpha}$.*
4. *If $\alpha \notin \Lambda_0$, then $\mathbb{Z}\alpha \cap \Lambda^1 = \pm\alpha$ and $\mathbb{Z}\alpha \cap \Lambda^0 = \emptyset$.*
5. *There exists a unique element $h_\alpha = [t_\alpha, t_{-\alpha}]$ with $\alpha(h_\alpha) = 2$.*
6. *For each $\beta \in \Lambda^1$ we have $\beta(h_\alpha) \in \mathbb{Z}$.*
7. *$\mathbb{K}\alpha \cap \Lambda^1 \subseteq (\frac{1}{2}\mathbb{Z})\alpha$.*
8. *$\mathbb{K}\alpha \cap \Lambda^1_i \subseteq \{\pm\frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha\}$.*

Proof. Since α is integrable, we find $t_{\pm\alpha} \in T_{\pm\alpha}$ such that

$$L(t_\alpha, t_{-\alpha}) \approx sl(2, \mathbb{K})$$

and $\mathcal{M}(t_{\pm\alpha})$ are locally nilpotent. We may w.l.o.g. assume that $\alpha([t_\alpha, t_{-\alpha}]) = 2$, and denote $h_\alpha := [t_\alpha, t_{-\alpha}]$.

1. and 2. We consider the $L(t_\alpha, t_{-\alpha})-$submodule of L

$$V := \mathbb{K}t_{-\alpha} + H^0 + \sum_{n=1}^{\infty} T_{(2n-1)\alpha} + \sum_{n=1}^{\infty} L^0_{2n\alpha}.$$

By Proposition 3.3, L is a locally finite $L(t_\alpha, t_{-\alpha})-$module and, as a submodule of a locally finite module, V is also locally finite. In particular V is a sum of finite dimensional simple $L(t_\alpha, t_{-\alpha})-$submodules by Weyl's theorem. Hence the representation theory of $sl(2, \mathbb{K})$ implies that the set of h_α-eigenvalues on V is symmetric with

$$dim V^\mu(h_\alpha) = dim V^{-\mu}(h_\alpha)$$

for each $\mu \in \mathbb{K}$. Now $V^{-2}(h_\alpha) = \mathbb{K}t_{-\alpha}$ implies that $dim V^2(h_\alpha) = dim T_\alpha = 1$ and furthermore that

$$dim V^{2(2n-1)}(h_\alpha) = dim T_{(2n-1)\alpha} = 0$$

for $n > 1$, and $dim V^{2(2n)}(h_\alpha) = dim L^0_{2n\alpha} = 0$ for $n \geq 1$. Likewise we obtain $dim T_{-(2n-1)\alpha} = 0$ for $n > 1$, and $dim L^0_{-2n\alpha} = 0$ for $n \geq 1$.

3. We have

$$V^{'} := T_\alpha + T_{-\alpha} + H_0 + \sum_{n=0}^{\infty} T_{\pm 2n\alpha} + \sum_{n=1}^{\infty} L^0_{\pm(2n-1)\alpha}$$

is an $L(t_\alpha, t_{-\alpha})-$submodule of L. By arguing as in 1., we prove 3.

4. Since by 1., $dim T_\alpha = dim T_{-\alpha} = 1$, and by 3. $L^0_\alpha = L^0_{-\alpha} = 0$, we can check that

$$V'' := T_\alpha + T_{-\alpha} + H_0 + \sum_{n=0}^{\infty} T_{-2n\alpha} + \sum_{n=2}^{\infty} L^0_{-(2n-1)\alpha}$$

is an $L(t_\alpha, t_{-\alpha})-$submodule of L. By arguing as in 1., we obtain

$$T_{\pm 2n\alpha} = L^0_{\pm(2n-1)\alpha} = 0$$

for $n \geq 1$. From 2. we complete the proof.

5. Since both spaces $T_{\pm\alpha}$ are one dimensional and do not commute, the space $[T_\alpha, T_{-\alpha}]$ is one dimensional. Hence the element h_α is uniquely determined by $\alpha(h_\alpha) = 2$.

6. Since L is a locally finite $L(t_\alpha, t_{-\alpha})-$module and $\beta(h_\alpha)$ is the eigenvalue of $ad(h_\alpha)$ on the root space T_β, this is a consequence of the finite dimensional representation theory of $sl(2, \mathbb{K})$

7. Let $\beta = c\alpha \in \Lambda^1$ with $c \in \mathbb{K}$. Then 5. and 6. imply that $c \in \frac{1}{2}\mathbb{Z}$.

8. If, in addition, β is integrable, then we also have that $\frac{1}{c} \in \frac{1}{2}\mathbb{Z}$ and thus $\frac{4}{2c} \in \mathbb{Z}$. Since $2c$ divides 4 it equals $1, 2$, or 4, so that we may have $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$.$\square$

We have seen that for integrable roots the root spaces $T_{\pm\alpha}$ are one dimensional, showing that the test algebras $L(t_\alpha, t_{-\alpha})$ do not depend on the choice of $t_{\pm\alpha}$.

Definition 3.4 *We say that a simple root α of T is abelian simple if $L^0_\alpha = 0$. If $L^0_\alpha \neq \{0\}$, (and by Proposition 3.4-3, $L^0_{-\alpha} \neq \{0\}$), α is called strongly simple if there exist nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that, for any $\epsilon \in \{+,-\}$, $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, and $[e^0_\alpha, e^0_{-\alpha}] \neq 0$. Finally, we say that a simple root α is weakly simple if $L^0_\alpha \neq \{0\}$ and it is not a strongly simple root. We call an abelian simple root α abelian integrable if α is also an integrable root. A strongly simple root α is said to be strongly integrable if α is also an integrable root and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0. We denote by $\Lambda^1_{a,i}$ (resp. $\Lambda^1_{s,i}$) the set of all abelian integrable (resp. strongly integrable) nonzero roots of T relative to H^0.*

It is clear that if $\alpha \in \Lambda^1_{a,i}$ (resp. $\alpha \in \Lambda^1_{s,i}$), then $-\alpha \in \Lambda^1_{a,i}$ (resp. $-\alpha \in \Lambda^1_{s,i}$).

Proposition 3.5 *Let $\alpha \in \Lambda^1_{s,i}$. Then there exist nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that $\alpha([e^0_\alpha, e^0_{-\alpha}]) \neq 0$ and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0.*

Proof. Let us fix nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that, for any $\epsilon \in \{+,-\}$, $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, $[e^0_\alpha, e^0_{-\alpha}] \neq 0$, and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0. By applying Jacobi identity to

$$[[e^0_{-\alpha}, t_\alpha], [e^0_\alpha, t_{-\alpha}]] \in [T_0, T_0] = 0,$$

we have

$$0 = [[[t_\alpha, t_{-\alpha}], e^0_\alpha], e^0_{-\alpha}] - [[[t_\alpha, e^0_\alpha], t_{-\alpha}], e^0_{-\alpha}] +$$
$$+ [[[t_{-\alpha}, e^0_{-\alpha}], e^0_\alpha], t_\alpha] + [[[e^0_{-\alpha}, e^0_\alpha], t_{-\alpha}], t_\alpha].$$

Therefore,

$$\alpha([t_\alpha, t_{-\alpha}])[e^0_\alpha, e^0_{-\alpha}] = \alpha([e^0_\alpha, e^0_{-\alpha}])[t_\alpha, t_{-\alpha}],$$

and from here the proposition follows.□

Proposition 3.6 *If $\alpha \in \Lambda^1_{s,i}$, then*

1. *$dim L^0_{\pm\alpha} = 1$.*
2. *$\mathbb{Z}\alpha \cap \Lambda^1 = \{\pm\alpha\}$.*
3. *$\mathbb{Z}\alpha \cap \Lambda^0 = \{\pm\alpha\}$.*

Proof. 1. From Proposition 3.5, α is an integrable root of the split Lie algebra L^0, (see Proposition 3.1), in the sense of [10, Definition I.3]. Then, by [10, Proposition I.6] we have $dim L^0_{\pm\alpha} = 1$.

2. and 3. As by 1. and Proposition 3.4-1, $dim T_{\pm\alpha} = dim L^0_{\pm\alpha} = 1$, we have $[T_\alpha, L^0_\alpha] = [T_{-\alpha}, L^0_{-\alpha}] = 0$.

Let us consider

$$V := T_0 + H^0 + T_\alpha + T_{-\alpha} + L^0_\alpha + L^0_{-\alpha} + \sum_{n=1}^{\infty} T_{-2n\alpha} + \sum_{n=2}^{\infty} L^0_{-(2n-1)\alpha}.$$

We can verify that V is an $L(t_\alpha, t_{-\alpha})$-module, and by arguing as in Proposition 3.4-1, we conclude $T_{-2n\alpha} = L^0_{-(2m-1)\alpha} = 0$, for $n \geq 1$ and $m \geq 2$. Proposition 3.4-2 completes the proof.□

Corollary 3.1 *Let $\alpha \in \Lambda^1_{s,i}$. Then we have:*

1. $[T_\alpha, L^0_{-\alpha}] \neq 0$
2. $[T_\alpha, T_0] \neq 0$
3. $[L^0_\alpha, T_0] \neq 0$

Proof. 1. and 2. Let us fix $t_{\pm\alpha} \in T_{\pm\alpha}$ such that $\alpha([t_\alpha, t_{-\alpha}]) = 1$, and $0 \neq e^0_\alpha \in L^0_\alpha$ satisfying $[t_\alpha, e^0_\alpha] = 0$.

Let us consider

$$V := \mathbb{K}[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}] \oplus \mathbb{K}[e^0_\alpha, t_{-\alpha}] \oplus \mathbb{K}e^0_\alpha.$$

We assert that V is an $L(t_\alpha, t_{-\alpha})$-submodule of L. Indeed, this can be verified taking into account the following identities:

Since $[H_0, T_0] = 0$,

$$[[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}], t_\alpha] = -[[t_{-\alpha}, t_\alpha], [e^0_\alpha, t_{-\alpha}]] - [[t_\alpha, [e^0_\alpha, t_{-\alpha}]], t_{-\alpha}] =$$
$$= -[[t_\alpha, [e^0_\alpha, t_{-\alpha}]], t_{-\alpha}] = [[[t_\alpha, t_{-\alpha}], e^0_\alpha], t_{-\alpha}] - [[[t_\alpha, e^0_\alpha], t_{-\alpha}], t_{-\alpha}] =$$
$$= [[[t_\alpha, t_{-\alpha}], e^0_\alpha], t_{-\alpha}] = [e^0_\alpha, t_{-\alpha}].$$

As $[t_\alpha, e^0_\alpha] = 0$,

$$[[e^0_\alpha, t_{-\alpha}], t_\alpha] = e^0_\alpha,$$

and finally, by Proposition 3.6-2,

$$[[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}], t_{-\alpha}] \subset T_{-2\alpha} = 0.$$

By arguing as in Proposition 3.4-1, we conclude that $[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}] \neq 0$, and from here 1. and 2.

3. From Proposition 3.5, we can fix $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, $\epsilon \in \{+,-\}$, and satisfying $\alpha([e^0_\alpha, e^0_{-\alpha}]) \neq 0$. By Jacobi identity we obtain $[[e^0_{-\alpha}, t_\alpha], e^0_\alpha] = -\alpha([e^0_\alpha, e^0_{-\alpha}])t_\alpha \neq 0$, and 3. is proved.□

Lemma 3.4 *Let $\alpha \in \Lambda^1_{s,i}$. Then there exist nonzero elements $t_\alpha \in T_\alpha$, $t_0 \in T_0$, $e^0_\alpha \in L^0_\alpha$ such that $[e^0_\alpha, t_0] = t_\alpha$ and $[t_\alpha, t_0] = e^0_\alpha$.*

Proof. By Corollary 3.1 there exist $t'_\alpha \in T_\alpha$, $t'_0, t''_0 \in T_0$ and $f^0_\alpha \in L^0_\alpha$ such that $[t'_\alpha, t'_0] \neq 0$ and $[f^0_\alpha, t''_0] \neq 0$. If $[f^0_\alpha, t'_0] \neq 0$ or $[t'_\alpha, t''_0] \neq 0$ the proof is clear taking into account that $dim T_\alpha = dim L^0_\alpha = 1$.

If $[f^0_\alpha, t'_0] = 0$ and $[t'_\alpha, t''_0] = 0$, let consider $t'''_0 := t'_0 + t''_0$. As $[t'_\alpha, t'''_0] = \lambda f^0_\alpha$ and $[f^0_\alpha, t'''_0] = \beta t'_\alpha$ with α, β nonzero elements of $\mathbb{K}$, it is easy to check that $t_0 := \sqrt{\frac{1}{\lambda\beta}} t'''_0$, $t_\alpha := \sqrt{\frac{\beta}{\lambda}} t'_\alpha$ and $e^0_\alpha := f^0_\alpha$ complete the proof.□

Proposition 3.7 *Let T be a split Lie triple system respect to H^0 and let $L = L^0 \oplus L^1$ be its standard embedding. Then $H^0 \oplus T_0$ is a MASA of L.*

Proof. It is easy to check, taking into account the definition of split Lie triple system and Proposition 3.1.□

Proposition 3.8 *Let T be a split Lie triple system respect to H^0, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1_{s,i}$. Then there exist two nonzero linear functionals $\alpha_1, \alpha_2 : H^0 \oplus T_0 \to \mathbb{K}$ such that $L^0_\alpha \oplus T_\alpha = L_{\alpha_1} \oplus L_{\alpha_2}$.*

Proof. By Lemma 3.4 we can choose nonzero elements $t_\alpha \in T_\alpha$, $t_0 \in T_0$, $e^0_\alpha \in L^0_\alpha$ such that $[e^0_\alpha, t_0] = t_\alpha$ and $[t_\alpha, t_0] = e^0_\alpha$. Since $dim T_\alpha = dim L^0_\alpha = 1$, we can determine, for any $t'_0 \in T_0$, the unique $\lambda_{t'_0}, \beta_{t'_0} \in \mathbb{K}$ such that $[t_\alpha, t'_0] = \lambda_{t'_0} e^0_\alpha$ and $[e^0_\alpha, t'_0] = \beta_{t'_0} t_\alpha$. As $[T_0, T_0] = 0$ we have

$$0 = [[t'_0, t_0], t_\alpha] = -[[t_0, t_\alpha], t'_0] - [[t_\alpha, t'_0]], t_0] = [e^0_\alpha, t'_0] - \lambda_{t'_0}[e^0_\alpha, t_0] = \beta_{t'_0} t_\alpha - \lambda_{t'_0} t_\alpha,$$

and therefore $\beta_{t'_0} = \lambda_{t'_0}$.

Thus we can define the mapping

$$\delta_\alpha : T_0 \to \mathbb{K} \quad \text{by} \quad \delta_\alpha(t'_0) := \lambda_{t'_0}.$$

Now, if we also define

$$\alpha_1, \alpha_2 : H^0 \oplus T_0 \to \mathbb{K}$$

by

$$\alpha_1(h^0 + t'_0) := \alpha(h^0) - \delta_\alpha(t'_0) \quad \text{and} \quad \alpha_2(h^0 + t'_0) := \alpha(h^0) + \delta_\alpha(t'_0),$$

we assert that $L_{\alpha_1} \oplus L_{\alpha_2} = L^0_\alpha \oplus T_\alpha$. Indeed,

$$[h^0 + t'_0, e^0_\alpha + t_\alpha] = \alpha(h^0)(e^0_\alpha + t_\alpha) - \lambda_{t'_0} t_\alpha - \lambda_{t'_0} e^0_\alpha =$$

$$= (\alpha(h^0) - \delta_\alpha(t'_0))(e^0_\alpha + t_\alpha) = \alpha_1(h^0 + t'_0)(e^0_\alpha + t_\alpha),$$

therefore $\mathbb{K}(e^0_\alpha + t_\alpha) \subset L_{\alpha_1}$.

In a similar way we can verify that $\mathbb{K}(e_\alpha^0 - t_\alpha) \subset L_{\alpha_2}$. Then we have $L_\alpha^0 \oplus T_\alpha = \mathbb{K}e_\alpha^0 + \mathbb{K}t_\alpha = \mathbb{K}(e_\alpha^0 + t_\alpha) + \mathbb{K}(e_\alpha^0 - t_\alpha) \subset L_{\alpha_1} \oplus L_{\alpha_2}$.

If $e_{\alpha_1} + e_{\alpha_2} \in L_{\alpha_1} \oplus L_{\alpha_2}$, we obtain that

$$[h^0, e_{\alpha_1} + e_{\alpha_2}] = \alpha_1(h^0)e_{\alpha_1} + \alpha_2(h^0)e_{\alpha_2} = \alpha(h^0)(e_{\alpha_1} + e_{\alpha_2}),$$

and therefore $e_{\alpha_1} + e_{\alpha_2} \in L_\alpha^0 \oplus T_\alpha$. The proof is complete.□

Theorem 3.1 *Let T be a split Lie triple system respect to H^0 such that $\Lambda^1 = \Lambda_{a,i}^1 \cup \Lambda_{s,i}^1$. Then its standard embedding L is split respect to $H^0 \oplus T_0$.*

Proof. First, if $T_0 = 0$ then H^0 is a MASA of L. We assert that in this case $\Lambda^0 \cap \Lambda^1 = \emptyset$. Indeed, if $\alpha \in \Lambda^0 \cap \Lambda^1$, α must be strongly integrable and by Corollary 3.1-1, $T_0 \neq 0$, a contradiction. Hence $\Lambda^0 \cup \Lambda^1$ is the root system of L respect to H^0, and we have the splitting decomposition

$$L = H^0 \oplus (\bigoplus_{\beta \in \Lambda^0} L_\beta^0) \oplus (\bigoplus_{\alpha \in \Lambda^1} T_\alpha).$$

Second, if $T_0 \neq 0$ then $H^0 \oplus T_0$ is a MASA of L, and: If either $\alpha \notin \Lambda^1$ and $\alpha \in \Lambda^0$, (then $[T_0, L_\alpha^0] = 0$), or α is an abelian integrable root of T, (then $[T_0, T_\alpha] = 0$); we can define the root $\alpha' : H^0 \oplus T_0 \to \mathbb{K}$ as $\alpha'(h_0, t_0) := \alpha(h_0)$ for any $(h_0, t_0) \in H^0 \oplus T_0$, the associated root space is $L_{\alpha'} = L_\alpha^0$ in the first case and $L_{\alpha'} = T_\alpha$ in the second one.

If α is a strongly integrable root of T then, by Proposition 3.8, there exist two nonzero roots of L respect to $H^0 \oplus T_0$, α_1' and α_2', such that $L_\alpha^0 \oplus T_\alpha = L_{\alpha_1'} \oplus L_{\alpha_2'}$. As by Proposition 3.1, $L = H^0 \oplus (\bigoplus_{\alpha \in \Lambda_0} L_\alpha^0) \oplus T_0 \oplus (\bigoplus_{\beta \in \Lambda_1} T_\beta)$, we conclude that L admits the splitting decomposition respect to $H^0 \oplus T_0$ given by

$$L = (H^0 \oplus T_0) \oplus (\bigoplus_{\alpha'} L_{\alpha'}).$$

□

4 LOCALLY FINITE SPLIT LIE TRIPLE SYSTEMS

Definition 4.1 *A Lie triple subsystem of a Lie triple system T is a submodule U such that $[U, U, U] \subseteq U$. We say that a Lie triple system T is locally finite if every finite subset of T is contained in a finite dimensional subtriple of T.*

We recall that a similar concept is defined for Lie algebras (c.f. [7] or [10]).

Proposition 4.1 *The standard embedding $L = L^0 \oplus L^1$ of a locally finite Lie triple system T is a locally finite Lie algebra.*

Proof. Let V^0 be a finite subset of L^0. By the construction of L^0, the subset V^0 can be described from a suitable finite subset V of T. As T is locally finite, there exists a finite dimensional subtriple of T, U, such that V is contained in U. The Lie subalgebra of L generated by U is finite dimensional and V^0 is contained in it. Therefore L^0 is locally finite. It is easy to conclude that L is also locally finite.$\Box$

Let us note that in a locally finite split Lie triple system all abelian (resp. strongly) simple roots of Λ^1 are abelian (resp. strongly) integrable.

Theorem 4.1 *Let T be a locally finite split simple Lie triple system such that $\Lambda^1 = \Lambda^1_{a,i} \cup \Lambda^1_{s,i}$. Then T is isomorphic to one of the followings:*

1. *$T = \mathcal{L}$ where $\mathcal{L}$ is a locally finite split simple Lie algebra.*
2. *$T = Skw(\mathcal{L}, \xi)$ with $\mathcal{L}$ as in the previous case and ξ an involutive automorphism of $\mathcal{L}$.*

Proof. By Propositions 2.1 and 2.2, either $T = L$ or $T = Skw(L', \xi)$, where L, L' are simple Lie algebras and ξ is an involutive automorphism of L'. By Proposition 4.1, L and L' are locally finite. By Proposition 3.1 L is split and finally by Theorem 3.1 L' is split. The proof is complete.$\Box$

Remark 4.1 *If T is of infinite dimension, following [8], the algebra $\mathcal{L}$ of Theorem 4.1, must be one of the following $sl(J, \mathbb{K})$, $d(J, \mathbb{K})$ or $sp(J, \mathbb{K})$ where J is an infinite set whose cardinality equals the dimension of the Lie algebra.*

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