On Infinite Dimensional Lie Algebras Having a Cartan Decomposition

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We introduce the concept of Cartan decomposition relative to a Cartan subalgebra H, in the sense of [2], for Lie algebras of arbitrary dimension. The class of simple Lie algebras having such decomposition is a wide class of Lie algebras closely related to the topologically simple L^*-algebras ([8, 9, 5]), the simple c-involutive Lie algebras ([6]) and the topologically simple compact Banach-Lie algebras ([4]). We give a description theorem for the complex ones, turning out to be the natural extension of the finite dimensional setup.

Key Words: Infinite dimensional Lie algebras, Cartan decompositions, roots and root spaces.

1. INTRODUCTION AND PREVIOUS DEFINITIONS

In order to make sense, the definition of Cartan subalgebra given below requieres a definition of semisimplicity for infinite dimensional Lie algebras. In this paper, a Lie algebra is defined to be *semisimple* if it has no non trivial abelian ideals. A critic of this definition is given in [6, apendix to chapter II]. But certainly, any other reasonable definition of semisimplicity will imply the absence of non trivial abelian ideals. We define the *Annihilator* of a Lie algebra L as the ideal given by $Ann(L) = \{x \in L : [x, y] = 0$ for all $y \in L\}$. It is clear that any semisimple Lie algebra L has zero annihilator, therefore L is also semisimple in the sense given in [8] for L^*-algebras. We shall say that a Lie algebra L is *simple* if the product is nonzero and has no nonzero proper ideals. Throughout this paper, Lie algebras L are considered of arbitrary dimension, semisimple and over an algebraically closed field $\mathbb{K}$ of characteristic 0. For subsets M, N of L we denote by $[M, N]$ the subspace spanned by $\{[m, n] : m \in M, n \in N\}$.

Y. Billing and A. Pianzola introduced in [2] the concept of Cartan subalgebra for Lie algebras of arbitrary dimension as follows:

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A subalgebra H of L is called a *Cartan subalgebra* if

1. The elements of H act locally ad-nilpotently on H.
2. H is its own normalizer in L, i.e., $N_L(H) = H$.

If L is finite dimensional, then H is nilpotent by Engel's theorem and the classical definition of Cartan subalgebra is recovered. A *root* of L relative to H is a mapping $\alpha : H \longrightarrow \mathbb{K}$ such that there exists $v_\alpha \in L$, $v_\alpha \neq 0$ satisfying $[h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in H$. The *root space* associated to α is the subspace $L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in H\}$. Given a set S of nonzero roots of L, we shall denote by $Sp_{\mathbb{Z}}S$ the set of mappings

$$Sp_{\mathbb{Z}}S = \{\sum_{i=1}^{n} p_i\alpha_i : p_i \in \mathbb{Z} \quad \text{and} \quad \alpha_i \in S\}.$$

We shall call that L has a *Cartan decomposition* relative to H if

1. $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$, where Λ is the set of all nonzero roots of L relative to H.
2. Each L_α, $\alpha \in \Lambda$, is finite dimensional.
3. For any finite set $S \subset \Lambda$ we have that $Sp_{\mathbb{Z}}S \cap \Lambda$ is also finite.
4. $[L_\alpha, L_{-\alpha}] \neq 0$.

Let us observe that in case that L is a simple complex Lie algebra with involution having a Cartan decomposition, Corollary 4.2 will imply that our definition of Cartan subalgebra coincides with the one given by Schue for L^*-algebras in [8], and the one given by L'Harpe for c-involutive Lie algebras in [6, definition 4 on page 32], by adding to H the condition of being also selfadjoint. Hence, our definition of Cartan decomposition recovers the ones given for simple finite dimensional Lie algebras and topologically simple L^*-algebras ([8]), c-involutive Lie algebras ([6])and compact Banach-Lie algebras ([4]).

It is easy to prove that the roots are linear mappings, the root space associated to the zero root is contained in the Cartan subalgebra and, by the Jacobi identity, that if $\alpha + \beta$ is a root then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and if $\alpha + \beta$ is not a root then $[L_\alpha, L_\beta] = 0$.

The paper is organized as follows. In the second section we study the roots and root spaces associated to a Cartan subalgebra of a Lie algebra of arbitrary dimension, our main results in this section is that the root spaces associated to nonzero roots are one-dimensional and that the only integral multiplies $k\alpha$ of a nonzero root α which are roots are 0 and $\pm\alpha$. In the third section we study the direct limit of a direct system of Lie algebras, we prove the existence of the direct limit (unique up to isomorphism), and we study some of its properties. Finally, in the fourth section we obtain a description theorem for infinite dimensional simple complex Lie algebras having a Cartan decomposition L, by approximating L as the direct limit of a suitable family of finite dimensional simple Lie subalgebras, we note that the introduction of the concept of connection of roots to construct an adequate direct system of Lie algebras is perhaps the most interesting novelty in this paper. It is shown that this class is written as simple Lie subalgebras of certain simple associative algebras. The result turn out to be the natural extension of the finite dimensional theory, the L^*-algebras theory ([8, 9, 5]) and the theory of compact Banach-Lie algebras ([4]). Specifically, the following theorem is proved

THEOREM 4.3. *Let L be an infinite dimensional simple complex Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H. Then L is isomorphic to one of the followings:*

1. *A^-, with A a simple associative algebra.*
2. *$Skw(A, \xi)$ with A as above and ξ an involutive antiautomorphism of A.*

2. ROOTS AND CARTAN DECOMPOSITIONS

LEMMA 2.1. *Let L be a semisimple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, and let α, β be two nonzero roots. Then we have:*

1. *If $\beta+\alpha$ and $\beta-\alpha$ are not roots then $\beta(h_\alpha) = 0$ for any $h_\alpha \in [L_\alpha, L_{-\alpha}]$.*
2. *$\beta(h_\alpha) = r\alpha(h_\alpha)$ with $r \in \mathbb{Q}$ and h_α as above.*

Proof. We obtain 1. as an easy consequence of Jacobi identity and the facts $[L_\alpha, L_\beta] = [L_\alpha, L_{-\beta}] = 0$.

2. Let fix $h_\alpha = [v_\alpha, v_{-\alpha}]$ with $v_\alpha \in L_\alpha$ and $v_{-\alpha} \in L_{-\alpha}$. Let consider $V = \mathcal{L}(L_{\beta+j\alpha} : j \in \mathbb{Z})$ the linear space generated by $\{L_{\beta+j\alpha} : j \in \mathbb{Z}\}$. It is clear that V is a finite dimensional vector space invariant for $ad(v_\alpha)$, $ad(v_{-\alpha})$ and $ad(h_\alpha) = ad(v_\alpha)ad(v_{-\alpha}) - ad(v_{-\alpha})ad(v_\alpha)$. The fact that the trace of $ad(h_\alpha)$ on V is 0 gives us the equation $m\beta(h_\alpha) + k\alpha(h_\alpha) = 0$ with $m \neq 0$ and $m, k \in \mathbb{Z}$. Hence $\beta(h_\alpha) = \frac{k}{m}\alpha(h_\alpha)$. ∎

COROLLARY 2.1. *Let L be a semisimple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, and let α be a nonzero root. Then $\alpha(h_\alpha) \neq 0$ for any $0 \neq h_\alpha \in [L_\alpha, L_{-\alpha}]$.*

Proof. Let suppose that $\alpha(h_\alpha) = 0$. Since $h_\alpha \in L_0$ and, by Lemma 2.1 -2, $\beta(h_\alpha) = 0$ for all root β, we have $[h_\alpha, L] = 0$. Hence, $h_\alpha \in Ann(L)$ and then $h_\alpha = 0$, a contradiction. ∎

THEOREM 2.1. *Let L be a semisimple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, and let α be a nonzero root. Then L_α is one-dimensional and the only integral multiplies $k\alpha$ which are roots are 0 and $\pm\alpha$.*

Proof. By Lemma 2.1-2 we can fix $0 \neq v_\alpha \in L_\alpha$ and $0 \neq v_{-\alpha} \in L_{-\alpha}$ such that $0 \neq h_\alpha = [v_\alpha, v_{-\alpha}]$. Let consider

$$L' = \mathbb{K}[v_\alpha, L_{-\alpha}] \oplus \mathbb{K}v_\alpha \oplus (\bigoplus_{j=1} L_{-j\alpha}).$$

We have that L' is a finite dimensional Lie algebra invariant for $ad(v_\alpha)$, $ad(v_{-\alpha})$, and $ad(h_\alpha) = [ad(v_\alpha), ad(v_{-\alpha})]$. Since the trace of $ad(h_\alpha)$ on L' is 0 and since $[h_\alpha, v_\alpha] = \alpha(h_\alpha)v_\alpha$, $[h_\alpha, v_{-j\alpha}] = -j\alpha(h_\alpha)v_{-j\alpha}$, and $[h_\alpha, h'] = 0$ for any $h' \in \mathbb{K}[v_\alpha, L_{-\alpha}]$, we have

$$\alpha(h_\alpha)(1 + \sum(-j)d_j) = 0$$

where d_j is the dimension of $L_{-j\alpha}$. By Corollary 2.1 $\alpha(h_\alpha) \neq 0$, hence $1 + \sum(-j)d_j = 0$. Thus $-2\alpha, -3\alpha, \cdots$ are not roots and $d_1 = 1$. Since we can replace α by $-\alpha$ in the argument, we have both conclusions of the theorem. ∎

The next corollary is now immediate

COROLLARY 2.2. *Under the hypothesis of Theorem 2.1, if α, β and $\alpha + \beta$ are nonzero roots of L relative to H then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*

3. DIRECT LIMITS OF LIE ALGEBRAS

Let $(I, \leq)$ be a direct set and let $\{L_i\}_{i\in I}$ be a family of Lie algebras such that for any $i, j \in I$ with $i \leq j$ there exists a monomorphism $e_{ji} : L_i \longrightarrow L_j$ such that $e_{ji} \circ e_{ik} = e_{jk}$ and $e_{ii} = Id$ for all i, j, k with $k \leq i \leq j$. Then we shall say that $S = (\{L_i\}_{i\in I}, \{e_{ji}\}_{i\leq j})$ is a *direct system* of Lie algebras.

Given S we define a *direct limit*, $\lim\limits_{\rightarrow} S$, as a couple $(L, \{e_i\}_{i\in I})$ where L is a Lie algebra, $e_i : L_i \longrightarrow L$ is a monomorphism that satisfies

$$e_i = e_j \circ e_{ji}$$

for any $j \geq i$ and $(L, \{e_i\}_{i\in I})$ is universal for this property in the sense that if $(L', \{t_i\}_{i\in I})$ is another such couple, then there exists a unique monomorphism $\theta : L \longrightarrow L'$ such that $t_i = \theta \circ e_i$ for all $i \in I$. It is clear that if a direct limit exists, then it is unique up to isomorphism. We define the concepts of direct system and direct limit for associative algebras in a similar way.

If A is an associative algebra then A^- will denote the Lie algebra whose product is given by $[x, y] = xy - yx$.

As in [3], we can prove that any direct system of Lie algebras S has a direct limit, and state the following results:

THEOREM 3.1. *Let* $S = (\{A_i\}_{i\in I}, \{e_{ji}\}_{i\leq j})$ *be a direct system of associative algebras. Then* $S^- = (\{A_i^-\}_{i\in I}, \{e_{ji}\}_{i\leq j})$ *is a direct system of Lie algebras and* $\lim\limits_{\rightarrow} S^- = (\lim\limits_{\rightarrow} S)^-$.

THEOREM 3.2. *Let* $S = (\{A_i\}_{i\in I}, \{e_{ji}\}_{i\leq j})$ *be a direct system of associative algebras, with* $\{\xi_i\}_{i\in I}$ *a family of involutive antiautomorphisms,* $\xi_i : A_i \to A_i$*, such that* $\xi_j \circ e_{ji} = e_{ji} \circ \xi_i$ *for* $i \leq j$*. Write* $A = \lim\limits_{\rightarrow} S$*, then:*

1. *There exists a unique involutive antiautomorphism* $\xi : A \to A$ *satisfying* $\xi \circ e_i = e_i \circ \xi_i$ *for any* $i \in I$.
2. *If we consider the Lie subalgebra of* A_i^-, $Skw(A_i, \xi_i)$ *then*

$$Skw(S,\xi) := (\{Skw(A_i,\xi_i)\}_{i\in I}, \{e_{ji}|_{Skw(A_i,\xi_i)}\}_{i\leq j})$$

is a direct system of Lie algebras and $\lim\limits_{\rightarrow}(Skw(S,\xi)) = Skw(\lim\limits_{\rightarrow} S, \xi)$.

4. THE DESCRIPTION THEOREM

We shall prove in this section that if L is an infinite dimensional simple complex Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, then L is isomorphic to $\lim\limits_{\rightarrow} S$, $S = (\{L_i\}_{i\in I}, \{i_{ji}\}_{i\leq j})$ being a direct system, with the inclusion, of simple finite dimensional Lie subalgebras of a same type A_l, B_l, C_l or D_l. Finally, the properties of the direct limits, shown in §3, will give us the description theorem.

In the remainder of this section L will denote an infinite dimensional semisimple complex Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, and Λ will be the set of nonzero roots of L relative to H.

A subset Λ_0 of Λ is called a *root system* (relative to H) if it satisfies the conditions: $\alpha \in \Lambda_0$ implies $-\alpha \in \Lambda_0$; and $\alpha, \beta \in \Lambda_0$, $\alpha + \beta \in \Lambda$ implies $\alpha+\beta \in \Lambda_0$. From now on, given $\alpha \in \Lambda$ we denote by h_α the unique nonzero element of L_0 of the form $h_\alpha = [e_\alpha, e_{-\alpha}]$ with $0 \neq e_\alpha \in L_\alpha$, $0 \neq -e_\alpha \in L_{-\alpha}$ and such that $\alpha(h_\alpha) = 1$, (by Corollary 2.1 there exists such h_α for any $\alpha \in \Lambda$). If we define $H_{\Lambda_0} = Sp\{h_\alpha : \alpha \in \Lambda_0\}$ and $V_{\Lambda_0} = \bigoplus\limits_{\alpha\in\Lambda_0} L_\alpha$, it is straightforward to verify that $L_{\Lambda_0} = H_{\Lambda_0} \oplus V_{\Lambda_0}$ is a Lie subalgebra of L, with Cartan subalgebra $H_{\Lambda_0} = H \cap L_{\Lambda_0}$, whose roots relative to H_{Λ_0} are precisely the roots in Λ_0. We shall say that L_{Λ_0} is the Lie subalgebra *associated* to the root system Λ_0.

Our next goal is to prove that the Lie subalgebra L_{Λ_0} associated to a finite root system Λ_0 in L is semisimple.

LEMMA 4.1. *Let* L_{Λ_0} *be the Lie subalgebra associated to a finite root system* Λ_0 *in* L*. Write* $< \cdot, \cdot >$ *the Killing form on* L*. Then we have:*

1. $< h_\alpha, h_\alpha > \neq 0$ *for any* $\alpha \in \Lambda_0$.

2. $< h, v_\alpha >= 0$ *for any* $h \in H_{\Lambda_0}$ *and* $v_\alpha \in L_\alpha$, $\alpha \in \Lambda_0$.

3. $< v_\alpha, v_\beta >= 0$ *for any* $v_\alpha \in L_\alpha$, $v_\beta \in L_\beta$, $\alpha, \beta \in \Lambda_0$ *and* $\beta \neq -\alpha$.

4. $< v_\alpha, v_{-\alpha} >\neq 0$ *for any* $0 \neq v_{i\alpha} \in L_{i\alpha}$, $i \in \{\pm 1\}$ *and* $\alpha \in \Lambda_0$.

Proof. 1. We have $< h_\alpha, h_\alpha >= trz(ad(h_\alpha) \circ ad(h_\alpha)) = \alpha(h_\alpha)^2 + \sum\limits_{\gamma \in \Lambda_0 - \{\alpha\}} \gamma(h_\alpha)^2$. As in the proof of Lemma 2.1-2 we obtain $\gamma(h_\alpha) = r_\gamma \alpha(h_\alpha)$ with $r_\gamma \in \mathbb{Q}$, and finally we conclude from $\alpha(h_\alpha) = 1$ that $< h_\alpha, h_\alpha >= 1 + \sum\limits_{\gamma \in \Lambda_0 - \{\alpha\}} r_\gamma^2 \neq 0$.

2. Since $< \cdot, \cdot >$ is invariant in the sense of [7, p. 69], we have

$$< h, v_\alpha >=< h, [h_\alpha, v_\alpha] >=< [h, h_\alpha], v_\alpha >= 0.$$

3. It is clear that $< v_\alpha, v_\beta >= trz(ad(v_\alpha) \circ ad(v_\beta)) = 0$.

4. Since

$$< h_\alpha, h_\alpha >=< [e_\alpha, e_{-\alpha}], h_\alpha >=< e_\alpha, [h_\alpha, e_{-\alpha}] >= - < e_\alpha, e_{-\alpha} >,$$

by applying 1. we have $< e_\alpha, e_{-\alpha} >\neq 0$. Hence, as $dim L_{\pm\alpha} = 1$ we conclude $< v_\alpha, v_{-\alpha} >\neq 0$. ∎

Lemma 4.2. *Under the hypothesis of Lemma 4.1, if* $< x, L_{\Lambda_0} >= 0$ *for some* $x \in L_{\Lambda_0}$ *then* $x \in H_{\Lambda_0}$.

Proof. Write $x = h + \sum\limits_{\alpha \in \Lambda_0} w_\alpha \in L_{\Lambda_0}$, with $h \in H_{\Lambda_0}$ and $w_\alpha \in L_\alpha$. Since $< x, v_{-\alpha} >= 0$ for any $v_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Lambda_0$, Lemma 4.1-2,3 shows that $< w_\alpha, v_{-\alpha} >= 0$, and therefore $w_\alpha = 0$ by Lemma 4.1-4. ∎

Proposition 4.1. *The Lie subalgebra* L_{Λ_0} *associated to a finite root system* Λ_0 *in* L *is semisimple.*

Proof. Let us firstly observe that $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$ and that

$$Rad(L_{\Lambda_0}) \subset H_{\Lambda_0}.$$

Indeed, if $x \in L_{\Lambda_0}$ then

$$x = \sum_{\alpha \in \Lambda_0} \lambda_\alpha h_\alpha + \sum_{\alpha \in \Lambda_0} v_\alpha = \sum_{\alpha \in \Lambda_0} \lambda_\alpha [e_\alpha, e_{-\alpha}] + \sum_{\alpha \in \Lambda_0} [h_\alpha, v_\alpha] \in [L_{\Lambda_0}, L_{\Lambda_0}]$$

and so $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$. Since the radical of a finite dimensional Lie algebra $L^{'}$ is characterized as the ideal $Rad(L^{'}) = \{x \in L^{'} :< x, [L^{'}, L^{'}] >= 0\}$, where $< \cdot, \cdot >$ denotes the Killing form (see [7, p.73]), the fact $[L_{\Lambda_0}, L_{\Lambda_0}] = L_{\Lambda_0}$ and Lemma 4.2 show $Rad(L_{\Lambda_0}) \subset H_{\Lambda_0}$.

Secondly, we assert that

$$Rad(L_{\Lambda_0}) = Ann(L_{\Lambda_0}).$$

Indeed, $Ann(L_{\Lambda_0})$ is a solvable ideal and therefore is included in $Rad(L_{\Lambda_0})$. If $h \in Rad(L_{\Lambda_0}) \subset H_{\Lambda_0}$ we have, by the character of ideal of $Rad(L_{\Lambda_0})$, $[h, v_\alpha] \in Rad(L_{\Lambda_0}) \subset H_{\Lambda_0}$ for any $0 \neq v_\alpha \in L_\alpha$ and $\alpha \in \Lambda_0$, therefore $\alpha(h) = 0$. Hence, we have for any $x \in L_{\Lambda_0}$,

$$[h, x] = [h, \sum_{\alpha \in \Lambda_0} \lambda_\alpha h_\alpha + \sum_{\alpha \in \Lambda_0} v_\alpha] = [h, \sum_{\alpha \in \Lambda_0} \lambda_\alpha h_\alpha] + \sum_{\alpha \in \Lambda_0} [h, v_\alpha] = 0$$

and so $h \in Ann(L_{\Lambda_0})$.

Finally, as by Levi's theorem, ([7, p.91]),

$$L_{\Lambda_0} = Rad(L_{\Lambda_0}) \oplus T_{\Lambda_0},$$

with T_{Λ_0} a semisimple subalgebra of L_{Λ_0}, we have

$$L_{\Lambda_0} = [L_{\Lambda_0}, L_{\Lambda_0}] = [Ann(L_{\Lambda_0}) \oplus T_{\Lambda_0}, Ann(L_{\Lambda_0}) \oplus T_{\Lambda_0}] \subset [T_{\Lambda_0}, T_{\Lambda_0}] \subset T_{\Lambda_0}$$

and so $L_{\Lambda_0} = T_{\Lambda_0}$, the proof is complete. ∎

We shall say that a finite set of nonzero roots $\{\alpha_i\}$ is *linearly independent* if the set $\{h_{\alpha_i}\}$ is linearly independent. J. R. Schue introduced for any non zero root α of a semisimple L^*−algebra L with inner product $< \cdot | \cdot >$ and with a Cartan decomposition relative to a Cartan subalgebra H, the elements $0 \neq h'_\alpha \in H$ such that $\alpha(h) =< h|h'_\alpha >$ for any $h \in H$. As any complex finite dimensional semisimple Lie algebra admits an, essentially unique, structure of L^*−algebra satisfying $(L_\alpha)^* = L_{-\alpha}$ (see [8]). Following [8, Definition on page 73] we have $h'_\alpha \in [L_\alpha, L^*_\alpha]$. Therefore $h'_\alpha = kh_\alpha$ with $0 \neq k \in \mathbb{C}$. Hence, [1, Lemma 1], [1, Corollary 2] and the observations in §1 give us the following two results:

Lemma 4.3. *Let L be a finite dimensional semisimple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H. Write L as $L = \bigoplus_{j=1}^{m} L_j$ where L_j are simple Lie algebras. If α is a nonzero root relative to H, then L_α belongs precisely to one L_j. If we denote by*

$$\Lambda_j = \{\alpha : L_\alpha \subseteq L_j\},$$

then $Sp\{[e_\alpha, e_{-\alpha}] : \alpha \in \Lambda_j\}$ is a Cartan subalgebra H_j of L_j and the restrictions to H_j of the $\alpha \in \Lambda_j$ are precisely the roots of L_j.

Corollary 4.1. *Under the hipotheses of Lemma 4.3, let us suppose $\{\alpha_1, ..., \alpha_n\}$ is a linearly independent set of nonzero roots of L. If there exists a root γ of L such that $\gamma = \sum_{i=1}^{n} c_i \alpha_i \quad c_i \neq 0$, then all α_i and γ are roots of the same simple component L_j.*

Lemma 4.4. *Let L be a finite dimensional semisimple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H. If α and β are two nonzero roots such that $\alpha \neq \pm\beta$ then α and β are linearly independent.*

Proof. Suppose α and β are not linearly independent, then $h_\alpha = ch_\beta$ with $0 \neq c \in \mathbb{C}$. Let consider L as an L^*-algebra with inner product $< \cdot|\cdot >$. By the above observation, there exist non zero elements $h'_\alpha, h'_\beta \in H$ such that $\alpha(h) =< h|h'_\alpha >$ and $\beta(h) =< h|h'_\beta >$ for any $h \in H$ and $h'_\alpha = k_\alpha h_\alpha$, $h'_\beta = k_\beta h_\beta$ with $k_\alpha, k_\beta \in \mathbb{C} - \{0\}$. Hence,

$$\alpha(h) =< h|k_\alpha h_\alpha >=< h|k_\alpha ch_\beta >=< h|k_\alpha ck_\beta^{-1}h'_\beta >= \overline{k_\alpha ck_\beta^{-1}}\beta(h)$$

for any $h \in H$. From the theory of finite dimensional split semisimple Lie algebras, this is only possible if $\alpha = \pm\beta$. ∎

Let α and β be two nonzero roots of a semisimple Lie algebra, we shall say that α and β are *connected* if there exist $\alpha_1, ...\alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, ..., \alpha_1 + ... + \alpha_{n-1} + \alpha_n\}$$

is a family of nonzero roots, α_1 is a fixed element of $\{\alpha, -\alpha\}$ and $\alpha_1 + ... + \alpha_{n-1} + \alpha_n = \beta$. We shall also say that $\{\alpha_1, ...\alpha_n\}$ is a *connection* from α to β. It is clear that $\alpha_p \neq \pm \sum\limits_{i=1}^{p-1} \alpha_i, \;\; p = 2, .., n$. We denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \alpha \text{ and } \beta \text{ are connected}\}$$

Let us observe that $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \in \Lambda_\alpha$.

LEMMA 4.5. *Under the hipotheses of Lemma 4.3, if furthermore α and β are two connected nonzero roots, then L_α and L_β belong to the same simple Lie algebra L_j.*

Proof. We have $\alpha_1, ..., \alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, ..., \alpha_1 + ... + \alpha_{n-1} + \alpha_n\}$$

are nonzero roots, $\alpha_1 \in \{\alpha, -\alpha\}$ and $\alpha_1 + ... + \alpha_{n-1} + \alpha_n = \beta$. If we consider α_1, α_2, and $\alpha_1 + \alpha_2$, by Lemma 4.4 and Corollary 4.1, we have that $L_{\alpha_1}, L_{\alpha_2}$and $L_{\alpha_1+\alpha_2}$ belong to the same simple Lie algebra L_j. The same argument with $\alpha_1 + \alpha_2$, α_3 and $\alpha_1 + \alpha_2 + \alpha_3$ gives us that $L_{\alpha_3}, L_{\alpha_1+\alpha_2+\alpha_3} \subset L_j$. Following this process we finally obtain that $L_\alpha, L_\beta \subset L_j$. ∎

PROPOSITION 4.2. *Let L be an infinite dimensional simple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H, and let α be a nonzero root. Then the following assertion hold:*

1. *Λ_α is a root system.*
2. *There exists $\beta \in \Lambda_\alpha$ such that $\beta \neq \pm\alpha$.*
3. *If γ is a nonzero root such that $\gamma \notin \Lambda_\alpha$, then $[L_\beta, L_\gamma] = 0$ and $\gamma(h_\beta) = 0$ for any $\beta \in \Lambda_\alpha$.*

Proof. 1. If $\beta \in \Lambda_\alpha$ then there exists a connection $\{\alpha_1, ..., \alpha_n\}$ from α to β. It is easy to check that $\{-\alpha_1, ..., -\alpha_n\}$ is a connection from α to $-\beta$ and therefore $-\beta \in \Lambda_\alpha$. If $\beta, \gamma \in \Lambda_\alpha$ and $\beta + \gamma \in \Lambda$, then there exists a connection $\{\alpha_1, ..., \alpha_n\}$ from α to β. Hence, $\{\alpha_1, ..., \alpha_n, \gamma\}$ is a connection from α to $\beta + \gamma$ and so $\beta + \gamma \in \Lambda_\alpha$.

2. Let us firstly observe that there exists a nonzero root $\gamma \in \Lambda$, $\gamma \neq \pm\alpha$ such that either $[L_\alpha, L_\gamma] \neq 0$ or $[L_{-\alpha}, L_\gamma] \neq 0$. Indeed, if we suppose $[L_\alpha, L_\gamma] = [L_{-\alpha}, L_\gamma] = 0$ for any $\gamma \in \Lambda$, $\gamma \neq \pm\alpha$, by Lemma 2.1-1 we have $\gamma(h_\alpha) = 0$ for any $\gamma \in \Lambda$, $\gamma \neq \pm\alpha$. Let us consider

$$I := \mathbb{C}h_\alpha \oplus L_\alpha \oplus L_{-\alpha}.$$

By the above, it is easy to prove that $[I, L] \subset I$, therefore I is a nonzero finite dimensional ideal of an infinite dimensional simple Lie algebra L, a contradiction. Hence, there exists a nonzero root $\gamma \neq \pm\alpha$ such that either $[L_\alpha, L_\gamma] \neq 0$ or $[L_{-\alpha}, L_\gamma] \neq 0$. In the first case, $\{\alpha, \gamma\}$ is a connection from α to $\beta := \alpha + \gamma$, therefore $\beta \in \Lambda_\alpha$ and $\beta \neq \pm\alpha$. In the second case we argue similarly.

3. Let us suppose there exists $\beta \in \Lambda_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$. If $\{\alpha_1, ..., \alpha_n\}$ is a connection from α to β, we have $\{\alpha_1, ..., \alpha_n, \gamma\}$ is a connection from α to $\beta + \gamma$. Since Λ_α is a root system then $\gamma \in \Lambda_\alpha$, a contradiction. Therefore $[L_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. As $-\beta \in \Lambda_\alpha$ for any $\beta \in \Lambda_\alpha$, we also have $[L_{-\beta}, L_\gamma] = 0$. Finally, by Lemma 2.1-1 we conclude $\gamma(h_\beta) = 0$ for any $\beta \in \Lambda_\alpha$. ∎

THEOREM 4.1. *Let L be an infinite dimensional simple Lie algebra having a Cartan decomposition $L = H \oplus (\bigoplus_{\alpha \in \Lambda} L_\alpha)$ relative to a Cartan subalgebra H. Then for any $\alpha, \beta \in \Lambda$ there exists a connection from α to β.*

Proof. Let consider the root system Λ_α and the Lie subalgebra

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$$

of L associated to Λ_α. We assert that L_{Λ_α} is a nonzero ideal of L. Indeed, we have by Proposition 4.2-3 that $[L_\beta, L_\gamma] = 0$ and that $[h_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. Hence,

$$[L_{\Lambda_\alpha}, L] = [\sum_{\beta \in \Lambda_\alpha} \mathbb{C}h_\beta + \sum_{\beta \in \Lambda_\alpha} L_\beta, H + (\sum_{\gamma \in \Lambda_\alpha} L_\gamma) + (\sum_{\gamma \notin \Lambda_\alpha} L_\gamma)] \subset L_{\Lambda_\alpha}.$$

The simplicity of L implies $L_{\Lambda_\alpha} = L$ and therefore $\Lambda_\alpha = \Lambda$. ∎

COROLLARY 4.2. *Under the hipotheses of Theorem 4.1,*

$$H = Sp\{h_\beta : \beta \in \Lambda\} = L_0.$$

Proof. By the proof of Theorem 4.1, $H = H_{\Lambda_\alpha} = Sp\{h_\beta : \beta \in \Lambda\}$, α being any nonzero root of Λ. As a consequence $H \subset L_0$ and the proof is complete. ∎

Let S be a non empty finite subset of Λ. As $Sp_{\mathbb{Z}} S \cap \Lambda$ is a finite root system we have, by Proposition 4.1, that the finite dimensional Lie subalgebra $L_{(Sp_{\mathbb{Z}} S\cap\Lambda)}$ of L associated to $Sp_{\mathbb{Z}} S \cap \Lambda$ is semisimple. For simplicity of notation we shall denote it by $L_S := L_{(Sp_{\mathbb{Z}} S\cap\Lambda)}$. It is well known from the theory of semisimple finite dimensional Lie algebras that L_S can be written

$$L_S = \bigoplus_{i=1}^{n_S} L_{S_i},$$

with L_{S_i}, $i = 1, ..., n_S$, simple finite dimensional Lie algebras. Hence, we can consider the family of simple finite dimensional Lie subalgebras of L, $\{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}$, where $\mathcal{F}$ denotes the family of all non empty finite subset of Λ.

THEOREM 4.2. *Let L be an infinite dimensional simple Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H. Then there exists a direct system, with the inclusion, $R = (\{L_j\}_{j\in J}, \{i_{kj}\}_{j\leq k})$ of simple finite dimensional Lie subalgebras of L such that:*

1. $L = \lim\limits_{\rightarrow} R$.
2. *The direct system may be chosen so that all the L_j are of a same type A_l, B_l, C_l or D_l.*

Proof. Let consider the family of simple finite dimensional Lie subalgebras of L, $\{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}$, given above and let $\{i_{S_i,T_j}\}$ be the inclusion mappings. In order to prove that

$$R := (\{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}, \{i_{S_i,T_j}\})$$

is the required direct system we assert that given

$$L_{S_i}, L_{T_j} \in \{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}},$$

there exists

$$L_{Q_{i_0}} \in \{L_{S_i}\}_{S\in\mathcal{F}, i\in\{1,\dots,n_S\}}$$

such that $L_{S_i}, L_{T_j} \subset L_{Q_{i_0}}$. Indeed, let us fix $\alpha \in S_i$. By Theorem 4.1, for any $\beta \in S_i \cup T_j$ there exists a connection from α to β, which we denote by $C_{\alpha,\beta}$. We have that $Q := \bigcup\limits_{\beta\in S_i\cup T_j} C_{\alpha,\beta}$ is a finite set of Λ and therefore we can consider the finite dimensional semisimple Lie subalgebra L_Q. Write $L_Q = \bigoplus\limits_{i=1}^{n_Q} L_{Q_i}$, L_{Q_i} being simple Lie subalgebras of L_Q. By

Lemma 4.3, there exists $L_{Q_{i_0}}$ such that $L_\alpha \subset L_{Q_{i_0}}$. Finally, by Lemma 4.5, $L_{S_i}, L_{T_j} \subset L_{Q_{i_0}}$. Therefore, R is a direct system with the inclusion.

1. Let us denote $\lim\limits_{\rightarrow} R = (L', \{e_j\}_j)$. As $(L, \{i_j\}_j)$, where i_j denotes the inclusion mapping, satisfies the conditions of the direct limit for R, the universal property of the direct limits shows the existence of a unique monomorphism $\Phi : L' \to L$ such that $\Phi \circ e_j = i_j$. Since $L' = \bigcup\limits_j e_j(L_j)$, (see §3 and [3]), we have $\Phi(L') = \Phi(\bigcup\limits_j e_j(L_j)) = \bigcup\limits_j L_j$, and therefore Φ is an isomorphism from L' onto $\bigcup\limits_j L_j$. Finally, we assert that $L = \bigcup\limits_j L_j$. Indeed, if $x \in L$, by the Cartan decomposition of L and Corollary 4.2 $x = \sum\limits_{i=1}^{n} \lambda_{\alpha_i} h_{\alpha_i} + \sum\limits_{j=1}^{m} v_{\beta_j}$ with $\alpha_i \in \Lambda$, $v_{\beta_j} \in L_{\beta_j}$ and $\lambda_{\alpha_i} \in \mathbb{C}$. Consider $T = \{\alpha_i : i = 1, ..., n\} \cup \{\beta_j : j = 1, ..., m\} \subset \Lambda$ and, following the above notation, $T' = \bigcup\limits_{\beta \in T} C_{\gamma,\beta}$, γ being a fixed element of T. We have T' is a finite set of Λ that gives us the semisimple finite dimensional Lie algebra L_T. Write $L_T = \bigoplus\limits_{i=1}^{r} L_{T_i}$, where L_{T_i}, $i = 1, ..., r$ are simple finite dimensional Lie algebras. As R is a direct system for the inclusion then there exists a finite dimensional simple Lie subalgebra L_P such that $\bigcup\limits_{i=1}^{r} L_{T_i} \subseteq L_P$ and therefore $x \in L_P$. The proof of 1, is complete.

To see 2., firstly notice that the infinite dimensional character of L let us remove the exceptional Lie algebras of R, and secondly that (i) each L_i is contained in one of type A_l or else (ii) there exists L_{i_0} so that $L_i \supset L_{i_0}$ implies that L_i is of type B_l, C_l or D_l. In each of the two cases is possible to define a subnet satisfying assertion 1. ∎

As consequence of Theorem 4.2 and the properties of the direct limits given in §3 we have the following theorem:

Theorem 4.3 (Main Theorem). *Let L be an infinite dimensional simple complex Lie algebra having a Cartan decomposition relative to a Cartan subalgebra H. Then L is isomorphic to one of the followings:*

1. *A^-, with A a simple associative algebra.*
2. *$Skw(A, \xi)$ with A as above and ξ an involutive antiautomorphism of A.*

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