Bergmann Transformations

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Abstract

The Martinez construction of fractions from a Jordan algebra requires a Jordan derivation involving certain quadratic multiplications on the original algebra. We study a general Bergmann construction of such structural transformations in the context of Jordan pairs. The Bergmann transformations corresponding to fractions are defined only on subpairs determined by sesquiprincipal inner ideals, dominions, and we give criterion for a creating structural transformations on them. These results will be applied to the creation of Jordan algebras of fractions, and the methods should have future application to the problem of creating fractions for Jordan pairs.[1]

Throughout, we consider algebraic systems over an arbitrary ring of scalars Φ. A *Jordan pair*, a pair $\mathcal{V} = (V^+, V^-)$ of Φ-modules with compositions $(x,a) \mapsto Q_x(a) \in V^\tau$ for $(x,a) \in V^\tau \times V^{-\tau}$, $\tau = \pm$, which are quadratic in x and linear in a, and satisfy the following axioms *strictly* (in all scalar extensions, equivalently, all their linearizations hold in $\mathcal{V}$ itself): for all $x, y \in V^\tau, a, b \in V^{-\tau}$

$$\text{(JP1)}\quad D_{x,a}Q_x = Q_x D_{a,x}, \quad \text{(JP2)}\quad D_{Q_x a,a} = D_{x,Q_a(x)}, \quad \text{(JP3)}\quad Q_{Q_x a} = Q_x Q_a Q_x,$$

where as usual we set $Q_{x,y} := Q_{x+y} - Q_x - Q_y$, which gives the trilinear product $\{x,a,y\} := Q_{x,y}(a) =: D_{x,a}(y)$ with $\{V^\tau V^{-\tau} V^\tau\} \subseteq V^\tau$. We will economize on superscripts and use typography instead, denoting, for a fixed $\tau = \pm$, elements of V^τ by x, y, z, w and elements of $V^{-\tau}$ by a, b, c. We can turn V^τ into a Jordan algebra $(V^\tau)^{(a)}$ via $U_x y := Q_x Q_a y$, $x^{(2,a)} := Q_x a$.

We will use [3] as reference for most results about Jordan pairs, with some results on homotopes, dominions, and universal envelopes from [5]. The following formulas are used frequently enough in the paper for us to display them:

$$\begin{aligned}
&(0.1.1)\quad D_{x,a}Q_y + Q_y D_{a,x} = Q_{\{x,a,y\},y},\\
&(0.1.2)\quad D_{x,Q_a y} = D_{\{x,a,y\},a} - D_{y,Q_a x} = D_{x,a}D_{y,a} - Q_{x,y}Q_a,\\
&\qquad\qquad D_{Q_a y,x} = D_{a,\{y,a,x\}} - D_{Q_a x,y} = D_{a,y}D_{a,x} - Q_a Q_{y,x},\\
&(0.1.3)\quad Q_{Q_x a,y} = Q_{x,y}D_{a,x} - D_{y,a}Q_x = D_{x,a}Q_{x,y} - Q_x D_{a,y},\\
&(0.1.4)\quad Q_{\{x,a,y\}} + Q_{Q_x Q_a y,y} = Q_x Q_a Q_y + Q_y Q_a Q_x + D_{x,a}Q_y D_{a,x},\\
&(0.1.5)\quad Q_{Q_x Q_a y, D_{x,a}y} = Q_x Q_a Q_y D_{a,x} + D_{x,a}Q_y Q_a Q_x,\\
&(0.1.6)\quad Q_{\alpha x + Q_x a} = B_{\alpha,x,a}Q_x = Q_x B_{\alpha,a,x},\ Q_{B_{\alpha,x,a}y} = B_{\alpha,x,a}Q_y B_{\alpha,a,x},\\
&\qquad\qquad (B_{\alpha,x,a} := \alpha^2\mathbf{1} + \alpha D_{x,a} + Q_x Q_a),\\
&(0.1.7)\quad x^{(n+1,a)} = Q_x a^{(n,x)}, \quad D_{x^{(n,a)},a^{(k,x)}} = D_{x,a^{(n+k-1,x)}} = D_{x^{(n+k-1,a)},a},\\
&(0.1.8)\quad D_{Q_x Q_a y, a^{(m-1,x)}} - D_{D_{x,a}y, a^{(m,x)}} + D_{y,a^{(m+1,x)}} = 0.
\end{aligned}$$

The *multiplication envelope* $\mathcal{M}(\mathcal{V})$ is the subalgebra $\mathrm{End}(\mathcal{V}) := \mathrm{End}(V^+ \oplus V^-)$ generated by all $Q_x, D_{x,a}$. We make use of the *universal multiplication envelope* $\mathcal{UME}(\mathcal{V})$, encoding the action of linear multiplication operators from $\mathcal{V}$ on all possible bimodules, and the *universal polynomial envelope* $\mathcal{UPE}(\mathcal{V})$, encoding the action of multiplications by $\mathcal{V}$ on all extensions $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$ [5]. The

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generic indeterminates in $\mathcal{UPE}(\mathcal{V})$ will be denoted by $\tilde{\tilde{x}}, \tilde{\tilde{a}}$ etc.; they can be specialized to elements of any extension. The universal multiplication envelope can be thought of as the generic polynomials in $\mathcal{UPE}(\mathcal{V})$ homogeneous of degree 1 in some $\tilde{\tilde{x}}, \tilde{\tilde{a}}$ in $\mathcal{UPE}(\mathcal{V})$ [5, 2.4].

If the element s *dominates* n in $V^{-\sigma}$ in the sense that there are $N^\tau, S^\tau \in \mathrm{End}(V^\tau)$ with $Q_n = N^{-\sigma}Q_s = Q_sN^\sigma$, $Q_{n,s} = S^{-\sigma}Q_s = Q_sS^\sigma$, then [5, 3.2] the **dominion** $K^{-\sigma}_{s\succ n} := \Phi n + \Phi s + Q_sV^\sigma$ is an inner ideal, whose elements $x := \gamma n + \alpha s + Q_sa$, $y := \alpha s + Q_sa$, $z := Q_sa$ all have Q-operators which can be "divided by Q_s",

$$
\begin{aligned}
&(0.2.1)\quad Q_{n,z} = M_a^{-\sigma}Q_s = Q_sM_a^\sigma,\ \big(M_a^{-\sigma}{:=}S^{-\sigma}D_{s,a}{-}D_{n,a},\ M_a^\sigma{:=}D_{a,s}S^{-\sigma}{-}D_{a,n}\big),\\
&(0.2.2)\quad Q_{n,y} = G^{-\sigma}Q_s = Q_sG^\sigma \quad \big(G^\tau = \alpha S^\tau + M_a^\tau\big),\\
&(0.2.3)\quad Q_x = X^{-\sigma}Q_s = Q_sX^\sigma \quad \big(X^\tau = \gamma^2N^\tau + \gamma G^\tau + B^\tau\big).
\end{aligned}
$$

1 Bergmann Triples and Pairs

A **structural pair** $\mathcal{T} = (T^+, T^-)$, or **structural transformation of a Jordan pair** $\mathcal{V}$, consists of two linear transformations $T^\tau \in \mathrm{End}(V^\tau)$ (the superscript indicates the domain and range) satisfying

$$(1.1.1)\qquad Q_{T^\tau(x)} = T^\tau Q_x T^{-\tau}.$$

for all $x \in V^\tau$ and $\tau = \pm$. The structural transformations of a pair form a submonoid, the *structure monoid* $Str(\mathcal{V})$, of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$ under $\mathcal{T}_1\mathcal{T}_2 := (T_1^+T_2^+, T_2^-T_1^-)$. A structural $\mathcal{T}$ induces a pair of inner ideals $(I^+, I^-) = (T^+(V^+), T^-(V^-))$ [since $Q_{I^\tau}V^{-\tau} \subseteq I^\tau$], as well as a **homotopic** Jordan pair $\mathcal{V}^{(\mathcal{T})} = (V^+, V^-)$ under $Q_x^{(\mathcal{T})\tau} := Q_xT^{-\tau}$, $D_{x,a}\mathcal{T} := D_{x,T^{-\tau}(a)}$. A second structural $\mathcal{S} = (S^\tau, S^{-\tau})$ then induces a homomorphism

$$(1.1.2)\qquad \mathcal{V}^{(\mathcal{STS})} \to \mathcal{V}^{(\mathcal{T})}.$$

Familiar examples of structural transformations are the pairs $\mathcal{T}_\varphi := (\varphi^+, (\varphi^-)^{-1})$ determined by an automorphism (φ^+, φ^-) of the pair, and the **Bergmann transformations** $\mathcal{B}_{\alpha,x,a}$, in particular **principal structural transformations** $\mathcal{T}_{x,a}$, determined by any $x \in V^\tau, a \in V^{-\tau}$,

$$\mathcal{B}_{\alpha,x,a} := (B_{\alpha,x,a}, B_{\alpha,a,x}),\quad \mathcal{T}_{x,a} := \mathcal{B}_{0,x,a} = (Q_xQ_a, Q_aQ_x)\quad \textit{for}\quad \alpha \in \Phi, x \in V^\tau, a \in V^{-\tau}$$

which result immediately from (JP3) and (0.1.6).

A structural transformation is **inner** if it is built from multiplications, $T^\pm \in \mathcal{M}(\mathcal{V})$. The most important structural transformations for us are the **generic structural transformations** $\mathcal{T} \in \mathcal{UME}(\mathcal{V})$ which are built generically out of multiplications from $\mathcal{V}$, satisfying (1.1.1) for generic variables $\tilde{\tilde{x}}$, so that (1.1.1) continues to hold for the action of T in all extensions of $\mathcal{V}$:

$$(1.1.1\mathit{Gen})\qquad T^\tau \in \mathcal{UME}(\mathcal{V}),\quad T^{-\tau} = (T^\tau)^*\quad \textit{with}\quad Q_{T^\tau(\tilde{\tilde{x}})} = T^\tau Q_{\tilde{\tilde{x}}}T^{\tau *} \in \mathcal{UME}(\mathcal{V}).$$

The basic examples of generic transformations are the Bergmann and principal transformations.

A **monostructural transformation** is a single transformation $T^\tau \in \mathrm{Hom}(V^{-\tau}, V^\tau)$ (the superscript indicates the codomain) satisfying

$$(1.2.1)\qquad Q_{T^\tau(a)} = T^\tau Q_a T^\tau.$$

for all $a \in V^{-\tau}$.[2] The product of two monostructural transformations T^+, T^- gives a structural transformation (T^+T^-, T^-T^+). Any monostructural T^τ with domain $V^{-\tau}$ yields an inner ideal $I^\tau = T^\tau(V^{-\tau})$, and turns the domain $V^{-\tau}$ into a **homotopic** Jordan triple system $(V^{-\tau})^{(T)}$ via

[2] These are odd in the sense of gradings: if we rewrite V^+, V^- as a $\mathbb{Z}_2$-grading $V_{\bar{0}}, V_{\bar{1}}$ then structural transformations are even, $\mathcal{T}(V_{\bar{i}}) \subseteq V_{\bar{i}}$, while monostructural transformations are odd, $T^\tau(V_{\bar{\tau}+\bar{1}}) \subseteq V_{\bar{\tau}}$.

$P_x y := Q_x T(y), L_{x,y} = D_{x,T(y)}$; in the important case of a *principal monostructural transformation* $T^\tau = Q_t$ for $t \in V^\tau$, the *elemental homotope* $(V^{-\tau})^{(t)} := (V^{-\tau})^{(Q_t)}$ becomes a Jordan *algebra* via $x^{(2,t)} := Q_x t$. For each element $z^{-\tau} \in V^{-\tau}$ a monostructural T^τ induces an algebra homomorphism

$$(1.2.2) \qquad T^\tau : (V^{-\tau})^{(T(z^{-\tau}))} \to (V^\tau)^{(z^{-\tau})}$$

of the elemental homotopes, since $T\big(U_x^{(T(z))} y\big) = T\big(Q_x Q_{T(z)} y\big) = TQ_x TQ_z Ty = Q_{T(x)} Q_z T(y) = U^{(z)}_{T(x)} T(y)$ and $T(x^{(2,T(z))}) = T(Q_x T(z)) = Q_{T(x)} z = T(x)^{(2,z)}$.

An **inner** monostructural transformation $T \in \mathcal{M}(\mathcal{V})$ is built out of multiplications from $\mathcal{V}$, while a **generic** monostructural $T \in \mathcal{UME}(\mathcal{V})$ satisfies (1.2.1) in $\mathcal{UME}(\mathcal{V})$. Note that because they are built out of multiplications, the inner T^τ send any ideal I into itself, and hence induce structural or monostructural transformations on any quotient $\overline{\mathcal{V}}$. All generic, and most inner, structural or monostructural transformations also induce structural or monostructural transformations on all extensions $\widetilde{\mathcal{V}} \supseteq \overline{\mathcal{V}}$.[3] The basic examples of generic monostructural transformations are the principal Q_t.

To put these structural transformations in a more general context, for any Jordan triple J the structure monoid $Str(J)$ consists of all pairs $(T_1, T_2) \in \mathrm{End}(J) \times \mathrm{End}(J)^{op}$ such that $P_{T_i(x)} = T_i P_x T_j$ $(i = 1, 2, j = 3 - i)$, with canonical reversal involution $(T_1, T_2)^* = (T_2, T_1)$. In the case $J = \mathcal{V}^{pol}$ of the polarized triple of a Jordan pair, if we set $G^{\sigma,\tau} := Str(J) \cap (\mathrm{Hom}(V^\tau, V^\sigma) \times \mathrm{Hom}(V^{-\sigma}, V^{-\tau})^{op})$ we obtain a 2×2-graded object with associative products $G^{\sigma,\tau} \dot{G}^{\tau,\rho} \subseteq G^{\sigma,\rho}$ via $(T_1, T_2) \cdot (S_1, S_2) = (T_1 S_1, S_2 T_2)$ and involution $G^{\sigma\tau *} = G^{-\tau,-\sigma}$. Here the structural transformations are $G^{+,+} = Str(\mathcal{V}) = G^{-,-*}$, and the monostructural transformations are the symmetric elements $(T_1, T_1) \in G^{\sigma,-\sigma}$.

Since the defining structural condition is quadratic in x, the structural transformations do not form a linear space. In some situations it is possible to glue two structural transformations together by means of "glue". The archetypal example for monostructural transformations is Q_{x_1}, Q_{x_2} being glued together by Q_{x_1,x_2} to form $Q_{x_1+x_2}$.

Bergmann Triple Proposition 1.3 *We say $\mathcal{G} = (G^+, G^-)$ consisting of two linear transformations $G^\tau : V^\tau \to V^\tau$ is* **structural glue** *for two structural pairs $\mathcal{T}_1, \mathcal{T}_2$, and call $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2)$ a* **Bergmann triple***, if the following two Bergmann Triple Gluing Relations hold for all $x \in V^\tau$:*

$$(1.3.1) \quad \textbf{(T-Glue 1):} \quad T_i^\tau Q_x G^{-\tau} + G^\tau Q_x T_i^{-\tau} = Q_{T_i^\tau(x), G^\tau(x)} \quad (i = 1, 2),$$

$$(1.3.2) \quad \textbf{(T-Glue 2):} \quad T_1^\tau Q_x T_2^{-\tau} + T_2^\tau Q_x T_1^{-\tau} + G^\tau Q_x G^{-\tau} = Q_{G^\tau(x)} + Q_{T_1^\tau(x), T_2^\tau(x)}.$$

In this case, for any scalars α_1, α_2 in a scalar extension Ω we we can glue the two pairs together via $\mathcal{G}$ obtain a **Bergmann transformation** *on $\mathcal{V}_\Omega$*

$$(1.3.3) \qquad \mathcal{X}_{\alpha_1,\alpha_2,\mathcal{T}_1,\mathcal{G},\mathcal{T}_2} : \quad X^\tau := \alpha_1^2 T_1^\tau + \alpha_1\alpha_2 G^\tau + \alpha_2^2 T_2^\tau.$$

Structurality of the T_i and the conditions T-Glue 1-2 *are necessary and sufficient for $\mathcal{X}_{\alpha_1,\alpha_2,\mathcal{T}_1,\mathcal{G},\mathcal{T}_2}$ to be structural for all scalars in all possible extensions Ω.*

Proof: On the one hand we have

[3] Most inner structural transformations are structural by the Jordan pair axioms, but it is possible for a transformation built from outer multiplications to be accidentally structural on $\mathcal{V}$ but not on all $\widetilde{\mathcal{V}}$: in $\mathcal{V} = \mathcal{V}(A^+)$, $\widetilde{\mathcal{V}} = \mathcal{V}(\widetilde{A}^+)$ for $A := \Phi\langle x, y\rangle / I$, $\widetilde{A} := \Phi\langle x, y, z\rangle / \widetilde{I} = A\langle z\rangle$ quotients of the free associative algebras on x, y, z by the ideals $I, \widetilde{I}$ generated by the elements $[[x, y], x], [[x, y], y]$ the multiplication $T = [V_x, V_y] = Ad_{[x,y]}$ is trivially structural on $\mathcal{V}$ since it vanishes identically ($[x, y]$ is by construction in the center of A), but not on $\widetilde{A}$ because $U_{T(z)} z = [x, y]zz[x, y]z + z[x, y]zz[x, y] - [xy]zzz[x, y] - z[x, y]z[x, y]z \notin \widetilde{A}[x, y] + [x, y]\widetilde{A} \supseteq T(\widetilde{A})$ since $z[x, y]z[x, y]z \notin \widetilde{A}[x, y] + [x, y]\widetilde{A}$.

$$
\begin{aligned}
Q_{X^\tau(x)} &= Q_{\alpha_1^2 T_1^\tau(x)} + Q_{\alpha_1^2 T_1^\tau(x),\alpha_1\alpha_2 G^\tau(x)} + Q_{\alpha_1\alpha_2 G^\tau(x)} + Q_{\alpha_1^2 T_1^\tau(x),\alpha_2^2 T_2^\tau(x)} \\
&\quad + Q_{\alpha_2^2 T_2^\tau(x),\alpha_1\alpha_2 G^\tau(x)} + Q_{\alpha_2^2 T_2^\tau(x)} \\
&= \alpha_1^4 Q_{T_1^\tau(x)} + \alpha_1^3\alpha_2 Q_{T_1^\tau(x),G^\tau(x)} + \alpha_1^2\alpha_2^2\big(Q_{G^\tau(x)} + Q_{T_1^\tau(x),T_2^\tau(x)}\big) \\
&\quad + \alpha_1\alpha_2^3 Q_{T_2^\tau(x),G^\tau(x)} + \alpha_2^4 Q_{T_2^\tau(x)},
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
X^\tau Q_x X^{-\tau} &= \alpha_1^4 T_1^\tau Q_x T_1^{-\tau} + \alpha_1^3\alpha_2\big(T_1^\tau Q_x G^{-\tau} + G^\tau Q_x T_1^{-\tau}\big) \\
&\quad + \alpha_1^2\alpha_2^2\big(T_1^\tau Q_x T_2^{-\tau} + T_2^\tau Q_x T_1^{-\tau} + G^\tau Q_x G^{-\tau}\big) \\
&\quad + \alpha_1\alpha_2^3\big(T_2^\tau Q_x G^{-\tau} + G^\tau Q_x T_2^{-\tau}\big) + \alpha_2^4 T_2^\tau Q_x T_2^{-\tau}.
\end{aligned}
$$

By identifying coefficients of like powers $\alpha_1^i\alpha_2^j$ we see structurality and the gluing conditions suffice for structurality of $\mathcal{X}$, and these conditions are necessary for structurality in the extensions where $\alpha_i = t_i$ are indeterminates in $\Omega = \Phi[t_1, t_2]$ by independence of the powers of t_1, t_2. ■

For generic structural transformations $T_i \in \mathcal{UME}(\mathcal{V})$ we have obvious notions of **generic glue, generic Bergmann triple, generic Bergmann transformation** satisfying (1.3) generically. The archetypal example of a generic Bergmann triple is the principal triple $(\mathcal{T}_{x_1,a}, \mathcal{G}, \mathcal{T}_{x_2,a})$ with generic glue $\mathcal{G} := (Q_{x_1,x_2}Q_a, Q_aQ_{x_1,x_2})$ for $x_i \in V^\tau, a \in V^{-\tau}$. Clearly these $\mathcal{T}_i$ are structural transformations, and $\mathcal{G}$ is structural glue as in (1.3.1-2) since for $\sigma = \pm\tau$ we have generically in $\approx\atop{x}$ that

$$
\begin{aligned}
&\text{(TG1)} \quad T_i^\sigma Q_{\widetilde{\widetilde{x}}} G^{-\sigma} + G^\sigma Q_{\widetilde{\widetilde{x}}} T_i^{-\sigma} = Q_{T_i^\sigma(\widetilde{\widetilde{x}}),G^\sigma(\widetilde{\widetilde{x}})}, \\
&\text{(TG2)} \quad T_i^\sigma Q_{\widetilde{\widetilde{x}}} T_j^{-\sigma} + T_j^\sigma Q_{\widetilde{\widetilde{x}}} T_i^{-\sigma} + G^\sigma Q_{\widetilde{\widetilde{x}}} G^{-\sigma} = Q_{G^\sigma(\widetilde{\widetilde{x}})} + Q_{T_i^\sigma(\widetilde{\widetilde{x}}),T_j^\sigma(\widetilde{\widetilde{x}})}
\end{aligned}
$$

which follow directly from the linearizations

$$
\begin{aligned}
&\text{(TG1}'\text{)} \quad Q_{x_i} Q_{\widetilde{\widetilde{c}}} Q_{x_i,x_j} + Q_{x_i,x_j} Q_{\widetilde{\widetilde{c}}} Q_{x_i} = Q_{Q_{x_i,x_j}(\widetilde{\widetilde{c}}),Q_{x_i}(\widetilde{\widetilde{c}})}, \\
&\text{(TG2}'\text{)} \quad Q_{x_i} Q_{\widetilde{\widetilde{c}}} Q_{x_j} + Q_{x_j} Q_{\widetilde{\widetilde{c}}} Q_{x_i} + Q_{x_i,x_j} Q_{\widetilde{\widetilde{c}}} Q_{x_i,x_j} = Q_{Q_{x_i,x_j}(\widetilde{\widetilde{c}})} + Q_{Q_{x_i}(\widetilde{\widetilde{c}}),Q_{x_j}(\widetilde{\widetilde{c}})}
\end{aligned}
$$

of (JP3) for all $\widetilde{\widetilde{c}} = Q_{\widetilde{\widetilde{a}}}\widetilde{\widetilde{x}}$ (when $\sigma = \tau$) and $\widetilde{\widetilde{c}} = \widetilde{\widetilde{x}}$ (when $\sigma = -\tau$). Here the resulting Bergmann operator $\mathcal{X}_{\alpha_1,\alpha_2,\mathcal{T}_1,\mathcal{G},\mathcal{T}_2} = \mathcal{T}_{\alpha_1x_1+\alpha_2x_2,a}$ is principal.

The Bergmann terminology comes from the fact that for the special case $\mathcal{T}_1 = (Id, Id), \mathcal{T}_2 = (Q_xQ_a, Q_aQ_x)$, $\mathcal{G} = (D_{x,a}, D_{a,x})$ the Bergmann transformation $\mathcal{X}$ is just the ordinary Bergmann transformation $\mathcal{B}_{x,a}$. Here it is a matter of gluing a structural transformation to the identity structural transformation using as glue a **Lie-structural transformation** or **Lie-structural pair**

$$
\text{(1.4)} \qquad \mathcal{D} = (D^+, D^-): \qquad Q_{D^\tau(x),x} = D^\tau Q_x + Q_x D^{-\tau} \quad (x \in V^\tau,\ \tau = \pm).
$$

Bergmann Pair Proposition 1.5 *We say a Lie-structural transformation $\mathcal{D}$ is* **structural glue** *for a structural transformation $\mathcal{T}$, and call $(\mathcal{D}, \mathcal{T})$ a* **Bergmann pair**, *if the following two Bergmann Pair Gluing Relations hold for all $x \in V^\tau$:*

(1.5.1) **(P-Glue 1):** $T^\tau Q_x D^{-\tau} + D^\tau Q_x T^{-\tau} = Q_{T^\tau(x),D(x)}$,

(1.5.2) **(P-Glue 2):** $T^\tau Q_x + Q_x T^{-\tau} + D^\tau Q_x D^{-\tau} = Q_{D^\tau(x)} + Q_{T^\tau(x),x}$.

In this case, for any scalars α_1, α_2 in a scalar extension Ω we we can glue the two pairs together via $\mathcal{G}$ obtain a **Bergmann transformation** *on $\mathcal{V}_\Omega$*

$$
\text{(1.5.3)} \qquad
\begin{aligned}
B_{\alpha_1,\alpha_2,\mathcal{D},\mathcal{T}} &:= \mathcal{X}_{\alpha_1^2\mathcal{I},\alpha_1\alpha_2\mathcal{D},\alpha_2^2\mathcal{T}} := \alpha_1^2\mathcal{I} + \alpha_1\alpha_2\mathcal{D} + \alpha_2^2\mathcal{T}, \\
B_{\alpha,\mathcal{D},\mathcal{T}} &:= B_{\alpha,1,\mathcal{D},\mathcal{T}} := \alpha^2\mathcal{I} + \alpha\mathcal{D} + \mathcal{T}.
\end{aligned}
$$

Structurality of T, Lie-structurality of D, and the conditions P-Glue 1-2 *are necessary and sufficient for $\mathcal{X}_{\alpha_1,\alpha_2}$ to be structural for all scalars in all possible extensions Ω.*

PROOF: This follows from the above Triple Proposition for $T_1 = \mathbf{1}_{\mathcal{V}}, T_2 = T$, $\mathcal{G} = \mathcal{D}$ since here T_1 is automatically structural, (T-Glue 1) for i=1 is precisely the Lie-structural condition (1.3.4), (T-Glue 1) for i=2 is (P-Glue 1), and (T-Glue 2) is (P-Glue 2). ■

The Lie-structural transformations form a subalgebra, the *Lie structure algebra $StrL(\mathcal{V})$*, of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)^{op}$ under $[\mathcal{D}_1, \mathcal{D}_2] := ([D_1^+, D_2^+], [D_2^-, D_1^-])$. We again have obvious notions of **generic Lie-structural transformation** $\mathcal{D}$ and **generic Bergmann pair** $(\mathcal{T}, \mathcal{D})$ for a generic structural transformation $\mathcal{T}$. The archetypal example of a generic Bergmann pair is, of course, the principal pair $\mathcal{D}_{x,a} = (D_{x,a}, D_{a,x})$, $\mathcal{T}_{x,a} = (Q_x Q_a, Q_a Q_x)$, with $\mathcal{B}_{\alpha,x,a} = (B_{\alpha,x,a}, B_{\alpha,a,x})$ the usual generic Bergmann transformation. If x happens to be invertible, then $\mathcal{B}_{\alpha,x,a}$ reduces to a principal structural transformation $\mathcal{T}_{x,\tilde{a}} = (Q_x Q_{\alpha x^{-1}+a},\ Q_{\alpha x^{-1}+a} Q_x)$.

2 Cancellation Operators

It is clear from the definitions that the restriction to an invariant subpair $\mathcal{V} \subseteq \widetilde{\mathcal{V}}$ of a structural transformation, monostructural transformation, Bergmann triple, or pair on $\widetilde{\mathcal{V}}$ remains such on $\mathcal{V}$, but the restriction of a generic structural transformation need not remain generic: being generic on $\mathcal{V}$ means it remains so on all extensions of $\mathcal{V}$, which may not be extensions of $\widetilde{\mathcal{V}}$ and so are not guaranteed to remain structural. On the other hand, if the monostructural transformation does *not* leave the subpair invariant, we can sometimes shove it down into the subpair. In the case of Ore fractions [4, 1, 2], $\tilde{q} = Q_s^{-1} n \in \widetilde{\mathcal{V}}$ does not leave $\mathcal{V}$ invariant, but for a reduced denominator $Q_s \widetilde{Q}_{\tilde{q}}, \widetilde{Q}_{\tilde{q}} Q_s, \widetilde{D}_{\tilde{q},s}, \widetilde{D}_{s,\tilde{q}}$ all leave $\mathcal{V}$ invariant. Here we begin a convention that our denominators s and our inner ideals K will always belong to $V^{-\sigma}$. We say a *transformation* $\widetilde{T}^\sigma \in \mathrm{Hom}(\widetilde{V}^{-\sigma}, \widetilde{V}^\sigma)$ has **denominator** $s \in V^{-\sigma}$ if Q_s shoves $\widetilde{T}$ (or rather, they shove each other) down to endomorphisms on $\mathcal{V}$ in the sense that

$$\text{(2.1)} \quad \text{(Transformation Denominator)}: N^{-\sigma} := Q_s \widetilde{T}^\sigma \in \mathrm{End}(V^{-\sigma}),\ N^\sigma := \widetilde{T}^\sigma Q_s \in \mathrm{End}(V^\sigma),$$

which implies that the N^τ are s-**cancellation operators** which result by *cancelling* Q_s on the right and left from the operator $T^{-\sigma} := Q_s \widetilde{T}^\sigma Q_s$,

$$\text{(2.2)} \qquad T^{-\sigma} = N^{-\sigma} Q_s = Q_s N^\sigma \in \mathrm{Hom}(V^\sigma, Q_s V^\sigma).$$

[More precisely, $Q_s \widetilde{T}^\sigma$, $\widetilde{T}^\sigma Q_s$, $Q_s \widetilde{T}^{-\sigma} Q_s$ leave $\mathcal{V}$ invariant, and $N^{-\sigma}, N^\sigma, T^{-\sigma}$ are their restrictions to $\mathcal{V}$.] If Q_s is actually invertible, then $\widetilde{T}^\sigma = Q_s^{-1} N^{-\sigma} = N^\sigma Q_s^{-1}$ (hence the description of Q_s as "denominator"). We say $\widetilde{T}^\sigma$ has **generic denominator** s if $N^\tau \in \mathcal{UME}(\mathcal{V})$ satisfy (2.1) generically in $\mathcal{UME}(\mathcal{V})$ and $N^{-\tau} = (N^\tau)^*$. Note that the cancellation operators are in general not uniquely determined, but in many important cases there are natural inner or generic operators N^τ, S^τ to choose. For example, (0.2) shows that if s dominates n then N^τ, S^τ, M_a, G^τ, X^τ are cancellation operators for $T^{-\sigma} = Q_n$, $Q_{n,s}$, $Q_{n,z}$, $Q_{n,y}, Q_x$.

If $\widetilde{T}^\sigma$ is monostructural to begin with, then the induced $T^{-\sigma}$ is monostructural on $\widetilde{\mathcal{V}}$ leaving $\mathcal{V}$ invariant, and its s-cancellation operator $(N^{-\sigma}, N^\sigma)$ is a structural transformation on $\mathcal{V}$,

$$\text{(2.3)} \qquad Q_{N^{-\sigma}(x)} = N^{-\sigma} Q_x N^\sigma, \quad Q_{N^\sigma(a)} = N^\sigma Q_a N^{-\sigma} \qquad (x \in V^{-\sigma}, a \in V^\sigma),$$

since $Q_{Q_s \widetilde{T}(x)} = Q_s Q_{\widetilde{T}(x)} Q_s = Q_s \widetilde{T} Q_x \widetilde{T} Q_s$ on $\widetilde{\mathcal{V}}$ becomes $Q_{N^{-\sigma}(x)} = N^{-\sigma} Q_x N^\sigma$ on $\mathcal{V}$, and dually $Q_{\widetilde{T} Q_s a} = \widetilde{T} Q_s Q_a Q_s \widetilde{T}$ becomes $Q_{N^\sigma(a)} = N^\sigma Q_a N^{-\sigma}$. If $\widetilde{T}^\sigma$ is a generic monostructural transformation, then (2.2) holds in $\mathcal{UME}(\widetilde{\mathcal{V}})$ (not necessarily in $\mathcal{UME}(\mathcal{V})$).

The archetypal example for our development is the restriction to $\mathcal{V}$ of a principal structural transformation by an element $\tilde{q}$ of an ambient pair $\widetilde{\mathcal{V}}$. Following the example of reduced Ore fractions, we say that an *element* $\tilde{q} \in \widetilde{\mathcal{V}}^\sigma$ for $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$ has **denominator** $s \in V^{-\sigma}$ if its multiplication operators $Q_s\widetilde{Q}_{\tilde{q}}$, $Q_{\tilde{q}}Q_s$, $D_{\tilde{q},s}$, $D_{s,\tilde{q}}$ all leave $\mathcal{V}$ invariant, and its homotope powers fall in $\mathcal{V}$:

$$(2.4) \qquad \text{(Elemental Denominator):} \quad \begin{array}{l} Q_s\widetilde{Q}_{\tilde{q}},\ D_{s,\tilde{q}} \in \mathrm{End}(V^{-\sigma}),\ \widetilde{Q}_{\tilde{q}}Q_s,\ D_{\tilde{q},s} \in \mathrm{End}(V^{\sigma}), \\ s_2 := Q_s(\tilde{q}) \in V^{-\sigma},\ q_2 := \widetilde{Q}_{\tilde{q}}(s),\ q_3 := \widetilde{Q}_{\tilde{q}}Q_s(\tilde{q}) \in V^{\sigma}. \end{array}$$

(this is a much stronger condition than merely $\widetilde{T} = \widetilde{Q}_{\tilde{q}}$ having denomiator s). We call $n := s_2$ the **numerator** of $\tilde{q}$ ($\tilde{q}$ has many numerators and denominators).

Pseudo-Principal Example 2.5 *Suppose $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$ and the element $\tilde{q} \in \widetilde{\mathcal{V}}^\sigma$ has denominator $s \in V^{-\sigma}$ as in (2.4), so by restriction we have cancellation operators $N^\tau, S^\tau \in \mathrm{End}(V^\tau)$ $(\tau = \pm\sigma)$ and elements s_2, q_2, q_3 given by*

$$\begin{array}{ll} (2.5.0) & n := s_2 := Q_s(\tilde{q}), \quad q_2 := Q_{\tilde{q}}(s),\ q_3 := Q_{\tilde{q}}(n) \in V^\sigma, \\ (2.5.1) & N^{-\sigma} := Q_sQ_{\tilde{q}}|_V, \quad N^\sigma := Q_{\tilde{q}}Q_s|_V, \\ (2.5.2) & S^{-\sigma} := D_{s,\tilde{q}}|_V, \qquad S^\sigma := D_{\tilde{q},s}|_V. \end{array}$$

Then for generic $z := Q_s\tilde{\tilde{a}}$, $y := \tilde{\tilde{\alpha}}s + z$, $x := \tilde{\tilde{\gamma}}n + y$, $\tilde{\tilde{a}} \in \tilde{\tilde{V}}^\sigma$, $\tilde{\tilde{w}} \in \tilde{\tilde{V}}^\tau$, $\tilde{\tilde{\alpha}}, \tilde{\tilde{\gamma}} \in \tilde{\tilde{\Phi}}$ the unrestricted generic maps N^τ, S^τ, $M^\tau_{\tilde{\tilde{a}}}$, B^τ, G^τ, X^τ are cancellation operators in $\mathcal{UME}(\widetilde{\mathcal{V}})$:

$$\begin{array}{ll} (2.5.3) & (n\text{-Cancellation}): \quad Q_n = N^{-\sigma}Q_s = Q_sN^\sigma, \quad (N^\tau)^* = N^{-\tau}, \\ (2.5.4) & (n,s\text{-Cancellation}): \quad Q_{n,s} = S^{-\sigma}Q_s = Q_sS^\sigma, \quad (S^\tau)^* = (S^{-\tau}), \\ (2.5.5) & (z\text{-Cancellation}): \quad Q_z = Q_sQ_{\tilde{\tilde{a}}}Q_s, \\ (2.5.6) & (n,z\text{-Cancellation}): \quad Q_{n,z} = M^{-\sigma}_{\tilde{\tilde{a}}}Q_s = Q_sM^\sigma_{\tilde{\tilde{a}}}, \quad (M^\tau_{\tilde{\tilde{a}}})^* = M^{-\tau}_{\tilde{\tilde{a}}}, \quad \textit{for} \\ & \quad M^{-\sigma}_{\tilde{\tilde{a}}} := Q_sQ_{\tilde{q},\tilde{\tilde{a}}} = S^{-\sigma}D^{-\sigma}_{s,\tilde{\tilde{a}}} - D_{n,\tilde{\tilde{a}}}, \quad M^\sigma_{\tilde{\tilde{a}}} := Q_{\tilde{q},\tilde{\tilde{a}}}Q_s = D^\sigma_{\tilde{\tilde{a}},s}S^\sigma - D_{\tilde{\tilde{a}},n}, \\ (2.5.7) & (y\text{-Cancellation}): Q_y = B^{-\sigma}_{\tilde{\tilde{\alpha}},s,\tilde{\tilde{a}}}Q_s = Q_sB^\sigma_{\alpha,\tilde{\tilde{a}},s}, \ (B_{\tilde{\tilde{\alpha}},\tilde{\tilde{a}},s})^* = B_{\tilde{\tilde{\alpha}},s,\tilde{\tilde{a}}} \\ (2.5.8) & (n,y\text{-Cancellation}): \quad Q_{n,y} = G^{-\sigma}Q_s = Q_sG^\sigma \quad \textit{for} \quad G^\tau := \tilde{\tilde{\alpha}}S^\tau + M^\tau_{\tilde{\tilde{a}}} = (G^{-\tau})^*, \\ (2.5.9) & (x\text{-Cancellation}): \quad Q_x = X^{-\sigma}Q_s = Q_sX^\sigma \quad \textit{for} \quad X^\tau = \gamma^2N^\tau + \gamma G^\tau + B^\tau = (X^{-\tau})^*, \end{array}$$

and satisfy the following relations in $\mathcal{UPE}(\widetilde{\mathcal{V}})$:[4]

$$\begin{array}{ll} (2.5.10) & (N\text{-Structure}): \quad Q_{N^\tau(\tilde{\tilde{w}})} = N^\tau Q_{\tilde{\tilde{w}}}N^{-\tau}, \\ (2.5.11) & (S\text{-Structure}): \quad S^\tau Q_{\tilde{\tilde{w}}} + Q_{\tilde{\tilde{w}}}S^{-\tau} = Q_{S^\tau(\tilde{\tilde{w}}),\tilde{\tilde{w}}}, \\ (2.5.12) & (\text{P-Glue 1}): \quad N^\tau Q_{\tilde{\tilde{w}}} + Q_{\tilde{\tilde{w}}}N^{-\tau} + S^\tau Q_{\tilde{\tilde{w}}}S^{-\tau} = Q_{S^\tau(\tilde{\tilde{w}})} + Q_{N^\tau(\tilde{\tilde{w}}),\tilde{\tilde{w}}}, \\ (2.5.13) & (\text{P-Glue 2}): \quad N^\tau Q_{\tilde{\tilde{w}}}S^{-\tau} + S^\tau Q_{\tilde{\tilde{w}}}N^{-\tau} = Q_{N^\tau(\tilde{\tilde{w}}),S^\tau(\tilde{\tilde{w}})}. \end{array}$$

If we denote the homotope powers in the Jordan algebras $(V^{-\sigma})^{(\tilde{q})}$, $(V^\sigma)^{(s)}$ by

$$(2.6) \qquad \begin{array}{c} \tilde{q}_1 = \tilde{q} \in \widetilde{V}^\sigma,\ q_2 := \widetilde{Q}_{\tilde{q}}(s),\ q_3 := \widetilde{Q}_{\tilde{q}}(n),\ q_k := \tilde{q}^{(k,s)} \quad (q_k \in V^\sigma \ \ \textit{for } k \geq 2), \\ s_1 = s,\ s_2 = n,\ s_k := s^{(k,\tilde{q})} \quad (s_k \in V^{-\sigma} \ \ \textit{for } k \geq 1), \end{array}$$

then we have generic power-shifting relations

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[4]It is not clear whether these hold generically in $\mathcal{UME}(\mathcal{V})$ without imposing further conditions on the relation between s and n. It is crucial for the arguments here that we consider only extensions $\mathcal{V}' \supseteq \mathcal{V}$ which contain at least $\tilde{q}$, even if not all of $\widetilde{\mathcal{V}}$. We will not bother to distinguish notationally between the generic $\tilde{\tilde{N}}^\sigma = Q_{\tilde{q}}Q_s \in \mathcal{UME}(\widetilde{\mathcal{V}})$ and the map $N^\sigma = Q_{\tilde{q}}Q_s|_{V^\sigma} \in \mathrm{End}(V^\sigma)$, or between $\tilde{\tilde{S}}^\tau$ and S^τ.

(2.6.1N) (N-Shifting): $N^\sigma(q_k) = q_{k+2}, \quad N^{-\sigma}(s_k) = s_{k+2},$

(2.6.1S) (S-Shifting): $S^\sigma(q_k) = 2q_{k+1}, \quad S^{-\sigma}(s_k) = 2s_{k+1},$

(2.6.1Q) (Q-Shifting): $q_{k+1} = \widetilde{Q}_{\tilde{q}}(s_k), \quad s_{k+1} = Q_s(q_k),$

(2.6.1I) (Inner Power Shifting): $D_{s_k,\tilde{q}_1} = D_{s_{k-1},q_2} = \ldots = D_{s_j,q_{k+1-j}},$

$$D_{\tilde{q}_1,s_k} = D_{q_2,s_{k-1}} = \ldots = D_{q_{k+1-j},s_j},$$

generic triality chains

(2.6.2ER) (Elemental Right Triality): $\{N^\sigma(\tilde{\tilde{a}}), s_k, \tilde{\tilde{c}}\} - \{S^\sigma(\tilde{\tilde{a}}), s_{k+1}, \tilde{\tilde{c}}\} + \{\tilde{\tilde{a}}, s_{k+2}, \tilde{\tilde{c}}\} = 0,$
(2.6.2IR) (Inner Right Triality): $D_{N^\sigma(\tilde{\tilde{a}}),s_k} - D_{S^\sigma(\tilde{\tilde{a}}),s_{k+1}} + D_{\tilde{\tilde{a}},s_{k+2}} = 0,$
(2.6.2OR) (Outer Right Triality): $D_{\tilde{\tilde{c}},s_k} N^\sigma - D_{\tilde{\tilde{c}},s_{k+1}} S^\sigma + D_{\tilde{\tilde{c}},s_{k+2}} = 0,$
(2.6.2EL) (Elemental Left Triality): $\{s_k, N^\sigma(\tilde{\tilde{a}}), \tilde{\tilde{x}}\} - \{s_{k+1}, S^\sigma(\tilde{\tilde{a}}), \tilde{\tilde{x}}\} + \{s_{k+2}, \tilde{\tilde{a}}, \tilde{\tilde{x}}\} = 0,$
(2.6.2IL) (Inner Left Triality): $D_{s_k,N^\sigma(\tilde{\tilde{a}})} - D_{s_{k+1},S^\sigma(\tilde{\tilde{a}})} + D_{s_{k+2},\tilde{\tilde{a}}} = 0,$
(2.6.2OL) (Outer Left Triality): $N^{-\sigma} D_{s_k,\tilde{\tilde{c}}} - S^{-\sigma} D_{s_{k+1},\tilde{\tilde{c}}} + D_{s_{k+2},\tilde{\tilde{c}}} = 0,$

and generic relations

(2.6.3) (Commutativity): $N^\tau S^\tau = S^\tau N^\tau = M^\tau_{q_2},$
(2.6.4) (Two N): $(S^\tau)^2 = 2N^\tau + D^\tau_2 \qquad (D^\sigma_2 := D_{q_2,s},\ D^{-\sigma}_2 := D_{s,q_2}),$
(Two Q): $S^{-\sigma} Q_{s,n} = Q_{s,n} S^\sigma = 2Q_n + Q_{s_3,s}.$

Here s dominates n (generically in $\widetilde{\mathcal{V}}$), and on the dominion $\tilde{\tilde{K}}_{s\succ n} = \tilde{\tilde{\Phi}} n + \tilde{\tilde{\Phi}} s + Q_s \tilde{\tilde{V}}^\sigma$ we have the generic action formulas

(2.7.1) $Q_{\tilde{q}}(\tilde{\tilde{\gamma}} n + \tilde{\tilde{\alpha}} s + Q_s \tilde{\tilde{a}}) = \tilde{\tilde{\gamma}} q_3 + \tilde{\tilde{\alpha}} q_4 + N^\sigma(\tilde{\tilde{a}}) \in \tilde{\tilde{V}}^\sigma,$

(2.7.2) $N^{-\sigma}(\tilde{\tilde{\gamma}} n + \tilde{\tilde{\alpha}} s + Q_s \tilde{\tilde{a}}) = \tilde{\tilde{\gamma}} s_4 + \tilde{\tilde{\alpha}} s_3 + Q_n \tilde{\tilde{a}} \in Q_s \tilde{\tilde{V}}^\sigma,$

(2.7.3) $S^{-\sigma}(\tilde{\tilde{\gamma}} n + \tilde{\tilde{\alpha}} s + Q_s \tilde{\tilde{a}}^\sigma) = 2\tilde{\tilde{\gamma}} s_3 + 2\tilde{\tilde{\alpha}} n + Q_{s,n} \tilde{\tilde{a}} \subseteq \tilde{\tilde{\Phi}} n + Q_s \tilde{\tilde{V}}^\sigma,\ S^{-\sigma}(Q_s \tilde{\tilde{V}}^\sigma) \subseteq Q_s \tilde{\tilde{V}}^\sigma.$

In the special case that the element $s \in V^{-\sigma}$ has an inverse $s^{-1} \in \widetilde{V}^\sigma$, we have $\tilde{q} := Q_{s^{-1}} n$, $z := Q_s \tilde{\tilde{a}}$, $y := Q_s \tilde{u} = \tilde{\tilde{\alpha}} s + Q_s \tilde{\tilde{a}}$, $x := Q_s \tilde{v} = \tilde{\tilde{\gamma}} n + \tilde{\tilde{\alpha}} s + Q_s \tilde{\tilde{a}}$ for $\tilde{u} := \alpha s^{-1} + a$, $\tilde{v} := \gamma \tilde{q} + \tilde{u}$ in $\tilde{\tilde{V}}^\sigma$, and the denominator conditions (2.5.1-2) *mean that $Q_{\tilde{q}}, Q_{\tilde{q},s^{-1}}, Q_{\tilde{q},\tilde{\tilde{a}}}, Q_{\tilde{q},\tilde{u}}, Q_{\tilde{v}}$ have denominator $s \in V^{-\sigma}$ with*

$$(2.8)\qquad \begin{aligned} &N^{-\sigma} = Q_{\tilde{q}} Q_{s^{-1}}|_V, && N^\sigma = Q_{s^{-1}} Q_{\tilde{q}}|_V, && \\ &S^{-\sigma} = Q_s Q_{\tilde{q},s^{-1}}|_V, && S^\sigma = Q_{\tilde{q},s^{-1}} Q_s|_V, && Q_{n,s} = Q_s Q_{\tilde{q},s^{-1}} Q_s|_V, \\ &M^\sigma_a := Q_{\tilde{q},a} Q_s|_V, && G^\sigma := Q_{\tilde{q},\tilde{u}} Q_s|_V, && X^\sigma := Q_{\tilde{v}} Q_s|_V, \\ &M^{-\sigma}_a := Q_s Q_{\tilde{q},a}|_V, && G^{-\sigma} := Q_s Q_{\tilde{q},\tilde{u}}|_V && X^{-\sigma} := Q_s Q_{\tilde{v}}|_V. \end{aligned}$$

In this case the monostructural transformations $\widetilde{T} = Q_{\tilde{q}}, Q_{\tilde{u}}, Q_{\tilde{v}}$ on $\widetilde{\mathcal{V}}$ determine monostructural transformations $T = Q_n, Q_y, Q_x$ on $\mathcal{V}$ whose s-cancellation operators $\mathcal{N} = \mathcal{T}_{\tilde{q},s}|_{\mathcal{V}}$, $\mathcal{B} := \mathcal{T}_{\tilde{u},s}|_{\mathcal{V}}$, $\mathcal{X} := \mathcal{T}_{\tilde{v},s}|_{\mathcal{V}}$ are structural transformations induced on $\mathcal{V}$ by restriction from principal structural transformations on $\widetilde{\mathcal{V}}$, and $\mathcal{G} := \mathcal{T}_{\tilde{q},\tilde{u};s}|_{\mathcal{V}}$ is structural glue, forming a Bergmann triple whose resulting Bergmann transformation is $\mathcal{X}_{\gamma^2 N,\gamma\mathcal{G},\mathcal{B}} = \gamma^2 \mathcal{N} + \gamma \mathcal{G} + \mathcal{B}$.

Proof: The denominator hypotheses on the element $\tilde{q}$ imply the denominator conditions (2.1) on the operators: (2.5.1-2) hold for N^τ, S^τ by $Q_n = Q_{Q_s \tilde{q}} = Q_s(Q_{\tilde{q}} Q_s) = (Q_s Q_{\tilde{q}}) Q_s$ by(JP3), and $Q_{n,s} = Q_{Q_s \tilde{q},s} = Q_s D_{\tilde{q},s} = D_{s,\tilde{q}} Q_s$ by (JP1). (2.5.3-4) are restatements of (2.5.1-2). (2.5.5) comes from (JP3), (2.5.6) from $Q_{n,z} = Q_{Q_s \tilde{q}, Q_s \tilde{\tilde{a}}} = Q_s Q_{\tilde{q},a} Q_s$ [by (JP3)] $= M^{-\sigma}_{\tilde{\tilde{a}}} Q_s = Q_s M^\sigma_{\tilde{\tilde{a}}}$ for $M^{-\sigma}_{\tilde{\tilde{a}}} = Q_s Q_{\tilde{q},\tilde{\tilde{a}}} = D_{s,\tilde{q}} D_{s,\tilde{\tilde{a}}} - D_{Q_s \tilde{q},\tilde{\tilde{a}}}$ [by (0.1.2)] $= D_{s,\tilde{q}} D_{s,\tilde{\tilde{a}}} - D_{n,\tilde{\tilde{a}}}$, and dually $M^\sigma_{\tilde{\tilde{a}}} = Q_{\tilde{q},\tilde{\tilde{a}}} Q_s =$

$D_{\widetilde{\widetilde{a}},s}D_{\tilde{q},s} - D_{\widetilde{\widetilde{a}},Q_s\tilde{q}} = D_{\widetilde{\widetilde{a}},s}S^{\sigma} - D_{\widetilde{\widetilde{a}},n}$. (2.5.7) comes from (0.1.6), (2.5.8) holds since $Q_{n,y} = \alpha Q_{n,s} + Q_{n,z}$, (2.5.9) since $Q_x = Q_{\gamma n+y} = \gamma^2 Q_n + \gamma Q_{n,y} + Q_y$. For (2.5.10), we know that the principal structural transformation $\mathcal{N} = \mathcal{T}_{\tilde{q},s}$ of the Principal Example (1.5) satisfies the identities of (10), and yields by restriction a structural transformation (not necessarily generic) on $\mathcal{V}$. (2.5.11) holds since $(D_{\tilde{q},s}, D_{s,\tilde{q}}) = (S^{\sigma}, S^{-\sigma})$ is always Lie-structural by (0.1.1). (2.5.12) comes from (0.1.4), (2.5.13) from (0.1.5).

For the relations (2.6.1-4) involving the homotope powers, note that $s_1, s_2 \in V^{-\sigma}$ by hypothesis and hence $s^{(k+2,\tilde{q})} = Q_s Q_{\tilde{q}} s^{(k,\tilde{q})} = N^{-\sigma}(s_k) \in V^{-\sigma}$, and by (2.6) q_2, q_3 (though not q_1) belong to V^{σ}, hence so does $\tilde{q}^{(k+2,s)} = Q_{\tilde{q}} Q_s \tilde{q}^{(k,s)} = N^{\sigma}(q_k)$ for $k \geq 2$. [5]

For N-shifting (2.6.1N), $N^{\sigma} = \widetilde{Q}_{\tilde{q}} Q_s = U_{\tilde{q}}^{(s)}$ takes $q_k = \tilde{q}^{(k,s)}$ to $\tilde{q}^{(k+2,s)} = q_{k+2}$, and dually $N^{-\sigma}(s_k) = Q_s \widetilde{Q}_{\tilde{q}} = U_s^{(\tilde{q})}$ takes s_k to s_{k+2}. Similarly, for S-shifting (2.6.1S) $S^{\sigma} = D_{\tilde{q},s} = V_{\tilde{q}}^{(s)}$ takes q_k to $2q_{k+1}$, and $S^{-\sigma} = D_{s,\tilde{q}} = V_s^{\tilde{q})}$ takes s_k to $2s_{k+1}$. Q-Shifting (2.6.1Q) and Inner Power Shifting (2.5.4I) follow from (0.1.7).

The terminology in the triality relations comes from whether s_k cancels a (hidden) $\tilde{q}$ from the right or the left. Inner Right Triality (2.6.2IR) follows from (0.1.8) since $N^{\sigma}(\widetilde{\widetilde{a}}) = Q_{\tilde{q}} Q_s,\ S^{\sigma} = D_{\tilde{q},s}$. Applying this operator to $\widetilde{\widetilde{c}}$ yields Elementary Right Triality (2.6.2ER), and interpreting Elementary Right Triality as an operator on $\widetilde{\widetilde{a}}$ yields Outer Right Triality (2.6.2OR). Since these hold as elements in $\mathcal{UPE}(\widetilde{\mathcal{V}})$ and as operators in $\mathcal{UME}(\widetilde{\mathcal{V}})$, we can apply the involution in $\mathcal{UME}(\widetilde{\mathcal{V}})$ to obtain Inner Left Triality (2.6.2IL) and Outer Left Triality (2.6.2OL), and then have Inner Left act on $\widetilde{\widetilde{x}}$ to obtain Elemental Left Triality (2.6.2EL).[6]

Commutativity (2.6.3) holds for $\tau = \sigma$ since $Q_{\tilde{q}} Q_s (D_{\tilde{q},s}) = (D_{\tilde{q},s}) Q_{\tilde{q}} Q_s$ [by (JP1) twice] $= Q_{Q_{\tilde{q}} s, \tilde{q}} Q_s$ [by (JP1)] $= Q_{q_2,\tilde{q}} = D_{q_2,s} D_{\tilde{q},s} - D_{q_2,Q_s\tilde{q}}$ [by (0.1.2)] $= D_{q_2,s} S^{\sigma} - D_{q_2,n} = M_{q_2}^{\sigma}$, and dually for $\tau = -\sigma$ we have $Q_s Q_{\tilde{q}}(D_{s,\tilde{q}}) = (D_{s,\tilde{q}}) Q_s Q_{\tilde{q}} = Q_s Q_{q_2,\tilde{q}} = D_{s,\tilde{q}} D_{s,q_2} - D_{Q_s\tilde{q},q_2} = S^{-\sigma} D_{s,q_2} - D_{n,q_2} = M_{q_2}^{-\sigma}$.

The first part *Two N* of Squaring (2.6.2) follows from $(S^{\sigma})^2 = (\widetilde{D}_{\tilde{q},s})^2 = 2\widetilde{Q}_{\tilde{q}} \widetilde{Q}_s + \widetilde{D}_{\widetilde{Q}_{\tilde{q}} s,s}$ [by (0.1.2)] $= 2N^{\sigma} + D_{q_2,s}$, and dually $(S^{-\sigma})^2 = (\widetilde{D}_{\tilde{q},s})^2 = 2\widetilde{Q}_s \widetilde{Q}_{\tilde{q}} + \widetilde{D}_{s,\widetilde{Q}_{\tilde{q}} s} = 2N^{-\sigma} + D_{s,q_2}$. For the second part *Two Q*, $S^{-\sigma} Q_{s,n} = (S^{-\sigma})^2 Q_s = \big[2N^{-\sigma} + D_{s,q_2}\big] Q_s$ [by the first part] $= 2Q_n + Q_{Q_s q_2, s}$ [by n-cancellation (2.5.1), (JP1)] $= 2Q_n + Q_{s_3,s}$.

By (0.2) we know $K_{s\succ n}$ is an inner ideal, and (2.7.1) holds piecewise by definition (2.6) of q_3, q_4, and the definition of N^{σ}. (2.7.2) holds by applying Q_s to (2.7.1) by (2.6.1N) and $Q_s N^{\sigma} = Q_n$. (2.7.3) likewise follows piecewise from (2.6.1S) and $S(Q_s \widetilde{\widetilde{a}}) = Q_{s,n} \widetilde{\widetilde{a}}$ [by (2.5.1)].

For the case (2.8) of invertible s, we can cancel Q_s from $Q_s Q_{\tilde{q},s^{-1}} Q_s = Q_{Q_s\tilde{q}, Q_s s^{-1}} = Q_{n,s} = Q_s S = S\, Q_s$, from $Q_s Q_{\tilde{q},a} Q_s = Q_{Q_s\tilde{q}, Q_s a} = Q_{n,z} = Q_s M_a = M_a Q_s$, from $Q_s Q_{\tilde{q},\tilde{u}} Q_s = Q_{Q_s\tilde{q}, Q_s\tilde{u}} = Q_{n,y} = Q_s G = G Q_s$, and from $Q_s Q_{\tilde{v}} Q_s = Q_{Q_s\tilde{v}} = Q_x = Q_s X = X Q_s$. ∎

This long list of ancillary relations provides the necessary tools in the following to make $\mathcal{N}$ structural, without the help of $\tilde{q}$ or $\widetilde{\mathcal{V}}$, and to motivate our treatment of Jordan derivations in [6]. They also played a role in the treatment of domination and tight domination in [5]. The operators

[5] The dominion $(K_{s\succ n}^{-\sigma})^{(\tilde{q})}$ is closed under the Jordan algebra structure on $(\widetilde{V}^{-\sigma})^{(\tilde{q})}$ despite $\tilde{q} \notin V^{\sigma}$, and the s_k are just powers of the element s in this subalgebra. V^{σ} is clearly a Jordan subalgebra of $(\widetilde{V}^{\sigma})^{(s)}$, but the q_k for $k \geq 2$ are not powers in that subalgebra since $q_1 \notin V^{\sigma}$.

[6] Note that Outer Left OL does *not* result by interpreting EL as an operator identity: EL on $\widetilde{\widetilde{x}}$ is IL, but EL on $\widetilde{\widetilde{a}}$ is the relation $Q_{s_k,\widetilde{\widetilde{x}}} N^{\sigma} - Q_{s_{k+1},\widetilde{\widetilde{x}}} + Q_{s_{k+2},\widetilde{\widetilde{x}}} = 0$, which we shall not use. The relations OR, OL will be the most important for us (cf. 5.1.1-2).

M_a^τ will play an important role in our development. The problem of creating "fractions" is the situation where $\tilde{q}, \tilde{u}, \tilde{v} \in \widetilde{\mathcal{V}}$ are merely figments of our imagination, and all that exists is their traces N, G, S on $\mathcal{V}$ and n, z, x in $\mathcal{V}$. The most general problem would be that of creating a "holomorph" $\widetilde{\mathcal{V}} \supseteq \mathcal{V}$ where all suitable structural transformations become "inner".

3 The Injective Case

In the construction of fractions $\tilde{q} = Q_s^{-1} n$, the operator Q_s begins its life in $\mathcal{V}$ as injective, and graduates to an invertible life in $\widetilde{\mathcal{V}}$. In the invertible case (2.8) the operators $N^\sigma = Q_{\tilde{q}} Q_s$, $N^{-\sigma} = Q_s Q_{\tilde{q}}$, $X^\sigma = Q_{\tilde{v}} Q_s$, $X^{-\sigma} = Q_s Q_{\tilde{v}}$ will be structural transformations on $\widetilde{\mathcal{V}}$ leaving $\mathcal{V}$ invariant, and $S^\sigma = \widetilde{D}_{s,\tilde{q}}$, $S^{-\sigma} = \widetilde{D}_{\tilde{q},s}$ will be Lie-structural transformations leaving $\mathcal{V}$ invariant, so they will have to be structural and Lie-structural on $\mathcal{V}$ to begin with. We wish to develop conditions guaranteeing this structurality.

Strengthening (0.2), we say $s \in V^{-\sigma}$ **structurally dominates** $n \in V^{-\sigma}$ **on** $\mathcal{V}$ if there is a structural $\mathcal{N} = (N^+, N^-)$ and a Lie-structural $\mathcal{S} = (S^\sigma, S^{-\sigma}))$ satisfying (for all $w \in V^\tau, a \in V^\sigma$, $\gamma, \alpha \in \Phi$, $\tau = \pm\sigma$) the cancellation relations

(3.1.1)	$Q_n = N^{-\sigma} Q_s = Q_s N^\sigma$	($\mathcal{N}$ *results by cancelling* Q_s),
(3.1.2)	$Q_{n,s} = S^{-\sigma} Q_s = Q_s S^\sigma$,	($\mathcal{S}$ *results by cancelling* Q_s),
(3.1.3)	$Q_{N^\tau(w)} = N^\tau Q_w N^{-\tau}$	($\mathcal{N}$ *is structural*),
(3.1.4)	$Q_{S^\tau(w),w} = S^\tau Q_w + Q_w S^{-\tau}$	(S *is Lie-structural*),
(3.1.5)	$(\mathcal{N}, \mathcal{G}, \mathcal{B})$	*is a Bergmann triple*,
(3.1.6)	$(\mathcal{S}, \mathcal{N})$	*is a Bergmann pair*.

Thus $\mathcal{N}, \mathcal{B}, \mathcal{X}$ as in (1.3) and (2.5.3,7,9) are all structural. (3.1.3) guarantees $\mathcal{N}$ is a structural pair ($\mathcal{B}$ always is by (0.1.6)), so (3.1.5) amounts to saying that $\mathcal{G} = \alpha\mathcal{S} + \mathcal{M}_a$ is structural glue.

We say s **generically structurally dominates** n if $N^\sigma = (N^{-\sigma})^* \in Q_V Q_V$, $S^\sigma = (S^{-\sigma})^* \in D_{V,V}$ satisfy (3.1.1-5) generically, i.e., it generically dominates as in [5, (3.1), hence (3.1.1-2) where (3.1.4) is automatic for an inner derivation] with generic structural transformations $\mathcal{N}, \mathcal{G}, \mathcal{B}$ on $\mathcal{V}$ forming a generic Bergmann triple.

In the presence of injectivity, the mere dominance (3.1.1-2) goes a long way towards structural domination.

Injectivity Theorem 3.2 *If the operator Q_s is injective on V^σ, then the structural conditions* (3.1.1-2) *alone guarantee that s structurally dominates n on the subpair $(V^\sigma, Q_s V^{-\sigma})$. Indeed, the structural conditions* (3.1.3-4) *and gluing condition* (3.1.5-6) *always hold for $\tau = -\sigma$ on the subpair, and* (3.1.1-6) *holds for $\tau = \sigma$ on the subpair if the map Q_s is injective.*

Proof: We are given dominance (3.1.1-2), and must verify that (3.1.3-4) and gluing (3.1.5-6) [i.e., T-Glue 1-2 (1.3.1-2) for $\mathcal{T}_1 = \mathcal{N}, \mathcal{T}_2 = \mathcal{B}, \mathcal{G} = \alpha\mathcal{S} + \mathcal{M}_a$ and P-Glue 1-2 (1.5.1-2) for $\mathcal{T}_1 = \mathcal{N}, \mathcal{T}_2 = \mathbf{1}, \mathcal{G} = \mathcal{D} = \mathcal{S}$] hold as maps on V^σ when $\tau = -\sigma$, and as maps on $Q_s V^\sigma$ when $\tau = \sigma$ *and we can cancel* Q_s.

We claim that *whenever* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{G}$ result by cancelling Q_s as in (2.2),

$$Q_s T_i^\sigma = T_i^{-\sigma} Q_s = Q_{t_i}, \quad Q_s G^\sigma = G^{-\sigma} Q_s = Q_{t_1,t_2} \tag{3.2.1}$$

then structurality (1.1) for $\mathcal{T}_i$ and (T-Glue 1-2) for $\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2$ will hold *on the subpair* $(V^\sigma, Q_s V^\sigma)$, so $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2)$ will be a Bergmann triple on the subpair [note that the inner ideal $Q_s V^\sigma$ is invariant under such $T = T_i^{-\sigma}, G^{-\sigma}$ since $T^{-\sigma}(Q_s V^\sigma) = Q_s(T^\sigma(V^\sigma)) \subseteq Q_s V^\sigma$]. This will apply to $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2) := (\mathcal{N}, \mathcal{G}, \mathcal{B})$ with $t_1 = n, t_2 = y$ as in (3.1.1), (3.1.3), (3.1.5), (0.1.6), (0.2.2) and to $(\mathcal{S}, \mathcal{N})$, i.e., to $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2) := (\mathcal{N}, \mathcal{S}, \mathbf{1})$ with $t_1 = n, t_2 = s$ as in (3.1.2), (3.1.4), (3.1.6).

To include G and S in the notation, we agree $Q_{T_i} := Q_{t_i}$, $Q_B := y$, $Q_I := Q_s$, and (not quite true Q_x-operators) $Q_G := Q_{t_1,t_2}$, $Q_S := Q_{t_1,t_2} := Q_{n,s}$, so that $Q_s T^\sigma = T^{-\sigma} Q_s = Q_T$ and for any two $T, T' \in \{T_1, T_2, B, I, G, S\}$ we have that the maps $Q_{T(x)}$, $Q_{T(x),\, T'(x)}, T Q_x T'$ satisfy

$$
\begin{aligned}
&Q_{T^{-\sigma}(Q_s b)} = Q_{Q_T(b)} = Q_{Q_s T^\sigma(b)} = Q_s\big(Q_{T^\sigma(b)}\big)Q_s,\\
(3.2.2)\qquad &T^{-\sigma}Q_{Q_s b}T'^\sigma = T^{-\sigma}Q_sQ_bQ_sT'^\sigma = Q_TQ_bQ_{T'} = Q_s\big(T^\sigma Q_b T'^{-\sigma}\big)Q_s,\\
&Q_{T^{-\sigma}(Q_s b),T'^\sigma(Q_s b)} = Q_{Q_T b, Q_{T'} b} = Q_{Q_s(T^\sigma(b)),Q_s(T'^\sigma(b))} = Q_s\big(Q_{T^\sigma(b),T'^\sigma(b)}\big)Q_s.
\end{aligned}
$$

First, structurality (1.1.1) of $\mathcal{T}$ holds whenever $\mathcal{T}$ results by cancelling Q_s from Q_t for a true element t [i.e., for $\mathcal{T} = \mathcal{T}_i, \mathcal{N}, \mathcal{B}, \mathcal{X}$ with $t = t_i, n, y, x$ as in (3.2.1),(3.1.1),(0.1.6),(0.2.3)]: setting $F^\tau(w) := Q_{T^\tau(w)} - T^\tau Q_w T'^{-\tau}$ for $T' = T$ we have by subtracting the second row of (2) from the first that $F^{-\sigma}(Q_s b) = Q_{Q_T b} - Q_T Q_b Q_{T'} = Q_s F^\sigma(b) Q_s = 0$ by (JP3) when $Q_T = Q_t$ for $t \in V$, which shows $F^{-\sigma}(w) = 0$ as map on V^σ when $w = Q_s b \in Q_s V^\sigma$, and $Q_s F^\sigma(w) = 0$ as map on $Q_s V^\sigma$ when $w = b \in V^\sigma$, so that *if we can cancel* Q_s then $F^\sigma(w) = 0$ on $Q_s V^\sigma$. In particular, $\mathcal{T} = \mathcal{T}_i, \mathcal{N}, \mathcal{B}, \mathcal{X}$ are structural.

The T-Gluing conditions (1.3.1-2) *for any such* $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2)$ [hence the P-Gluing conditions (1.5.1-2) for $(\mathcal{N}, \mathcal{S})$ via $(\mathcal{T}_1, \mathcal{G}, \mathcal{T}_2) = (\mathcal{N}, \mathcal{S}, \mathbf{1})$] can be formulated in terms of $F_i^{1,\tau}(x) := T_i^\tau Q_x G^{-\tau} + G^\tau Q_x T_i^{-\tau} - Q_{T_i^\tau(x), G^\tau(x)}$, $F^{2,\tau}(x) := T_1 Q_x T_2^{-\tau} + T_2^\tau Q_x T_1^{-\tau} + G^\tau Q_x G^{-\tau} - Q_{G^\tau(x)} - Q_{T_1^\tau(x), T_2^\tau(x)}$, where for $F_i^{1,\tau}$ subtracting the third row of (2) when $T, T' = T_i, G$ from the sum of the second rows when $T, T' = T_i, G$ and G, T_i [where $Q_{T_i} = Q_{t_i}$, $Q_G = Q_{t_1,t_2}$] guarantees that $F^{-\sigma}(Q_s b) = Q_s F^\sigma(b) Q_s$ equals $Q_{t_i} Q_b Q_{t_1,t_2} + Q_{t_1,t_2} Q_b Q_{t_i} - Q_{Q_{t_i}(b), Q_{t_1,t_2}(b)} = 0$ by linearized (JP3). Similarly, for $F^{2,\tau}$ taking the sum of three second rows of (2) when $T, T' = T_1, T_2'$ and T_2, T_1' and G, G and then subtracting off the sum of the third row of (2) when $T, T' = T_1, T_2$ and the first row of (2) when $T = G$ [with $Q_{T_i} = Q_{t_i}$, $Q_G = Q_{t_1,t_2}$ as before] guarantees that $F^{-\sigma}(Q_s b) = Q_s F^\sigma(b) Q_s$ equals $Q_{t_1} Q_b Q_{t_2} + Q_{t_2} Q_b Q_{t_1} + Q_{t_1,t_2} Q_b Q_{t_1,t_2} - Q_{Q_{t_1,t_2}(b)} - Q_{Q_{t_1}(b), Q_{t_2}(b)} = 0$ by linearized (JP3). Thus again $F^{-\sigma}(Q_s V^\sigma) = 0$ on V^σ and $F(V^\sigma) = 0$ on $Q_s V^\sigma$ as long as we can cancel Q_s. In particular, $(\mathcal{N}, \mathcal{G}, \mathcal{B})$ and $(\mathbf{1}, \mathcal{S}, \mathcal{N})$ are Bergmann triples, so $(\mathcal{S}, \mathcal{N})$ is a Bergmann pair on $(V^\sigma, Q_s V^{-\sigma})$.[7] ■

We have not been able to obtain Inner Shifting (2.6.1I), Inner Right Triality (2.6.2IL), or Squaring (2.6.4) directly from injectivity.

4 The Gluing Conditions

We will spend the rest of the paper finding conditions (suitable for application to fractions) that guarantee $(\mathcal{N}, \mathcal{G}, \mathcal{B})$ is a Bergmann triple and $(\mathcal{S}, \mathcal{N})$ is a Bergmann pair on the entire pair $(V^\sigma, V^{-\sigma})$. Besides structurality (3.1.1-4) we still need glue (3.1.5) = (1.3) and (1.5). This modest proposal about glue translates, by T-Glue n (1.3.n) ($n = 1, 2$), into conditions $n_T^{\tau,k}$ on S and N, which we will group according to the glue number $n = 1, 2$, the parity $\tau = \pm\sigma$, the power k of the scalar α^k, and (in T-Glue 1 1.3.1) the structural transformation $T = N, B$, together with the pair conditions P-Glue 1-2, for a total of 18 conditions.[8] For T-Glue 1 (1.3.1) we first demand conditions relating $\mathcal{N}$ to the glue $\mathcal{G}$. We make heavy use of the operators $M^\tau \in \mathrm{End}(V^\tau)$ ($M_a^\sigma := D_{a,s} S^\sigma - D_{a,n}$, $M_a^{-\sigma} := S^{-\sigma} D_{s,a} - D_{n,a}$).

N-Gluing Conditions (4N): for all $b, a \in V^\sigma, x \in V^{-\sigma}$

$$N^\tau Q_w(\alpha S^{-\tau} + M_a^{-\tau}) + (\alpha S^{-\tau} + M_a^{-\tau}) Q_q N^{-\tau} - Q_{N^\sigma(w),(\alpha S^{-\tau} + M_a^{-\tau})} = 0.$$

$$
\begin{aligned}
&(1_N^{\sigma,0}) && N^\sigma Q_b M_a^{-\sigma} + M_a^\sigma Q_b N^{-\sigma} - Q_{N^\sigma(b), M_a^\sigma(b)} = 0,\\
&(1_N^{-\sigma,0}) && N^{-\sigma} Q_x M_a^\sigma + M_a^{-\sigma} Q_x N^\sigma - Q_{N^{-\sigma}(x), M_a^{-\sigma}(x)} = 0,\\
&(1_N^{\sigma,1}) && N^\sigma Q_b S^{-\sigma} + S^\sigma Q_b N^{-\sigma} - Q_{N^\sigma(b), S^\sigma(b)} = 0,\\
&(1_N^{-\sigma,1}) && N^{-\sigma} Q_x S^\sigma + S^{-\sigma} Q_x N^\sigma - Q_{N^{-\sigma}(x), S^{-\sigma}(x)} = 0.
\end{aligned}
\Bigg\}\ \text{P-Glue 1}
$$

[7] Actually, the third row of (2) for $T = S, T' = \mathbf{1}$ shows that $\mathcal{S}$ is Lie-structural on the subpair $(V^\sigma, Q_s V^{-\sigma})$ whenever it results from cancelling Q_s from $Q_{s,n}$ for *any* n as in (3.1.2) [not assuming (3.1.1)].

[8] The reader may well be thinking of the scene in *Independence Day* when the alien is cut loose from its spacesuit.

Next we require conditions relating $\mathcal{B}$ to the glue $\mathcal{G}$. In a following proposition we will see that these conditions hold automatically for a Lie-structural transformation S connected to N by (3.1.1-4), though by a lengthy detailed computation, and so do not require anything new in the condition (3.1.5).

B-Gluing Conditions (4B): for all $b, a \in V^\sigma, x \in V^{-\sigma}$

$$B^\tau_{\alpha,a}Q_w(\alpha S^{-\tau} + M^{-\tau}_a) + (\alpha S^\tau + M^\tau_a)Q_w B^{-\tau}_{\alpha,a} - Q_{N^\tau(w),(\alpha S^\tau + M^\tau_a)(w)} = 0.$$

$$(1^{\sigma,0}_B) \qquad Q_aQ_sQ_bM^{-\sigma}_a + M^\sigma_a Q_bQ_sQ_a - Q_{Q_aQ_sb,M^\sigma_a(b)} = 0,$$

$$(1^{-\sigma,0}_B) \qquad Q_sQ_aQ_xM^\sigma_a + M^{-\sigma}_a Q_xQ_aQ_s - Q_{Q_sQ_ax,M^{-\sigma}_a(x)} = 0,$$

$$(1^{\sigma,1}_B) \qquad D_{a,s}Q_bM^{-\sigma}_a + M^\sigma_a Q_bD_{s,a} + Q_aQ_sQ_bS^{-\sigma} + S^\sigma Q_bQ_sQ_a$$
$$- Q_{D_{a,s}b,D_{a,s}S^\sigma(b)} + Q_{D_{a,s}b,D_{a,n}b} - Q_{Q_aQ_sb,S^\sigma(b)} = 0,$$

$$(1^{-\sigma,1}_B) \qquad D_{s,a}Q_xM^\sigma_a + M^{-\sigma}_a Q_xD_{a,s} + Q_sQ_aQ_xS^\sigma + S^{-\sigma}Q_xQ_aQ_s$$
$$- Q_{D_{s,a}x,S^{-\sigma}D_{s,a}(x)} + Q_{D_{s,a}x,D_{n,a}x} - Q_{Q_sQ_ax,S^{-\sigma}(x)} = 0,$$

$$(1^{\sigma,2}_B) \qquad Q_bM^{-\sigma}_a + M^\sigma_a Q_b + D_{a,s}Q_bS^{-\sigma} + S^\sigma Q_bD_{s,a} - Q_{b,M^\sigma_a(b)} - Q_{D_{a,s}b,S^\sigma(b)} = 0,$$

$$(1^{-\sigma,2}_B) \qquad Q_xM^\sigma_a + M^{-\sigma}_a Q_x + D_{s,a}Q_xS^\sigma + S^{-\sigma}Q_xD_{a,s} - Q_{x,M^{-\sigma}_a(x)} - Q_{D_{s,a}x,S^{-\sigma}(x)} = 0,$$

$$\left.\begin{array}{ll} (1^{\sigma,3}_B) & Q_bS^{-\sigma} + S^\sigma Q_b - Q_{S^\sigma(b),b} = 0, \\ (1^{-\sigma,3}_B) & Q_xS^\sigma + S^{-\sigma}Q_x - Q_{S^{-\sigma}(x),x} = 0. \end{array}\right\} \text{Lie Structural}$$

Finally, for T-Glue 2 (1.3.2) we require conditions relating $\mathcal{B}$ to the glue $\mathcal{G}$.

N-B-Gluing Conditions (4NB): for all $b, a \in V^\sigma, x \in V^{-\sigma}$

$$N^\tau Q_wB^{-\tau}_{\alpha,a} + B^\tau_{\alpha,a}Q_wN^{-\tau} + (\alpha S^\tau + M^\tau_a)Q_w(\alpha S^{-\tau} + M^{-\tau}_a) - Q_{(\alpha S^\tau + M^\tau_a)(w)} - Q_{N^\tau(w),(\alpha S^\tau + M^\tau_a)(w)} = 0.$$

$$(2^{\sigma,0}) \qquad N^\sigma Q_bQ_sQ_a + Q_aQ_sQ_bN^{-\sigma} + M^\sigma_a Q_bM^{-\sigma}_a - Q_{M^\sigma_a(b)} - Q_{N^\sigma(b),Q_aQ_s(b)} = 0,$$

$$(2^{-\sigma,0}) \qquad N^{-\sigma}Q_xQ_aQ_s + Q_sQ_aQ_xN^\sigma + M^{-\sigma}_a Q_xM^\sigma_a - Q_{M^{-\sigma}_a(x)} - Q_{N^{-\sigma}(x),Q_sQ_s(x)} = 0,$$

$$(2^{\sigma,1}) \qquad N^\sigma Q_bD_{s,a} + D_{a,s}Q_bN^{-\sigma} + M^\sigma_a Q_bS^{-\sigma} + S^\sigma Q_bM^{-\sigma}_a - Q_{M^\sigma_a(b),S^\sigma(b)} - Q_{N^\sigma(b),D_{a,s}(b)} = 0,$$

$$(2^{-\sigma,1}) \qquad N^{-\sigma}Q_xD_{a,s} + D_{s,a}Q_xN^\sigma + M^{-\sigma}_a Q_xS^\sigma + S^{-\sigma}Q_xM^\sigma_a - Q_{M^{-\sigma}_a(x),S^{-\sigma}(x)} - Q_{N^{-\sigma}(x),D_{s,a}(x)} = 0,$$

$$\left.\begin{array}{ll} (2^{\sigma,2}) & N^\sigma Q_b + Q_bN^{-\sigma} + S^\sigma Q_bS^{-\sigma} - Q_{S^\sigma(b)} - Q_{N^\sigma(b),b} = 0, \\ (2^{-\sigma,2}) & N^{-\sigma}Q_x + Q_xN^\sigma + S^{-\sigma}Q_xS^\sigma - Q_{S^{-\sigma}(x)} - Q_{N^{-\sigma}(x),x} = 0. \end{array}\right\} \text{P-Glue 2}$$

The major goal of our paper is to determine a small number of conditions besides (3.1.1-4) that will guarantee these 18 gluing conditions. As alluded to earlier, the conditions (4B) are easily disposed of: the operators $N^{\pm\sigma}$ do not appear, and the B-Gluing formulas $(1^{\pm}_B\sigma, k)$ hold automatically for any suitable derivation. One suspects this follows immediately from properties of the Bergmann operator, but I could only prove it by breaking the operator into its constituent pieces.

Bergmann Glue Proposition 4.1 *The B-Gluing Conditions* (4B) *hold for any Lie-structural transformation S of $\mathcal{V}$ as in* (3.1.4) *connected with n by the cancellation relation* (3.1.2).

Proof: To help the reader through the following tortuous verifications, we indicate the migration of terms via superscripts, with $\blacktriangle, \blacktriangledown, \bullet, \blacklozenge$ denoting a term which about be cancelled out by its twin, and we also create terms $*$ and their anti-terms $**$ at will.

Formula $(1^{\sigma,0}_B)$ follows (omitting superscripts, which are clear by context) from

$$
\begin{aligned}
&Q_aQ_sQ_b\big(SD^{(1)}_{s,a}-D^{(2)}_{n,a}\big)+\big(D^{(3)}_{a,s}S-D^{(4)}_{a,n}\big)Q_bQ_sQ_a-Q^{(5)}_{Q_aQ_sb,D_{a,s}S^\sigma(b)}+Q^{(6)}_{Q_aQ_sb,D_{a,n}(b)}\\
&=Q_aQ_s\big[Q_bS^{(1)}+SQ^{(7*)}_b\big]D_{s,a}+D_{a,s}\big[SQ^{(3)}_b+Q_bS^{(8*)}\big]Q_sQ_a-Q_a\big[Q_sS^{(7**)}\big]Q_bD_{s,a}\\
&\qquad -D_{a,s}Q_b\big[SQ^{(8**)}_s\big]Q_a+\big[-Q_aQ_sQ_bD^{(2)}_{n,a}-D_{a,n}Q_bQ_sQ^{(4)}_a+Q^{(6)}_{Q_aQ_s(b),D_{a,n}(b)}\big]\\
&\qquad -\big[Q^{(5)}_{Q_aQ_s(b),D_{a,s}S(b)}+Q^{9*)}_{Q_aQ_sS(b),D_{a,s}(b)}\big]+Q^{(9**)}_{Q_a(Q_sS)(b),D_{a,s}(b)}\\
&=\big[Q_aQ_sQ_{S(b),b}D^{(1/7*)}_{s,a}+D_{a,s}Q_{S(b),b}Q_sQ^{(2/8*)}_a-Q^{(5)}_{Q_aQ_s(b),D_{a,s}(S(b))}-Q^{(9*)}_{Q_aQ_s(S(b)),D_{a,s}(b)}\big]\\
&\ -\big[Q_aQ_sQ_bD^{(2)}_{n,a}+Q_aQ_{s,n}Q_bD^{(7**)}_{s,a}+D_{a,n}Q_bQ_sQ^{(4)}_a+D_{a,s}Q_bQ_{s,n}Q^{(8**)}_a-Q^{(6)}_{Q_aQ_s(b),D_{a,n}(b)}\\
&\ -Q^{(9**)}_{Q_aQ_{s,n}(b),D_{a,s}(b)}\big],
\end{aligned}
$$

[by structurality (3.1.4) on (1/7*),(2/8*); cancellation (3.1.2) on (7**),(8**),(9**)] which vanishes by linearizations $b\to S(b),b$ and $s\to s,n$ of (0.1.5).

Formula $(1^{-\sigma,0}_B)$ follows dually (though not by a dual proof, since the formulas (1_B) are all symmetric under reversal; because of asymmetry ($s,n\in V^{-\sigma}$) the "dual" proofs are really "inside-out"). We compute

$$
\begin{aligned}
&Q_sQ_aQ_x\big[D^{(1)}_{a,s}S-D^{(2)}_{a,n}\big]+\big[SD^{(3)}_{s,a}-D^{(4)}_{n,a}\big]Q_xQ_aQ_s-Q^{(5)}_{Q_sQ_ax,SD_{s,a}(x)}+Q^{(6)}_{Q_sQ_ax,D_{n,a}(x)}\\
&=\big[Q_sQ_aQ_xD^{(1)}_{a,s}\big]S+S\big[D_{s,a}Q_xQ_aQ^{(3)}_s\big]-\big[Q^{(5)}_{Q_sQ_a(x),S(D_{s,a}(x))}+Q^{(7*)}_{S(Q_sQ_a(x)),D_{s,a}(x)}\big]\\
&\qquad +Q^{(7**)}_{(SQ_s)Q_a(x),D_{s,a}(x)}-\big[Q_sQ_aQ_xD^{(2)}_{a,n}+D_{n,a}Q_xQ_aQ^{(4)}_s-Q^{(6)}_{Q_sQ_a(x),D_{n,a}(x)}\big]\\
&=\big[Q_sQ_aQ_xD^{(1)}_{a,s}-Q^{(5)}_{Q_sQ_a(x),D_{s,a}(x)}\big]S+S\big[D_{s,a}Q_xQ_aQ^{(3)}_s-Q^{(7*)}_{Q_sQ_a(x),D_{s,a}(x)}\big]\\
&\qquad +Q^{(7**)\blacktriangle}_{(Q_{s,n})Q_a(x),D_{s,a}(x)}+\big[Q_{s,n}Q_aQ_xD^{(2)}_{a,s}+D_{s,a}Q_xQ_aQ^{(4)}_{s,n}-Q^{(6)\blacktriangle}_{Q_{s,n}Q_a(x),D_{s,a}(x)}\big]
\end{aligned}
$$

[by structurality (3.1.4) on (5/7*); cancel (3.1.2) on (7**); linearized (0.1.4) $s\to s,n$ on (2/4/6)]

$$
=\ -\big[D_{s,a}Q_xQ_a(Q_sS)\big]^{(1/5)\blacktriangledown}-\big[(SQ_s)Q_aQ_xD_{a,s}\big]^{(3/7*)\bullet}+\big[D_{s,a}Q_xQ_aQ^{(4)\blacktriangledown}_{s,n}+Q_{s,n}Q_aQ_xD^{(2)\bullet}_{a,s}\big]
$$

[by (0.1.5) on (1/5), (3/7*)], which vanishes by cancellation (3.1.2).

The formula $(1^{\sigma,1}_B)$ is

$$
\begin{aligned}
&D_{a,s}Q_b\big[SD^{(1)}_{s,a}-D^{(2)}_{n,a}\big]+\big[D^{(3)}_{a,s}S-D^{(4)}_{a,n}\big]Q_bD_{s,a}+Q_aQ_sQ^{(5)}_bS+SQ_bQ_sQ^{(6)}_a\\
&\qquad -Q^{(7)}_{D_{a,s}b,D_{a,s}S(b)}+Q^{(8)}_{D_{a,s}b,D_{a,n}b}-Q^{(9)}_{Q_aQ_sb,S(b)}\\
&=D_{a,s}\big[Q^{(1)}_bS+SQ^{(3)}_b\big]D_{s,a}+Q_aQ_s\big[Q^{(5)}_bS+SQ^{(10*)}_b\big]+\big[Q^{(11*)}_bS+SQ^{(6)}_b\big]Q_sQ_a-Q_a\big[Q^{(10**)}_sS\big]Q_b\\
&\qquad -Q_b\big[SQ^{(11**)}_s\big]Q_a-Q^{(7)}_{D_{a,s}(b),D_{a,s}(S(b))}-\big[Q^{(9)}_{Q_aQ_s(b),S(b)}+Q^{(12*)}_{Q_aQ_s(S(b)),b}\big]+Q^{(12**)}_{Q_a(Q_sS)(b),b}\\
&\qquad +\big[-D_{a,s}Q_bD^{(2)}_{n,a}-D_{a,n}Q_bD^{(4)}_{s,a}+Q^{(8)}_{D_{a,s}(b),D_{a,n}(b)}\big]\\
&=\Big[D_{a,s}Q_{S(b),b}D^{(1/3)}_{a,s}+Q_aQ_sQ^{(5/10*)}_{S(b),b}+Q_{S(b),b}Q_sQ^{(6/11*)}_a\\
&\qquad -Q^{(7)}_{D_{a,s}(b),D_{a,s}(S(b))}-Q^{(9)}_{Q_aQ_s(b),S(b)}-Q^{(12*)}_{Q_aQ_s(S(b)),b}\Big]\\
&\qquad -\Big[D_{a,s}Q_bD^{(2)}_{n,a}+D_{a,n}Q_bD^{(4)}_{s,a}+Q_aQ_{n,s}Q^{(11**)}_b+Q_bQ_{n,s}Q^{(10**)}_a-Q^{(8)}_{D_{a,s}(b),D_{a,n}(b)}-Q^{(12**)}_{Q_aQ_{n,s}(b),b}\Big]
\end{aligned}
$$

[by structurality (3.1.4) on (1/3),(5/10*),(6/11*); cancellation (3.1.2) on (11**),(12**)], which vanishes by the linearizations $b \to S(b), b$ and $s \to s, n$ of (0.1.4).

Dually, the formula $(1_B^{-\sigma,1})$ becomes

$$\begin{aligned}
&D_{s,a}Q_x\big[D^{(1)}_{a,s}S - D^{(2)}_{a,n}\big] + \big[SD^{(3)}_{s,a} - D^{(4)}_{n,a}\big]Q_xD_{a,s} + Q_sQ_aQ^{(5)}_xS + SQ_xQ_aQ^{(6)}_s \\
&\qquad -Q^{(7)}_{D_{s,a}x,SD_{s,a}(x)} + Q^{(8)}_{D_{s,a}x,D_{n,a}x} - Q^{(9)}_{Q_sQ_ax,S(x)} \\
&= \big[D_{s,a}Q_xD^{(1)}_{a,s} + Q_sQ_aQ^{(5)}_x\big]S + S\big[D_{s,a}Q_xD^{(3)}_{a,s} + Q_xQ_aQ^{(6)}_s\big] \\
&\qquad -\big[D_{s,a}Q_xD^{(2)}_{a,n} + D_{n,a}Q_xD^{(4)}_{a,s} - Q^{(8)}_{D_{s,a}(x),D_{n,a}(x)}\big] - Q^{(7)}_{D_{s,a}x,S(D_{s,a}x)} - Q^{(9)}_{Q_sQ_ax,S(x)} \\
&= [D_{s,a}Q_xD^{(1)}_{a,s} + Q_sQ_aQ^{(5)}_x - Q^{(7a)}_{D_{s,a}x} - Q^{(9a)}_{Q_sQ_ax,x}\big]S + S\big[D_{s,a}Q_xD^{(3)}_{a,s} + Q_xQ_aQ^{(6)}_s - Q^{(7b)}_{D_{s,a}x} - Q^{(9b)}_{Q_sQ_ax,x}\big] \\
&\qquad -\big[D_{s,a}Q_xD_{a,n} + D_{n,a}Q_xD_{a,s} - Q_{D_{s,a}(x),D_{n,a}(x)}\big]^{(2/4/8)} + Q^{(9c)}_{S(Q_sQ_ax),x} \\
&\hfill \text{[by structurality (3.1.4) on (7),(9)]} \\
&= \big[Q_xQ_aQ_s\big]^{(1/5/7a/9a)\blacktriangle}S + S\big[-Q_sQ_aQ_x\big]^{(3/6/7b/9b)\blacktriangledown} - \big[-Q_aQ_xQ^{\blacktriangle}_{s,n} - Q_{s,n}Q_xQ^{\blacktriangledown}_a + Q^{\blacktriangleright}_{Q_{s,n}Q_ax,x}\big]^{(2/4/8)} \\
&\qquad + Q^{(9c)\blacktriangleright}_{(SQ_s)Q_ax,x} \qquad \text{[by linearized (0.1.4) } x, a, y \to \{s, n\}, a, x \text{ on (2/4/8)]}
\end{aligned}$$

which vanishes by cancellation (3.1.2) on $\blacktriangle, \blacktriangledown, \blacktriangleright$.

The formula $(1_B^{\sigma,2})$ becomes $Q_b\big[SD_{s,a} - D_{n,a}\big] + \big[D_{a,s}S - D_{a,n}\big]Q_b + D_{a,s}Q_bS + SQ_bD_{s,a} - Q_{b,[D_{a,s}S-D_{a,n}](b)} - Q_{D_{a,s}b,S(b)} = \big[Q_bS + SQ_b\big]D_{s,a} + D_{a,s}\big[Q_bS + SQ_b\big] - Q_{b,D_{a,s}(S(b))} - \big[Q_bD_{n,a} + D_{a,n}Q_b - Q_{b,D_{a,n}b}\big]^{\blacktriangle} - Q_{S(b),D_{a,s}(b)} = Q_{S(b),b}D_{s,a} + D_{a,s}Q_{S(b),b} - Q_{D_{a,s}(S(b)),b} - Q_{D_{a,s}(b),S(b)}$ [by structurality (3.1.4) twice, (0.1.1) on $\blacktriangle$], which vanishes by the linearization $b \to b, S(b)$ of (3.1.4).

Dually, the formula $(1_B^{-\sigma,2})$ becomes $Q_x\big[D_{a,s}S - D_{a,n}\big] + \big[SD_{s,a} - D_{n,a}\big]Q_x + D_{s,a}Q_xS + SQ_xD_{a,s} - Q_{x,[SD_{s,a}-D_{n,a}](x)} - Q_{D_{s,a}x,S^{-\sigma}(x)} = \big[Q_xD_{a,s} + D_{s,a}Q_x\big]S + S\big[D_{s,a}Q_x + Q_xD_{a,s}\big] - Q_{x,S(D_{s,a}(x))} - Q_{D_{s,a}(x),S(x)} - \big[Q_xD_{a,n} + D_{n,a}Q_x - Q_{x,D_{n,a}x}\big]^{\blacktriangle} = Q_{D_{s,a}(x),x}S + SQ_{D_{s,a}(x),x} - Q_{S(D_{s,a}x),x} - Q_{D_{s,a}x,S(x)}$ [by (0.1.1)] on $\blacktriangle$, which vanishes by the linearization $x \to x, D_{s,a}x$ of structurality (3.1.4).

Note that the final conditions $(1_B^{\pm\sigma,3})$ are just the conditions (3.1.4) that S be a Lie-structural transformation on $\mathcal{V}$. ■

5 The Main Theorem

Our main result is that the 18 Gluing Conditions (4N, 4B, 4NB) which guarantee that $(\mathcal{N}, \mathcal{G}, \mathcal{B})$ is a Bergmann triple and $(\mathcal{S}, \mathcal{N})$ is a Bergmann pair, in particular that $\mathcal{X}$ is structural, can be reduced to a small number of connections between N and S.

Structural Domination Theorem 5.1 *The Gluing Conditions* (4N), (4B), (4NB) *for* (3.1.5) *will follow from the Structural Domination Conditions* (3.1.1), (3.1.2), (3.1.4) *on* s, n *and the two Gluing Conditions* (P-Glue 1) $= (1_N^{\pm\sigma,1})$, (P-Glue 2) $= (2^{\pm\sigma,2})$,

(5.1.1) (P-Glue 1) $N^\tau Q_w S^{-\tau} + S^\tau Q_w N^{-\tau} = Q_{N^\tau(w),S^\tau(w)}$,

(5.1.2) (P-Glue 2) $N^\tau Q_w + Q_w N^{-\tau} + S^\tau Q_w S^{-\tau} = Q_{S^\tau(w)} + Q_{N^\tau(w),w}$,

if we assume that the following additional conditions hold for elements $q_2, q_3 \in V^\sigma$ *with* $s_{i+1} := Q_s q_i$ *and all* $a \in V^\sigma$:

(5.1.3) (Two N) $S^\tau S^\tau = 2N^\tau + D_2^\tau \qquad (D_2^\sigma := D_{q_2,s}, D_2^{-\sigma} := D_{s,q_2})$,

(5.1.4) (Right Triality $(k=1)$ $D_{a,s}N^\sigma - D_{a,n}S^\sigma + D_{a,s_3} = 0$,

(5.1.5) (Right Triality $(k=2)$ $D_{a,n}N^\sigma - D_{a,s_3}S^\sigma + D_{a,s_4} = 0$,

(5.1.4*) (Outer Left Triality $(k=1)$ $N^{-\sigma}D_{s,a} - S^{-\sigma}D_{n,a} + D_{s_3,a} = 0$,

(5.1.5*) (Outer Left Triality $(k=2)$ $N^{-\sigma}D^{-\sigma}_{n,a} - S^{-\sigma}D^{-\sigma}_{s_3,a} + D^{-\sigma}_{s_4,a} = 0$.

If these hold generically in a, b rather than as maps on $\mathcal{V}$, then we can omit (5.1.4*–5*).

Thus $(\mathcal{N}, \mathcal{G}, \mathcal{B})$ will be a Bergmann triple and $(\mathcal{S}, \mathcal{N})$ a Bergmann pair if (P-Glue 1-2), (3.1.1-4), *and* (5.1.3-5*) *all hold.*

Proof: The dual (5.1.4*) holds automatically if (5.1.4) is generic, and similarly for (5.1.5). We already know by the Bergmann Glue Proposition 4.1 that (4B) follows from (3.1.2), (3.1.4), so we are concerned only to derive (4N), (4NB). We noted in the proof of (2.6.4) that (3.1.1) and Two N imply Two Q (5.3.4) (which is also the case $w = s, \tau = -\sigma$ of (5.1.2)):

(5.1.3′) (Two Q) $SQ_{s,n} = Q_{s,n}S = 2Q_n + Q_{s_3,s}$.

We first show that the N-Gluing Conditions (4N) $(1_N^{\pm\sigma,k})$, $k = 0, 1$, follow from the above (5.1-5*). For $k = 1$ the relations $(1_N^{\pm\sigma,1})$ are just (P-Glue 1). For $k = 0$ the relation $(1_N^{\sigma,0})$ follows from (3.1.4), (5.1.1-2)$^\sigma$, (5.1.4*-5*), (5.1.4-5) since it reduces to

$$
\begin{aligned}
&NQ_b\big[SD_{s,a} - D_{n,a}\big] + \big[D_{a,s}S - D_{a,n}\big]Q_bN - Q_{N(b),D_{a,s}S(b)} - Q_{N(b),D_{a,n}(b)}\\
&= \big(NQ_bS\big)^{(1)}D_{s,a} - \big(NQ_b\big)^{(2)}D_{n,a} + D_{a,s}\big(SQ_bN\big)^{(3)} - D_{a,n}\big(Q_bN\big)^{(4)}\\
&\qquad - Q^{(5)}_{N(b),D_{a,s}(S(b))} - Q^{(6)}_{N(b),D_{a,n}(b)}\\
&= \big[-SQ_b^{(1a)}N + Q^{(1b)\blacktriangle}_{N(b),S(b)}\big]D_{s,a} - \big[-Q_bN^{(2a)} - SQ_b^{(2b)}S + Q^{(2c)}_{S(b)} + Q^{(2d)\blacktriangledown}_{N(b),b}\big]D_{n,a}\\
&\qquad + D_{a,s}\big[-NQ_b^{(3a)} + Q^{(3b)\blacklozenge}_{N(b),S(b)}\big] - D_{a,n}\big[-NQ_b^{(4a)} - SQ_b^{(4b)}S + Q^{(4c)}_{S(b)} + Q^{(4d)\bullet}_{N(b),b}\big]\\
&\qquad \big[D_{a,s}Q^{(5a)\blacklozenge)}_{N(b),S(b)} + Q^{(5b)\blacktriangle}_{N(b),S(b)}D_{s,a} - Q^{(5c)}_{[D_{a,s}N](b),S(b)}\big]\\
&\qquad - \big[-Q^{(6a)}_{[D_{a,n}N](b),b} + D_{a,n}Q^{(6b)\bullet}_{N(b),b} + Q^{(6c)\blacktriangledown}_{N(b),b}D_{n,a}\big]
\end{aligned}
$$

[by (5.1.1)$^\sigma$ for (1),(3), (5.1.2)$^\sigma$ for (2),(4), and linearized (0.1.1) for (5),(6)]

$$
\begin{aligned}
&= SQ_b\big[SD^{(1a1)\blacktriangle}_{n,a} - D^{(1a2)}_{s_3,a}\big] + Q_b\big[SD^{(2a1)}_{s_3,a} - D^{(2a2)}_{s_4,a}\big] - SQ_b^{(2b)\blacktriangle}SD_{n,a}\\
&\qquad + \big[-D^{(3a1)\blacktriangledown)}_{a,n}S + D^{(3a2)}_{a,s_3}\big]Q_bS - \big[Q_{S(b)}D^{(2c)\blacklozenge}_{n,a} + D_{a,n}Q^{(4c)\bullet}_{S(b)}\big] - \big[-D^{(4a1)}_{a,s_3}S + D^{(4a2)}_{a,s_4}\big]Q_b\\
&\qquad + D_{a,n}SQ_b^{(4b)\blacktriangledown}S + \big[Q^{(5c1)\blacklozenge}_{D_{a,n}S(b),S(b)} - Q^{(5c2)}_{D_{a,s_3}(b),S(b)}\big] + \big[-Q^{(6a1)}_{D_{a,s_3}S(b),b} + Q^{(6a2)}_{D_{a,s_4}(b),b}\big]
\end{aligned}
$$

[by (5.1.4*) for (1a), (5.1.4) for (3a),(5c), (5.1.5*) for (2a), (5.1.5) for (4a),(6a)]

$$
\begin{aligned}
&= \big[SQ_b^{(1a2)} + Q_b^{(2a1)}S\big]D_{s_3,a} + \big[-Q_bD^{(2a2)\blacklozenge)}_{s_4,a} - D^{(4a2)\blacklozenge}_{a,s_4}Q_b + Q^{(6a2)\blacklozenge}_{D_{a,s_4}(b),b}\big]\\
&\qquad + D_{a,s_3}\big[SQ_b^{(4a1)} + Q_b^{(3a2)}S\big] - Q^{(6a1)}_{D_{a,s_3}(S(b)),b} - Q^{(5b2)}_{D_{a,s_3}(b),S(b)}
\end{aligned}
$$

$$
= Q_{S(b),b}D_{s_3,a} + D_{a,s_3}Q_{S(b),b} - Q_{D_{a,s_3}(S(b)),b} - Q_{D_{a,s_3}(b),S(b)}
$$

[by (3.1.4) on (1a2/2a1),(4a1/3a2); (0.1.1) on $\blacklozenge$], which vanishes by linearized (0.1.1)].

The formula $(1_N^{-\sigma,0})$ follows dually from (3.1.4), $(5.1.1\text{-}2)^{-\sigma}$, (5.1.4*-5*), (5.1.4-5):

$$NQ_xD^{(1)}_{a,s}S+NQ_xD^{(2)}_{a,n}-SD_{s,a}Q^{(3)}_xN+D_{n,a}Q^{(4)}_xN+Q^{(5)}_{N(x),SD_{s,a}(x)}-Q^{(6)}_{N(x),D^{(1)}_{n,a}(x)}$$
$$=\big[NQ^{(1a)}_{\{s,a,x\},x}S-SQ^{(3a)}_{\{s,a,x\},x}N\big]^{(7)}+\big[NQ^{(2a)}_{\{s,a,x\},x}+Q^{(4a)}_{\{s,a,x\},x}N\big]^{(8)}$$
$$-\big(ND_{s,a}\big)^{(1b)}Q_xS+SQ_x\big(D_{a,s}N\big)^{(3b)}-\big(ND_{n,a}\big)^{(2b)}Q_x-Q_x\big(D_{a,n}N\big)^{(4b)}$$
$$+Q^{(5)}_{N(x),S(\{s,a,x\})}-Q^{(6)}_{N(x),D_{n,a}(x)} \qquad \text{[by (0.1.1) on (1),(2),(3),(4)]}$$
$$=\big[-Q^{(7a)}_{N(\{s,a,x\}),S(x)}-Q^{(7b)\blacktriangle}_{N(x),S(\{s,a,x\})}\big]$$
$$+\big[SQ^{(8a)}_{\{n,a,x\},x}S+Q^{(8b)}_{S\{n,a,x\},S(x)}+Q^{(8c)}_{N(\{n,a,x\}),x}+Q^{(8d)\blacktriangle}_{N(x),\{n,a,x\}}\big]^{(8)}$$
$$-\big[SD^{(1b1)}_{n,a}-D^{(1b2)}_{s_3,a}\big]Q_xS-SQ_x\big[D_{a,n}S^{(3b1)}+D^{(3b2)}_{a,s_3}\big]-\big[SD^{(2b1)}_{s_3,a}-D^{(2b2)}_{s_4,a}\big]Q_x$$
$$+Q_x\big[D^{(4b1)}_{a,s_3}+D^{(4b2)}_{a,s_4}\big]+Q^{(5)\blacktriangle}_{N(x),S(\{s,a,x\})}-Q^{(6)\blacktriangledown}_{N)x),\{n,a,x\}} \qquad \text{[by } (5.1.1)^{-\sigma} \text{ on (7),}$$

$(5.1.2)^{-\sigma}$ on (8), (5.1.4*) on (1b), (5.1.5*) on (2b), (5.1.4) on (3b), (5.1.5) on (4b)]

$$=Q^{(7a8b)}_{\big[-ND_{s,a}+SD_{n,a}\big](x),S(x)}+Q^{(8c)}_{ND_{n,a}(x),x}+S\big[-D_{n,a}Q^{(1b1)}_x-Q_xD^{(3b1)}_{a,n}+Q^{(8a)}_{\{n,a,x\},x}\big]^{(9)}S$$
$$-S\big[D_{s_3,a}Q^{(2b1)}_x+Q_xD^{(3b2)}_{s_3,a}\big]^{(10)}+\big[Q_xD^{(4b1)}_{a,s_3}+D_{s_3,a}Q^{(1b2)}_x\big]^{(11)}S+\big[D_{s_4,a}Q^{(2b2)}_x+Q_xD^{(4b2)}_{a,s_4}\big]^{(12)}$$
$$=Q^{(7a8b)\blacktriangle}_{D_{s_3,a}(x),S(x)}+\big[Q^{(8c1)\blacktriangle}_{SD_{s_3,a}(x),x}-Q^{(8c2)\blacktriangledown}_{D_{a,s_4}(x),x}\big]$$
$$-S\big[Q_{\{s_3,a,x\},x}\big]^{(10)\blacktriangle}+\big[Q_{\{s_3,a,x\},x}\big]^{(11)\blacktriangle}S+Q^{(12)\blacktriangledown}_{\{s_4,a,x\},x}$$

[by (5.1.4*) on (7a8b), (5.1.5*) on (8c), and (0.1.1) on (9),(10),(11),(12)], which vanishes by linearized (3.1.4).

Now we turn to the remaining condition (4NB): $(2^{\pm\sigma,k})$. The case $k=2$ is our assumption (5.1.2). For $k=0$ the formula $(2^{\sigma,0})$ follows from (3.1.1), (3.1.2), (3.1.4), $(5.1.2)^\sigma$, (5.1.4*), (5.1.4) (5.1.3′) via

$$NQ_bQ_sQ^{(1)}_a+Q_aQ_sQ_bN^{(2)}+D_{a,s}\big[SQ_bS\big]^{(3)}D_{s,a}+Q^{(4)}_{D_{a,s}(S(b))}-D_{a,s}\big[SQ_b\big]^{(5)}D_{n,a}$$
$$-D_{a,n}\big[Q_bS\big]^{(6)}D_{s,a}+Q^{(7)}_{D_{a,s}S(b),D_{a,n}(b)}+\big[D_{a,n}Q_bD_{n,a}-Q_{D_{a,n}(b)}\big]^{(8)}-Q^{(9)}_{N(b),Q_aQ_s(b)}$$
$$=NQ_bQ_sQ^{(1)}_a+Q_aQ_sQ_bN^{(2)}+D_{a,s}\big[-NQ^{(3a)}_b-Q_bN^{(3b)}+Q^{(3c)\blacktriangle}_{S(b)}+Q^{(3d)}_{N(b),b}\big]D_{s,a}$$
$$+\big[Q^{(4a)}_{Q_aQ_sS(b),S(b)}-Q_aQ_sQ^{(4b)}_{S(b)}-Q_{S(b)}Q_sQ^{(4c)}_a-D_{a,s}Q^{(4d)\blacktriangle}_{S(b)}D_{s,a}\big]$$
$$+D_{a,s}\big[-Q^{(5a)}_{S(b),b}+Q_bS^{(5b)}\big]D_{n,a}+D_{a,n}\big[-Q^{(6a)}_{S(b),b}+SQ^{(6b)}_b\big]D_{s,a}$$
$$+\big[Q^{(7)}_{D_{a,s}S(b),D_{a,n}(b)}+Q^{(7*)}_{D_{a,n}S(b),D_{a,s}(b)}-Q^{(7**)}_{(D_{a,n}S)(b),D_{a,s}(b)}\big]$$
$$+\big[-Q_aQ_nQ^{(8a)}_b-Q_bQ_nQ^{(8b)}_a+Q^{(8c)}_{Q_aQ_n(b),b}\big]+\big[-Q^{(9)}_{N(b),Q_aQ_s(b)}-Q^{(9*)}_{b,Q_aQ_sN(b)}+Q^{(9**)}_{b,Q_aQ_n(b)}\big]$$

[by $(5.1.2)^\sigma$ on (3); (0.1.4) on (4),(8); (3.1.4) on (5),(6); (3.1.1) on (9*)]

$$=\big[NQ^{(1)}_b-Q^{(4c)}_{S(b)}\big]Q_sQ_a+Q_aQ_s\big[Q_bN^{(2)}-Q^{(4b)}_{S(b)}\big]-\big[\big(D_{a,s}N\big)Q_bD^{(3a)}_{s,a}+D_{a,s}Q_b\big(ND_{s,a}\big)^{(3b)}\big]$$
$$-\big[-Q^{(4a)}_{Q_aQ_{s,n}b,S(b)}+D_{a,s}Q_{S(b),b}D^{(5a)}_{n,a}+D_{a,n}Q_{S(b),b}D^{(6a)}_{s,a}-Q^{(7)}_{D_{a,s}(S(b)),D_{a,n}(b)}$$
$$-Q^{(7*)}_{D_{a,s}(b),D_{a,n}(S(b))}\big]^{(11)}-Q^{(7**)}_{[D_{a,n}S](b),D_{a,s}(b)}+\big[-Q_aQ_nQ^{(8a)}_b-Q_bQ_nQ^{(8b)}_a+2Q^{(8c9**)}_{Q_aQ_n(b),b}\big]$$
$$+\big[D_{a,s}Q_{N(b),b}D^{(3d)}_{s,a}-Q^{(9*)}_{Q_aQ_sN(b),b}-Q^{(9)}_{Q_aQ_s(b),N(b)}\big]^{(12)}+D_{a,s}Q_b\big[SD_{n,a}\big]^{(5b)}$$

$$+\big[D_{a,n}S\big]^{(6b)}Q_bD_{a,s} \qquad \text{[by (3.1.2) for (4a)]}$$

$$= \big[-Q_bN^{(1a)}-SQ_b^{(1b)}S+Q_{N(b),b}^{(1c)\blacktriangle}\big]Q_sQ_a + Q_aQ_s\big[-NQ_b^{(2a)}-SQ_b^{(2b)}S+Q_{N(b),b}^{(2c)\blacktriangledown}\big]$$
$$+\big[\big(-D_{a,n}S^{(3a1)\blacklozenge}+D_{a,s_3}^{(3a2)}\big)Q_bD_{s,a}+D_{a,s}Q_b\big(-SD_{n,a}^{(3b1)\bullet}+D_{s_3,a}^{(3b2)}\big)\big] + \big[Q_aQ_{n,s}Q_{S(b),b}^{(11a)}$$
$$+Q_{S(b),b}Q_{n,s}Q_a^{(11b)}-Q_{Q_aQ_{s,n}S(b),b}^{(11c)}\big] + \big[-Q_aQ_nQ_b^{(8a)}-Q_bQ_nQ_a^{(8b)} + 2Q_{Q_aQ_nb,b}^{(8c9**)}\big]$$
$$+\big[-Q_aQ_sQ_{N(b),b}^{(12a)\blacktriangledown}-Q_{N(b),b}Q_sQ_a^{(12b)\blacktriangle}+Q_{D_{a,s}(N(b)),D_{a,s}(b)}^{(12c)\blacktriangleleft}\big] + D_{a,s}Q_b\big[SD_{n,a}^{(5b)\bullet}\big]$$
$$+\big[D_{a,n}S^{(6b)\blacklozenge}\big]Q_bD_{s,a} - \big[Q_{D_{a,s}(b),D_{a,s}N(b)}^{(7**a)\blacktriangleleft} + Q_{D_{a,s}(b),D_{a,s_3}(b)}^{(7**b)}\big]$$

[by $(5.1.2)^\sigma$ for (1),(2) and the linearization of (0.1.4) $y \to b, S(b), a \to s, n$ for (11), $y \to b, S(b)$ for (12); (5.1.4) for (3a), (7**); (5.1.4*) for (3b)]

$$= \big[D_{a,s_3}Q_bD_{s,a}^{(3a2)}+D_{a,s}Q_bD_{s_3,a}^{(3b2)} - Q_{D_{a,s}(b),D_{a,s_3}(b)}^{(7**b)}\big]^{(13)}$$
$$+\big[-\big(SQ_b^{(1b)}+Q_{S(b),b}^{(11b)}\big)Q_{s,n}Q_a-Q_aQ_{s,n}\big(-Q_bS^{(2b)}+Q_{S(b),b}^{(11a)}\big)-Q_{Q_aQ_{s,n}S(b),b}^{(11c)}\big]^{(14)}$$
$$+2\big[-Q_bQ_nQ_a^{(1a8b)}-Q_aQ_nQ_b^{(2a8a)}+Q_{Q_aQ_n(b),b}^{(8c9**)}\big]^{(15)}$$

[by (3.1.1) for (1a),(2a), (3.1.2) for (1b),(2b)]

$$= \big[-Q_aQ_{s_3,s}Q_b-Q_bQ_{s_3,s}Q_a+Q_{Q_aQ_{s_3,s}(b),b}\big]^{(13)} + \big[Q_b\big(SQ_{s,n}\big)Q_a+Q_a\big(Q_{s,n}S\big)Q_b$$
$$-Q_{Q_aQ_{s,n}S(b),b}\big]^{(14)} + 2\big[-Q_bQ_nQ_a-Q_aQ_nQ_b+Q_{Q_aQ_n(b),b}\big]^{(15)}$$

[by (3.1.4) for (14), linearized (0.1.4) $s \to s, s_3$ for (13)]

$$= Q_a\big[-Q_{s_3,s}+Q_{s,n}S-2Q_n\big]Q_b+Q_b\big[-Q_{s_3,s}+SQ_{s,n}-2Q_n\big]Q_a+Q_{Q_a[Q_{s_3,s}-Q_{s,n}S+2Q_n](b),b}$$

which vanishes by assumption $(5.1.3')$.

Formula $(2^{-\sigma,0})$ follows dually by an equally tortuous computation: it follows from (3.1.1), (3.1.2), (3.1.4), $(5.1.2)^{-\sigma}$, $(5.1.4^*)$, (5.1.4), $(5.1.3')$ since it reduces to

$$\big[NQ_xQ_aQ_s+Q_sQ_aQ_xN\big]^{(1)} + S\big[D_{s,a}Q_xD_{a,s}\big]^{(2)}S + Q_{SD_{a,s}(x)}^{(3)}$$
$$+\big[-D_{n,a}Q_x^{(4)}D_{a,s}S-SD_{s,a}Q_x^{(5)}D_{a,n}+Q_{SD_{s,a}(x),D_{n,a}(x)}^{(6)}\big] + \big[D_{n,a}Q_xD_{a,n}-Q_{D_{n,a}(x)}\big]^{(7)}$$
$$-Q_{N(x),Q_sQ_a(x)}^{(8)}$$

$$= \big[NQ_xQ_a^{(1a)}Q_s+Q_sQ_a^{(1b)}Q_xN\big] + S\big[-Q_sQ_a^{(2a)}Q_x-Q_xQ_a^{(2b)}Q_s+Q_{D_{s,a}(x)}^{(2c)\blacktriangle}+Q_{Q_sQ_a(x),x}^{(2d)}\big]S$$
$$+\big[-SQ_{D_{s,a}(x)}^{(3a)\blacktriangle}S+Q_{ND_{s,a}(x),D_{s,a}(x)}^{(3b)} - NQ_{D_{s,a}(x)}^{(3c)} - Q_{D_{s,a}(x)}^{(3d)}N\big]$$
$$+\big[D_{n,a}Q_x^{(4)}D_{a,s}-Q_{D_{s,a}(x),D_{n,a}(x)}^{(6a)}\big]S + S\big[-D_{s,a}Q_x^{(5)}D_{a,n}+Q_{D_{s,a}(x),D_{n,a}(x)}^{(6b)}\big]$$
$$-Q_{D_{s,a}(x),(SD_{n,a})(x)}^{(6c)} + \big[-Q_nQ_a^{(7a)}Q_x-Q_xQ_a^{(7b)}Q_n+Q_{Q_nQ_a(x),x}^{(7c)}\big]$$
$$-\big[Q_{N(x),Q_sQ_a(x)}^{(8)}+Q_{N(Q_sQ_ax),x}^{(8*)}\big] + Q_{Q_nQ_a(x),x}^{(8**)}$$

[by $(5.1.2)^{-\sigma}$ on (3); (0.1.4) on (2), (7); (3.1.4) on (6); (3.1.1) on (8**)]

$$= N\big[Q_xQ_aQ_s^{(1a)}-Q_{D_{s,a}(x)}^{(3c)}\big] + \big[Q_sQ_a^{(1b)}Q_x - Q_{D_{s,a}(x)}^{(3d)}\big]N + \big[Q_{s,n}Q_a^{(2a)}Q_x+D_{n,a}Q_x^{(4)}D_{a,s}$$
$$-Q_{D_{s,a}(x),D_{n,a}(x)}^{(6a)}\big]^{(9)}S - S\big[Q_xQ_a^{(2b)}Q_{s,n} + D_{s,a}Q_x^{(5)}D_{a,n} - Q_{D_{s,a}(x),D_{n,a}(x)}^{(6b)}\big]^{(10)}$$

$$
\begin{aligned}
&+\big[-NQ^{(2d1)}_{Q_sQ_ax,x}-Q^{(2d2)}_{Q_sQ_ax,x}N+Q^{(2d3)}_{S(Q_sQ_ax),S(x)}+Q^{(2d4)\blacktriangle}_{N(Q_sQ_ax),x}+Q^{(2d5)\blacktriangledown}_{N(x),Q_sQ_ax}\big]\\
&+\big[-Q_nQ^{(7a)}_aQ_x-Q_xQ^{(7b)}_aQ_n+2Q^{(7c8**)}_{Q_nQ_ax,x}\big]-\big[Q^{(8)\blacktriangledown}_{N(x),Q_sQ_ax}+Q^{(8*)\blacktriangle}_{N(Q_sQ_ax),x}\big]\\
&\quad+Q^{(3b6c)}_{[ND_{s,a}-SD_{n,a}](x),D_{s,a}(x)} \qquad \text{[by linearized } (5.1.2)^{-\sigma} \text{ on (2d); (3.1.2) on (2a), (2b)]}
\end{aligned}
$$

$$
\begin{aligned}
&=N\big[Q_xQ^{(1a)}_aQ_s-Q^{(3c)}_{D_{s,a}(x)}-Q^{(2d1)}_{Q_sQ_ax,x}\big]^{(11)}+\big[Q_sQ^{(1b)}_aQ_x-Q^{(3d)}_{D_{s,a}(x)}-Q^{(2d2)}_{Q_sQ_ax,x}\big]^{(12)}N\\
&+\big[Q_xQ^{(9a)}_aQ_{s,n}+D_{s,a}Q^{(9b)}_xD_{a,n}-Q^{(9c)}_{Q_{s,n}Q_ax,x}\big]S+S\big[Q_{s,n}Q^{(10a)}_aQ_x+D_{n,a}Q^{(10b)}_xD_{a,s}\\
&-Q^{(10c)}_{Q_{s,n}Q_ax,x}\big]+\big[-Q_nQ^{(7a)}_aQ_x-Q_xQ^{(7b)}_aQ_n+2Q^{(7c8**)}_{Q_nQ_ax,x}\big]+Q^{(2d3)}_{Q_{s,n}Q_ax,S(x)}-Q^{(3b6c)}_{D_{s_3,a}(x),D_{s,a}(x)}
\end{aligned}
$$

[by linearized (0.1.4) $s\to s,n$ on (9),(10); (3.1.2) on (2d3); and (5.1.4*) on (3b6c)]

$$
\begin{aligned}
&=-N\big[D_{s,a}Q^{(11a)\blacktriangle}_xD_{a,s}+Q_sQ^{(11b)}_aQ_x\big]-\big[D_{s,a}Q^{(12a)\blacktriangledown}_xD_{a,s}+Q_xQ^{(12b)}_aQ_s\big]N+Q_xQ_a\big[2Q^{(9a1)}_n+Q^{(9a2)}_{s_3,s}\big]\\
&+\big[2Q^{(10a1)}_n+Q^{(10a2)}_{s_3,s}\big]Q_aQ_x+D_{s,a}Q_x\big[D_{a,s}N^{(9b1)\blacktriangledown}+D^{(9b2)}_{a,s_3}\big]+\big[ND^{(10b1)\blacktriangle}_{s,a}+D^{(10b2)}_{s_3,a}\big]Q_xD_{a,s}\\
&-\big[Q^{(9c10c1)}_{S(Q_{s,n}Q_ax),x}+Q^{(9c10c2)\blacklozenge}_{Q_{s,n}Q_ax,S(x)}\big]+\big[-Q_nQ^{(7a)}_aQ_x-Q_xQ^{(7b)}_aQ_n+2Q^{(7c8**)}_{Q_nQ_ax,x}\big]\\
&\quad+Q^{(2d3)\blacklozenge}_{Q_{s,n}Q_ax,S(x)}-Q^{(3a6c)}_{D_{s_3,a}(x),D_{s,a}(x)} \qquad \text{[by (0.1.4) on (11),(12);}
\end{aligned}
$$

(5.1.3′) on (9a),(10a); (3.1.4) on (9c),(10c); (5.1.4) on (9b) and (5.1.4*) on (10b)]

$$
\begin{aligned}
&=\big[-NQ^{(11b)}_s+2Q^{(10a1)}_n-Q^{(7a)}_n\big]^{(13)\blacktriangle}Q_aQ_x+Q_xQ_a\big[-Q_sN^{(12b)}+2Q^{(9a1)}_n-Q^{(7b)}_n\big]^{(14)\blacktriangledown}\\
&+\big[Q_xQ^{(9a2)}_aQ_{s_3,s}+Q_{s_3,s}Q^{(10a2)}_aQ_x+D_{s,a}Q^{(9b2)}_xD_{a,s_3}+D_{s_3,a}Q^{(10b2)}_xD_{a,s}-Q^{(3a6c)}_{D_{s_3,a}(x),D_{s,a}(x)}\big]^{(15)}\\
&\quad-Q^{(9c10c1)}_{[2Q_n+Q_{s_3,s}]Q_ax,x}+2Q^{(7c8c*)}_{Q_nQ_ax,x} \qquad \text{[by (5.1.3}'\text{) on (9c10c1)]}
\end{aligned}
$$

$$
=-Q_{[2Q_n+Q_{s_3,s}]Q_ax,x}+\big[Q_{Q_{s_3,s}Q_ax,x}\big]+2Q_{Q_nQ_ax,x}=0
$$

[by (3.1.1) on (13), (14), and by linearized (0.1.4) $s\to s,s_3$ on (15)].

For $k=1$ the formulas $(2^{\pm\sigma,1})$ are much easier. $(2^{\sigma,1})$ follows from (3.1.4), $(5.1.2)^\sigma$, (5.1.4*), (5.1.4) since it reduces to

$$
\begin{aligned}
&NQ_bD^{(1)}_{s,a}+D^{(2)}_{a,s}Q_bN+D_{a,s}SQ^{(3)}_bS+SQ^{(4)}_bSD_{s,a}-D_{a,n}Q^{(5)}_bS-SQ^{(6)}_bD_{n,a}\\
&\quad-Q^{(7)}_{D_{a,s}(S(b)),S(b)}+Q^{(8)}_{D_{a,n}(b),S(b)}-Q^{(9)}_{N(b),D_{a,s}(b)}\\
&=\big[NQ^{(1)}_b+SQ^{(4)}_bS-Q^{(7a)}_{S(b)}\big]^{(10)}D_{s,a}+D_{a,s}\big[Q^{(2)}_bN+SQ^{(3)}_bS-Q^{(7b)}_{S(b)}\big]^{(11)}\\
&\quad-\big[Q^{(9)}_{D_{a,s}(b),N(b)}+Q^{(9*)}_{D_{a,s}(N(b)),b}\big]^{(12)}+Q^{(9**)}_{(D_{a,s}N)(b),b}+D_{a,n}\big[SQ^{(5a)}_b-Q^{(5b)}_{S(b),b}\big]\\
&\quad+\big[-Q^{(6a)}_{Sb,b}+Q^{(6b)}_bS\big]D_{n,a}+\big[Q^{(8)}_{D_{a,n}(b),S(b)}+Q^{(8*)}_{D_{a,n}(S(b)),b}\big]-Q^{(8**)}_{(D_{a,n}S)(b),b}
\end{aligned}
$$

[by (3.1.4) for (5),(6), (0.1.1) for (7)]

$$
\begin{aligned}
&=\big[-Q_bN^{\blacktriangleright}+Q^{\blacktriangle}_{N(b),b}\big]^{(10)}D_{s,a}+D_{a,s}\big[-NQ^{\blacktriangledown}_b+Q^{\blacktriangleleft}_{N(b),b}\big]^{(11)}-\big[D_{a,s}Q^{\blacktriangleleft}_{b,N(b)}+Q^{\blacktriangle}_{N(b),b}D_{s,a}\big]^{(12)}\\
&\quad-Q^{(8**9**)\bullet}_{D_{a,s_3}(b),b}+\big[D^{(5a1)\blacktriangledown}_{a,s}N+D^{(5a2)\bullet}_{a,s_3}\big]Q_b+Q_b\big[ND^{(6b1)\blacktriangleright}_{s,a}+D^{(6b2)\bullet}_{s_3,a}\big]\\
&\quad+\big[-D_{a,n}Q^{(5b)}_{S(b),b}-Q^{(6a)}_{S(b),b}D_{n,a}+Q^{(8)}_{D_{a,n}(b),S(b)}+Q^{(8*)}_{D_{a,n}(S(b)),b}\big]^{\blacklozenge}
\end{aligned}
$$

[by $(5.1.2)^\sigma$ for (10),(11), (0.1.1) for (12), (5.1.4) for (5a), (8**9**), (5.1.4*) for (6b)], which vanishes by (0.1.1) on $\bullet$, $\blacklozenge$.

Dually, formula $(2^{-\sigma,1})$ follows from (3.1.4), $(5.1.2)^{-\sigma}$, $(5.1.4^*)$, (5.1.4) since

$$
\begin{aligned}
&NQ_x^{(1)}D_{a,s}+D_{s,a}^{(2)}Q_xN+S\big[D_{s,a}Q_x+Q_xD_{a,s}\big]^{(3)}S\\
&\qquad-\big[D_{n,a}Q_x^{(4)}S+SQ_x^{(5)}D_{a,n}-Q^{(6)}_{D_{n,a}(x),S(x)}\big]-Q^{(7)}_{SD_{s,a}(x),S(x)}-Q^{(8)}_{N(x),D_{s,a}(x)}\\
&=N\big(Q^{(1a)\blacktriangledown}_{D_{s,a}(x),x}-D_{s,a}Q_x^{(1b)}\big)+\big(Q^{(2a)\blacktriangledown}_{D_{s,a}(x),x}-Q_xD_{s,a}^{(2b)}\big)N+S\big[Q^{(3)\blacktriangledown}_{D_{s,a}(x),x}\big]S\\
&\ +\big(Q_xD_{a,n}^{(4a)}-Q^{(4b)\blacktriangle}_{D_{n,a}(x),x}\big)S+S\big(D_{n,a}Q_x^{(5a)}-Q^{(5b)\blacktriangle}_{D_{n,a}(x),x}\big)+Q^{(6)\blacktriangle}_{D_{n,a}(x),S(x)}\\
&\qquad-\big[Q^{(7)\blacktriangledown}_{SD_{s,a}(x),S(x)}+Q^{(8)\blacktriangledown}_{N(x),D_{s,a}(x)}+Q^{(9*)\blacktriangledown}_{ND_{s,a}(x),x}\big]+Q^{(10*)\blacktriangle}_{S(D_{n,a}x),x}\\
&\qquad\quad+\big[Q^{(10**)}_{-SD_{n,a}(x),x}+Q^{(9**)}_{ND_{s,a}(x),x}\big]^{(11)}\qquad\qquad\text{[by (0.1.1) for (1)-(5)]}\\
&=\ -\big[ND_{s,a}^{(1b)\bullet}Q_x+Q_x^{(2b)\blacklozenge}D_{s,a}N\big]+Q_x\big[D_{a,s}^{(4a1)\blacklozenge}N+D_{a,s_3}^{(4a2)}\big]+\big[ND_{s,a}^{(5a1)\bullet}+D_{s_3,a}^{(5a2)}\big]Q_x-Q^{(11)}_{D_{s_3,a}(x),x}
\end{aligned}
$$

[by (3.1.4) for $\blacktriangle$, linearized $(5.1.2)^{-\sigma}$ for $\blacktriangledown$, $(5.1.4^*)$ for (5a), (11), (5.1.4) for (4a)], which vanishes by (0.1.1) for $D_{s_3,a}$. This completes the verification that the Gluing Conditions (4N), (4B), (4NB) follow from (5.1.2-5). ■

6 Redundancy

In the presence of scalars $\frac{1}{2}$ and $\frac{1}{3}$, the complicated gluing conditions conditions become redundant.

Redundancy Theorem 6.1 *Conditions* (5.1.3), (3.1.4) *imply* 2 (5.1.2), *so when* $\frac{1}{2}\in\Phi$ *Condition* (5.1.2) [P-Glue 2] *is a consequence of the other axioms in Structural Domination Theorem* 5.1.

If N,S *satisfy for* $\tau=\pm\sigma$ *the two conditions*

(6.1.1) (M condition): $S^\tau N^\tau=N^\tau S^\tau=M^\tau_{q_2}$,

(6.1.2) (Powers 2): $S^\sigma(q_2)=2q_3,\ S^{-\sigma}(s_i)=2s_{i+1},\ D_{q_3,s_i}=D_{q_2,s_{i+1}},\ D_{s_i,q_3}=D_{s_{i+1},q_2}$

(*where* $i=1,2$, *and* $s_1:=s, s_2:=n, s_3:=Q_sq_2$),

then these together with (5.1.3), $(5.1.2)^\tau$ *imply* $3\,(5.1.1)^\tau$, *so if* $\frac{1}{3}\in\Phi$ *we can replace condition* (5.1.1) [P-Glue 1] *in Theorem* 5.1 *by these conditions.*

Here the M*-condition* (6.1.1) *is equivalent to two conditions*

$(6.1.1)'$ (Commutativity): $N^\tau S^\tau=S^\tau N^\tau$,

$(6.1.1)''$ (Cube Condition): $S^\tau S^\tau S^\tau=D_2'^{\,\tau}+3S^\tau N^\tau\quad\big(D_2'^{\,\sigma}:=D_{q_2,n},\ D_2'^{\,-\sigma}:=D_{n,q_2}\big)$.

Conditions (5.1.3), (6.1.2) *imply* $2[N^\tau,S^\tau]=0$, *so that if* $\frac{1}{2}\in\Phi$ *then automatically* N^τ,S^τ *commute and Commutativity* $(6.1.1)'$ *is satisfied.*

Thus when $\frac{1}{6}\in\Phi$ *we can replace the P-gluing conditions* (P-Glue 1-2) = (5.1.1), (5.1.2) *by the elemental conditions* (6.1.1-2) *and the triality conditions* (5.1.4-5*).

Proof: First we redundify P-Glue 2 (5.1.2) by showing that 2(5.1.2) vanishes as a consequence of Two N (5.1.3) and Lie-structurality (3.1.4): with the abbreviation $D_2^\sigma=D_{q_2,s}, D_2^{-\sigma}=D_{s,q_2}$ from (5.1.3) and omitting superscripts for generic $\tau=\pm\sigma$, we compute the formula $2(5.1.2)^\tau$ as

$$
\begin{aligned}
&2\big[NQ_w+Q_wN+SQ_wS-Q_{S(w)}-Q_{N(w),w}\big]\\
&=(S^2-\widehat{D})Q_w+Q_w(S^2-D_2)+2SQ_wS-Q_{S(w),S(w)}-Q_{(S^2-D_2)(w),w}\quad\text{[using Two } N\text{ (5.1.3) thrice]}\\
&=\big[S^2Q_w+Q_wS^2+2SQ_wS-Q_{S(w),S(w)}-Q_{S^2(w),w}\big]-\big[D_2Q_w+Q_wD_2-Q_{D(w),w}\big]^\bullet\\
&=S\big[Q^{(1)}_{S(w),w}-Q_w^{(2)\blacktriangle}S\big]+Q_w^{(3)}S^2+2^\blacktriangle SQ_w^{(4)}S-Q^{(5)}_{S(w),S(w)}-Q^{(6)}_{S^2(w),w}
\end{aligned}
$$

$$\text{[since } \mathcal{S} \text{ on (1/2), } \mathcal{D} \text{ on } \bullet \text{ are Lie-structural by (3.1.4), (0.1.1) on } \bullet]$$

$$= \big[Q^{(1a)\blacktriangledown}_{S^2(w),w} + Q^{(1b)\blacktriangleright}_{S(w),S(w)} - Q^{(1c)}_{S(w),w}S\big] + Q^{(3)}_w S^2 + SQ^{(2,4)}_w S - Q^{(5)\blacktriangleright}_{S(w),S(w)} - Q^{(6)\blacktriangledown}_{S^2(w),w} \qquad [S \text{ is structural on (1)}]$$

$$= Q^{(1c)\blacktriangleleft}_{S(w),w}S + Q^{(3)\blacklozenge}_w S^2 - \big[Q^{(2,4a)\blacktriangleleft}_{Sw,w} + Q_w S^{(2,4b)\blacklozenge}\big]S = 0. \qquad [S \text{ is structural on (2),(4)}]$$

We always have

$$\text{(6.2)} \qquad (S^\sigma)^3 - D_2'^{\sigma} - 3N^\sigma S^\sigma = M^\sigma_{q_2} - N^\sigma S^\sigma, \ (S^{-\sigma})^3 - D_2'^{-\sigma} - 3S^{-\sigma}N^{-\sigma} = M^{-\sigma}_{q_2} - S^{-\sigma}N^{-\sigma},$$

since for $\tau = \sigma$ we have by Two N (5.1.3) that $(S^\sigma)^3 - D_2'^{\sigma} - 3N^\sigma S^\sigma = (2N^\sigma + D_2^\sigma)S^\sigma - D_{q_2,n} - 3N^\sigma S^\sigma = (D_{q_2,s}S^\sigma - D_{q_2,n}) - N^\sigma S^\sigma = M^\sigma_{q_2} - N^\sigma S^\sigma$, while for $\tau = -\sigma$ we have $(S^{-\sigma})^3 - D_2'^{-\sigma} - 3S^{-\sigma}N^{-\sigma} = S^{-\sigma}(2N^{-\sigma} + D_2^{-\sigma}) - D_{n,q_2} - 3S^{-\sigma}N^{-\sigma} = (S^{-\sigma}D_{s,q_2} - D_{n,q_2}) - S^{-\sigma}N^{-\sigma} = M^{-\sigma}_{q_2} - S^{-\sigma}N^{-\sigma}$.

From structurality (3.1.4) and Power (6.1.2) we obtain

$$\text{(6.3)} \quad [S^\tau, D_2^\tau] = [S^\tau, D_2'^{\tau}] = 0,$$

$$\text{(6.4)} \quad M^\tau_{q_2} = D_2^\tau S^\tau - D_2'^{\tau} = S^\tau D_2^\tau - D_2'^{\tau}.$$

For (6.3), $[S^\sigma, D_{q_2,s_i}] = D_{S(q_2),s_i} - D_{q_2,S(s_i)}$ [by structurality] $= 2D_{q_3,s_i} - 2D_{q_2,s_{i+1}} = 0$ [by Power], and dually $[S^{-\sigma}, D_{s_i,q_2}] = 0$. For (6.4), by (0.1.1) we have $M^\sigma_{q_2} = D_{q_2,s}S^\sigma - D_{q_2,n} = D_2^\sigma S^\sigma - D_2'^{\sigma}$, and dually $M^{-\sigma} := D^{-\sigma}_{q_2} = S^{-\sigma}D_{s,q_2} - D_{n,q_2} = D_{s,q_2}S^{-\sigma} - D_{n,q_2}$ [by (6.2)] $= D_{s,q_2}S^{-\sigma} - D_2'^{-\sigma}$.

Clearly (6.1.1) implies (6.1.1)$'$, and (6.1.1)$''$ [by (6.4)]. Conversely, by (6.4) Cubing (6.1.1)$''$ implies $M^\sigma_{q_2} = N^\sigma S^\sigma$, $M^{-\sigma} = S^{-\sigma}N^{-\sigma}$, so together with Commutativity (6.1.1)$'$ they imply (6.1.1). These guarantee that the M-condition (6.1.1) is equivalent to (6.1.1)$'$ with (6.1.1)$''$.

Now we can redundify (5.1.1) with the help of (5.1.3), (6.1.1), (6.1.2): we compute $3(5.1.1)^\tau$ as

$$\big[NQ_wS + SQ_wN - Q_{N(w),S(w)}\big] + \big[(2N)Q_wS + SQ_w(2N) - Q_{(2N)(w),S(w)}\big]$$

$$= \big[(NQ_w + Q_wN)^{(1)}S - Q^{(2)}_w M_{q_2}\big] + \big[S(NQ_w + Q_wN)^{(3)} - M_{q_2}Q^{(4)}_w\big] - Q^{(5)}_{N(w),S(w)}$$
$$+ \big[(SS^{(6)} - D_2^{(7)})Q_wS - SQ_w(SS^{(8)} - D_2^{(9)})\big] - \big[Q^{(10)}_{SS(w),w} - Q^{(11)}_{D_2(w),S(w)}\big]$$

$$\text{[by M (6.1.1) in (2),(4) and (5.1.3) for } 2N]$$

$$= \big[-SQ^{(1a)\blacktriangle}_w S + Q^{(1b)\bullet}_{S(w)} + Q^{(1c)\blacklozenge}_{N(w),w}\big]S + S\big[-SQ^{(3a)\blacktriangledown}_w S + Q^{(3b)\bullet}_{S(w)} + Q^{(3c)\blacklozenge}_{N(w),w}\big]$$
$$+ M_{q_2}Q^{(2)}_w + Q_wM^{(4)}_{q_2} - Q^{(5)\blacklozenge}_{N(w),S(w)} + \big[SSQ^{(6)\blacktriangledown}_w S + SQ^{(8)\blacktriangle}_w SS$$
$$+ D_2Q^{(7)}_w S + SQ^{(9)}_w D_2\big] + \big[-Q^{(10)\bullet}_{SS(w),S(w)} + Q^{(11)}_{D_2(w),S(w)}\big] \qquad \text{[by P-Glue 2 (5.1.2) on (1), (2)]}$$

$$= Q^{(1c3c5)\blacklozenge}_{SN(w),w} + M_{q_2}Q^{(4)}_w - Q^{(2)}_w M_{q_2} + D\big[SQ^{(7a)}_w - Q^{(7b)\blacktriangleright}_{S(w),w}\big] + \big[-Q^{(9a)\blacktriangleright}_{S(w),w} + Q^{(9b)}_w S\big]D$$
$$+ Q^{(11)\blacktriangleright}_{D_2(w),S(w)} \qquad [S \text{ is structural (3.1.4) on } \bullet, \blacklozenge, (7), (9)]$$

$$= \big[D_2^\tau S^\tau - M^\tau_{q_2}\big]Q_w + Q_w\big[S^{-\tau}D_2^{-\tau} - M^{-\tau}_{q_2}\big] - Q_{[M_{q_2} + D_2S](w),w}$$

$$\text{[by M (6.1.1) on } \blacklozenge, D_2 \text{ structural on } \blacktriangleright]$$

$$= \big[D_2'^{\tau}\big]Q_w + Q_w\big[D_2'^{-\tau}\big] - Q_{[D_2'^{\tau}](w),w}, \qquad \text{[by (6.1.4)]}$$

which vanishes since D_2' is also structural by (0.1.1). This shows $3\,(5.1.1)^\tau$ does indeed vanish.

Finally, note that Two N (5.1.3), (6.3) imply $2[N^\tau, S^\tau] = [S^\tau S^\tau - D_2^\tau, S^\tau] = [S^\tau, D_2^\tau]$ vanishes by (6.3). ■

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