JORDAN SUPERALGEBRAS DEFINED BY BRACKETS

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Abstract

Jordan superalgebras defined by brackets on associative commutative superalgebras are studied. It is proved that any such a superalgebra is imbedded into a superalgebra defined by Poisson brackets. In particular, all Jordan superalgebras of brackets are i-special. The speciality of these superalgebras is also examined, and it is proved, in particular, that the Cheng—Kac superalgebra is special.

Introduction

Let $\Gamma = \Gamma_0 + \Gamma_1$ be an associative commutative superalgebra over a ground field F, $chF \neq 2$, with a bracket $\{,\} : \Gamma \times \Gamma \to \Gamma$. Consider the direct sum of two copies of the vector space Γ

$$J = \Gamma + \Gamma x$$

with the product

$$a \cdot b = ab, \ a \cdot bx = (ab)x,$$
$$(bx) \cdot a = (-1)^{|a|}(ba)x, \ ax \cdot bx = (-1)^{|b|}\{a, b\}$$

where $a, b \in \Gamma_0 \cup \Gamma_1$, juxtaposition stands for the product in Γ and $(-1)^{|a|} = (-1)^k$ if $a \in \Gamma_k$. We will refer to $J = J(\Gamma, \{,\})$ as a Kantor double of $(\Gamma, \{,\})$.

A bracket $\{,\}$ is called Jordan if the Kantor double J is a Jordan superalgebra. I. L. Kantor [K] proved that every Poisson bracket is Jordan.

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In [KMZ] it was shown that all Jordan superalgebras that correspond to the so called "superconformal algebras" are Kantor doubles or are embeddable in Kantor doubles (the Cheng-Kac series). Moreover, the same is true for all simple finite-dimensional Jordan superalgebras over a field of prime characteristic with a non-semisimple even part (see [MZ]).

A Jordan superalgebra J is called *special* if it can be embedded into a Jordan superalgebra of the type A^+, which is obtained from an associative superalgebra A via the new multiplication $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$. Furthermore, J is called *i-special* if it is a homomorphic image of a special one.

In [S2] it was proved that for every Poisson bracket the corresponding Jordan Kantor double $J(\Gamma, \{,\})$ is i-special. Moreover, in this case $J(\Gamma, \{,\})$ is special if and only if $\{\{\Gamma, \Gamma\}, \Gamma\} = 0$.

In this paper we consider the speciality and i-speciality problems for the general Jordan Kantor doubles. First, we prove that all Jordan brackets are embeddable into Poisson brackets. In view of [S2] this implies that all Jordan Kantor doubles are i-special. Then we examine speciality of Jordan superalgebras from [KMZ], [MZ]. In particular, we show that the Jordan superalgebras of the Cheng-Kac series are special.

1 Embedding of Jordan brackets into Poisson brackets

We start with some definitions and notations that will be used in the rest of the paper.

By a superalgebra we mean a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A = A_0 + A_1$

Example. Let V be a vector space. The Grassmann (or exterior) algebra $G(V)$ is the quotient of the tensor algebra $T(V)$ modulo the ideal generated by symmetric tensors $v \otimes w + w \otimes v;\ v, w \in V$. Clearly $G(V) = G_{\bar{0}} + G_{\bar{1}}$, where $G_{\bar{0}}$ (resp. $G_{\bar{1}}$) is spanned by products of elements of V of even (resp. odd) length.

Let V be a vector space of countable dimension. By the Grassmann envelope of a superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ we mean the subalgebra $G(A) = A_{\bar{0}} \otimes G_{\bar{0}} + A_{\bar{1}} \otimes G_{\bar{1}}$ of the tensor product $A \otimes G(V)$.

Let $\mathcal{V}$ be a homogeneous variety of algebras, that is, a class of F-algebras satisfying a certain set of homogeneous identities and all their partial linearizations (see [ZSSS]).

Definition 1.1 *A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is called a $\mathcal{V}$-superalgebra if the Grassmann envelope $G(A)$ lies in $\mathcal{V}$.*

In particular, if $A = A_{\bar{0}} + A_{\bar{1}}$ is a $\mathcal{V}$-superalgebra, then $A_{\bar{0}} \in \mathcal{V}$ and $A_{\bar{1}}$ is a $\mathcal{V}$-bimodule over $A_{\bar{0}}$ (see [J]).

In this way one can define Lie superalgebras, Jordan superalgebras, etc. Clearly, associative superalgebras are just $\mathbb{Z}/2\mathbb{Z}$-graded associative algebras.

If A is an associative superalgebra then the vector space A with a new operation $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ is a Jordan superalgebra which is denoted $A^{(+)}$. A Jordan superalgebra is said to be *special* if it is embeddable into a superalgebra of type $A^{(+)}$. Otherwise J is said to be *exceptional.* If $J \subseteq A^{(+)}$ and the associative superalgebra A is generated by the subspace J then A is said to be an associative enveloping superalgebra of J. A Jordan superalgebra is *i-special* if it is a homomorphic image of a special Jordan superalgebra.

Let $\Gamma = \Gamma_0 + \Gamma_1$ be a unital associative commutative superalgebra with a skewsymmetric bilinear mapping $\{,\} : \Gamma \times \Gamma \to \Gamma$, $\{\Gamma_i, \Gamma_j\} \subseteq \Gamma_{i+j}$ which we call a bracket.

Consider the Kantor double $J = \Gamma + \Gamma x$ (see the Introduction) with the multiplication

$$a \cdot b = ab,\ a \cdot bx = (ab)x,$$
$$(bx) \cdot a = (-1)^{|a|}(ba)x,\ ax \cdot bx = (-1)^{|b|}\{a, b\},$$

where $a, b \in \Gamma_0 \cup \Gamma_1$ and juxtaposition stands for the multiplication in Γ. The $\mathbb{Z}/2\mathbb{Z}$-gradation on Γ can be extended to a $\mathbb{Z}/2\mathbb{Z}$-gradation on J via $J_0 = \Gamma_0 + \Gamma_1 x$, $J_1 = \Gamma_1 + \Gamma_0 x$.

A bracket $\{,\}$ is said to be *Jordan* if the Kantor double J is a Jordan superalgebra. For a Jordan bracket $\{,\}$ the mapping $D :\mapsto \{a, 1\}$ is a derivation of Γ (see [M]). Moreover, in [M], [KM] it was proved that a bracket $\{,\}$ is Jordan if and only if for arbitrary elements $a, b, c \in \Gamma_0 \cup \Gamma_1$, $x \in \Gamma_1$ hold the identities

$$\begin{aligned}
&\{a, b\} = -(-1)^{|a||b|}\{b, a\}, && (1)\\
&\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\} - D(a)bc, && (2)\\
&J(a, b, c) := \{\{a, b\}c\} + (-1)^{|a||b|+|a||c|}\{\{b, c\}, a\} + (-1)^{|a||c|+|b||c|}\{\{c, a\}, b\} &&\\
&\ = -\{a, b\}D(c) + (-1)^{|a||b|+|a||c|}\{b, c\}D(a) + (-1)^{|a||c|+|b||c|}\{c, a\}D(b), && (3)\\
&\{\{x, x\}, x\} = -\{x, x\}D(x). && (4)
\end{aligned}$$

Identity (4) is needed only in characteristic 3 case, otherwise it follows from (3) (see [KM1]).

A Jordan bracket $\{,\}$ is called a *Poisson bracket* if $D(a) = 0$ for any $a \in \Gamma$. Observe that in this case Γ is a Lie superalgebra with respect to the bracket.

We will now prove that every Jordan bracket can be embedded into a Poisson bracket.

Proposition 1.1 *Let $\{,\}$ be a Jordan bracket on Γ. On the superalgebra of Laurent polynomials $\Gamma[t^{-1},t]$ consider a bracket*

$$< t^i a, t^j b >= t^{i+j-2}(-(i-1)aD(b) + (-1)^{|a||b|}(j-1)bD(a) + \{a,b\})$$

where $a,b \in \Gamma_0 \cup \Gamma_1$. Then $<,>$ is a Poisson bracket. Moreover, the Jordan superalgebra $J = (\Gamma, \{,\})$ is isomorphic to a subsuperalgebra $\Gamma + t\Gamma x$ of the Kantor double $J = (\Gamma[t^{-1},t], <,>)$.

Proof: First of all observe that $< ta, tb >= \{a,b\}$. Hence $\Gamma + t\Gamma x$ is a subsuperalgebra of $J(\Gamma[t^{-1},t], <,>)$ which is isomorphic to $J(\Gamma, \{,\})$.

Passing to Grassmann envelopes we see that it is sufficient to prove that $<,>$ is a Poisson bracket only for the case $\Gamma = \Gamma_0$.

Let $a,b,c \in \Gamma = \Gamma_0$, and let us check that $<,>$ satisfies the Leibniz identity

$$\begin{aligned} < t^{i+j}ab, t^k c > &= t^{i+j+k-2}(-(i+j-1)abD(c) + (k-1)cD(ab) + \{ab,c\}) \\ &= t^{i+j+k-2}(-(i+j-1)abD(c) + (k-1)caD(b) + \\ &\quad + (k-1)cbD(a) + \{a,c\}b + a\{b,c\} + abD(c)) \\ &= t^{i+j+k-2}a(-(j-1)bD(c) + (k-1)cD(b) + \{b,c\}) \\ &\quad + t^{i+j+k-2}b(-(i-1)aD(c) + (k-1)cD(a) + \{a,c\}) \\ &= t^i a < t^j b, t^k c > + t^j b < t^i a, t^k c > . \end{aligned}$$

Now let us check the Jacobi identity. We have

$$\begin{aligned} &<< t^i a, t^j b >, t^k c >=< t^{i+j-2}(-(i-1)aD(b) + (j-1)bD(a) + \{a,b\}), t^k c > \\ &= t^{i+j+k-4}(-(i+j-3)(-(i-1)aD(b) + (j-1)bD(a) + \{a,b\})D(c) \\ &\quad + (k-1)(-(i-1)D(a)D(b) - (i-1)aD^2(b) + (j-1)D(b)D(a) \\ &\quad + (j-1)bD^2(a) + D(\{a,b\}))c \\ &\quad - (i-1)\{aD(b),c\} + (j-1)\{bD(a),c\} + \{\{a,b\},c\}) \\ &= t^{i+j+k-4}((i+j-4)(i-1)aD(b)D(c) - (i+j-4)(j-1)bD(a)D(c) \\ &\quad + (k-1)(j-i)cD(a)D(b) - (k-1)(i-1)aD^2(b)c \\ &\quad + (k-1)(j-1)D^2(a)bc - (i+j-3)\{a,b\}D(c) \\ &\quad + (k-1)D(\{a,b\})c - (i-1)\{a,c\}D(b) - (i-1)a\{D(b),c\} \\ &\quad + (j-1)\{b,c\}D(a) + (j-1)b\{D(a),c\} + \{\{a,b\},c\}). \end{aligned}$$

Let us compute the coefficients of all summands in $\frac{1}{t^{i+j+k-4}} J(t^i a, t^j b, t^k c)$.
The summand $aD(b)D(c)$ has the coefficient

$$(i+j-4)(i-1) - (k+i-4)(i-1) + (i-1)(k-j) = 0.$$

The summand $aD^2(b)c$ has the coefficient

$$-(k-1)(i-1)+(i-1)(k-1)=0.$$

By symmetry, all the summands in $J(t^ia, t^jb, t^kc)$ which do not contain the bracket $\{,\}$ are zero.

Now the difference $-(i-1)a\{D(b),c\}+(j-1)b\{D(a),c\}$ after cyclic summing gives zero.

The summand $D(c)\{a,b\}$ has the coefficient

$$-(i+j-3)+(j-1)+(i-1)=1,$$

and therefore

$$\begin{aligned}\frac{1}{t^{i+j+k-4}}J(t^ia,t^jb,t^kc) &= D(c)\{a,b\}+D(b)\{c,a\}+D(a)\{b,c\}\\ &\quad+\{\{a,b\}c\}+\{\{b,c\},a\}+\{\{c,a\},b\}=0.\end{aligned}$$

By the property (4) of Jordan brackets Proposition is proved.

In [S2] it was shown that a Kantor double of a Poisson bracket is always i-special. This and Proposition 1.1 immediately imply the following corollary.

Corollary 1.1 *For an arbitrary Jordan bracket $\{,\}$ on an associative commutative superalgebra Γ the superalgebra $J(\Gamma,\{,\})$ is i-special.*

2 On speciality of bracket superalgebras

Let $J=J_0+J_1$ be a Jordan superalgebra. For an arbitrary element $x\in J$ let $R(x)$ denote the operator of right multiplication by x. If $x\in J_1$ then $R(x)^2$ is a derivation of the superalgebra J. Let $D(J_1,J_1)$ denote the linear span of all derivations $R(x)^2$, $x\in J_1$. Clearly $D(J_1,J_1)$ is a Lie algebra.

Now our aim will be to prove the following proposition.

Proposition 2.1 *Let $J=J_0+J_1$ be a special Jordan superalgebra, I a nilpotent ideal of the even part J_0 which is finitely generated as an ideal and is invariant under $D(J_1,J_1)$. Then the ideal $id_J(I)$ of the superalgebra J generated by I is nilpotent.*

Corollary 2.1 *Let $J=J_0+J_1$ be a Jordan superalgebra which does not contain nonzero nilpotent ideals. If J_0 contains a nonzero nilpotent ideal which is $D(J_1,J_1)$-invariant and finitely generated as an ideal then J is exceptional.*

Let $J = J_0 + J_1$ be a special Jordan superalgebra with an associative enveloping superalgebra $R = R_0 + R_1$. Let I be an ideal of J_0 which is invariant under $D(J_1, J_1)$. Let I_s denote the linear span of all products $a_1 \cdots a_k,\ k \geq s, a_i \in I$, in the algebra R.

Lemma 2.1 *$I_s R I_s R I_s \subseteq R I_{2s} R$.*

Proof: First let us notice that

$$I_s R_0 \subseteq R_0 I_s.$$

Indeed, the algebra R_0 is generated by J_0 and $J_1 J_1$. For arbitrary elements $u_1, \ldots, u_k \in I,\ k \geq s,\ a \in J_0;\ x, y \in J_1$ we have $u_1 \cdots u_k a = u_1 \cdots u_{k-1}(u_k a + a u_k) - u_1 \cdots u_{k-2}(u_{k-1} a + a u_{k-1}) u_k + \cdots + (-1)^k a u_1 \cdots u_k \in R I_s$. Furthermore, $xy = \frac{1}{2}([x, y] + (xy + yx))$. The commutator $[x, y]$, lies in J_0 whereas the operator

$$J \to J$$

$$u \to [u, xy + yx]$$

lies in $D(J_1, J_1)$.

Hence $u_1 \cdots u_k(xy + yx) = (xy + yx) u_1 \cdots u_k + [u_1 \cdots u_k, xy + yx] \in R I_s$.

This implies that

$$I_s R_0 I_s \subseteq R_0 I_{2s}.$$

Let us show that $I_s R_i I_s R_j I_s \subseteq R I_{2s} R$.

If $i = 0$ or $j = 0$ then the inclusion was proved above.

If $i = j = 1$ then $R_i I_s R_j \subseteq R_0$ and it remains to refer to the inclusion above. Lemma is proved.

Corollary 2.2 *In the assumptions of Lemma 2.1, for any $k \geq 0$ holds the inclusion*

$$(R I_s R)^{3^k} \subseteq R I_{2^k s} R.$$

Proof: The Lemma gives the base of induction for $k = 1$. Assume that the assertion is true for $k - 1$, then

$$(R I_s R)^{3^k} = ((R I_s R)^{3^{k-1}})^3 \subseteq (R I_{2^{k-1} s} R)^3 \subseteq R I_{2 \cdot 2^{k-1} s} R = R I_{2^k s} R.$$

For elements $a, b \in R$ denote $a \cdot b = \frac{1}{2}(ab + ba)$. Let $I^{\cdot n}$ denote the n-th power of the ideal I in J.

Lemma 2.2 *For an arbitrary $k \geq 1$ we have $\underbrace{I^{\cdot 2} \cdots I^{\cdot 2}}_{k} \subseteq R I^{\cdot k+1}$, where juxtaposition stands for the multiplication in R.*

Proof: The assertion is obvious for $k=1$. By the induction assumption $\underbrace{I^{\cdot 2}\cdots I^{\cdot 2}}_{k}\subseteq RI^{\cdot k}I^{\cdot 2}$. If $u\in I^{\cdot k},\ a,b\in I$, then $u(a\cdot b)=u\cdot(a\cdot b)+\frac{1}{2}[u,a\cdot b]$ and clearly $u\cdot(a\cdot b)\in I^{\cdot k+1}$. Now, $[u,a\cdot b]=[u\cdot a,b]+[u\cdot b,a]$, $u\cdot a\in I^{\cdot k+1},\ u\cdot b\in I^{\cdot(k+1)}$ and it remains to notice that $(u\cdot a)b=2(u\cdot a)\cdot b-b(u\cdot a)\in RI^{\cdot k+1}$. Lemma is proved.

Lemma 2.3 *Suppose that the ideal I of J_0 is generated (as an ideal) by m elements $a_1,\ldots,a_m$. Then $I_{3(m+2)}\subseteq R_0I^{\cdot 3}$.*

Proof: Let A be the subalgebra of R_0 generated by J_0. Let us show first that I_{m+2} lies in the ideal of A generated by $I^{\cdot 2}$. Let $u_1,\ldots,u_{m+2}\in I$. We have $u_1\cdots u_{m+2}=\frac{1}{2}u_1\cdots u_m[u_{m+1},u_{m+2}]=\pm u_1\cdots\hat{u}_i\cdots u_{m+1}[u_i,u_{m+2}]$ mod $id_A(I^{\cdot 2})$.

If $a,b\in I,\ x\in J_0$ then $[a\cdot x,b]=[a,b\cdot x]+[x,a\cdot b]=[a,b\cdot x]$ mod $id_A(I^{\cdot 2})$. Hence, without loss of generality we can assume that $u_1,\ldots,u_{m+1}\in\{a_1,\ldots,a_m\}$ and therefore $u_i=u_j$ for some $1\leq i\neq j\leq m+1$. This implies that $u_1\cdots u_{m+1}\in id_A(I^{\cdot 2})$.

Now let us show that

$$(id_A(I^{\cdot 2}))^3\subseteq id_A(I^{\cdot 3})$$

Since for arbitrary elements $a,b\in J_0$ we have $ab=a\cdot b+\frac{1}{2}[a,b]$ and $[I^{\cdot 2},[a,b]]\subseteq I^{\cdot 2}$ it is sufficient to prove that

$$I^{\cdot 2}J_0I^{\cdot 2}J_0I^{\cdot 2}\subseteq id_A(I^{\cdot 3}).$$

Choose arbitrary elements $u_1,u_2,u_3\in I^{\cdot 2}$ and $a,b\in J_0$.

We have $\{I^{\cdot 2},J_0,I^{\cdot 2}\}\subseteq(I^{\cdot 2}\cdot J_0)\cdot I^{\cdot 2}+(I^{\cdot 2}\cdot I^{\cdot 2})\cdot J_0\subseteq I^{\cdot 3}$. This implies that the expression $u_1au_2bu_3+id_A(I^{\cdot 3})/id_A(I^{\cdot 3})$ is skew symmetric in u_1,u_2,u_3.

We have also $I^{\cdot 2}I^{\cdot 2}\subseteq I^{\cdot 2}\cdot I^{\cdot 2}+[I^{\cdot 2}\cdot I,I]\subseteq id_A(I^{\cdot 3})$ and $I^{\cdot 2}II^{\cdot 2}\subseteq(I^{\cdot 2}\cdot I)I^{\cdot 2}+II^{\cdot 2}I^{\cdot 2}\subseteq id_A(I^{\cdot 3})$.

Hence $u_1(au_2b+bu_2a)u_3\subseteq I^{\cdot 2}II^{\cdot 2}\subseteq id_A(I^{\cdot 3})$.

Therefore the expression $u_1au_2bu_3+id_A(I^{\cdot 3})/id_A(I^{\cdot 3})$ is skewsymmetric in a and b.

Now $u_1au_2bu_3=4(u_1\cdot a)(u_2\cdot b)u_3=4(u_2\cdot b)(u_1\cdot a)u_3$ mod $id_A(I^{\cdot 3})$.

Finally, $u_1au_2bu_3=4((u_1\cdot a)\cdot(u_2\cdot b))u_3\in I^{\cdot 2}I^{\cdot 2}=(0)$ mod $id_A(I^{\cdot 3})$.

The inclusions $I_{m+2}\subseteq id_A(I^{\cdot 2})$, $(id_A(I^{\cdot 2}))^3\subseteq id_A(I^{\cdot 3})$ imply $I_{3(m+2)}\subseteq(I_{m+2})^3\subseteq(id_A(I^{\cdot 2}))^3\subseteq id_A(I^{\cdot 3})$.

Lemma is proved.

Proof of Proposition 2.1. Suppose that the ideal I of J_0 is $D(J_1,J_1)$-invariant and generated (as an ideal) by m elements. Suppose further that $I^{\cdot n}=(0),\ n\geq 2$.

Find integers s_1, s_2 such that $2^{s_1} \geq n-1,\ 2^{s_2} \geq 3(m+2)$. By Corollary 2.2 applied to the ideal $I^{\cdot 3}$ we have

$$\begin{aligned}(RI^{\cdot 3}R)^{3^{s_1}} &\subseteq R(I^{\cdot 3})^{2^{s_1}}R \subseteq R(I^{\cdot 3})^{n-1}R \\ &\subseteq R(I^{\cdot 2})^{n-1}R \subseteq RI^{\cdot n}R = (0).\end{aligned}$$

Similarly, by Lemma 2.3,

$$(RIR)^{3^{s_2}} \subseteq RI_{2^{s_2}}R \subseteq RI_{3(m+2)}R \subseteq RI^{\cdot 3}R.$$

Let $s = s_1 + s_2$. Then

$$(RIR)^{3^s} = ((RIR)^{3^{s_2}})^{3^{s_1}} \subseteq (RI^{\cdot 3}R)^{3^{s_1}} = (0),$$

which proves the Proposition.

Corollary 2.3 *Let $\Gamma = \Gamma_0 + \Gamma_1$ be an associative commutative finitely generated superalgebra such that $\Gamma_1\Gamma_1 \neq (0)$. Let $\{,\}$ be a Jordan bracket on Γ. Suppose that the Kantor double superalgebra $J = J(\Gamma, \{,\})$ does not contain nonzero nilpotent ideals. Then the superalgebra J is exceptional.*

Proof: Let I the ideal of J_0 generated by $\Gamma_1\Gamma_1 \neq (0)$. Since $\Gamma_1\Gamma_1$ is the product in the superalgebra J, to check that I is $D(J_1, J_1)$-invariant it is sufficient to prove that $\Gamma_1 D(J_1, J_1) \subseteq \Gamma_1$. We have

$$D(J_1, J_1) = D(\Gamma_1, \Gamma_1) + D(\Gamma_1, \Gamma_0 x) + D(\Gamma_0 x, \Gamma_0 x).$$

It is easy to see that $\Gamma_1 D(\Gamma_1, \Gamma_1) = \Gamma_1 D(\Gamma_1, \Gamma_0 x) = (0)$ and that $\Gamma_1 D(\Gamma_0 x, \Gamma_0 x) \subseteq \Gamma \cap J_1 = \Gamma_1$.

If Γ is generated by m even elements $a_1, \ldots, a_m$ and n odd elements $b_1, \ldots, b_n$, then I is generated (as a J_0-ideal) by all products $b_i b_j,\ 1 \leq i < j \leq n$.

It is easy to see that $I = \Gamma_1\Gamma_1 + \Gamma_1\Gamma_1\Gamma_1 x$ and that for any k

$$I^{\cdot k} = \underbrace{\Gamma_1\Gamma_1 \cdots \Gamma_1}_{2k} + \underbrace{\Gamma_1 \cdots \Gamma_1}_{2k+1} x.$$

Hence, if $2k \geq n+1$ then $I^{\cdot k} = (0)$. By Corollary 1 the superalgebra J is exceptional.

Let now F be an algebraically closed field of characteristic $p > 2$, and let $O(m) = F[a_1, \ldots, a_m \,|\, a_1^p = \cdots = a_m^p = 0]$ be the algebra of truncated polynomials. Furthermore, let $G(n) =< 1, \xi_1, \ldots, \xi_n \,|\, \xi_i\xi_j + \xi_j\xi_i = 0,\ 1 \leq i, j \leq n >$ be the Grassmann algebra on n variables. The tensor product

$$O(m,n) = O(m) \otimes_F G(n) = O(m) \otimes_F G_0(n) + O(m) \otimes_F G_1(n)$$

is an associative commutative superalgebra.

In [MZ] it was proved that an arbitrary simple finite dimensional Jordan superalgebra J over F with a nonsemisimple part J_0 is isomorphic to one of the following superalgebras:

1) a Kantor double $J = J(O(m,n), \{,\})$ of an appropriate Jordan bracket on $O(m,n)$,

2) a superalgebra of a Cheng-Kac type (see below for more details).

Corollary 2 implies that for $n \geq 2$ the simple superalgebras $J(O(m,n), \{,\})$ are exceptional.

A classification along the same lines: Kantor doubles and Cheng-Kac superalgebras, is developed in [KMZ] for infinite dimensional Jordan superalgebras of zero characteristic that correspond to the so called "superconformal algebras" ([KL], [CK]). Speciality problems for these superalgebras can be treated similarly.

Let Γ be again an arbitrary associative commutative superalgebra with a Jordan bracket $\{,\}, D(a) = \{a, 1\}$. The bracket $\{,\}$ is said to be of vector type if $\{a, b\} = D(a)b - aD(b)$. In [M],[KM] and in [S1] it was proved that the Kantor double of a bracket of vector type is a special superalgebra. Furthermore, in [M],[KM] two important examples of classical and Grassmann Poisson brackets were treated and it was showed that in both cases the Kantor double superalgebras are exceptional. Since both classical and Grassman Poisson brackets are not nilpotent, the last result follows from the following speciality criterium for the Kantor double of a Poisson bracket obtained in [S2]: a Poisson bracket $\{,\} : \Gamma \times \Gamma \to \Gamma$ is special if and only if $\{\{\Gamma, \Gamma\}, \Gamma\} = (0)$.

Proposition 2.2 *Let $\Gamma = \Gamma_0 + \Gamma_1$ be a finitely generated associative commutative superalgebra with a Jordan bracket $\{,\}$ such that the superalgebra $J(\Gamma, \{,\})$ does not contain nonzero nilpotent ideals. Suppose that either (i) $\Gamma_1 = (0)$ or (ii) Γ_1 contains an element ξ such that $\Gamma_1 = \Gamma_0\xi$ and $\{\Gamma_0, \xi\} = (0)$, $\{\xi, \xi\} = -1$. Then the superalgebra $J(\Gamma, \{,\})$ is special if and only if the restriction of $\{,\}$ to Γ_0 is of vector type.*

Remark. Proposition 2.2 takes care of all superalgebras of the type $J(O(m,n), \{,\})$, $n = 0$ or 1, from the classification list of [MZ].

Proof: Let $\Gamma = \Gamma_0 + \Gamma_1$ be a finitely generated associative commutative superalgebra with a Jordan bracket $\{,\}$. Suppose that the Kantor double superalgebra $J = \Gamma + \Gamma x$ is special and does not contain nonzero nilpotent ideals. Let $R = R_0 + R_1$ be an associative enveloping superalgebra of J, $J \subseteq R^{(+)}$. Factoring out the Baer radical of R we can assume that R does not contain nonzero nilpotent ideals. For elements $u, v \in R$ let uv denote their product in R and let $[u, v] = uv - vu$, $u \circ v = uv + vu$. We will also denote by $\cdot$ the product in the superalgebra J.

We will use the following identities from [S2] which can be verified in an associative algebra by a straightforward computation:

$$\begin{aligned} [a \circ x, b \circ x] &= [a, x^2] \circ b - [b, x^2] \circ a + ([a, b] \circ x) \circ x + [[a, x], [b, x]], \quad (5) \\ [u, [r, s]] &= ((u \circ r) \circ s - (u \circ s) \circ r). \quad (6) \end{aligned}$$

Let $a, b \in J_0$. Our aim is to prove that $< a, b >= \{a, b\} - D(a) \cdot b + a \cdot D(b) = 0$. It is easy to see that $< a, b >= - < b, a >$ and $<, >$ satisfies the Leibniz identity, that is, $< a, b^{\cdot 2} >= 2 < a, b > \cdot b$. Now, for an arbitrary element $u \in J$ we have by (6)

$$[u, [a, b]] = 4((u \cdot a) \cdot b - (u \cdot b) \cdot a) = 0,$$

Hence $[a, b]$ lies in the center of the algebra R. Moreover,

$$2[a, b]^2 = [a, [a, b^{\cdot 2}]] - [a, [a, b]] \circ b = 0.$$

Since R does not contain nonzero nilpotent ideals, it follows that $[a, b] = 0$.

Furthermore, observe that $[a, x^2] = 2[a \cdot x, 1 \cdot x] = 4\{a, 1\} = 4D(a)$ and $[a \cdot x, b \cdot x] = 2\{a, b\}$. Since $[a, b] = 0$, we get from (5)

$$2\{a, b\} = 2D(a) \cdot b - 2D(b) \cdot a + 1/4\, [[a, x], [b, x]],$$

which yields

$$< a, b >= \frac{1}{8}\, [[a, x], [b, x]].$$

Consider $2[a, x]^2 = [a, [a, x^2]] - [a, [a, x]] \circ x$. The second summond is zero by (6), and the first one is equal to $4[a, D(a)] = 0$. Hence $[a, x]^2 = 0$ and $[a, x][b, x] + [b, x][a, x] = 0$. Now by (6) again,

$$16 < a, b >^2 = [[[a, x]^2, [b, x]], [b, x]] - [a, x] \circ [[[a, x], [b, x]], [b, x]] = 0.$$

Let I be the ideal generated in J_0 by all the elements $< a, b >$; $a, b \in \Gamma_0$. Since the algebra Γ is finitely generated it implies that the even part Γ_0 is also finitely generated. Let $\Gamma_0 =< a_1, \ldots, a_m >$. Then the ideal I is generated (as an ideal) by elements $< a_i, a_j >$, $1 \leq i < j \leq m$.

If $\Gamma = \Gamma_0$ then $J_0 = \Gamma_0$. If $\Gamma = \Gamma_0 + \Gamma_0\xi$, $\{\xi, \xi\} = -1$, then $J_0 = \Gamma_0 + \Gamma_0\xi x$, $(\xi x) \cdot (\xi x) = 1$. In the latter case $J_0 \simeq \Gamma_0 \oplus \Gamma_0$. In both cases J_0 is an associative commutative algebra. This implies that the ideal I is nilpotent.

For elements $a, b, c \in \Gamma_0$ consider

$$< a, b >_c = (a \cdot cx) \cdot (b \cdot cx) - ((a \cdot cx) \cdot cx) \cdot b + a \cdot ((b \cdot cx) \cdot cx)$$

$$= \{ac, bc\} - \{ac, c\}b + a\{bc, c\} =< a, b > c^2 \in I.$$

Since Γ_0 and $\Gamma_0 x$ are invariant under $D(\Gamma_0 x, \Gamma_0 x)$ this implies that the ideal I is invariant under $D(\Gamma_0 x, \Gamma_0 x)$.

If $\Gamma = \Gamma_0$ then $ID(J_1, J_1) \subseteq I$. In the case $\Gamma = \Gamma_0 + \Gamma_0 \xi$ we have $D(J_1, J_1) = D(\Gamma_0 x, \Gamma_0 x) + D(\Gamma_0 x, \Gamma_0 \xi) + D(\Gamma_0 \xi, \Gamma_0 \xi)$.

From $\Gamma_0 D(\Gamma_0 \xi, \Gamma_0 \xi) = \Gamma_0 D(\Gamma_0 x, \Gamma_0 \xi) = (0)$ it follows that $ID(J_1, J_1) \subseteq I$.

By Proposition 2.1 we have $I = (0)$, that is,

$$\{a, b\} = D(a)b - aD(b)$$

for arbitrary elements $a, b \in \Gamma_0$.

Now suppose that the restriction $\{,\}|_{\Gamma_0}$ is of vector type. If $\Gamma = \Gamma_0$ then speciality of $J(\Gamma, \{,\})$ follows from the results of [M], [S1].

Let $\Gamma = \Gamma_0 + \Gamma_0 \xi$, $\{\Gamma_0, \xi\} = (0)$, $\{\xi, \xi\} = -1$. Denote $D = R(x)^2$. In this case the superalgebra $J(\Gamma, \{,\})$ is embeddable into the Cheng-Kac superalgebra $CK(\Gamma_0, D)$ (see below), which is special by Proposition 3.1 (also below). Proposition is proved.

3 Cheng-Kac Superalgebras

Recall the definition of Cheng-Kac Jordan superalgebras (see [MZ]). Let Z be an associative commutative algebra with a derivation $D : Z \to Z$.

Let $A = Z + \sum_{i=1}^{3} Zw_i$, $M = xZ + \sum_{i=1}^{3} x_i Z$ be two free Z-modules of rank 4. The multiplication on A is Z-linear, $w_i w_j = 0$ for $i \neq j$, $w_1^2 = w_2^2 = 1$, $w_3^2 = -1$.

Denote $x_{i \times i} = 0$, $x_{1 \times 2} = -x_{2 \times 1} = x_3$, $x_{1 \times 3} = -x_{3 \times 1} = x_2$, $-x_{2 \times 3} = x_{3 \times 2} = x_1$.

The bimodule structure $A \times M \to M$ is defined via

	g	$w_j g$
xf	$x(fg)$	$x_j(fg^d)$
$x_i f$	$x_i(fg)$	$x_{i \times j}(fg)$

The bracket on M is defined via

	xg	x_jg
xf	$f^dg - fg^d$	$-w_j(fg)$
x_if	$w_i(fg)$	0

In [MZ] it was proved that the superalgebra $CK(Z,D) = A + M$ is Jordan.

If $\Gamma = \Gamma_0 + \Gamma_0\xi$, $\{\Gamma_0, \xi\} = (0)$, $\{\xi, \xi\} = -1$, $D(a) = \{a, 1\}$, then the Kantor double superalgebra $J = J(\Gamma, \{,\})$ is generated by Γ_0 and the elements $w_1 = \xi x$, x with $w_1^2 = 1$. Clearly, J is embeddable into $CK(\Gamma_0, D)$.

Proposition 3.1 *A Cheng-Kac Jordan superalgebra $CK(Z,D)$ is special.*

Proof: Let W denote the algebra of differential operators on Z, that is, the algebra of linear transformations on Z generated by all right multiplications $R(a)$, $a \in Z$, and by the derivation D. Consider the algebra $R = M_4(W)$ of 4×4 matrices over W with the $\mathbb{Z}/2\mathbb{Z}$-gradation

$$R_0 = \begin{pmatrix} W & 0 & W & 0 \\ 0 & W & 0 & W \\ W & 0 & W & 0 \\ 0 & W & 0 & W \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0 & W & 0 & W \\ W & 0 & W & 0 \\ 0 & W & 0 & W \\ W & 0 & W & 0 \end{pmatrix}.$$

Our embedding of the superalgebra $CK(Z,D)$ will extend the King-McCrimmon embedding of the Kantor double of the vector type bracket $\{a,b\} = D(a)b - aD(b)$ into $M_2(W)$ (see [KM]).

For an element $a \in Z$ let

$$\varphi(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad \varphi(w_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\varphi(w_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \varphi(w_3) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\varphi(x) = \begin{pmatrix} 0 & 0 & 0 & 2D \\ 0 & 0 & -1 & 0 \\ 0 & -2D & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \varphi(x_1) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(x_2) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \varphi(x_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A straightforward computation shows that φ extends to a homomorphism. Proposition is proved.

Corollary 3.1 *The Cheng-Kac Lie superalgebra (see [CK]) embedds into a superalgebra of* 8×8 *matrices over the Weyl algebra* $< t^{-1}, t, \frac{d}{dt} >$.

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