LIE SUPERALGEBRAS GRADED BY THE ROOT SYSTEM B(m,n)

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Abstract. We determine the Lie superalgebras that are graded by the root system $\mathrm{B}(m,n)$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1,2n)$.

§1. Introduction

The notion of a Lie algebra graded by a finite root system has proved to be a very useful one for studying many important classes of Lie algebras such as the affine and toroidal Lie algebras and various generalizations of them: for example, the extended affine Lie algebras of [AABGP] or the intersection matrix Lie algebras of [Sl], which arise in the study of singularities. Any finite-dimensional simple Lie algebra of characteristic zero containing an ad-nilpotent element (or equivalently a copy of $\mathfrak{sl}_2$) is a Lie algebra graded by a finite root system (see [S]). Thus, this concept encompasses a diverse array of Lie algebras under one unifying theme - they all exhibit a grading by a finite (possibly nonreduced) root system.

Our focus here and in [BE2] is on Lie superalgebras graded by the root systems of the finite-dimensional basic classical simple Lie superalgebras $\mathrm{A}(m,n)$, $\mathrm{B}(m,n)$, $\mathrm{C}(n)$, $\mathrm{D}(m,n)$, $\mathrm{D}(2,1;\alpha), (\alpha \neq 0,-1)$, $\mathrm{G}(3)$, and $\mathrm{F}(4)$ (a standard reference for results on simple superalgebras is Kac's ground-breaking paper [K1]). In this work, we investigate Lie superalgebras graded by the root system $\mathrm{B}(m,n)$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1,2n)$. These superalgebras differ from the others by their complexity and most closely resemble the Lie algebras graded by the nonreduced root systems BC_r. The others have some similarities with the Lie algebras graded by the reduced root systems, but there are many striking differences as seen in [BE2].

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Our main results are

- (i) the determination of the $\mathrm{B}(m,n)$-graded Lie superalgebras for $m \geq 1$ in Theorem 4.9 and a realization of them by a Tits construction in Theorem 4.21;
- (ii) a classification of the $\mathrm{B}(0,n)$-graded Lie superalgebras for $n \geq 2$ (Theorem 5.21);
- (iii) a description of the $\mathrm{B}(0,1)$-graded Lie superalgebras (Theorem 6.20);
- (iv) a realization of all $\mathrm{B}(m,n)$-graded Lie superalgebras as unitary Lie superalgebras of hermitian forms, except for the $\mathrm{B}(0,1)$-graded Lie superalgebras whose coordinate superalgebra is not associative (Section 7).

When $m = 0$ exceptional behavior occurs, and when $m = 0$, $n = 1$ additional degeneracies appear, which render these cases both more interesting and more challenging, and which require them to be treated separately.

To set the stage for our investigations, we begin with a brief review of the Lie algebra case. Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic zero with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$. Such a Lie algebra $\mathfrak{g}$ is the $\mathbb{F}$-analogue of a finite-dimensional complex simple Lie algebra. Berman and Moody [BM] initiated the study of Lie algebras graded by the root system Δ and following them we say,

Definition 1.1. *A Lie algebra L over $\mathbb{F}$ is* **graded by the (reduced) root system Δ** *or is* **Δ-graded** *if*

(ΔG1) *L contains as a subalgebra a finite-dimensional split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;*

(ΔG2) *$L = \bigoplus_{\mu\in\Delta\cup\{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h,x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$ for $\mu \in \Delta \cup \{0\}$; and*

(ΔG3) *$L_0 = \sum_{\mu\in\Delta} [L_\mu, L_{-\mu}]$.*

In [ABG2] this concept was expanded to include the nonreduced root systems BC_r. Let $\mathfrak{g}$ be a split "simple"[3] Lie algebra whose root system relative to a split Cartan subalgebra $\mathfrak{h}$ is of type B_r, C_r, or D_r for some $r \geq 1$. Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta_X} \mathfrak{g}_\mu$ where $X = \mathrm{B}, \mathrm{C}$, or D, and

$$\begin{aligned}
\Delta_B &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\epsilon_i \mid i = 1, \dots, r\} \\
\Delta_C &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm 2\epsilon_i \mid i = 1, \dots, r\} \qquad (1.2)\\
\Delta_D &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\}.
\end{aligned}$$

The set

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\epsilon_i,\ \pm 2\epsilon_i \mid i = 1, \dots, r\}, \qquad (1.3)$$

which is contained in dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, is a root system of type BC_r in the sense of Bourbaki [B, Chapitre VI].

[3]In all cases but two, $\mathfrak{g}$ is simple. When $\mathfrak{g}$ is of type $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$ then $\mathfrak{g} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and when $\mathfrak{g}$ is of type D_1, then $\mathfrak{g} = \mathfrak{h}$, which is 1-dimensional.

Definition 1.4. *A Lie algebra L over a field $\mathbb{F}$ of characteristic zero is* **graded by the root system** BC_r *or is* BC_r**-graded** *if*

(i) *L contains as a subalgebra a finite-dimensional split "simple" Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta_X} \mathfrak{g}_\mu$ whose root system relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Δ_X (as in (1.2)) for $X = \mathrm{B}, \mathrm{C},$ or D;*

(ii) *($\Delta G2$) and ($\Delta G3$) of Definition 1.1 hold for L relative to the root system Δ of type BC_r in (1.3).*

In both the reduced and nonreduced cases, the subalgebra $\mathfrak{g}$ is called **grading subalgebra** of L.

When viewed as a module under the adjoint action of $\mathfrak{g}$, a Lie algebra L graded by a finite reduced root system is a sum of finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weights are the highest long root, highest short root, or 0. Therefore L decomposes into (possibly infinitely many) copies of $\mathfrak{g}$, copies of W, and one-dimensional trivial $\mathfrak{g}$-modules, where W is the irreducible $\mathfrak{g}$-module whose highest weight is the highest short root. When $\Delta = \mathrm{A}_r$, D_r, E_6, E_7, or E_8, then $W = 0$, and when $\Delta = \mathrm{B}_r$, C_r, F_4, or G_2, then $\dim W = 2r+1,\ r(2r-1)-1,\ 26,$ or 7, respectively. By collecting isomorphic summands, we may assume that there are $\mathbb{F}$-vector spaces A, B, D so that

$$L = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D, \tag{1.5}$$

where D is the sum of all the trivial $\mathfrak{g}$-modules.

A BC_r-graded Lie algebra L with grading subalgebra $\mathfrak{g}$ has a similar decomposition into a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. There is one possible isotypic component corresponding to each root length and one corresponding to 0 (the sum of the trivial $\mathfrak{g}$-modules). Thus, if $r \geq 2$, there are up to four isotypic components (compared to two or three when Δ is reduced). There is one exception - when $\mathfrak{g}$ has type D_2 there are five possible isotypic components. The larger number of components in the BC_r-cases presents many complications not seen in the reduced cases. When $r \geq 3$, the components can be parametrized by subspaces A, B, C, and D, so that the decomposition is given by

$$L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D. \tag{1.6}$$

Here V is an n-dimensional vector space over a field $\mathbb{F}$ of characteristic zero, and $(\,|\,)$ is a nondegenerate bilinear form on V which is symmetric of maximal Witt index or is skew-symmetric. Set $\rho = 1$ if the form is symmetric, and $\rho = -1$ if it is skew-symmetric, so that

$$(v|u) = \rho(u|v) \qquad \text{for all } u, v \in V.$$

Then

$$\begin{aligned}
\mathfrak{g} &= \{x \in \mathrm{End}_{\mathbb{F}}(V) \mid (x.u|v) = -(u|x.v) \text{ for all } u, v \in V\},\\
\mathfrak{s} &= \{s \in \mathrm{End}_{\mathbb{F}}(V) \mid (s.u|v) = (u|s.v) \text{ for all } u, v \in V \text{ and } \mathfrak{tr}(s) = 0\},
\end{aligned}$$

where $\mathfrak{tr}$ denotes the trace, and $\mathfrak{g}$ is a split simple Lie algebra. When

(i) $n = 2r+1$ and $\rho = 1$, then $\mathfrak{g}$ has type B_r;
(ii) $n = 2r$ and $\rho = -1$, then $\mathfrak{g}$ has type C_r; and
(iii) $n = 2r$ and $\rho = 1$, then $\mathfrak{g}$ has type D_r.

The classification of the Lie algebras L graded by finite root systems amounts to describing the coordinate algebra $\mathfrak{a} = A \oplus B$, the coordinate subspace C (in type BC_r), and the Lie subalgebra D of L, which acts as derivations on $\mathfrak{a}$ (on $\mathfrak{b} = A \oplus B \oplus C$ in the BC_r-case), and to determining the multiplication in L. This has been accomplished in the papers [BM], [BZ], [N], [ABG1], [ABG2], [BS].

§2. Lie Superalgebras Graded by Finite Root Systems

Let $\mathfrak{g}$ be a finite-dimensional split simple basic classical Lie superalgebra over a field $\mathbb{F}$ of characteristic zero with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$. Thus, $\mathfrak{g}$ is an analogue over $\mathbb{F}$ of a complex simple Lie superalgebra whose root system Δ is a root system of type $\mathrm{A}(m,n)$, $\mathrm{B}(m,n)$, $\mathrm{C}(n)$, $\mathrm{D}(m,n)$, $\mathrm{D}(2,1;\alpha), (\alpha \neq 0,-1)$, $\mathrm{G}(3)$, and $\mathrm{F}(4)$. These Lie superalgebras can be characterized by the properties of being simple, having reductive even part, and having a nondegenerate even supersymmetric bilinear form. Mimicking the definitions of the previous section, we say

Definition 2.1. *(Compare [GN, Sec. 4.7].) A Lie superalgebra L over $\mathbb{F}$ is* **graded by the root system Δ** *or is* **Δ-graded** *if*

(i) *L contains as a subsuperalgebra a finite-dimensional split simple basic classical Lie superalgebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;*
(ii) *($\Delta G2$) and ($\Delta G3$) of Definition 1.1 hold for L relative to the root system Δ.*

We would like to view L as a $\mathfrak{g}$-module in order to determine the structure of L. However, a major obstacle encountered in the superalgebra case is that $\mathfrak{g}$-modules need not be completely reducible. We circumvent this roadblock below (and in [BE2]) by showing that a Δ-graded Lie superalgebra L must be completely reducible as a module for its grading subalgebra $\mathfrak{g}$ in all cases except when Δ is of type $\mathrm{A}(n,n)$. To prove this, the following result is instrumental.

Lemma 2.2. *Let L be a Δ-graded Lie superalgebra, and let $\mathfrak{g}$ be its grading subsuperalgebra. Then L is locally finite as a module for $\mathfrak{g}$.*

Proof. By (ΔG2) it is enough to check that the module generated by any $x_\mu \in L_\mu$ is finite-dimensional. Once we fix an ordering of the roots of $\mathfrak{g}$, there is a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$, and the module generated by x_μ is $U(\mathfrak{g})x_\mu = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)x_\mu$. But the dimension of $U(\mathfrak{n}^+)x_\mu$ is finite, since $\dim U(\mathfrak{n}^+)_\nu$ is finite for any $\nu \in \mathbb{Z}\Delta$ and L has only finitely many $\mathfrak{h}$-weight spaces. Also, $U(\mathfrak{h})U(\mathfrak{n}^+)x_\mu = U(\mathfrak{n}^+)x_\mu$ because the action of $U(\mathfrak{h})$ is diagonalizable, and again $\dim U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)x_\mu$ is finite by the same weight argument as above. □

§3. B(m,n)-graded Lie superalgebras

Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a $\mathbb{Z}_2$-graded vector space over a field $\mathbb{F}$ of characteristic zero, with $\dim V_{\bar 0} = 2m+1$ and $\dim V_{\bar 1} = 2n$, with $m \geq 0$ and $n \geq 1$. We assume $(\,|\,)$ is a nondegenerate supersymmetric bilinear form of maximal Witt index on V. Thus, we may suppose there is a basis $\{u_0, u_1, \dots, u_{2m}\}$ of $V_{\bar 0}$ and a basis $\{v_1, \dots, v_{2n}\}$ of $V_{\bar 1}$ such that

$$\begin{aligned} (u_0 \mid u_0) = 1, \qquad (u_i \mid u_{i+m}) &= 1 = (u_{i+m} \mid u_i) \qquad (i = 1, \dots, m) \\ (v_j \mid v_{j+n}) &= 1 = -(v_{j+n} \mid v_j) \qquad (j = 1, \dots, n), \end{aligned} \tag{3.1}$$

and all other products are 0.

The space $\mathrm{End}_{\mathbb{F}}(V)$ of transformations on V inherits a $\mathbb{Z}_2$-grading: $\mathrm{End}_{\mathbb{F}}(V) = \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 0} \oplus \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 1}$ where $x.u \in V_{a+b}$ (subscripts read mod 2) whenever $x \in \big(\mathrm{End}_{\mathbb{F}}(V)\big)_a$ and $u \in V_b$. Setting

(3.2)
$$\begin{aligned} \mathfrak{g} &= \{x \in \mathrm{End}_{\mathbb{F}}(V) \mid (x.u \mid v) = -(-1)^{\bar x \bar u}(u \mid x.v) \text{ for all } u, v \in V\}, \\ \mathfrak{s} &= \{s \in \mathrm{End}_{\mathbb{F}}(V) \mid (s.u \mid v) = (-1)^{\bar s \bar u}(u \mid s.v) \text{ for all } u, v \in V \text{ and } \mathfrak{str}(s) = 0\}, \end{aligned}$$

we have that $\mathfrak{g}$ is the orthosymplectic split simple Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$. (In displays such as (3.2), we assume all elements shown are homogeneous, and our convention is that $\bar u = b$ (viewed as an element of $\mathbb{Z}_2$) whenever $u \in V_b$.) The transformations $s \in \mathfrak{s}$ are supersymmetric relative to the form on V and have supertrace 0. Thus, $\mathfrak{str}(s) = \mathfrak{tr}_{V_{\bar 0}}(s) - \mathfrak{tr}_{V_{\bar 1}}(s) = 0$ whenever $s \in \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 0}$, and the supertrace is automatically 0 for all transformations in $\big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 1}$. The spaces $\mathfrak{g}, \mathfrak{s}, V, \mathbb{F}$ are all $\mathfrak{g}$-modules; they are irreducible and nonisomorphic (unless $m = 0$ and $n = 1$, where $\mathfrak{s} \cong V$); and according to the next result, each B(m,n)-graded Lie superalgebra is a sum of copies of them.

Theorem 3.3. *A Lie superalgebra L graded by the root system* B(m,n) *has a decomposition as a module for $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$ as follows:*

$$L \cong \begin{cases} (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D & \text{when } m \geq 1, \\ (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D & \text{when } m = 0 \text{ and } n \geq 2, \\ (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus D & \text{when } m = 0 \text{ and } n = 1. \end{cases} \tag{3.4}$$

where D is the sum of the trivial $\mathfrak{g}$-submodules of L, V is as above the natural $(2m+1+2n)$-dimensional defining representation of $\mathfrak{g}$, and $\mathfrak{s}$ is as in (3.2).

Proof. Using the basis in (3.1), we may identify linear transformations with their matrices. The diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$. The corresponding even and odd roots are

$$\begin{aligned} \Delta_{\bar 0} &= \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm\delta_r \pm \delta_s, \pm 2\delta_r \mid 1 \leq i < j \leq m,\ 1 \leq r < s \leq n\}, \\ \Delta_{\bar 1} &= \{\pm\delta_r, \pm\epsilon_i \pm \delta_r \mid 1 \leq i \leq m,\ 1 \leq r \leq n\}, \end{aligned}$$

where for $h = \mathrm{diag}(0, b_1, \dots, b_m, -b_1, \dots, -b_m, c_1, \dots, c_n, -c_1, \dots, -c_n) \in \mathfrak{h}$, $\epsilon_i(h) = b_i$ and $\delta_r(h) = c_r$ for any i, r. When $m \geq 1$, a system of simple roots is

$$\Pi = \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m\},$$

and the corresponding Cartan matrix is

$$\begin{pmatrix} & & & & 0 & & & & \\ & A_{n-1} & & & \vdots & & 0 & & \\ & & & & 0 & & & & \\ & & & & -1 & & & & \\ 0 & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ & & & & -1 & & & & \\ & & & & 0 & & & & \\ & 0 & & & \vdots & & B_m & & \\ & & & & 0 & & & & \end{pmatrix};$$

while for $m = 0$,

$$\Pi = \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n\}$$

and the Cartan matrix is

$$\begin{pmatrix} & & & & 0 \\ & A_{n-1} & & & \vdots \\ & & & & 0 \\ & & & & -1 \\ 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

Let $t_1, \dots, t_{n+m} \in \mathfrak{h}$ be the dual basis to $\delta_1, \dots, \delta_n, \epsilon_1, \dots, \epsilon_m$. Then relative to this basis of $\mathfrak{h}$, the coroots $h_1, \dots, h_{n+m}$ ($\alpha_i(h_j)$ is the (j, i)-entry of the Cartan matrix) have the following expressions:

$$\begin{aligned} h_i &= t_i - t_{i+1} && (1 \leq i \leq n-1) \\ h_n &= t_n + t_{n+1} && (\text{if } \ m \geq 1) \\ h_{n+j} &= t_{n+j} - t_{n+j+1} && (1 \leq j \leq m-1) \\ h_{n+m} &= 2t_{n+m}. \end{aligned}$$

Now, suppose

$$\lambda = \sum_{i=1}^{n} \pi_i \delta_i + \sum_{j=1}^{m} \mu_j \epsilon_j,$$

and $\lambda(h_i) = a_i$ in Kac's notation. The conditions for λ to be the highest weight of a finite-dimensional irreducible module are given in [K1, Thm. 8]. The first condition is that $a_i \in \mathbb{Z}_{\geq 0}$ for $i \neq n$. Thus, putting in the above expressions for h_i and using the duality, we have

$$\begin{aligned} \pi_i - \pi_{i+1} &= a_i \in \mathbb{Z}_{\geq 0} && i = 1, \dots, n-1 \\ \mu_j - \mu_{j+1} &= a_{n+j} \in \mathbb{Z}_{\geq 0} && j = 1, \dots, m-1 \\ 2\mu_m &= a_{n+m} \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

The second requirement is $a_n - (a_{n+1} + \dots a_{n+m-1} + \frac{1}{2}a_{n+m}) = k \in \mathbb{Z}_{\geq 0}$ if $m \geq 1$, and $\frac{1}{2}a_n = k \in \mathbb{Z}_{\geq 0}$ if $m = 0$. Thus, $a_n = k + \mu_1$ if $m \geq 1$ and $a_n = 2k$ if $m = 0$, which implies $\pi_n = k \in \mathbb{Z}_{\geq 0}$ for all m, and hence that $\pi_1 \geq \dots \geq \pi_n \geq 0$ is a partition. The final condition is that when $k < m$, then $a_{n+k+1} = \cdots = a_{n+m} = 0$, which says $\mu_{k+1} = \cdots = \mu_m = 0$. This happens only if $\mu_m \in \mathbb{Z}_{\geq 0}$. Thus the only roots that give finite-dimensional irreducible modules when they are the highest weight are

$$\begin{aligned}
&2\delta_1, \quad \delta_1 + \delta_2, \quad \delta_1 &&\text{if } n \geq 2,\\
&2\delta_1, \quad \delta_1 + \epsilon_1, \quad \delta_1 &&\text{if } n = 1 \text{ and } m > 0,\\
&2\delta_1, \quad \delta_1 &&\text{if } n = 1 \text{ and } m = 0.
\end{aligned}$$

Now $2\delta_1$ is the highest root, so the corresponding irreducible module is the adjoint module $\mathfrak{g}$; $\delta_1 + \delta_2$ (or $\delta_1 + \epsilon_1$ if $n = 1$) is the highest weight of $\mathfrak{s}$; and δ_1 is the highest weight of the natural module V. However, when $m \geq 1$, $\pm 2\epsilon_i$ is a weight of $\mathfrak{s}$, which is not a root of $\mathfrak{g}$. On the other hand, the nonzero weights of V are roots, and they are the nonzero weights of $\mathfrak{s}$ when $m = 0$. Finally, if $m = 0$ and $n = 1$, then $\mathfrak{s}$ and V are isomorphic.

It follows from Lemma 2.2 that any Lie superalgebra L graded by the root system B(m,n) is, when viewed as a $\mathfrak{g}$-module, a sum of finite-dimensional modules such that, in any composition series of these modules, only copies of $\mathfrak{g}$, $\mathfrak{s}$ (for $m = 0$ and $n \geq 2$), V and the trivial module appear, possibly with the parity changed. The final step in the proof is to argue that L is a completely reducible $\mathfrak{g}$-module, and to check this, it is enough to prove the following:

"Let X be a $\mathfrak{g}$-module with a diagonalizable action of $\mathfrak{h}$ and with a submodule Y such that both X/Y and Y are one of the modules listed above (possibly with the parity changed). Then there is a submodule Z such that $X = Y \oplus Z$."

Since any finite-dimensional B$(0,n)$-module is typical and hence any finite dimensional B$(0,n)$-module is completely reducible (see [K2]), it will be assumed that $m \geq 1$ and that the modules involved then are only $\mathfrak{g}$, V, and the trivial module. By diagonalizability of the action of $\mathfrak{h}$ on X, if X/Y and Y are isomorphic (possibly with the parity changed) with highest weight μ, take linearly independent elements $x_\mu, y_\mu \in X_\mu$ so that $X = U(\mathfrak{g})x_\mu + U(\mathfrak{g})y_\mu$. But $U(\mathfrak{g})x_\mu$ and $U(\mathfrak{g})y_\mu$ are strictly contained in X (the dimension of their highest weight spaces is 1), and both X/Y and Y are irreducible. The only possibility is that both submodules are irreducible and that $X = U(\mathfrak{g})x_\mu \oplus U(\mathfrak{g})y_\mu$, so that X is completely reducible.

Up to a nonzero factor, the Casimir operator C of $\mathfrak{g}$ (see [K1, Sec. 5.2]) acts as $1 + 2(n-m)$ times the identity on $\mathfrak{g}$, and as $n - m$ times the identity on V. Hence if X/Y is trivial and Y is adjoint, or if X/Y adjoint and Y trivial, it follows that $X = \ker C \oplus \mathrm{im} C$ is completely reducible. Also, if $n \geq m$, then $1+2(n-m) \neq n-m$, so if X/Y is adjoint and $Y \cong V$ or conversely, then $X = Y \oplus Z$, where Y and Z are the two different eigenspaces of the action of C.

If X/Y is trivial and Y is the natural module, then as modules for the semisimple Lie algebra $\mathfrak{g}_{\bar{0}} = \mathfrak{o}_{2m+1} \oplus \mathfrak{sp}_{2n}$, $X \cong \mathbb{F}v \oplus V_{\bar{0}} \oplus V_{\bar{1}}$ with $\mathfrak{g}_{\bar{0}}v = 0$. Then $\mathfrak{g}_{\bar{1}}v = 0$, since $\mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{g}_{\bar{1}}, V_{\bar{0}}) = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{g}_{\bar{1}}, V_{\bar{1}}) = 0$, because $\mathfrak{g}_{\bar{1}}, V_{\bar{0}}, V_{\bar{1}}$ are nonisomorphic irreducible $\mathfrak{g}_{\bar{0}}$-modules. Thus $\mathfrak{g}v = 0$ and $X \cong \mathbb{F}v \oplus V$ as $\mathfrak{g}$-modules, so X is completely reducible. In case Y is trivial and X/Y is natural, again $X \cong \mathbb{F}v \oplus V_{\bar{0}} \oplus V_{\bar{1}}$

as $\mathfrak{g}_{\bar 0}$-modules, with $\mathfrak{g}v=0$. Now, $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 0},\mathbb{F})=\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 1},\mathbb{F})=0$, so $\mathfrak{g}_{\bar 1}V_i\subseteq V_{i+1}$ (indices modulo 2) and again X is completely reducible.

If $Y\cong V$ and $X/Y\cong\mathfrak{g}$, a similar argument can be used because

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 1},V_{\bar 0})&=\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(V_{\bar 0}\otimes V_{\bar 1}\otimes V_{\bar 0}\otimes V_{\bar 1},V_{\bar 0})\\&\cong\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(V_{\bar 0}\otimes V_{\bar 0},V_{\bar 0})\otimes\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(V_{\bar 1}\otimes V_{\bar 1},V_{\bar 0})\\&=0,\end{aligned}$$

because $\mathrm{Hom}_{\mathfrak{o}_{2m+1}}(V_{\bar 0}\otimes V_{\bar 0},V_{\bar 0})=0$ unless $m=1$ (see, for instance [BZ, Appendix]), but then $n\geq m$. Similarly, $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 1},V_{\bar 1})=0$ and $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 0},V_{\bar 0})=0=\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 0},V_{\bar 1})$. Therefore, as $\mathfrak{g}_{\bar 0}$-modules, $X\cong\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}\oplus V_{\bar 0}\oplus V_{\bar 1}$ with $Y\cong V_{\bar 0}\oplus V_{\bar 1}$, and by the above, $Z=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ is a submodule with $X\cong Y\oplus Z$.

Finally, if Y is adjoint and X/Y is natural, again as $\mathfrak{g}_{\bar 0}$-modules $X\cong\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}\oplus V_{\bar 0}\oplus V_{\bar 1}$ with $Y\cong\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$. Now, $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 0},\mathfrak{g}_{\bar 0})\cong\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 0}\otimes V_{\bar 0},\mathbb{F})\cong\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes\mathfrak{g}_{\bar 0},V_{\bar 0})=0$ and, in the same vein, $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 0},\mathfrak{g}_{\bar 1})=0$ (unless $m=1$, but then $n\geq m$) and $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 1},\mathfrak{g}_{\bar 0})=0=\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1}\otimes V_{\bar 1},\mathfrak{g}_{\bar 1})$. Thus, any $\mathrm{B}(m,n)$-graded Lie superalgebra is a completely reducible $\mathfrak{g}$-module.

By collecting irreducible summands of $\mathfrak{g}$ which are isomorphic up to a change of parity, we may suppose that there are superspaces A, B, C, and D over $\mathbb{F}$ such that L has a decomposition as in the statement of the theorem. □

In the decomposition (3.4), we identify the grading subalgebra $\mathfrak{g}$ with $\mathfrak{g}\otimes 1$ where $1\in A_{\bar 0}$.

The $\mathfrak{g}$-module homomorphisms among the modules $\mathfrak{g},\mathfrak{s},V,\mathbb{F}$ (as listed in Table 3.5 below) play a key role in our investigations.

Table 3.5

the homomorphism	forms a basis for
$x\otimes y\mapsto[x,y]$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})$
$x\otimes y\mapsto x\circ y$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{s})$
$x\otimes s\mapsto x\circ s$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s},\mathfrak{g})$
$x\otimes s\mapsto[x,s]$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s},\mathfrak{s})$
$s\otimes t\mapsto[s,t]$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s},\mathfrak{g})$
$s\otimes t\mapsto s\circ t$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s},\mathfrak{s})\quad(m=0)$
$x\otimes u\mapsto x.u$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V,V)$
$s\otimes u\mapsto s.u$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes V,V)$
$u\otimes v\mapsto\gamma_{u,v}$	$\mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathfrak{g})$
$u\otimes v\mapsto\sigma_{u,v}$	$\mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathfrak{s})$
$x\otimes y\mapsto\mathfrak{str}(xy)$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathbb{F})$
$s\otimes t\mapsto\mathfrak{str}(st)$	$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s},\mathbb{F})$
$u\otimes v\mapsto(u\mid v)$	$\mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathbb{F})$

Besides the homomorphisms in this table or those obtained by symmetry (for instance, $s\otimes x \mapsto s\circ x$ generates $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{g},\mathfrak{g})$), there are no other homomorphisms among these modules. Thus, for example $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V,\mathfrak{g}) = 0$ unless $(m,n) = (0,1)$, where $V \cong \mathfrak{s}$.

Appearing in this table are the supercommutator product $[\,,]$ and the circle product $\circ$ on $\mathrm{End}_{\mathbb{F}}(V)$, and two special transformations $\gamma_{u,v}$ and $\sigma_{u,v}$ on V, which are associated to any two homogeneous elements $u, v \in V$. Their definitions are given by

$$
\begin{aligned}
&[x,y] = xy - (-1)^{\bar{x}\bar{y}}yx\\
&x\circ y = xy + (-1)^{\bar{x}\bar{y}}yx - \frac{2}{2m+1-2n}\mathfrak{str}\big(xy\big)\\
&\gamma_{u,v}(w) = u(v\mid w) - (-1)^{\bar{u}\bar{v}}v(u\mid w)\\
&\widetilde{\sigma}_{u,v}(w) = u(v\mid w) + (-1)^{\bar{u}\bar{v}}v(u\mid w)\\
&\sigma_{u,v} = \widetilde{\sigma}_{u,v} - \frac{1}{2m+1-2n}\mathfrak{str}(\widetilde{\sigma}_{u,v})\,\mathrm{I}
\end{aligned}
\tag{3.6}
$$

for all homogeneous $x, y \in \mathrm{End}_{\mathbb{F}}(V)$, $u, v \in V$. Observe that

$$
\begin{aligned}
&[\gamma_{u,v},\gamma_{u',v'}] = \gamma_{\gamma_{u,v}(u'),v'} + (-1)^{(\bar{u}+\bar{v})\bar{u}'}\gamma_{u',\gamma_{u,v}(v')}\\
&[\gamma_{u,v},\sigma_{u',v'}] = \sigma_{\gamma_{u,v}(u'),v'} + (-1)^{(\bar{u}+\bar{v})\bar{u}'}\sigma_{u',\gamma_{u,v}(v')}.
\end{aligned}
\tag{3.7}
$$

Proof of the assertions in Table 3.5. It is easy to check that all the maps given in the table are indeed nonzero homomorphisms. Also the three last lines of the table are clear.

a) Notice first that $\mathfrak{g}$ has a $\mathbb{Z}$-grading as in [K1, 2.1.2], $\mathfrak{g} = \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2$, with $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{-2}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_1$. The spaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are contragredient irreducible $\mathfrak{g}_0$-modules and $\mathfrak{g}_2\otimes\mathfrak{g}_{-2}$ generates $\mathfrak{g}\otimes\mathfrak{g}$ as $\mathfrak{g}$-module. Moreover, $\mathfrak{g}_0 = \mathbb{F}c\oplus[\mathfrak{g}_0,\mathfrak{g}_0]$ where $[\mathfrak{g}_0,\mathfrak{g}_0] = \mathfrak{o}_{2m+1}\oplus\mathfrak{sl}_n$, $[c,x_i] = ix_i$ for any $x_i\in\mathfrak{g}_i$ and $i = -2,-1,0,1,2$, and $\mathfrak{o}_{2m+1}$ centralizes $\mathfrak{g}_2\oplus\mathfrak{g}_{-2}$.

Once a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$, hence of $\mathfrak{g}$, and a system of simple roots are fixed, we may take a highest weight vector $v\in\mathfrak{g}_2$ and a lowest weight vector w of $\mathfrak{g}_{-2}$ (as $\mathfrak{g}_0$-modules). Then $v\otimes w$ generates $\mathfrak{g}\otimes\mathfrak{g}$ as a $\mathfrak{g}$-module. Hence any $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})$ is determined by $\varphi(v\otimes w)$, which belongs to $\mathfrak{h}$ (the zero weight space). By its invariance under $\mathrm{ad}\,c$, φ respects the $\mathbb{Z}$-grading. Since $\mathfrak{o}_{2m+1}$ centralizes $\mathfrak{g}_2\otimes\mathfrak{g}_{-2}$, $\varphi(v\otimes w)$ is centralized by $\mathfrak{o}_{2m+1}$. Then φ restricts to $\tilde\varphi\in\mathrm{Hom}_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}},\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}} = \mathfrak{g}_{-2}\oplus(\mathbb{F}c\oplus\mathfrak{sl}_n)\oplus\mathfrak{g}_2 = \mathfrak{sp}_{2n}$. Since $\dim\mathrm{Hom}_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}) = 1$ (see, for instance, [BZ, Appendix]), it must be that $\dim\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}) = 1$ too, and the assertion of the first line of Table 3.5 follows.

b) Let $x_{2\delta_1}$ (resp. $x_{-2\delta_1}$) be a highest (resp. lowest) weight vector of $\mathfrak{g}$ (notation as in the Proof of Theorem 3.3). Then $x_{2\delta_1}\otimes x_{-2\delta_1}$ generates $\mathfrak{g}\otimes\mathfrak{g}$ as a $\mathfrak{g}$-module. Since any $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},V)$ takes $x_{2\delta_1}\otimes x_{-2\delta_1}$ to the zero weight space of V and $\varphi(x_{2\delta_1}\otimes x_{-2\delta_1})$ is annihilated by the ideal $\mathfrak{o}_{2m+1}$ of $\mathfrak{g}_{\bar 0}$, it follows that $\varphi = 0$

unless $m = 0$. If $m = 0$, then φ is even and $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g}, V) \le 1$. Hence for $n = 1$, $V \cong \mathfrak{s}$ and we obtain the second line of the table. However, for $n \ge 2$, if $\varphi \ne 0$, it can be assumed that $\varphi(x_{\bar 0}\otimes y_{\bar 0}) = \mathfrak{str}(x_{\bar 0}y_{\bar 0})u_0$ for any $x_{\bar 0}, y_{\bar 0} \in \mathfrak{g}_{\bar 0}$, because $\mathrm{Hom}_{\mathfrak{sp}(V_{\bar 1})}(\mathfrak{sp}(V_{\bar 1})\otimes\mathfrak{sp}(V_{\bar 1}), V_{\bar 1}) = 0$ and $\mathrm{Hom}_{\mathfrak{sp}(V_{\bar 1})}(\mathfrak{sp}(V_{\bar 1})\otimes\mathfrak{sp}(V_{\bar 1}), \mathbb{F})$ is spanned by the supertrace form ([BZ, Appendix]). Take $x_{\bar 0} = \gamma_{v_1,v_{n+1}}$ and $x_{\bar 1} = \gamma_{u_0,v_2}$. Then $[x_{\bar 1}, x_{\bar 0}] = 0$, so

$$\begin{aligned} 0 &= \varphi\big([x_{\bar 1}, x_{\bar 0}]\otimes x_{\bar 0}\big) + \varphi\big(x_{\bar 0}\otimes [x_{\bar 1}, x_{\bar 0}]\big)\\ &= x_{\bar 1}.\varphi(x_{\bar 0}\otimes x_{\bar 0})\\ &= \mathfrak{str}(x_{\bar 0}^2)x_{\bar 1}.u_0 \ne 0, \end{aligned}$$

a contradiction. Thus $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g}, V) = 0$ unless $(m, n) = (0, 1)$. Notice that, since all the modules involved are contragredient, $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g}, V) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g}\otimes V, \mathbb{F}) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V, \mathfrak{g})$.

c) $\mathfrak{g}\otimes\mathfrak{s}$ is generated as a $\mathfrak{g}$-module by $\mathfrak{g}_{2\delta_1}\otimes\mathfrak{s}_\mu$, where μ is the lowest weight of $\mathfrak{s}$. Thus, $\mu = -(\delta_1+\delta_2)$ if $n \ge 2$, $\mu = -(\delta_1+\epsilon_1)$ if $m \ge 1$ and $n = 1$, and $\mu = -\delta_1$ if $m = 0$, $n = 1$. Any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, \mathfrak{g})$ takes $\mathfrak{g}_{2\delta_1}\otimes\mathfrak{s}_\mu$ to the weight space $\mathfrak{g}_{2\delta_1+\mu}$, which is one-dimensional. This establishes the third line of the table, and since $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, \mathfrak{g}) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g}, \mathfrak{s})$ as in b), it also proves the validity of the second row.

d) The same argument as in c) proves that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, \mathfrak{s})$ and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V, V)$ are one-dimensional. Using $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, \mathfrak{s}) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s}, \mathfrak{g})$ and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V, V) \cong \mathrm{Hom}_{\mathfrak{g}}(V\otimes V, \mathfrak{g})$, we obtain lines 4, 5, 7 and 9 of the table. Also, this argument shows that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, V) = 0$ (since $2\delta_1 - (\delta_1+\delta_2)$ or $2\delta_1 - (\delta_1+\epsilon_1)$ for $m = 0$ is not a weight of V) unless $(m, n) = (0, 1)$. Hence, $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{s}, V) = 0 = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V, \mathfrak{s}) = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes V, \mathfrak{g})$ unless $(m, n) = (0, 1)$.

e) Since $V\otimes V \cong V^*\otimes V \cong \mathrm{End}_{\mathbb{F}}(V) = \mathfrak{g}\oplus\mathfrak{s}\oplus\mathbb{F}\,I$, it follows that $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V, \mathfrak{s})$ is one-dimensional, as is $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes V, V)$, proving lines 8 and 10 of the table. Moreover, $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V, V)$ is trivial unless $(m, n) = (0, 1)$.

f) Finally, to find the dimension of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s}, \mathfrak{s})$ for $m = 0$, we may assume $n \ge 2$ by e). Then any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s}, \mathfrak{s})$ is determined by the image of $\sigma_{v_1,v_2}\otimes\sigma_{v_{n+1},v_{n+2}} \in \mathfrak{g}_{2\delta_1}\otimes\mathfrak{g}_{-2\delta_1}$, which belongs to $\mathfrak{s}_0 = \mathrm{span}_{\mathbb{F}}\{\sigma_{v_j,v_{n+j}} \mid j = 1, \dots, n\}$. Since both σ_{v_1,v_2} and $\sigma_{v_{n+1},v_{n+2}}$ are annihilated by γ_{u_0,v_j}, $j = 3, \dots, n$, and by $\gamma_{v_1,v_{n+2}}$, it follows that $\varphi(\sigma_{v_1,v_2}\otimes\sigma_{v_{n+1},v_{n+2}}) \in \mathbb{F}(\sigma_{v_1,v_{n+1}} - \sigma_{v_2,v_{n+2}})$. Thus $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{s}\otimes\mathfrak{s}, \mathfrak{s}) \le 1$, as required. $\square$

§4. B(m,n)-graded Lie superalgebras with $m \ge 1$ and the Tits construction

To facilitate the investigation of B(m,n)-graded Lie superalgebras with $m \ge 1$, we use the following result which, in more generality, was first proved in the Lie algebra context in [BZ, Prop. 2.7]. With slight modifications to accommodate for the parity of terms and some changes in the notation to make the statements more compatible with the results of the next sections we have:

Lemma 4.1. *Let L be a Lie superalgebra over $\mathbb{F}$ with a perfect subsuperalgebra $\mathfrak{g}$. Under the adjoint action of $\mathfrak{g}$ on L, assume that L is a direct sum of*

(1) *copies of the adjoint module $\mathfrak{g}$,*
(2) *copies of some nontrivial module V,*
(3) *copies of the trivial module $\mathbb{F}$.*

Assume that

(1') $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}) = 1 = \dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V, V)$
(2') $\dim \mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathfrak{g}) = 1 = \dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathbb{F}) = \dim \mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathbb{F})$
(3') $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V, V) = 0 = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V,\mathfrak{g}) = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V,\mathbb{F})$.

Suppose that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes\mathfrak{g},\mathbb{F}) = \mathbb{F}\kappa$, $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathbb{F}) = \mathbb{F}\lambda$, $\mathrm{Hom}_{\mathfrak{g}}(V\otimes V,\mathfrak{g}) = \mathbb{F}\pi$ *(where κ,λ,π are supersymmetric or superskewsymmetric), and the following conditions hold:*

(i) *There exist $f,g\in\mathfrak{g}_{\bar 0}\cup\mathfrak{g}_{\bar 1}$ such that $[g,g]=0$, $\kappa(f,g)=0=\kappa(g,f)$ and $[[f,g],g]\neq 0$.*
(ii) *There exist $f,g\in\mathfrak{g}_{\bar 0}\cup\mathfrak{g}_{\bar 1}$, and $u\in V_{\bar 0}\cup V_{\bar 1}$ such that $f.(g.u)\neq 0$, $g.(f.u)=0$ and $0=\kappa(g,f)$.*
(iii) *There exists $f\in\mathfrak{g}_{\bar 0}\cup\mathfrak{g}_{\bar 1}$ and $u,v\in V_{\bar 0}\cup V_{\bar 1}$ such that $\pi(f.u,v)=0=\pi(u,f.v)$ and $\lambda(u,v)f\neq 0$*
(iv) *The mappings $\mathfrak{g}\otimes V\otimes V\to\mathfrak{g}$ given by $f\otimes u\otimes v\mapsto \pi(f.u,v)$ and $f\otimes u\otimes v\mapsto (-1)^{\bar u\bar f}\pi(u,f.v)$ are linearly independent;*
(v) *There exists $0\neq\vartheta\in\mathbb{F}$ such that $\vartheta\kappa(\pi(u,v),f) = (-1)^{(\bar u+\bar v)\bar f}\lambda(f.u,v)$ for all $u,v\in V$, and all $f\in\mathfrak{g}$.*

Then $L\cong(\mathfrak{g}\otimes A)\oplus(V\otimes C)\oplus D$ where

(a) *A is a unital (super)commutative associative $\mathbb{F}$-superalgebra;*
(b) *C is a unital associative A-module;*
(c) *D is a trivial $\mathfrak{g}$-module and a Lie superalgebra;*
(d) *Multiplication in L is given by*

$$
\begin{aligned}
&[f\otimes a, g\otimes a'] = (-1)^{\bar a\bar g}\Big([f,g]\otimes aa' + \kappa(f,g)\langle a,a'\rangle\Big)\\
&[f\otimes a, u\otimes c] = (-1)^{\bar a\bar u} f.u\otimes a.c\\
&[u\otimes c, v\otimes c'] = (-1)^{\bar c\bar v}\Big(\pi(u,v)\otimes\chi(c,c') + \lambda(u,v)\langle c,c'\rangle\Big)\\
&[d, f\otimes a] = (-1)^{\bar d\bar f} f\otimes da, \qquad [d, u\otimes c] = (-1)^{\bar d\bar u} u\otimes dc,\\
&[d,d'] \quad \textit{(is the product in } D)
\end{aligned}
\tag{4.2}
$$

for all $f,g\in\mathfrak{g}$, $a,a'\in A$, $u,v\in V$, $c,c'\in C$, $d,d'\in D$, where

- $\chi : C\otimes C\to A$ *is an A-bilinear form which is supersymmetric (resp. superskewsymmetric) if* π *is superskewsymmetric (resp. supersymmetric)*
- $\mathfrak{b} = A\oplus C$ *is an A-algebra under*
  $(a+c)(a'+c') = \big(aa' + \vartheta^{-1}\chi(c,c')\big) + a.c' + (-1)^{\bar{a}'\bar{c}}a'.c$
- D *is a subsuperalgebra of* L
- $\langle\, ,\,\rangle : \mathfrak{b}\times\mathfrak{b}\to D,\quad (a+c,a'+c')\mapsto\langle a,a'\rangle+\langle c,c'\rangle$ *is* $\mathbb{F}$*-bilinear, with its restriction to* $A\times A$ *being always superskewsymmetric and its restriction to* $C\times C$ *being supersymmetric (resp. superskewsymmetric) if* λ *is superskewsymmetric (resp. supersymmetric)*
- $[d,\langle\beta,\beta'\rangle] = \langle d\beta,\beta'\rangle + (-1)^{\bar{d}\bar{\beta}}\langle\beta,d\beta'\rangle$ *holds for* $d\in D,\ \beta,\beta'\in\mathfrak{b}$*.*
  In particular, $\langle\mathfrak{b},\mathfrak{b}\rangle = \langle A,A\rangle+\langle C,C\rangle$ *is an ideal of* D
- $\phi : D\to \mathrm{Der}_{\mathbb{F}}(\mathfrak{b}),\ d\mapsto\phi(d)$ *where* $\phi(d): a\to da,\ \phi(d): c\to dc,$ *is a representation with* $\langle A,A\rangle\subseteq\ker\phi$*, and such that* $\langle\mathfrak{b},\mathfrak{b}\rangle$ *acts trivially on* A
- $\kappa(\, ,\,)$ *is supersymmetric, and the relations*
  $(-1)^{\bar{a}\bar{a}''}\langle a\cdot a',a''\rangle + (-1)^{\bar{a}'\bar{a}}\langle a'\cdot a'',a\rangle + (-1)^{\bar{a}''\bar{a}'}\langle a''\cdot a,a'\rangle = 0$
  $\langle a,cc'\rangle = \langle a.c,c'\rangle - (-1)^{\bar{a}\bar{c}}\langle c,a.c'\rangle$ *(recall that* $cc' = \vartheta^{-1}\chi(c,c')$ *)*
  hold for all $a,a',a''\in A$ *and* $c,c'\in C$*. In particular, if* $\langle A,A\rangle = 0$*, then* $\langle a.c,c'\rangle = (-1)^{\bar{a}\bar{c}}\langle c,a.c'\rangle$*.*
- *Moreover, the following relation must hold for* $w_1,w_2,w_3\in V,\ c_1,c_2,c_3\in C$*:*

$$0 = \sum_{\circlearrowleft}(-1)^{\bar{w}_1\bar{w}_3+\bar{c}_1\bar{c}_3}\Big(\lambda(w_1,w_2)w_3\otimes\langle c_1,c_2\rangle c_3 + \pi(w_1,w_2)\cdot w_3\otimes\chi(c_1,c_2)\cdot c_3\Big) \tag{4.3}$$

Conversely, the above conditions are sufficient for $L = (\mathfrak{g}\otimes A)\oplus(V\otimes C)\oplus D$*, satisfying (a)–(d), to be a Lie superalgebra.*

The notation "$\sum_{\circlearrowleft}$" in (4.3) signifies summation over the cyclic permutation of the variables.

Lemma 4.4. *The hypotheses (1)-(3), (1')-(3'), and (i)-(v) of Lemma 4.1 are satisfied by every* $\mathrm{B}(m,n)$*-graded Lie superalgebra* $(m\geq 1)$ *with respect to* $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$*,* V *its natural* $(2m+1+2n)$*-dimensional module with a nondegenerate supersymmetric bilinear form* $(\,|\,)$ *as in (3.1),* $\vartheta = \frac{1}{2}$*, and the mappings*

$$\begin{aligned} \pi(u,v) &= \gamma_{u,v}\\ \lambda(u,v) &= (u\mid v)\\ \kappa(x,y) &= \mathfrak{str}(xy). \end{aligned} \tag{4.5}$$

Proof. By Theorem 3.3, we may assume that a Lie superalgebra L graded by B(m,n) for $m \geq 1$ has a decomposition of the form

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D,$$

where $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$, and V is its natural $(2m+1+2n)$-dimensional module. Thus, assumptions (1)-(3) hold. The results on the dimensions of the homomorphism spaces in (1)'-(3)' are shown in Table 3.5.

Observe that

$$\mathfrak{str}\big(u(v \mid \_)\big) = (-1)^{\bar{u}}(v \mid u) = (u \mid v) \qquad \text{for all} \quad u, v \in V,$$

so that

$$\sigma_{u,v}(w) = u(v \mid w) + (-1)^{\bar{u}\bar{v}} v(u \mid w) - \frac{2(u \mid v)}{2m+1-2n}\,\mathrm{I}\,. \tag{4.6}$$

Then for any $f \in \mathfrak{g}$,

$$\begin{aligned}
\mathfrak{str}(\gamma_{u,v} f) &= \mathfrak{str}\big(u(v \mid f_)\big) - (-1)^{\bar{u}\bar{v}}\mathfrak{str}\big(v(u \mid f_)\big) \\
&= -(-1)^{\bar{v}\bar{f}}\mathfrak{str}\big(u(f.v \mid _)\big) + (-1)^{\bar{u}\bar{v}+\bar{u}\bar{f}}\mathfrak{str}\big(v(f.u \mid _)\big) \\
&= -(-1)^{\bar{v}\bar{f}}(u \mid f.v) + (-1)^{\bar{u}(\bar{v}+\bar{f})}(v \mid f.u) \\
&= 2(-1)^{(\bar{u}+\bar{v})\bar{f}}(f.u \mid v) = -2(-1)^{\bar{v}\bar{f}}(u \mid f.v).
\end{aligned} \tag{4.7}$$

In particular,

(4.8)
$$\begin{aligned}
\mathfrak{str}(\gamma_{u,v}\gamma_{u',v'}) &= -2(-1)^{\bar{v}(\bar{u'}+\bar{v'})}(u \mid \gamma_{u',v'}.v) \\
&= -2(-1)^{\bar{v}(\bar{u'}+\bar{v'})}(u \mid u')(v' \mid v) \;+\; 2(-1)^{\bar{v}(\bar{u'}+\bar{v'})+\bar{u'}\bar{v'}}(u \mid v')(u' \mid v) \\
&= 2\Big((-1)^{\bar{v'}(\bar{v}+\bar{u'})}(u \mid v')(v \mid u') \;-\; (-1)^{\bar{v}\bar{u'}}(u \mid u')(v \mid v')\Big)
\end{aligned}$$

Conditions (i)-(v) of Lemma 4.1 assert the existence of elements of L which satisfy certain relations. In verifying that these conditions are met relative to the mappings in (4.5), we use the basis of (3.1) and equation (3.7).

(i) Take $f = \gamma_{u_{m+1},v_1} \in \mathfrak{g}_{\bar{1}}$ and $g = \gamma_{u_0,u_1} \in \mathfrak{g}_{\bar{0}}$. Then $[g,g] = 0$, $\kappa(f,g) = \mathfrak{str}(fg) = 0$. Also by (4.8), $\kappa(g,g) = \mathfrak{str}(\gamma^2_{u_0,u_1}) = 2\Big((u_0 \mid u_1)(u_1 \mid u_0) - (u_0 \mid u_0)(u_1 \mid u_1)\Big) = 0$. But

$$\begin{aligned}
&[f,g] = -[g,f] = -\gamma_{\gamma_{u_0,u_1}(u_{m+1}),v_1} = -\gamma_{u_0,v_1} \qquad \text{and} \\
&[[f,g],g] = [g,\gamma_{u_0,v_1}] = \gamma_{\gamma_{u_0,u_1}(u_0),v_1} = -\gamma_{u_1,v_1} \neq 0.
\end{aligned}$$

(ii) Set $f = \gamma_{u_0,v_1}$, $g = \gamma_{v_2,v_2}$, and $u = v_1$. Then $f.u = 0$, $g.u = -2v_2$, $f.(g.u) = -2f.v_2 = -2u_0 \neq 0$, while $\kappa(f,g) = 2\Big((u_0 \mid v_2)(v_1 \mid v_2) + (u_0 \mid v_2)(v_1 \mid v_2)\Big) = 0$.

(iii) For this one, let $f = \gamma_{v_1,v_{n+1}}$, and $u = v = u_0$. Then $\pi(f.u,v) = \gamma_{f.u,v} = 0 = \pi(u,f.v)$, and $\lambda(u,v)f = (u \mid v)f = f \neq 0$.

(iv) This requires checking that the maps $\mathfrak{g}\otimes V\otimes V \to \mathfrak{g}$ given by $f\otimes u\otimes v \mapsto \pi(f.u,v) = \gamma_{f.u,v}$ and $f\otimes u\otimes v \mapsto (-1)^{\bar u\bar f}\pi(u,f.v) = (-1)^{\bar u\bar f}\gamma_{u,f.v}$ are linearly independent. Letting $f = \gamma_{u_0,v_1}$, $u = u_0$, and $v = u_1$, we have

$$\gamma_{\gamma_{u_0,v_1}(u_0),u_1} = -\gamma_{v_1,u_1}, \qquad \gamma_{u_0,\gamma_{u_0,v_1}(u_1)} = 0,$$

and clearly $f\otimes u\otimes v \mapsto \gamma_{u,f.v}$ is not 0.

(v) We need to show there is some scalar $\vartheta \in \mathbb{F}$ such that $\vartheta\kappa(\pi(u,v),f) = \vartheta\mathfrak{str}(\gamma_{u,v}f) = (-1)^{(\bar u+\bar v)\bar f}(f.u \mid v) = (-1)^{(\bar u+\bar v)\bar f}\lambda(f.u,v)$ for all homogeneous $f\in\mathfrak{g}$, $u,v\in V$. But according to (4.7), this can be accomplished by taking $\vartheta = \frac{1}{2}$.

As a result, all the hypotheses of Lemma 4.1 are satisfied in any B(m,n)-graded Lie superalgebra (with $m \geq 1$). Therefore the conclusions given in the statements labelled by $\bullet$ in Lemma 4.1 must hold for these superalgebras. $\square$

Therefore, we have the following description of the B(m,n)-graded Lie superalgebras ($m \geq 1$):

Theorem 4.9. *Assume $L = (\mathfrak{g}\otimes A)\oplus(V\otimes C)\oplus D$ is a superalgebra over a field $\mathbb{F}$ of characteristic zero with $\mathfrak{g} = \mathfrak{osp}(2m+1,2n)$ for $m \geq 1$, with V as in Section 3, and with $\mathbb{F}$-superspaces A, C, D satisfying the following conditions:*

(a) *A is a unital (super)commutative $\mathbb{F}$-superalgebra;*
(b) *C is a left unital A-module;*
(c) *$\chi : C\times C \to A$ is an $\mathbb{F}$-bilinear supersymmetric form and hence, $\mathfrak{b} = A\oplus C$ is a unital $\mathbb{F}$-algebra with multiplication:*

$$(a+c)(a'+c') = \big(aa' + 2\chi(c,c')\big) + \big(a.c' + (-1)^{\bar{a'}\bar c}a'.c\big);$$

(d) *D is a trivial $\mathfrak{g}$-module and a Lie superalgebra, and there is a linear map $\phi : D \to \mathrm{End}_{\mathbb{F}}(\mathfrak{b})$ with $d\mapsto\phi_d$ such that if $d(\beta) := \phi_d(\beta)$ for all $\beta\in\mathfrak{b}$, then $d(A)\subseteq A$, $d(C)\subseteq C$ for all $d\in D$;*
(e) *there is a bilinear superskewsymmetric map $\langle\,|\,\rangle : \mathfrak{b}\times\mathfrak{b}\to D$ with $\langle A \mid C\rangle = 0$;*
(f) *the product in L is given by:*

$$\begin{aligned}
&[z\otimes a, z'\otimes a'] = (-1)^{\bar a\bar{z'}}\Big([z,z']\otimes aa' + \mathfrak{str}(zz')\langle a \mid a'\rangle\Big)\\
&[z\otimes a, u\otimes c] = (-1)^{\bar a\bar u}z.u\otimes a.c\\
&[u\otimes c, v\otimes c'] = (-1)^{\bar c\bar v}\Big(\gamma_{u,v}\otimes\chi(c,c') + (u \mid v)\langle c \mid c'\rangle\Big)\\
&[d, p\otimes\beta] = (-1)^{\bar d\bar p}p\otimes d\beta\\
&[d,d'] \qquad \textit{(is the product in } D).
\end{aligned} \tag{4.10}$$

Then L is a Lie superalgebra if and only if

- *A is an associative superalgebra;*
- *C is an associative module for A;*
- *D is a Lie subsuperalgebra of L and $\phi : D \to \mathrm{Der}_{\mathbb{F}}(\mathfrak{b})$ is a representation of D as superderivations on the algebra $\mathfrak{b}$;*
- $[d, \langle \beta \mid \beta' \rangle] = \langle d\beta \mid \beta' \rangle + (-1)^{\bar{d}\bar{\beta}} \langle \beta \mid d\beta' \rangle$ *for $d \in D$, $\beta, \beta' \in \mathfrak{b}$;*
- $\sum_{\circlearrowleft} (-1)^{\bar{\beta}_1 \bar{\beta}_3} \langle \beta_1 \mid \beta_2 \beta_3 \rangle = 0$ *for $\beta_1, \beta_2, \beta_3 \in \mathfrak{b}$;*
- $\langle A \mid A \rangle \subseteq \ker \phi$ *and* $\langle \mathfrak{b} \mid \mathfrak{b} \rangle A = 0$*;*
- $\langle c \mid c' \rangle c'' = (-1)^{\bar{c}(\bar{c'} + \bar{c''})} \chi(c', c'').c - (-1)^{\bar{c'}\bar{c''}} \chi(c, c'').c'$ *for $c, c', c'' \in C$.*

Moreover, the B(m,n)*-graded Lie superalgebras for $m \geq 1$ are exactly these Lie superalgebras with the added constraint that*

$$D = \langle A \mid A \rangle + \langle C \mid C \rangle.$$

Note that the last condition "•" above is a consequence of (4.3).

Suppose $L = (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D$ is a B(m,n)-graded Lie superalgebra for $m \geq 1$. Thus, A, C, D are $\mathbb{F}$-superspaces satisfying the constraints of Theorem 4.9. As in that theorem, let $\mathfrak{b} = A \oplus C$ be the algebra with multiplication prescribed by (c), and define $D_{\mathfrak{b},\mathfrak{b}} \subseteq \mathrm{Der}(\mathfrak{b})$ by

$$\begin{aligned} &D_{c,c'}(A) = 0 \\ &D_{c,c'}(c'') = (-1)^{\bar{c}(\bar{c'} + \bar{c''})} \chi(c', c'').c - (-1)^{\bar{c'}\bar{c''}} \chi(c, c'').c' \quad \text{for } c, c', c'' \in C \\ &D_{A,A} = D_{A,C} = D_{C,A} = 0 \end{aligned} \tag{4.11}$$

Then for L modulo its center $Z(L) = \{\ell \in L \mid [\ell, L] = 0\}$, we have

$$L/Z(L) \cong \mathfrak{L}(\mathfrak{b}), \qquad \text{where} \tag{4.12}$$

$$\mathfrak{L}(\mathfrak{b}) \stackrel{\text{def}}{=} (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D_{\mathfrak{b},\mathfrak{b}}. \tag{4.13}$$

The multiplication on $\mathfrak{L}(\mathfrak{b})$ is that given by (4.10) with D replaced by $D_{\mathfrak{b},\mathfrak{b}}$ and $\langle \beta \mid \beta' \rangle = D_{\beta,\beta'}$ for all $\beta, \beta' \in \mathfrak{b}$.

Remark 4.14. Note that it follows from the relation

$$[E, D_{\beta,\beta'}] = D_{E\beta,\beta'} + (-1)^{\bar{E}\bar{\beta}} D_{\beta,E\beta'}$$

which holds for $E \in D_{\mathfrak{b},\mathfrak{b}}$, $\beta, \beta' \in \mathfrak{b}$, that $D_{\mathfrak{b},\mathfrak{b}}$ is a subsuperalgebra of $\mathfrak{gl}(\mathfrak{b})$. Hence all the conditions of Theorem 4.9 are satisfied by $\mathfrak{L}(\mathfrak{b})$, and as a consequence, $\mathfrak{L}(\mathfrak{b})$ is a B(m,n)-graded Lie superalgebra. Furthermore, any B(m,n)-graded Lie superalgebra with coordinate superalgebra $\mathfrak{b}$ is a cover of $\mathfrak{L}(\mathfrak{b})$ in the following sense.

Recall that a *central extension* of a Lie superalgebra L is a pair $(\widetilde{L},\pi)$ consisting of a Lie superalgebra $\widetilde{L}$ and a surjective Lie superalgebra homomorphism $\pi:\widetilde{L}\to L$ (preserving the grading) whose kernel lies in the center of $\widetilde{L}$. If $\widetilde{L}$ is perfect ($\widetilde{L}=[\widetilde{L},\widetilde{L}]$), then $\widetilde{L}$ is said to be a *cover* or *covering* of L. Any perfect Lie superalgebra L has a unique (up to isomorphism) universal central extension $(\widehat{L},\widehat{\pi})$ which is also perfect, called the *universal covering superalgebra* of L. Two perfect Lie superalgebras L_1 and L_2 are said to be *centrally isogenous* if $L_1/Z(L_1)\cong L_2/Z(L_2)$.

We present two different constructions of the $\mathrm{B}(m,n)$-graded Lie superalgebras for $m\geq 1$, – a Tits construction and then a unitary construction in Section 7.

The Tits Construction.

In [BZ, 3.28] a (generalized) Tits construction is presented starting with the Jordan algebras of two symmetric nondegenerate bilinear forms. This construction has a super-counterpart which we discuss next. (There is a slight discrepancy in the notation used here compared to [BZ], as we have switched the roles of A and $\mathfrak{A}$. This change is inconsequential.)

Assume $\mathfrak{A}$ is a unital (super)commutative associative superalgebra over the field $\mathbb{F}$ (of characteristic zero). Let $V=V_{\bar 0}\oplus V_{\bar 1}$ be a superspace over $\mathfrak{A}$ endowed with an even supersymmetric $\mathfrak{A}$-bilinear form $(\ ,\ )_V$. Let $J(V):=\mathfrak{A}\oplus V$, the Jordan superalgebra of the form. Thus, the product in $J(V)$ is given by

$$(\alpha+v)(\beta+v')=\bigl(\alpha\beta+(v\mid v')_V\bigr)+\alpha.v'+(-1)^{\bar\beta\bar v}\beta.v\,. \tag{4.15}$$

The space of $\mathfrak{A}$-linear derivations of $J(V)$ which map V to V is just the Lie superalgebra $\mathcal{D}(J(V))$ of $\mathfrak{A}$-linear skew symmetric transformations $E:V\to V$ such that $(Ev\mid v')_V=-(-1)^{\bar E\bar v}(v\mid Ev')_V$ for all homogeneous $E\in\mathrm{End}_{\mathfrak{A}}(V)$, $v,v'\in V$ viewed under the supercommutator product. The mapping

$$D_{v,v'}(v'')=(-1)^{\bar{v'}\bar{v''}}(v\mid v'')_V.v'-(-1)^{\bar v(\bar{v'}+\bar{v''})}(v'\mid v'')_V.v \tag{4.16}$$

belongs to $\mathcal{D}(J(V))$ and satisfies

$$\begin{aligned} D_{v,v'}&=-(-1)^{\bar v\bar{v'}}D_{v',v}\\ [E,D_{v,v'}]&=D_{Ev,v'}+(-1)^{\bar E\bar v}D_{v,Ev'} \end{aligned} \tag{4.17}$$

for all $E\in\mathcal{D}(J(V))$, $v,v'\in V$. In the special case that $\mathfrak{A}=\mathbb{F}$, and the form $(\,|\,)_V$ on V is nondegenerate, the Lie superalgebra $\mathcal{D}(J(V))$ is simply $\mathfrak{osp}(V)$.

Similarly, we assume $W=W_{\bar 0}\oplus W_{\bar 1}$ is a module for a unital (super)commutative associative superalgebra A endowed with an even supersymmetric A-bilinear form

$(\,|\,)_W$. We let $J(W) := A \oplus W$, and set $\mathcal{D}(J(W)) := \{e \in \mathrm{End}_A(W) \mid (ew \mid w')_W + (-1)^{\bar{e}\bar{w}}(w \mid ew')_W = 0$ for all $w, w' \in W\}$. We define the analogue of (4.16):

$$d_{w,w'}(w'') = (-1)^{\bar{w'}\bar{w''}}(w \mid w'')_W.w' - (-1)^{\bar{w}(\bar{w'}+\bar{w''})}(w' \mid w'')_W.w \tag{4.18}$$

for all $w, w', w'' \in W$.

Proposition 4.19. *Assume $\mathfrak{A}$ and A are unital, (super)commutative associative $\mathbb{F}$-superalgebras. Let $J(V) = \mathfrak{A} \oplus V$ (resp. $J(W) = A \oplus W$) be the Jordan superalgebra corresponding to an even supersymmetric $\mathfrak{A}$-bilinear (resp. A-bilinear) form on the $\mathfrak{A}$-module V (resp. on the A-module W). Let $\tilde{\mathcal{D}}(J(V))$ and $\tilde{\mathcal{D}}(J(W))$ be subsuperalgebras of $\mathcal{D}(J(V))$ and $\mathcal{D}(J(W))$ containing respectively $D_{V,V} = \{D_{v,v'} \mid v, v' \in V\}$ and $d_{W,W} = \{d_{w,w'} \mid w, w' \in W\}$. Then*

$$\mathcal{T}(J(V)/\mathfrak{A}, J(W)/A) = \big(\tilde{\mathcal{D}}(J(V)) \otimes A\big) \oplus (V \otimes W) \oplus \big(\mathfrak{A} \otimes \tilde{\mathcal{D}}(J(W))\big)$$

with a superanticommutative multiplication given by

(4.20)
- $[D \otimes a, v \otimes w] = (-1)^{\bar{a}\bar{v}} Dv \otimes a.w\,, \qquad [\alpha \otimes d, v \otimes w] = (-1)^{\bar{d}\bar{v}} \alpha.v \otimes dw$
- $[v \otimes w, v' \otimes w'] = (-1)^{\bar{w}\bar{v'}} \Big(D_{v,v'} \otimes (w \mid w')_W + (v \mid v')_V \otimes d_{w,w'}\Big)$
- $[D \otimes a, \alpha \otimes d] = 0$
- $[D \otimes a, D' \otimes a'] = (-1)^{\bar{a}\bar{D'}}[D, D'] \otimes aa'$,
- $[\alpha \otimes d, \alpha' \otimes d'] = (-1)^{\bar{\alpha'}\bar{d}} \alpha\alpha' \otimes [d, d']$

for any homogeneous elements $D \in \tilde{\mathcal{D}}(J(V))$, $d \in \tilde{\mathcal{D}}(J(W))$, $a, a' \in A$, $\alpha, \alpha' \in \mathfrak{A}$, $v, v' \in V$, $w, w' \in W$, where $D_{v,v'}$ and $d_{w,w'}$ are as in (4.16) and (4.18), is a Lie superalgebra.

Proof. Because $\big(\tilde{\mathcal{D}}(J(V)) \otimes A\big) \oplus \big(\mathfrak{A} \otimes \tilde{\mathcal{D}}(J(W)\big)$ is a Lie superalgebra and $V \otimes W$ is a module for it, the split null extension $\big(\tilde{\mathcal{D}}(J(V)) \otimes A\big) \oplus \big(\mathfrak{A} \otimes \tilde{\mathcal{D}}(J(W)) \oplus (V \otimes W)$ is a Lie superalgebra. Therefore, as in [BZ, Prop. 3.9], to show that $\mathcal{T} = \mathcal{T}(J(V)/\mathfrak{A}, J(W)/A)$ is a Lie superalgebra amounts to verifying that the Jacobi sum $\mathcal{J}(\ell_1, \ell_2, \ell_3) = \sum_{\circlearrowleft} (-1)^{\bar{\ell_1}\bar{\ell_3}}[\ell_1, [\ell_2, \ell_3]]$ is 0 for the triples

$$\{\ell_1, \ell_2, \ell_3\} = \begin{cases} \{E \otimes a, v \otimes w, v' \otimes w'\} \\ \{\alpha \otimes e, v \otimes w, v' \otimes w'\} \\ \{v \otimes w, v' \otimes w', v'' \otimes w''\}. \end{cases}$$

The argument in [BZ, 3.29], with appropriate parity signs inserted, provides a proof of this proposition. $\square$

Lemma 4.1 enables us to realize any B(m,n)-graded Lie superalgebra for $m \geq 1$ via this construction, just by repeating the arguments in [BZ, Thm. 3.53].

Theorem 4.21. *Assume that L is a Lie superalgebra graded by $\mathrm{B}(m,n)$ for $m \geq 1$ over a field $\mathbb{F}$ of characteristic zero. Then there exists a unital, (super)commutative associative superalgebra A and a Jordan algebra $J(W) = A \oplus W$ associated with an A-module W having a supersymmetric A-bilinear form such that L is centrally isogenous to*

$$\mathcal{T}(J(V)/\mathbb{F}, J(W)/A) = (\mathfrak{g} \otimes A) \oplus (V \otimes W) \oplus d_{W,W}$$

where $\mathfrak{g}$ is the split simple Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$ and V is its natural $(2m+1+2n)$-dimensional module having a nondegenerate symmetric bilinear form $(\,|\,)$ as in (3.1) relative to which $\mathfrak{g}$ is the space $\mathcal{D}(J(V))$ of skew-symmetric transformations. The multiplication in $\mathcal{T}(J(V)/\mathbb{F}, J(W)/A)$ is that given in (4.20) above.

Remark 4.22. The translation between this realization of $\mathrm{B}(m,n)$-graded Lie superalgebras ($m \geq 1$) and the description in Theorem 4.9 is easy to establish. In the superalgebra $\mathcal{T}(J(V)/\mathbb{F}, J(W)/A)$ of Theorem 4.21, set $(v \mid v') = (v \mid v')_V$, $C = W$, $\chi(c,c') = -(c \mid c')_W$ for all $c, c' \in C = W$, and $\langle A \mid A\rangle = 0$. Note that $\gamma_{u,v} = -D_{u,v}$ for $u, v \in V$, and $d_{c,c'} = \langle c \mid c'\rangle$. Then for example, the relation $[v \otimes c, v' \otimes c'] = (-1)^{\bar{c}\bar{v'}}\Big(D_{v,v'} \otimes (c \mid c')_W + (v \mid v')_V \otimes d_{c,c'}\Big)$ in (4.20) just says $[v \otimes c, v' \otimes c'] = (-1)^{\bar{c}\bar{v'}}\Big(\gamma_{v,v'} \otimes \chi(c,c') + (v \mid v')\langle c \mid c'\rangle\Big)$ as in (4.10).

§5. $\mathrm{B}(0,n)$-graded Lie superalgebras ($n \geq 2$)

By Theorem 3.3, a Lie superalgebra L graded by the root system $\mathrm{B}(0,n)$ of $\mathfrak{g} = \mathfrak{osp}(1,2n)$ decomposes as

$$L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D$$

for suitable superspaces A, B, C, D. The grading subsuperalgebra $\mathfrak{g}$ is identified with $\mathfrak{g} \otimes 1$ where $1 \in A_{\bar{0}}$. Because the $\mathfrak{g}$-homomorphisms in Table 3.5 are the super versions of corresponding ones in the ungraded case (compare [ABG2, Chap. 2]), we may suppose that following hold:

- there is a unital multiplication on $\mathfrak{a} = A \oplus B$ (with $1 \in A_{\bar{0}}$) together with a superinvolution η such that $\eta\,|_A = \mathrm{I}$ and $\eta\,|_B = -\mathrm{I}$;
- there are even bilinear mappings

$$\begin{aligned} &\mathfrak{a} \times C \to C, && (\alpha, c) \mapsto \alpha.c \qquad \text{with } \ 1.c = c \\ &C \times C \to \mathfrak{a}, && (c,c') \mapsto \chi(c,c') = c * c' + c \diamond c' \end{aligned}$$

with $(c,c') \mapsto c*c' \in A$ supersymmetric and $(c,c') \mapsto c\diamond c' \in B$ superskewsymmetric so that $\chi(c,c')^\eta = (-1)^{\bar{c}\bar{c'}}\chi(c',c)$, and hence there is a product on $\mathfrak{b} = A \oplus B \oplus C$ given by

$$(5.1) \qquad (\alpha + c)(\alpha' + c') = \Big(\alpha\alpha' + 2\chi(c,c')\Big) + \Big(\alpha.c' + (-1)^{\bar{\alpha'}\bar{c}}(\alpha')^\eta.c\Big);$$

- there is a bilinear superskewsymmetric map $\langle\,|\,\rangle : \mathfrak{b}\times\mathfrak{b}\to D$ with $\langle A \mid B\rangle = \langle A \mid C\rangle = \langle B \mid C\rangle = 0$;
- there is a superanticommutative product on D and a linear map $\phi : D \to \mathrm{End}_{\mathbb{F}}(\mathfrak{b})$ with $d \mapsto \phi_d$ such that if $d(\beta) := \phi_d(\beta)$ for all $\beta\in\mathfrak{b}$, then $d(A)\subseteq A,\ d(B)\subseteq B,\ d(C)\subseteq C$ for all $d\in D$;
- the multiplication on L is given by

(5.2)
$$\begin{aligned}
&[z\otimes\alpha, z'\otimes\alpha'] = (-1)^{\bar{\alpha}\bar{z'}}\Big([z,z']\otimes\frac{1}{2}(\alpha\circ\alpha') + z\circ z'\otimes\frac{1}{2}[\alpha,\alpha'] + \mathfrak{str}(zz')\langle\alpha \mid \alpha'\rangle\Big)\\
&[z\otimes\alpha, u\otimes c] = (-1)^{\bar{\alpha}\bar{u}} z.u\otimes\alpha.c\\
&[u\otimes c, v\otimes c'] = (-1)^{\bar{c}\bar{v}}\Big(\gamma_{u,v}\otimes c * c' + \sigma_{u,v}\otimes c\diamond c' + (u\mid v)\langle c\mid c'\rangle\Big)\\
&[d, p\otimes\beta] = (-1)^{\bar{d}\bar{p}} p\otimes d\beta\\
&[d,d'] \quad \text{(is the product in } D\text{)},
\end{aligned}$$

for all $z\otimes\alpha,\ z'\otimes\alpha' \in (\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$, $u\otimes c, v\otimes c'\in V\otimes C,\ d\in D,\ \ p\otimes\beta\in(\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)\cup(V\otimes C)$, where

$$\begin{aligned}
&z\circ z' = zz' + (-1)^{\bar{z}\bar{z'}}z'z - \frac{2}{1-2n}\mathfrak{str}(zz')\,\mathrm{I} \qquad \forall\ z,z'\in\mathfrak{g}\cup\mathfrak{s}\\
&\alpha\circ\alpha' = \alpha\alpha' + (-1)^{\bar{\alpha}\bar{\alpha'}}\alpha'\alpha, \qquad [\alpha,\alpha'] = \alpha\alpha' - (-1)^{\bar{\alpha}\bar{\alpha'}}\alpha'\alpha \quad \forall\alpha,\alpha'\in\mathfrak{a}
\end{aligned} \tag{5.3}$$

and $\gamma_{u,v}\in\mathfrak{g}$ and $\sigma_{u,v}\in\mathfrak{s}$ are as in (3.6) and (4.6).

The arguments in [ABG2] adapted to the super case tell us that L is a Lie superalgebra if and only if the conditions derived in (i)-(v) below hold:

(i) The validity of the Jacobi superidentity with at least one of the elements taken from D is equivalent to the following: Relative to its product, D is a Lie superalgebra (a subsuperalgebra of L), $\ \phi : D\to \mathrm{Der}_{\mathbb{F}}(\mathfrak{b})$ is a representation of D as superderivations on the algebra $\mathfrak{b}$ relative to the product in (5.1), and

$$[d,\langle\beta,\beta'\rangle] = \langle d\beta,\beta'\rangle + (-1)^{\bar{d}\bar{\beta}}\langle\beta \mid d\beta'\rangle$$

for $d\in D,\ \beta,\beta'\in\mathfrak{b}$. In particular, $\langle A\mid A\rangle,\ \langle B\mid B\rangle,\ \langle C\mid C\rangle$ are ideals of D.

(ii) For $z_1\otimes\alpha_1,\ z_2\otimes\alpha_2,\ z_3\otimes\alpha_3 \in(\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$, the Jacobi superidentity is equivalent to the two relations

(5.4)
$$\begin{aligned}
0 =&\mathfrak{str}(z_1z_2z_3)\Bigg(\sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}\langle\alpha_1 \mid \alpha_2\alpha_3\rangle\Bigg)\\
&\qquad - (-1)^{\bar{z_2}\bar{z_3}}\mathfrak{str}(z_1z_3z_2)\Bigg(\sum_{\circlearrowleft}(-1)^{(\bar{\alpha_1}+\bar{\alpha_2})\bar{\alpha_3}}\langle\alpha_1 \mid \alpha_3\alpha_2\rangle\Bigg)
\end{aligned}$$

(5.5)

$$\begin{aligned}
0 = &-\sum_{\circlearrowleft}(-1)^{\bar{z_1}\bar{z_3}+\bar{\alpha_1}\bar{\alpha_3}} z_1z_2z_3 \otimes (\alpha_1,\alpha_2,\alpha_3)\\
&+\sum_{\circlearrowleft}(-1)^{(\bar{z_1}+\bar{z_2})\bar{z_3}+(\bar{\alpha_1}+\bar{\alpha_2})\bar{\alpha_3}} z_1z_3z_2 \otimes (\alpha_1,\alpha_3,\alpha_2)\\
&-\sum_{\circlearrowleft}(-1)^{\bar{z_1}\bar{z_3}+\bar{\alpha_1}\bar{\alpha_3}} \mathfrak{str}(z_1z_2)z_3 \otimes \Big(\langle\alpha_1 \mid \alpha_2\rangle\alpha_3 - \frac{1}{1-2n}[[\alpha_1,\alpha_2],\alpha_3]\Big)\\
&-(-1)^{\bar{z_1}\bar{z_3}}\frac{\mathfrak{str}(z_1z_2z_3)}{1-2n}\,\mathrm{I}\otimes\left(\sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}[\alpha_1,\alpha_2\alpha_3]\right)\\
&+(-1)^{(\bar{z_1}+\bar{z_2})\bar{z_3}}\frac{\mathfrak{str}(z_1z_3z_2)}{1-2n}\,\mathrm{I}\otimes\left(\sum_{\circlearrowleft}(-1)^{(\bar{\alpha_1}+\bar{\alpha_2})\bar{\alpha_3}}[\alpha_1,\alpha_3\alpha_2]\right),
\end{aligned}$$

where $(\alpha_1,\alpha_2,\alpha_3) = (\alpha_1\alpha_2)\alpha_3 - \alpha_1(\alpha_2\alpha_3)$, (the associator). The first corresponds to the D-component and the second to the $(\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)$-component.

To examine (5.4), suppose first that $z_i \in \mathfrak{g}$ (and so $\alpha_i \in A$) for all i. Then $z_2 \circ z_3 \in \mathfrak{s}$, so $\mathfrak{str}(z_1(z_2\circ z_3)) = 0$ as $\mathfrak{str}(\mathfrak{g}\ \mathfrak{s}) = 0$, and $\mathfrak{str}(z_1z_2z_3) = -(-1)^{\bar{z_2}\bar{z_3}}\mathfrak{str}(z_1z_3z_2)$. Equation (5.4) reduces to

$$0 = \mathfrak{str}(z_1z_2z_3)\left(\sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}\langle\alpha_1 \mid \alpha_2\circ\alpha_3\rangle\right)$$

in this case. As we can find $z_1,z_2,z_3 \in \mathfrak{g}$ with $\mathfrak{str}(z_1z_2z_3) \neq 0$, it must be that $0 = \sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}\langle\alpha_1 \mid \alpha_2\circ\alpha_3\rangle$ for all $\alpha_i \in A$. However, $\alpha_2\alpha_3 = \frac{1}{2}\alpha_2\circ\alpha_3 + \frac{1}{2}[\alpha_2,\alpha_3]$, and when $\alpha_2,\alpha_3 \in A$, then $[\alpha_2,\alpha_3] \in B$. Because $\langle A \mid B\rangle = 0$, we see that $\langle\alpha_1 \mid \alpha_2\circ\alpha_3\rangle = 2\langle\alpha_1 \mid \alpha_2\alpha_3\rangle$. Thus

$$0 = \sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}\langle\alpha_1 \mid \alpha_2\alpha_3\rangle \tag{5.6}$$

for $\alpha_i \in A$ ($i = 1,2,3$).

Assuming next that $z_1,z_2 \in \mathfrak{g}$ and $z_3 \in \mathfrak{s}$ gives $[z_2,z_3] \in \mathfrak{s}$ so that $\mathfrak{str}(z_1[z_2,z_3]) = 0$ and $\mathfrak{str}(z_1z_2z_3) = (-1)^{\bar{z_2}\bar{z_3}}\mathfrak{str}(z_1z_3z_2)$. Thus, (6.2) becomes

$$0 = \mathfrak{str}(z_1z_2z_3)\left(\sum_{\circlearrowleft}(-1)^{\bar{\alpha_1}\bar{\alpha_3}}\langle\alpha_1 \mid [\alpha_2,\alpha_3]\rangle\right)$$

Using the relations $[A,B]\subseteq A$, $A\circ B\subseteq B$, $[A,A]\subseteq B$, and $A\circ A\subseteq A$, we determine that (5.6) holds whenever $\alpha_1,\alpha_2 \in A$, and $\alpha_3 \in B$.

The other remaining possibilities for $z_1,z_2,z_3 \in \mathfrak{g}\cup\mathfrak{s}$ can be treated in exactly the same way to show that (5.6) holds for any $\alpha_1,\alpha_2,\alpha_3 \in A\cup B$. Thus, (5.4) is equivalent to (5.6).

Using the basis in (3.1) (with $m = 0$) we identify linear transformations with matrices whose rows and columns are numbered from 0 to $2n$, and we let $E_{i,j}$ be the matrix with 1 in the (i,j)-position and 0's elsewhere.

For (5.5) we assume now that $z_1 = E_{2,1} - \zeta_1 E_{n+1,n+2}$, $z_2 = E_{1,1} - \zeta_2 E_{n+1,n+1} + (1-\zeta_2)E_{0,0}$, $z_3 = E_{1,0} - \zeta_3 E_{0,n+1}$, ($\zeta_i = \pm 1$, $z_i \in \mathfrak{g}$ when $\zeta_i = 1$ and $z_i \in \mathfrak{s}$ when $\zeta_i = -1$). Then all the elements $z_1z_2 = E_{2,1}$, $z_2z_3 = E_{1,0} - (1-\zeta_2)\zeta_3 E_{0,n+1}$, $z_3z_1 = \zeta_1\zeta_3 E_{0,n+2}$ have zero supertrace, and

$$\begin{aligned} &z_1z_2z_3 = E_{2,0}, \qquad z_2z_3z_1 = (1-\zeta_2)\zeta_1\zeta_3 E_{0,n+2}, \qquad z_3z_1z_2 = 0 \\ &z_1z_3z_2 = (1-\zeta_2)E_{2,0}, \qquad z_2z_1z_3 = 0, \qquad z_3z_2z_1 = -\zeta_1\zeta_2\zeta_3 E_{0,n+2}. \end{aligned}$$

Therefore (5.5) with this substitution becomes

$$\begin{aligned} 0 = &-(-1)^{\bar{\alpha_1}\bar{\alpha}_3} E_{2,0} \otimes (\alpha_1,\alpha_2,\alpha_3) - (-1)^{\bar{\alpha_2}\bar{\alpha}_1}(1-\zeta_2)\zeta_1\zeta_3 E_{0,n+2} \otimes (\alpha_2,\alpha_3,\alpha_1) \\ &\quad + (-1)^{(\bar{\alpha_1}+\bar{\alpha_2})\bar{\alpha_3}}(1-\zeta_2)E_{2,0} \otimes (\alpha_1,\alpha_3,\alpha_2) \\ &\qquad - (-1)^{(\bar{\alpha_3}+\bar{\alpha_1})\bar{\alpha_2}}\zeta_1\zeta_2\zeta_3 E_{0,n+2} \otimes (\alpha_3,\alpha_2,\alpha_1), \end{aligned}$$

which implies

$$(\alpha_1,\alpha_2,\alpha_3) = (-1)^{\bar{\alpha_2}\bar{\alpha}_3}(1-\zeta_2)(\alpha_1,\alpha_3,\alpha_2).$$

In particular, with $\zeta_2 = 1$, $(\mathfrak{a}, A, \mathfrak{a}) = 0$, and with $\zeta_2 = -1$, $(\mathfrak{a}, B, A) = (\mathfrak{a}, A, B) = 0$. If $\alpha_2, \alpha_3 \in B$, ($\zeta_2 = \zeta_3 = -1$), $(\alpha_1,\alpha_2,\alpha_3) = 2(-1)^{\bar{\alpha_2}\bar{\alpha}_3}(\alpha_1,\alpha_3,\alpha_2) = 4(\alpha_1,\alpha_2,\alpha_3)$ so that $(\mathfrak{a}, B, B) = 0$ too. These results combine to say that $(\mathfrak{a},\mathfrak{a},\mathfrak{a}) = 0$ that is

(5.7) $\mathfrak{a}$ is associative.

When $\mathfrak{a}$ is associative, equation (5.5) reduces to

$$0 = \sum_{\circlearrowleft} (-1)^{\bar{z_1}\bar{z_3}+\bar{\alpha_1}\bar{\alpha_3}} \mathfrak{str}(z_1z_2)z_3 \otimes \Big(\langle \alpha_1 \mid \alpha_2\rangle\alpha_3 - \frac{1}{1-2n}[[\alpha_1,\alpha_2],\alpha_3]\Big).$$

Then from the substitution $z_1 = E_{1,0} - \zeta_1 E_{0,n+1}$, $z_2 = E_{0,1} + \zeta_2 E_{n+1,0}$, and $z_3 = E_{2,2} - \zeta_3 E_{n+2,n+2} + (1-\zeta_3)E_{0,0}$, we deduce $\mathfrak{str}(z_2z_3) = \mathfrak{str}(z_3z_1) = 0$, and $\mathfrak{str}(z_1z_2) = -(1+\zeta_1\zeta_2)$, so that

$$\langle \alpha_1 \mid \alpha_2\rangle\alpha_3 = \begin{cases} 0 & \alpha_1 \in A \text{ and } \alpha_2 \in B, \text{ or } \alpha_2 \in A \text{ and } \alpha_1 \in B, \\ \frac{1}{1-2n}[[\alpha_1,\alpha_2],\alpha_3] & \text{otherwise.} \end{cases}$$

Alternatively, we may write

$$\langle \alpha_1 \mid \alpha_2\rangle\alpha_3 = \frac{1}{2(1-2n)}[[\alpha_1,\alpha_2] + [\alpha_1^\eta,\alpha_2^\eta],\alpha_3] \tag{5.8}$$

for all $\alpha_1,\alpha_2,\alpha_3 \in \mathfrak{a}$. Therefore, the validity of Jacobi superidentity in this case (which is the same as (5.4) and (5.5)) is equivalent to the three conditions (5.6), (5.7), and (5.8).

(iii) The Jacobi superidentity for $z\otimes\alpha,\ z'\otimes\alpha'\in(\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$, and $u\otimes c\in V\otimes C$, gives the relation

(5.9)
$$
\begin{aligned}
0 =&z.(z'.u)\otimes\big(\alpha.(\alpha'.c)-(\alpha\alpha').c\big)-(-1)^{\bar z\bar z'+\bar\alpha\bar\alpha'}z'.(z.u)\otimes\big(\alpha'.(\alpha.c)-(\alpha'\alpha).c\big)\\
&-\mathfrak{str}(zz')u\otimes\Big(\langle\alpha\mid\alpha'\rangle c-\frac{1}{1-2n}[\alpha,\alpha'].c\Big).
\end{aligned}
$$

Letting $z=E_{2,1}-\zeta E_{n+1,n+2}$, $z'=E_{1,0}-\zeta' E_{0,n+1}$, where $\zeta,\zeta'\in\{\pm1\}$, and $u=u_0$ (as in (3.1)) gives $z.(z'.u)=z.v_1=v_2$, $z'.(z.u)=0$, and $\mathfrak{str}(zz')=0$, so that (5.9) simplifies to $v_2\otimes\big(\alpha.(\alpha'.c)-(\alpha\alpha').c\big)$ in this case. Thus,

$$
(\alpha\alpha').c=\alpha.(\alpha'.c) \tag{5.10}
$$

for all $\alpha,\alpha'\in\mathfrak{a}$, $c\in C$; in other words, C is an associative $\mathfrak{a}$-module. Moreover, if $z=E_{1,0}-\zeta E_{0,n+1}$, $z'=E_{0,1}+\zeta' E_{n+1,0}$, and $u=u_0$, then $\mathfrak{str}(zz')=-(1+\zeta\zeta')$, so that we obtain the conclusion

$$
\langle\alpha,\alpha'\rangle c=\begin{cases}0 & \alpha\in A\text{ and }\alpha'\in B,\text{ or }\alpha'\in A\text{ and }\alpha\in B,\\ \frac{1}{1-2n}[\alpha,\alpha'].c & \text{otherwise.}\end{cases}
$$

This last condition may be subsumed into one expression as

$$
\langle\alpha,\alpha'\rangle c=\frac{1}{2(1-2n)}\Big([\alpha,\alpha']+[\alpha^\eta,(\alpha')^\eta]\Big).c \tag{5.11}
$$

for all $\alpha,\alpha'\in\mathfrak{a}$, $c\in C$. Thus, the conditions for the Jacobi superidentity to hold are (5.10) and (5.11) here.

(iv) Suppose $u\otimes c, v\otimes c'\in V\otimes C$, $x\in\mathfrak{g}$, $s\in\mathfrak{s}$, $z\in\mathfrak{g}\cup\mathfrak{s}$, $\alpha\in\mathfrak{a}$, $a\in A$, and $b\in B$. Then various substitutions in the Jacobi superidentity show that in order to have a Lie superalgebra the following must hold:

(a)
$$
\begin{aligned}
0\ =&\ [x,\gamma_{u,v}]\otimes\frac12 a\circ(c*c')+x\circ\sigma_{u,v}\otimes\frac12[a,c\diamond c']\\
&-(-1)^{\bar a(\bar c+\bar c')}(u\mid v)x\otimes\langle c\mid c'\rangle a-\gamma_{x.u,v}\otimes(a.c)*c'\\
&-(-1)^{\bar x\bar u+\bar a\bar c}\gamma_{u,x.v}\otimes c*(a.c')
\end{aligned}
$$

(b)
$$
\begin{aligned}
0\ =&\ [s,\gamma_{u,v}]\otimes\frac12 b\circ(c*c')+s\circ\sigma_{u,v}\otimes\frac12[b,c\diamond c']\\
&-(-1)^{\bar b(\bar c+\bar c')}(u\mid v)s\otimes\langle c\mid c'\rangle b-\sigma_{s.u,v}\otimes(b.c)\diamond c'\\
&-(-1)^{\bar s\bar u+\bar b\bar c}\sigma_{u,s.v}\otimes c\diamond(b.c')
\end{aligned}
$$

$$
\begin{aligned}
&\text{(c)} && 0 = x\circ\gamma_{u,v}\otimes\frac{1}{2}[a,c*c'] + [x,\sigma_{u,v}]\otimes\frac{1}{2}a\circ(c\diamond c') \\
&&& \quad - \sigma_{x.u,v}\otimes(a.c)\diamond c' - (-1)^{\bar x\bar u+\bar a\bar c}\sigma_{u,x.v}c\diamond(a.c') \\
&\text{(d)} && 0 = s\circ\gamma_{u,v}\otimes\frac{1}{2}[b,c*c'] + [s,\sigma_{u,v}]\otimes\frac{1}{2}b\circ(c\diamond c') \\
&&& \quad - \gamma_{s.u,v}\otimes(b.c)*c' - (-1)^{\bar s\bar u+\bar b\bar c}\gamma_{u,s.v}\otimes c*(b.c') \\
&\text{(e)} && 0 = \mathfrak{str}(z\gamma_{u,v})\langle\alpha\mid c*c'\rangle + \mathfrak{str}(z\sigma_{u,v})\langle\alpha\mid c\diamond c'\rangle \\
&&& \quad - (z.u\mid v)\langle\alpha.c\mid c'\rangle - (-1)^{\bar z\bar u+\bar\alpha\bar c}(u\mid z.v)\langle c\mid\alpha.c'\rangle.
\end{aligned}
$$

Let us consider (e) first. If $z\in\mathfrak{g}$ and $\alpha\in A$, then $\mathfrak{str}(z\sigma_{u,v})=0$, while $\mathfrak{str}(z\gamma_{u,v})=(z.u\mid v)-(-1)^{\bar u\bar v}(z.v\mid u)=(z.u\mid v)+(-1)^{(\bar u+\bar z)\bar v}(v\mid z.u)=2(z.u\mid v)$. Thus (e) is equivalent to $2\langle\alpha\mid c*c'\rangle=\langle\alpha.c\mid c'\rangle-(-1)^{\bar\alpha\bar c}\langle c\mid\alpha.c'\rangle$, and since $\langle A\mid B\rangle=0$, to $2\langle\alpha\mid\chi(c,c')\rangle=\langle\alpha.c\mid c'\rangle-(-1)^{\bar\alpha\bar c}\langle c\mid\alpha.c'\rangle$. If instead $z\in\mathfrak{s}$ and $\alpha\in B$, then $\mathfrak{str}(z\gamma_{u,v})=0$, while $\mathfrak{str}(z\sigma_{u,v})=2(z.u\mid v)$ and $(u\mid z.v)=(-1)^{\bar u\bar z}(z.u\mid v)$. So we obtain $2\langle\alpha\mid\chi(c,c')\rangle=\langle\alpha.c\mid c'\rangle+(-1)^{\bar\alpha\bar c}\langle c\mid\alpha.c'\rangle$. The combined result is that

$$
2\langle\alpha\mid\chi(c,c')\rangle=\langle\alpha.c\mid c'\rangle-(-1)^{\bar\alpha\bar c}\langle c\mid\alpha^{\eta}.c'\rangle \tag{5.12}
$$

for $\alpha\in\mathfrak{a}$, $c,c'\in C$. Using the multiplication in $\mathfrak{b}$ from (5.1), we see that equation (5.12) can be rewritten as

$$
0=\langle\alpha\mid c\cdot c'\rangle+(-1)^{\bar{c'}(\bar\alpha+\bar c)}\langle c'\mid\alpha\cdot c\rangle+(-1)^{\bar\alpha(\bar c+\bar{c'})}\langle c\mid c'\cdot\alpha\rangle. \tag{5.13}
$$

Now we tackle (a)-(d). For these it is helpful to quote the following relations (compare (3.7)):

(5.14)

$$
\begin{aligned}
&\text{(i)} && [x,\gamma_{u,v}]=\gamma_{x.u,v}+(-1)^{\bar x\bar u}\gamma_{u,x.v} \\
&\text{(ii)} && x\circ\sigma_{u,v}=x\sigma_{u,v}+(-1)^{\bar x(\bar u+\bar v)}\sigma_{u,v}x \qquad (\text{as } \mathfrak{str}(\mathfrak{g}\ \mathfrak{s})=0) \\
&&& \quad =\gamma_{x.u,v}-(-1)^{\bar x\bar u}\gamma_{u,x.v}-\tfrac{2(u|v)}{1-2n}x \quad (\text{as } u(v\mid x\_)=(-1)^{\bar x\bar v}u(x.v\mid \_)) \\
&\text{(iii)} && x\circ\gamma_{u,v}=\sigma_{x.u,v}-(-1)^{\bar x\bar u}\sigma_{u,x.v} \\
&\text{(iv)} && [x,\sigma_{u,v}]=\sigma_{x.u,v}+(-1)^{\bar x\bar u}\sigma_{u,x.v} \\
&\text{(v)} && [s,\gamma_{u,v}]=\sigma_{s.u,v}-(-1)^{\bar s\bar u}\sigma_{u,s.v} \\
&\text{(vi)} && s\circ\sigma_{u,v}=\sigma_{s.u,v}+(-1)^{\bar s\bar u}\sigma_{u,s.v}-\tfrac{2(u|v)}{1-2n}s \\
&\text{(vii)} && s\circ\gamma_{u,v}=\gamma_{s.u,v}+(-1)^{\bar s\bar u}\gamma_{u,s.v} \\
&\text{(viii)} && [s,\sigma_{u,v}]=\gamma_{s.u,v}-(-1)^{\bar s\bar u}\gamma_{u,s.v}.
\end{aligned}
$$

Applying the first two of these, we see that (a) becomes

$$
\begin{aligned}
0 &= \gamma_{x.u,v}\otimes\left(\frac{1}{2}a\circ(c*c')+\frac{1}{2}[a,c\diamond c']-(a.c)*c'\right)\\
&\qquad+(-1)^{\bar{x}\bar{u}}\gamma_{u,x.v}\otimes\left(\frac{1}{2}a\circ(c*c')-\frac{1}{2}[a,c\diamond c']-(-1)^{\bar{a}\bar{c}}c*(a.c')\right)\\
&\qquad-(u\mid v)x\otimes\left((-1)^{\bar{a}(\bar{c}+\bar{c'})}\langle c\mid c'\rangle a-\frac{2}{1-2n}[a,c\diamond c']\right).
\end{aligned}\tag{5.15}
$$

Setting $u=v_1,\ v=v_2$, and $x=\gamma_{u_0,v_1}$ gives $(u\mid v)=0,\ x.v=0,\ \gamma_{x.u,v}\neq 0$, so that (5.15) implies

$$(a.c)*c'=\frac{1}{2}a\circ(c*c')+\frac{1}{2}[a,c\diamond c'].$$

Similarly,

$$(-1)^{\bar{a}\bar{c}}c*(a.c')=\frac{1}{2}a\circ(c*c')-\frac{1}{2}[a,c\diamond c']$$

and

$$\langle c\mid c'\rangle a=\frac{2}{1-2n}[c\diamond c',a]=\frac{1}{1-2n}[\chi(c,c')-\chi(c,c')^{\eta},a].$$

Analogously, (iii) and (iv) can be used to show equation (c) is equivalent to

$$
\begin{aligned}
(a.c)\diamond c' &= \frac{1}{2}[a,c*c']+\frac{1}{2}a\circ(c\diamond c')\\
(-1)^{\bar{a}\bar{c}}c\diamond(a.c') &= -\frac{1}{2}[a,c*c']+\frac{1}{2}a\circ(c\diamond c').
\end{aligned}
$$

Therefore, together (a) and (c) are equivalent to the three relations:

$$
\begin{aligned}
&\chi(a.c,c')=a\chi(c,c')\\
&\chi(c,a.c')=(-1)^{\bar{a}\bar{c'}}\chi(c,c')a\\
&\langle c,c'\rangle a=\frac{1}{1-2n}[\chi(c,c')-\chi(c,c')^{\eta},a]
\end{aligned}\tag{5.17}
$$

for $a\in A,\ c,c'\in C$.

Equations (b) and (d) can be dealt with the same way. Using (v) and (vi) we have that (b) is equivalent to

$$
\begin{aligned}
(b.c)\diamond c' &= \frac{1}{2}b\circ(c*c')+\frac{1}{2}[b,c\diamond c']\\
(-1)^{\bar{b}\bar{c}}c\diamond(b.c') &= -\frac{1}{2}b\circ(c*c')+\frac{1}{2}[b,c\diamond c']\\
(-1)^{\bar{b}(\bar{c}+\bar{c'})}\langle c\mid c'\rangle b &= -\frac{2}{1-2n}[b,c\diamond c'],
\end{aligned}
$$

while from (vii) and (viii) we have that (d) is equivalent to

$$(b.c) * c' = \frac{1}{2}[b, c * c'] + \frac{1}{2} b \circ (c \diamond c')$$
$$(-1)^{\bar{b}\bar{c}} c * (b.c') = \frac{1}{2}[b, c * c'] - \frac{1}{2} b \circ (c \diamond c').$$

Consequently, (b) and (d) combined are equivalent to

$$\begin{aligned} \chi(b.c, c') &= b\chi(c, c') \\ \chi(c, b.c') &= -(-1)^{\bar{b}\bar{c'}} \chi(c, c') b \\ \langle c, c' \rangle b &= \frac{1}{1-2n}[\chi(c, c') - \chi(c, c')^\eta, b] \end{aligned} \tag{5.18}$$

for $b \in B$, $c, c' \in C$. As a result, conditions (a)-(d) are equivalent to

$$\begin{aligned} \chi(\alpha.c, c') &= \alpha\chi(c, c') \\ \chi(c, \alpha.c') &= (-1)^{\bar{\alpha}\bar{c'}} \chi(c, c') \alpha^\eta \\ \langle c, c' \rangle \alpha &= \frac{1}{1-2n}[\chi(c, c') - \chi(c, c')^\eta, \alpha] \end{aligned} \tag{5.19}$$

for $\alpha \in \mathfrak{a}$, $c, c' \in C$. Thus, the Jacobi superidentity with two elements from $V \otimes C$ and one from $(\mathfrak{g} \otimes A) \cup (\mathfrak{s} \otimes B)$ is equivalent to (5.12) and (5.19) holding.

Recall that $\chi(c, c')^\eta = (-1)^{\bar{c}\bar{c'}} \chi(c', c)$. A form satisfying that property and the first two relations of (5.19) is said to be *η-superhermitian.*

(v) What remains to be determined is when the Jacobi superidentity holds for three elements $w_i \otimes c_i \in V \otimes C$ $(i = 1, 2, 3)$. First of all, we have

$$0 = \sum_{\circlearrowleft} (-1)^{\bar{w_1}\bar{w_3} + \bar{c_1}\bar{c_3}} \Big(\gamma_{w_1,w_2}.w_3 \otimes (c_1 * c_2).c_3 + \sigma_{w_1,w_2}.w_3 \otimes (c_1 \diamond c_2).c_3 + (w_1 \mid w_2) w_3 \otimes \langle c_1 \mid c_2 \rangle c_3 \Big) \quad \text{or}$$

$$\begin{aligned} 0 = \sum_{\circlearrowleft} (-1)^{\bar{w_1}\bar{w_3} + \bar{c_1}\bar{c_3}} \Big(& (w_2 \mid w_3) w_1 \otimes \chi(c_1, c_2) c_3 \\ & - (-1)^{\bar{w_1}\bar{w_2}} (w_1 \mid w_3) w_2 \otimes \chi(c_1, c_2)^\eta.c_3 \\ & + (w_1 \mid w_2) w_3 \otimes \Big(\langle c_1 \mid c_2 \rangle c_3 - \frac{2}{1-2n} (c_1 \diamond c_2).c_3 \Big) \Big) \quad \text{or} \end{aligned}$$

$$\begin{aligned} 0 = \sum_{\circlearrowleft} (-1)^{\bar{w_1}\bar{w_3} + \bar{c_1}\bar{c_3}} \Big((w_1 \mid w_2) w_3 \otimes \Big(& \langle c_1 \mid c_2 \rangle c_3 - \frac{2}{1-2n} (c_1 \diamond c_2).c_3 \\ & - (-1)^{\bar{c_3}(\bar{c_1} + \bar{c_2})} \chi(c_3, c_1).c_2 \\ & - (-1)^{\bar{c_1}(\bar{c_2} + \bar{c_3})} \chi(c_2, c_3)^\eta.c_1 \Big) \Big). \end{aligned}$$

But we may choose w_1, w_2, w_3 with the property that $(w_1 \mid w_2) \neq 0,\ (w_1 \mid w_3) = 0 = (w_2 \mid w_3)$ to conclude that the Jacobi superidentity is equivalent to

(5.20)
$$\begin{aligned}\langle c_1 \mid c_2\rangle c_3 = \frac{1}{1-2n}\Bigl(\chi(c_1,c_2) - \chi(c_1,c_2)^\eta\Bigr).c_3 \\ - (-1)^{\bar{c_2}\bar{c_3}}\chi(c_1,c_3)^\eta.c_2 + (-1)^{\bar{c_1}(\bar{c_2}+\bar{c_3})}\chi(c_2,c_3)^\eta.c_1.\end{aligned}$$

We have completed the analysis of when the Jacobi superidentity holds for superalgebras having a $\mathrm{B}(0,n)$-decomposition. Putting this all together, we have the following classification result for Lie superalgebras graded by the root system $\mathrm{B}(0,n)$, $(n \geq 2)$.

Theorem 5.21. *Assume $L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D$ is superalgebra over a field $\mathbb{F}$ of characteristic zero with $\mathfrak{g} = \mathfrak{osp}(1,2n)$ for $n \geq 2$, $\mathfrak{s}$ and V as in Section 3, $\mathbb{F}$-superspaces A, B, C, D, and with multiplication as in (5.2). Then L is a Lie superalgebra if and only if*

- *$\mathfrak{a} = A \oplus B$ is a unital associative superalgebra together with a superinvolution η such that $\eta\mid_A = \mathrm{I}$ and $\eta\mid_B = -\mathrm{I}$;*
- *C is a left unital associative module for $\mathfrak{a}$;*
- *$\chi : C \times C \to \mathfrak{a}$ is η-superhermitian;*
- *D is a Lie subsuperalgebra of L and $\phi : D \to \mathrm{Der}_{\mathbb{F}}(\mathfrak{b})$ is a representation of D as superderivations on the algebra $\mathfrak{b} = \mathfrak{a} \oplus C$ with product*
$$(\alpha + c)(\alpha' + c') = \Bigl(\alpha\alpha' + 2\chi(c,c')\Bigr) + \Bigl(\alpha.c' + (-1)^{\bar{\alpha'}\bar{c}}(\alpha')^\eta.c\Bigr)$$
such that $d(A) \subseteq A$, $d(B) \subseteq B$, and $d(C) \subseteq C$;
- *$[d, \langle \beta \mid \beta'\rangle] = \langle d\beta \mid \beta'\rangle + (-1)^{\bar{d}\bar{\beta}}\langle \beta \mid d\beta'\rangle$ for $d \in D$, $\beta, \beta' \in \mathfrak{b}$;*
- *$\sum_{\circlearrowleft}(-1)^{\bar{\beta}_1\bar{\beta}_3}\langle \beta_1 \mid \beta_2\beta_3\rangle = 0$ for $\beta_1, \beta_2, \beta_3 \in \mathfrak{b}$;*
- *$\langle \alpha \mid \alpha'\rangle\alpha'' = \frac{1}{2(1-2n)}[[\alpha,\alpha'] - [\alpha,\alpha']^\eta, \alpha'']$ for all $\alpha, \alpha', \alpha'' \in \mathfrak{a}$;*
- *$\langle \alpha \mid \alpha'\rangle c = \frac{1}{2(1-2n)}\Bigl([\alpha,\alpha'] - [\alpha,\alpha']^\eta\Bigr).c$ for all $\alpha, \alpha' \in \mathfrak{a}$, $c \in C$;*
- *$\langle c \mid c'\rangle\alpha = \frac{1}{1-2n}[\chi(c,c') - \chi(c,c')^\eta, \alpha]$ for all $\alpha \in \mathfrak{a}$, $c, c' \in C$;*
- *$\langle c \mid c'\rangle c'' = \frac{1}{1-2n}\Bigl(\chi(c,c') - \chi(c,c')^\eta\Bigr).c''$
$+(-1)^{\bar{c}(\bar{c'}+\bar{c''})}\chi(c',c'')^\eta.c - (-1)^{\bar{c'}\bar{c''}}\chi(c,c'')^\eta c'$ for $c, c', c'' \in C$.*

Moreover, the $\mathrm{B}(0,n)$-graded Lie superalgebras for $n \geq 2$ are exactly these Lie superalgebras with the added constraint that
$$D = \langle A \mid A\rangle + \langle B \mid B\rangle + \langle C \mid C\rangle.$$

Remark 5.22. A Lie superalgebra $L = (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus(V\otimes C)\oplus D$ graded by the root system $\mathrm{B}(0,n)$ for $n\geq 2$ can be decomposed relative to the even subalgebra $\mathfrak{g}_{\bar 0}$ of $\mathfrak{g}$, which is a simple Lie algebra of type C_n, to obtain

$$L = (\mathfrak{g}_{\bar 0}\otimes A)\oplus(\widetilde{\mathfrak{s}}_{\bar 0}\otimes B)\oplus\big(V_{\bar 1}\otimes(A\oplus B\oplus C)\big)\oplus(B\oplus C\oplus D),$$

where $V_{\bar 1}$, (the space of odd elements of V) is the natural $2n$-dimensional module for $\mathfrak{g}_{\bar 0}$, and $\widetilde{\mathfrak{s}}_{\bar 0}$ is the subspace of $\mathfrak{s}_{\bar 0}$ of elements of zero trace. From this it is evident that L has a BC_n-grading with grading subalgebra of type C_n in the sense of [ABG2]. By passing to the Grassmann envelope, results from [ABG2] can be quoted to obtain that $\mathfrak{a} = A\oplus B$ is associative (but only when $n\geq 4$).

Remark 5.23. For $\mathfrak{b} = \mathfrak{a}\oplus C$, the coordinate superalgebra of a $B(0,n)$-graded Lie superalgebra with $n\geq 1$, we define $D_{\mathfrak{b},\mathfrak{b}}\subseteq \mathrm{Der}(\mathfrak{b})$ by the formulas

$$\begin{aligned}
(5.24)\quad & D_{\alpha,\alpha'}(\alpha'') = \tfrac{1}{2(1-2n)}[[\alpha,\alpha']-[\alpha,\alpha']^\eta,\alpha''] \quad \text{for all } \alpha,\alpha',\alpha''\in\mathfrak{a};\\
& D_{\alpha,\alpha'}(c) = \tfrac{1}{2(1-2n)}\Big([\alpha,\alpha']-[\alpha,\alpha']^\eta\Big).c \quad \text{for all } \alpha,\alpha'\in\mathfrak{a},\ \ c\in C;\\
& D_{c,c'}(\alpha) = \tfrac{1}{1-2n}[\chi(c,c')-\chi(c,c')^\eta,\alpha] \quad \text{for all } \alpha\in\mathfrak{a},\ c,c'\in C;\\
& D_{c,c'}(c'') = \tfrac{1}{1-2n}\Big(\chi(c,c')-\chi(c,c')^\eta\Big).c''\\
&\qquad\qquad +(-1)^{\bar c(\bar c'+\bar c'')}\chi(c',c'')^\eta.c-(-1)^{\bar c'\bar c''}\chi(c,c'')^\eta c',\ \ c,c',c''\in C.\\
& D_{\mathfrak{a},C} = 0 = D_{C,\mathfrak{a}}.
\end{aligned}$$

Then $L/Z(L)\cong\mathfrak{L}(\mathfrak{b})$ where

$$(5.25)\qquad \mathfrak{L}(\mathfrak{b}) \stackrel{\text{def}}{=} (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus(V\otimes C)\oplus D_{\mathfrak{b},\mathfrak{b}}.$$

The product on $\mathfrak{L}(\mathfrak{b})$ is that in (5.2) with $D_{\mathfrak{b},\mathfrak{b}}$ in place of D and with $\langle\beta\mid\beta'\rangle = D_{\beta,\beta'}$ for all $\beta,\beta'\in\mathfrak{b}$. As in Remark 4.14, $D_{\mathfrak{b},\mathfrak{b}}$ can be shown to be a Lie subsuperalgebra of $\mathfrak{gl}(\mathfrak{b})$; hence all the conditions of Theorem 5.21 are satisfied, and $\mathfrak{L}(\mathfrak{b})$ is a Lie superalgebra. Every $\mathrm{B}(0,n)$-graded Lie superalgebra with coordinate superalgebra $\mathfrak{b}$ is a cover of $\mathfrak{L}(\mathfrak{b})$.

§6. B(0,1)-graded Lie superalgebras

According to Theorem 3.3, a Lie superalgebra L graded by the root system $\mathrm{B}(0,1)$ of $\mathfrak{osp}(1,2)$ has a decomposition

$$L = (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus D$$

relative to its grading subalgebra $\mathfrak{g} = \mathfrak{osp}(1,2)$. Just as for $\mathrm{B}(0,n)$-graded superalgebras we have

- $\mathfrak{a} = A\oplus B$ is a unital superalgebra with $1\in A_{\bar 0}$ together with a superinvolution η such that $\eta\mid_A = \mathrm{I}$ and $\eta\mid_B = -\mathrm{I}$;
- D is a Lie subsuperalgebra of L acting on $\mathfrak{a}$ by superderivations such that $d(A)\subseteq A$, and $d(B)\subseteq B$;

- $\langle\ |\ \rangle : \mathfrak{a}\times\mathfrak{a}\to D$ is a bilinear supersymmetric map with $\langle A \mid B\rangle = 0$ such that
$$[d,\langle\alpha\mid\alpha'\rangle] = \langle d\alpha\mid\alpha'\rangle + (-1)^{\bar d\bar\alpha}\langle\alpha\mid d\alpha'\rangle$$
for $d\in D$, $\alpha,\alpha'\in\mathfrak{a}$.
- The multiplication in L is given by

(6.1)
$$\begin{aligned}
&[z\otimes\alpha, z'\otimes\alpha'] = (-1)^{\bar\alpha\bar z'}\Big([z,z']\otimes\frac{1}{2}(\alpha\circ\alpha') + z\circ z'\otimes\frac{1}{2}[\alpha,\alpha'] + \mathfrak{str}(zz')\langle\alpha\mid\alpha'\rangle\Big)\\
&[d, z\otimes\alpha] = (-1)^{\bar d\bar\alpha} z\otimes d\alpha\\
&[d,d'] \quad \text{(is the product in } D),
\end{aligned}$$

for all $z\otimes\alpha,\ z'\otimes\alpha'\in(\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$, $d\in D$, where the notation used is that of (5.3).

The properties of D mentioned above are those which result from applying the Jacobi superidentity with at least one element from D. We want to examine next what conditions are imposed by setting $\mathcal{J}(\ell_1,\ \ell_2,\ \ell_3)$ equal to 0 with the substitutions $\ell_i = z_i\otimes\alpha_i\in(\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$. The starting point is equations (5.4) and (5.5).

(i) The D-portion of $\mathcal{J}(z_1\otimes\alpha_1,\ z_2\otimes\alpha_2,\ z_3\otimes\alpha_3) = 0$ gives (5.4), or equivalently (5.6), $\quad 0 = \sum_{\circlearrowleft}(-1)^{\bar\alpha_1\bar\alpha_3}\langle\alpha_1\mid\alpha_2\alpha_3\rangle$.

(ii) The part of $\mathcal{J}(z_1\otimes\alpha_1,\ z_2\otimes\alpha_2,\ z_3\otimes\alpha_3)$ in $(\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\subseteq\mathfrak{gl}(1,2)\otimes\mathfrak{a}$ leads to equation (5.5). Our analysis of that equation amounts to determining the linear independence of the multilinear maps $\mathfrak{sl}(1,2)^3 = \mathfrak{sl}(1,2)\times\mathfrak{sl}(1,2)\times\mathfrak{sl}(1,2)\to\mathfrak{gl}(1,2)$ given below when they are restricted to $\mathfrak{g}^3$, $\mathfrak{g}^2\times\mathfrak{s}$, $\mathfrak{g}\times\mathfrak{s}^2$, and $\mathfrak{s}^3$:

Φ_1	Φ_2	Φ_3
$(-1)^{\bar z_1\bar z_3}z_1z_2z_3$	$(-1)^{\bar z_2\bar z_1}z_2z_3z_1$	$(-1)^{\bar z_3\bar z_2}z_3z_1z_2$
Ψ_1	Ψ_2	Ψ_3
$(-1)^{(\bar z_1+\bar z_2)\bar z_3}z_1z_3z_2$	$(-1)^{(\bar z_2+\bar z_3)\bar z_1}z_2z_1z_3$	$(-1)^{(\bar z_3+\bar z_1)\bar z_2}z_3z_2z_1$
Ω_1	Ω_2	Ω_3
$(-1)^{\bar z_1\bar z_3}\mathfrak{str}(z_1z_2)z_3$	$(-1)^{\bar z_2\bar z_1}\mathfrak{str}(z_2z_3)z_1$	$(-1)^{\bar z_3\bar z_2}\mathfrak{str}(z_3z_1)z_2$
Υ_1	Υ_2	
$(-1)^{\bar z_1\bar z_3}\mathfrak{str}(z_1z_2z_3)\,\mathrm{I}$	$(-1)^{(\bar z_1+\bar z_2)\bar z_3}\mathfrak{str}(z_1z_3z_2)\,\mathrm{I}$	

In this table, the image of a map is directly under it. Thus, for example, $\Psi_2(z_1,z_2,z_3) = (-1)^{(\bar z_2+\bar z_3)\bar z_1}z_2z_1z_3$.

(I) Assume first that $z_1,z_2,z_3\in\mathfrak{g}$. As in the analysis of (5.4), we know that $\Upsilon_2 = -\Upsilon_1$. Suppose a linear combination of these maps equals 0:

$$\sum_{i=1}^3\eta_i\Phi_i + \sum_{i=1}^3\theta_i\Psi_i + \sum_{i=1}^3\mu_i\Omega_i + \nu\Upsilon_1 = 0. \tag{6.2}$$

Then with $z_1 = E_{1,0} - E_{0,2}$, $z_2 = E_{2,1}$ and $z_3 = E_{1,2}$, the values of these maps are

Φ_1	Φ_2	Φ_3	Ψ_1	Ψ_2	Ψ_3	Ω_1	Ω_2	Ω_3	Υ_1
$-E_{0,2}$	0	0	0	0	$E_{1,0}$	0	$-(E_{1,0} - E_{0,2})$	0	0

while for $z_1 = z_2 = E_{1,0} - E_{0,2}$ and $z_3 = E_{2,1}$, the values are

Φ_1	Φ_2	Φ_3	Ψ_1	Ψ_2	Ψ_3	Ω_1	Ω_2	Ω_3	Υ_1
$-E_{1,1}$	$E_{0,0}$	$-E_{2,2}$	$-E_{0,0}$	$E_{1,1}$	$E_{2,2}$	0	0	0	I

From these values we obtain the following linear relations among the coefficients in (6.2):

1) $\eta_1 = \mu_2$, $\theta_3 = \mu_2$, and permuting these cyclically, $\eta_2 = \mu_3 = \theta_1$, $\eta_3 = \mu_1 = \theta_2$.

2) $-\eta_1 + \theta_2 + \nu = 0 = \eta_2 - \theta_1 + \nu = -\eta_3 + \theta_3 + \nu$ and their cyclic permutations.

Since $\eta_2 = \theta_1 = \mu_3$, we have $\nu = 0$ and $\eta_1 = \theta_2 = \eta_3 = \theta_3 = \mu_1 = \mu_2$, plus their cyclic permutations. Hence $\eta_1 = \eta_2 = \eta_3 = \theta_1 = \theta_2 = \theta_3 = \mu_1 = \mu_2 = \mu_3$ and $\nu = 0$. Thus, there is at most one linear dependence relation among the maps Φ_i's, Ψ_i's, Ω_i's and Υ_1 on $\mathfrak{g}$, namely:

$$\sum_{i=1}^{3} \Phi_i + \sum_{i=1}^{3} \Psi_i + \sum_{i=1}^{3} \Omega_i = 0, \tag{6.3}$$

which is easily checked to hold using some symbolic package like *Mathematica*.

Therefore, using (6.3) to express Ω_3 in terms of the other maps, and omitting the arguments of the maps for brevity, we see that because of (5.5) the part of the Jacobi superidentity $\mathcal{J}(z_1 \otimes a_1, z_2 \otimes a_2, z_3 \otimes a_3) = 0$ in $\mathfrak{gl}(1,2) \otimes \mathfrak{a}$ becomes

(6.4)

$$\begin{aligned}
0 = &- \Phi_1 \otimes \Big((-1)^{\bar{a_1}\bar{a_3}} (a_1, a_2, a_3) - (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&- \Phi_2 \otimes \Big((-1)^{\bar{a_2}\bar{a_1}} (a_2, a_3, a_1) - (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&- \Phi_3 \otimes \Big((-1)^{\bar{a_3}\bar{a_2}} (a_3, a_1, a_2) - (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&+ \Psi_1 \otimes \Big((-1)^{(\bar{a_1}+\bar{a_2})\bar{a_3}} (a_1, a_3, a_2) + (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&+ \Psi_2 \otimes \Big((-1)^{(\bar{a_2}+\bar{a_3})\bar{a_1}} (a_2, a_1, a_3) + (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&+ \Psi_3 \otimes \Big((-1)^{(\bar{a_3}+\bar{a_1})\bar{a_2}} (a_3, a_2, a_1) + (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&- \Omega_1 \otimes \Big((-1)^{\bar{a_1}\bar{a_3}} \big(\langle a_1 \mid a_2 \rangle + [[a_1, a_2], a_3] \big) \\
&\qquad\qquad\qquad - (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&- \Omega_2 \otimes \Big((-1)^{\bar{a_2}\bar{a_1}} \big(\langle a_2 \mid a_3 \rangle a_1 + [[a_2, a_3], a_1] \big) \\
&\qquad\qquad\qquad - (-1)^{\bar{a_3}\bar{a_2}} \big(\langle a_3 \mid a_1 \rangle a_2 + [[a_3, a_1], a_2] \big) \Big) \\
&+ \Upsilon_1 \otimes \sum_{\circlearrowleft} (-1)^{\bar{a_1}\bar{a_3}} [a_1, a_2 \circ a_3] \, .
\end{aligned}$$

By linear independence, the coefficients of $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Psi_3, \Omega_1, \Omega_2$ and Υ_1 must be 0, which allows us to deduce the following:

$$
\left\{
\begin{array}{l}
(a_1, a_2, a_3) \text{ is superskewsymmetric in its arguments} \\
(a_1, a_2, a_3) = \langle a_1 \mid a_2 \rangle a_3 + [[a_1, a_2], a_3] \quad \text{(coefficient of } \Phi_3) \\
\sum_{\circlearrowleft} (-1)^{\bar{a_1}\bar{a_3}} [a_1, a_2 \circ a_3] = 0
\end{array}
\right.
\tag{6.5}
$$

for any homogeneous $a_1, a_2, a_3 \in A$. But the last condition is a consequence of the first one, since $\sum_{\circlearrowleft} (-1)^{\bar{a_1}\bar{a_3}} [a_1, a_2 \circ a_3] = -\sum_{\circlearrowleft} (-1)^{\bar{a_1}\bar{a_3}} \Big((a_1, a_2, a_3) + (-1)^{\bar{a_2}\bar{a_3}} (a_1, a_3, a_2) \Big)$ in any superalgebra.

(II) Assume next that $z_1, z_2 \in \mathfrak{g}$ and $z_3 \in \mathfrak{s}$. Then $\mathfrak{str}(z_2 z_3) = 0 = \mathfrak{str}(z_3 z_1)$, so $\Omega_2 = \Omega_3 = 0$. Also $[z_2, z_3] \in \mathfrak{s}$, so $\mathfrak{str}(z_1 [z_2, z_3]) = 0$ and $\Upsilon_1 = \Upsilon_2$.

Again start with a trivial linear combination,

$$
\sum_{i=1}^{3} \eta_i \Phi_i + \sum_{i=1}^{3} \theta_i \Psi_i + \mu \Omega_1 + \nu \Upsilon_1 = 0, \tag{6.6}
$$

and substitute into it the following values of the z_i's:

a) $z_1 = E_{1,2}, \ z_2 = E_{2,1}, \ z_3 = E_{1,0} + E_{0,2}$,
b) $z_1 = E_{2,1}, \ z_2 = E_{1,0} - E_{0,2}, \ z_3 = E_{1,0} + E_{0,2}$.

The maps above then take the following values:

	Φ_1	Φ_2	Φ_3	Ψ_1	Ψ_2	Ψ_3	Ω_1	Υ_1
a)	$E_{1,0}$	0	0	0	0	$E_{0,2}$	$-(E_{1,0} + E_{0,2})$	0
b)	$E_{2,2}$	$E_{1,1}$	$-E_{0,0}$	$E_{2,2}$	$-E_{0,0}$	$E_{1,1}$	0	$-\mathrm{I}$

Thus:

1) $\eta_1 = \theta_3 = \mu$, and interchanging z_1 and z_2, $\theta_2 = \eta_3 = \mu$.

2) $\eta_1 + \theta_1 = \nu = \eta_2 + \theta_3 = -\eta_3 - \theta_2$.

Therefore $\eta_1 = \eta_3 = \theta_2 = \theta_3 = \mu$, $\theta_1 = \eta_2 = -3\mu$, $\nu = -2\mu$ and, up to scalars, there is at most one linear dependence relation, namely:

$$
\Phi_1 - 3\Phi_2 + \Phi_3 - 3\Psi_1 + \Psi_2 + \Psi_3 + \Omega_1 - 2\Upsilon_1 = 0, \tag{6.7}
$$

which again can be verified using *Mathematica*.

Hence, using (6.7) to express Ω_1 in terms of the other maps, the part of the Jacobi superidentity $\mathcal{J}(z_1\otimes a_1, z_2\otimes a_2, z_3\otimes b)=0$ in $\mathfrak{gl}(1,2)\otimes\mathfrak{a}$ becomes by (5.5):

(6.8)
$$\begin{aligned}
0 = &-\Phi_1\otimes\Bigl((-1)^{\bar{a_1}\bar{b}}(a_1,a_2,b)-(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&-\Phi_2\otimes\Bigl((-1)^{\bar{a_2}\bar{a_1}}(a_2,b,a_1)+3(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&-\Phi_3\otimes\Bigl((-1)^{\bar{b}\bar{a_2}}(b,a_1,a_2)-(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&+\Psi_1\otimes\Bigl((-1)^{(\bar{a_1}+\bar{a_2})\bar{b}}(a_1,b,a_2)-3(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&+\Psi_2\otimes\Bigl((-1)^{(\bar{a_2}+\bar{b})\bar{a_1}}(a_2,a_1,b)+(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&+\Psi_3\otimes\Bigl((-1)^{(\bar{b}+\bar{a_1})\bar{a_2}}(b,a_2,a_1)+(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)\\
&+\Upsilon_1\otimes\Bigl(\bigl(\sum_{\circlearrowleft}(-1)^{\bar{a_1}\bar{b}}[a_1,[a_2,b]]\bigr)-2(-1)^{\bar{a_1}\bar{b}}\bigl(\langle a_1\mid a_2\rangle b+[[a_1,a_2],b]\bigr)\Bigr)
\end{aligned}$$

and, therefore, for any homogeneous elements $a_1,a_2\in A$ and $b\in B$:

$$\begin{cases}(a_1,a_2,b)=-(-1)^{\bar{a_1}\bar{a_2}}(a_2,a_1,b)=(-1)^{(\bar{a_1}+\bar{a_2})\bar{b}}(b,a_1,a_2)\\ \qquad\qquad\;=-(-1)^{\bar{a_1}\bar{a_2}+\bar{a_1}\bar{b}+\bar{a_2}\bar{b}}(b,a_2,a_1)\\ (a_1,b,a_2)=3(-1)^{\bar{a_2}\bar{b}}(a_1,a_2,b)\\ \langle a_1\mid a_2\rangle b=(a_1,a_2,b)-[[a_1,a_2],b].\end{cases}\tag{6.9}$$

There is one additional condition,

$$\sum_{\circlearrowleft}(-1)^{\bar{a_1}\bar{b}}[a_1,[a_2,b]]=2(-1)^{\bar{a_1}\bar{b}}(a_1,a_2,b),$$

which is a consequence of the prior ones, since in any superalgebra its left-hand side is

$$-\sum_{\circlearrowleft}(-1)^{\bar{a_1}\bar{b}}\Bigl((a_1,a_2,b)-(-1)^{\bar{a_1}\bar{a_2}}(a_2,a_1,b)\Bigr),$$

which equals $2(-1)^{\bar{a_1}\bar{b}}(a_1,a_2,b)$ by the identities above.

(III) Assume now that $z_1,z_2\in\mathfrak{s}$ and $z_3\in\mathfrak{g}$. Then $\Omega_2=\Omega_3=0$ too and $\Upsilon_1=-\Upsilon_2$.

Starting with a trivial linear combination

$$\sum_{i=1}^{3}\eta_i\Phi_i+\sum_{i=1}^{3}\theta_i\Psi_i+\mu\Omega_1+\nu\Upsilon_1=0,\tag{6.10}$$

and taking the following values of the z_i's:

a) $z_1=E_{1,0}+E_{0,2},\ z_2=E_{2,0}-E_{0,1},\ z_3=E_{2,1}$,
b) $z_1=E_{2,0}-E_{0,1},\ z_2=E_{1,0}+E_{0,2},\ z_3=E_{2,1}$,
c) $z_1=z_2=E_{1,0}+E_{0,2},\ z_3=E_{2,1}$,
d) $z_1=2E_{0,0}+E_{1,1}+E_{2,2},\ z_2=E_{1,0}+E_{0,2},\ z_3=E_{2,1}$,

the maps above assume the following values:

	Φ_1	Φ_2	Φ_3	Ψ_1	Ψ_2	Ψ_3	Ω_1	Υ_1
a)	0	0	$-E_{2,1}$	0	$-E_{2,1}$	0	$2E_{2,1}$	0
b)	$E_{2,1}$	0	0	0	0	$E_{2,1}$	$-2E_{2,1}$	0
c)	$E_{1,1}$	$-E_{0,0}$	$E_{2,2}$	$E_{0,0}$	$-E_{1,1}$	$-E_{2,2}$	0	$-\mathrm{I}$
d)	$2E_{0,1}$	$E_{0,1}$	$E_{2,0}$	$E_{2,0}$	$E_{0,1}$	$2E_{2,0}$	0	0

Thus:

1) $\eta_3 + \theta_2 = 2\mu = \eta_1 + \theta_3$,

2) $\eta_1 - \theta_2 = \nu = -\eta_2 + \theta_1 = \eta_3 - \theta_3$,

3) $2\eta_1 + \eta_2 + \theta_2 = 0 = \eta_3 + \theta_1 + 2\theta_3$,

which can be solved in terms of η_1 and θ_2 giving:

$$\begin{gathered}\eta_2 = -2\eta_1 - \theta_2, \qquad \nu = \eta_1 - \theta_2, \qquad \theta_1 = -\eta_1 - 2\theta_2,\\ \eta_3 = \eta_1, \qquad \theta_3 = \theta_2, \qquad 2\mu = \eta_1 + \theta_2.\end{gathered}$$

Therefore, our zero linear combination (6.10) becomes:

$$\begin{aligned}0 = &\eta_1\big(\Phi_1 - 2\Phi_2 + \Phi_3 - \Psi_1 + \frac{1}{2}\Omega_1 + \Upsilon_1\big)\\ &+ \theta_2\big(- \Phi_2 - 2\Psi_1 + \Psi_2 + \Psi_3 + \frac{1}{2}\Omega_1 - \Upsilon_1\big).\end{aligned}$$

With *Mathematica* one confirms that indeed:

$$\begin{aligned}&\Phi_1 - 2\Phi_2 + \Phi_3 - \Psi_1 + \frac{1}{2}\Omega_1 + \Upsilon_1 = 0\\ &- \Phi_2 - 2\Psi_1 + \Psi_2 + \Psi_3 + \frac{1}{2}\Omega_1 - \Upsilon_1 = 0.\end{aligned} \tag{6.11}$$

Actually, the second relation above follows from the first one by interchanging z_1 and z_2 and multiplying by $(-1)^{\bar z_1 \bar z_2}$, so that Φ_1 becomes Ψ_2, Φ_2 becomes Ψ_1, Φ_3 becomes Ψ_3 and Ω_1 and Υ_1 remain fixed. Hence it is enough to check the first relation in (6.11).

Using (6.11) to express Φ_3 and Ψ_3 in terms of the other maps, equation (5.5) (the Jacobi superidentity) reduces to

(6.12)

$$
\begin{aligned}
0 = &- \Phi_1 \otimes \Big((-1)^{\bar{b_1}\bar{a}}(b_1,b_2,a) - (-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2)\Big)\\
&- \Phi_2 \otimes \Big((-1)^{\bar{b_1}\bar{b_2}}(b_2,a,b_1) + 2(-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2) - (-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1)\Big)\\
&+ \Psi_1 \otimes \Big((-1)^{(\bar{b_1}+\bar{b_2})\bar{a}}(b_1,a,b_2) - (-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2) + 2(-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1)\Big)\\
&+ \Psi_2 \otimes \Big((-1)^{(\bar{a}+\bar{b_2})\bar{b_1}}(b_2,b_1,a) - (-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1)\Big)\\
&- \Omega_1 \otimes \Big((-1)^{\bar{b_1}\bar{b_2}}\big(\langle b_1 \mid b_2\rangle a + [[b_1,b_2],a]\big) - \frac{1}{2}(-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2)\\
&\qquad\qquad + \frac{1}{2}(-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1)\Big)\\
&+ \Upsilon_1 \otimes \Big(\big(\sum_{\circlearrowleft}(-1)^{\bar{b_1}\bar{a}}[b_1,b_2\circ a]\big) + (-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2) + (-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1)\Big).
\end{aligned}
$$

This produces the conditions:

$$
\begin{cases}
(b_1,b_2,a) = (-1)^{(\bar{b_1}+\bar{b_2})\bar{a}}(a,b_2,b_1)\\
(b_1,a,b_2) = (-1)^{\bar{b_1}\bar{a}}(a,b_1,b_2) - 2(-1)^{\bar{b_1}(\bar{a}+\bar{b_2})}(a,b_2,b_1)\\
\langle b_1 \mid b_2\rangle a = \frac{1}{2}(b_1,b_2,a) - \frac{1}{2}(-1)^{\bar{b_1}\bar{b_2}}(b_2,b_1,a) - [[b_1,b_2],a]
\end{cases}
\tag{6.13}
$$

and a last condition

$$
\Big(\sum_{\circlearrowleft}(-1)^{\bar{b_1}\bar{a}}[b_1,b_2\circ a]\Big) + (-1)^{\bar{a}\bar{b_2}}(a,b_1,b_2) + (-1)^{(\bar{a}+\bar{b_1})\bar{b_2}}(a,b_2,b_1) = 0,
$$

which again can be derived from the previous ones.

(IV) Finally, assume that $z_i \in \mathfrak{s}$ for $i = 1,2,3$. In this case, $\Upsilon_1 = \Upsilon_2$. Setting the linear combination in (6.2) equal to 0 and making the following substitutions:

a) $z_1 = 2E_{00} + E_{1,1} + E_{2,2},\ z_2 = z_3 = E_{1,0} + E_{0,2},$

b) $z_1 = 2E_{00} + E_{1,1} + E_{2,2},\ z_2 = E_{1,0} + E_{0,2},\ z_3 = E_{2,0} - E_{0,1},$

we obtain the values:

	Φ_1	Φ_2	Φ_3	Ψ_1	Ψ_2
a)	$E_{1,2}$	$E_{1,2}$	$-2E_{1,2}$	$-E_{1,2}$	$2E_{1,2}$
b)	$2E_{0,0} - E_{1,1}$	$2E_{0,0} - E_{1,1}$	$E_{0,0} - 2E_{2,2}$	$2E_{0,0} - E_{2,2}$	$E_{0,0} - 2E_{1,1}$

	Ψ_3	Ω_1	Ω_2	Ω_3	Υ_1
a)	$-E_{1,2}$	0	0	0	0
b)	$2E_{0,0} - E_{2,2}$	0	$4E_{0,0} + 2E_{1,1} + 2E_{2,2}$	0	$3\,\mathrm{I}$

The substitution a) above gives us the linear equation

$$\eta_1 + \eta_2 - 2\eta_3 - \theta_1 + 2\theta_2 - \theta_3 = 0$$

but also two more equations obtained by cyclic symmetry. In the same way, the second substitution provides us three equations, and six more from cyclic symmetry. The nullspace of the homogeneous system given by these 12 equations is the linear span of

$$\begin{aligned}
&(10, -4, 10, -14, 0, 0, -7, 0, 0, 2)\\
&(4, 0, 2, -4, -2, 0, -1, 0, 1, 0)\\
&(0, -2, 2, -2, 2, 0, -1, 1, 0, 0)\\
&(-1, -1, -1, 1, 1, 1, 0, 0, 0, 0)
\end{aligned} \tag{6.14}$$

or in a more symmetric form, the linear span of

$$\begin{aligned}
&(1, 1, 1, 1, 1, 1, -7, -7, -7, 6)\\
&(2, -2, 0, -2, 0, 2, -1, 0, 1, 0)\\
&(0, 2, -2, 2, -2, 0, 1, -1, 0, 0)\\
&(1, 1, 1, -1, -1, -1, 0, 0, 0, 0)
\end{aligned}$$

Again, applying *Mathematica*, all these basic solutions can be shown to give valid relations among the maps involved, namely:

$$\Phi_1 + \Phi_2 + \Phi_3 + \Psi_1 + \Psi_2 + \Psi_3 - 7(\Omega_1 + \Omega_2 + \Omega_3) + 6\Upsilon_1 = 0, \tag{6.15}$$

$$2(\Phi_1 - \Phi_2 - \Psi_1 + \Psi_3) - \Omega_1 + \Omega_3 = 0, \tag{6.16}$$

$$2(\Phi_2 - \Phi_3 + \Psi_1 - \Psi_2) + \Omega_1 - \Omega_2 = 0,$$

$$\Phi_1 + \Phi_2 + \Phi_3 - (\Psi_1 + \Psi_2 + \Psi_3) = 0. \tag{6.17}$$

The third relation above is obtained by cyclically permuting the arguments in (6.16), so it does not give anything new.

Then from (6.14) it follows that:

$$\begin{cases}
2\Upsilon_1 = -10\Phi_1 + 4\Phi_2 - 10\Phi_3 + 14\Psi_1 - 7\Omega_1,\\
\Omega_3 = -4\Phi_1 - 2\Phi_3 + 4\Psi_1 + 2\Psi_2 + \Omega_1,\\
\Omega_2 = 2\Phi_2 - 2\Phi_3 + 2\Psi_1 - 2\Psi_2 + \Omega_1,\\
\Psi_3 = \Phi_1 + \Phi_2 + \Phi_3 - \Psi_1 - \Psi_2,
\end{cases}$$

so (5.5) becomes here

(6.18)

$$
\begin{aligned}
0 = &- \Phi_1 \otimes \Big((-1)^{\bar{b_1}\bar{b_3}} (b_1, b_2, b_3) - (-1)^{(\bar{b_1}+\bar{b_3})\bar{b_2}} (b_3, b_2, b_1) \\
&\qquad - 4(-1)^{\bar{b_2}\bar{b_3}} \big(\langle b_3 \mid b_1 \rangle b_2 + [[b_3, b_1], b_2] \big) + 5 \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] \Big) \\
&- \Phi_2 \otimes \Big((-1)^{\bar{b_2}\bar{b_1}} (b_2, b_3, b_1) - (-1)^{(\bar{b_1}+\bar{b_3})\bar{b_2}} (b_3, b_2, b_1) \\
&\qquad + 2(-1)^{\bar{b_1}\bar{b_2}} \big(\langle b_2 \mid b_3 \rangle b_1 + [[b_2, b_3], b_1] \big) - 2 \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] \Big) \\
&- \Phi_3 \otimes \Big((-1)^{\bar{b_3}\bar{b_2}} (b_3, b_1, b_2) - (-1)^{(\bar{b_1}+\bar{b_3})\bar{b_2}} (b_3, b_2, b_1) \\
&\qquad - 2(-1)^{\bar{b_1}\bar{b_2}} \big(\langle b_2 \mid b_3 \rangle b_1 + [[b_2, b_3], b_1] \big) \\
&\qquad\quad - 2(-1)^{\bar{b_2}\bar{b_3}} \big(\langle b_3 \mid b_1 \rangle b_2 + [[b_3, b_1], b_2] \big) + 5 \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] \Big) \\
&+ \Psi_1 \otimes \Big((-1)^{(\bar{b_1}+\bar{b_2})\bar{b_3}} (b_1, b_3, b_2) - (-1)^{(\bar{b_1}+\bar{b_3})\bar{b_2}} (b_3, b_2, b_1) \\
&\qquad - 2(-1)^{\bar{b_1}\bar{b_2}} \big(\langle b_2 \mid b_3 \rangle b_1 + [[b_2, b_3], b_1] \big) \\
&\qquad\quad - 4(-1)^{\bar{b_2}\bar{b_3}} \big(\langle b_3 \mid b_1 \rangle b_2 + [[b_3, b_1], b_2] \big) + 7 \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] \Big) \\
&+ \Psi_2 \otimes \Big((-1)^{\bar{b_1}(\bar{b_2}+\bar{b_3})} (b_2, b_1, b_3) - (-1)^{(\bar{b_1}+\bar{b_3})\bar{b_2}} (b_3, b_2, b_1) \\
&\qquad\qquad + 2(-1)^{\bar{b_1}\bar{b_2}} \big(\langle b_2 \mid b_3 \rangle b_1 + [[b_2, b_3], b_1] \big) \\
&\qquad\qquad\quad - 2(-1)^{\bar{b_2}\bar{b_3}} \big(\langle b_3 \mid b_1 \rangle b_2 + [[b_3, b_1], b_2] \big) \Big) \\
&- \Omega_1 \otimes \Big((-1)^{\bar{b_1}\bar{b_3}} \big(\langle b_1 \mid b_2 \rangle b_3 + [[b_1, b_2], b_3] \big) + (-1)^{\bar{b_1}\bar{b_2}} \big(\langle b_2 \mid b_3 \rangle b_1 + [[b_2, b_3], b_1] \big) \\
&\qquad + (-1)^{\bar{b_2}\bar{b_3}} \big(\langle b_3 \mid b_1 \rangle b_2 + [[b_3, b_1], b_2] \big) + \frac{7}{2} \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] \Big)
\end{aligned}
$$

The linear independence of $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2$ and Ω_1 imply that all the coefficients in (6.18) must be trivial. Now we may use the fact that in any superalgebra

$$
\sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} [b_1, [b_2, b_3]] = - \sum_{\circlearrowleft} (-1)^{\bar{b_1}\bar{b_3}} \Big((b_1, b_2, b_3) - (-1)^{\bar{b_2}\bar{b_3}} (b_1, b_3, b_2) \Big)
$$

and label things as follows:

$$
\begin{aligned}
X_1 &= (-1)^{\bar{b_1}\bar{b_3}}\big(\langle b_1 \mid b_2\rangle b_3 + [[b_1,b_2],b_3]\big),\\
X_2 &= (-1)^{\bar{b_2}\bar{b_1}}\big(\langle b_2 \mid b_3\rangle b_1 + [[b_2,b_3],b_1]\big),\\
X_3 &= (-1)^{\bar{b_3}\bar{b_2}}\big(\langle b_3 \mid b_1\rangle b_2 + [[b_3,b_1],b_2]\big),\\
Y_1 &= (-1)^{(\bar{b_1}+\bar{b_2})\bar{b_3}}(b_1,b_3,b_2),\\
Y_2 &= (-1)^{(\bar{b_2}+\bar{b_3})\bar{b_1}}(b_2,b_1,b_3),\\
Y_3 &= (-1)^{(\bar{b_3}+\bar{b_1})\bar{b_2}}(b_3,b_2,b_1),\\
Z_1 &= (-1)^{\bar{b_1}\bar{b_3}}(b_1,b_2,b_3),\\
Z_2 &= (-1)^{\bar{b_2}\bar{b_1}}(b_2,b_3,b_1),\\
Z_1 &= (-1)^{\bar{b_3}\bar{b_2}}(b_3,b_1,b_2).
\end{aligned}
$$

Then the annihilation of the coefficient of Φ_1 in (6.18) gives

$$-4X_3 + 5Y_1 + 5Y_2 + 4Y_3 - 4Z_1 - 5Z_2 + 5Z_3 = 0,$$

and so all the cyclic permutations are equal to 0.

Proceeding in the same way with the other coefficients and computing the row reduced echelon form of the coefficient matrix obtained, it can be shown that the relationships among the X's, Y's and Z's are all consequences of

$$
\begin{aligned}
6X_1 + Z_1 - 2Z_2 + Z_3 &= 0\\
3Y_1 - 2Z_1 + Z_2 - 2Z_3 &= 0
\end{aligned}
$$

and their cyclic permutations. Therefore, the $\mathfrak{gl}(1,2)\otimes\mathfrak{a}$ portion of the Jacobi superidentity $\mathcal{J}(z_1\otimes b_1, z_2\otimes b_2, z_3\otimes b_3) = 0$ is valid in this case if and only if

$$
(6.19)\qquad
\begin{cases}
6\big(\langle b_1 \mid b_2\rangle b_3 + [[b_1,b_2],b_3]\big) + (b_1,b_2,b_3)\\
\qquad\qquad - 2(-1)^{\bar{b_1}(\bar{b_2}+\bar{b_3})}(b_2,b_3,b_1) + (-1)^{(\bar{b_1}+\bar{b_2})\bar{b_3}}(b_3,b_1,b_2) = 0\\
3(-1)^{\bar{b_2}\bar{b_3}}(b_1,b_3,b_2) = 2(b_1,b_2,b_3) - (-1)^{\bar{b_1}(\bar{b_2}+\bar{b_3})}(b_2,b_3,b_1)\\
\qquad\qquad + 2(-1)^{(\bar{b_1}+\bar{b_2})\bar{b_3}}(b_3,b_1,b_2)
\end{cases}
$$

We arrive at the main theorem

Theorem 6.20. *The Lie superalgebras L graded by the root system* $\mathrm{B}(0,1)$ *of* $\mathfrak{g} = \mathfrak{osp}(1,2)$ *are up to isomorphism the Lie superalgebras*

$$L = (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus D$$

with multiplication given by

$$[z\otimes\alpha, z'\otimes\alpha'] = (-1)^{\bar\alpha\bar z'}\Big([z,z']\otimes\frac{1}{2}\alpha\circ\alpha' + z\circ z'\otimes\frac{1}{2}[\alpha,\alpha'] + \mathfrak{str}(zz')\langle\alpha\mid\alpha'\rangle\Big)$$
$$[d, z\otimes\alpha] = (-1)^{\bar d\bar z} z\otimes d(\alpha)$$

where $z\otimes\alpha$, $z'\otimes\alpha'$ *are homogeneous elements in* $(\mathfrak{g}\otimes\alpha)\cup(\mathfrak{s}\cup B)$, $d\in D$; $z\circ z' = zz' + (-1)^{\bar z\bar z'}z'z + 2\mathfrak{str}(zz')1$*; and*

- $\mathfrak{a} = A\oplus B$ *is a unital superalgebra with* $1\in A_{\bar 0}$ *together with a superinvolution* η *such that* $\eta\mid_A = \mathrm{I}$ *and* $\eta\mid_B = -\mathrm{I}$*;*

- D *is a Lie subsuperalgebra of* L *acting on* $\mathfrak{a}$ *by superderivations commuting with* η *by means of the action* $(d,\alpha)\mapsto d\alpha$*;*

- $\langle\ \mid\ \rangle : \mathfrak{a}\times\mathfrak{a}\to D$ *is a bilinear supersymmetric map with* $\langle A\mid B\rangle = 0$ *such that* $D = \langle A\mid A\rangle + \langle B\mid B\rangle$ *and*

$$[d,\langle\alpha\mid\alpha'\rangle] = \langle d\alpha\mid\alpha'\rangle + (-1)^{\bar d\bar\alpha}\langle\alpha\mid d\alpha'\rangle$$

for $d\in D$, $\alpha,\alpha'\in\mathfrak{a}$

- *Relative to the associator* $(\ ,\ ,\ )$ *on* $\mathfrak{a}$*, for any homogeneous elements* $a,a',a''\in A$ *and* $b,b',b''\in B$*:*

  (a,a',a'') *is superskew (in particular,* $(A,\circ)$ *is a Jordan algebra),*

$$\langle a\mid a'\rangle(a'') = (a,a',a'') - [[a,a'],a'']$$
$$\begin{aligned}(a,a',b) &= -(-1)^{\bar a\bar a'}(a',a,b) = (-1)^{(\bar a+\bar a')\bar b}(b,a,a')\\ &= -(-1)^{\bar a\bar a'+\bar a\bar b+\bar a'\bar b}(b,a',a)\end{aligned}$$
$$(a,b,a') = 3(-1)^{\bar a'\bar b}(a,a',b)$$
$$\langle a\mid a'\rangle(b) = (a,a',b) - [[a,a'],b]$$
$$(b,b',a) = (-1)^{(\bar b+\bar b')\bar a}(a,b,b')$$
$$(b,a,b') = (-1)^{\bar b\bar a}(a,b,b') - 2(-1)^{\bar b(\bar a+\bar b')}(a,b',b)$$
$$\langle b\mid b'\rangle(a) = \tfrac{1}{2}(b,b',a) - \tfrac{1}{2}(-1)^{\bar b\bar b'}(b',b,a) - [[b,b'],a]$$
$$\begin{aligned}\langle b\mid b'\rangle(b'') = \tfrac{1}{6}\Big(&-(b,b',b'') + 2(-1)^{\bar b(\bar b'+\bar b'')}(b',b'',b) - (-1)^{(\bar b+\bar b')\bar b''}(b'',b,b')\Big)\\ &-[[b,b'],b'']\end{aligned}$$
$$\begin{aligned}3(-1)^{\bar b'\bar b''}(b,b'',b') = 2(b,b',b'') &- (-1)^{\bar b(\bar b'+\bar b'')}(b',b'',b)\\ &+2(-1)^{(\bar b+\bar b')\bar b''}(b'',b,b')\end{aligned}$$

In the course of our computations proving Theorem 6.20, we have derived relations (6.3), (6.7), (6.11), (6.15), (6.16) and (6.17), which show that the following identities hold for any homogeneous elements $x_i\in\mathfrak{g}$, $s_i\in\mathfrak{s}$ $(i=1,2,3)$:

(I1)
$$\sum_{\circlearrowleft}(-1)^{\bar{x_1}\bar{x_3}}\Big(x_1x_2x_3+(-1)^{\bar{x_2}\bar{x_3}}x_1x_3x_2+\mathfrak{str}(x_1x_2)x_3\Big)=0$$

(I2)
$$\begin{aligned}&x_1x_2s_3-3(-1)^{\bar{x_1}(\bar{x_2}+\bar{s_3})}x_2s_3x_1+(-1)^{(\bar{x_1}+\bar{x_2})\bar{s_3}}s_3x_1x_2-3(-1)^{\bar{x_2}\bar{s_3}}x_1s_3x_2\\&\quad+(-1)^{\bar{x_1}\bar{x_2}}x_2x_1s_3+(-1)^{\bar{x_1}\bar{x_2}+\bar{x_1}\bar{s_3}+\bar{x_2}\bar{s_3}}s_3x_2x_1+\mathfrak{str}(x_1x_2)s_3-2\mathfrak{str}(x_1x_2s_3)\,\mathrm{I}=0\end{aligned}$$

(I3)
$$\begin{aligned}&s_1s_2x_3-2(-1)^{\bar{s_1}(\bar{s_2}+\bar{x_3})}s_2x_3s_1+(-1)^{(\bar{s_1}+\bar{s_2})\bar{x_3}}x_3s_1s_2-(-1)^{\bar{s_2}\bar{x_3}}s_1x_3s_2\\&\qquad\qquad+\frac{1}{2}\mathfrak{str}(s_1s_2)x_3+\mathfrak{str}(s_1s_2x_3)\,\mathrm{I}=0\end{aligned}$$

(I4)
$$\Big(\sum_{\circlearrowleft}(-1)^{\bar{s_1}\bar{s_3}}\Big(s_1s_2s_3+(-1)^{\bar{s_2}\bar{s_3}}s_1s_3s_2-7\mathfrak{str}(s_1s_2)s_3\Big)\Big)+6\mathfrak{str}(s_1s_2s_3)\,\mathrm{I}=0$$

(I5)
$$\begin{aligned}&2\Big(s_1s_2s_3-(-1)^{\bar{s_1}(\bar{s_2}+\bar{s_3})}s_2s_3s_1-(-1)^{\bar{s_2}\bar{s_3}}s_1s_3s_2+(-1)^{\bar{s_1}\bar{s_2}+\bar{s_1}\bar{s_3}+\bar{s_2}\bar{s_3}}s_3s_2s_1\Big)\\&\qquad\qquad-\mathfrak{str}(s_1s_2)s_3+(-1)^{\bar{s_2}\bar{s_3}}\mathfrak{str}(s_1s_3)s_2=0\end{aligned}$$

(I6)
$$\sum_{\circlearrowleft}(-1)^{\bar{s_1}\bar{s_3}}\Big(s_1s_2s_3-(-1)^{\bar{s_2}\bar{s_3}}s_1s_3s_2\Big)=0.$$

Remark 6.21. As in Remarks 4.14 and 5.23, given a superalgebra $\mathfrak{a}=A\oplus B$ satisfying the conditions in Theorem 6.20, a Lie subsuperalgebra $D_{\mathfrak{a},\mathfrak{a}}\subseteq\mathrm{Der}(\mathfrak{a})$ can be defined by the formulas

(6.22)
$$\begin{aligned}&D_{a,a'}(\alpha)=(a,a',\alpha)-[[a,a'],\alpha]\quad\text{for all } a,a'\in A,\ \alpha\in\mathfrak{a};\\&D_{b,b'}(a)=\frac{1}{2}(b,b',a)-\frac{1}{2}(-1)^{\bar{b}\bar{b}'}(b',b,a)-[[b,b'],a]\quad\text{for all } b,b'\in B,\ a\in A;\\&D_{b,b'}(b'')=\frac{1}{6}\Big(-(b,b',b'')+2(-1)^{\bar{b}(\bar{b}'+\bar{b}'')}(b',b'',b)-(-1)^{(\bar{b}+\bar{b}')\bar{b}''}(b'',b,b')\Big)\\&\qquad\qquad-[[b,b'],b'']\quad\text{for all } b,b'b''\in B;\\&D_{A,B}=0=D_{B,A}.\end{aligned}$$

Then, if $\mathfrak{a}$ is the coordinate superalgebra of a B(0,1)-graded Lie superalgebra L, $L/Z(L)\cong\mathfrak{L}(\mathfrak{a})$ where

$$\mathfrak{L}(\mathfrak{a})\stackrel{\text{def}}{=}(\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus D_{\mathfrak{a},\mathfrak{a}}.\tag{6.23}$$

The product on $\mathfrak{L}(\mathfrak{a})$ is that in (6.1) with $D_{\mathfrak{a},\mathfrak{a}}$ in place of D and with $\langle\alpha\mid\alpha'\rangle=D_{\alpha,\alpha'}$ for all $\alpha,\alpha'\in\mathfrak{a}$. Thus every B(0,1)-graded Lie superalgebra with coordinate superalgebra $\mathfrak{a}$ is a cover of $\mathfrak{L}(\mathfrak{a})$.

§7. Models of the B(m,n)-graded Lie superalgebras

We conclude the paper by describing a realization of B(m,n)-graded Lie superalgebras as unitary Lie superalgebras associated with hermitian forms. This construction yields all B(m,n)-graded Lie superalgebras up to central extension except for the B$(0,1)$-graded Lie superalgebras whose coordinate superalgebra $\mathfrak{a}$ is not associative. At the end, we present some explicit examples of B$(0,1)$-graded Lie superalgebras (namely $\mathfrak{psl}(2,2)$ and forms of the exceptional simple Lie superalgebra F(4)) with $\mathfrak{a}$ not associative to illustrate that such superalgebras do in fact exist. Seligman [S] applied an analogue of this unitary construction to obtain models of the finite-dimensional simple Lie algebras which are BC_r-graded for $r \geq 3$. In [ABG2], it was shown that all BC_r-graded Lie algebras for $r \geq 3$ (except those whose grading subalgebra is type C_3) can be realized as unitary Lie algebras. Algebraic group versions of this construction have played an important role in describing finite-dimensional simple algebraic groups over nonalgebraically closed fields (see [T]).

Let V be the superspace considered in Section 3 so that $\dim V = N = 2m+1+2n$. Suppose $\mathfrak{a} = A \oplus B$ is a unital associative superalgebra over $\mathbb{F}$ with a superinvolution η such that $\eta\,|_A = \mathrm{I}$ and $\eta\,|_B = -\mathrm{I}$. Let C be a superspace which is a unital right $\mathfrak{a}$-module with action ".". (In previous sections we have worked with left $\mathfrak{a}$-modules, but using η it is easy to pass from a right $\mathfrak{a}$-module C to a left $\mathfrak{a}$-module C by defining $\alpha \cdot c = (-1)^{\bar{\alpha}\bar{c}} c \cdot \alpha^\eta$ for $\alpha \in \mathfrak{a}$ and $c \in C$.) We assume C is equipped with an η-superhermitian form $\chi : C \times C \to \mathfrak{a}$ (compare 5.19):

$$
\begin{aligned}
&\chi(c', c) = (-1)^{\bar{c}\bar{c'}} \chi(c, c')^\eta \\
&\chi(c.\alpha, c') = (-1)^{\bar{\alpha}\bar{c}} \alpha^\eta \chi(c, c') \\
&\chi(c, c'.\alpha) = \chi(c, c')\alpha.
\end{aligned}
\tag{7.1}
$$

Then we may identify $\mathrm{End}_{\mathfrak{a}}\big((V \otimes \mathfrak{a}) \oplus C\big)$ in a natural way with the algebra of 2×2 matrices

$$
\mathcal{E} = \begin{pmatrix} \mathrm{End}_{\mathfrak{a}}(V \otimes \mathfrak{a}) & \mathrm{Hom}_{\mathfrak{a}}(C, V \otimes \mathfrak{a}) \\ \mathrm{Hom}_{\mathfrak{a}}(V \otimes \mathfrak{a}, C) & \mathrm{End}_{\mathfrak{a}}(C) \end{pmatrix}
\tag{7.2}
$$

whose components have the following realizations:

$$
\begin{aligned}
\mathrm{End}_{\mathbb{F}}(V) \otimes \mathfrak{a} &\cong \mathrm{End}_{\mathfrak{a}}(V \otimes \mathfrak{a}) \\
z \otimes \alpha &\mapsto M_{z \otimes \alpha} \big(\, : v \otimes \alpha' \mapsto (-1)^{\bar{\alpha}\bar{v}} z.v \otimes \alpha\alpha' \,\big) \\
V \otimes C^* &\cong \mathrm{Hom}_{\mathfrak{a}}(C, V \otimes \mathfrak{a}) \\
v \otimes \lambda &\mapsto Y_{v \otimes \lambda} \big(\, : c \mapsto v \otimes \lambda(c) \big) \\
V \otimes C &\cong \mathrm{Hom}_{\mathfrak{a}}(V \otimes \mathfrak{a}, C) \\
v \otimes c &\mapsto X_{v \otimes c} \big(\, : u \otimes \alpha \mapsto (-1)^{\bar{c}\bar{u}} (v \mid u) c.\alpha \big)
\end{aligned}
\tag{7.3}
$$

where $C^* = \mathrm{Hom}_{\mathfrak{a}}(C, \mathfrak{a})$ carries a natural left $\mathfrak{a}$-module (and a right $\mathrm{End}_{\mathfrak{a}}(C)$-module) structure.

The composition of maps in $\mathrm{End}_{\mathfrak{a}}\big((V\otimes\mathfrak{a})\oplus C\big)$ becomes the multiplication given by

$$\begin{pmatrix} M_{z\otimes\alpha} & Y_{v\otimes\lambda}\\ X_{w\otimes c} & P\end{pmatrix}\begin{pmatrix} M_{z'\otimes\alpha'} & Y_{v'\otimes\lambda'}\\ X_{w'\otimes c'} & P'\end{pmatrix} =$$
$$\begin{pmatrix} (-1)^{\bar\alpha\bar z'}M_{zz'\otimes\alpha\alpha'}+(-1)^{\bar\lambda\bar w'}M_{v(w'|-)\otimes\lambda(c')} & (-1)^{\bar\alpha\bar z'}Y_{z.v'\otimes\alpha\lambda'}+Y_{v\otimes\lambda P}\\ (-1)^{(\bar w+\bar c)\bar z'}X_{(z')^*w\otimes c.\alpha}+(-1)^{\bar P\bar w'}X_{w'\otimes P(c)} & (-1)^{\bar c\bar v'}(w\mid v')c.\lambda'+PP'\end{pmatrix},$$

where $z^*\in\mathrm{End}_{\mathbb{F}}(V)$ and $c.\lambda$ are defined by

$$\begin{cases} (v\mid z.w)+(-1)^{\bar z\bar v}(z^*.v\mid w)=0 & \forall v,w\in V,\ \forall z\in\mathrm{End}_{\mathbb{F}}(V)\\ (c.\lambda)(d)=c.\lambda(d) & \forall c,d\in C,\ \forall\lambda\in C^*.\end{cases}$$

We consider the η-hermitian superform on $V\otimes\mathfrak{a}$ given by

$$\omega(u\otimes\alpha,v\otimes\beta)=(-1)^{\bar\alpha\bar v}(u\mid v)\alpha^\eta\beta,$$

and the η-hermitian superform $\xi:=\omega\perp-\chi$ on $(V\otimes\mathfrak{a})\oplus C$. Then

(7.4)
$$\begin{aligned}\mathfrak{U}=\mathfrak{u}(\xi)\overset{\text{def}}{=}\{T\in\mathrm{End}_{\mathfrak{a}}\big((V\otimes\mathfrak{a})\oplus C\big)\mid \xi(T\underline{z},\underline{z}')+(-1)^{\bar T\bar z}\xi(\underline{z},T\underline{z}')=0\\ \forall\underline{z},\underline{z}'\in(V\otimes\mathfrak{a})\oplus C\},\end{aligned}$$

is a Lie subsuperalgebra of $\mathrm{End}_{\mathfrak{a}}\big((V\otimes\mathfrak{a})\oplus C\big)$ under the supercommutator $[T,T']=TT'-(-1)^{\bar T\bar T'}T'T$, called the *unitary Lie superalgebra* of $\xi=\omega\perp-\chi$.

Writing $T=\begin{pmatrix} M & Y\\ X & P\end{pmatrix}$ as before, we deduce that

(i) $$\begin{aligned}M\in\mathfrak{u}(\omega)&=\{E\in\mathrm{End}_{\mathfrak{a}}(V\otimes\mathfrak{a})\mid\omega(E\underline{z},\underline{z}')+(-1)^{\bar E\bar z}\omega(\underline{z},E\underline{z}')=0\\ &\qquad\forall\underline{z},\underline{z}'\in V\otimes\mathfrak{a}\}\\ &=M_{\mathfrak{g}\otimes A}\oplus M_{\mathfrak{s}\otimes B}\oplus M_{\mathrm{I}\otimes B}.\end{aligned}$$

(ii) $P\in\mathfrak{u}(\chi)=\{F\in\mathrm{End}_{\mathfrak{a}}(C)\mid\omega(F\underline{z},\underline{z}')+(-1)^{\bar F\bar z}\omega(\underline{z},F\underline{z}')=0\ \forall\underline{z},\underline{z}'\in C\}$.

(iii) $\chi(X\underline{z},\underline{z}')-(-1)^{\bar T\bar{\underline{z}}}\omega(\underline{z},Y\underline{z}')=0\quad\forall\underline{z}\in V\otimes\mathfrak{a},\ \underline{z}'\in C$ (and the same happens when $\underline{z}\in C$ and $\underline{z}'\in V\otimes\mathfrak{a}$).

It follows from the nondegeneracy of ω that for any $X\in\mathrm{Hom}_{\mathfrak{a}}(V\otimes\mathfrak{a},C)$, there is a unique $Y\in\mathrm{Hom}_{\mathfrak{a}}(C,V\otimes\mathfrak{a})$ satisfying (iii). Moreover, when $X=X_{v\otimes c}$ in (iii), then $Y=Y_{v\otimes\chi_c}$, where $\chi_c=\chi(c,-)\in C^*$.

For any homogeneous $v\in V$ and $c\in C$, let

$$T_{v\otimes c}=\begin{pmatrix} 0 & Y_{v\otimes\chi_c}\\ X_{v\otimes c} & 0\end{pmatrix}$$

and identify any $M \in \mathrm{End}_{\mathfrak{a}}(V \otimes \mathfrak{a})$ with $\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}_{\mathfrak{a}}\big((V \otimes \mathfrak{a}) \oplus C\big)$ (resp. $P \in \mathrm{End}_{\mathfrak{a}}(C)$ with $\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$).

Notice that $\mathfrak{g}$ is embedded in $\mathfrak{U}$ as $M_{\mathfrak{g}\otimes 1}$. Then items (i)–(iii) above show that $\mathfrak{U}$ decomposes as a $\mathfrak{g}$-module into a direct sum

$$\begin{aligned} \mathfrak{U} &= M_{\mathfrak{g}\otimes A} \oplus M_{\mathfrak{s}\otimes B} \oplus T_{V\otimes C} \oplus D_{\mathfrak{U}} \\ &\cong (\mathfrak{g}\otimes A) \oplus (\mathfrak{s}\otimes B) \oplus (V \otimes C) \oplus D_{\mathfrak{U}}, \end{aligned}$$

where

$$D_{\mathfrak{U}} = \begin{pmatrix} M_{\mathrm{I}\otimes B} & 0 \\ 0 & \mathfrak{u}(\chi) \end{pmatrix}$$

is a trivial module for $\mathfrak{g}$.

Note that for $z\otimes\alpha, z'\otimes\alpha' \in (\mathfrak{g}\otimes A)\cup(\mathfrak{s}\otimes B)$, $v, v' \in V$ and $c, c' \in C$:

(7.5)

$$\begin{aligned} [M_{z\otimes\alpha}, M_{z'\otimes\alpha'}] &= (-1)^{\bar{\alpha}\bar{z}'}\Big(M_{zz'\otimes\alpha\alpha'} - (-1)^{\bar{z}\bar{z}'+\bar{\alpha}\bar{\alpha}'} M_{z'z\otimes\alpha'\alpha}\Big) \\ &= (-1)^{\bar{\alpha}\bar{z}'}\Big(M_{[z,z']\otimes\frac12(\alpha\circ\alpha')} + M_{z\circ z'\otimes\frac12[\alpha,\alpha']} + \frac{\mathfrak{str}(zz')}{2m+1-2n} M_{\mathrm{I}\otimes[\alpha,\alpha']}\Big) \\ [M_{z\otimes\alpha}, T_{v\otimes c}] &= (-1)^{\bar{\alpha}\bar{v}} T_{z.v\otimes\alpha.c} \\ [T_{v\otimes c}, T_{v'\otimes c'}] &= (-1)^{\bar{c}\bar{v}'}\Big(M_{v(v'|-)\otimes\chi(c,c')} + (v\mid v')c.\chi_{c'} \\ &\qquad - (-1)^{\bar{c}\bar{c}'+\bar{v}\bar{v}'}\big(M_{v'(v|-)\otimes\chi(c,c')} + (v'\mid v)c'.\chi_c\big) \\ &= (-1)^{\bar{c}\bar{v}'}\Big(\big(M_{v(v'|-)\otimes\chi(c,c')} - (-1)^{\bar{v}\bar{v}'} M_{v'(v|-)\otimes\chi(c,c')^\eta}\big) \\ &\qquad + (v\mid v')\big(c.\chi_{c'} - (-1)^{\bar{c}\bar{c}'} c'.\chi_c\big)\Big) \\ &= (-1)^{\bar{c}\bar{v}'}\Big(\big(M_{\gamma_{v,v'}\otimes\frac12(\chi(c,c')+\chi(c,c')^\eta)} + M_{\tilde{\sigma}_{v,v'}\otimes\frac12(\chi(c,c')-\chi(c,c')^\eta)}\big) \\ &\qquad + (v\mid v')\big(c.\chi_{c'} - (-1)^{\bar{c}\bar{c}'} c'.\chi_c\big) \\ &= (-1)^{\bar{c}\bar{v}'}\Big(\big(M_{\gamma_{v,v'}\otimes\chi^+(c,c')} + M_{\sigma_{v,v'}\otimes\chi^-(c,c')}\big) \\ &\qquad + (v\mid v')\chi_{c,c'} + \frac{2(v\mid v')}{2m+1-2n} M_{\mathrm{I}\otimes\chi^-(c,c')}\Big) \end{aligned}$$

where $\chi^+(c,c') = \frac12\big(\chi(c,c') + \chi(c,c')^\eta\big)$, $\chi^-(c,c') = \frac12\big(\chi(c,c') - \chi(c,c')^\eta\big)$ and $\chi_{c,c'} = c.\chi_{c'} - (-1)^{\bar{c}\bar{c}'} c'.\chi_c$.

In order for $\mathfrak{U}$ to be B(m,n)-graded when $m \geq 1$, we must have $\eta = \mathrm{I}$ and $B = 0$ (so no $\mathfrak{s}\otimes B$ term occurs). Even if we impose that condition when $m \geq 1$, the Lie superalgebra $\mathfrak{U}$ still may not be B(m,n)-graded since it may fail to satisfy condition (ΔG3) of Definition 2.1. To remedy this situation we need to pass to the Lie subsuperalgebra $\mathfrak{L} = \mathfrak{L}(\mathfrak{u}(\xi))$ of $\mathfrak{U}$ generated by the nonzero weights relative to the usual Cartan subalgebra of $\mathfrak{g}$, or equivalently, to the Lie subsuperalgebra generated by $M_{\mathfrak{g}\otimes A}$, $M_{\mathfrak{s}\otimes B}$ and $T_{V\otimes C}$. Then with the above identifications, we have that

$$\mathfrak{L} = (\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus(V\otimes C)\oplus(D_{\mathfrak{U}}\cap\mathfrak{L})$$

and from the equations in (7.5), $D_{\mathfrak{U}}\cap\mathfrak{L}$ is the sum of $\mathrm{I}\otimes([A,A]+[B,B])$ and the span of the maps

$$D_{c,c'} = \begin{pmatrix} \frac{2}{2m+1-2n}M_{\mathrm{I}\otimes\chi^-(c,c')} & 0 \\ 0 & \chi_{c,c'} \end{pmatrix}.$$

Then (7.5) shows (assuming $B=0$ if $m\geq 1$) that $\mathfrak{L}$ is $\mathrm{B}(m,n)$-graded, with coordinate superalgebra $\mathfrak{b}=\mathfrak{a}\oplus C$. We refer to $\mathfrak{L}=\mathfrak{L}(\mathfrak{u}(\xi))$ as the *$B(m,n)$-graded unitary Lie superalgebra of the η-superhermitian form ξ*. Note that $\mathfrak{L}/Z(\mathfrak{L})\cong\mathfrak{L}(\mathfrak{b})$ as in (4.12) and (5.23). Hence we have the following:

Theorem 7.6. *A Lie superalgebra L over $\mathbb{F}$ is $B(m,n)$-graded for $(m,n)\neq(0,1)$ if and only if there exist a unital associative superalgebra $\mathfrak{a}=A\oplus B$ over $\mathbb{F}$ with a superinvolution η such that $\eta\,|_A=\mathrm{I}$ and $\eta\,|_B=-\mathrm{I}$, and such that $B=0$ if $m\geq 1$; and a unital $\mathfrak{a}$-module C equipped with an η-superhermitian form χ so that L is centrally isogenous to the $B(m,n)$-graded unitary Lie superalgebra $\mathfrak{L}(\mathfrak{u}(\xi))$ of the η-superhermitian form $\xi=\omega\perp-\chi$ on the $\mathfrak{a}$-module $\mathfrak{a}^N\oplus C$, $(N=2m+1+2n)$.*

Remark 7.7. By using the basis in (3.1) for V, we may obtain a matrix realization of the unitary Lie superalgebra $\mathfrak{U}=\mathfrak{u}(\xi)$ similar to that for BC_r-graded Lie algebras in [ABG2, Example 1.23].

First, we identify $\mathrm{End}_{\mathbb{F}}(V)$ with $\mathfrak{M}_N(\mathbb{F})$ and then the associative superalgebra $\mathrm{End}_{\mathbb{F}}(V)\otimes\mathfrak{a}$ (having multiplication $(z\otimes\alpha)(z'\otimes\alpha')=(-1)^{\bar\alpha\bar z'}zz'\otimes\alpha\alpha'$) with $\mathfrak{M}_N(\mathfrak{a})$; however, a word of caution is needed here. The elements of $\mathfrak{M}_N(\mathfrak{a})$ are linear combinations of the elements $E_{i,j}\alpha$ $(0\leq i,j\leq N-1)$, but the multiplication in $\mathfrak{M}_N(\mathfrak{a})$ is given by

$$(E_{i,j}\alpha)(E_{r,s}\alpha') = (-1)^{\bar\alpha \bar{E_{r,s}}}\delta_{j,r}E_{i,s}\alpha\alpha', \tag{7.8}$$

where $\bar{E_{r,s}}=0$ if either $0\leq r,s\leq 2m$ or $2m+1\leq r,s\leq N-1$, and $\bar{E_{r,s}}=1$ otherwise. For any $z\in\mathfrak{M}_N(\mathbb{F})$ and $\alpha\in\mathfrak{a}$, let $z\alpha$ denote the (image of the) element $z\otimes\alpha\in\mathrm{End}_{\mathbb{F}}(V)\otimes\mathfrak{a}$ in $\mathfrak{M}_N(\mathfrak{a})$.

In the same vein, V is identified with $\mathbb{F}^N=\mathfrak{M}_{N\times 1}(\mathbb{F})$, where the first $2m+1$ coordinates are even and the last $2n$ coordinates are odd. Thus $V\otimes\mathfrak{a}$ is identified with $\mathfrak{a}^N=\mathfrak{M}_N(\mathfrak{a})$, and $v\otimes\alpha$ with $v\alpha$.

Similar considerations apply to $\mathrm{End}_{\mathbb{F}}(V)\otimes W\cong\mathfrak{M}_N(W)$ and to $V\otimes W\cong W^N$ for any superspace W.

Now we make the following identifications: $\mathrm{Hom}_{\mathfrak{a}}(V\otimes\mathfrak{a},C)\cong\mathrm{Hom}_{\mathfrak{a}}(\mathfrak{a}^N,C)\cong\mathfrak{M}_{1\times N}(C)$ where, according to our conventions, the N-tuple $\underline{c}^t=(c_0,\dots,c_{N-1})$ takes the element $\underline{\alpha}=\begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_{N-1}\end{pmatrix}$ to the element

$$\underline{c}^t\underline{\alpha}=\sum_{i=0}^{2m}c_i.\alpha_i+\sum_{i=2m+1}^{N-1}(-1)^{\bar{c_i}}c_i\alpha_i.$$

For $v \in V \cong \mathbb{F}^N$, $v = \begin{pmatrix} \nu_0 \\ \vdots \\ \nu_{N-1} \end{pmatrix}$, and $c \in C$, $v^t c$ denotes the element $(\nu_0 c, \dots, \nu_{N-1} c)$. Then the map $X_{v\otimes c}$ in (7.3) corresponds to $(v^{\mathrm{t}} J)c$.

With all these identifications in hand, we have

$$\mathfrak{U} = \left\{ \begin{bmatrix} M & \chi_{\underline{c}} \\ \underline{c}^{\mathrm{t}} J & P \end{bmatrix} \;\Big|\; M \in \mathfrak{M}_N(\mathfrak{a}),\ \omega(M\underline{\alpha}, \underline{\alpha}') + (-1)^{\bar{M}\bar{\underline{\alpha}}} \omega(\underline{\alpha}, M\underline{\alpha}') = 0, \right.$$
$$\left. \underline{c} \in C^N,\ P \in \mathfrak{u}(\chi) \right\},$$

These matrices can be considered as matrices over the associative superalgebra $\mathfrak{c} = \begin{pmatrix} \mathfrak{a} & C^* \\ C & \mathrm{End}_{\mathfrak{a}}(C) \end{pmatrix}$ with its natural multiplication. Thus, $\mathfrak{U}$ can be considered as a Lie subsuperalgebra of $\mathfrak{M}_{N+1}(\mathfrak{c})$, which is an associative superalgebra with the product in (7.8), but now $\bar{E_{r,s}} = 0$ if either $r, s \in \{0, \dots, 2m, N\}$ or $r, s \in \{2m+1, \dots, N-1\}$, and $\bar{E_{r,s}} = 1$ otherwise.

$\mathfrak{psl}(2,2)$ as a B$(0,1)$-graded Lie superalgebra.

In this part we describe how the Lie superalgebra $L = \mathfrak{psl}(2,2) = \mathfrak{sl}(2,2)/\mathbb{F}\,\mathrm{I}$, is B$(0,1)$-graded. Let $\mathfrak{g} = \mathfrak{osp}(1,2)$, and assume V is the natural 3-dimensional module for $\mathfrak{g}$. Then there is a homomorphism of Lie superalgebras

$$L \cong (\mathfrak{g} \otimes \mathbb{F}1) \oplus (V \otimes B)$$

where $B = \mathbb{F}h \oplus \mathbb{F}x \oplus \mathbb{F}y$, by means of the identifications

$$\mathfrak{g} \otimes 1 \hookrightarrow L$$
$$\begin{pmatrix} 0 & \nu & -\mu \\ \mu & \alpha & \beta \\ \nu & \gamma & -\alpha \end{pmatrix} \otimes 1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & -\mu \\ 0 & \mu & \alpha & \beta \\ 0 & \nu & \gamma & -\alpha \end{pmatrix} + \mathbb{F}\,\mathrm{I}$$

$$V \otimes h \hookrightarrow L$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes h \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a & -c & b \\ 0 & b & a & 0 \\ 0 & c & 0 & a \end{pmatrix} + \mathbb{F}\,\mathrm{I}$$

$$V \otimes x \hookrightarrow L$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes x \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix} + \mathbb{F}\,\mathrm{I}$$

$$V \otimes y \hookrightarrow L$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes y \mapsto \begin{pmatrix} 0 & -a & c & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathbb{F}\,\mathrm{I}$$

All these are readily verified to be homomorphisms of $\mathfrak{g}$-modules.

Recall for $\mathfrak{osp}(1,2)$, that V can be identified with $\mathfrak{s}$ by means of

$$\begin{pmatrix} a\\ b\\ c\end{pmatrix} \mapsto \begin{pmatrix} 2a & -c & b\\ b & a & 0\\ c & 0 & a\end{pmatrix}$$

Thus, we may write $L = (\mathfrak{g}\otimes\mathbb{F}1)\oplus(\mathfrak{s}\otimes B)$ so that the bracket in L then becomes

$$[z\otimes\alpha, z'\otimes\alpha'] = [z,z']\otimes\frac{1}{2}\alpha\circ\alpha' + z\circ z'\otimes\frac{1}{2}[\alpha,\alpha']$$

for any homogeneous $z\otimes\alpha\in(\mathfrak{g}\otimes\mathbb{F}1)\cup(\mathfrak{s}\otimes B)$. The coordinate algebra is $\mathfrak{a} = \mathbb{F}1\oplus B$, where

$$\begin{aligned}
&1 \text{ is the unit element}\\
&h^2 = 1\\
&hx = \frac{1}{3}x = -xh, \quad hy = -\frac{1}{3}y = -yh\\
&xy = -\frac{1}{2} - \frac{1}{6}h, \quad yx = -\frac{1}{2} + \frac{1}{6}h, \quad x^2 = y^2 = 0
\end{aligned}$$

Suppose $Q = \mathfrak{M}_2(\mathbb{F}) = \mathbb{F}\,\mathrm{I}\oplus\mathfrak{sl}_2$ with the product

$$(\alpha\,\mathrm{I}+u)(\beta\,\mathrm{I}+v) = \bigl(\alpha\beta + \frac{1}{2}\mathfrak{tr}(uv)\bigr)\,\mathrm{I} + \bigl(\alpha v + \beta u + \frac{1}{2}[u,v]\bigr)$$

Define the algebra $Q_{\frac{1}{3}} = (Q,\cdot)$ using the multiplication

$$(\alpha\,\mathrm{I}+u)\cdot(\beta\,\mathrm{I}+v) = \bigl(\alpha\beta + \frac{1}{2}\mathfrak{tr}(uv)\bigr)\,\mathrm{I} + \bigl(\alpha v + \beta u + \frac{1}{6}[u,v]\bigr)$$

Then the map

$$\begin{aligned}
\mathfrak{a} &\to Q_{\frac{1}{3}}\\
1 &\mapsto \mathrm{I}\\
h &\mapsto E_{1,1} - E_{2,2}\\
x &\mapsto E_{1,2}\\
y &\mapsto -E_{2,1}
\end{aligned}$$

is an algebra isomorphism.

One can check (it is straightforward but lengthy) that $Q_{\frac{1}{3}}$ satisfies all the conditions of our coordinatization result (Theorem 6.20) for B(0,1)-graded Lie superalgebras with $\langle\,|\,\rangle = 0$. To do this, it is enough to verify that for any $b, b', b''\in B\cong [Q_{\frac{1}{3}}, Q_{\frac{1}{3}}]^{\cdot}$:

$$\begin{aligned}
(b,b',b'')^{\cdot} &= \frac{4}{9}\bigl(\mathfrak{tr}(bb')b'' - \mathfrak{tr}(b'b'')b\bigr)\\
[[b,b']^{\cdot}, b'']^{\cdot} &= \frac{-2}{9}\bigl(\mathfrak{tr}(bb'')b' - \mathfrak{tr}(b'b'')b\bigr)\,.
\end{aligned}$$

Forms of $\mathbf{F}(4)$ as $\mathbf{B}(0,1)$-graded Lie superalgebras.

In [BZ, Example 4.6] and [BE1, Thm. 2.5], certain forms of the exceptional simple Lie superalgebra F(4) are realized by the Tits construction,

$$\mathcal{T}(\mathcal{C},J)=\mathrm{Der}_{\mathbb{F}}(\mathcal{C})\oplus(\mathcal{C}^0\otimes J^0)\oplus\mathrm{Der}_{\mathbb{F}}(J). \tag{7.9}$$

Here $\mathcal{C}$ is a Cayley-Dickson algebra over $\mathbb{F}$ and $\mathcal{C}^0$ is the set of elements of trace 0 relative to the standard trace "$\mathfrak{tr}$" on $\mathcal{C}$. Thus, $\mathrm{Der}_{\mathbb{F}}(\mathcal{C})$ is a form of the simple Lie algebra G_2, and $\mathcal{C}^0$ is its 7-dimensional simple module. In addition, J is the simple Jordan superalgebra commonly denoted D_2. Thus, $J=J_{\bar 0}\oplus J_{\bar 1}$, where $J_{\bar 0}=\mathbb{F}e\oplus\mathbb{F}f$ and $J_{\bar 1}=\mathbb{F}x\oplus\mathbb{F}y$, and

$$\begin{gathered} e^2=e, \qquad f^2=f, \qquad ef=0\\ xy=e+2f=-yx, \qquad ex=\tfrac{1}{2}x=fx, \qquad ey=\tfrac{1}{2}y=fy. \end{gathered} \tag{7.10}$$

Observe that $e+f=1$. The space J^0 is spanned by the elements $x,y,e-2f$, (it is the set of elements of J of trace 0 relative to a trace "$\mathfrak{t}$" on J and $\mathfrak{t}(1)=1$).

The multiplication in (7.9) is given by ([BE1, (1.4)]):

$$\begin{aligned} &\mathrm{Der}_{\mathbb{F}}(\mathcal{C}) \ \text{ and } \ \mathrm{Der}_{\mathbb{F}}(J) \text{ are commuting subsuperalgebras,}\\ &[D,a\otimes x]=Da\otimes x \quad\text{and}\quad [d,a\otimes x]=a\otimes dx\\ &[a\otimes x,b\otimes y]=\mathfrak{t}(xy)D_{a,b}+[a,b]\otimes x*y+2\mathfrak{tr}(ab)d_{x,y} \end{aligned} \tag{7.11}$$

for any $D\in\mathrm{Der}_{\mathbb{F}}(\mathcal{C})$, $d\in\mathrm{Der}_{\mathbb{F}}(J)$, $a,b\in\mathcal{C}^0$ and $x,y\in J^0$. In (7.11), $x*y=xy-\mathfrak{t}(xy)1$ and $d_{x,y}(z)=x(yz)-(-1)^{\bar x\bar y}y(xz)$ for $x,y\in J^0$, $z\in J$; and for any $a,b\in\mathcal{C}^0,\ c\in\mathcal{C},\ D_{a,b}(c)=[[a,b],c]-3(a,b,c)$.

The Lie superalgebra $\mathrm{Der}_{\mathbb{F}}(J)$ of superderivations of J kills $\mathbb{F}1$ and leaves J^0 invariant. It follows that $\mathrm{Der}_{\mathbb{F}}(J)$ is isomorphic to $\mathfrak{osp}(J^0,\mathfrak{t})$ (superskewsymmetric maps relative to the trace form $\mathfrak{t}(xy)$), which is isomorphic naturally to $\mathfrak{g}=\mathfrak{osp}(1,2)$ (see for example, [BE1, Lemma 2.4]), and J^0 is a simple $\mathfrak{g}$-module isomorphic to the three-dimensional natural $\mathfrak{g}$-module V. When $\mathcal{C}$ is the split Cayley-Dickson (octonion) algebra, then $\mathcal{T}(\mathcal{C},J)\cong\mathrm{F}(4)$.

Now reading (7.9) right to left we obtain

$$\mathcal{T}(\mathcal{C},J)\cong\mathfrak{g}\oplus(V\otimes\mathcal{C}^0)\oplus\mathrm{Der}_{\mathbb{F}}(\mathcal{C})\cong(\mathfrak{g}\otimes\mathbb{F}1)\oplus(V\otimes\mathcal{C}^0)\oplus\mathrm{Der}_{\mathbb{F}}(\mathcal{C}). \tag{7.12}$$

From this it is evident that the Lie superalgebra $\mathcal{T}(\mathcal{C},J)$ is B$(0,1)$-graded.

For any $x\in J^0$, denote by ℓ_x the left multiplication map $y\mapsto x*y$ on J^0. Identifying $\mathfrak{g}$ with $\mathfrak{osp}(J^0,\mathfrak{t})$, we can identify J^0 with $\mathfrak{s}$ (as a $\mathfrak{g}$-module) by means of the map $x\mapsto 6\ell_x$. Moreover, the maps $J^0\otimes J^0\to\mathfrak{s}$ given by $x\otimes y\mapsto\ell_{x*y}$ and $x\otimes y\mapsto\ell_x\circ\ell_y=\ell_x\ell_y-(-1)^{\bar x\bar y}\ell_y\ell_x+2\mathfrak{str}(\ell_x\ell_y)\,\mathrm{I}$ are proportional, since they are $\mathfrak{g}$-invariant, and a bit of computation gives $\ell_x\circ\ell_y=3\ell_{x*y}$ for any $x,y\in J^0$. Similarly, one proves that $d_{x,y}=9[\ell_x,\ell_y]$ and $\mathfrak{str}(\ell_x\ell_y)=\frac{1}{4}\mathfrak{t}(xy)$ for any $x,y\in J^0$. Then (7.12) becomes

$$\mathcal{T}(\mathcal{C},J) \cong (\mathfrak{g}\otimes\mathbb{F}1)\oplus(\mathfrak{s}\otimes\mathcal{C}^0)\oplus\mathrm{Der}_{\mathbb{F}}(\mathcal{C}) \tag{7.13}$$

and the multiplication in (7.11) translates to

(7.14)
$$\begin{aligned}
&\mathfrak{g}\cong\mathfrak{g}\otimes\mathbb{F}1 \ \text{ and } \ \mathrm{Der}_{\mathbb{F}}(\mathcal{C}) \text{ are commuting subsuperalgebras,}\\
&[D,\ell_x\otimes a]=\ell_x\otimes Da \quad\text{ and }\quad [d\otimes 1,\ell_x\otimes a]=\ell_{dx}\otimes a,\\
&[\ell_x\otimes a,\ell_y\otimes b]=[\ell_x,\ell_y]\otimes\frac{1}{2}\mathfrak{tr}(ab)1 \,+\, \ell_x\circ\ell_y\otimes\frac{1}{18}[a,b] \,+\, \frac{1}{9}\mathfrak{str}(\ell_x\ell_y)D_{a,b}
\end{aligned}$$

for any $D\in\mathrm{Der}_{\mathbb{F}}(\mathcal{C})$, $d\in\mathfrak{g}=\mathrm{Der}_{\mathbb{F}}(J)$, $a,b\in\mathcal{C}^0$ and $x,y\in J^0$.

Comparing (5.2) and (7.14) gives that the coordinate algebra $\mathfrak{a}$ of $\mathcal{T}(\mathcal{C},J)$ is $(\mathcal{C}_{\frac{1}{9}},\cdot)$, where $\mathcal{C}_{\frac{1}{9}}:=\mathbb{F}1\oplus\mathcal{C}^0$ with the multiplication

$$(\alpha 1+a)\cdot(\beta 1+b)=\bigl(\alpha\beta+\frac{1}{2}\mathfrak{tr}(ab)\bigr)1+\bigl(\alpha b+\beta a+\frac{1}{18}[a,b]\bigr) \tag{7.15}$$

for any $\alpha,\beta\in\mathbb{F}$ and $a,b\in\mathcal{C}$. This is an algebra satisfying the constraints of Theorem 6.20. Moreover, in the notation of that theorem, $\langle 1\mid 1\rangle=0=\langle 1\mid\mathcal{C}^0\rangle$, and for $a,b\in\mathcal{C}^0$, $\langle a\mid b\rangle=\frac{1}{9}D_{a,b}$. The original multiplication in the Cayley-Dickson algebra $\mathcal{C}$ is obtained by substituting $\frac{1}{2}$ for $\frac{1}{18}$ in (7.15), so the coordinate algebra $\mathcal{C}_{\frac{1}{9}}$ is a deformation of the Cayley-Dickson algebra $\mathcal{C}$ in just the same way that $Q_{\frac{1}{3}}$ is a deformation of the quaternion algebra Q in the $\mathfrak{psl}(2,2)$ example.

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