Grothendieck's inequalities for real and complex JBW*-triples

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Abstract

We prove that, if $M > 4(1+2\sqrt{3})$ and $\varepsilon > 0$, if $\mathcal{V}$ and $\mathcal{W}$ are complex JBW*-triples (with preduals $\mathcal{V}_*$ and $\mathcal{W}_*$, respectively), and if U is a separately weak*-continuous bilinear form on $\mathcal{V}\times\mathcal{W}$, then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$ and $\psi_1, \psi_2 \in \mathcal{W}_*$ satisfying

$$|U(x,y)| \leq M \ \|U\| \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}} \ \left(\|y\|^2_{\psi_2} + \varepsilon^2 \ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V}\times\mathcal{W}$. Here, for a norm-one functional φ on a complex JB*-triple $\mathcal{V}$, $\|.\|_\varphi$ stands for the prehilertian seminorm on $\mathcal{V}$ associated to φ in [BF1]. We arrive in this "Grothendieck's inequality" through results of C-H. Chu, B. Iochum, and G. Loupias [CIL], and a corrected version of the "Little Grothendieck's inequality" for complex JB*-triples due to T. Barton and Y. Friedman [BF1]. We also obtain extensions of these results to the setting of real JB*-triples.

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Introduction

In this paper we pay tribute to the important works of T. Barton and Y. Friedman [BF1] and C-H. Chu, B. Iochum, and G. Loupias [CIL] on the

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generalization of "Grothendieck's inequalities" to complex JB*-triples. Of course, the Barton-Friedman-Chu-Iochum-Loupias techniques are strongly related to those of A. Grothendieck [Gro], G. Pisier (see [P1], [P2], and [P3]), and U. Haagerup [H], leading to the classical "Grothendieck's inequalities" for C*-algebras. One of the most important facts contained in the Barton-Friedman paper is the construction of "natural" prehilbertian seminorms $\|.\|_\varphi$, associated to norm-one continuous linear functionals φ on complex JB*-triples, in order to play, in Grothendieck's inequalities, the same role as that of the prehilbertian seminorms derived from states in the case of C*-algebras. This is very relevant because JB*-triples need not have a natural order structure.

A part of Section 1 of the present paper is devoted to review the main results in [BF1], and the gaps in their proofs (some of which are also subsumed in [CIL]). We note that those gaps consist in assuming that separately weak*-continuous bilinear forms on dual Banach spaces, as well as weak*-continuous linear operators between dual Banach spaces, attain their norms. Section 1 also contains quick partial solutions of the gaps just mentioned. These solutions are obtained by applying theorems of J. Lindenstrauss [L] and V. Zizler [Z] on the abundance of weak*-continuous linear operators attaining their norms (see Theorems 1.4 and 1.6, respectively).

We begin Section 2 by proving a deeper correct version of the Barton-Friedman "Little Grothendieck's Theorem" for complex JB*-triples [BF1, Theorem 1.3] (see Theorem 2.1). Roughly speaking, our result assures that the assertion in [BF1, Theorem 1.3] is true whenever we replace the prehilbertian seminorm $\|.\|_\phi$ arising in that assertion with $\|.\|_{\varphi_1,\varphi_2} := \sqrt{\|.\|_{\varphi_1}^2 + \|.\|_{\varphi_2}^2}$, where φ_1, φ_2 are suitable norm-one continuous linear functionals. It is worth mentioning that in fact our Theorem 2.1 deals with complex JBW*-triples and weak*-continuous operators, and that, in such a case, the functionals φ_1, φ_2 above can be chosen weak*-continuous. Among the consequences of Theorem 2.1 we emphasize appropriate "Little Grothendieck's inequalities" for JBW-algebras and von Neumann algebras (see Corollary 2.5 and Remark 2.7, respectively). Corollary 2.5 allows us to adapt an argument in [P] in order to extend Theorem 2.1 to the real setting (Theorem 2.9).

Section 3 contains the main results of the paper, namely the "Big Grothendieck's inequalities" for complex and real JBW*-triples (Theorems 3.1 and 3.4, respectively). Indeed, given $M > 4(1 + 2\sqrt{3})$ (respectively, $M > 4(1 + 2\sqrt{3})\ (1 + 3\sqrt{2})^2)$, $\varepsilon > 0$, V, W complex (respectively, real) JBW*-

triples, and a separately weak*-continuous bilinear form U on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V_*$ and $\psi_1, \psi_2 \in W_*$ satisfying

$$|U(x,y)| \leq M \; \|U\| \; \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \; \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \; \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \; \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in V \times W$.

The concluding section of the paper (Section 4) deals with some applications of the results previously obtained. We give a complete solution to a gap in the proof of the results of [R1] on the strong* topology of complex JBW*-triples, and extend those results to the real setting. We also extend to the real setting the fact proved in [R2] that the strong* topology of a complex JBW*-triple $\mathcal{W}$ and the Mackey topology $m(\mathcal{W}, \mathcal{W}_*)$ coincide on bounded subsets of $\mathcal{W}$. From this last result we derive a Jarchow-type characterization of weakly compact operators from (real or complex) JB*-triples to arbitrary Banach spaces.

1 Discussing previous results

We recall that a complex JB*-triple is a complex Banach space $\mathcal{E}$ with a continuous triple product $\{.,.,.\} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

1. (Jordan Identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x, L(b,a)y, z\} + \{x,y,L(a,b)z\}$ for all a,b,c,x,y,z in $\mathcal{E}$, where $L(a,b)x := \{a,b,x\}$;
2. The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with nonnegative spectrum for all a in $\mathcal{E}$;
3. $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathcal{E}$.

Complex JB*-triples have been introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces (see [K1], [K2] and [U]).

If $\mathcal{E}$ is a complex JB*-triple and $e \in \mathcal{E}$ is a tripotent ($\{e,e,e\} = e$) it is well known that there exists a decomposition of $\mathcal{E}$ into the eigenspaces of $L(e,e)$, the Peirce decomposition,

$$\mathcal{E} = \mathcal{E}_0(e) \oplus \mathcal{E}_1(e) \oplus \mathcal{E}_2(e),$$

where $\mathcal{E}_k := \{x \in \mathcal{E} : L(e,e)x = \frac{k}{2}x\}$. The natural projection $P_k(e) : \mathcal{E} \to \mathcal{E}_k(e)$ is called the Peirce k-projection. A tripotent $e \in \mathcal{E}$ is called complete if $\mathcal{E}_0(e) = 0$. By [KU, Proposition 3.5] we know that the complete tripotents in $\mathcal{E}$ are exactly the extreme points of its closed unit ball.

By a complex JBW*-triple we mean a complex JB*-triple which is a dual Banach space. We recall that the triple product of every complex JBW*-triple is separately weak*-continuous [BT], and that the bidual $\mathcal{E}^{**}$ of a complex JB*-triple $\mathcal{E}$ is a JBW*-triple whose triple product extends the one of $\mathcal{E}$ [Di].

Given a complex JBW*-triple $\mathcal{W}$ and a norm-one element φ in the predual $\mathcal{W}_*$ of $\mathcal{W}$, we can construct a prehilbert seminorn $\|.\|_\varphi$ as follows (see [BF1, Proposition 1.2]). By the Hahn-Banach theorem there exists $z \in \mathcal{W}$ such that $\varphi(z) = \|z\| = 1$. Then $(x,y) \mapsto \varphi\{x,y,z\}$ becomes a positive sesquilinear form on $\mathcal{W}$ which does not depend on the point of support z for φ. The prehilbert seminorm $\|.\|_\varphi$ is then defined by $\|x\|_\varphi^2 := \varphi\{x,x,z\}$ for all $x \in \mathcal{W}$. If $\mathcal{E}$ is a complex JB*-triple and φ is a norm-one element in $\mathcal{E}^*$, then $\|.\|_\varphi$ acts on $\mathcal{E}^{**}$, hence in particular it acts on $\mathcal{E}$.

In [BF1, Theorem 1.4], J. T. Barton and Y. Friedman claim that for every pair of complex JB*-triples $\mathcal{E}, \mathcal{F}$, and every bounded bilinear form V on $\mathcal{E} \times \mathcal{F}$, there exist norm-one functionals $\varphi \in \mathcal{E}^*$ and $\psi \in \mathcal{F}^*$ such that the inequality

$$|V(x,y)| \leq (3 + 2\sqrt{3})\ \|V\|\ \|x\|_\varphi\ \|y\|_\psi \tag{1.1}$$

holds for every $(x,y) \in \mathcal{E} \times \mathcal{F}$. This result is called "Grothendieck's inequality for JB*-triples". However, the beginning of the Barton-Friedman proof assumes that the two following assertions are true.

1. For $\mathcal{E}, \mathcal{F}$ and V as above, there exists a separately weak*-continuous extension of V to $\mathcal{E}^{**} \times \mathcal{F}^{**}$.

2. Again for $\mathcal{E}, \mathcal{F}$ and V as above, every separately weak*-continuous extension of V to $\mathcal{E}^{**} \times \mathcal{F}^{**}$ attains its norm (at a couple of complete tripotents).

We have been able to verify Assertion 1, but only by applying the fact, later proved by C-H. Chu, B. Iochum and G. Loupias [CIL, Lemma 5], that every bounded linear operator from a complex JB*-triple to the dual of another complex JB*-triple factors through a complex Hilbert space. Actually, this fact is also claimed in the Barton-Friedman paper (see [BF1, Corollary 3.2]), but their proof relies on their alleged [BF1, Theorem 1.4].

Lemma 1.1 *Let $\mathcal{E}$ and $\mathcal{F}$ be complex JB*-triples. Then every bounded bilinear form V on $\mathcal{E}\times\mathcal{F}$ has a separately weak*-continuous extension to $\mathcal{E}^{**}\times\mathcal{F}^{**}$.*

Proof. Let V be a bounded bilinear form on $\mathcal{E}\times\mathcal{F}$. Let F denote the unique bounded linear operator from $\mathcal{E}$ to $\mathcal{F}^*$ which satisfies

$$V(x,y) =< F(x), y >$$

for every $(x,y)\in\mathcal{E}\times\mathcal{F}$. By [CIL, Lemma 5], F factors through a Hilbert space, and hence is weakly compact. By [HP, Lemma 2.13.1], we have $F^{**}(\mathcal{E}^{**})\subset\mathcal{F}^*$. Then the bilinear form $\widetilde{V}$ on $\mathcal{E}^{**}\times\mathcal{F}^{**}$ given by

$$\widetilde{V}(\alpha,\beta) =< F^{**}(\alpha), \beta >$$

extends V and is weak*-continuous in the second variable. But $\widetilde{V}$ is also weak*-continuous in the first variable because, for $(\alpha,\beta)\in\mathcal{E}^{**}\times\mathcal{F}^{**}$, the equality

$$< F^{**}(\alpha), \beta >=< \alpha, F^*(\beta) >$$

holds. □

Unfortunately, as the next example shows, Assertion 2 above is not true.

Example 1.2 *Take $\mathcal{E}$ and $\mathcal{F}$ equal to the complex ℓ_2 space, and consider the bounded bilinear form on $\mathcal{E}\times\mathcal{F}$ defined by $V(x,y):=(S(x)|\sigma(y))$ where S is the bounded linear operator on ℓ_2 whose associated matrix is*

$$\begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 & \dots \\ 0 & \frac{2}{3} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{n}{n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

and σ is the conjugation on ℓ_2 fixing the elements of the canonical basis. Then V does not attain its norm.

It is worth mentioning that, although the bilinear form V above does not attain its norm, it satisfies inequality 1.1 for every $x, y \in \ell_2$ and every norm-one elements $\varphi, \psi \in \ell_2^*$. Therefore it does not become a counterexample to the Barton-Friedman claim. In fact we do not know if Theorem 1.4 of [BF1] is true.

Now that we know that Assertion 2 is not true, we prove that it is "almost" true.

Lemma 1.3 *Let $\mathcal{E}, \mathcal{F}$ be complex JB*-triples. Then the set of bounded bilinear forms on $\mathcal{E}\times\mathcal{F}$ whose separately weak*-continuous extensions to $\mathcal{E}^{**}\times\mathcal{F}^{**}$ attain their norms is norm-dense in the space $\mathcal{L}(^2(\mathcal{E}\times\mathcal{F}))$ of all bounded bilinear forms on $\mathcal{E}\times\mathcal{F}$.*

Proof. Let V be in $\mathcal{L}(^2(\mathcal{E}\times\mathcal{F}))$. Denote by $\widetilde{V}$ the (unique) separately weak*-continuous extension of V to $\mathcal{E}^{**}\times\mathcal{F}^{**}$. By the proof of Lemma 1.1, we can assure the existence of a bounded linear operator $F_V : \mathcal{E} \rightarrow \mathcal{F}^*$ satisfying $F_V^{**}(\mathcal{E}^{**}) \subset \mathcal{F}^*$ and

$$\widetilde{V}(\alpha, \beta) = < F_V^{**}(\alpha), \beta >$$

for every $(\alpha, \beta) \in \mathcal{E}^{**}\times\mathcal{F}^{**}$. It follows that $\widetilde{V}$ attains its norm whenever F_V^{**} does. Since the mapping $V \mapsto F_V$, from $\mathcal{L}(^2(\mathcal{E}\times\mathcal{F}))$ into the Banach space of all bounded linear operators from $\mathcal{E}$ to $\mathcal{F}^*$, is a surjective isometry, the result follows from [L, Theorem 1]. □

An alternative proof of the above Lemma can be given taking as a key tool [A, Theorem 1].

Now note that, if X and Y are dual Banach spaces, and if U is a separately weak*-continuous bilinear form on $X \times Y$ which attains its norm, then U actually attains its norm at a couple of extreme points of the closed unit balls of X and Y (hence at a couple of complete tripotents in the case that X and Y are complex JB*-triples). Since the Barton-Friedman proof of their claim actually shows that the inequality (1.1) holds (for suitable norm-one functionals $\varphi \in \mathcal{E}^*$ and $\psi \in \mathcal{F}^*$) whenever the separately weak*-continuous extension of V given by Lemma 1.1 attains its norm at a couple of complete tripotents, the next theorem follows from Lemma 1.3.

Theorem 1.4 *Let $\mathcal{E}, \mathcal{F}$ be complex JB*-triples. Then the set of all bounded bilinear forms V on $\mathcal{E}\times\mathcal{F}$ such that there exist norm-one functionals $\varphi \in \mathcal{E}^*$ and $\psi \in \mathcal{F}^*$ satisfying*

$$|V(x, y)| \leq (3 + 2\sqrt{3})\ \|V\|\ \|x\|_\varphi\ \|y\|_\psi$$

for every $(x,y) \in \mathcal{E} \times \mathcal{F}$*, is norm dense in* $\mathcal{L}(^2(\mathcal{E} \times \mathcal{F}))$*.*

Another alleged proof of the Barton-Friedman claim [BF1, Theorem 1.4] (with constant $3+2\sqrt{3}$ replaced with $4(1+2\sqrt{3})$) appears in the Chu-Iochum-Loupias paper already quoted (see [CIL, Theorem 6]). Such a proof relies on the Barton-Friedman version of the so called "Little Grothendieck's Theorem" for complex JB*-triples [BF1, Theorem 1.3]. However, the Barton-Friedman argument for this "Little Grothendieck's Theorem" also has a gap (see [P]).

Several authors (the second author of the present paper among others) subsumed the gap in the proof of Theorem 1.3 of [BF1] just commented, and formulated daring claims like the following (see [R1, Proposition 1] and the proof of Lemma 4 of [CM]). For every complex JBW*-triple $\mathcal{W}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{W} \to \mathcal{H}$, there exists a norm-one functional $\varphi \in \mathcal{W}_*$ such that the inequality

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ \|x\|_\varphi \tag{1.2}$$

holds for all $x \in \mathcal{W}$. As in the case of the Barton-Friedman big Grothendieck's inequality, we do not know if the above claim is true. In any case, the next lemma is implicitly shown in the proof of Theorem 1.3 of [BF1].

Lemma 1.5 *Let* $\mathcal{W}$ *be a complex JBW*-triple,* $\mathcal{H}$ *a complex Hilbert space, and* T *a weak*-continuous linear operator from* $\mathcal{W}$ *to* $\mathcal{H}$ *which attains its norm. Then* T *satisfies inequality* (1.2) *for a suitable norm-one functional* $\varphi \in \mathcal{W}_*$*.*

We note that, for $\mathcal{W}$ and $\mathcal{H}$ as in the above lemma, weak*-continuous linear operators from $\mathcal{W}$ to $\mathcal{H}$ need not attain their norms (see the introduction of [P]). Now, from Lemma 1.5 and [Z] we obtain the following result.

Theorem 1.6 *Let* $\mathcal{W}$ *be a complex JBW*-triple and* $\mathcal{H}$ *a complex Hilbert space. Then the set of weak*-continuous linear operators* T *from* $\mathcal{W}$ *to* $\mathcal{H}$ *such that there exists a norm-one functional* $\varphi \in \mathcal{W}_*$ *satisfying*

$$\|T(x)\| \leq \sqrt{2}\ \|T\|\ \|x\|_\varphi$$

for all $x \in \mathcal{W}$*, is norm dense in the space of all weak*-continuous linear operators from* $\mathcal{W}$ *to* $\mathcal{H}$*.*

2 Little Grothendieck‘s Theorem for JBW*-triples

In this section we prove appropriate versions of "Little Grothendieck's inequality" for real and complex JBW*-triples. We begin by considering the complex case, where the key tools are the Barton-Friedman result collected in Lemma 1.5, and a fine principle on approximation of operators by operators attaining their norms, due to R. A. Poliquin and V. E. Zizler [PZ].

Theorem 2.1 *Let $K > \sqrt{2}$ and $\varepsilon > 0$. Then, for every complex JBW*-triple $\mathcal{W}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{W} \to \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that the inequality*

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}$$

holds for all $x \in \mathcal{W}$.

Proof. Without loss of generality we can suppose $\|T\| = 1$. Take $\delta > 0$ such that $\delta \leq \varepsilon^2$ and $\sqrt{2((1+\delta)^2+\delta)} \leq K$. By [PZ, Corollary 2] there is a rank one weak*-continuous linear operator $T_1 : \mathcal{W} \to \mathcal{H}$ such that $\|T_1\| \leq \delta$ and $T - T_1$ attains its norm. Since T_1 is of rank one and weak*-continuous, it also attains its norm. By Lemma 1.5, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that

$$\|T_1(x)\| \leq \sqrt{2} \ \|T_1\| \ \|x\|_{\varphi_1},$$

$$\|(T-T_1)(x)\| \leq \sqrt{2} \ \|T-T_1\| \ \|x\|_{\varphi_2}$$

for all $x \in \mathcal{W}$. Therefore for $x \in \mathcal{W}$ we have

$$\|T(x)\| \leq \|(T-T_1)(x)\| + \|T_1(x)\|$$

$$\leq \sqrt{2} \ \|T-T_1\| \ \|x\|_{\varphi_2} + \sqrt{2} \ \|T_1\| \ \|x\|_{\varphi_1}$$

$$\leq \sqrt{2} \ (1+\delta) \ \|x\|_{\varphi_2} + \sqrt{2\delta} \ \sqrt{\delta} \ \|x\|_{\varphi_1}$$

$$\leq \sqrt{2((1+\delta)^2+\delta)} \ \left(\|x\|_{\varphi_2}^2 + \delta \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}$$

$$\leq K \ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}.$$

□

Given a complex JBW*-triple $\mathcal{W}$ and norm-one elements $\varphi_1, \varphi_2 \in \mathcal{W}_*$ we denote by $\|.\|_{\varphi_1,\varphi_2}$ the prehilbert seminorm on $\mathcal{W}$ given by $\|x\|^2_{\varphi_1,\varphi_2} := \|x\|^2_{\varphi_1} + \|x\|^2_{\varphi_2}$. The next result follows straightforwardly from Theorem 2.1.

Corollary 2.2 *Let $\mathcal{W}$ be a complex JBW*-triple and T a weak*-continuous linear operator from $\mathcal{W}$ to a complex Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}_*$ such that, for every $x \in \mathcal{W}$, we have*

$$\|T(x)\| \leq 2\|T\|\|x\|_{\varphi_1,\varphi_2}.$$

We recall that a JB*-algebra is a complete normed Jordan complex algebra (say $\mathcal{A}$) endowed with a conjugate-linear algebra involution * satisfying $\|U_x(x^*)\| = \|x\|^3$ for every $x \in \mathcal{A}$. Here, for every Jordan algebra $\mathcal{A}$, and every $x \in \mathcal{A}$, U_x denotes the operator on $\mathcal{A}$ defined by $U_x(y) := 2x\circ(x\circ y) - x^2 \circ y$, for all $y \in \mathcal{A}$. We note that every JB*-algebra can be regarded as a complex JB*-triple under the triple product given by

$$\{x, y, z\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$$

(see [BKU] and [Y]). By a JBW*-algebra we mean a JB*-algebra which is a dual Banach space. Every JBW*-algebra $\mathcal{A}$ has a unit $\mathbf{1}$ [Y], so that the binary product of $\mathcal{A}$ can be rediscovered from the triple product by means of the equality $x \circ y = \{x, \mathbf{1}, y\}$.

Theorem 2.3 *Let $M > 2$. Then, for every JBW*-algebra $\mathcal{A}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T : \mathcal{A} \to \mathcal{H}$, there exists a norm-one positive functional $\xi \in \mathcal{A}_*$ such that the inequality*

$$\|T(x)\| \leq M \ \|T\| \ \left(\xi(x \circ x^*)\right)^{\frac{1}{2}}$$

holds for all $x \in \mathcal{A}$.

Proof. Taking $K := \sqrt{M}$ and $\varepsilon := \sqrt{\frac{M-2}{2}}$ in Theorem 2.1, we find norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{A}_*$ such that

$$\|T(x)\| \leq K \ \|T\| \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$. Let $i = 1, 2$. We choose $e_i \in \mathcal{A}$ with $\varphi_i(e_i) = \|e_i\| = 1$, and denote by ξ_i the mapping $x \mapsto \varphi_i(x \circ e_i)$ from $\mathcal{A}$ to $\mathbb{C}$. Clearly ξ_i is a norm-one weak*-continuous linear functional on $\mathcal{A}$. Moreover, from the identity

$$\{x,x,e_i\} + \{x^*,x^*,e_i\} = 2e_i \circ (x \circ x^*)$$

we obtain that ξ_i is positive and that the equality $\|x\|^2_{\varphi_i} + \|x^*\|^2_{\varphi_i} = 2\xi_i(x \circ x^*)$ holds. Therefore we have $\|x\|^2_{\varphi_i} \leq 2\xi_i(x \circ x^*)$ and hence

$$\|T(x)\| \leq \sqrt{2}K\ \|T\|\ \left(\xi_2(x \circ x^*) + \varepsilon^2\ \xi_1(x \circ x^*)\right)^{\frac{1}{2}}.$$

Finally, putting $\xi := \frac{1}{1+\varepsilon^2}(\xi_2 + \varepsilon^2\ \xi_1)$, ξ becomes a norm-one positive functional in $\mathcal{A}_*$ and for $x \in \mathcal{A}$ we have

$$\|T(x)\| \leq \sqrt{2\ (1+\varepsilon^2)}K\ \|T\|\ \ (\xi(x \circ x^*))^{\frac{1}{2}} = M\ \|T\|\ \ (\xi(x \circ x^*))^{\frac{1}{2}}.$$

□

We recall that the bidual of every JB*-algebra $\mathcal{A}$ is a JBW*-algebra containing $\mathcal{A}$ as a JB*-subalgebra.

Corollary 2.4 *Let $\mathcal{A}$ be a JB*-algebra and T a bounded linear operator from $\mathcal{A}$ to a complex Hilbert space. Then there exists a norm-one positive functional $\xi \in \mathcal{A}^*$ satisfying*

$$\|T(x)\| \leq 2\|T\|\,(\xi(x \circ x^*))^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$.

Proof. By Theorem 2.3, for $n \in \mathbb{N}$ there is a norm-one positive functional $\xi_n \in \mathcal{A}^*$ satisfying

$$\|T(x)\| \leq (2 + \frac{1}{n})\|T\|\,(\xi_n(x \circ x^*))^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$. Take in $\mathcal{A}^*$ a weak* cluster point η of the sequence ξ_n. Then η is a positive functional with $\|\eta\| \leq 1$, and the inequality

$$\|T(x)\| \leq 2\|T\|\,(\eta(x \circ x^*))^{\frac{1}{2}}$$

holds for all $x \in \mathcal{A}$. If $\eta = 0$, then $T = 0$ and nothing has to be proved. Otherwise take $\xi := \frac{1}{\|\eta\|}\eta$. □

For background about JB- and JBW-algebras the reader is referred to [HS]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [W] (respectively, [E]).

Corollary 2.5 *Let $K > 2\sqrt{2}$. Then, for every JBW-algebra A, every real Hilbert space H, and every weak*-continuous linear operator $T : A \to H$, there exists a norm-one positive functional $\xi \in A_*$ such that*

$$\|T(x)\| \leq K \ \|T\| \ \left(\xi(x^2)\right)^{\frac{1}{2}}$$

for all $x \in A$.

Proof. Let $\widehat{A}$ denote the JBW*-algebra whose self-adjoint part is equal to A, and $\widehat{H}$ be the Hilbert space complexification of H. Consider the complex-linear operator $\widehat{T} : \widehat{A} \to \widehat{H}$, which extends T. Clearly we have $\|\widehat{T}\| \leq \sqrt{2}\|T\|$. By Theorem 2.3 there exists a norm-one positive functional $\xi \in \widehat{A}_*$ such that

$$\|T(x)\| = \|\widehat{T}(x)\| \leq \frac{K}{\sqrt{2}} \ \|\widehat{T}\| \ \left(\xi(x^2)\right)^{\frac{1}{2}} \leq K \ \|T\| \ \left(\xi(x^2)\right)^{\frac{1}{2}}$$

for all $x \in A$. Since ξ is positive, $\xi|_A$ is in fact a norm-one positive functional in A_*. $\square$

The next result follows from the above corollary in the same way that Corollary 2.4 was derived from Theorem 2.3.

Corollary 2.6 *[P, Theorem 3.2]*

Let A be a JB-algebra, H a real Hilbert space, and $T : A \to H$ a bounded linear operator. Then there is a norm-one positive linear functional $\varphi \in A^*$ such that

$$\|T(x)\| \leq 2\sqrt{2}\|T\|\left(\varphi(x^2)\right)^{\dfrac{1}{2}}$$

for all $x \in A$.

Remark 2.7 1.– *Since every C*-algebra becomes a JB*-algebra under the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$, it follows from Theorem 2.3 that, given $M > 2$, a von Neumann algebra $\mathcal{A}$, and a weak*-continuous linear operator T from $\mathcal{A}$ to a complex Hilbert space, there exists a norm-one positive functional $\varphi \in \mathcal{A}_*$ satisfying*

$$\|T(x)\| \leq M \ \|T\| \ \left(\varphi(\frac{1}{2}(xx^* + x^*x))\right)^{\frac{1}{2}}$$

for all $x \in \mathcal{A}$. A lightly better result can be derived from [H, Proposition 2.3].

2.– *As is asserted in [CIL], Corollary 2.4 can be proved by translating verbatim Pisier's arguments for the case of C*-álgebras [P2, Theorem 9.4]. We note that actually Corollary 2.4 contains Pisier's result. Moreover, it is worth mentioning that our proof of Corollary 2.4 avoids any use of ultraproducts techniques.*

Following [IKR], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. In [IKR] it is shown that every real JB*-triple E can be regarded as a real form of a complex JB*-triple. Indeed, given a real JB*-triple E there exists a unique complex JB*-triple structure on the complexification $\widehat{E} = E \oplus i\ E$, and a unique conjugation (i.e., conjugate-linear isometry of period 2) τ on $\widehat{E}$ such that $E = \widehat{E}^{\tau} := \{x \in \widehat{E} : \tau(x) = x\}$. The class of real JB*-triples includes all JB-algebras [HS], all real C*-algebras [G], and all J*B-algebras [Al].

By a real JBW*-triple we mean a real JB*-triple whose underlying Banach space is a dual Banach space. As in the complex case, the triple product of every real JBW*-triple is separately weak*-continuous [MP], and the bidual $\mathcal{E}^{**}$ of a real JB*-triple $\mathcal{E}$ is a real JBW*-triple whose triple product extends the one of $\mathcal{E}$ [IKR]. Noticing that every real JBW*-triple is a real form of a complex JBW*-triple [IKR], it follows easily that, if W is a real JBW*-triple and if φ is a norm-one element in W_*, then, for $z \in W$ such that $\varphi(z) = \|z\| = 1$, the mapping $x \mapsto (\varphi\{x, x, z\})^{\frac{1}{2}}$ is a prehilbert seminorm on W (not depending on z). Such a seminorm will be denoted by $\|.\|_{\varphi}$.

Now we proceed to deal with "Little Grothendieck's inequality" for real JBW*-triples. We begin by showing the appropriate version of Lemma 1.5 for real JBW*-triples. Such a version is obtained by adapting the proof of a recent result of the first author for real JB*-triples (see [P]) to the setting of real JBW*-triples.

Lemma 2.8 *Let $M > 1 + 3\sqrt{2}$. Then, for every real JBW*-triple W, every real Hilbert space H, and every weak*-continuous linear operator $T : W \to H$ which attains its norm, there exists a norm one functional $\varphi \in W_*$ such that*

$$\|T(x)\| \leq M\ \|T\|\ \|x\|_{\varphi}$$

for all $x \in W$.

Proof. We follow with minors changes the line of proof of [P, Theorem 4.3]. Without loss of generality we can suppose $\|T\| = 1$. Write

$$K = [2\sqrt{2}(\frac{M^2}{1+3\sqrt{2}} - (1+\sqrt{2}))]^{\frac{1}{2}} > 2\sqrt{2}$$

and $\rho = \frac{2\sqrt{2}}{1+\sqrt{2}}$. By [IKR, Lemma 3.3], there exists a complete tripotent $e \in W$ with $1 = \|T(e)\|$. Then denoting by ξ the linear functional on W given by $\xi(x) := (T(x)|T(e))$ for every $x \in W$, ξ belongs to W_* and satisfies $\|\xi\| = \xi(e) = 1$. Moreover, when in the proof of [P, Theorem 4.3] Corollary 2.5 replaces [P, Theorem 3.2], we obtain the existence of a norm-one functional $\psi \in W_*$ with $\psi(e) = 1$ such that

$$\|T(x)\| \leq K\|x\|_\psi + (1+\sqrt{2})\ \|x\|_\xi$$

for all $x \in W$. Setting $\varphi := \frac{1}{1+\rho}(\xi + \rho\ \psi)$, φ is a norm-one functional in W_* with $\varphi(e) = 1$, and we have

$$\|T(x)\| \leq \sqrt{(1+\sqrt{2})^2 + \frac{K^2}{\rho}}\sqrt{\|x\|_\xi^2 + \rho\ \|x\|_\psi^2}$$

$$= \left([(1+\sqrt{2})^2 + \frac{K^2}{\rho}](1+\rho)\right)^{\frac{1}{2}}\ \|x\|_\varphi = M\ \|x\|_\varphi$$

for all $x \in W$. $\square$

When in the proof of Theorem 2.1 Lemma 2.8 replaces Lemma 1.5, we arrive in the following result.

Theorem 2.9 *Let $K > 1+3\sqrt{2}$ and $\varepsilon > 0$. Then, for every real JBW*-triple W, every real Hilbert space H, and every weak*-continuous linear operator $T : W \to H$, there exist norm-one functionals $\varphi_1, \varphi_2 \in W_*$ such that the inequality*

$$\|T(x)\| \leq K\ \|T\|\ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2\ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}}$$

holds for all $x \in W$.

For norm-one elements φ_1, φ_2 in the predual of a given real JBW*-triple W, we define the prehilbert seminorm $\|.\|_{\varphi_1,\varphi_2}$ on W verbatim as in the complex case.

Corollary 2.10 *Let W be a real JBW*-triple and T a weak*-continuous linear operator from W to a real Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in W_*$ such that, for every $x \in W$, we have*

$$\|T(x)\| \leq 6\|T\|\|x\|_{\varphi_1,\varphi_2}.$$

3 Grothendieck's Theorem for JBW*-triples

In this section we prove "Grothendieck's inequality" for separately weak*-continuous bilinear forms defined on the cartesian product of two JBW*-triples.

Theorem 3.1 *Let $M > 4(1+2\sqrt{3})$ and $\varepsilon > 0$. For every couple $(\mathcal{V}, \mathcal{W})$ of complex JBW*-triples and every separately weak*-continuous bilinear form V on $\mathcal{V} \times \mathcal{W}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$, and $\psi_1, \psi_2 \in \mathcal{W}_*$ satisfying*

$$|V(x,y)| \leq M \ \|V\| \ \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \ \|x\|_{\varphi_1}^2\right)^{\frac{1}{2}} \ \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \ \|y\|_{\psi_1}^2\right)^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$.

Proof. We begin by noticing that a bilinear form U on $\mathcal{V} \times \mathcal{W}$ is separately weak*-continuous if and only if there exists a weak*-to-weak-continuous linear operator $F_U : \mathcal{V} \to \mathcal{W}_*$ such that the equality

$$U(x,y) = < F_U(x), y >$$

holds for every $(x,y) \in \mathcal{V} \times \mathcal{W}$.

Put $T := F_V : \mathcal{V} \to \mathcal{W}_*$ in the sense of the above paragraph. By [CIL, Lemma 5] there exist a Hilbert space $\mathcal{H}$ and bounded linear operators $S : \mathcal{V} \to \mathcal{H}$, $R : \mathcal{H} \to \mathcal{W}_*$ satisfying $T = R\ S$ and $\|R\| \ \|S\| \leq 2(1+2\sqrt{3}) \ \|T\|$. Notice that in fact we can enjoy such a factorization in such a way that R is injective. Indeed, take $\mathcal{H}^{'}$ equals to the orthogonal complement of $Ker(R)$ in $\mathcal{H}$, $R^{'} := R|_{\mathcal{H}^{'}}$ and $S^{'} := \pi_{\mathcal{H}^{'}} \ S$, where $\pi_{\mathcal{H}^{'}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}^{'}$, to have $T = R^{'} \ S^{'}$ with $R^{'}$ injective and $\|R^{'}\| \ \|S^{'}\| \leq 2(1+2\sqrt{3}) \ \|T\|$.

Next we show that S is weak*-continuous. By [DS, Corollary V.5.5] it is enough to prove that S is weak*-continuous on bounded subsets of $\mathcal{V}$. Let x_λ

be a bounded net in $\mathcal{V}$ weak*-convergent to zero. Take a weak cluster point h of $S(x_\lambda)$ in $\mathcal{H}$. Then $R(h)$ is a weak cluster point of $T(x_\lambda) = R\ S(x_\lambda)$ in $\mathcal{W}_*$. Moreover, since T is weak*-to-weak-continuous, we have $T(x_\lambda) \to 0$ weakly. It follows $R(h) = 0$ and hence $h = 0$ by the injectivity of R. Now, zero is the unique weak cluster point in $\mathcal{H}$ of the bounded net $S(x_\lambda)$, and therefore we have $S(x_\lambda) \to 0$ weakly.

Now that we know that the operator S is weak*-continuous, we apply Theorem 2.1 with $K = \sqrt{\frac{M}{2(1+2\sqrt{3})}} > \sqrt{2}$ to find norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$, and $\psi_1, \psi_2 \in \mathcal{W}_*$ satisfying

$$\|S(x)\| \leq K\ \|S\|\ \left(\|x\|^2_{\varphi_2} + \varepsilon^2\ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}} \quad \text{and}$$

$$\|R^*(y)\| \leq K\ \|R^*\|\ \left(\|y\|^2_{\psi_2} + \varepsilon^2\ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}$$

for all $x \in \mathcal{V}$ and $y \in \mathcal{W}$. Therefore

$$|V(x,y)| = |< T(x), y >| = |< S(x), R^*(y) >|$$

$$\leq \frac{M}{2(1+2\sqrt{2})}\ \|R\|\ \|S\|\ \left(\|x\|^2_{\varphi_2} + \varepsilon^2\ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}\ \left(\|y\|^2_{\psi_2} + \varepsilon^2\ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}$$

$$\leq M\ \|V\|\ \left(\|x\|^2_{\varphi_2} + \varepsilon^2\ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}\ \left(\|y\|^2_{\psi_2} + \varepsilon^2\ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}},$$

for all $(x,y) \in \mathcal{V} \times \mathcal{W}$. $\square$

In the same way that Theorem 2.3 was derived from Theorem 2.1, we can obtain from Theorem 3.1 that, given $M > 8\ (1+2\sqrt{3})$, JBW*-algebras $\mathcal{A}, \mathcal{B}$, and a separately weak*-continuous bilinear form V on $\mathcal{A} \times \mathcal{B}$, there exist norm-one positive functionals $\varphi \in \mathcal{A}_*$ and $\psi \in \mathcal{B}_*$ satisfying

$$|V(x,y)| \leq M\|V\|\ (\varphi(x \circ x^*))^{\frac{1}{2}}\ (\psi(y \circ y^*))^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{A} \times \mathcal{B}$. As a relevant particular case we obtain the following result.

Corollary 3.2 *Let $M > 8(1+2\sqrt{3})$. For every couple $(\mathcal{A}, \mathcal{B})$ of von Neumann algebras and every separately weak*-continuous bilinear form V on $\mathcal{A} \times \mathcal{B}$, there exist norm-one positive functionals $\varphi \in \mathcal{A}_*$ and $\psi \in \mathcal{B}_*$ satisfying*

$$|V(x,y)| \leq M\|V\|\ (\varphi(\frac{1}{2}(xx^* + x^*x)))^{\frac{1}{2}}\ (\psi(\frac{1}{2}(yy^* + y^*y)))^{\frac{1}{2}}$$

for all $(x,y) \in \mathcal{A} \times \mathcal{B}$.

A refined version of the above corollary can be found in [H, Proposition 2.3].

Now we proceed to deal with Grothendieck's Theorem for real JBW*-triples. The following lemma generalizes [CIL, Lemma 5] to the real case.

Lemma 3.3 *Let E and F be real JB*-triples and $T : E \to F^*$ a bounded linear operator. Then T has a factorization $T = R\ S$ through a real Hilbert space with $\|R\|\ \|S\| \leq 4(1+2\sqrt{3})\ \|T\|$*

Proof.

Let us consider the JB*-complexifications $\widehat{E}$ and $\widehat{F}$ of E and F, respectively, and denote by $\widehat{T} : \widehat{E} \to \widehat{F}^*$ the complex linear extension of T, so that we easily check that $\|\widehat{T}\| \leq 2\|T\|$. As we have mentioned before, $\widehat{T}$ has a factorization $\widehat{T} = \widehat{R}\widehat{S}$ through a complex Hilbert space $\mathcal{H}$, with $\|\widehat{R}\|\ \|\widehat{S}\| \leq 2(1+2\sqrt{3})\ \|\widehat{T}\|$.

Since $\widehat{T}$ is the complex linear extension of T, the inclusion $\widehat{T}(E) \subseteq F^*$ holds. Put $H := \overline{\widehat{S}(E)}$, the closure of $\widehat{S}(E)$ in $\mathcal{H}$. Then H is a real Hilbert space and we have $\widehat{R}(H) \subseteq \overline{\widehat{R}(\widehat{S}(E))} = \overline{\widehat{T}(E)} \subseteq F^*$.

Finally we define the bounded linear operators $S := \widehat{S}|_{E} : E \to H$ and $R := \widehat{R}|_{H} : H \to F^*$. It is easy to see that $T = R\ S$ and

$$\|R\|\ \|S\| \leq \|\widehat{R}\|\ \|\widehat{S}\| \leq 2(1+2\sqrt{3})\ \|\widehat{T}\| \leq 4(1+2\sqrt{3})\ \|T\|.$$

□

When in the proof of Theorem 3.1 Lemma 3.3 and Theorem 2.9 replace [CIL, Lemma 5] and Theorem 2.1, respectively, we obtain the following theorem.

Theorem 3.4 *Let $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ and $\varepsilon > 0$. For every couple (V, W) of real JBW*-triples and every separately weak*-continuous bilinear form U on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V_*$, and $\psi_1, \psi_2 \in W_*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|^2_{\varphi_2} + \varepsilon^2\ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}\ \left(\|y\|^2_{\psi_2} + \varepsilon^2\ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}$$

for all $(x, y) \in V \times W$.

Thanks to Lemma 3.3, Lemma 1.1 remains true when real JB*-triples replace complex ones. Then Theorems 3.4 and 3.1 give rise to the real and complex cases, respectively, of the result which follows.

Corollary 3.5 *Let $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ (respectively, $M > 4(1+2\sqrt{3})$) and $\varepsilon > 0$. Then for every couple (E, F) of real (respectively, complex) JB*-triples and every bounded bilinear form U on $E\times F$ there exist norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and $\psi_1, \psi_2 \in F^*$ satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \left(\|x\|^2_{\varphi_2} + \varepsilon^2\ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}}\ \left(\|y\|^2_{\psi_2} + \varepsilon^2\ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}$$

for all $(x, y) \in E \times F$.

Remark 3.6 *In the complex case of the above corollary, the interval of variation of the constant M can be enlarged by arguing as follows. Let $M > 3 + 2\sqrt{3}$, $\varepsilon > 0$, $\mathcal{E}$ and $\mathcal{F}$ be complex JB*-triples, and U a norm-one bounded bilinear form on $\mathcal{E}\times\mathcal{F}$. Consider the separately weak*-continuous bilinear form $\widetilde{U}$ on $\mathcal{E}^{**}\times\mathcal{F}^{**}$ which extends U, and take a weak*-to-weak continuous linear operator $T : E^{**} \to F^*$ satisfying*

$$\widetilde{U}(\alpha,\beta) = < T(\alpha), \beta >$$

for all $(\alpha,\beta) \in \mathcal{E}^{**}\times\mathcal{F}^{**}$. Choose $\delta > 0$ such that $\delta \leq \varepsilon^2$ and $(3+2\sqrt{3})(1+\delta) \leq M$. By [PZ, Corollary 2] there is a rank one weak-to-weak continuous linear operator $T_1 : \mathcal{E}^{**} \to \mathcal{F}^*$ such that $\|T_1\| \leq \delta$ and $T_2 := T - T_1$ attains its norm. Since T_1 is of rank one and weak*-continuous, it also attains its norm. For $i = 1, 2$, consider the separately weak*-continuous bilinear form $\widetilde{U}_i$ on $\mathcal{E}^{**}\times\mathcal{F}^{**}$ defined by*

$$\widetilde{U}_i(\alpha,\beta) = < T_i(\alpha), \beta >,$$

and put $U_i = \widetilde{U}_i|_{\mathcal{E}\times\mathcal{F}}$, so that U_i is a bounded bilinear form on $\mathcal{E}\times\mathcal{F}$ whose separately weak-continuous extension to $\mathcal{E}^{**}\times\mathcal{F}^{**}$ attains its norm. By the proof of [BF1, Theorem 1.4], there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ and $\psi_1, \psi_2 \in \mathcal{F}^*$ such that*

$$|U_i(x,y)| \leq (3+2\sqrt{3})\ \|U_i\|\ \|x\|_{\varphi_i}\|y\|_{\psi_i},$$

for all $(x, y) \in \mathcal{E}\times\mathcal{F}$ and $i = 1, 2$.

Therefore

$$
\begin{aligned}
|U(x,y)| &\le |U_2(x,y)| + |U_1(x,y)| \\
&\le (3+2\sqrt{3})(\|U_2\| \ \|x\|_{\varphi_2}\|y\|_{\psi_2} + \|U_1\| \ \|x\|_{\varphi_1}\|y\|_{\psi_1}) \\
&\le (3+2\sqrt{3})((1+\delta) \ \|x\|_{\varphi_2}\|y\|_{\psi_2} + \delta \ \|x\|_{\varphi_1}\|y\|_{\psi_1}) \\
&\le (3+2\sqrt{3})(1+\delta) \ (\|x\|_{\varphi_2}\|y\|_{\psi_2} + \delta \ \|x\|_{\varphi_1}\|y\|_{\psi_1}) \\
&\le (3+2\sqrt{3})(1+\delta) \ \sqrt{\|x\|^2_{\varphi_2} + \delta\|x\|^2_{\varphi_1}}\sqrt{\|y\|^2_{\psi_2} + \delta\|y\|^2_{\psi_1}} \\
&\le M \ \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \ \|x\|^2_{\varphi_1}\right)^{\frac{1}{2}} \ \left(\|y\|^2_{\psi_2} + \varepsilon^2 \ \|y\|^2_{\psi_1}\right)^{\frac{1}{2}}
\end{aligned}
$$

for all $(x,y) \in E \times F$.

We do not know if the value $\varepsilon = 0$ is allowed in Theorems 3.1 and 3.4. In any case, as the next result shows, the value $\varepsilon = 0$ is allowed for a "big quantity" of separately weak*-continuous bilinear forms.

Theorem 3.7 *Let* $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ *(respectively,* $M > 4(1+2\sqrt{3})$*) and* V, W *be real (respectively, complex) JBW*-triples. Then the set of all separately weak*-continuous bilinear forms* U *on* $V \times W$ *such that there exist norm-one functionals* $\varphi \in V_*$ *and* $\psi \in W_*$ *satisfying*

$$|U(x,y)| \le M \ \|U\| \ \|x\|_\varphi \ \|y\|_\psi$$

for all $(x,y) \in V \times W$*, is norm dense in the set of all separately weak*-continuous bilinear forms on* $V \times W$.

Proof. Let U a non zero separately weak*-continuous bilinear form on $V \times W$. By the proof of Theorem 3.4 (respectively, Theorem 3.1) there exists a real (respectively, complex) Hilbert space H such that for all $(x,y) \in V \times W$ we have

$$U(x,y) := < F(x), G(y) >,$$

where $F : V \to H$ and $G : W \to H^*$ are weak*-continuous linear operators satisfying $\|F\| \ \|G\| \le L \ \|U\|$ with $L = 4(1+2\sqrt{3})$ (respectively, $L = 2(1+2\sqrt{3})$).

By [Z], there are sequences $\{F_n : V \to H\}$ and $\{G_n : W \to H^*\}$ of weak*-continuous linear operators, converging in norm to F and G, respectively, and such that F_n and G_n attain their norms for every n. Then, putting

$$U_n(x,y) := < F_n(x), G_n(y) > \quad ((n,x,y) \in \mathbb{N} \times V \times W),$$

$\{U_n\}$ becomes a sequence of separately weak*-continuous bilinear forms on $V \times W$, converging in norm to U. Take $\sqrt{\frac{M}{L}} > K > 1 + 3\sqrt{2}$ (respectively, $\sqrt{\frac{M}{L}} > K > \sqrt{2}$). Applying Lemma 2.8 (respectively, Lemma 1.5), for $n \in \mathbb{N}$ we find norm-one functionals $\varphi_n \in V_*$ and $\psi_n \in W_*$ satisfying

$$\|F_n(x)\| \leq K \ \|F_n\| \ \|x\|_{\varphi_n} \ \text{ and}$$
$$\|G_n(y)\| \leq K \ \|G_n\| \ \|y\|_{\psi_n}$$

for all $(x, y) \in V \times W$.

Set

$$\delta = \frac{\frac{M}{K^2} - L}{1 + L} \, \frac{\|U\|}{2} > 0,$$

and take $m \in \mathbb{N}$ such that the inequalities

$$| \ \|F_n\| \ \|G_n\| - \|F\| \ \|G\| \ | < \delta,$$
$$| \ \|U_n\| - \|U\| \ | < \delta, \ \text{ and}$$
$$\|U_n\| \geq \frac{\|U\|}{2}$$

hold for every $n \geq m$.

Now for $n \geq m$ and $(x, y) \in V \times W$ we have

$$\begin{aligned}
|U_n(x, y)| &\leq K^2 \ \|F_n\| \ \|G_n\| \ \|x\|_{\varphi_n} \|y\|_{\psi_n} \\
&\leq K^2 \ (\|F\| \ \|G\| + \delta) \ \|x\|_{\varphi_n} \|y\|_{\psi_n} \\
&\leq K^2 \ (L \ \|U\| + \delta) \ \|x\|_{\varphi_n} \|y\|_{\psi_n} \\
&\leq K^2 \ (L \ \|U_n\| + \delta \ (1 + L)) \ \|x\|_{\varphi_n} \|y\|_{\psi_n} \\
&= K^2 \ (L \ \|U_n\| + (\frac{M}{K^2} - L) \frac{\|U\|}{2}) \ \|x\|_{\varphi_n} \|y\|_{\psi_n} \\
&\leq M \ \|U_n\| \ \|x\|_{\varphi_n} \|y\|_{\psi_n}.
\end{aligned}$$

□

As we noticed before Corollary 3.5, Lemma 1.1 remains true in the real setting. Then, given real or complex JB*-triples E, F, the mapping sending each element $U \in \mathcal{L}(^2(E \times F))$ to its unique separately weak*-continuous bilinear extension $\widetilde{U}$ to $E^{**} \times F^{**}$ is an isometry from $\mathcal{L}(^2(E \times F))$ onto the Banach space of all separately weak*-continuous bilinear forms on $E^{**} \times F^{**}$. Therefore we obtain the following corollary.

Corollary 3.8 *Let* $M > 4(1+2\sqrt{3})\ (1+3\sqrt{2})^2$ *(respectively,* $M > 4(1+2\sqrt{3})$*) and* E, F *be real (respectively, complex) JB*-triples. Then the set of all bounded bilinear forms* U *on* $E \times F$ *such that there exist norm-one functionals* $\varphi \in E^*$ *and* $\psi \in F^*$ *satisfying*

$$|U(x,y)| \leq M\ \|U\|\ \|x\|_\varphi\ \|y\|_\psi$$

for all $(x,y) \in E \times F$*, is norm dense in* $\mathcal{L}(^2(E \times F))$*.*

We note that Theorem 1.4 is finer than the complex case of the above corollary. However, since Theorem 1.4 depends on the proof of [BF1, Theorem 1.4], it is much more difficult.

Remark 3.9 *We do not know if the value* $\varepsilon = 0$ *is allowed in Theorems 2.1 and 2.9 (respectively, in Theorems 3.1 and 3.4) for some value of the constant* K *(respectively,* M*). Concerning this question, it is worth mentioning that the following three assertions are equivalent:*

1. *There is a universal constant* G *such that, for every real (respectively, complex) JBW*-triple* W *and every couple* (φ_1, φ_2) *of norm-one functionals in* $W_* \times W_*$*, we can find a norm-one functional* $\varphi \in W_*$ *satisfying*
$$\|x\|_{\varphi_i} \leq G\|x\|_\varphi$$
for every $x \in W$ *and* $i = 1, 2$*.*

2. *There is a universal constant* $\widehat{G}$ *such that for every couple of real (respectively, complex) JBW*-triples* (V, W) *and every separately weak*-continuous bilinear form* U *on* $V \times W$*, there are norm-one functionals* $\varphi \in V_*$*, and* $\psi \in W_*$ *satisfying*
$$|U(x,y)| \leq \widehat{G}\ \|U\|\ \|x\|_\varphi \|y\|_\psi$$
for all $(x,y) \in V \times W$*.*

3. *There is a universal constant* $\widetilde{G}$ *such that for every real (respectively, complex) JBW*-triple* W *and every weak*-continuous linear operator* T *from* W *to a real (respectively, complex) Hilbert space, there exists a norm-one functional* $\varphi \in W_*$ *satisfying*
$$\|T(x)\| \leq \widetilde{G}\ \|T\|\ \|x\|_\varphi$$
for all $x \in W$*.*

The implication $1 \Rightarrow 2$ *follows from Theorems 3.1 and 3.4.*

Assume that Assertion 2 *above is true. Let* W *be a real (respectively, complex) JBW*-triple,* H *a real (respectively, complex) Hilbert space, and* $T : W \to H$ *a weak*-continuous linear operator. Consider the separately weak*-continuous bilinear form* U *on* $W \times H$ *given by* $U(x,y) := (T(x)|y)$ *(respectively,* $U(x,y) := (T(x)|\sigma(y))$*, where* σ *is a conjugation on* H*). Regarding* H *as a JBW*-triple under the triple product* $\{x,y,z\} := \frac{1}{2}((x|y)z + (z|y)x)$*, and applying the assumption, we find norm-one functionals* $\varphi \in W_*$ *and* $\psi \in H_*$ *satisfying*

$$|U(x,y)| \leq \widehat{G}\ \|U\|\ \|x\|_\varphi \|y\|_\psi$$
$$\leq \widehat{G}\ \|T\|\ \|x\|_\varphi \|y\|$$

for all $(x,y) \in W \times H$*. Taking* $y = T(x)$ *(respectively,* $y = \sigma(T(x))$*) we obtain*

$$\|T(x)\| \leq \widehat{G}\|T\|\|x\|_\varphi$$

for all $x \in W$*. In this way Assertion* 3 *holds.*

Finally let us assume that Assertion 3 *is true. Let* W *be a real (respectively, complex) JBW*-triple and* φ_1, φ_2 *norm-one functionals in* W_**. Since* $\|.\|_{\varphi_1,\varphi_2}$ *comes from a suitable separately weak*-continuous positive sesquilinear form* $< .,. >$ *on* W *by means of the equality* $\|x\|^2_{\varphi_1,\varphi_2} =< x,x >$*, it follows from the proof of [R1, Corollary] that there exists a weak*-continuous linear operator* T *from* W *to a real (respectively, complex) Hilbert space satisfying* $\|x\|_{\varphi_1,\varphi_2} = \|T(x)\|$ *for all* $x \in W$ *(which implies* $\|T\| \leq \sqrt{2}$*). Now applying the assumption we find a norm one functional* $\varphi \in W_*$ *such that*

$$\|x\|_{\varphi_1,\varphi_2} = \|T(x)\| \leq \widetilde{G}\|T\|\|x\|_\varphi \leq \sqrt{2}\widetilde{G}\|x\|_\varphi$$

for all $x \in W$*. As a consequence, for* $i = 1,2$ *we have*

$$\|x\|_{\varphi_i} \leq \sqrt{2}\widetilde{G}\|x\|_\varphi$$

for all $x \in W$*.*

4 Some Applications

We define the strong*-topology $S^*(W, W_*)$ of a given real or complex JBW*-triple W as the topology on W generated by the family of seminorms $\{\|.\|_\varphi :$

$\varphi \in W_*,\ \|\varphi\| = 1\}$. In the complex case, the above notion has been introduced by T. J. Barton and Y. Friedman in [BF2]. When a JBW*-algebra $\mathcal{A}$ is regarded as a complex JBW*-triple, $S^*(\mathcal{A}, \mathcal{A}_*)$ coincides with the so-called "algebra-strong* topology" of $\mathcal{A}$, namely the topology on $\mathcal{A}$ generated by the family of seminorms of the form $x \mapsto \sqrt{\xi(x \circ x^*)}$ when ξ is any positive functional in $\mathcal{A}_*$ [R1, Proposition 3]. As a consequence, when a von Neumann algebra $\mathcal{M}$ is regarded as a complex JBW*-triple, $S^*(\mathcal{M}, \mathcal{M}_*)$ coincides with the familiar strong*-topology of $\mathcal{M}$ (compare [S, Definition 1.8.7]).

We note that, if $\mathcal{W}$ is a complex JBW*-triple, then, denoting by $\mathcal{W}_{\mathbb{R}}$ the realification of $\mathcal{W}$ (i.e., the real JBW*-triple obtained from $\mathcal{W}$ by restriction of scalar to $\mathbb{R}$), we have $S^*(\mathcal{W}, \mathcal{W}_*) = S^*(\mathcal{W}_{\mathbb{R}}, (\mathcal{W}_{\mathbb{R}})_*)$. Indeed, the mapping $\varphi \mapsto \Re e\, \varphi$ identifies $\mathcal{W}_*$ with $(\mathcal{W}_{\mathbb{R}})_*$, and, when φ has norm one, the equality $\|x\|_\varphi = \|x\|_{\Re e\, \varphi}$ holds for every $x \in \mathcal{W}$.

Proposition 4.1 *Let W be a real (respectively, complex) JBW*-triple. The following topologies coincide in W:*

1. *The strong*-topology of W.*

2. *The topology on W generated by the family of seminorms of the form $x \mapsto \sqrt{< x, x >}$, where $< .,. >$ is any separately weak*-continuous positive sesquilinear form on W.*

3. *The topology on W generated by the family of seminorms $x \mapsto \|T(x)\|$, when T runs over all weak*-continuous linear operators from W to arbitrary real (respectively, complex) Hilbert spaces.*

Proof. Let us denote by τ_1, τ_2, and τ_3 the topologies arising in paragraphs 1, 2, and 3, respectively. The inequality $\tau_1 \geq \tau_3$ follows from Corollary 2.10 (respectively, Corollary 2.2). Since the proof of [R1, Corollary 1] shows that for every separately weak*-continuous positive sesquilinear form $< .,. >$ on W there exists a weak*-continuous linear operator T from W to a real (respectively, complex) Hilbert space satisfying $\sqrt{< x, x >} = \|T(x)\|$ for all $x \in W$, we have $\tau_3 \geq \tau_2$. Finally, since for every norm-one functional $\varphi \in W_*$ there is a separately weak*-continuous positive sesquilinear form $< .,. >$ satisfying $\|x\|_\varphi = \sqrt{< x, x >}$ for all $x \in W$, the inequality $\tau_2 \geq \tau_1$ follows. □

For every Banach space X, B_X will stand for the closed unit ball of X. For every dual Banach space X (with a fixed predual denoted by X_*), we denote by $m(X, X_*)$ the Mackey topology on X relative to its duality with X_*.

Corollary 4.2 *Let W be a real or complex JBW*-triple. Then the strong*-topology of W is compatible with the duality (W, W_*).*

Proof. We apply the characterization of $S^*(W, W_*)$ given by paragraph 3 in Proposition 4.1. Clearly $S^*(W, W_*)$ is stronger than the weak*-topology $\sigma(W, W_*)$ of W. On the other hand, if T is a weak*-continuous linear operator from W to a Hilbert space H, and if we put $T = S^*$ for a suitable bounded linear operator $S : H_* \to W_*$, then $S(B_{H_*})$ is an absolutely convex and weakly compact subset of W_* and we have $\|T(x)\| = \sup | < x, S(B_{H_*}) > |$. This shows that $S^*(W, W_*)$ is weaker than $m(W, W_*)$. □

The complex case of the above corollary is due to T. J. Barton and Y. Friedman [BF2]. The complex case of Proposition 4.1 is claimed in [R1, Corollary 2] (see also [R2, Proposition D.17]), but the proof relies on [R1, Proposition 1], which subsumes a gap from [BF1] (see the comments before Lemma 1.5). Now that we have saved [R1, Corollary 2], all subsequent results in [R1] concerning the strong*-topology of complex JBW*-triples are valid. Moreover, keeping in mind Proposition 4.1 and Corollary 4.2, some of those results remain true for real JBW*-triples with verbatim proof. For instance, the following assertions hold:

1. Linear mappings between real JBW*-triples are strong*-continuous if and only if they are weak*-continuous (compare [R1, Corollary 3]).

2. If W is a real JBW*-triple, and if V is a weak*-closed subtriple, then the inequality $S^*(W, W_*)|_V \leq S^*(V, V_*)$ holds, and in fact $S^*(W, W_*)|_V$ and $S^*(V, V_*)$ coincide on bounded subsets of V (compare [R1, Proposition 2]).

It follows from the first part of Assertion 2 above and a new application of Proposition 4.1 that, if W is a real JBW*-triple, and if V is a weak*-complemented subtriple of W, then we have $S^*(W, W_*)|_V = S^*(V, V_*)$. Since every real JBW*-triple V is weak*-complemented in the realification of a complex JBW*-triple $\mathcal{W}$ (see V as a real form of its JB*-complexification), and $S^*(\mathcal{W}, \mathcal{W}_*) = S^*(\mathcal{W}_{\mathbb{R}}, (\mathcal{W}_{\mathbb{R}})_*)$, the results [R1, Theorem] and [R2, Theorem D.21] for complex JBW*-triples can be transferred to the real setting, providing the following result.

Theorem 4.3 *Let W be a real JBW*-triple. Then the triple product of W is jointly $S^*(W,W_*)$-continuous on bounded subsets of W, and the topologies $m(W,W_*)$ and $S^*(W,W_*)$ coincide on bounded subsets of W.*

Our concluding goal in this paper is to establish, in the setting of real JB*-triples, a result on weakly compact operators originally due to H. Jarchow [J] in the context of C*-algebras, and later extended to complex JB*-triples by C-H. Chu and B. Iochum [CI]. This could be made by transferring the complex results to the real setting by a complexification method. However, we prefer to do it in a more intrinsic way, by deriving the result from the second assertion in Theorem 4.3 according to some ideas outlined in [R2, pp. 142-143].

Proposition 4.4 *Let X be a dual Banach space (with a fixed predual X_*). Then the Mackey topology $m(X,X_*)$ coincides with the topology on X generated by the family of semi-norms $x \mapsto \|T(x)\|$, where T is any weak*-continuous linear operator from X to a reflexive Banach space.*

Proof. Let us denote by τ the second topology arising in the statement. As in the proof of Corollary 4.2, if T is a weak*-continuous linear operator from X to a reflexive Banach space, then there exists an absolutely convex and weakly compact subset D of X_* such that the equality

$$\|T(x)\| = \sup |< x, D >|$$

holds for every $x \in X$. This shows that $\tau \leq m(X,X_*)$.

Let D be an absolutely convex and weakly compact subset of X_*. Consider the Banach space $\ell_1(D)$ and the bounded linear operator

$$F : \ell_1(D) \to X_*$$

given by

$$F(\{\lambda_\varphi\}_{\varphi\in D}) := \sum_{\varphi\in D} \lambda_\varphi \varphi.$$

Then we have $F(B_{\ell_1(D)}) = D$, and hence F is weakly compact. By [DFJP] there exists a reflexive Banach space Y together with bounded linear operators $S : \ell_1(D) \to Y$, $R : Y \to X_*$ such that $F = R\,S$. Then, for $x \in X$, we have

$$\sup |< x, D >| = \sup |< x, F(B_{\ell_1(D)}) >|$$

$$= \sup | < x, R(S(B_{\ell_1(D)})) > | \leq \|S\| \ \sup | < x, R(B_Y) > |$$
$$= \|S\| \ \|R^*(x)\|.$$

Since D is an arbitrary absolutely convex and weakly compact subset of X_*, and R^* is a weak*-continuous linear operator from X to the reflexive Banach space Y^*, the inequality $m(X, X_*) \leq \tau$ follows.□

Let X be a dual Banach space (with a fixed predual X_*). In agreement with Proposition 4.1, we define the strong*-topology of X, denoted by $S^*(X, X_*)$, as the topology on X generated by the family of semi-norms $x \mapsto \|T(x)\|$, where T is any weak*-continuous linear operator from X to a Hilbert space.

Proposition 4.5 *Let X be a dual Banach space (with a fixed predual X_*). Then the following assertions are equivalent:*

1. *The topologies $m(X, X_*)$ and $S^*(X, X_*)$ coincide on bounded subsets of X.*

2. *For every weak*-continuous linear operator F from X to a reflexive Banach space, there exists a weak*-continuous linear operator G from X to a Hilbert space satisfying $\|F(x)\| \leq \|G(x)\| + \|x\|$ for all $x \in X$.*

3. *For every weak*-continuous linear operator F from X to a reflexive Banach space, there exist a weak*-continuous linear operator G from X to a Hilbert space and a mapping $N : (0, \infty) \to (0, \infty)$ satisfying*

$$\|F(x)\| \leq N(\varepsilon) \ \|G(x)\| + \varepsilon \ \|x\|$$

for all $x \in X$ and $\varepsilon > 0$.

Proof. $1 \Rightarrow 2$.– Let F be a weak*-continuous linear operator from X to a reflexive Banach space. Then, by Proposition 4.4

$$\mathcal{O} := \{y \in B_X : \|F(y)\| \leq 1\}$$

is a $m(X, X_*)|_{B_X}$-neighborhood of zero in B_X. By assumption, there exist Hilbert spaces $H_1, \ldots, H_n$ and weak*-continuous linear operators $G_i : X \to H_i$ $(i : 1, \ldots, n)$ such that

$$\mathcal{O} \supseteq \cap_{i=1}^n \{y \in B_X : \|G_i(y)\| \leq 1\}.$$

Now set $H := (\bigoplus_{i=1}^{n} H_i)_{\ell_2}$, and consider the weak*-continuous linear operator $G : X \to H$ defined by $G(x) := (G_1(x), \dots , G_n(x))$. Notice that

$$\{y \in B_X : \|G(y)\| \leq 1\} \subseteq \cap_{i=1}^{n}\{y \in B_X : \|G_i(y)\| \leq 1\} \subseteq \mathcal{O}.$$

Finally, if $x \in X \setminus \{0\}$, then $\frac{1}{\|x\|+\|G(x)\|}\, x$ lies in $\{y \in B_X : \|G(y)\| \leq 1\} \subseteq \mathcal{O}$, and hence $\|F(\frac{1}{\|x\|+\|G(x)\|}\, x)\| \leq 1$.

2 ⇒ 3.– Let F be a weak*-continuous linear operator from X to a reflexive Banach space. By assumption, for every $n \in \mathbb{N}$ there exists a Hilbert space H_n and a weak*-continuous linear operator G_n from X to H_n such that $\|nF(x)\| \leq \|G_n(x)\| + \|x\|$ for all $x \in X$. Now set $H := (\bigoplus_{n\in\mathbb{N}} H_n)_{\ell_2}$, and consider the bounded linear operator $G : X \to H$ defined by $G(x) := \{\frac{1}{n\|G_n\|}\, G_n(x)\}$ and the mapping $N : \varepsilon \to \|G_{n(\varepsilon)}\|$ (where $n(\varepsilon)$ denotes the smallest natural number satisfying $n > \frac{1}{\varepsilon}$). Then G is weak*-continuous. Indeed, given $y = \{h_n\} \in H$, we can take for $n \in \mathbb{N}$ α_n in X_* satisfying $(G_n(x)|h_n) =< x, \alpha_n >$ for every $x \in X$, so that we have

$$\sum_{n\in\mathbb{N}} \|\frac{\alpha_n}{n\|G_n\|}\| \leq \sum_{n\in\mathbb{N}} \frac{\|h_n\|}{n} \leq \sqrt{\sum_{n\in\mathbb{N}} \|h_n\|^2}\ \sqrt{\sum_{n\in\mathbb{N}} \frac{1}{n^2}} < \infty,$$

and hence $\alpha := \sum_{n\in\mathbb{N}} \frac{\alpha_n}{n\ \|G_n\|}$ is an element of X_* satisfying $(G(x)|h) =< x, \alpha >$ for all $x \in X$. Moreover, for all $\varepsilon > 0$ and $x \in X$ we have

$$\|F(x)\| \leq \frac{1}{n(\varepsilon)}\|G_{n(\varepsilon)}(x)\| + \frac{1}{n(\varepsilon)}\|x\|$$

$$\leq \|G_{n(\varepsilon)}\|\ \|G(x)\| + \frac{1}{n(\varepsilon)}\|x\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|.$$

3 ⇒ 1.– Let x_λ be a net in B_X converging to zero in the topology $S^*(X, X_*)$. Let F be a weak*-continuous linear operator from X to a reflexive Banach space, and $\varepsilon > 0$. By assumption, there exist a weak*-continuous linear operator G from X to a Hilbert space and a mapping $N : (0, \infty) \to (0, \infty)$ satisfying

$$\|F(x)\| \leq N(\frac{\varepsilon}{2})\ \|G(x)\| + \frac{\varepsilon}{2}\ \|x\|$$

for all $x \in X$. Take λ_0 such that $\|G(x_\lambda)\| \leq \frac{\varepsilon}{2\ N(\frac{\varepsilon}{2})}$ whenever $\lambda \geq \lambda_0$. Then we have $\|F(x_\lambda)\| \leq \varepsilon$ for all $\lambda \geq \lambda_0$. By Proposition 4.4, x_λ $m(X, X_*)$-converges to zero. □

We can now state the following characterization of weakly compact operators on JB*-triples.

Theorem 4.6 *Let E be a real (respectively, complex) JB*-triple, X a real (respectively, complex) Banach space, and $T : E \to X$ a bounded linear operator. The following assertions are equivalent:*

1. *T is weakly compact.*

2. *There exist a bounded linear operator G from E to a real (respectively, complex) Hilbert space and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.

3. *There exist norm one functionals $\varphi_1, \varphi_2 \in E^*$ and a function $N : (0, +\infty) \to (0, +\infty)$ such that*
$$\|T(x)\| \leq N(\varepsilon)\ \|x\|_{\varphi_1,\varphi_2} + \varepsilon\|x\|$$
for all $x \in E$ and $\varepsilon > 0$.

Proof. The implication $2 \Rightarrow 3$ follows from Corollary 2.10 (respectively, Corollary 2.2). The implication $3 \Rightarrow 2$ holds because, for norm-one functionals $\varphi_1, \varphi_2 \in E^*$, $\|.\|_{\varphi_1,\varphi_2}$ is a prehilbert seminorm on E, and hence there exists a bounded linear operator G from E to a Hilbert space satisfying $\|G(x)\| = \|x\|_{\varphi_1,\varphi_2}$ for all $x \in E$. On the other hand, the implication $2 \Rightarrow 1$ is known to be true, even if E is an arbitrary Banach space (see for instance [J, Theorem 20.7.3]). To conclude the proof, let us show that 1 implies 2. Assume that Assertion 1 holds. Then, by [DFJP], there exist a reflexive Banach space Y and bounded linear operators $F : E \to Y$ and $S : Y \to X$ such that $T = S\ F$ and $\|S\| \leq 1$. By Theorem 4.3 and Proposition 4.5, there exist a weak*-continuous linear operator $\widetilde{G}$ from E^{**} to a Hilbert space and a mapping $N : (0, \infty) \to (0, \infty)$ satisfying
$$\|F^{**}(\alpha)\| \leq N(\varepsilon)\ \|\widetilde{G}(\alpha)\| + \varepsilon\ \|\alpha\|$$
for all $\alpha \in E^{**}$ and $\varepsilon > 0$. By putting $G := \widetilde{G}|_E$, the inequality in Assertion 2 follows. □

The complex case of the above theorem is established in [CI, Theorem 11], with $\|.\|_{\varphi_1,\varphi_2}$ in Assertion 3 replaced with $\|.\|_{\varphi}$ for a single norm-one functional $\varphi \in E^*$. As we have noticed in similar occasions, we do not know if such a replacement is correct.

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